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ON GROWTH FUNCTION OF PETRI NET

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Abstract. In this paper we introduce the growth function of a Petri net. We show that the growth function of any Petri net is bounded by a certain polynomial. There are relations between the growth function and the representative complexity of the language which is accepted by a Petri net.

1. Introduction

Petri net was introduced in 1962 by C. Petri in connection with a theory proposed to model the parallel and distributed processing systems. From then onwards, the theory of Petri net was developed extensively by many authors (see, for example, [4, 5, 13, 14]).

In a Petri net each place describes a local state, and each marking describes a global state of the net. Since the number of tokens which may be assigned to a place can be unbounded, there is an infinity of markings for a Petri net. The set of all markings for a Petri net with n places is simply the set N^n of all n-vectors. This set, although infinite, is of course denumerable. From this point of view a Petri net could be seen as an infinite state machine.

In order to study this infinite state machine, we propose a new tool: the notion of state growth speed, which is called the growth function of the machine. An analogous growth function for Lindenmayer systems was earlier considered by some authors (see [10, 15]). As we shall see in the sequel, in the theory of growth function only the state growth speed of the system matters and no attention is paid to the states themselves. This implies that many problems which are in general very hard for the infinite state machine could become solvable for the growth function. The obtained

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results on growth function of Petri nets could shed some light to problems concerning the capacity of Petri nets.

The purpose of this paper is to study the growth function of Petri nets and its applications.

The definitions of Petri net and of Petri net language are recalled in Section 2. Section 3 deals with the notion of growth function of a Petri net. The main result of this part is the growth speed theorem which shows that the growth function of any Petri net is bounded by a certain polynomial. Section 4 is devoted to the relations between the growth function of a Petri net and the representative complexity of the language which is accepted by this Petri net.

2. Preliminaries

We first recall some necessary notions and definitions. For a finite alphabet Σ, Σ^* (resp. Σ^r, $\Sigma^{\leq r}$) denotes the set of all words (resp. of all words of length r, of length at most r) on the alphabet Σ. The empty word is denoted by Λ. For any word $\omega \in \Sigma^*$, $l(\omega)$ denotes the length of ω. Every subset $L \subseteq \Sigma^*$ is called a language over the alphabet Σ. Let N be the set of all non-negative integers and $N^+ = N \setminus \{0\}$.

Definition 1. *A (free-labeled) Petri net* $\mathcal{N}$ is given by a list

$$\mathcal{N} = (P, T, I, O, \mu_0, M_f),$$

where

$P = \{p_1, \ldots, p_n\}$ is a finite set of *places*;
$T = \{t_1, \ldots, t_m\}$ is a finite set of *transitions*, $P \cap T = \emptyset$;
$I : P \times T \to N$ is the *input function*;
$O : T \times P \to N$ is the *output function*;
$\mu_0 : P \to N$ is the *initial marking*;
$M_f = \{\mu_{f_1}, \ldots, \mu_{f_k}\}$ is a finite set of *final marking*.

Definition 2. *A marking* μ (global state) of a Petri net $\mathcal{N}$ is a function

$$\mu : P \to N.$$

The marking μ can also be defined as a n-vector $\mu = (\mu_1, \ldots, \mu_n)$ with $\mu_i = \mu(p_i)$ and $|P| = n$.

Definition 3. A transition $t \in T$ is said to be *firable at the marking* μ if:

$$\forall p \in P : \mu(p) \geq I(p, t).$$

Let t be firable at μ and if t fires, then the Petri net $\mathcal{N}$ shall change its state from the marking μ to a new marking μ' which is defined as follows:

$$\forall p \in P : \mu'(p) = \mu(p) - I(p,t) + O(t,p).$$

We set $\delta(\mu, t) = \mu'$ and the function δ is said to be *the function of changing state of the net*.

A firing sequence can be defined as a sequence of transitions such that the firing of each its prefix will be led to a marking at which the following transition will be firable. By $\mathcal{F}_{\mathcal{N}}$ we denote the set of all firing sequences of $\mathcal{N}$.

We now extend the function δ for a firing sequence by induction as follows.

