SERRE CONJECTURE II FOR PSEUDO-REDUCTIVE GROUPS

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Abstract. The Serre conjecture II predicts that every torsor under a semisimple, simply connected, algebraic group over a field of cohomological dimension at most 2 and of degree of imperfection at most 1 has a rational point. We generalize this conjecture to pseudo-reductive groups and prove their equivalence. In particular, we show that every torsor under a pseudo-semisimple, simply connected group over a global function field or a non-archimedean local field always has a rational point.

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1. Serre conjecture II

In 1962, Serre [Ser62, Conjecture II, page 65] predicted that every torsor under a semisimple, simply connected, algebraic group over a perfect field of cohomological dimension at most 2 should be trivial. Serre [Ser94, Remarques, page 146] also formulated the conjecture over an imperfect field. He anticipated that over a field of cohomological dimension at most 2 and of degree of imperfection at most 1, every torsor under a semisimple, simply connected, algebraic group should be trivial. This statement is also known as Serre conjecture II. The conjecture was settled for global function fields by Harder [Har75, Satz A] and non-archimedean local fields by Bruhat–Tits [BT87, Théorèm 4.7(ii)]. See the survey [Gil10] for a detailed review of the state of the art on the Serre conjecture II. The goal of this article is to extend the results of Harder and Bruhat–Tits to pseudo-reducitve groups. In fact, we prove a more general result: we formulate a Serre conjecture II for pseudo-reductive groups and prove this is equivalent to the original Serre conjecture II.

The arithmetic analogue of semisimple algebraic groups are pseudo-semisimple groups. Recall that a *pseudo-reductive* k-group is a connected, affine, smooth k-group that has no nontrivial unipotent normal k-subgroups and a *pseudo-semisimple* group is a perfect pseudo-reductive group, where k is a field. The analogies between pseudo-semisimple groups and semisimple algebraic groups is a foundational topic in the theory of pseudo-reductive groups, developed by Tits and Conrad-Gabber-Prasad [CGP15, CP16]. We find that the notion of simply connectedness for connected, affine, smooth algebraic groups proposed in [BČS25, 2.5.2] is a perfect analogy for the notion of

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simply connectedness for semisimple groups, in particular, to formulate a Serre conjecture II for pseudo-reductive groups: for a field k, an affine, connected, smooth k-group G is *simply connected* if the $\overline{k}$-group $G_{\overline{k}}/\mathscr{R}_{\mathrm{u},\overline{k}}(G)$ is semisimple and simply connected, where $\mathscr{R}_{\mathrm{u},\overline{k}}(G_{\overline{k}})$ denotes the geometric unipotent radical of G.

Conjecture 1.1 (Serre conjecture II for pseudo-reductive groups)**.** *Let k be a field of cohomological dimension at most 2 and of degree of imperfection at most 1. For a pseudo-semisimple, simply connected k-group G, then*

$$\mathrm{H}^1(k, G) = \{*\}.$$

Theorem 1.2 (Corollary 3.4)**.** *Let k be a field of cohomological dimension at most 2 and of degree of imperfection at most 1. The following are equivalent*

(i) $\mathrm{H}^1(k, G) = \{*\}$ *for every semisimple, simply connected k-group G;*

(ii) $\mathrm{H}^1(k, G) = \{*\}$ *for every pseudo-semisimple, simply connected k-group G.*

By the known cases of the classical Serre conjecture II, we deduce a new vanishing theorem.

Corollary 1.3. *Let k be a field and G a pseudo-semisimple, simply connected k-group.*

(a) *If k is a non-archimedean local field, then $\mathrm{H}^1(k, G) = \{*\}$.*

(b) *If k is a global function field, then $\mathrm{H}^1(k, G) = \{*\}$.*

1.4. An overview of the proof and of the paper. The first step of the proof is to reduce Theorem 1.2 to absolutely pseudo-simple k-groups. We utilize the map from the product of minimal pseudo-simple normal k-subgroups of G and the fact that every pseudo-semisimple, simply connected k-group G is always of minimal type to reduce to the case when G is pseudo-simple. By a standard Galois descent argument, we even reduce to the case when G is absolutely pseudo-simple.

