An equality on the complex Monge-Ampère measures

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Abstract

In this paper we extend an equality on the complex Monge-Ampère measures of Bedford-Taylor which holds in the class of locally bounded plurisubharmonic functions to the class $\mathcal{E}(\Omega)$ introduced and investigated by Cegrell in [5] and the class of plurisubharmonic functions which are bounded near the boundary.

1 Introduction

Let Ω be an open set in $\mathbb{C}^n$ and u, v be locally bounded plurisubharmonic functions on Ω. As in [3], Bedford-Taylor have shown that if $\mathcal{O} \subset \Omega$ is a plurifinely open set and if $u = v$ on $\mathcal{O}$ then

$$(dd^c u)^n|_{\mathcal{O}} = (dd^c v)^n|_{\mathcal{O}},$$

(see Corollary 4.3 in [3]). Here the plurifine topology on an open set Ω in $\mathbb{C}^n$ is the coarsest topology on Ω that makes all plurisubharmonic functions on Ω are continuous. The plurifine topology has been investigated by some authors, for example, Wiegerinck, El Marzguioui, El Kadiri, Fuglede v. v. v. In order to look for essential results concerning to the plurifine topology and plurifinely plurisubharmonic functions, plurifine holomorphic functions we refer readers to the papers [13], [10], [9]. Next, in the case Ω is a bounded hyperconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{E}(\Omega)$, $v \in PSH^-(\Omega)$ and the set $\mathcal{O} = \{u > v\}$, in [14], Khue-Hiep have proved that

$$(dd^c \max(u, v))^n|_{\{u>v\}} = (dd^c u)^n|_{\{u>v\}},$$

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(see Theorem 4.1 in [14]) where $\mathcal{E}(\Omega)$ is the class of negative plurisubharmonic functions introduced and investigated by Cegrell in [5] and [6] which we will recall in the next section and $PSH^-(\Omega)$ denotes the set of negative plurisubharmonic functions on Ω. Very recent, in [9] when to study plurifinely plurisubharmonic functions and to construct the Monge-Ampère operator in this class, Kadiri and Wiegerinck extended the above result of Bedford-Taylor to the class of finite plurifinely plurisubharmonic functions. They proved that if u and v are finite plurifinely plurisubharmonic functions on a plurifinely open subset $U \subset \Omega$, Ω is open in $\mathbb{C}^n$ and if $u = v$ on a plurifinely open subset $\mathcal{O} \subset U$ then

$$(dd^c u)^n|_{\mathcal{O}} = (dd^c v)^n|_{\mathcal{O}},$$

(see Theorem 4.8 in [9]). To continue the direction of the above investigations in the note we establish these equalities for the class $\mathcal{E}(\Omega)$ on the sets of the form $\mathcal{O} = \{\varphi_1 > \psi_1\} \cap \{\varphi_2 > \psi_2\} \cap \cdots \cap \{\varphi_m > \psi_m\}$ where $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_m$ are plurisubharmonic functions on Ω and u, v are plurisubharmonic functions in the class $\mathcal{E}(\Omega)$. Namely, the first result of this note is the following

Theorem 1.1. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and $\varphi_1, \ldots, \varphi_m$, $\psi_1, \ldots, \psi_m$ are plurisubharmonic functions on Ω. Let $\mathcal{O} = \{\varphi_1 > \psi_1\} \cap \{\varphi_2 > \psi_2\} \cap \cdots \cap \{\varphi_m > \psi_m\}$. Assume that $u, v \in \mathcal{E}(\Omega)$. If $u = v$ on $\mathcal{O}$ then*

$$(dd^c u)^n|_{\mathcal{O}} = (dd^c v)^n|_{\mathcal{O}}.$$

Next, we extend this result for the class of plurisubharmonic functions which are bounded near the boundary of Ω. We have the following.