Let $t \in T^*, t_j \in T, \mu$ be a marking at which tt_j is a firing sequence. Then

$$\begin{cases} \delta(\mu, \Lambda) & = & \mu, \\ \delta(\mu, tt_j) & = & \delta(\delta(\mu, t), t_j). \end{cases}$$

Definition 4. *The language acceptable by a (free-labeled) Petri net $\mathcal{N}$* is the set

$$L(\mathcal{N}) = \big\{ t \in T^* | (t \in \mathcal{F}_{\mathcal{N}}) \wedge (\delta(\mu_0, t) \in M_f) \big\}.$$

The set of all (free-labeled) Petri net languages is denoted by $\mathcal{L}^f$.

3. The growth function of a Petri net

Let $\mathcal{N} = (P, T, I, O, \mu_0, M_f)$ be a Petri net. We set

$$S_r = \big\{ \mu | (\exists t \in \mathcal{F}_{\mathcal{N}}) \wedge (t \in T^r) \wedge \mu = \delta(\mu_0, t) \big\},$$
$$S_{\leq r} = \big\{ \mu | (\exists t \in \mathcal{F}_{\mathcal{N}}) \wedge (t \in T^{\leq r}) \wedge \mu = \delta(\mu_0, t) \big\},$$

which are the sets of all reachable markings of $\mathcal{N}$ by firing r, resp. at most r transitions.

Definition 5. *The growth functions $h_{\mathcal{N}}$, $g_{\mathcal{N}}$ of a Petri net $\mathcal{N}$* are defined by:

$$h_{\mathcal{N}}(r) = |S_r|,$$
$$g_{\mathcal{N}}(r) = |S_{\leq r}|.$$

Now we remark that an exact estimate of $g_{\mathcal{N}}(r)$ or $h_{\mathcal{N}}(r)$ will certainly be a complicated function of r. However, it is almost the case that for large value of r, $g_{\mathcal{N}}(r)$ or $h_{\mathcal{N}}(r)$ can be closely approximated by a much simpler function which tells us about the state of the growth speed of $\mathcal{N}$.

In the sequel, we use the notations and definitions of the theory of computational complexity:

Definition 6. If f and g are functions defined on the positive integers, then

(1) $f = O(g)$ if there is a $C > 0$ and an $n_0 > 0$ such that $|f(n)| \leq C|g(n)|$ for all $n \geq n_0$,

(2) $f = \Omega(g)$ if $g = O(f)$,

(3) $f = \Omega(\mathcal{C})$, where $\mathcal{C}$ is a class of functions, if $f = \Omega(g)$ for all $g \in \mathcal{C}$.

The following theorem gives us an upper bound of the growth speed for any Petri net.

Theorem 1 (The growth speed theorem). *If $\mathcal{N}$ is a Petri net with m transitions and n places, $k = \min\{m, n\}$ and P_k is any polynomial of degree k, then*

$$h_{\mathcal{N}} = O(P_k),$$

$$g_{\mathcal{N}} = O(P_k).$$

Thus the growth funtion of any Petri net is bounded by a certain polynomial. This is an essential limitation of the Petri nets.

Proof. Let $\mathcal{N} = (P, T, I, O, \mu_0, M_f)$ be a Petri net with $|T| = m$, $|P| = n$. We now estimate $|S_{\leq r}|$. There are two ways for doing it.

Firstly, we prove that $|S_{\leq r}| \leq P_n(r)$ with $|P| = n$. Let

$$\mu_0 = (a_1, \ldots, a_n),$$

$$a = \max a_i, \quad 1 \leq i \leq n,$$

$$l = \max |O(t_j, p_i) - I(p_i, t_j)|, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m.$$

Let $t = t_{j_1} t_{j_2} \ldots t_{j_p}$, $p \leq r$, be any firing sequence of $\mathcal{N}$. The function of changing state by t can be determined as follows:

$\delta(\mu_0, t_{j_1}) = \mu'$ with

$$\mu'(p_i) = \mu_0(p_i) + (O(t_{j_1}, p_i) - I(p_i, t_{j_1})),$$

$$\mu'(p_i) \quad \leq \quad a + l, \quad \forall p_i \in P$$

$\delta(\mu_0, t_{j_1} \ldots t_{j_p}) = \mu^{(p)}$ with

$$\mu^{(p)}(p_i) = \mu^{(p-1)}(p_i) + (O(t_{j_p}, p_i) - I(p_i, t_{j_p})),$$

$$\mu^{(p)}(p_i) \quad \leq \quad a + p.l \quad \leq \quad a + l.r, \quad \forall p_i \in P.$$

Therefore

$$|S_{\leq r}| \leq (a + lr)^n = P_n(r), \quad \forall r \in N^+.$$

Secondly, we show that $|S_{\leq r}| \leq P_m(r)$ with $|T| = m$. We define the matrices I^-, O^+, D by

$$\begin{aligned} I^-[j,i] &= (I(p_i, t_j))_{m\times n}, \\ O^+[j,i] &= (O(t_j, p_i))_{m\times n}, \\ D &= O^+ - I^-. \end{aligned}$$

and set

$$e[j] = (0, \cdots, 0, \underbrace{1}_{j-th\, place}, 0, \cdots, 0)_{1\times m}.$$

Let $t = t_{j_1}t_{j_2}\dots t_{j_p}$, $p \leq r$, be any firing sequence of $\mathcal{N}$. Firing t, the function of changing state is also determined by another way as follows:

$$\delta(\mu_0, t_{j_1}) = \mu' = \mu_0 + e[j_1]D,$$

$$\delta(\mu_0, t_{j_1}\dots t_{j_p}) = \mu^{(p)} = \mu^{(p-1)} + e[j_p]D.$$

We obtain

$$\delta(\mu_0, t_{j_1}\dots t_{j_p}) = \mu_0 + e[j_1]D + \cdots + e[j_p]D.$$

We set $e[j]D = \nu_j$, $j = 1, \cdots, m$. Let f_j be the number of occurences of the transition t_j in t. We can now express the function of changing state in the form

$$\begin{cases} \mu^{(p)} & = \mu_0 + \sum\limits_{j=1}^{m} f_j\nu_j, \\ \sum\limits_{j=1}^{m} f_j & \leq r. \end{cases}$$

It follows that $|S_{\leq r}|$ equals at most the number of the non-negative integral solutions of the inequality $\sum\limits_{j=1}^{m} f_j \leq r$. In [8] we have proved that this number equals $C^r_{m+r} = \dfrac{(m+r)!}{r!m!} \leq (m+r)^m$. Therefore

$$|S_{\leq r}| \leq (m + r)^m = P_m(r), \quad \forall r \in N^+.$$

Combining the above estimates of $|S_{\leq r}|$ we obtain

$$|S_{\leq r}| \leq P_k(r), \quad k = \min\{m, n\}.$$

Finally, from the property $|S_r| \leq |S_{\leq r}|$, $\forall r \in N$ it follows that $|S_r| \leq P_k(r)$. So we obtain $h_{\mathcal{N}} = O(P_k)$, $g_{\mathcal{N}} = O(P_k)$.

Now we consider the growth function for some special classes of Petri nets. Let $\mathcal{S} = \bigcup S_r$, $r \geq 0$. Then $\mathcal{S}$ is the set of all reachable markings of net.

A Petri net $\mathcal{N}$ is *safe* if $\forall \mu \in \mathcal{S}$, $\forall p_i \in P : \mu(p_i) \leq 1$, i.e. the number of token in any place is either 0 or 1. Safeness is an important property of hardware devices. If $|P| = n$, then $|\mathcal{S}| \leq 2^n = C$. Therefore for any $r \in N^+$,

$$h_{\mathcal{N}}(r) \leq g_{\mathcal{N}}(r) \leq C.$$

A Petri net is *bounded* if there exists a contant K such that for $\forall \mu \in \mathcal{S}$, $\forall p_i \in P : \mu(p_i) \leq K$. It is easy to see that if $\mathcal{N}$ is bounded and $|P| = n$, then

$$|\mathcal{S}| \leq (K+1)^n = C.$$

Therefore, for any $r \in N^+$ we also have

$$h_{\mathcal{N}}(r) \leq g_{\mathcal{N}}(r) \leq C.$$

A Petri net is *conservative* if $\forall \mu \in \mathcal{S}$, $|P| = n$:

$$\sum_{i=1}^{n} \mu(p_i) = \sum_{i=1}^{n} \mu_0(p_i).$$

Since μ_0 is given, $\sum\limits_{i=1}^{n} \mu_0(p_i) = K$. This implies that $\mu(p_i) \leq K$, i.e. $\mathcal{N}$ is bounded and we also obtain

$$h_{\mathcal{N}}(r) \leq g_{\mathcal{N}}(r) \leq C.$$

Thus, the growth function of either a safe or bounded or conservative Petri net is bounded by a contant.

4. The growth function and representative complexity

In [1, 3] we have examined the representative complexity of language defined as follows.

Let $L \subseteq \Sigma^*$. We define

$$x_1 E_{\leq r} x_2(\text{mod } L) \Leftrightarrow \forall \omega \in \Sigma^* : x_1\omega \in L \leftrightarrow x_2\omega \in L, \quad \forall x_1, x_2 \in \Sigma^{\leq r}.$$
$$x_1 E_r x_2(\text{mod} L) \Leftrightarrow \forall \omega \in \Sigma^* : x_1\omega \in L \leftrightarrow x_2\omega \in L, \quad \forall x_1, x_2 \in \Sigma^r.$$

It is easy to show that the relations $E_{\leq r}(\text{mod} L)$ and $E_r(\text{mod } L)$ are reflexive, symmetric and transitive. Therefore, they are equivalence relations.

Let

$$G_L(r) = \text{Rank} E_{\leq r}(\text{mod} L),$$
$$H_L(r) = \text{Rank} E_r(\text{mod} L).$$

be the representative complexity characteristics of the language L over $\Sigma^{\leq r}$ and Σ^r, respectively. There is a nice relation between the growth functions of a Petri net and the representative complexities of the language which is accepted by this Petri net.

Theorem 2 (The supply-demand theorem). *Let $L = L(\mathcal{N})$, where $\mathcal{N}$ is a Petri net. Then for any $r \in N^+$,*

$$\begin{aligned} H_L(r) &\leq h_{\mathcal{N}}(r) + 1, \\ G_L(r) &\leq g_{\mathcal{N}}(r) + 1. \end{aligned}$$

Proof. We first extend the partial function δ to a total function over $T^{\leq r}$ by adding a new marking μ_ε defined as follows:

$$\tilde{\delta}(\mu, x) = \delta(\mu, x)$$

if x is a firing sequence of $\mathcal{N}$ at μ,

$$\tilde{\delta}(\mu, x) = \mu_\varepsilon$$

if x is not a firing sequence of $\mathcal{N}$ at μ.

For all $x \in T^{\leq r}$, $\tilde{\delta}(\mu_\varepsilon, x) = \mu_\varepsilon \notin M_f$.

Set $\tilde{S}_{\leq r} = S_{\leq r} \cup \{\mu_\varepsilon\}$ and $|\tilde{S}_{\leq r}| = |S_{\leq r}| + 1$. We shall prove that if $L = L(\mathcal{N})$ then $G_L(r) \leq |\tilde{S}_{\leq r}|$. Assume to the contrary that $G_L(r) > |\tilde{S}_{\leq r}|$. There exist $x_1, x_2 \in T^{\leq r}$ such that $x_1 \overline{E}_{\leq r} x_2(\text{mod} L)$ but $\tilde{\delta}(\mu_0, x_1) = \tilde{\delta}(\mu_0, x_2)$. It follows from the last equation that both x_1, x_2 are (or are not) firing sequences and we can verify that

$$\forall \omega \in T^* : x_1\omega \in L \leftrightarrow x_2\omega \in L.$$

According to the definition, this implies that $x_1 E_{\leq r} x_2 (\mathrm{mod} L)$ which conflicts with the hypothesis $x_1 \overline{E}_{\leq r} x_2 (\mathrm{mod} L)$. Therefore,

$$G_L(r) \leq |\tilde{S}_{\leq r}| = |S_{\leq r}| + 1 = g_{\mathcal{N}}(r) + 1.$$

By an analogous argument we also obtain $H_L(r) \leq h_{\mathcal{N}}(r) + 1$.

