ACTA MATHEMATICA VIETNAMICA
Volume 36, Number 2, 2011, pp. 331–344

ON CHARACTERISTIC SYSTEMS FOR GENERAL MULTIDIMENSIONAL MONGE-AMPÈRE EQUATIONS

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Dedicated to Tran Duc Van on the occasion of his sixtieth birthday

Abstract. Like the characteristic method in solving the Cauchy problem for a first-order partial differential equation, we reduce the Cauchy problem for a general multidimensional Monge-Ampère equation to that for a normal first-order system of nonlinear partial differential equations, that could be called characteristic system for a given multidimensional Monge-Ampère equation.

1. Introduction

It is well known that the Cauchy problem for a first-order partial differential equation in $\mathbb{R}^n$ can be reduced to that for the characteristic first-order system of $(2n+1)$ ordinary differential equations. In this paper, we try to do the same for the Cauchy problem for a general multidimensional Monge-Ampère equation, that is in the following form:

$$\begin{vmatrix} z_{x_1x_1}+a_{11} & z_{x_1x_2}+a_{12} & ... & z_{x_1x_n}+a_{1n} \\ z_{x_2x_1}+a_{21} & z_{x_2x_2}+a_{22} & ... & z_{x_2x_n}+a_{2n} \\ ... & ... & ... & ... \\ z_{x_nx_1}+a_{n1} & z_{x_nx_2}+a_{12} & ... & z_{x_nx_n}+a_{nn} \end{vmatrix} = b, \tag{1.1}$$

where $z=z(x_1,x_2,...,x_n)$ is an unknown function defined for $x=(x_1,x_2,...,x_n)\in \Omega\subset\mathbb{R}^n; a_{jk}=a_{jk}(x,z(x),z_x(x)), b=b(x,z(x),z_x(x)),\ z_x(x)=(z_{x_1}(x),\ z_{x_2}(x),...\ z_{x_n}(x))$. All functions in this paper are assumed to be smooth and real-valued. We will denote $p=(p_1,p_2,...,p_n)$, $z_{xx}(x)\equiv[z_{x_jx_k}(x)]_{n\times n}$, $A(x,z,p)\equiv[a_{jk}(x,z,p)]_{n\times n}$.

The Cauchy problem for multidimensional Monge-Ampère equation (1.1) can be described as follows:

Received January 12, 2011.

2000 *Mathematics Subject Classification.* Primary: 35L70, 35K55; Secondary 53C21.

Key words and phrases. General multidimensional Monge-Ampère equation, Characteristic system, Noncharacteristic Cauchy problem.

This work is partially supported by NAFOSTED, Vietnam. Grant code: 101.01.23.09.

Suppose that in $\mathbb{R}^n_x$ there is an $(n-1)$-dimensional surface Γ, that is given by equations:

$$\begin{cases} x_1 &= X^0_1(y') \\ x_2 &= X^0_2(y') \\ &\dots \\ x_n &= X^0_n(y'), \end{cases} \tag{1.2}$$

where

$$\text{rank } [\frac{DX^0(y')}{Dy'}] = n-1,$$

and $X^0(y') \equiv (X^0_1(y'), X^0_2(y'), \dots, X^0_n(y'))$.
Here and in what follows we put

$$y' = (y_1, y_2, \dots, y_{n-1}) \in \omega \subset \mathbb{R}^{n-1}_{y'}.$$

Suppose also that we are given $(n+1)$ functions $Z^0(y'); P^0_j(y'), j = 1, 2, \dots, n$.

The Cauchy problem for equation (1.1) consists in looking for $z = z(x) \in C^2$ that is a solution of (1.1) such that

$$\begin{cases} z(x)\big|_{x=X^0(y')} &= Z^0(y'), \\ z_x(x)\big|_{x=X^0(y')} &= P^0(y'), \end{cases} \tag{1.3}$$

where $P^0(y') \equiv (P^0_1(y'), P^0_2(y'), \dots, P^0_n(y'))$. From (1.3) we have the following necessary consistency conditions for the initial Cauchy data:

$$\frac{\partial Z^0(y')}{\partial y_k} = \sum_{j=1}^{n} P^0_j(y') \frac{\partial X^0_j(y')}{\partial y_k} \quad , y' \in \omega, \ k = 1, \dots, n-1, \tag{1.4}$$

which are assumed to be fulfilled.

In the case $b(x, z, p) \equiv 0$, we considered in [6] the Cauchy problem for the general multidimensional Monge-Ampère equations (1.1) and we have reduced the Cauchy problem $(1.1), (1.3)$ to that for a normal system of $(2n+1)$ first-order nonlinear partial differential equations. In this paper we try to do the same for the case when $b(x, z, p)$ may be not zero. Namely, our main result is Theorem 3.5 in section 3, which states that any solution $z(x) \in C^2$ of the Cauchy problem $(1.1), (1.3)$ will generate a solution to the corresponding Cauchy problem for the normal system (3.20) of $(2n+1)$ first-order nonlinear partial differential equations. Moreover, at the end of section 4 we will establish also the so-called *non-characteristic condition* for the Cauchy problem $(1.1), (1.3)$, that has been established in [6] for the case $b(x, z, p) \equiv 0$.

