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ON CONJUGATE MAPS AND DIRECTIONAL DERIVATIVES OF CONVEX VECTOR FUNCTIONS

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ABSTRACT. In this paper, the concepts and the properties of conjugate maps and directional derivatives of convex vector functions from a subset of R^n to R^m with respect to a convex, closed and pointed cone are presented on the base of the notions of Pareto-supremum and Pareto-infimum. Some well-known results in the scalar case are generalized to the vector case. An application of conjugate maps to the dual problem is shown.

1. INTRODUCTION

The directional derivative and the conjugate function of a scalar convex function are basic concepts of convex analysis. They play an important role in the optimization theory. In the vector case, different approaches to conjugate maps were proposed by several authors. They can be categorized in three types:

(i) approach based on efficiency [4],

(ii) approach based on weak efficiency [5], [6],

(iii) approach based on strong supremum [7],

where the order in the spaces under consideration is assumed to be generated by the positive orthant.

In this paper, we shall consider the space R^m with the order generated by a convex, closed and pointed cone. The Pareto-supremum of a subset is defined as the Pareto-minimum of the set of the upper bounds. We will obtain necessary and sufficient conditions for the Pareto-supremum of a subset to be nonempty. By this, we define the notion of conjugate maps of set-valued maps from R^n to R^m, which is an extension of (iii), and obtain some results concerning the conjugate maps of convex vector functions similar to the scalar case. In addition, based on the notions of

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Pareto-supremum and Pareto-infimum we define the notion of directional derivative of a convex vector function and prove several results extending those in the scalar case.

The paper is organized as follows. In the next section, we introduce the concept of Pareto-supremum and Pareto-infimum of subsets of R^m with respect to an order generated by a convex cone. Necessary and sufficient conditions for the existence of supremum and infimum are established. Some basic properties of the supremum and infimum sets, which are needed in the sequel, are also presented. Section 3 is devoted to conjugate maps of convex vector functions. We define conjugate maps in a standard way and investigate their properties, especially the extension of the Fenchel-Moreau Theorem to the vector case. Section 4 deals with directional derivatives of convex vector functions. As in the scalar case, we show that the directional derivative of a convex vector function f at $x \in \mathrm{ri}(\mathrm{dom} f)$ coincides with the support map of the set $\partial f(x)$. The last section is intended to give an application of the conjugate maps to the dual problems of vector optimization problems. We establish a weak duality theorem for the dual problem by using the conjugate maps.

2. Supremum and Infimum

Let $C \subseteq R^m$ be a convex cone. Define a partial order $' \succeq_C'$ on R^m as follows.

$$x, y \in R^m, \;\; x \succeq_C y \Leftrightarrow x - y \in C.$$

Sometimes we write $' \succeq'$ instead of $' \succeq_C'$ if it is clear which cone is under consideration. We recall the following definition.

Definition 2.1 ([1, Chapter 2, Definition 2.1]). Let $A \subseteq R^m$ be nonempty. We say that

i) $x \in A$ is an ideal minimal point of A with respect to C if $x \preceq_C y$, for every $y \in A$. The set of all ideal minimal points of A is denoted by $\mathrm{IMin}(A|C)$.

ii) $x \in A$ is a Pareto-minimal point of A with respect to C if $y \preceq_C x$, for some $y \in A$, then $x \preceq_C y$. The set of all Pareto-minimal points of A is denoted by $\mathrm{Min}(A|C)$.

The concepts of IMax and Max are defined dually.

The cone C is said to be pointed if $C \cap (-C) = \{0\}$. The following result from [1] will be needed in the sequel.

Lemma 2.2 [1, Chapter 2, Definition 2.2]. *If* $\mathrm{IMin}(A|C) \neq \emptyset$, *then* $\mathrm{IMin}(A|C) = \mathrm{Min}(A|C)$ *and it is a point whenever* C *is pointed.*

Let $A \subseteq R^m$ be nonempty. We say that $x \in R^m$ is an upper bound of A with respect to C if $x \succeq_C y$, for every $y \in A$. The set of all upper bounds of A is denoted by $\text{Ub}(A|C)$.

The concept of lower bound is defined dually. The set of all lower bounds of A with respect to C is denoted by $\text{Lb}(A|C)$. We say that A is bounded above (below) with respect to C if $\text{Ub}(A|C) \neq \emptyset$ ($\text{Lb}(A|C) \neq \emptyset$).

