The torsion subgroup of the Whitehead group of graded and valued division algebras

Huynh Viet Khanh

Joint work with Nguyen Duc Anh Khoa and Adrian R. Wadsworth

Abstract. Let D be a division algebra finite-dimensional over its centre K. The reduced Whitehead group $\mathrm{SK}_1(D)$ has long been studied by valuation-theoretic methods. In particular, Hazrat and Wadsworth showed that, for tame division algebras over Henselian fields, computations of $\mathrm{SK}_1(D)$ can be reduced to the associated graded division algebra. In this talk I will discuss the analogous questions for the torsion subgroup $\mathrm{TK}_1(D) = \tau(K_1(D))$ of the Whitehead group.

The first part of the talk will be about graded division algebras. Let E be a graded division algebra finite-dimensional over its centre T, with torsion-free grade group. Replacing the reduced-norm-one subgroup $E^{(1)}$ by the subgroup E^{μ} whose image in $K_1(E)$ is torsion, we obtain exact sequences for $\mathrm{TK}_1(E)$ parallel to the Hazrat–Wadsworth exact sequences for $\mathrm{SK}_1(E)$. These yield formulas in the unramified, totally ramified, and semiramified cases. In the semiramified case, for example, a torsion-norm analogue of the usual Tate cohomology term appears.

The second part concerns valued division algebras. For a tame division algebra D over a Henselian centre, we determine the principal-unit obstruction to the congruence theorem for $\mathrm{TK}_1(D)$ and obtain an exact sequence relating $\mathrm{TK}_1(D)$ to $\mathrm{TK}_1(\mathrm{gr}(D))$. This also gives the corrected form of the obstruction computation in earlier work of Khanh and Khoa.

Finally, I will explain a stability theorem. If $Q = q(E)$ is the quotient division algebra of a graded division algebra E, then the natural map $K_1(E) \to K_1(Q)$ has torsion-free cokernel. Consequently $\mathrm{SK}_1(E) \cong \mathrm{SK}_1(Q)$ and $\mathrm{TK}_1(E) \cong \mathrm{TK}_1(Q)$. This allows one to transfer several results on ordinary division algebras, including Motiee's primary decomposition and injectivity results, to the graded setting.

References.

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