In the next section we discuss some existing characteristic methods for the Cauchy problem $(1.1), (1.3)$, that can be applied only to rather narrow classes of equations (1.1). Section 3 is the main one, where in any dimension n and for any right-hand side $b(x, z, p)$ we drive the so-called characteristic system (3.20) for equation (1.1). As in [6], at the last section in Theorem 4.1, we will prove the converse assertion of Theorem 3.5, which states that any solution to the corresponding Cauchy problem for system (3.20) will give us a solution $z(x) \in C^2$

of the Cauchy problem (1.1), (1.3). This explains us, why system (3.20) could be called *characteristic system* for the Monge-Ampère equation (1.1).

2. Characteristic method for classical hyperbolic Monge-Ampère equations

Before formulating characteristic system for equation (1.1), we resume the characteristic method for classical hyperbolic Monge-Ampère equations. When $n = 2$, the classical Monge-Ampère equation (1.1) is of the form

$$\begin{vmatrix} z_{x_1x_1} + a_{11} & z_{x_1x_2} + a_{12} \\ z_{x_2x_1} + a_{21} & z_{x_2x_2} + a_{22} \end{vmatrix} = b. \tag{2.1}$$

The characteristic equation for (2.1) is ([3])

$$\lambda^2 + (a_{12} + a_{21})\lambda + (a_{11}a_{22} - b) = 0. \tag{2.2}$$

We set

$$\Delta \equiv (a_{12} + a_{21})^2 - 4(a_{11}a_{22} + b) = (a_{12} - a_{21})^2 - 4b. \tag{2.3}$$

When $\Delta > 0$, equation (2.1) is called hyperbolic. In this case we denote the roots of (2.2) by λ_1, λ_2. When $\Delta < 0$, equation (2.1) is called elliptic, and when $\Delta \equiv 0$, it is called parabolic. At the end of the 19-th century, G. Darboux and E. Goursat in [1] and [2] introduced the *characteristic method* for solving the Cauchy problem for hyperbolic equation (2.1), that reduces this problem to that for a first-order partial differential equation in $\mathbb{R}^2$. But this method requires existence of two independent first integrals for equation (2.1). A function $\varphi(x_1, x_2, z, p_1, p_2)$ is called a first integral for equation (2.1) if it satisfies the following system of equations:

$$\begin{cases} \frac{\partial\varphi}{\partial x_1} + \frac{\partial\varphi}{\partial z} - a_{11}\frac{\partial\varphi}{\partial p_1} - \lambda_1\frac{\partial\varphi}{\partial p_2} &= 0, \\ \frac{\partial\varphi}{\partial x_2} + \frac{\partial\varphi}{\partial z} - \lambda_2\frac{\partial\varphi}{\partial p_1} - a_{22}\frac{\partial\varphi}{\partial p_2} &= 0. \end{cases}$$

The characteristic method of G. Darboux and E. Goursat then have been developed in [3] by M. Tsuji to the Cauchy problem for general multidimensional Monge-Ampère equations (1.1) with $b(x, z, p) \equiv 0$, provided that this equation possesses n independent first integrals. But, the condition on existence of two independent first integrals for equation (2.1) is rather strict one, because in [5] H. T. Ngoan. D. Kong and M. Tsuji have showed that there are many Monge-Ampère equations (2.1), that do not possess this property. To avoid this difficulty, M. Tsuji in [4] has reduced the Cauchy problem for the hyperbolic equation (2.1) to that for the following first-order system of quasilinear equations

$$\begin{cases} \frac{\partial Z}{\partial y_1} - P_1\frac{\partial X_1}{\partial y_1} - P_2\frac{\partial X_2}{\partial y_1} &= 0, \\ \frac{\partial P_1}{\partial y_1} + a_{11}\frac{\partial X_1}{\partial y_1} + \lambda_1\frac{\partial X_2}{\partial y_1} &= 0, \\ \frac{\partial P_2}{\partial y_1} + \lambda_2\frac{\partial X_1}{\partial y_1} + a_{22}\frac{\partial X_2}{\partial y_1} &= 0, \\ \frac{\partial P_1}{\partial y_2} + a_{11}\frac{\partial X_1}{\partial y_2} + \lambda_2\frac{\partial X_2}{\partial y_2} &= 0, \\ \frac{\partial P_2}{\partial y_2} + \lambda_1\frac{\partial X_1}{\partial y_2} + a_{22}\frac{\partial X_2}{\partial y_2} &= 0. \end{cases} \tag{2.4}$$

We see that system (2.4) of quasilinear partial differential equations is not a normal one. Moreover, this method of reduction cannot be applied neither to the case of dimension greater than 2, nor to elliptic, parabolic Monge-Ampère equations.

We would like to emphasis that our characteristic system (3.20) is a normal one and our method of reduction to the characteristic system can be applied for any dimension n and for any general Monge-Ampère equation with arbitrary values of $b(x, z, p)$.

3. Reduction of the multidimensional Monge-Ampère equation to characteristic system

3.1. **Change of variables in equation** (1.1)**.** Suppose $z(x)$ is a C^2-solution of the Cauchy problem (1.1), (1.3). In equation (1.1) we change variables

$$x = X(y) \equiv (X_1(y), X_2(y), ..., X_n(y)), \tag{3.1}$$

where $y \equiv (y_1, y_2, ..., y_n); y' \equiv (y_1, y_2, ..., y_{n-1}) \in \omega$ is a chosen local coordinate of the surface $\Gamma \subset \mathbb{R}^n$ defined by (1.2). We assume that the vector-function $X(y)$ satisfies the following conditions:

$$X(y', 0) = X^0(y'), y' \in \omega \tag{3.2}$$

and

$$\det\left[\frac{DX(y)}{Dy}\right] \neq 0, y = (y', y_n) \in \omega \times (-\delta, \delta), \tag{3.3}$$

where $X^0(y')$ is the vector-function defined in (1.2) and

$$\frac{DX}{Dy} \equiv \begin{bmatrix} \frac{\partial X_1}{\partial y_1} & \frac{\partial X_2}{\partial y_1} & \cdots & \frac{\partial X_n}{\partial y_1} \\ \frac{\partial X_1}{\partial y_2} & \frac{\partial X_2}{\partial y_2} & \cdots & \frac{\partial X_n}{\partial y_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial X_1}{\partial y_n} & \frac{\partial X_2}{\partial y_n} & \cdots & \frac{\partial X_n}{\partial y_n} \end{bmatrix}.$$