The concept of supremum is defined as follows.

Definition 2.3. Let $A \subseteq R^m$ be nonempty. We say that

i) $x \in R^m$ is an ideal supremal point of A with respect to C if $x \in \text{Ub}(A|C)$ and $x \preceq_C y$, for every $y \in \text{Ub}(A|C)$. The set of all ideal supremal points of A is denoted by $\text{ISup}(A|C)$.

ii) $x \in R^m$ is a Pareto-supremal point of A with respect to C if $x \in \text{Ub}(A|C)$ and $y \preceq_C x$ for some $y \in \text{Ub}(A|C)$ implies $x \preceq_C y$. The set of all Pareto-supremal points of A is denoted by $\text{Sup}(A|C)$.

The concepts of IInf and Inf are defined dually. Sometimes we write $\text{IMin}A$, $\text{Min}A$, $\text{ISup}A$, $\text{Sup}A$,... instead of $\text{IMin}(A|C)$, $\text{Min}(A|C)$, $\text{ISup}(A|C)$, $\text{Sup}(A|C)$,... if it is clear which cone is under consideration.

Remark 2.4. i) From definition we have

$$\begin{aligned} \text{ISup}A &= \text{IMin}(\text{Ub}A). \\ \text{Sup}A &= \text{Min}(\text{Ub}A). \\ \text{IInf}A &= \text{IMax}(\text{Lb}A). \\ \text{Inf}A &= \text{Max}(\text{Lb}A). \end{aligned}$$

ii) When $m = 1$ and $C = R_+$, the concepts of ISup (IInf) and Sup (Inf) are equivalent and they are precisely the usual concept of supremum (infimum) in R.

The following simple result will be needed in the sequel

Lemma 2.5. *Let A be a nonempty subset of R^m. Then*

i) $\text{Lb}A = -\text{Ub}(-A)$*.*

ii) $\text{Inf}A = -\text{Sup}(-A)$*.*

iii) $\text{IInf}A = -\text{ISup}(-A)$*.*

Proof. This is immediate from definition. □

Lemma 2.6. *Let A be a nonempty subset of R^m. If* $\text{ISup}A \neq \emptyset$*, then* $\text{ISup}A = \text{Sup}A$ *and it is a point whenever C is pointed.*

Proof. This is immediate from Remark 2.4 and Lemma 2.2. □

From now on, the cone C is assumed to be convex, closed and pointed. As usual, we shall write $x \succ y$ whenever $x \succeq y$ and $x \neq y$, for every $x, y \in R^m$. When int$C \neq \emptyset$, we write $x \gg y$ if $x - y \in$int$C$. To simplify the presentation, sometimes a set which has only one element is identified with that element.

Now, we shall establish necessary and sufficient conditions for the existence of supremum and infimum.

A sequence $(x_k) \subseteq R^m$ is said to be increasing if

$$x_{k+1} \succeq x_k, \quad (\forall k \in N).$$

A sequence $(x_k) \subseteq R^m$ is said to be bounded from above if the set $\{x_k : k \in N\}$ is bounded from above.

Lemma 2.7. *If a sequence $(x_k) \subseteq R^m$ is convergent and bounded above by $a \in R^m$, then $\lim x_k \preceq a$.*

Proof. Since $(x_k) \subseteq a - C$ and C is closed, $\lim x_k \in a - C$. Hence

$$\lim x_k \preceq a. \qquad \square$$

Lemma 2.8. *If a sequence $(x_k) \subseteq R^m$ is increasing and bounded from above then it is convergent and*

$$\lim x_k = \text{ISup}\{x_k : k \in N\}.$$

Proof. Let a be an arbitrary upper bound of $\{x_k : k \in N\}$. Then

$$(x_k) \subseteq (x_1 + C) \cap (a - C).$$

By [1, Chapter 1, Proposition 1.8], $(x_1 + C) \cap (a - C)$ is compact. Then there exists a convergent subsequence $(x_{k_l}) \subseteq (x_k)$. Put $x = \lim\limits_{l\to\infty} x_{k_l}$. Since $(x_{k_l})_l$ is increasing then, for every $l, p \in N$, we have

$$x_{k_{l+p}} \in x_{k_l} + C.$$

Let $p \to \infty$. Since C is closed,

$$x \in x_{k_l} + C.$$

Hence,

$$x \in x_k + C, \quad (\forall k \in N). \tag{1}$$

Let $\varepsilon > 0$ be given. Since $x_{k_l} \to x$, by [1, Chapter 1, Proposition 1.8], there exists $L \in N$ such that

$$(x_{k_L} + C) \cap (x - C) \subseteq B(x, \varepsilon).$$

Set $K = k_L$. Then for every $k > K$, one has

$$x_k \in (x_K + C) \cap (x - C) \subseteq B(x, \varepsilon).$$

Hence, $\lim x_k = x$.