For an absolutely pseudo-simple, simply connected k-group G, the comparison map

$$i_G \colon G \to \mathrm{Res}_{k'/k}(G_{k'}/\mathscr{R}_{\mathrm{u},k'}(G_{k'}))$$

is an isomorphism when $\mathrm{char}(k) > 3$. We recall the Shapiro lemma [Oes84, Chapitre IV, 2.3, Lemme] (see also the more general result from [BČS25, Lemma 4.1.1]): for a finite field extension k'/k and a smooth, affine k'-group G',

$$\mathrm{H}^1(k, \mathrm{Res}_{k'/k}(G')) = \mathrm{H}^1(k', G'),$$

where $\mathrm{Res}_{k'/k}(G')$ denotes the restriction of scalar of G' over k'/k. Thus, we could reduce to the classical Serre conjecture II. When $\mathrm{char}(k) < 3$, we will need the study of basic exotic groups and basic non-reduced groups. More precisely, using the classification theory of pseudo-reductive groups developed in [CGP15, CP16], we could show that if i_G is not an isomorphism then there is a finite field extension k'/k and a k'-group G' such that

$$G \simeq \mathrm{Res}_{k'/k}(G'),$$

where G' is either basic exotic or basic non-reduced. The conclusion is then followed by the triviality of torsor under basic non-reduced groups and a bijection between the set of torsors under basic exotic groups and that of semisimple, simply connected groups.

This article is organized as follows. In section 2, we review the classification of pseudo-reductive groups. In section 3, we prove a decomposition result and a structure result for pseudo-semisimple, simply connected groups. As an application, we derive the main theorem 1.2.

1.5. Conventions and notation. For a field k, let k^s be a choice of its separable closure. The *degree of imperfection* of k when $p := \mathrm{char}(k) > 0$, is a nonnegative integer r (if it exists) such that

k/k^p is of degree p^r. If $\mathrm{char}(k) = 0$, the degree of imperfection is 0. For a prime ℓ, the *ℓ-cohomological dimension* of k is the smallest nonnegative integer n such that for every torsion $\mathrm{Gal}(k^s/k)$-module A, the ℓ-primary torsion subgroup of its ith Galois cohomology group $\mathrm{H}^i(k^s/k, A)$ is trivial for all $i > n$. If no such n exists, the *ℓ-cohomological dimension* of k is $+\infty$. The *cohomological dimension* of k is the supremum of all of its ℓ-cohomological dimensions for different primes ℓ.

The *k-unipotent radical* of a connected, affine, smooth k-group G, denoted by $\mathscr{R}_{\mathrm{u},k}(G)$, is the maximal connected, smooth, unipotent normal k-subgroup of G. A connected, affine, smooth k-group G is

- *pseudo-reductive* if $\mathscr{R}_{\mathrm{u},k}(G) = 1$;
- *perfect* if it is equal to its derived subgroup;
- *simply connected* if the reductive $\overline{k}$-group $G_{\overline{k}}/\mathscr{R}_{\mathrm{u},\overline{k}}(G_{\overline{k}})$ is perfect and simply connected.

A pseudo-reductive k-group is *pseudo-semisimple* if it is perfect. A connected, affine, smooth k-group is *pseudo-simple* if it is non-commutative and it does not contain any connected, smooth, normal subgroup except 1 and itself. A k-group G is *absolutely pseudo-simple* if G_{k^s} is pseudo-simple.

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2. Recollections on pseudo-reductive groups

Throughout this section, k is a field and $p := \mathrm{char}(k)$. The key result (cf. Lemma 2.2) is that every absolutely pseudo-simple, simply connected k-group is isomorphic to the restriction of scalars of some semisimple, simply connected group defined over a finite field extension of k, when $\mathrm{char}(k) > 3$. The mechanism to measure how a pseudo-reductive group is near to being a restriction of scalars of a reductive group is the comparision map. We will review the comparison map and the key result in subsection 2.1.

2.1. The comparison map i_G. Let G be a connected, affine, smooth k-group and let k' be the field of definition of $\mathscr{R}_{\mathrm{u},\overline{k}}(G_{\overline{k}})$. There is a natural map

$$i_G \colon G \to \mathrm{Res}_{k'/k}(G^{\mathrm{red}}_{k'})$$

corresponding to the quotient map $G_{k'} \to G^{\mathrm{red}}_{k'} := G_{k'}/\mathscr{R}_{\mathrm{u},k'}(G_{k'})$. We call i_G the *comparison map* of G. A pseudo-reductive k-group G is *of minimal type* if $\ker(i_G) \cap Z_G(T) = 1$ for some maximal k-torus $T \subset G$ (equivalently, for every maximal k-torus $T \subset G$, cf. [CGP15, Proposition 9.4.2]). By [CP16, Remark 4.3.2, Proposition 5.3.3], every pseudo-semisimple, simply connected k-group is of minimal type except possibly when $p = 2$ and $[k : k^2] \geqslant 4$.