We denote

$$y = Y(x) \equiv (Y_1(x), Y_2(x), ..., Y_n(x)), \tag{3.4}$$

that is the inverse to (3.1). We set

$$Z(y) \equiv z(X(y)), \tag{3.5}$$

$$P(y) \equiv z_x(X(y)) = (P_1(y), P_2(y), ..., P_n(y)). \tag{3.6}$$

Remark 3.1. From (3.2), (3.5), (3.6) and (1.3) it follows that $(X(y), Z(y), P(y))$ satisfy the following "initial" conditions:

$$\begin{cases} X(y', 0) &= X^0(y'), \\ Z(y', 0) &= Z^0(y'), \qquad y' \in \omega \\ P(y', 0) &= P^0(y'), \end{cases} \tag{3.7}$$

where $X^0(y'), Z^0(y'), P^0(y')$ are given in (1.2) and (1.3) as Cauchy data for equation (1.1).

From (3.5) and (3.6) it follows that

$$\frac{\partial Z(y)}{\partial y_k} = \sum_{j=1}^{n} P_j(y) \frac{\partial X_j(y)}{\partial y_k}, \; k = 1, \dots, n. \tag{3.8}$$

From (3.5) and (3.6) it follows also that

$$z_{xx}(X(y)) = \frac{DY}{Dx} \frac{DP}{Dy}. \tag{3.9}$$

Proposition 3.2. *Suppose $X(y) \in C^1$ is given so that condition* (3.3) *holds; $z(x) \in C^2$ is a solution of* (1.1) *and the functions $Z(y), P(y)$ are defined respectively by* (3.5) *and* (3.6). *Then equation* (1.1) *is equivalent to the following relation:*

$$\det\left[\frac{DP}{Dy} + \frac{DX}{Dy} A(X, Z, P)\right] = b \det\left[\frac{DX}{Dy}\right]. \tag{3.10}$$

Proof. From (3.9) we have

$$\begin{aligned}
&\det[z_{xx} + A(x, z(x), z_x(x))] \\
&= \det\left[\frac{DY}{Dx}\frac{DP}{Dy} + A(X, Z, P)\right] \\
&= \det\left[\frac{DY}{Dx}\left(\frac{DP}{Dy} + (\frac{DY}{Dx})^{-1} A(X, Z, P)\right)\right] \\
&= \det\left[\frac{DX}{Dy}\right]^{-1} \det\left[\frac{DP}{Dy} + \frac{DX}{Dy} A(X, Z, P)\right] = b,
\end{aligned}$$

from which the equivalence of (1.1) and (3.10) follows. □

3.2. **Some row-vectors.** We introduce now some row-vectors, that will play important roles in reducing equation (1.1) to a characteristic system.

$$\frac{\partial X(y)}{\partial y_j} \equiv \left(\frac{\partial X_1(y)}{\partial y_j}, \frac{\partial X_2(y)}{\partial y_j}, \dots, \frac{\partial X_n(y)}{\partial y_j}\right), j = 1, 2, \dots, n-1, \tag{3.11}$$

$$\frac{\partial P(y)}{\partial y_j} \equiv \left(\frac{\partial P_1(y)}{\partial y_j}, \frac{\partial P_2(y)}{\partial y_j}, \dots, \frac{\partial P_n(y)}{\partial y_j}\right), j = 1, 2, \dots, n-1, \tag{3.12}$$

$$\vec{v}_j(y) \equiv \frac{\partial P(y)}{\partial y_j} + \frac{\partial X}{\partial y_j} A(X(y), Z(y), P(y)) = (v_{j1}(y), v_{j2}(y), \dots, v_{jn}(y)), \quad j = 1, 2, \dots, n-1 \tag{3.13}$$

$$\vec{g} \equiv \vec{v}_1(y) \times \vec{v}_2(y) \times \cdots \times \vec{v}_{n-1}(y) = (g_1, g_2, \dots, g_n), \tag{3.14}$$

$$\vec{h} \equiv \frac{\partial X(y)}{\partial y_1} \times \frac{\partial X(y)}{\partial y_2} \times \cdots \times \frac{\partial X(y)}{\partial y_{n-1}} = (h_1, h_2, \dots, h_n), \tag{3.15}$$

where

$$\vec{v}_1 \times \vec{v}_2 \times \cdots \times \vec{v}_{n-1} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_{n-1} & \vec{e}_n \\ v_{11} & v_{12} & \dots & v_{1,n-1} & v_{1,n} \\ v_{21} & v_{22} & \dots & v_{2,n-1} & v_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-1,1} & v_{n-1,2} & \dots & v_{n-1,n-1} & v_{n-1,n} \end{vmatrix} \in R^n, \tag{3.16}$$