Finally, from (1) and Lemma 2.7 it implies $x = \text{ISup}\{x_k : k \in N\}$. The proof is complete. $\square$

Proposition 2.9. *Let A be a nonempty and linearly ordered subset of R^m. Then* $\text{ISup}A \neq \emptyset$ *if and only if A is bounded from above.*

Proof. The 'only if' part is obvious. For the 'if' part, since C is convex, closed and pointed, by [1, Chapter 1, Proposition 1.10], $\text{int}C' \neq \emptyset$. Let $\xi \in \text{int}C'$. One has

$$\xi(c) > 0, \quad (\forall c \in C \setminus \{0\}). \tag{2}$$

Pick any $a \in \text{Ub}A$. Then $\xi(x) \leq \xi(a)$, for every $x \in A$. Hence $\text{Sup}\xi(A) < +\infty$. Therefore, we can find a sequence $(x_k) \subseteq A$ such that $(\xi(x_k))$ is increasing and $\lim \xi(x_k) = \text{Sup}\xi(A)$. Since A is linearly ordered, by (2), (x_k) is increasing. From Lemma 2.8, (x_k) converges and $\lim x_k = \text{ISup}\{x_k\}$. By Lemma 2.7, $\text{ISup}\{x_k\} \preceq a$. We shall show $\text{ISup}\{x_k\} \in \text{Ub}A$ and this will complete the proof. Indeed, let $x \in A$ be arbitrary. Since A is linearly ordered, one of the following two cases holds.

i) There exists k such that $x \preceq x_k$. In this case, obviously $x \preceq \text{ISup}\{x_k\}$.

ii) $x \succ x_k$, for every $k \in N$. By Lemma 2.7, $\text{ISup}\{x_k\} \preceq x$. Observe that $\xi(x) = \text{Sup}\xi(A) = \xi(\text{ISup}\{x_k\})$. Then by (2), $x = \text{ISup}\{x_k\}$.

Thus $x \preceq \text{ISup}\{x_k\}$ for every $x \in A$. The Proposition is proved. $\square$

Remark 2.10. From the proof of Proposition 2.9, we see that if a nonempty, linearly ordered subset A of R^m is bounded from above then there exists an increasing sequence $(x_k) \subseteq A$ converging to $\text{ISup}A$.

Corollary 2.11. *Let A be a nonempty and linearly ordered subset of R^m. Then* $\mathrm{IInf}A \neq \emptyset$ *if and only if A is bounded below.*

Proof. This is immediate from Proposition 2.9 and Lemma 2.5. □

The following lemma characterizes ISup and IInf of linearly ordered subsets.

Proposition 2.12. *Let A be a nonempty, linearly ordered subset of R^m. Then*

i)

$$a = \mathrm{ISup}A \Leftrightarrow \begin{cases} a \in \mathrm{Ub}A. \\ A \cap B(a, \varepsilon) \neq \emptyset, \quad (\forall \varepsilon > 0). \end{cases}$$

ii)

$$b = \mathrm{IInf}A \Leftrightarrow \begin{cases} b \in \mathrm{Lb}A. \\ A \cap B(b, \varepsilon) \neq \emptyset, \quad (\forall \varepsilon > 0). \end{cases}$$

Proof. i) The 'only if' part is immediate from Remark 2.10. For the 'if' part, let $x \in \mathrm{Ub}A$ be arbitrary. From the hypothesis, there is a sequence $(a_k) \subseteq A$ converging to a. Since $(a_k) \subseteq x - C$ and since C is closed, $a \preceq x$. Hence, $a = \mathrm{ISup}A$.

ii) This is immediate from i) and Lemma 2.5.

The proof is complete. □

When $\mathrm{int}C \neq \emptyset$ Proposition 2.12 can be rewritten as follows.