Using the above relation we get some corollaries and applications.

Corollary 1. *If L is a language with either $H_L = \Omega(\mathcal{P}_k)$ or $G_L = \Omega(\mathcal{P}_k)$, where $\mathcal{P}_k$ is the class of all polynominals of degree k, then L is not acceptable by any Petri net whose numbers of transitions and of places are less than k.*

Proof. Assume on the contrary that L is acceptable by a Petri net $\mathcal{N}$ with $k = \min\{|T|, |P|\}$. Applying the Theorem 2 and the Theorem 1 we obtain

$$H_L(r) \leq h_{\mathcal{N}}(r) + 1 = O(P_k),$$
$$G_L(r) \leq g_{\mathcal{N}}(r) + 1 = O(P_k).$$

This conflicts with the hypothesis either $H_L = \Omega(\mathcal{P}_k)$ or $G_L = \Omega(\mathcal{P}_k)$. Therefore L is not acceptable by any Petri net whose numbers of transitions and of places are less than k.

Corollary 2. *If L is a language with either $H_L = \Omega(\mathcal{P})$ or $G_L = \Omega(\mathcal{P})$, where $\mathcal{P}$ is the class of all polynominals, then L is not acceptable by any Petri net.*

Proof. The proof is analogous to the one of Corollary 1.

By Corollaries 1 and 2, we can give a lot of rather simple languages not being acceptable by either any Petri net or a Petri net whose number of transitions and number of places are less than a given contant.

Example 1. Let $|\Sigma| = k \geq 2$, $c \notin \Sigma$ and

$$L = \{xcx \mid x \in \Sigma^+\}.$$

It can be verified that if $x_1, x_2 \in \Sigma^{\leq r}$, $x_1 \neq x_2$, then $x_1 \overline{E}_{\leq r} x_2 (\mathrm{mod} L)$. Therefore

$$G_L(r) = |\Sigma^{\leq r}| = \frac{k(k^r - 1)}{(k-1)} = \Omega(\mathcal{P}).$$

According to Corollary 2, L is not acceptable by any Petri net.

Example 2. Let $\Sigma = \{0, 1\}$, $c \notin \Sigma$, $k \geq 2$, and

$$L_k = \big\{xcx \,|\, x \in \Sigma^*\,,\ |x|_1 = k\big\},$$

where $|x|_1$ denotes the number of occurences of 1 in x. We shall prove that for any $r \geq k$, $H_{L_k}(r) \geq P_k(r)$. Set:

$$W_r = \big\{x \in \Sigma^* \,|\, l(x) = r,\ |x|_1 = k\big\},$$

where $l(x)$ is the length of x. It is easy to show that

$$|W_r| = C_r^k = \frac{r!}{k!(r-k)!} = \frac{r(r-1)\cdots(r-k+1)}{k!} = P_k(r).$$

For any $x_1, x_2 \in W_r$ we prove that if $x_1 \neq x_2$ then $x_1\overline{E}_r x_2\,(\mathrm{mod} L_k)$. In fact, if we choose $\omega = cx_1$, then $x_1\omega = x_1cx_1 \in L_k$, but $x_2\omega = x_2cx_1 \notin L_k$. It follows that $x_1\overline{E}_r x_2\,(\mathrm{mod} L_k)$. Therefore,

$$H_{L_k}(r) \geq |W_r| = P_k(r).$$

According to Corollary 1, this implies that L_k is not acceptable by a Petri net whose numbers of transitions and of places are less than k.

Theorem 3. *Let $\mathcal{N}$ be a Petri net with $g_{\mathcal{N}}(r) \leq C$, then $L = L(\mathcal{N})$ is regular.*

Proof. Recall that the Myhill-Nerode's equivalence relation $E(\mathrm{mod} L)$ is defined by

$$x_1 E x_2 (\mathrm{mod} L) \Leftrightarrow \forall \omega \in \Sigma^* : x_1\omega \in L \leftrightarrow x_2\omega \in L, \quad \forall x_1, x_2 \in \Sigma^*.$$

Let $I_L = \mathrm{Rank}\, E(\mathrm{mod} L)$. Myhill and Nerode have proved that L is regular if and only if $I_L \leq C$.