$\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ are unit row-vectors on coordinate axes $Ox_1, Ox_2, \dots, Ox_n$ respectively. From the definitions of the vectors $\vec{g}$ and $\vec{h}$, it is easy to get the following propositions

Proposition 3.3. *The following properties are valid for the vector $\vec{g}$:*
1) *The vector $\vec{g}$ is orthogonal to all vectors $\vec{v}_k, k = 1, 2, ..., n-1$, i.e.*

$$\langle \vec{g}, \vec{v}_k \rangle = 0, k = 1, 2, ..., n-1; \tag{3.17}$$

2) *$\vec{g} \neq \vec{0}$ if and only if the vectors $\vec{v}_k, k = 1, 2, ..., n-1$ are linearly independent;*
3) *Each of $g_j(y)$ depends on $(X(y), Z(y), P(y))$, that are in functions $a_{ik}(X(y), Z(y), P(y))$. All components $g_j(y)$ are homogeneous polynomials of degree $(n-1)$ with respect to derivatives $\frac{\partial X(y)}{\partial y_k}, \frac{\partial P(y)}{\partial y_k}, k = 1, 2, \dots, n-1$.*

Proposition 3.4. *The following properties are valid for the vector $\vec{h}$:*
1) *Vector $\vec{h}$ is orthogonal to all vectors $\frac{\partial X(y)}{\partial y_k} = 0, k = 1, 2, ..., n-1$, i.e.*

$$\langle \vec{h}, \frac{\partial X(y)}{\partial y_k} \rangle = 0, k = 1, 2, ..., n-1; \tag{3.18}$$

2) *$\vec{h} \neq \vec{0}$ if and only if the vectors $\frac{\partial X(y)}{\partial y_k}, k = 1, 2, ..., n-1$ are linearly independent. In this case, if there is a vector $\vec{m}$, such that $\langle \vec{m}, \frac{\partial X(y)}{\partial y_k} \rangle = 0, k = 1, 2, ..., n-1$, then there exists a real number $c \in \mathbb{R}$ such that $\vec{m} = c\vec{h}$;*
3) *Each of $h_j(y)$ is a homogeneous polynomial of degree $(n-1)$ with respect to derivatives $\frac{\partial X(y)}{\partial y_k}, k = 1, 2, \dots, n-1$.*

3.3. **The main result.** Our main result is the following one.

Theorem 3.5. *Suppose that following conditions hold:*
1) *$X(y) \in C^2$ and satisfies (3.3),*
2) *$z(x) \in C^2$ is a solution of equation (1.1),*
3) *Functions $Z(y), P(y)$ are defined respectively by (3.5) and (3.6),*
4) *The vector-function $X(y)$ satisfies the following system of relations:*

$$\sum_{k=1}^{n} \frac{\partial X(y)}{\partial y_k} = \vec{g}, y \in \omega \tag{3.19}$$

where the vector-function $\vec{g}$ is defined by (3.14).

Then $(X(y), Z(y), P(y))$ *is a* C^1*-solution of the following first-order normal system of* $(2n+1)$ *nonlinear partial differential equations:*

$$\begin{cases} \sum_{k=1}^n \frac{\partial X}{\partial y_k} &= \vec{g}, \\ \sum_{k=1}^n \frac{\partial Z}{\partial y_k} &= \langle \vec{g}, P \rangle, \\ \sum_{k=1}^n \frac{\partial P}{\partial y_k} &= -\vec{g} A^T(X,Z,P) + b(X,Z,P)\vec{h}, \end{cases} \tag{3.20}$$

where the vector-function $\vec{h}$ *is defined by* (3.15).

Proof. We begin the proof of Theorem 3.1 by proving some lemmas.

Lemma 3.6. *Suppose* $X(y) \in C^2$ *and satisfies* (3.19). *Then we have*

$$\det\left[\frac{DX(y)}{Dy}\right] = (-1)^{(n-1)} \langle \vec{g}, \vec{h} \rangle, \tag{3.21}$$

where the vector-functions $\vec{g}, \vec{h}$ *are defined respectively by* (3.14) *and* (3.15).

Proof. By definition

$$\det\left[\frac{DX(y)}{Dy}\right] = \begin{vmatrix} \frac{\partial X_1}{\partial y_1} & \frac{\partial X_2}{\partial y_1} & \cdots & \frac{\partial X_n}{\partial y_1} \\ \frac{\partial X_1}{\partial y_2} & \frac{\partial X_2}{\partial y_2} & \cdots & \frac{\partial X_n}{\partial y_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial X_1}{\partial y_{n-1}} & \frac{\partial X_2}{\partial y_{n-1}} & \cdots & \frac{\partial X_n}{\partial y_{n-1}} \\ \frac{\partial X_1}{\partial y_n} & \frac{\partial X_2}{\partial y_n} & \cdots & \frac{\partial X_n}{\partial y_n} \end{vmatrix}. \tag{3.22}$$

