Slice Geometry, Fox Linearization, and Surjectivity of Commutator-Type Word Maps

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Abstract

Let $F_2 = \langle a, b\rangle$ and consider commutator-type words $w = [a, u(a, b)]$. On SU(2) we prove an exact fixed-slice criterion and a quantitative image theorem: a parameter $\Delta(u)$ controls an explicit geodesic ball contained in the image, with $\Delta(u) = 0$ equivalent to surjectivity and $\Delta(u) \leq 2$ forcing the trace-zero class. Combined with the image-ball structure of arbitrary SU(2) word maps, this gives a variational reformulation of the Belovs–Montoya–Yakaryılmaz conjecture and an outer-commutator certificate for exact two-state quantum word separation. Algebraically, the boundary degree and dihedral exponent are the specializations $F_u(1)$ and $F_u(-1)$ of one Fox polynomial, and we prove $p_{[a,u]}(t) = (t-1)F_u(t)$, yielding cyclotomic surjectivity criteria on SU(n). Finally, SU(2)-surjectivity transfers to SO(n) for $n \not\equiv 2 \pmod 4$, with full transfer for nontrivial abelianization, while in dimension four scalar trace-surjectivity, word-surjectivity, and SU(2)-surjectivity are equivalent.

Keywords. Word map; trace polynomial; SU(2); SO(n); commutator; maximal torus; Fox calculus; word image; quantum finite automata.
2020 Mathematics Subject Classification. 20G20, 22E46, 15A15, 20F10.

1 Introduction

Let $F_r = \langle x_1, \ldots, x_r\rangle$ be the free group of rank r. For a group G and a word $w \in F_r$, evaluation gives the word map $w_G : G^r \to G$. Understanding the image of such maps is a classical problem linking group theory, algebraic geometry, representation theory, and Lie theory. Borel proved that every nontrivial word map on a connected semisimple algebraic group over an algebraically closed field is dominant [7], but surjectivity on compact real forms is much more rigid. Gotô's theorem that every element of a compact connected semisimple Lie group is a commutator [13] is the basic model, and Elkasapy–Thom later extended the Weyl-group mechanism to broad classes of words on SU(n) through Laurent polynomials attached to the word [9]. More recently, the subject has also developed in a quantitative direction through word measures and distributional questions; see, for example, the small-ball and Fourier estimates of Avni–Glazer–Larsen [3] and Avni–Glazer [2].

Rank one permits a complementary and much more explicit description. On SU(2), conjugacy classes are completely determined by trace, while Fricke–Klein–Horowitz coordinates turn traces of words into polynomial functions. Khoi and Toan used this structure to obtain a necessary and sufficient trace-polynomial criterion for surjectivity of arbitrary word maps on SU(2) and to treat several double-commutator families [17]. Related work of Jezernik–Sánchez and Bien–Ramezan-Nassab–Trung studies surjectivity for PSL_2 over algebraically closed fields, with the latter

emphasizing the derived and metabelian layers [16, 6]; Rani has recently obtained further explicit SU(2) and $\mathrm{SL}_2(\mathbb{C})$ families by trace methods [19]. At the same time, Gordeev and Plotkin have highlighted the relation between word maps and representation varieties of one-relator groups [12]. These developments suggest that rank-one trace geometry, derived-series information, and higher-rank Weyl geometry should be viewed as complementary descriptions rather than separate techniques.

A closely related motivation comes from quantum finite automata. Belovs–Montoya–Yakaryılmaz conjectured that every pair of distinct words can be separated with zero error by a two-state measure-once quantum finite automaton [5]. Their SU(2) formulation shows that, for $\omega = xy^{-1}$, exact separation of x and y is equivalent to the word image of ω meeting the trace-zero conjugacy class. The image-ball principle proved below turns this into the exact variational condition $m(\omega) \le 0$, where $m(\omega)$ is the minimum trace of the word map; hence the Belovs–Montoya–Yakaryılmaz conjecture itself becomes the assertion $m(xy^{-1}) \le 0$ for every distinct pair $x, y \in \{a, b\}^*$. Chen and Wu recently developed a complementary slice-based approach using Fox calculus, metabelian polynomials, and explicit dihedral and quaternionic specializations to produce trace-zero witnesses for broad classes of positive-word differences [8]. Their work emphasizes precisely the distinction relevant here: trace-zero intersection is weaker than global surjectivity, but is already sufficient for exact quantum word separation.

The present paper focuses on the outer-commutator class $w(a, b) = [a, u(a, b)]$, with the convention $[x, y] = xyx^{-1}y^{-1}$. The central observation is that the outer commutator forces a remarkably rigid rank-one geometry. If $A = i\sigma_3$, then $[a, u]$ is surjective on SU(2) precisely when the one-variable slice $u(A, \mathrm{SU}(2))$ meets the great circle of purely off-diagonal matrices. In trace coordinates this becomes the simultaneous vanishing of two explicit polynomials on the character disk. More importantly, the same slice carries a quantitative parameter $\Delta(u)$ for which the image contains every conjugacy class of trace at least $\Delta(u) - 2$. The two natural thresholds are therefore sharply separated: $\Delta(u) = 0$ is exactly the surjectivity threshold, while $\Delta(u) \le 2$ is a sufficient trace-zero threshold. Consequently, when a positive-word difference xy^{-1} has the form $[a, u]$, the inequality $\Delta(u) \le 2$ gives a direct certificate for the Belovs–Montoya–Yakaryılmaz conclusion, even in cases where full surjectivity is not known.

The same outer-commutator structure also explains why two apparently different SU(2) mechanisms in this paper belong to one algebraic picture. On a commuting boundary slice the relevant invariant is the exponent sum $\exp_b(u)$, detected by a topological degree, whereas on an anticommuting slice the relevant quantity is the alternating exponent $N_{\mathrm{alt}}(u)$. For an alternating word $u = a^{m_1}b^{n_1}\cdots a^{m_k}b^{n_k}$, both arise from the one-variable Fox polynomial $F_u(t) = \sum_j n_j t^{m_1+\cdots+m_j}$: one has $F_u(1) = \exp_b(u)$ and $F_u(-1) = N_{\mathrm{alt}}(u)$. The key structural identity proved below is $p_{[a,u]}(t) = (t-1)F_u(t)$, where p_w is the Laurent polynomial appearing in the Gotô–Elkasapy–Thom method. Consequently, the dihedral criterion $F_u(-1) \neq 0$ is exactly the nontrivial second-root specialization of the higher-rank Weyl criterion, while the boundary-degree criterion reads the first derivative at the forced root $t = 1$, since $p'_{[a,u]}(1) = F_u(1)$. This factorization also yields a direct cyclotomic extension: if $F_u(\zeta) \neq 0$ at every nontrivial nth root of unity, then $[a, u]$ is surjective on SU(n). In particular, one recovers the SU(2) classification for $[a, [a^m, b^q]]$ as the rank-one member of the higher-dimensional condition $q \neq 0$ and $\gcd(m, n) = 1$.

The Fox description also makes clear where first-order methods stop. For the higher-order family $[a, [[a, b], [a, b^q]]]$, the inner word belongs to the second derived subgroup and the Fox polynomial vanishes identically, so both the metabelian polynomial and its rank-one specializations are blind. Nevertheless, when $q \equiv 2 \pmod 3$ a finite-order slice still gives surjectivity on SU(2). This provides a concrete transition from the first Fox/metabelian layer to genuinely higher-order slice geometry and motivates one of the open problems at the end of the paper.

The second part of the paper studies how the rank-one information propagates to orthogonal groups. For an arbitrary word, surjectivity on SU(2) implies surjectivity on SO(n) for every

$n \geq 3$ with $n \not\equiv 2 \pmod 4$, by combining the coverings of SO(3) and SO(4) with block-diagonal maximal tori. If the abelianization of the word is nontrivial, an SO(2) block supplies the missing dimensions and gives transfer to every $\mathrm{SO}(n)$. Dimension four is exceptional in a useful way: the decomposition $\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$ implies that SU(2)-surjectivity, SO(4)-surjectivity, and scalar trace-surjectivity on SO(4) are equivalent. Beyond rank one and this exceptional rank-two case, scalar trace is no longer a complete conjugacy invariant. We therefore use traces of powers together with a Pfaffian invariant to formulate full word-surjectivity through complete conjugacy coordinates, and we exhibit a family whose scalar trace is surjective on SO(5) although the same word is not surjective on SU(2).