By Theorem 2, $G_L(r) \leq g_{\mathcal{N}}(r) + 1$. It follows that $G_L(r) \leq C$. Since $G_L(r)$ is non-decreasing and bounded, there exists $\lim G_L(r) = q$, $q =$ const, when $r \to \infty$. Since the values of $G_L(r)$ are integers, there is a constant r_0 such that $\forall r \geq r_0 : \quad G_L(r) = q$.

Assume to the contrary that L is not regular. By Myhill-Nerode's theorem, $I_L = +\infty$. Therefore, there is an infinite sequence $x_1, x_2, \ldots, x_k, \ldots$ with $x_i \in \Sigma^*$, $x_i \neq x_j$ and $x_i\overline{E}x_j(\mathrm{mod} L)$. From this sequence we pick up a finite sequence $x_1, x_2, \ldots, x_q, x_{q+1}$ and set $k = \max\big\{l(x_1), \ldots, l(x_{q+1})\big\}$.

Choose $r = \max\{k, r_0\}$. Then $x_i \overline{E}_{\leq r} x_j (\mathrm{mod} L)$ for $i \neq j$. It follows that $G_L(r) \geq q + 1$. Thus, there is $r \geq r_0$ with $G_L(r) \neq q$. This contradicts the property that $\forall r \geq r_0,\ \ G_L(r) = q$. So L is regular.

Corollary 3. *If the Petri net $\mathcal{N}$ is safe or bounded or conservative, then $L = L(\mathcal{N})$ is regular.*

Proof. We have proved that under the above assumption the growth function is bounded. According to Theorem 3, this implies that $L(\mathcal{N})$ is regular.

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References

1. P. T. An, *On a necessary condition for free-labeled Petri net languages*, Proceedings of the Fifth Vietnamese Mathematical Conference, Science and Technics Publishing House, Hanoi, 1999, 73-80.
2. P. T. An, *A complexity characteristic of Petri net languages*, Acta Math. Vietnam. **24** (1999) 2, 157-167.
3. P. T. An and P. V. Thao, *On capacity of labeled Petri net languages*, Vietnam J. Math. **27** (1999) 3, 231-240.
4. W. Brauer, W. Reisig and G. Rozenberg (Eds.), *Petri nets: Central models and their properties*, LNCS **254**, Springer-Verlag, Berlin, 1987.
5. W. Brauer, W. Reisig and G. Rozenberg (Eds.), *Petri nets: Applications and relationships to other models of concurrency*, LNCS **255**, Springer-Verlag, Berlin, 1987.
6. S. N. Cole, *Real-time computation by n-dimensional iterative arrays of finite state machines*, IEEE Trans. Comp. **C-18** (1969) 4, 349-365.
7. P. D. Dieu, *On a complexity characteristic of languages*, EIK **8** (1972) 8/9, 447-460.
8. J. E. Hopcroft and J. D. Ullman, *Introduction to Automata Theory, Languages and Computation*, Addison-Wesley, New York, 1979.
9. M. Jantzen, *Language theory of Petri nets*, LNCS **254**, Springer-Verlag, Berlin, 1987, 397-412.
10. A. Paz and A. Salomaa, *Integral sequential word functions and growth equivalence of Lindenmayer systems*, Information and Control **23** (1973) 4, 313-343.
11. J. L. Peterson, *Petri Net Theory and the Modeling of Systems*, Prentice-Hall, New York, 1981.
12. G. Rozenberg, *Behaviour of elementary net systems*, LNCS **254**, Springer-Verlag, Berlin, 1987, 60-94.
13. G. Rozenberg (Ed.), *Advances in Petri nets 1988*, LNCS **340**, Springer-Verlag, Berlin, 1988.

14. G. Rozenberg (Ed.), *Advances in Petri nets 1989*, LNCS **424**, Springer-Verlag, Berlin, 1990.
15. A. Salomaa, *On exponential growth in Lindenmayer systems*, Indagationes Math. **35** (1973) 1, 23-30.

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