In the determinant at the right-hand side we sum up $(n-1)$ first rows and then add them to the last row. Then using (3.19) and definitions of $\vec{g}$ and $\vec{h}$, we have

$$\begin{aligned} \det\left[\frac{DX(y)}{Dy}\right] &= \begin{vmatrix} \frac{\partial X_1}{\partial y_1} & \frac{\partial X_2}{\partial y_1} & \cdots & \frac{\partial X_n}{\partial y_1} \\ \frac{\partial X_1}{\partial y_2} & \frac{\partial X_2}{\partial y_2} & \cdots & \frac{\partial X_n}{\partial y_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial X_1}{\partial y_{n-1}} & \frac{\partial X_2}{\partial y_{n-1}} & \cdots & \frac{\partial X_n}{\partial y_{n-1}} \\ g_1 & g_2 & \cdots & g_n \end{vmatrix} \\ &= (-1)^{(n-1)} \begin{vmatrix} g_1 & g_2 & \cdots & g_n \\ \frac{\partial X_1}{\partial y_1} & \frac{\partial X_2}{\partial y_1} & \cdots & \frac{\partial X_n}{\partial y_1} \\ \frac{\partial X_1}{\partial y_2} & \frac{\partial X_2}{\partial y_2} & \cdots & \frac{\partial X_n}{\partial y_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial X_1}{\partial y_{n-1}} & \frac{\partial X_2}{\partial y_{n-1}} & \cdots & \frac{\partial X_n}{\partial y_{n-1}} \end{vmatrix} \\ &= (-1)^{(n-1)} \langle \vec{g}, \vec{h} \rangle. \end{aligned}$$

The lemma is proved. □

Corollary 3.7. *Suppose* $X(y) \in C^2$ *and satisfies* (3.19). *Then the condition* (3.3) *holds if and only if*

$$\langle \vec{g}, \vec{h} \rangle \neq 0. \tag{3.23}$$

Lemma 3.8. *Suppose all conditions of Theorem* 3.1 *hold. Then the second equation of* (3.20) *holds, i.e.*

$$\sum_{k=1}^{n} \frac{\partial Z}{\partial y_k} = \langle \vec{g}, P \rangle. \tag{3.24}$$

Proof. The relation (3.24) follows easily from (3.8) and (3.19). □

Lemma 3.9. *Suppose all conditions of Theorem* 3.1 *hold. We set*

$$\sum_{k=1}^{n} \frac{\partial P}{\partial y_k} = \vec{f}. \tag{3.25}$$

Then the following relations hold

$$\langle \vec{f}, \frac{\partial X}{\partial y_k} \rangle = \langle \vec{g}, \frac{\partial P}{\partial y_k} \rangle, k = 1, 2...., (n-1). \tag{3.26}$$

Proof. From (3.8) we have for $k, m = 1, 2, ..., n$

$$\frac{\partial Z}{\partial y_k} = \sum_{\ell=1}^{n} P_\ell \frac{\partial X_\ell}{\partial y_k}, \tag{3.27}$$

$$\frac{\partial Z}{\partial y_m} = \sum_{\ell=1}^{n} P_\ell \frac{\partial X_\ell}{\partial y_m}. \tag{3.28}$$

Differentiating both sides of $(3.27), (3.28)$ with respect to y_m and y_k respectively, we get

$$\frac{\partial^2 Z}{\partial y_k \partial y_m} = \sum_{\ell=1}^{n} \frac{\partial P_\ell}{\partial y_m} \frac{\partial X_\ell}{\partial y_k} + \sum_{\ell=1}^{n} P_\ell \frac{\partial^2 X_\ell}{\partial y_k \partial y_m}, \tag{3.29}$$

$$\frac{\partial^2 Z}{\partial y_m \partial y_k} = \sum_{\ell=1}^{n} \frac{\partial P_\ell}{\partial y_k} \frac{\partial X_\ell}{\partial y_m} + \sum_{\ell=1}^{n} P_\ell \frac{\partial^2 X_\ell}{\partial y_m \partial y_k}. \tag{3.30}$$

Since $Z(y), X(y) \in C^2$, from $(3.29), (3.30)$ we get

$$\sum_{\ell=1}^{n} \frac{\partial P_\ell}{\partial y_m} \frac{\partial X_\ell}{\partial y_k} = \sum_{\ell=1}^{n} \frac{\partial P_\ell}{\partial y_k} \frac{\partial X_\ell}{\partial y_m}. \tag{3.31}$$

Summing up both sides of (3.31) with respect to m from 1 to n, we get (3.26). □

Lemma 3.10. *Suppose that* (3.19) *and* (3.25) *hold, where* $\vec{g}$ *is defined by* (3.14). *Then the following assertions are equivalent*

$$\langle \vec{f}, \frac{\partial X}{\partial y_k} \rangle = \langle \vec{g}, \frac{\partial P}{\partial y_k} \rangle, k = 1, 2...., (n-1), \tag{3.32}$$

$$\langle \vec{f} + \vec{g} A^T(X, Z, P), \frac{\partial X}{\partial y_k} \rangle = 0, k = 1, 2...., (n-1). \tag{3.33}$$

Proof. a) Suppose that (3.32) holds. Since $\langle \vec{g}, \vec{v}_k\rangle = 0$, from (3.13) we have for any $k = 1, 2...., n-1$

$$\begin{aligned}\langle \vec{f}, \frac{\partial X}{\partial y_k}\rangle &= \langle \vec{g}, \frac{\partial P}{\partial y_k}\rangle \\ &= \langle \vec{g}, \vec{v}_k\rangle - \langle \vec{g}, \frac{\partial X}{\partial y_k}A\rangle \\ &= -\langle \vec{g}, \frac{\partial X}{\partial y_k}A\rangle \\ &= -\langle \vec{g}A^T, \frac{\partial X}{\partial y_k}\rangle,\end{aligned}$$

from which (3.33) follows.