The resulting picture is therefore unified rather than a collection of unrelated criteria. Rank-one trace geometry produces an exact fixed-slice test and a quantitative image bound; Fox linearization identifies the algebraic content of the two basic slice mechanisms; the factorization with the Gotô polynomial turns that rank-one information into cyclotomic criteria on $\mathrm{SU}(n)$; and low-dimensional coverings and maximal tori then transfer a substantial part of the picture to $\mathrm{SO}(n)$. The higher-order examples show at the same time that the Fox layer is not the end of the story, leaving a concrete class of genuinely nonlinear problems beyond the first metabelian approximation.

2 Preliminaries

2.1 Trace polynomials and the $\mathrm{SU}(2)$ character disk

Throughout, $[x, y] = xyx^{-1}y^{-1}$. For $A, B \in \mathrm{SL}_2(\mathbb{C})$, put $x = \operatorname{Tr} A, \quad y = \operatorname{Tr} B, \quad z = \operatorname{Tr}(AB)$. A classical theorem of Fricke–Klein–Horowitz implies that for every $v \in F_2$ there is a unique polynomial $P_v(x, y, z) \in \mathbb{Z}[x, y, z]$ such that $\operatorname{Tr}(v(A, B)) = P_v(x, y, z)$; see [11, 15, 20]. We shall repeatedly use $P_{uv} = P_{vu}, \quad P_{uv} + P_{uv^{-1}} = P_u P_v$, and the Fricke identity

$$P_{[u,v]} = P_u^2 + P_v^2 + P_{uv}^2 - P_u P_v P_{uv} - 2. \tag{1}$$

Write

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

For $A, B \in \mathrm{SU}(2)$, the trace coordinates (x, y, z) fill the compact character region $\mathcal{T} = \{(x, y, z) \in [-2, 2]^3 : x^2 + y^2 + z^2 - xyz \leq 4\}$. Fix $A = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Then $\operatorname{Tr} A = 0$, and for $B = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$ one has $y = \operatorname{Tr} B = 2\Re\alpha$ and $z = \operatorname{Tr}(AB) = -2\Im\alpha$. Hence the slice $x = 0$ is exactly $D_2 = \{(y, z) \in \mathbb{R}^2 : y^2 + z^2 \leq 4\}$.

2.2 The extremal trace criterion on $\mathrm{SU}(2)$

We recall two elementary facts which underlie the paper.

Lemma 2.1. *For $X, Y \in \mathrm{SU}(2)$, $[X, Y] = -I$ if and only if $\operatorname{Tr} X = \operatorname{Tr} Y = \operatorname{Tr}(XY) = 0$.*

Proof. If $[X, Y] = -I$, then $XY = -YX$. Hence $YXY^{-1} = -X$, and therefore $\operatorname{Tr} X = 0$; similarly $\operatorname{Tr} Y = 0$. Thus $X^2 = Y^2 = -I$, and conjugating XY by X sends XY to $-XY$, so $\operatorname{Tr}(XY) = 0$. Conversely, trace zero in SU(2) implies $X^2 = Y^2 = (XY)^2 = -I$. Hence $XYXY = -I$, and using $X^{-1} = -X$ and $Y^{-1} = -Y$ gives $[X, Y] = -I$. □

Proposition 2.2 (Extremal trace criterion)**.** *For an arbitrary word $w \in F_2$, the following are equivalent:* (i) *$w : \mathrm{SU}(2)^2 \to \mathrm{SU}(2)$ is surjective;* (ii) *the trace map $\operatorname{Tr} w$ attains the value -2; and* (iii) *the minimum of $\operatorname{Tr}(w(A, B))$ over $A, B \in \mathrm{SU}(2)$ is -2.*

Proof. The image of $\operatorname{Tr} w$ is a compact interval because $\mathrm{SU}(2)^2$ is connected. It contains 2, since $w(I, I) = I$. If it also contains -2, then it contains all of $[-2, 2]$. Conjugacy classes in $\mathrm{SU}(2)$ are determined by the trace and the image of a word map is conjugation invariant. Thus every conjugacy class occurs. □

This is the rank-one trace mechanism exploited systematically in [17].

Proposition 2.3 (Image-ball principle)**.** *Let $w \in F_r$, and let $m(w)$ denote the minimum of $\operatorname{Tr} w(A_1, \dots, A_r)$ over $(A_1, \dots, A_r) \in \mathrm{SU}(2)^r$. Then $w(\mathrm{SU}(2)^r) = \{U \in \mathrm{SU}(2) : \operatorname{Tr} U \geq m(w)\}$. Equivalently, under the standard identification $\mathrm{SU}(2) \simeq S^3$, the image of every word map is a closed geodesic ball centered at the identity.*

Proof. The continuous image of the connected space $\mathrm{SU}(2)^r$ under $\operatorname{Tr} \circ w$ is an interval. It contains 2 because $w(I, \dots, I) = I$, and compactness shows that it is exactly $[m(w), 2]$. The image of a word map is invariant under simultaneous conjugation of the inputs, hence under conjugation of the output. Since conjugacy classes in $\mathrm{SU}(2)$ are determined by trace, every conjugacy class with trace in $[m(w), 2]$ is contained in the image, and no other class can occur. □

Proposition 2.4 (Variational form of the Belovs–Montoya–Yakaryılmaz conjecture)**.** *Let $x, y \in \{a, b\}^*$ be distinct words and put $\omega = xy^{-1}$. In the standard two-state measure-once quantum finite automaton model, x and y can be separated exactly if and only if $m(\omega) \leq 0$, where $m(\omega) = \min_{A,B \in \mathrm{SU}(2)} \operatorname{Tr} \omega(A, B)$. Consequently, the Belovs–Montoya–Yakaryılmaz conjecture is equivalent to the family of inequalities $m(xy^{-1}) \leq 0$ for all distinct $x, y \in \{a, b\}^*$.*

Proof. Belovs–Montoya–Yakaryılmaz show that exact separation of x and y by a two-state measure-once quantum finite automaton is equivalent to the image of the word map associated with $\omega = xy^{-1}$ containing a rotation through angle $\pi/2$, equivalently an element of trace zero [5]. By Proposition 2.3, the trace values attained by ω form the interval $[m(\omega), 2]$. This interval meets zero if and only if $m(\omega) \leq 0$, which proves both assertions. □

Corollary 2.5 (Exact Haar size of the image)**.** *Write $m(w) = 2\cos\theta_w, \qquad 0 \leq \theta_w \leq \pi$, and let μ be normalized Haar measure on $\mathrm{SU}(2) \simeq S^3$. Then $\mu\big(w(\mathrm{SU}(2)^r)\big) = \frac{\theta_w - \frac{1}{2}\sin(2\theta_w)}{\pi}$.*

Proof. In polar coordinates about $I \in S^3$, the volume density is proportional to $\sin^2\theta$. Hence the normalized volume of a geodesic ball of radius θ_w is

$$\frac{\int_0^{\theta_w} \sin^2\theta \, d\theta}{\int_0^{\pi} \sin^2\theta \, d\theta} = \frac{\theta_w - \frac{1}{2}\sin(2\theta_w)}{\pi}.$$

□

This elementary reformulation places the yes/no surjectivity question inside the broader study of the size and distribution of word images on compact groups; compare the recent word-measure results in [3, 2].

2.3 Maximal tori and trace ranges in $\mathrm{SO}(n)$

Let

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Every element of $\mathrm{SO}(2r)$ is conjugate to $\operatorname{diag}(R(\theta_1), \dots, R(\theta_r))$, and every element of $\mathrm{SO}(2r+1)$ is conjugate to $\operatorname{diag}(R(\theta_1), \dots, R(\theta_r), 1)$. Therefore

$$\operatorname{Tr}(\mathrm{SO}(n)) = \begin{cases} [-n, n], & n \text{ even}, \\ [2-n, n], & n \text{ odd}. \end{cases} \tag{2}$$

We write $m_n = -n$ for even n and $m_n = 2 - n$ for odd n.

3 Exact fixed-slice criteria on $\mathrm{SU}(2)$

Throughout this section, $w(a,b) = [a, u(a,b)]$.