Let k' be a finite reduced k-algebra. Let G' be a k'-group such that its fiber over each factor field of k' is reductive and let T' be a maximal k'-torus of G'. The conjugation action of T' on G' induces an action of $T'/Z_{G'}$ on G'. Via the functoriality, $\mathrm{Res}_{k'/k}(T'/Z_{G'})$ naturally acts on $\mathrm{Res}_{k'/k}(G)$. Given a pair (C, ϕ) of a commutative, pseudo-reductive k-group C and a k-homomorphism $\phi \colon \mathrm{Res}_{k'/k}(T') \to C$ such that the k-homomorphism $\mathrm{Res}_{k'/k}(T') \to \mathrm{Res}_{k'/k}(T'/Z_{G'})$ factors through ϕ, the k-group C acts on $\mathrm{Res}_{k'/k}(G')$ through the k-homomorphism $C \to \mathrm{Res}_{k'/k}(T'/Z_{G'})$. Consider the inclusion $\iota \colon \mathrm{Res}_{k'/k}(T') \hookrightarrow \mathrm{Res}_{k'/k}(G')$ and the twisted diagonal map

$$\alpha \colon \mathrm{Res}_{k'/k}(T') \to \mathrm{Res}_{k'/k}(G') \rtimes C, \quad t' \mapsto (\iota(t')^{-1}, \phi(t')).$$

By [CGP15, Proposition 1.4.3], $\mathrm{coker}(\alpha)$ is a pseudo-reductive k-group. Every k-group G that arises as $\mathrm{coker}(\alpha)$ for some (G', T', C, ϕ) is called *standard*. The next lemma states that if an absolutely pseudo-simple, simply connected k-group G is standard and the root system of G_{k^s} is reduced then G is isomorphic to a restriction of scalars of an absolutely simple, simply connected k'-group for some purely inseparable finite extension k' of k. We also recall here some criteria for standardness and reducedness of root system.

Lemma 2.2. *Let G be a pseudo-reductive k-group and set $\overline{G} := G^{\mathrm{red}}_{\overline{k}}$.*

(a) *The root system of G_{k^s} is reduced unless k is imperfect, $p = 2$ and $\overline{G}$ has a connected, simple, semisimple, simply connected normal $\overline{k}$-subgroup of type C_n with $n \geqslant 1$.*

(b) *The k-group G is standard except possibly when $p \in \{2, 3\}$ and the Dynkin diagram of $\overline{G}$ either contains an edge with multiplicity p or has an isolated vertex when $p = 2$.*

(c) *If G is an absolutely pseudo-simple, simply connected, standard and the root system of G_{k^s} is reduced, then the comparison map i_G is an isomorphism.*

Proof. The claim (a) on reducedness of the root system of G_{k^s} is [CGP15, Theorem 2.3.10] and the claim (b) on standardness of G is [CGP15, Theorem 5.1.1(1)]. The last claim (c) is [CP16, Proposition 3.2.7]. □

By Lemma 2.2, when $p > 3$, the comparison map of an absolutely pseudo-simple, simply connected k-group is an isomorphism. The next subsections briefly review the theory of absolutely pseudo-simple, simply connected groups such that the corresponding comparison maps are not isomorphisms. By Lemma 2.2, these groups arise when the group itself is not standard or the root system over k^s is not reduced.

2.3. Exotic and basic exotic groups. A pseudo-reductive k-group that is not standard could decompose into a product of a standard k-group and an *exotic k-group* provided $p = 3$ (cf. Lemma 2.4).

Suppose that $p \in \{2, 3\}$. Let G be an absolutely simple, simply connected k-group with Dynkin diagram having an edge with multiplicity p. By [CGP15, Lemma 7.1.2], there is a simply connected, absolutely simple k-group $\widetilde{G}$ such that the relative Frobenius isogeny $F_{G/k}\colon G \to G^{(p)}$ is factorized uniquely into the composition of two k-isogenies

$$G \xrightarrow{\pi} \widetilde{G} \to G^{(p)}$$

such that π is non-central and has no nontrivial factorization. We call $\pi : G \to \widetilde{G}$ a *very special k-isogeny* of G.