Proposition 2.13. *Assume that* $\mathrm{int}C \neq \emptyset$ *and A is a nonempty, linearly ordered subset of R^m. Then*

i)

$$a = \mathrm{ISup}A \Leftrightarrow \begin{cases} a \in \mathrm{Ub}A. \\ \forall c \gg 0,\ \exists x \in A \ \textit{ such that } \ x \gg a - c. \end{cases}$$

ii)

$$b = \mathrm{IInf}A \Leftrightarrow \begin{cases} b \in \mathrm{Lb}A. \\ \forall c \gg 0,\ \exists x \in A \ \text{ such that } \ x \ll b + c. \end{cases}$$

Proof. i) For the 'only if' part, pick any $c \gg 0$. Then $a \in \mathrm{int}(a - c + C)$. Hence, by Proposition 2.12, there exists $x \in A$ such that $x \in \mathrm{int}(a - c + C)$, i.e. $x \gg a - c$.

For the 'if' part, let $\epsilon > 0$ be arbitrary. From [1, Chapter 1, Proposition 1.8], there exists $c \gg 0$ such that

$$(a - c + C) \cap (a - C) \subseteq B(a, \varepsilon).$$

By the hypothesis, $A \cap [(a-c+C) \cap (a-C)] \neq \emptyset$. Hence, $A \cap B(a,\epsilon) \neq \emptyset$. Then by Proposition 2.12, $a = \mathrm{ISup} A$.

ii) This is immediate from i) and Lemma 2.5.

The proof is complete. $\square$

Corollary 2.14. *Let $S \subseteq R$ be nonemty and bounded from above. Then for every $c \in C$, we have*

$$\mathrm{ISup}(Sc) = (\mathrm{Sup} S)c.$$

Proof. This is immediate from Proposition 2.12. $\square$

Now, let g be a function from $(\alpha,\beta) \subseteq R$ to R^m. We say that g is increasing on (α,β) if for every $t,t' \in (\alpha,\beta)$, one has

$$t \leq t' \Rightarrow g(t) \preceq g(t').$$

Corollary 2.15. *Let $g : (\alpha,\beta) \subseteq R \to R^m$ be increasing and bounded. Then the limits $\lim_{t \downarrow \alpha} g(t)$, $\lim_{t \uparrow \beta} g(t)$ exist and*

$$\lim_{t \downarrow \alpha} g(t) = \operatorname*{IInf}_{t \in (\alpha,\beta)} g(t).$$
$$\lim_{t \uparrow \beta} g(t) = \operatorname*{ISup}_{t \in (\alpha,\beta)} g(t).$$

Proof. The set $\{g(t) : t \in (\alpha,\beta)\}$ is linearly ordered and bounded below then by Corollary 2.11, there exists $\operatorname*{IInf}_{t \in (\alpha,\beta)} g(t)$. Put $a = \operatorname*{IInf}_{t \in (\alpha,\beta)} g(t)$. Let $\varepsilon > 0$ be given. From [1, Chapter 1, Proposition 1.8], there is $\varepsilon' > 0$ such that

$$(B(a,\epsilon') + C) \cap (B(a,\epsilon') - C) \subseteq B(a,\varepsilon).$$

By Proposition 2.12, there exists $t_0 \in (\alpha,\beta)$ such that $g(t_0) \in B(a,\varepsilon')$. Then for every $t \in (\alpha,t_0)$, one has

$$g(t) \in (a+C) \cap (g(t_0) - C) \subseteq (B(a,\varepsilon') + C) \cap (B(a,\varepsilon') - C) \subseteq B(a,\varepsilon).$$

Thus, $\lim_{t \downarrow \alpha} g(t) = \operatorname*{IInf}_{t \in (\alpha,\beta)} g(t)$.

Similarly, we have $\lim_{t \uparrow \beta} g(t) = \operatorname*{ISup}_{t \in (\alpha,\beta)} g(t)$. The proof is complete. $\square$

The following theorem is the main result of this section.