b) Suppose that (3.33) holds. Then from (3.33) and (3.13) we have for any $k = 1, 2...., n-1$

$$\begin{aligned}\langle \vec{f}, \frac{\partial X}{\partial y_k}\rangle &= -\langle \vec{g}A^T, \frac{\partial X}{\partial y_k}\rangle \\ &= -\langle \vec{g}, \frac{\partial X}{\partial y_k}A\rangle \\ &= \langle \vec{g}, \vec{v}_k\rangle - \langle \vec{g}, \frac{\partial X}{\partial y_k}A\rangle \\ &= \langle \vec{g}, \vec{v}_k - \frac{\partial X}{\partial y_k}A\rangle = \langle \vec{g}, \frac{\partial P}{\partial y_k}\rangle.\end{aligned}$$

The lemma is proved. ☐

Lemma 3.11. *Suppose that* (3.19) *and* (3.25) *hold, where* $\vec{g}$ *is defined by* (3.14). *Then the following assertions are equivalent*

$$\det\left[\frac{DP}{Dy} + \frac{DX}{Dy}A\right] = b\det\left[\frac{DX}{Dy}\right]. \tag{3.34}$$

$$\langle \vec{f} + \vec{g}A(X, Z, P), \vec{g}\rangle = b(X, Z, P)\langle \vec{g}, \vec{h}\rangle. \tag{3.35}$$

Proof. a) Suppose that (3.34) holds. We rewrite (3.34) as follows

(3.36)

$$\begin{vmatrix} \frac{\partial P_1}{\partial y_1} + \sum_{\ell=1}^n \frac{\partial X_\ell}{\partial y_1}a_{\ell 1} & \dots & \frac{\partial P_n}{\partial y_1} + \sum_{\ell=1}^n \frac{\partial X_\ell}{\partial y_1}a_{\ell n} \\ \vdots & \vdots & \vdots \\ \frac{\partial P_1}{\partial y_{n-1}} + \sum_{\ell=1}^n \frac{\partial X_\ell}{\partial y_{n-1}}a_{\ell 1} & \dots & \frac{\partial P_n}{\partial y_{n-1}} + \sum_{\ell=1}^n \frac{\partial X_\ell}{\partial y_{n-1}}a_{\ell n} \\ \frac{\partial P_1}{\partial y_n} + \sum_{\ell=1}^n \frac{\partial X_\ell}{\partial y_n}a_{\ell 1} & \dots & \frac{\partial P_n}{\partial y_n} + \sum_{\ell=1}^n \frac{\partial X_\ell}{\partial y_n}a_{\ell n} \end{vmatrix} = b\begin{vmatrix} \frac{\partial X_1}{\partial y_1} & \dots & \frac{\partial X_n}{\partial y_1} \\ \vdots & \vdots & \vdots \\ \frac{\partial X_1}{\partial y_{n-1}} & \dots & \frac{\partial X_n}{\partial y_{n-1}} \\ \frac{\partial X_1}{\partial y_n} & \dots & \frac{\partial X_n}{\partial y_n} \end{vmatrix}$$

We will verify the validity of (3.35). To do this, within each determinant at both sides of (3.36) we sum up $(n-1)$ first rows and then add them to the last row.

From (3.19) we have for any $j = 1, 2...., n$

$$\sum_{k=1}^{n}\sum_{\ell=1}^{n}\frac{\partial X_\ell}{\partial y_k}a_{\ell j} = \sum_{\ell=1}^{n}(\sum_{k=1}^{n}\frac{\partial X_\ell}{\partial y_k})\ a_{\ell j}$$

$$= \sum_{\ell=1}^{n} g_\ell\ a_{\ell j} \tag{3.37}$$

$$= (\vec{g}A)_j.$$

From (3.36), (3.37) and (3.25) it follows that
(3.38)

$$\begin{vmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ v_{n-1,1} & v_{n-1,2} & \dots & v_{n-1,n} \\ (\vec{f}+\vec{g}A)_1 & (\vec{f}+\vec{g}A)_2 & \dots & (\vec{f}+\vec{g}A)_n \end{vmatrix} = b \begin{vmatrix} \frac{\partial X_1}{\partial y_1} & \frac{\partial X_2}{\partial y_1} & \dots & \frac{\partial X_n}{\partial y_1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial X_1}{\partial y_{n-1}} & \frac{\partial X_2}{\partial y_{n-1}} & \dots & \frac{\partial X_n}{\partial y_{n-1}} \\ g_1 & g_2 & \dots & g_n \end{vmatrix}.$$

We can rewrite (3.38) as follows
(3.39)

$$\begin{vmatrix} (\vec{f}+\vec{g}A)_1 & (\vec{f}+\vec{g}A)_2 & \dots & (\vec{f}+\vec{g}A)_n \\ v_{11} & v_{12} & \dots & v_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ v_{n-1,1} & v_{n-1,2} & \dots & v_{n-1,n} \end{vmatrix} = b \begin{vmatrix} g_1 & g_2 & \dots & g_n \\ \frac{\partial X_1}{\partial y_1} & \frac{\partial X_2}{\partial y_1} & \dots & \frac{\partial X_n}{\partial y_1} \\ \vdots & \vdots & \vdots & \\ \frac{\partial X_1}{\partial y_{n-1}} & \frac{\partial X_2}{\partial y_{n-1}} & \dots & \frac{\partial X_n}{\partial y_{n-1}} \end{vmatrix}.$$

From the definitions of vectors $\vec{g}$ and $\vec{h}$, we can rewrite (3.39) in the form

$$\langle \vec{f} + \vec{g}A(X, Z, P), \vec{g}\rangle = b(X, Z, P)\langle \vec{g}, \vec{h}\rangle,$$

that means (3.35) is proved.