Theorem 1.2. *Let Ω be a bounded open set in $\mathbb{C}^n$ and $\varphi_1, \ldots, \varphi_m$, $\psi_1, \ldots, \psi_m$ are plurisubharmonic functions on Ω, T be a closed positive current of bidegree $(n-1, n-1)$ on Ω. Let $\mathcal{O} = \{\varphi_1 > \psi_1\} \cap \{\varphi_2 > \psi_2\} \cap \cdots \cap \{\varphi_m > \psi_m\}$. Assume that $u, v \in PSH \cap L^\infty_{loc}(\Omega \setminus K)$ where $K \Subset \Omega$. If $u = v$ on $\mathcal{O}$ then*

$$dd^c u \wedge T|_{\mathcal{O}} = dd^c v \wedge T|_{\mathcal{O}}.$$

Note that the set $\mathcal{O}$ in Theorem 1.1 and Theorem 1.2 are plurifinely open which are easy to image then a plurifinely open set in the general form. To get the proof of Theorem 1.2, in the fourth section of this paper we will construct the wedge product of a plurisubharmonic function which is bounded near the boundary of Ω with a closed positive current T.
The paper is organized as follows. In section 2 we recall some notions of pluripotential theory and Cegrell classes $\mathcal{F}(\Omega), \mathcal{E}(\Omega)$ in a bounded hyperconvex domain Ω of $\mathbb{C}^n$. Section 3 is devoted to the proof of Theorem 1.1 and in Section 4 we give the proof of Theorem 1.2.

2 The Cegrell classes

Some elements of pluripotential theory that will be used throughout the paper can be found in [1], [2], [3], [5], [6], [12], [14]. As usually, we denote by $d = \partial + \overline{\partial}$ the exterior differential and by $d^c = i(\overline{\partial} - \partial)$ the conjugate differential. Then $dd^c = 2i\partial\overline{\partial}$. Let Ω be an open set in $\mathbb{C}^n$. By $PSH^-(\Omega)$ we denote the set of negative plurisubharmonic functions on Ω. A domain $\Omega \subset \mathbb{C}^n$ is said to be hyperconvex if there exists $\varphi \in PSH^-(\Omega)$ such that $\{\varphi < -\varepsilon\} \Subset \Omega$ for every $\varepsilon > 0$.

2.1 First we recall the Cegrell classes $\mathcal{E}_0 = \mathcal{E}_0(\Omega), \mathcal{F} = \mathcal{F}(\Omega)$ and $\mathcal{E} = \mathcal{E}(\Omega)$ introduced and investigated in [5] and [6] in the case Ω is a bounded hyperconvex domain in $\mathbb{C}^n$. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. As in [5] and [6] we recall the following subclasses of $PSH^-(\Omega)$:

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \{\varphi \in PSH^-(\Omega) \cap \mathrm{L}^\infty(\Omega) : \lim_{z \to \partial\Omega} \varphi(z) = 0, \quad \int_\Omega (dd^c\varphi)^n < \infty\},$$

$$\mathcal{F} = \mathcal{F}(\Omega) = \big\{\varphi \in PSH^-(\Omega) : \exists \ \ \mathcal{E}_0 \ni \varphi_j \searrow \varphi, \ \ \sup_j \int_\Omega (dd^c\varphi_j)^n < \infty\big\},$$

and

$$\begin{aligned}\mathcal{E} = \mathcal{E}(\Omega) = \big\{\varphi \in PSH^-(\Omega) : \forall z_0 \in \Omega, \exists \ \text{ a neighbourhood } \ \omega \ni z_0,\\ \mathcal{E}_0 \ni \varphi_j \searrow \varphi \ \text{ on } \ \omega, \sup_j \int_\Omega (dd^c\varphi_j)^n < \infty\big\}.\end{aligned}$$

As in [5], we note that if $u \in PSH^-(\Omega)$ then $u \in \mathcal{E}(\Omega)$ if and only if for every $\omega \Subset \Omega$, there exists $v \in \mathcal{F}(\Omega)$ such that $v \geq u$ on Ω and $v = u$ on ω.

3 Proof of Theorem 1.1.

Write $\mathcal{O} = \bigcap_{j=1}^m \{\varphi_j > \psi_j\} = \bigcup_{c_j \in \mathbb{Q}} \bigcap_{j=1}^m \{\varphi_j > c_j > \psi_j\}$, where $\mathbb{Q}$ denotes the set of rational numbers. It is enough to show that if $u = v$ on $\mathcal{O}$ then

$$(dd^c u)^n|_{\bigcap_{j=1}^m \{\varphi_j > c_j > \psi_j\}} = (dd^c v)^n|_{\bigcap_{j=1}^m \{\varphi_j > c_j > \psi_j\}},$$