3.1 The off-diagonal criterion

Define

$$E = \left\{ \begin{pmatrix} 0 & \beta \\ -\overline{\beta} & 0 \end{pmatrix} : |\beta| = 1 \right\}.$$

Theorem 3.1 (Off-diagonal criterion)**.** *Let $A = i\sigma_3$. Then $[a,u]$ is surjective on $\mathrm{SU}(2)$ if and only if $u(A, \mathrm{SU}(2)) \cap E \neq \varnothing$. Equivalently, there is $B \in \mathrm{SU}(2)$ such that*

$$u(A,B) = \begin{pmatrix} 0 & \beta \\ -\overline{\beta} & 0 \end{pmatrix}, \qquad |\beta| = 1.$$

Proof. Assume first that w is surjective. By Proposition 2.2, there are $A_0, B_0 \in \mathrm{SU}(2)$ such that $[A_0, u(A_0,B_0)] = -I$. Lemma 2.1 gives $\operatorname{Tr} A_0 = 0$. Every trace-zero element of $\mathrm{SU}(2)$ is conjugate to $A = i\sigma_3$, so after simultaneous conjugation we may suppose $A_0 = A$. Put

$$U = u(A,B_0) = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}.$$

Again by Lemma 2.1, $\operatorname{Tr} U = 0, \qquad \operatorname{Tr}(AU) = 0$. But $\operatorname{Tr} U = 2\Re\alpha, \qquad \operatorname{Tr}(AU) = -2\Im\alpha$. Hence $\alpha = 0$, so $U \in E$. Conversely, if $U = u(A,B) \in E$, then $\operatorname{Tr} A = \operatorname{Tr} U = \operatorname{Tr}(AU) = 0$. Lemma 2.1 gives $[A,U] = -I$, and Proposition 2.2 yields surjectivity. □

The criterion shows why global surjectivity of u is not the relevant condition.

Remark 3.2. The implication $u(\mathrm{SU}(2)^2) = \mathrm{SU}(2) \implies [a,u]$ surjective is false: for $u = a^m$, $m \neq 0$, the map u is surjective but $[a,u] = I$. On the other hand, if there exists $A_0 \in \mathrm{SU}(2)$ with $\operatorname{Tr} A_0 = 0, \qquad u(A_0, \mathrm{SU}(2)) = \mathrm{SU}(2)$, then $[a,u]$ is surjective. Indeed, after conjugating A_0 to $i\sigma_3$, the corresponding slice contains E.

3.2 Trace-polynomial reformulation

Define $R_u(y,z) = P_u(0,y,z), \qquad S_u(y,z) = P_{au}(0,y,z)$.

Proposition 3.3. *For $w = [a,u]$, the following are equivalent:*
(i) *w is surjective on $\mathrm{SU}(2)$;*
(ii) *$u(i\sigma_3, \mathrm{SU}(2)) \cap E \neq \varnothing$;*
(iii) *$Z(R_u) \cap Z(S_u) \cap D_2 \neq \varnothing$.*

Proof. Only (ii)⇔(iii) remains. For $(y,z) \in D_2$, choose $B \in \mathrm{SU}(2)$ with $y = \operatorname{Tr} B, \quad z = \operatorname{Tr}(AB)$. Then $R_u(y,z) = \operatorname{Tr}(u(A,B)), \quad S_u(y,z) = \operatorname{Tr}(Au(A,B))$. By the proof of Theorem 3.1, these two traces vanish simultaneously exactly when $u(A,B) \in E$. □

Let $Q_w(y,z) = P_w(0,y,z)$.

Corollary 3.4. *For $w = [a,u]$, $Q_w(y,z) = R_u(y,z)^2 + S_u(y,z)^2 - 2$. Consequently, w is surjective if and only if $\min_{(y,z)\in D_2} Q_w(y,z) = -2$.*

Proof. Apply (1) to A and $u(A,B)$, using $\operatorname{Tr} A = 0$. □

3.3 Quantitative fixed-slice image geometry

Define $\Delta(u) = \min_{(y,z)\in D_2}\big(R_u(y,z)^2 + S_u(y,z)^2\big)$. By Corollary 3.4, $0 \le \Delta(u) \le 4$ and the minimum trace attained on the fixed slice $A = i\sigma_3$ is $\Delta(u) - 2$.

Theorem 3.5 (Fixed-slice image ball)**.** *For $w = [a,u]$, the image contains every $U \in \mathrm{SU}(2)$ with $\operatorname{Tr} U \ge \Delta(u) - 2$; equivalently,*

$$\{U \in \mathrm{SU}(2) : \operatorname{Tr} U \ge \Delta(u) - 2\} \subseteq w(\mathrm{SU}(2)^2).$$

In particular, $\Delta(u) = 0$ if and only if w is surjective on $\mathrm{SU}(2)$. If $\theta_u = \arccos(\Delta(u)/2 - 1)$, then the normalized Haar measure satisfies $\mu(w(\mathrm{SU}(2)^2)) \ge (\theta_u - \frac{1}{2}\sin(2\theta_u))/\pi$.

Proof. The map $B \longmapsto \operatorname{Tr}[A, u(A,B)]$ is continuous on the connected group $\mathrm{SU}(2)$. Its minimum is $\Delta(u) - 2$. At $B = I$, the value $u(A,I)$ is a power of A, hence commutes with A, so the trace is 2. Therefore the fixed-slice trace image is the full interval $[\Delta(u) - 2, 2]$. Simultaneous conjugation of the pair (A,B) conjugates the word value, and conjugacy classes in $\mathrm{SU}(2)$ are determined by trace. This proves the inclusion. The equivalence with surjectivity is Corollary 3.4, and the Haar-measure estimate follows from Corollary 2.5. □

Corollary 3.6 (Trace-zero threshold)**.** *If $\Delta(u) \le 2$, then there exists $B \in \mathrm{SU}(2)$ such that $\operatorname{Tr}[A, u(A,B)] = 0$.*

Proof. The interval $[\Delta(u) - 2, 2]$ contains 0 exactly when $\Delta(u) \le 2$. □

Corollary 3.7 (Outer-commutator BMV certificate)**.** *Let $x, y \in \{a,b\}^*$ be distinct words and suppose that their difference word satisfies $xy^{-1} = [a,u]$ in F_2. If $\Delta(u) \le 2$, then x and y are exactly separable by a two-state measure-once quantum finite automaton. In particular, every criterion in this paper that proves surjectivity of $[a,u]$ also certifies exact separation for such a pair.*

Proof. If $\Delta(u) \le 2$, Corollary 3.6 gives a trace-zero value of $[a,u]$. Proposition 2.4 then yields exact separation. The final assertion follows because surjectivity is equivalent to $\Delta(u) = 0$. □

Remark 3.8 (Relation to recent trace-zero methods). Chen–Wu treat general positive-word differences with equal abelianization and develop several Fox- and slice-based mechanisms for producing trace-zero witnesses [8]. Corollary 3.7 isolates a structured outer-commutator subclass for which one scalar quantity $\Delta(u)$ controls both the quantum-separation threshold $\Delta(u) \le 2$ and the stronger surjectivity threshold $\Delta(u) = 0$.

Corollary 3.9 (Universal trace-zero first coordinate)**.** *If $[a,u]$ is surjective on $\mathrm{SU}(2)$, then every $C \in \mathrm{SU}(2)$ admits a representation $C = [A_0, u(A_0, B_0)]$ with $\operatorname{Tr} A_0 = 0$.*

Proof. Surjectivity is equivalent to $\Delta(u) = 0$, so Theorem 3.5 shows that the fixed slice $[A, u(A,B)]$ meets every conjugacy class. If $X = [A, u(A,B)]$ has the same trace as C, choose $Q \in \mathrm{SU}(2)$ with $C = QXQ^{-1}$ and set $A_0 = QAQ^{-1}$ and $B_0 = QBQ^{-1}$. Equivariance of word evaluation then gives $C = [A_0, u(A_0, B_0)]$, while $\operatorname{Tr} A_0 = \operatorname{Tr} A = 0$. □

4 Great circles, degree, and combinatorial criteria

4.1 A great-circle criterion

Identify $\mathrm{SU}(2)$ with the unit sphere $S^3 \subset \mathbb{R}^4$ by $U = x_0 I + i(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)$, $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$. For $A = i\sigma_3$, $\operatorname{Tr} U = 2x_0$, $\operatorname{Tr}(AU) = -2x_3$. Hence $E = S^3 \cap E_0$, $E_0 = \operatorname{span}_{\mathbb{R}}\{i\sigma_1, i\sigma_2\}$.