b) Suppose that (3.35) holds. Then we can rewrite it as (3.39), then we get (3.38). Within each determinant in both sides of (3.38) we multiply all $(n - 1)$ first rows by (-1) and then add to the last row. Then, from the obtained equality and from (3.19), (3.25), (3.37) we get (3.36). The lemma is proved. □

Continuation of the proof of Theorem 3.5. To end the proof of Theorem 3.5 we need to prove that

$$\vec{f} = -\vec{g}A^T(X, Z, P) + b\vec{h}. \tag{3.40}$$

Indeed, from Lemmas 3.9 and 3.10, (3.33) follows. Then from (3.33) and from Proposition 3.4 it follows that there exists $c \in \mathbb{R}$ such that

$$\vec{f} + \vec{g}A^T(X, Z, P) = c\vec{h}. \tag{3.41}$$

Then from Proposition 3.2 and from (3.35), (3.41) it follows that

$$\langle \vec{g}(A - A^T), \vec{g}\rangle + \langle c\vec{h}, \vec{g}\rangle = b\langle \vec{g}, \vec{h}\rangle. \tag{3.42}$$

Since $\langle \vec{g}(A - A^T), \vec{g}\rangle = 0$ and $\langle \vec{g}, \vec{h}\rangle \neq 0$, from (3.42) we get $c = b$. Theorem 3.5 is proved. □

3.4. **Necessary and sufficient condition for validity of** (3.19)**.** In this subsection we will discuss the necessary and sufficient condition for validity of (3.19). From (3.9) and (3.12) it follows that

$$\frac{\partial P(y)}{\partial y_j} = \frac{\partial X(y)}{\partial y_j}[z_{xx}(X(y), z(X(y)), z_x(X(y)))]. \tag{3.43}$$

We set for $j = 1, 2, \dots, n-1$

$$\vec{w}_j \equiv \frac{\partial X(y)}{\partial y_j}[z_{xx}(X(y), z(X(y)), z_x(X(y))) + A(X(y), z(X(y)), z_x(X(y)))] \tag{3.44}$$

and

$$\vec{G} \equiv \vec{w}_1 \times \vec{w}_2 \times \cdots \times \vec{w}_{n-1} = (G_1, G_2, ..., G_n). \tag{3.45}$$

Note that each component G_j is a polynomial of degree $(n-1)$ with respect to $\frac{\partial X_k(y)}{\partial y_\ell}, k = 1, 2, ..., n; \ell = 1, 2, \dots, (n-1)$.

Theorem 3.12. *Suppose that following conditions hold:*
1) $X(y) \in C^2$ *and satisfies* (3.3),
2) $z(x) \in C^2$ *is a solution of equation* (1.1),
3) *Functions* $Z(y), P(y)$ *are defined respectively by* (3.5) *and* (3.6).

Then the vector-function $X(y)$ *satisfies condition* (3.19) *if and only if it satisfies the following first-order system of n nonlinear partial differential equations*

$$\sum_{k=1}^{n} \frac{\partial X(y)}{\partial y_k} = \vec{G}, y \in \omega, \tag{3.46}$$

where the vector-function $\vec{G}$ *is defined by* (3.45).

Proof. Since $Z(y) = z(X(y)), P(y) = z_x(X(y))$, from (3.43), (3.44) and (3.13), (3.14) we get

$$\vec{w}_j = \vec{v}_j, j = 1, 2, ..., n-1, \tag{3.47}$$

and therefore

$$\vec{G} = \vec{g}. \tag{3.48}$$

So, it follows from (3.48) that (3.46) is equivalent to (3.19). The theorem is proved. □

3.5. **Noncharacteristic condition for the Cauchy problem.** We finish this section by giving a definition of *noncharacteristic condition* for the Cauchy problem (1.1), (1.3), that was given in [6] in the case $b(x, z, p) \equiv 0$. The idea of introducing the noncharacteristic condition is that the value of determinant $\left|\frac{DX(y)}{Dy}\right|$ must not vanish at $y_n = 0$. So, from Lemma 3.6 we come to the following:

Definition 3.13. The Cauchy problem (1.1), (1.3) is called *noncharacteristic* if the following condition holds:

$$\langle \vec{g}^0(y'), \vec{h}^0(y') \rangle \neq 0, \forall y' \in \omega, \tag{3.49}$$

where

$$\vec{h}^0(y') \equiv \frac{\partial X^0(y')}{\partial y_1} \times \frac{\partial X^0(y')}{\partial y_2} \times ... \times \frac{\partial X^0(y')}{\partial y_{n-1}}, \tag{3.50}$$

$$\vec{g}^0(y') \equiv \vec{v}_1^0(y') \times \vec{v}_2^0(y') \times ... \times \vec{v}_{n-1}^0(y'), \tag{3.51}$$

$$\vec{v}_k^0(y') \equiv \frac{\partial P^0(y')}{\partial y_k} + \frac{\partial X^0(y')}{\partial y_k} A(X^0(y'), Z^0(y'), P^0(y')), \tag{3.52}$$

where $X^0(y'), Z^0(y'), P^0(y')$ are given in (1.2) and (1.3) as Cauchy data for equation (1.1).

4. Cauchy problem for the characteristic system

We now formulate corresponding Cauchy problem for the characteristic system (3.20): Look for $(X(y), Z(y), P(y))$ that is a C^1-solution of (3.20) and satisfies the following initial conditions:

$$\begin{cases} X(y', 0) & = X^0(y'), \\ Z(y', 0) & = Z^0(y'), \\ P(y', 0) & = P^0(y'), \end{cases} \tag{4.1}$$

where $X^0(y'), Z^0(y'), P^0(y')$ are given in (1.2) and (1.3) as Cauchy data for equation (1.1).