Assume that k is imperfect and $p \in \{2, 3\}$. A smooth k-group G is *basic exotic* (cf. [CGP15, Definition 7.2.6, Proposition 7.3.1]) if there exists a finite field extension k'/k with $k'^p \subset k$, a very special k'-isogeny $\pi'\colon G' \to \widetilde{G'}$ and a Levi k-subgroup $\overline{G}$ of $\mathrm{Res}_{k'/k}(\widetilde{G'})$ such that $G \simeq f^{-1}(\overline{G})$, where $f := \mathrm{Res}_{k'/k}(\pi')$. By [CGP15, Lemma 7.2.1], for a reductive k'-group G', a k-group $\mathrm{Res}_{k'/k}(G')$ admits a Levi k-subgroup if and only if G' is defined over k. By [CGP15, Lemma 7.2.1, Theorem 7.2.3, Proposition 7.3.1], every basic exotic k-group is absolutely pseudo-simple. Basic exotic k-groups are never standard [CGP15, Proposition 8.1.1].

An *exotic k-group* is a k-group of the form $\mathrm{Res}_{k'/k}(G')$ for a finite reduced k-algebra k' and a k'-group G' whose fibers are basic exotic. There is a decomposition lemma [CGP15, Theorem 8.2.10] for pseudo-reductive groups over fields of characteristic 3.

Lemma 2.4. *For $p = 3$ and every pseudo-reductive k-group G, there is a unique decomposition*

$$G = G_1 \times G_2$$

such that G_1 is standard and G_2 is either trivial or exotic.

When $k' = k^{1/p}$ and k is of finite degree of imperfection, the following result [CGP15, Proposition 7.3.3(1)] reduces the study of torsors under basic exotic k-groups to that of semisimple, simply connected k-groups.

Lemma 2.5. *Let k be an imperfect field of finite degree of imperfection and $p \in \{2, 3\}$. Let G be a basic exotic k-group arising from $(G', k', \overline{G})$ such that $k' = k^{1/p}$. Then the natural map*

$$\mathrm{H}^1(k, G) \to \mathrm{H}^1(k, \overline{G})$$

is bijective.

2.6. Totally non-reduced and basic non-reduced groups. A pseudo-reductive k-group G is *totally non-reduced* if G is perfect and every irreducible component of a root system of G_{k^s} is not reduced. A *basic non-reduced* k-group is an absolutely pseudo-simple k-group such that the root system of G_{k^s} is not reduced and the field of definition of $\mathscr{R}_{\mathrm{u},\overline{k}}(G_{\overline{k}})$ is quadratic over k. By 2.2(a), these groups occur only when k is imperfect and $p = 2$.

Suppose from now that $p = 2$ and $[k : k^2] = 2$. By [CP16, Proposition 10.1.4], every totally non-reduced k-group G is isomorphic to $\mathrm{Res}_{k'/k}(G')$ for a finite reduced k-algebra k' and a k'-group G' whose fibers are basic non-reduced, which is analogous to the concept of exotic and basic-exotic groups. We also have the decomposition lemma [CGP15, Proposition 10.1.6] for pseudo-reductive groups that has non-reduced root system over k^s.

Lemma 2.7. *For $p = 2$, $[k : k^2] = 2$ and a pseudo-reductive k-group G such that the root system of G_{k^s} is not reduced, there is a unique decomposition*

$$G = G_1 \times G_2$$

such that G_1 is totally non-reduced and $(G_2)_{k^s}$ has a reduced root system.

We conclude with the vanishing lemma [CGP15, Proposition 9.9.4] of $\mathrm{H}^1(k, G)$.

Lemma 2.8. *For $p = 2$, $[k : k^2] = 2$ and an absolutely pseudo-simple k-group G such that the root system of G_{k^s} is not reduced,*

$$\mathrm{H}^1(k, G) = \{*\}.$$

3. Structure of pseudo-semisimple, simply connected groups

Throughout this section, k is a field and $p := \mathrm{char}(k)$.

The common strategy to study semisimple, simply connected is to use its decomposition into simple factors. We adapt this strategy to "pseudo-reductive" setting.

Lemma 3.1. *Suppose that $[k : k^2] \leqslant 2$ when $p = 2$. For every pseudo-semisimple, simply connected k-group G, there is an isomorphism of k-groups*

$$G \simeq \prod_{i=1}^{n} G_i$$

such that each G_i is pseudo-simple, simply connected. Moreover, the k-groups G_i's are unique up to permutation and isomorphism.