Proposition 4.1 (Great-circle criterion)**.** *Suppose $u(A, \mathrm{SU}(2))$ contains a great circle $C = S^3 \cap P$, where $P \subset \mathbb{R}^4$ is a two-dimensional real subspace. If $P \cap E_0 \neq \{0\}$, then $[a, u]$ is surjective on* $\mathrm{SU}(2)$*. If $\{U_0, U_1\}$ is an orthonormal basis of P, the intersection condition is equivalent to*

$$\det \begin{pmatrix} \operatorname{Tr} U_0 & \operatorname{Tr} U_1 \\ \operatorname{Tr}(AU_0) & \operatorname{Tr}(AU_1) \end{pmatrix} = 0.$$

Proof. A nonzero vector in $P \cap E_0$, after normalization, belongs both to C and to E. The conclusion follows from Theorem 3.1. The determinant condition is precisely the solvability condition for a nonzero linear combination sU_0+tU_1 satisfying $\operatorname{Tr}(sU_0+tU_1) = \operatorname{Tr}(A(sU_0+tU_1)) = 0$. □

Corollary 4.2. *If $u(A, \mathrm{SU}(2))$ contains a great circle C lying in a great two-sphere which also contains E, then $[a, u]$ is surjective. In particular, if for some unit $X \in \operatorname{span}_{\mathbb{R}}\{i\sigma_1, i\sigma_2\}$ the slice contains the one-parameter subgroup $\{\exp(tX) : t \in \mathbb{R}\}$, then $[a, u]$ is surjective.*

4.2 Alternating words and the dihedral exponent

Write a reduced word in block form $u(a, b) = a^{m_1}b^{n_1} \cdots a^{m_k}b^{n_k}$, and set $M_j = m_1 + \cdots + m_j$, $M = M_k$, and $N_{\mathrm{alt}}(u) = \sum_{j=1}^{k} (-1)^{M_j} n_j$.

Proposition 4.3 (Alternating-word criterion)**.** *If $N_{\mathrm{alt}}(u) \neq 0$, then $[a, u]$ is surjective on* $\mathrm{SU}(2)$*.*

Proof. Let $B_\theta = \exp(i\theta\sigma_1)$. Since $AB_\theta A^{-1} = B_\theta^{-1}$, we have $A^m B_\theta^n = B_\theta^{(-1)^m n} A^m$. Moving all powers of A to the right gives $u(A, B_\theta) = B_\theta^{N_{\mathrm{alt}}(u)} A^M$. If M is even, this runs through the great circle $S^3 \cap \operatorname{span}_{\mathbb{R}}\{I, i\sigma_1\}$; if M is odd, it runs through $S^3 \cap \operatorname{span}_{\mathbb{R}}\{i\sigma_3, i\sigma_2\}$. When $N_{\mathrm{alt}}(u) \neq 0$, the entire corresponding great circle is covered, and each of these circles meets E. Proposition 4.1 applies. □

The invariant $N_{\mathrm{alt}}(u)$ is intrinsic. Indeed, consider the infinite dihedral group $D_\infty = \langle s, t : s^2 = 1,\ sts = t^{-1} \rangle$ and the homomorphism $F_2 \to D_\infty$ defined by $a \mapsto s$, $b \mapsto t$. The image of u has the unique form $t^{N_{\mathrm{alt}}(u)} s^{M \bmod 2}$.

4.3 A boundary-degree criterion

Let $N(u) = \exp_b(u)$ be the total exponent sum of b in u.

Theorem 4.4 (Boundary-degree criterion)**.** *If $N(u) \neq 0$, then $Z(R_u) \cap Z(S_u) \cap D_2 \neq \varnothing$. Consequently, $[a, u]$ is surjective on* $\mathrm{SU}(2)$*.*

Proof. On the boundary of D_2, use the commuting slice $B_t = \exp(it\sigma_3)$. Then $(y, z) = (2\cos t, -2\sin t)$ parametrizes ∂D_2. If $M = \exp_a(u)$, $\quad N = N(u)$, then A and B_t commute, so $u(A, B_t) = A^M B_t^N$. Writing $\phi(t) = \frac{M\pi}{2} + Nt$, we obtain $R_u(2\cos t, -2\sin t) = 2\cos\phi(t)$ and $S_u(2\cos t, -2\sin t) = -2\sin\phi(t)$. Thus the normalized boundary map $\frac{(R_u, S_u)}{\sqrt{R_u^2+S_u^2}} : \partial D_2 \longrightarrow S^1$ has degree $\pm N$, with the sign depending only on the chosen orientation. If $N \neq 0$, this map cannot extend to a map $D_2 \to S^1$. Hence (R_u, S_u) must vanish somewhere in D_2. □

Theorem 4.4 and Proposition 4.3 are complementary: the former uses a commuting boundary slice, while the latter uses an anticommuting interior slice. Define

$$E(u) = \sum_{\substack{1 \le j \le k \\ M_j \text{ even}}} n_j, \qquad O(u) = \sum_{\substack{1 \le j \le k \\ M_j \text{ odd}}} n_j.$$

Then $N(u) = E(u) + O(u)$, $\quad N_{\mathrm{alt}}(u) = E(u) - O(u)$.

Corollary 4.5 (Combined combinatorial criterion)**.** *For $u = a^{m_1}b^{n_1}\cdots a^{m_k}b^{n_k}$, each of the conditions $N(u) \neq 0, \qquad N_{\mathrm{alt}}(u) \neq 0$ is sufficient for the surjectivity of $[a, u]$ on* SU(2)*. Equivalently, it is sufficient that $(E(u), O(u)) \neq (0, 0)$.*

Remark 4.6. The simultaneous vanishing $N(u) = N_{\mathrm{alt}}(u) = 0$ does not imply non-surjectivity. Section 5 shows that these two quantities are the specializations $F_u(1)$ and $F_u(-1)$ of one Fox Laurent polynomial. Their simultaneous vanishing therefore means that the two rank-one specializations are blind; it does not force F_u itself to vanish. When $F_u \equiv 0$, equivalently the Gotô–Elkasapy–Thom polynomial of $[a, u]$ vanishes, the word lies beyond this first metabelian layer and genuinely higher-order information is required.

4.4 Products of commutators

Suppose $u(a, b) = \prod_{i=1}^{k}[a, u_i(a, b)]$. Write each u_i in alternating block form and let $N_{\mathrm{alt}}(u_i)$ be as above.

Proposition 4.7 (Product-of-commutators criterion)**.** *If $\sum_{i=1}^{k} N_{\mathrm{alt}}(u_i) \neq 0$, then $\left[a, \prod_{i=1}^{k}[a, u_i]\right]$ is surjective on* SU(2)*.*

Proof. On the slice $B_\theta = e^{i\theta\sigma_1}$, $u_i(A, B_\theta) = B_\theta^{N_{\mathrm{alt}}(u_i)}A^{M_i}$. Hence $[A, u_i(A, B_\theta)] = B_\theta^{-2N_{\mathrm{alt}}(u_i)}$. All these factors lie in the same one-parameter subgroup, so $u(A, B_\theta) = B_\theta^{-2\sum_i N_{\mathrm{alt}}(u_i)}$. If the exponent is nonzero, this is the full great circle $\{\exp(it\sigma_1) : t \in \mathbb{R}\}$, which meets E. □

Corollary 4.8. *For integers $q_1, \ldots, q_k$, $[a, [a, b^{q_1}]\cdots[a, b^{q_k}]]$ is surjective on* SU(2) *whenever $q_1 + \cdots + q_k \neq 0$. In particular, $[a, [a, b][a, b^q]]$ is surjective for every $q \in \mathbb{Z} \setminus \{-1\}$.*

4.5 Selected explicit families

The combined criterion shortens several computations which otherwise require separate matrix constructions.

Proposition 4.9. *The following hold.*

(a) *For $u = [a^m, b^n]$, the word $[a, u]$ is surjective on* SU(2) *if and only if m is odd and $n \neq 0$.*
(b) *For $u = [a^m, b^n]^k$, the word $[a, u]$ is surjective if and only if m is odd and $nk \neq 0$.*
(c) *For $u = ab^rab^s$, the word $[a, u]$ is surjective if and only if $(r, s) \neq (0, 0)$.*

Proof. For (a), $N(u) = 0, \qquad N_{\mathrm{alt}}(u) = \big((-1)^m - 1\big)n$. Thus m odd and $n \neq 0$ imply surjectivity. If m is even, then $A^m = \pm I$, so $u(A, B) = I$ for every B; if $n = 0$ the word is trivial on the slice. Theorem 3.1 gives necessity. Part (b) follows similarly, since taking the kth power multiplies the nonzero dihedral exponent by k. For (c), $N(u) = r + s, \qquad N_{\mathrm{alt}}(u) = s - r$. If $(r, s) \neq (0, 0)$, at least one of these numbers is nonzero, so Corollary 4.5 applies. If $r = s = 0$, then $u = a^2$ and $[a, u] = I$. □

We next record a higher-order example, where both $N(u)$ and $N_{\mathrm{alt}}(u)$ vanish.