for all $c_j \in \mathbb{Q}$. Now, for each $k \geq 1$, set $u_k = \max(u, -k)$ and $v_k = \max(v, -k)$. Then $u_k, v_k \in PSH(\Omega) \cap L^\infty(\Omega)$, $u_k \searrow u, v_k \searrow v$ as $k \to \infty$. On the other hand, from $u = v$ on $\mathcal{O}$ then we also have $u_k = v_k$ on $\mathcal{O}$. Corollary 4.3 in [3] implies that

$$(dd^c u_k)^n|_{\mathcal{O}} = (dd^c v_k)^n|_{\mathcal{O}}.$$

Thus it follows that

$$\min_{1 \leq j \leq m} \Big(\max(\varphi_j, \psi_j, c_j) - \max(\psi_j, c_j)\Big)(dd^c u_k)^n$$

$$= \min_{1\le j\le m}\Big(\max(\varphi_j,\psi_j,c_j)-\max(\psi_j,c_j)\Big)(dd^cv_k)^n,$$

for all $k\ge 1$. By Corollary 3.2 in [15], letting $k\to\infty$, we deduce that

$$\min_{1\le j\le m}\Big(\max(\varphi_j,\psi_j,c_j)-\max(\psi_j,c_j)\Big)(dd^cu)^n$$

$$= \min_{1\le j\le m}\Big(\max(\varphi_j,\psi_j,c_j)-\max(\psi_j,c_j)\Big)(dd^cv)^n.$$

However, $\min\limits_{1\le j\le m}\Big(\max(\varphi_j,\psi_j,c_j)-\max(\psi_j,c_j)\Big)=\min\limits_{1\le j\le m}\{\varphi_j-c_j\}>0$ on the set $\bigcap\limits_{j=1}^{m}\{\varphi_j>c_j>\psi_j\}$ and Lemma 4.2 in [14] implies that

$$(dd^cu)^n=(dd^cv)^n,$$

on the set $\bigcap\limits_{j=1}^{m}\{\varphi_j>c_j>\psi_j\}$ and the desired conclusion follows.

Now we give a corollary of Theorem 1.1 concerning to the comparison principle of the log canonical threshold. As in [8], Demailly and Kollár introduced the log canonical threshold of a plurisubharmonic function u at $z\in\Omega$ as-follows. Let u be a plurisubharmonic function on an open set Ω in $\mathbb{C}^n$ and $z\in\Omega$. By $c_u(z)$ we denote the log canonical threshold of u at z and as in [8] it is defined by:

$$c_u(z)=\sup\big\{c>0:\ e^{-2cu}\text{ is }L^1\text{ on a neighborhood of }z\big\}.$$

Using Theorem 1.1, we obtain the following result.

Corollary 3.1. *Let Ω be a domain in $\mathbb{C}^n$ and $\mathcal{O}$ be a plurifinely open subset in Ω. Assume that $u,v\in PSH(\Omega)$ such that $u\ge v$ on $\mathcal{O}$. Then $c_u(z)\ge c_v(z)$ for all $z\in\mathcal{O}$.*

Proof. Take $z_0\in\mathcal{O}$. By Theorem 2.3 in [3], there exists a plurisubharmonic function φ on a neighborhood $D=\{z\in\mathbb{C}^n:\ ||z-z_0||<r\}$ of z_0 such that $z_0\in\{z\in D:\varphi(z)>0\}\subset\mathcal{O}$. Since $u\ge v$ on $\mathcal{O}$, we have $\max(u,v,g)=\max(u,g)$ on $\{z\in D:\varphi(z)>0\}$, for all $g\in PSH(D)\cap L^\infty_{loc}(D\backslash\{z_0\})$. Using Theorem 1.1, we have

$$\int\limits_{\{z_0\}}(dd^c\max(u,v,g))^n=\int\limits_{\{z_0\}}(dd^c\max(u,g))^n,$$

for all $g\in PSH(D)\cap L^\infty_{loc}(D\backslash\{z_0\})$. By Theorem 3.3 in [11], we obtain $c_u(z_0)\ge c_v(z_0)$. □

4 Proof of Theorem 1.2.

In order to prove Theorem 1.2 we need to construct the wedge product of a plurisubharmonic function which is bounded near the boundary of Ω with a closed nonnegative current T of bidegree $(p,p), p<n$. Namely, we prove the following proposition.