Proof. Every pseudo-semisimple, simply connected k-group G is of minimal type (cf. 2.1). The decomposition is then followed by [CP16, Proposition 5.3.4]. ☐

Corollary 3.2. *Suppose that $[k : k^2] \leqslant 2$ when $p = 2$. For every pseudo-semisimple, simply connected k-group G, there is a finite étale k-algebra k' and an affine, smooth k'-group G' whose fiber over each factor field of k' is absolutely pseudo-simple, simply connected such that there is an isomorphism of k-groups*

$$G \simeq \mathrm{Res}_{k'/k}(G').$$

This follows from a standard Galois descent argument. However, for the sake of completion, we give a complete proof here.

Proof. Applying Lemma 3.1 for G_{k^s}, there are pseudo-simple, simply connected k^s-groups G_i such that

$$G_{k^s} \simeq \prod_{i=1}^{n} G_i.$$

The Galois group $\mathrm{Gal}(k^s/k)$ acts on the set $\{G_i : 1 \leqslant i \leqslant n\}$ by the uniqueness of the decomposition. For $1 \leqslant i \leqslant n$, the stabilizer of G_i is $\mathrm{Gal}(k^s/k_i)$ for some finite field extension of k_i/k. So G_i is defined over k_i and the product of simple factors in its orbit is isomorphic to

$$\prod_{\sigma \in \mathrm{Gal}(k^s/k)/\mathrm{Gal}(k^s/k_i)} G_i^{\sigma} \simeq (\mathrm{Res}_{k_i/k}(G_i))_{k_i}.$$

In other words, such product descends to a k-group $\mathrm{Res}_{k_i/k}(G_i)$. This gives the desired presentation since G_i is absolutely pseudo-simple over k_i. ☐

After reducing to absolutely pseudo-simple, simply connected k-group, we prove the following structure theorem for absolutely pseudo-simple, simply connected groups.

Theorem 3.3. *Suppose that if $p = 2$ then $[k : k^2] \leqslant 2$. For an absolutely pseudo-simple, simply connected k-group G, there is a finite extension k'/k and a k'-group G' such that*

$$G \simeq \mathrm{Res}_{k'/k}(G'),$$

where G' is either

- *a semisimple, simply connected k'-group;*
- *a basic exotic k'-group (exists only when $\mathrm{char}(k) \in \{2, 3\}$);*
- *a basic non-reduced k'-group (exists only when $\mathrm{char}(k) = 2$).*

Proof. If $p > 3$, then this follows by Lemma 2.2. If $p = 3$, then by Lemma 2.4, G is either standard or exotic. If G is standard, then we could utilize Lemma 2.2(c). If G is exotic, then by pseudo-simplicity, $G = \mathrm{Res}_{k'/k}(G')$ for some finite field extension k'/k and basic exotic k'-group G'. Suppose now that $p = 2$. By Lemma 2.7, either G is totally non-reduced or G_{k^s} has a reduced root system. In the former case, $G \simeq \mathrm{Res}_{k'/k}(G')$ for some finite extension k'/k and basic non-reduced k'-group G' (cf. 2.6). One may assume that G_{k^s} has a reduced root system. If G is standard, we could utilize Lemma 2.2(c) again. Otherwise, by [CGP15, Proposition 11.1.4], there is a finite extension k'/k and a basic exotic k'-group G' such that $G \simeq \mathrm{Res}_{k'/k}(G')$. ☐

We conclude with the proof of the main result of this article.

Corollary 3.4. *Let k be a field of cohomological dimension at most 2 and has degree of imperfection at most 1. The following are equivalent*

(i) $\mathrm{H}^1(k, G) = \{*\}$ *for every semisimple, simply connected k-group G;*

(ii) $\mathrm{H}^1(k, G) = \{*\}$ *for every pseudo-semisimple, simply connected k-group G.*

Proof. Recall that cohomological dimension of algebraic extension of k is not increasing and the imperfection degree is insensitive passing to finite extensions. Notice also that if k'/k is a finite field extension and G' is a smooth, affine k'-group, then

$$\mathrm{H}^1(k, \mathrm{Res}_{k'/k}(G')) = \mathrm{H}^1(k', G').$$

Let G be a pseudo-semisimple, simply connected k-group. Thanks to Corollary 3.2, one could assume that G is absolutely pseudo-simple, simply connected. By the structure theorem 3.3, we are reducing to the case where G is either semisimple, simply connected k-group or basic exotic or basic non-reduced. By Lemma 2.5 and Lemma 2.8, we reduce to the semisimple, simply connected case, as desired. □

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