Proposition 4.10. *Let $v_q(a, b) = [[a, b], [a, b^q]], \qquad w_q = [a, v_q]$. If $q \equiv 2 \pmod 3$, then w_q is surjective on* SU(2)*.*

Proof. Let $A = i\sigma_3, \qquad B = \frac{1}{2}I + \frac{i}{2}(\sigma_1 + \sqrt{2}\,\sigma_3)$. Then $B^3 = -I$. If $q = 3k + 2$, centrality of $(-I)^k$ gives $[A, B^q] = [A, B^2]$. Put

$$C_1 = [A, B] = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} - i \\ -\sqrt{2} - i & 1 \end{pmatrix},$$

$$C_2 = [A, B^2] = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} + i \\ -\sqrt{2} + i & 1 \end{pmatrix}.$$

A direct multiplication gives

$$[C_1, C_2] = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma_1 \in E.$$

Thus $v_q(A, B) \in E$, and Theorem 3.1 implies the surjectivity of w_q. □

Remark 4.11. For $q = 0, 1$, $v_q = I$, so w_q is trivial. Proposition 4.10 illustrates the genuinely higher-order regime of Remark 4.6: both first-order combinatorial invariants vanish, but a special finite-order slice still forces the off-diagonal intersection. No claim for the unresolved values outside the congruence class $q \equiv 2 \pmod 3$ is used in this paper.

5 Fox linearization and cyclotomic $\mathrm{SU}(n)$ criteria

The two SU(2) invariants above admit a common algebraic description. We use Fox's free differential calculus [10]. Let $\mathrm{ab} : \mathbb{Z}[F_2] \longrightarrow \mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$ be the ring homomorphism induced by $a \mapsto t$ and $b \mapsto s$, and define

$$F_u(t) = \left(\frac{\partial u}{\partial b}\right)^{\mathrm{ab}}\Bigg|_{s=1} \in \mathbb{Z}[t^{\pm 1}].$$

We call F_u the one-variable Fox polynomial of the inner word u relative to b.

Proposition 5.1 (Fox unification of the two slice invariants)**.** *Suppose $u = a^{m_1}b^{n_1}\cdots a^{m_k}b^{n_k}$ and put $M_j = m_1 + \cdots + m_j$. Then $F_u(t) = \sum_{j=1}^{k} n_j t^{M_j}$, so in particular $F_u(1) = \exp_b(u)$ and $F_u(-1) = N_{\mathrm{alt}}(u)$. Let $p_w(t)$ be the Laurent polynomial attached by Elkasapy–Thom [9] to $w \in F_2'$. For $w = [a, u]$ one has*

$$p_{[a,u]}(t) = (t-1)F_u(t).$$

In particular, $p'_{[a,u]}(1) = F_u(1) = \exp_b(u)$, whereas $p_{[a,u]}(-1) = -2F_u(-1) = -2N_{\mathrm{alt}}(u)$.

Proof. For $P_j = a^{m_1}b^{n_1}\cdots a^{m_{j-1}}b^{n_{j-1}}a^{m_j}$, Fox's product rule gives $\frac{\partial u}{\partial b} = \sum_{j=1}^{k} P_j \frac{\partial b^{n_j}}{\partial b}$. After abelianization and the specialization $s = 1$, one has $P_j \mapsto t^{M_j}$ and $\left(\frac{\partial b^{n_j}}{\partial b}\right)^{\mathrm{ab}}\Big|_{s=1} = n_j$ for every $n_j \in \mathbb{Z}$. Hence $F_u(t) = \sum_j n_j t^{M_j}$, and the identities at $t = \pm 1$ follow immediately. For the factorization, write $w = [a, u] = aua^{-1}u^{-1}$. Since $\frac{\partial u^{-1}}{\partial b} = -u^{-1}\frac{\partial u}{\partial b}$, we obtain $\frac{\partial w}{\partial b} = a\frac{\partial u}{\partial b} - w\frac{\partial u}{\partial b}$. After abelianization, w maps to 1, and after setting $b = 1$ this becomes $(t-1)F_u(t)$. On the free generators $[a^n, b^m]$ of the first derived subgroup used in [9], the same abelianized Fox derivative equals $m(t^n - 1)$, which is exactly the defining value of p_w. Both assignments are homomorphisms from F_2' to the additive group $\mathbb{Z}[t^{\pm 1}]$, so they agree on all of F_2'. Differentiating at $t = 1$ and evaluating at $t = -1$ give the final formulas. □

Proposition 5.1 makes the relation between the two rank-one arguments precise. The anticommuting condition $F_u(-1) \neq 0$ is exactly the $n = 2$ cyclotomic nonvanishing condition in Gotô's method. The commuting boundary-degree condition instead measures the first derivative at the root $t = 1$, which is forced by the outer commutator. Thus the boundary-degree theorem detects information that the usual nontrivial-root test does not see.

Theorem 5.2 (Cyclotomic Fox criterion)**.** *Let $n \geq 2$ and $w = [a, u]$. If $F_u(\zeta) \neq 0$ for every nontrivial nth root of unity ζ, then the word map $w : \mathrm{SU}(n)^2 \to \mathrm{SU}(n)$ is surjective.*

Proof. For every nontrivial nth root of unity ζ, $p_w(\zeta) = (\zeta - 1)F_u(\zeta) \neq 0$. The conclusion is exactly the Gotô–Elkasapy–Thom criterion [9, Lemma 3.1]. □

Corollary 5.3 (Explicit $\mathrm{SU}(n)$ families)**.** *The following hold.*

(a) *If* $q \neq 0$ *and* $\gcd(m,n) = 1$*, then* $[a,[a^m,b^q]]$ *is surjective on* $\mathrm{SU}(n)$*.*
(b) *More generally, if* $qk \neq 0$ *and* $\gcd(m,n) = 1$*, then* $[a,[a^m,b^q]^k]$ *is surjective on* $\mathrm{SU}(n)$*.*
(c) *For* $u = ab^r ab^s$*, if* $r \neq s$*, then* $[a,u]$ *is surjective on* $\mathrm{SU}(n)$ *for every* $n \geq 2$*. If* $r = s \neq 0$*, the cyclotomic criterion gives surjectivity for every odd* n*; independently, Proposition 4.9 gives surjectivity for* $n = 2$*.*

Proof. For $u = [a^m, b^q]$, Proposition 5.1 gives $F_u(t) = q(t^m - 1)$. Hence F_u has a nontrivial zero among the nth roots of unity exactly when $\gcd(m,n) > 1$. Taking the kth power multiplies F_u by k, proving (a) and (b). For $u = ab^r ab^s$, $F_u(t) = rt + st^2 = t(r + st)$. A zero on the unit circle can occur only when $|r| = |s|$. If $r = -s \neq 0$, that zero is $t = 1$, which is excluded from the cyclotomic test. If $r = s \neq 0$, the only unit-circle zero is $t = -1$, so the test succeeds precisely for odd n. The remaining $n = 2$ statement is already contained in Proposition 4.9. □

Corollary 5.4 (Infinitely many unitary ranks)**.** *If* $F_u \not\equiv 0$*, then there exists* $K = K(u)$ *such that* $[a,u]$ *is surjective on* $\mathrm{SU}(n)$ *whenever the least prime factor of* n *is at least* K*. In particular,* $[a,u]$ *is surjective on* $\mathrm{SU}(n)$ *for infinitely many* n*.*

Proof. By Proposition 5.1, $p_{[a,u]} = (t-1)F_u$ is nonzero. Apply [9, Corollary 3.2]. □

Remark 5.5 (The Fox-degenerate higher-order regime)*.* If $F_u \equiv 0$, then $p_{[a,u]} \equiv 0$, so the first metabelian Gotô polynomial cannot certify surjectivity. This is exactly what happens in Proposition 4.10: there $u = [[a,b],[a,b^q]] \in F_2''$, so $F_u \equiv 0$, yet for $q \equiv 2 \pmod 3$ a finite-order slice still proves that $[a,u]$ is surjective on $\mathrm{SU}(2)$. This example separates the first Fox/metabelian layer from genuinely higher-order slice geometry.

6 Transfer from $\mathrm{SU}(2)$ to orthogonal groups

The next theorem applies to an arbitrary word $w \in F_2$, not only to commutator-type words.