The following theorem is the converse one of Theorem 3.5. We show that a solution $(X(y), Z(y), P(y))$ of the Cauchy problem (3.20), (4.1) will give us a solution $z(x)$ of the Cauchy problem (1.1), (1.3). In the case $b(x, z, p) \equiv 0$ this theorem has been proved in [6].

Theorem 4.1. *Suppose* $(X(y), Z(y), P(y))$ *is a* C^2*-solution of the Cauchy problem* (3.20), (4.1) *and conditions* (1.4) *and* (3.3) *hold. Then the following function*

$$z(x) = Z(Y(x)), \tag{4.2}$$

where $Y(x)$ *is defined by* (3.4), *is a solution of the Cauchy problem* (1.1), (1.3). *Moreover, we have also*

$$z_x(x) = P(Y(x)). \tag{4.3}$$

Proof. From system (3.20) we set

$$\vec{f} \equiv -\vec{g}A^T(X(y), Z(y), P(y)) + b(X(y), Z(y), P(y))\vec{h}. \tag{4.4}$$

Then it is obvious that (3.25) and (3.35) hold. Since $(\vec{h}, \frac{\partial X}{\partial y_k}) = 0, k = 1, 2, ..., n-1$, from (4.4) we have

$$\langle \vec{f} + \vec{g}A^T(X(y), Z(y), P(y)), \frac{\partial X}{\partial y_k} \rangle = 0, k = 1, 2, ..., n-1. \tag{4.5}$$

From (4.5) and from Lemma 3.10 it follows for $k = 1, 2, ..., n-1$ that

$$\langle \vec{f}, \frac{\partial X}{\partial y_k} \rangle = \langle \vec{g}, \frac{\partial P}{\partial y_k} \rangle. \tag{4.6}$$

From conditions (1.4), (4.6) and from the proof of Theorem 3 in [6], it follows that relations (3.8) hold, i.e. we have

$$\frac{\partial Z(y)}{\partial y_k} = \sum_{j=1}^{n} P_j(y) \frac{\partial X_j(y)}{\partial y_k}\,,\ k = 1, \ldots, n. \tag{4.7}$$

Since condition (3.3) holds, relation (4.2) is equivalent to the following one

$$Z(y) = z(X(y)). \tag{4.8}$$

From (4.7), (4.8) and condition (3.3), (4.3) follows
Now, from (3.35), Lemma 3.11 and Proposition 3.2 it follows that the function $z(x)$ is a solution of the Cauchy problem (1.1), (1.3). Theorem 4.1 is proved. □

Theorem 4.1 can be considered as *characteristic method* for solving the Cauchy problem (1.1), (1.3). This means that instead of solving the Cauchy problem (1.1), (1.3) we try to solve the corresponding Cauchy problem (3.20), (4.1). But, to study the solvability of the Cauchy problem (3.20), (4.1), we should classify well the first-order normal systems (3.20) of nonlinear partial differential equations. In the case $2 \leqslant n \leqslant 5$ and $b(x, z, p) \equiv 0$, we have proved in [7] that system (3.20) is weekly hyperbolic and it is hyperbolic if and only if $n = 2$ and $a_{12}(x, z, p) \neq a_{21}(x, z, p)$. Moreover, in the case $n = 2$ we have given in [8] some sufficient conditions for solvability of the Cauchy problem (3.20), (4.1) for weekly hyperbolic systems and then have applied them to get solvability conditions of the corresponding Cauchy problem (1.1), (1.3) for classical weekly hyperbolic Monge-Ampère equations in two variables.

References

[1] G. Darboux, *Lecons Sur la Théorie Générale des Surfaces*, tome 3, Gauthier-Villars, Paris, 1894.

[2] E. Goursat, *Lecons sur l'Intégration des Équations aux Dérivées Partielles du Second Ordre*, tome 1, Hermann, Paris, 1896.

[3] M. Tsuji, Prolongation of classical solutions and singularities of generalized solutions, *Inst. H. Poincare-Analyse nonlineaire* **7** (1990), 505-523.

[4] M. Tsuji, Monge-Ampère equations and surfaces with negative Gaussian curvature, *Banach Center Publications* **39** (1997), 161-170.

[5] Ha Tien Ngoan, D. Kong and M. Tsuji, Intergration of Monge-Ampère equations and surfaces with negative Gaussian curvature, *Ann. Scoula Normal Sup. Pisa* **27** (1999), 309-330.

[6] Ha Tien Ngoan and Nguyen Thi Nga, On the Cauchy problem for multidimesional Monge-Ampère equations, *Acta Math. Vietnam.* **29** (3) (2004), 281-298.

[7] Ha Tien Ngoan and Nguyen Thi Nga, On the hyperbolicity of some systems of nonlinear first-order partial differential equations, *Vietnam J. Math.* **34** (1) (2006), 109-128.

[8] Ha Tien Ngoan and Nguyen Thi Nga, On the Cauchy problem for a quasilinear weakly hyperbolic systems in two variables and applications to that for weakly hyperbolic classical Monge-Ampère equations, *Advances in Deterministic and Stochastic Analysis*, Edited by N. M. Chuong, P. G. Cierlet, P. Lax, D. Mumford and D. H. Phong, World Scientific Publisher, 2007, pp. 177-196

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