Proposition 4.1. *Let Ω be an open subset in $\mathbb{C}^n$ and T be a closed positive current of bidegree (p, p) on Ω $(p < n)$. Assume that u is a plurisubharmonic function which is bounded near the boundary of Ω. Then the current uT has locally finite mass in Ω.*

Remark 4.2. To differ the above proposition to Proposition 2.1 in [7] is in our proposition we remove the hypothesis Ω is covered by a family of Stein open sets $X \Subset \Omega$ whose boundaries ∂X do not intersect $L(u) \cap SuppT$ where $L(u)$ denotes the unbounded locus of u (see the hypothesis (b) of Proposition 2.1 in [7]).

Proof. Without loss of generality, we may assume that $u \in PSH^-(\Omega)$. Let $\Omega'' \Subset \Omega' \Subset \Omega$ be such that

$$M = \sup \|u\|_{L^\infty(\Omega \setminus \Omega'')} < +\infty.$$

Take a sequence $\{u_j\}_{j\geq 1} \subset PSH \cap C^\infty(\Omega)$ such that $u_j \searrow u$ on Ω'. First, we show that

$$\sup_{j\geq 1} \int\limits_{\Omega'} dd^c u_j \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} < +\infty.$$

Indeed, take a function $\phi \in C_0^\infty(\Omega)$ such that $\phi = 1$ on $\overline{\Omega''}$ and $supp\phi \subset \Omega'$. We choose $C > 0$ such that $-Cdd^c\|z\|^2 \leq dd^c\phi \leq Cdd^c\|z\|^2$. By Stoke's theorem, we have

$$\begin{aligned}
\int\limits_{\Omega'} dd^c u_j \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} &\leq \int\limits_{\Omega} \phi dd^c u_j \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} \\
&\leq \int\limits_{\Omega} u_j dd^c\phi \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} \\
&= \int\limits_{\Omega'\setminus\Omega''} u_j dd^c\phi \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} \\
&\leq C \int\limits_{\Omega'\setminus\Omega''} -u_j T \wedge (dd^c\|z\|^2)^{n-p} \\
&\leq CM \int\limits_{\Omega'\setminus\Omega''} T \wedge (dd^c\|z\|^2)^{n-p}.
\end{aligned}$$

Hence,

$$\sup_{j\geq 1} \int\limits_{\Omega'} dd^c u_j \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} < +\infty.$$

Next, we prove that

$$\sup_{j\geq 1} \int\limits_{\Omega'} -u_j T \wedge (dd^c\|z\|^2)^{n-p} < +\infty.$$

Indeed, by Stoke's theorem we have

$$\begin{aligned}
\int\limits_{\Omega'} -u_j T \wedge (dd^c\|z\|^2)^{n-p} &\le \int\limits_{\Omega'} -\phi u_j T \wedge (dd^c\|z\|^2)^{n-p} \\
&= \int\limits_{\Omega} \|z\|^2 dd^c(-\phi u_j) \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} \\
&\le \int\limits_{\Omega} -\|z\|^2 \phi dd^c u_j \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} \\
&- 2\int\limits_{\Omega} \|z\|^2 du_j \wedge d^c\phi \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} \\
&+ \int\limits_{\Omega} -\|z\|^2 u_j dd^c\phi \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} \\
&\le -2\int\limits_{\Omega} \|z\|^2 du_j \wedge d^c\phi \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} \\
&+ \int\limits_{\Omega} -\|z\|^2 u_j dd^c\phi \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} \\
&= 2\int\limits_{\Omega} u_j d\|z\|^2 \wedge d^c\phi \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} \\
&+ \int\limits_{\Omega} u_j \|z\|^2 dd^c\phi \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} \\
&= 2\int\limits_{\Omega'\setminus\Omega''} u_j d\|z\|^2 \wedge d^c\phi \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} \\
&+ \int\limits_{\Omega'\setminus\Omega''} u_j \|z\|^2 dd^c\phi \wedge T \wedge (dd^c\|z\|^2)^{n-p-1} \\
&\le (2\sup_{z\in\Omega'} |d\phi(z)| + C\sup_{z\in\Omega'} \|z\|^2) M \int\limits_{\Omega'\setminus\Omega''} T \wedge (dd^c\|z\|^2)^{n-p}.
\end{aligned}$$

Hence,

$$\sup_{j\ge 1} \int\limits_{\Omega'} -u_j T \wedge (dd^c\|z\|^2)^{n-p} < +\infty.$$

This implies that

$$\int\limits_{\Omega'} |u| T \wedge (dd^c\|z\|^2)^{n-p} < +\infty.$$

The proof is complete. □

Definition 4.3. From Proposition 4.1 we can define the current $dd^c u \wedge T$ by

$$dd^c u \wedge T = dd^c(uT),$$

for the case T is a closed nonegative current of bidegree (p,p) $(p < n)$ and u is a plurisubharmonic function which bounded near the boundary of Ω.