Theorem 6.1 (Transfer theorem)**.** *Assume* $w : \mathrm{SU}(2)^2 \longrightarrow \mathrm{SU}(2)$ *is surjective. Then* $w : \mathrm{SO}(n)^2 \longrightarrow \mathrm{SO}(n)$ *is surjective for every* $n \geq 3, \quad n \not\equiv 2 \pmod 4$*.*

Proof. The double covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$ immediately gives surjectivity on $\mathrm{SO}(3)$. Next, $\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$. The componentwise word map is surjective on $\mathrm{SU}(2) \times \mathrm{SU}(2)$, and therefore surjectivity descends through the double covering $\mathrm{Spin}(4) \to \mathrm{SO}(4)$. It remains to use block-diagonal maximal tori. For $n = 4r$, a standard maximal torus of $\mathrm{SO}(4r)$ is contained in the subgroup $\mathrm{SO}(4)^r$. Since w is surjective on each $\mathrm{SO}(4)$ factor, its image on $\mathrm{SO}(4r)$ contains this maximal torus. The image of a word map is conjugation invariant, and every element of $\mathrm{SO}(4r)$ is conjugate to an element of the torus; hence w is surjective. For $n = 4r + 1$, use $\mathrm{SO}(4)^r \oplus 1$. For $n = 4r + 3$, use $\mathrm{SO}(4)^r \oplus \mathrm{SO}(3)$. These subgroups contain standard maximal tori of the ambient orthogonal groups, and the same argument applies. □

The dimensions $4r + 2$ are exactly those for which the above block decomposition leaves an $\mathrm{SO}(2)$ factor. For arbitrary words this gap can be filled whenever the abelianization is nontrivial.

Corollary 6.2 (Full transfer with nontrivial abelianization)**.** *Let* $w \in F_2$ *be surjective on* $\mathrm{SU}(2)$*, and put* $p = \exp_a(w), \quad q = \exp_b(w)$*. If* $(p,q) \neq (0,0)$*, then* w *is surjective on* $\mathrm{SO}(n)$ *for every* $n \geq 2$*.*

Proof. On the abelian group $\mathrm{SO}(2) \simeq S^1$, $w(z_1,z_2) = z_1^p z_2^q$. If $(p,q) \neq (0,0)$, this map is surjective. Theorem 6.1 already handles $n \not\equiv 2 \pmod 4$. For $n = 4r + 2$, a standard maximal torus is contained in $\mathrm{SO}(4)^r \times \mathrm{SO}(2)$. The word map is surjective on each factor, hence its image contains the full maximal torus. Conjugation invariance then gives surjectivity on $\mathrm{SO}(4r+2)$. □

Remark 6.3. For every derived word, and in particular for every commutator-type word $[a, u]$, one has $\exp_a(w) = \exp_b(w) = 0$. Thus Corollary 6.2 does not address the genuinely difficult $4r + 2$ case for the principal class studied in this paper. The obstruction is to the present block mechanism, not a proof of non-surjectivity.

Corollary 6.4. *Every* SU(2)*-surjectivity result in Section 4 automatically yields surjectivity on* $\mathrm{SO}(n), \quad n \geq 3, \quad n \not\equiv 2 \pmod 4$.

6.1 Low-dimensional converses

For SO(3), scalar trace determines conjugacy classes.

Proposition 6.5. *For an arbitrary word* w, $w : \mathrm{SO}(3)^2 \to \mathrm{SO}(3)$ *is surjective* $\iff$ $-1 \in \mathrm{Tr}\, w(\mathrm{SO}(3)^2)$. *Equivalently, after lifting to* SU(2)*, this is equivalent to the existence of a lifted value of trace zero.*

Proof. The trace range on SO(3) is $[-1, 3]$. The image of the trace map is an interval containing 3, and conjugacy classes in SO(3) are determined by trace. The statement follows. Under the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$, a trace-zero element maps to a rotation through angle π, whose SO(3) trace is -1. □

Example 6.6. Let $w(a, b) = [a, [a^2, b]]$. Then w is surjective on SO(3), but not on SU(2).

Proof. Take $A = \exp\left(\frac{i\pi}{4}\sigma_3\right), \qquad B = \exp\left(\frac{i\pi}{4}\sigma_1\right)$. Then $A^2 = i\sigma_3$, and $[A^2, B] = -i\sigma_1$. The latter inverts A, hence $w(A, B) = [A, -i\sigma_1] = A^2 = i\sigma_3$, which has trace zero. Proposition 6.5 gives surjectivity on SO(3). If w were surjective on SU(2), it would attain $-I$. But $[A, [A^2, B]] = -I$ forces $\mathrm{Tr}\, A = 0$, hence $A^2 = -I$, so $[A^2, B] = I$, a contradiction. □

The case SO(4) behaves differently and is particularly rigid.

Theorem 6.7 (SO(4) equivalence)**.** *For an arbitrary word* $w \in F_2$*, the following are equivalent:*

(i) $w : \mathrm{SU}(2)^2 \to \mathrm{SU}(2)$ *is surjective;*
(ii) $w : \mathrm{SO}(4)^2 \to \mathrm{SO}(4)$ *is surjective;*
(iii) *the scalar trace map* $\mathrm{Tr}\, w : \mathrm{SO}(4)^2 \longrightarrow [-4, 4]$ *is surjective.*

Proof. The implication (i)⇒(ii) is Theorem 6.1, and (ii)⇒(iii) is immediate. Assume (iii). Then there exist $A, B \in \mathrm{SO}(4)$ with $\mathrm{Tr}(w(A, B)) = -4$. An orthogonal 4×4 matrix has trace -4 only when it equals $-I_4$, so $w(A, B) = -I_4$. Consider the double covering $\pi : \mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2) \longrightarrow \mathrm{SO}(4)$. Choose lifts $\widetilde{A} = (A_1, A_2), \qquad \widetilde{B} = (B_1, B_2)$. Since word evaluation commutes with homomorphisms, $\pi\big(w(\widetilde{A}, \widetilde{B})\big) = -I_4$. Under the standard covering $\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$, the inverse image of $-I_4$ is $\{(-I, I), (I, -I)\}$. Hence one of $w(A_1, B_1)$ and $w(A_2, B_2)$ equals $-I$. Proposition 2.2 then implies that w is surjective on SU(2). Thus (iii)⇒(i). □

Remark 6.8. Theorem 6.7 is an exceptional low-dimensional phenomenon. In SO(3) a scalar trace already determines a conjugacy class; in SO(4) the exceptional isomorphism $\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$ restores an equivalence that scalar trace does not provide in general higher rank. Starting with SO(5), scalar trace is not a complete class coordinate, and trace-surjectivity by itself should not be identified with word-surjectivity.

7 Conjugacy coordinates and higher-rank surjectivity

Scalar trace ceases to determine conjugacy classes once the rank is at least two. We record a complete replacement.

7.1 Odd dimension

For $g \in \mathrm{SO}(2r+1)$, define $c_j(g) = \frac{\mathrm{Tr}(g^j)-1}{2}, \qquad j = 1, \dots, r$, and $\chi_{2r+1}(g) = (c_1(g), \dots, c_r(g))$. If $g \sim \mathrm{diag}(R(\theta_1), \dots, R(\theta_r), 1)$, then $c_j(g) = \sum_{\ell=1}^{r} \cos(j\theta_\ell)$.

Proposition 7.1. *The map χ_{2r+1} separates conjugacy classes in $\mathrm{SO}(2r+1)$.*

Proof. Put $x_\ell = \cos\theta_\ell$. Since $\cos(j\theta_\ell) = T_j(x_\ell)$, the coordinates $c_1, \dots, c_r$ determine recursively the power sums $\sum_\ell x_\ell^j$, $1 \le j \le r$. Newton identities then determine the multiset $\{x_1, \dots, x_r\}$. The Weyl group of type B_r permits arbitrary permutations and sign changes of the angles, so this multiset determines the conjugacy class. □

7.2 Even dimension

For $g \in \mathrm{SO}(2r)$, set $c_j(g) = \frac{\mathrm{Tr}(g^j)}{2}, \qquad j = 1, \dots, r$, and $\eta(g) = 2^{-r}\,\mathrm{Pf}(g - g^T)$. Define $\chi_{2r}(g) = (c_1(g), \dots, c_r(g), \eta(g))$. On the standard maximal torus,

$$c_j(g) = \sum_{\ell=1}^{r} \cos(j\theta_\ell), \qquad \eta(g) = \pm \prod_{\ell=1}^{r} \sin\theta_\ell,$$

where the global sign depends only on the convention for $R(\theta)$.

Proposition 7.2. *The map χ_{2r} separates conjugacy classes in $\mathrm{SO}(2r)$.*

Proof. The trace coordinates determine the multiset $\{\cos\theta_1, \dots, \cos\theta_r\}$. Thus only the parity of the number of angle sign changes remains. The Weyl group of type D_r allows an even number of sign changes, while the Pfaffian coordinate records the remaining parity whenever all sines are nonzero. If one sine vanishes, the apparent splitting of the two sign-parity orbits disappears. This is the standard type-D conjugacy distinction; compare the invariant-theoretic description in [1]. □

Let $\Omega_n = \chi_n(\mathrm{SO}(n)), \qquad \Phi_w = \chi_n \circ w : \mathrm{SO}(n)^2 \to \Omega_n$.

Theorem 7.3 (Conjugacy-coordinate criterion)**.** *For every $n \ge 3$, the word map $w : \mathrm{SO}(n)^2 \to \mathrm{SO}(n)$ is surjective if and only if $\Phi_w(\mathrm{SO}(n)^2) = \Omega_n$.*

Proof. The forward implication is immediate. Conversely, if $g \in \mathrm{SO}(n)$, choose A, B such that $\chi_n(w(A,B)) = \chi_n(g)$. Propositions 7.1 and 7.2 show that the two elements are conjugate. Since the image of a word map is conjugation invariant, g lies in the image. □

7.3 Semialgebraic formulation

Proposition 7.4. *For fixed n and fixed word w, the set $\Phi_w(\mathrm{SO}(n)^2) \subseteq \Omega_n$ is compact and semialgebraic. The equality $\Phi_w(\mathrm{SO}(n)^2) = \Omega_n$ is decidable, in principle, by quantifier elimination over the real closed field $\mathbb{R}$.*

Proof. Write the matrix entries of A, B as real variables. The constraints $A^TA = I, \quad \det A = 1, \qquad B^TB = I, \quad \det B = 1$ are polynomial. On orthogonal matrices, inverses are transposes, so each entry of $w(A,B)$ is polynomial in the entries of A, B. Traces of powers and $\mathrm{Pf}(W - W^T)$ are polynomial functions. Thus the graph of Φ_w is semialgebraic, and its image is semialgebraic by the Tarski–Seidenberg theorem. Compactness follows from compactness of $\mathrm{SO}(n)^2$. Equality with Ω_n can therefore be expressed as a first-order sentence over $\mathbb{R}$, and real quantifier elimination is effective; see [4]. □

Conjugation invariance reduces the number of variables: $\Phi_w(\mathrm{SO}(n)^2) = \Phi_w(T_n \times \mathrm{SO}(n))$ for any fixed maximal torus T_n. For SO(5) the criterion is especially concrete. Put $t = \operatorname{Tr} g, \quad s = \operatorname{Tr}(g^2)$. If $g \sim \operatorname{diag}(R(\theta_1), R(\theta_2), 1), \quad x = \cos\theta_1, \quad y = \cos\theta_2$, then $t = 1 + 2(x + y), \quad s = 4(x^2 + y^2) - 3$. It follows that the conjugacy region in these coordinates is

$$\Omega_5 = \left\{ (t, s) : -3 \le t \le 5, \quad \frac{(t-1)^2}{2} - 3 \le s \le (t-1)^2 + 5 - 4|t-1| \right\}. \tag{3}$$

Thus on SO(5) scalar trace-surjectivity is only surjectivity of the first projection of the two-dimensional class image $(\operatorname{Tr} w, \operatorname{Tr}(w^2))$.

8 Trace-surjectivity for $w = [a, u]$ on $\mathrm{SO}(n)$

We now study the scalar trace map separately: $\operatorname{Tr} w : \mathrm{SO}(n)^2 \longrightarrow [m_n, n]$.

8.1 A general slice criterion

For $A \in \mathrm{SO}(n)$, define $E_A^{(n)} = \{U \in \mathrm{SO}(n) : \operatorname{Tr}[A, U] = m_n\}$.

Proposition 8.1 (Trace-slice criterion)**.** *For $w = [a, u]$, the map $\operatorname{Tr} w$ is surjective onto $\operatorname{Tr}(\mathrm{SO}(n))$ if and only if there exists $A \in \mathrm{SO}(n)$ such that $u(A, \mathrm{SO}(n)) \cap E_A^{(n)} \neq \varnothing$. Equivalently, there exist $A, B \in \mathrm{SO}(n)$ such that $\operatorname{Tr}[A, u(A, B)] = m_n$.*

Proof. The image of $\operatorname{Tr} w$ is a compact interval because $\mathrm{SO}(n)^2$ is connected. It contains n, since $w(I, I) = I$. Hence it is the full trace range $[m_n, n]$ exactly when the minimum value m_n is attained. □

8.2 Even and odd dimensions

Proposition 8.2 (Even-dimensional criterion)**.** *Let $n = 2r$. The following are equivalent:*

(i) $\operatorname{Tr}[a, u]$ *is surjective onto* $[-2r, 2r]$*;*
(ii) *there exist* $A, B \in \mathrm{SO}(2r)$ *such that* $[A, u(A, B)] = -I_{2r}$*;*
(iii) *there exist* $A, B \in \mathrm{SO}(2r)$ *such that* $Au(A, B) = -u(A, B)A$.

Proof. The unique element of $\mathrm{SO}(2r)$ with trace $-2r$ is $-I_{2r}$. Moreover, $[A, U] = -I \iff AU = -UA$. Apply Proposition 8.1. □

Proposition 8.3 (Odd-dimensional criterion)**.** *Let $n = 2r + 1$. Then $\operatorname{Tr}[a, u]$ is surjective if and only if there exist $A, B \in \mathrm{SO}(2r + 1)$ and $v \in S^{2r}$ such that $[A, u(A, B)] = 2vv^T - I_{2r+1}$.*

Proof. The minimum trace $1 - 2r$ occurs exactly on the conjugacy class of $\operatorname{diag}(-I_{2r}, 1)$. This class is $\{2vv^T - I : v \in S^{2r}\}$. Now apply Proposition 8.1. □

8.3 A fixed inversion slice

Let

$$A_* = \begin{cases} \operatorname{diag}(R(\pi/2), \ldots, R(\pi/2)), & n = 2r, \\ \operatorname{diag}(R(\pi/2), \ldots, R(\pi/2), 1), & n = 2r + 1. \end{cases}$$

Then $\operatorname{Tr}(A_*^2) = m_n$.

Proposition 8.4 (Inversion criterion)**.** *If there exists $B \in \mathrm{SO}(n)$ such that $U = u(A_*, B)$ satisfies $UA_*U^{-1} = A_*^{-1}$, then $\operatorname{Tr}[a, u]$ is surjective on $\mathrm{SO}(n)$.*

Proof. The inversion relation gives $[A_*, U] = A_*UA_*^{-1}U^{-1} = A_*^2$. Since $\operatorname{Tr}(A_*^2) = m_n$, Proposition 8.1 applies. □

The inversion set is nonempty for all odd n and for $n \equiv 0 \pmod 4$. Indeed, with $D = \mathrm{diag}(1,-1)$, $DR(\theta)D^{-1} = R(-\theta)$, one may take $B_0 = \mathrm{diag}(D,\dots,D,(-1)^r)$ in dimension $2r+1$, and $B_0 = \mathrm{diag}(D,\dots,D)$ in dimension $2r$ when r is even.

Corollary 8.5. *Let $u = a^{m_1}b^{n_1}\cdots a^{m_k}b^{n_k}$. If the total b-exponent $N_b(u) = n_1+\cdots+n_k$ is odd, then $\mathrm{Tr}[a,u]$ is surjective on $\mathrm{SO}(n)$ for every $n \geq 3$, $n \not\equiv 2 \pmod 4$.*

Proof. For the inversion element B_0 above, the coset of $u(A_*, B_0)$ modulo the maximal torus is determined by the parity of the total exponent of B_0. Oddness of $N_b(u)$ places $u(A_*, B_0)$ in the inversion coset, so Proposition 8.4 applies. □

Remark 8.6. For alternating words, $N_b(u) \equiv N_{\mathrm{alt}}(u) \pmod 2$. Thus the hypothesis of Corollary 8.5 already implies $N_{\mathrm{alt}}(u) \neq 0$, hence word-surjectivity on SU(2) and, by Theorem 6.1, word-surjectivity on the same orthogonal groups. The value of Corollary 8.5 is that it gives a direct orthogonal-group explanation in terms of the inversion slice.

8.4 A trace-surjective family on $\mathrm{SO}(5)$

The following family shows that scalar trace-surjectivity on SO(5) can occur even when the same word is not surjective on SU(2).