As the proof of Corollary 2.3 in [7] and Corollary 3.2 in [15], we have the following proposition.

Proposition 4.4. *Let Ω be an open subset in $\mathbb{C}^n$ and T be a closed positive current of bidegree (p,p) on Ω $(p < n)$. Assume that u_j, u are plurisubharmonic functions which are bounded near the boundary of Ω. If $u_j \searrow u$ then the current $h(\varphi_1, ..., \varphi_m) dd^c u_j \wedge T \to h(\varphi_1, ..., \varphi_m) dd^c u \wedge T$ weakly for all $\varphi_1, ..., \varphi_m \in PSH \cap L^\infty_{loc}(\Omega)$, $h \in C(\mathbb{R}^m)$.*

Proposition 4.5. *Let $\Omega \subset \mathbb{C}^n$ be an open set and T be a closed positive current of bidegree $(n-1, n-1)$ on Ω. Assume that $u, v \in PSH \cap L^\infty_{loc}(\Omega)$. Then*

$$(dd^c \max(u,v)) \wedge T|_{\{u>v\}} = dd^c u \wedge T|_{\{u>v\}}.$$

Proof. Take $\Omega' \Subset \Omega$. We choose a sequence $u_j \in PSH \cap C^\infty(\Omega')$ such that $u_j \searrow u$ on Ω'. Since $\{u_j > v\}$ is an open set, we have

$$(dd^c \max(u_j, v) \wedge T|_{\{u_j>v\}} = dd^c u_j \wedge T|_{\{u_j>v\}}.$$

Moreover, since $\{u > v\} \subset \{u_j > v\}$, we get

$$(dd^c \max(u_j, v)) \wedge T|_{\{u>v\}} = dd^c u_j \wedge T|_{\{u>v\}}.$$

Hence

$$[\max(u,v) - v](dd^c \max(u_j, v)) \wedge T = [\max(u,v) - v] dd^c u_j \wedge T.$$

By Proposition 4.4, letting $j \to \infty$, we get

$$[\max(u,v) - v](dd^c \max(u,v)) \wedge T = [\max(u,v) - v] dd^c u \wedge T.$$

By Lemma 4.2 in [14], we deduce that

$$(dd^c \max(u,v)) \wedge T|_{\{u>v\}} = dd^c u \wedge T|_{\{u>v\}}.$$

□

As a corollary of the above fact we get the following.

Proposition 4.6. *Let $\Omega \subset \mathbb{C}^n$ be an open set and T be a closed positive current of bidegree $(n-1, n-1)$ on Ω. Assume that $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_m \in PSH(\Omega)$ and $u, v \in PSH \cap L^\infty_{loc}(\Omega)$ such that $u = v$ on $\mathcal{O} = \{\varphi_1 > \psi_1\} \cap \{\varphi_2 > \psi_2\} \cap \cdots \cap \{\varphi_m > \psi_m\}$. Then*

$$dd^c u \wedge T|_{\mathcal{O}} = dd^c v \wedge T|_{\mathcal{O}}.$$

Proof. First, we will prove that

$$(dd^c \max(u,v)) \wedge T|_{\mathcal{O}} = dd^c u \wedge T|_{\mathcal{O}}.$$

Write $\mathcal{O} = \bigcap_{j=1}^{m} \{\varphi_j > \psi_j\} = \bigcup_{c_j \in \mathbb{Q}} \bigcap_{j=1}^{m} \{\varphi_j > c_j > \psi_j\}$. It is enough to show that if $u = v$ on $\mathcal{O}$ then

$$(dd^c \max(u,v)) \wedge T|_{\bigcap_{j=1}^{m}\{\varphi_j > c_j > \psi_j\}} = dd^c u \wedge T|_{\bigcap_{j=1}^{m}\{\varphi_j > c_j > \psi_j\}}.$$