Proposition 8.7. *For every integer $m \geq 1$, let $w_m(a,b) = [a,[a^{2m},b]]$. Then $\mathrm{Tr}\, w_m : \mathrm{SO}(5)^2 \longrightarrow [-3,5]$ is surjective. On the other hand, $w_m : \mathrm{SU}(2)^2 \longrightarrow \mathrm{SU}(2)$ is not surjective.*

Proof. Put

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & D & 0 \\ D & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(5),$$

where the block sizes are $2,2,1$. Set $\alpha_m = \pi/(4m)$ and $\beta_m = \pi - \pi/(4m)$, and let $A_m = \mathrm{diag}(R(\alpha_m), R(\beta_m), 1)$. Since $\alpha_m + \beta_m = \pi$, the commutator is $[A_m, U] = J_5 := \mathrm{diag}(-I_2, -I_2, 1)$, whose trace is -3. Moreover, $A_m^{2m} = T_0 := \mathrm{diag}(R(\pi/2), R(-\pi/2), 1)$, independently of m. Take

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \in \mathrm{SO}(5).$$

A direct calculation gives $BT_0B^{-1} = UT_0$. Since U commutes with T_0 and $U^2 = I$, $[T_0, B] = U$. Therefore $w_m(A_m, B) = [A_m, [A_m^{2m}, B]] = [A_m, U] = J_5$. Thus $\mathrm{Tr}\, w_m$ attains -3. Since its image is an interval containing 5, it is all of $[-3,5]$. Finally, if w_m attained $-I$ on SU(2), then $[A,[A^{2m},B]] = -I$ would imply $\mathrm{Tr}\, A = 0$, hence $A^2 = -I$. Therefore A^{2m} would be central and $[A^{2m}, B] = I$, a contradiction. □

Remark 8.8. Proposition 8.7 does not decide whether the word map $w_m : \mathrm{SO}(5)^2 \to \mathrm{SO}(5)$ itself is surjective. In view of (3), this amounts to determining whether the two-dimensional conjugacy-coordinate image $\big(\mathrm{Tr}\, w_m, \mathrm{Tr}(w_m^2)\big)(\mathrm{SO}(5)^2)$ fills all of Ω_5, rather than merely projecting onto the full interval $[-3,5]$.

9 Summary of criteria and examples

The principal criteria fit naturally into two complementary hierarchies. The first is algebraic. For an alternating inner word u, the two elementary SU(2) invariants are the special values $\exp_b(u) = F_u(1)$ and $N_{\mathrm{alt}}(u) = F_u(-1)$. Nonvanishing of either value implies surjectivity on

SU(2) and, by the transfer theorem, on every SO(n) with $n \geq 3$ and $n \not\equiv 2 \pmod 4$. More generally, if $F_u(\zeta) \neq 0$ for every nontrivial nth root of unity, the cyclotomic criterion gives surjectivity on SU(n). Thus, for example, $[a, [a^m, b^q]]$ is surjective on SU(n) whenever $q \neq 0$ and $\gcd(m, n) = 1$, and the same conclusion holds for $[a, [a^m, b^q]^k]$ when $qk \neq 0$ and $\gcd(m, n) = 1$. For $u = ab^r ab^s$, the polynomial is $F_u(t) = t(r + st)$; hence $r \neq s$ gives surjectivity on every SU(n), while $r = s \neq 0$ gives all odd ranks by the cyclotomic test and $n = 2$ separately by the boundary-degree theorem.

The second hierarchy is geometric and quantitative. For an arbitrary word ω, Proposition 2.4 identifies exact two-state quantum word separation with the variational threshold $m(\omega) \leq 0$. For the outer-commutator class, the parameter $\Delta(u)$ gives an explicit fixed-slice certificate for that threshold: $\Delta(u) \leq 2$ forces a trace-zero value, whereas the stronger condition $\Delta(u) = 0$ is equivalent to full SU(2)-surjectivity. Thus the same calculation can be used at three levels: to bound the size of the word image, to certify the Belovs–Montoya–Yakaryılmaz conclusion for a structured positive-word difference, or to prove global surjectivity.

The explicit examples show why both hierarchies are needed. Products of commutators are efficiently handled by the great-circle mechanism when the corresponding alternating exponents do not cancel, while the family $[a, [[a, b], [a, b^q]]]$ with $q \equiv 2 \pmod 3$ lies in the Fox-degenerate regime $F_u \equiv 0$ and therefore requires a genuinely higher-order finite-order slice. On the orthogonal side, scalar trace is a complete surjectivity coordinate on SU(2) and, exceptionally, remains equivalent to full word-surjectivity on SO(4). For higher orthogonal rank this is no longer true at the level of conjugacy coordinates, and the family $w_m = [a, [a^{2m}, b]]$ provides a concrete testing ground: its scalar trace is surjective on SO(5), while full surjectivity on SO(5) remains open.

10 Further problems

The results above isolate several concrete open directions.

Problem 10.1 (The outer-commutator BMV threshold). Find effective criteria for $\Delta(u) \leq 2$ that do not force the stronger condition $\Delta(u) = 0$. In particular, determine how much of the trace-zero threshold can be detected directly from Fox or metabelian data when the corresponding word $[a, u]$ is not known to be surjective.

Problem 10.2 (The derived $4r + 2$ transfer problem). Let $w \in [F_2, F_2]$ be surjective on SU(2). Determine useful conditions implying that w is surjective on SO($4r + 2$). More specifically, study the class $w = [a, u]$, for which the exponent sums vanish identically and Corollary 6.2 cannot supply the missing SO(2) block.

Problem 10.3 (Trace versus word surjectivity on SO(5)). Determine whether the words $w_m = [a, [a^{2m}, b]]$ are surjective on SO(5). More generally, construct an explicit word of the form $[a, u]$ whose scalar trace map is surjective on SO(5) but whose word map is not.

Problem 10.4 (Higher-rank spectral images). For a fixed word w, describe $\Phi_w(\mathrm{SO}(n)^2) \subseteq \Omega_n$ by explicit inequalities. In particular, find practical semialgebraic criteria, substantially simpler than general quantifier elimination, for $\Phi_w(\mathrm{SO}(n)^2) = \Omega_n$.

11 Concluding remarks

The outer-commutator class $w = [a, u]$ is a setting in which three descriptions of word maps become visibly compatible. At rank one, fixing $A = i\sigma_3$ turns SU(2)-surjectivity into a two-equation intersection problem on the character disk and, through the parameter $\Delta(u)$, gives a quantitative geodesic ball contained in the image. At the metabelian level, a single Fox polynomial records both basic slice mechanisms through the identities $F_u(1) = \exp_b(u)$ and $F_u(-1) = N_{\mathrm{alt}}(u)$. At higher unitary rank, the factorization $p_{[a,u]}(t) = (t - 1)F_u(t)$ inserts the same invariant directly into the Gotô–Elkasapy–Thom criterion. In this sense the commuting

degree argument and the anticommuting great-circle argument are not isolated tricks; they are rank-one manifestations of a common Fox/Weyl structure.

This viewpoint also clarifies the hierarchy of questions one can ask about a word image. Full surjectivity is only the extremal case. Proposition 2.4 shows that the Belovs–Montoya–Yakaryılmaz conjecture itself is a variational statement about the sign of the minimum trace of positive-word differences. Within the outer-commutator class, the fixed slice refines that statement quantitatively: $\Delta(u) \leq 2$ is a sufficient one-qubit separation threshold, while $\Delta(u) = 0$ is the stronger surjectivity threshold. This places the present deterministic rank-one calculation naturally beside recent work on the size and distribution of word images and beside the trace-zero witness program in quantum finite automata [5, 3, 2, 8].

The passage to orthogonal groups reveals a different phenomenon. Surjectivity on SU(2) propagates through low-dimensional coverings and block maximal tori to all $\mathrm{SO}(n)$ with $n \not\equiv 2 \pmod 4$, and to every orthogonal dimension when the abelianization of the word is nontrivial. The group SO(4) is exceptional because $\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$ makes scalar trace-surjectivity, SO(4)-surjectivity, and SU(2)-surjectivity equivalent. Beyond that point, scalar trace no longer controls conjugacy and the appropriate object is the full conjugacy-coordinate image.

The remaining difficult cases are now sharply located. When $F_u \not\equiv 0$, cyclotomic information already yields surjectivity in infinitely many unitary ranks. When $F_u \equiv 0$, however, the first metabelian Gotô polynomial and both elementary rank-one specializations vanish simultaneously, while surjectivity may still survive, as shown by the higher-order double-commutator family treated above. Understanding this Fox-degenerate layer, together with the unresolved $4r + 2$ transfer problem for derived words and the distinction between scalar trace-surjectivity and full word-surjectivity on SO(5), appears to be the natural next stage of the theory.

Acknowledgment

The third author would like to thank the International Centre of Research and Postgraduate Training in Mathematics (ICRTM) for providing the necessary research facilities and support. This work is partially supported by VAST under grant number CSCL01/27-28. We are grateful to Professor Vu T. Khoi for many valuable discussions.

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