By Proposition 4.5, we have

$$(dd^c \max(u+\epsilon, v)) \wedge T|_{\{u+\epsilon > v\}} = dd^c u \wedge T|_{\{u+\epsilon > v\}},$$

for all $\epsilon > 0$. Moreover, since $\mathcal{O} \subset \{u + \epsilon > v\}$, we get

$$(dd^c \max(u+\epsilon, v)) \wedge T|_{\mathcal{O}} = dd^c u \wedge T|_{\mathcal{O}},$$

for all $\epsilon > 0$. Hence

$$\begin{aligned}\min_{1\le j\le m}&\Big(\max(\varphi_j, \psi_j, c_j) - \max(\psi_j, c_j)\Big)(dd^c \max(u+\epsilon, v)) \wedge T \\ &= \min_{1\le j\le m}\Big(\max(\varphi_j, \psi_j, c_j) - \max(\psi_j, c_j)\Big) dd^c u \wedge T.\end{aligned}$$

for all $\epsilon > 0$. By Proposition 4.4, letting $\epsilon \searrow 0$, we get

$$\begin{aligned}\min_{1\le j\le m}&\Big(\max(\varphi_j, \psi_j, c_j) - \max(\psi_j, c_j)\Big)(dd^c \max(u, v)) \wedge T = \\ &\min_{1\le j\le m}\Big(\max(\varphi_j, \psi_j, c_j) - \max(\psi_j, c_j)\Big) dd^c u \wedge T.\end{aligned}$$

Lemma 4.2 in [14] implies that

$$(dd^c \max(u,v)) \wedge T|_{\bigcap_{j=1}^{m}\{\varphi_j > c_j > \psi_j\}} = dd^c u \wedge T|_{\bigcap_{j=1}^{m}\{\varphi_j > c_j > \psi_j\}}.$$

Similarly, we have

$$(dd^c \max(u,v)) \wedge T|_{\mathcal{O}} = dd^c v \wedge T|_{\mathcal{O}}.$$

Therefore

$$dd^c u \wedge T|_{\mathcal{O}} = dd^c v \wedge T|_{\mathcal{O}}.$$

□

Proof of Theorem 1.2. Now, by using the same the notations as in proof of Theorem 1.1 we get the following

$$\begin{aligned}&\min_{1\le j\le m}\Big(\max(\varphi_j, \psi_j, c_j) - \max(\psi_j, c_j)\Big) dd^c u_k \wedge T \\ &= \min_{1\le j\le m}\Big(\max(\varphi_j, \psi_j, c_j) - \max(\psi_j, c_j)\Big) dd^c v_k \wedge T,\end{aligned}$$

for all $k \geq 1$. Proposition 4.4 implies that

$$\min_{1\leq j\leq m}\Big(\max(\varphi_j,\psi_j,c_j)-\max(\psi_j,c_j)\Big)dd^cu\wedge T$$
$$=\min_{1\leq j\leq m}\Big(\max(\varphi_j,\psi_j,c_j)-\max(\psi_j,c_j)\Big)dd^cv\wedge T,$$

Next, by repeating the same argument as in the proof of Theorem 1.1 we finish the proof of Theorem 1.2.

Remark 4.7. The above result still holds for the following case. Let Ω be a bounded open set in $\mathbb{C}^n$ and $\varphi_1,\dots,\varphi_m$, $\psi_1,\dots,\psi_m$ are plurisubharmonic functions on Ω, T be a closed positive current of bidegree $(n-q,n-q)$ on Ω $(0<q<n)$. Let $\mathcal{O}=\{\varphi_1>\psi_1\}\cap\{\varphi_2>\psi_2\}\cap\cdots\cap\{\varphi_m>\psi_m\}$. Assume that $u_1,\dots u_q,v_1,\dots v_q\in PSH\cap L^\infty_{loc}(\Omega\setminus K)$ where $K\Subset\Omega$. If $u_k=v_k, 1\leq k\leq q$ on $\mathcal{O}$ then

$$dd^cu_1\wedge\cdots\wedge dd^cu_q\wedge T|_{\mathcal{O}}=dd^cv_1\wedge\cdots\wedge dd^cv_q\wedge T|_{\mathcal{O}}.$$

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