Theorem 2.16. *Let A be a nonempty subset of R^m. Then* $\mathrm{Sup}A \neq \emptyset$ *if and only if A is bounded from above.*

Proof. The 'only if' part is obvious. For the 'if' part, let $b \in \mathrm{Ub}A$ be arbitrary. Put $B = \mathrm{Ub}A \cap (b - C)$. Since $\mathrm{Min}B \subseteq \mathrm{Min}(\mathrm{Ub}A)$ then to complete the proof it remains to show $\mathrm{Min}B \neq \emptyset$. Indeed, let S be a nonempty and linearly ordered subset of B. Pick any $x \in A$, one has

$$S \subseteq (x + C) \cap (b - C).$$

Then by Corollary 2.11, there exists $\mathrm{IInf}S$. Since C is closed then by Proposition 2.12, $\mathrm{IInf}S \in (x+C)\cap(b-C)$, for every $x \in A$, i.e., $\mathrm{IInf}S \in B$. Hence, by Zorn's lemma, $\mathrm{Min}B \neq \emptyset$. The theorem is proved. □

Corollary 2.17. *Let A be a nonempty subset of R^m. Then* $\mathrm{IInf}A \neq \emptyset$ *if and only if A is bounded below.*

Proof. This is immediate from Theorem 2.16 and Lemma 2.5. □

Remark 2.18. From the proof of Theorem 2.16 and Lemma 2.5, for every nonempty subset A of R^m, one has

$$\mathrm{Ub}A = \mathrm{Sup}A + C. \tag{3}$$

$$\mathrm{Lb}A = \mathrm{Inf}A - C. \tag{4}$$

Corollary 2.19. *Let A be a nonempty subset of R^m. Then*

i) *If* $\mathrm{Sup}A \neq \emptyset$ *and has only one element then* $\mathrm{Sup}A = \mathrm{ISup}A$.

ii) *If* $\mathrm{Inf}A \neq \emptyset$ *and has only one element then* $\mathrm{Inf}A = \mathrm{IInf}A$.

Proof. This is immediate from (3) and (4). □

The rest of this section is intended to establish some basic properties of supremum and infimum which will be needed in the sequel.

Proposition 2.20. *Let A be a nonempty subset of R^m. Then*

i) $\mathrm{Ub}A$ *is closed and convex.*

ii) $\mathrm{Ub}A = \mathrm{Ub}\overline{(coA)}$.

iii) $\mathrm{Sup}A = \mathrm{Sup}\overline{(coA)}$.

iv) $\mathrm{ISup}A = \mathrm{ISup}\overline{(coA)}$.

(Here $\overline{coA}$ denotes the closure of the convex hull of A.)

Proof. i) This is immediate from definitions and from the fact C is closed.

ii) It is clear that $\mathrm{Ub}\overline{(coA)} \subseteq \mathrm{Ub}A$. For the converse inclusion, let $a \in \mathrm{Ub}A$ be arbitrary. Then $A \subseteq (a - C)$. Since C is closed and convex, $\overline{coA} \subseteq (a - C)$, i.e. $a \in \mathrm{Ub}\overline{(coA)}$.

iii) This is immediate from the definition of Sup and ii).

iv) This is immediate from iii) and Corollary 2.19. □

Corollary 2.21. *Let $A \subseteq R^m$. If $\mathrm{Ub}A \cap \overline{coA}$ is nonempty then*

$$\mathrm{Ub}A \cap \overline{coA} = \mathrm{ISup}A.$$

Proof. From Proposition 2.20, one has

$$\mathrm{Ub}A \cap \overline{coA} = \mathrm{Ub}\overline{(coA)} \cap \overline{coA}.$$

Since C is pointed and $\mathrm{Ub}A \cap \overline{coA} \neq \emptyset$,

$$\mathrm{Ub}\overline{(coA)} \cap \overline{coA} = \mathrm{ISup}\overline{coA}$$

Hence, by Proposition 2.20,

$$\mathrm{Ub}A \cap \overline{coA} = \mathrm{ISup}A.$$

The proposition is proved. □

Proposition 2.22. *Let A, B be nonempty subsets of R^m. Then*

i) $\mathrm{Sup}(tA) = t\mathrm{Sup}A$*, for every* $t > 0$.

ii) *If* $A \subseteq B$ *then* $\mathrm{Sup}B \subseteq \mathrm{Sup}A + C$.

iii) $\mathrm{Sup}A + \mathrm{Sup}B \subseteq \mathrm{Sup}(A+B) + C$. *If in addition,* $\mathrm{ISup}A \cup \mathrm{ISup}B \neq \emptyset$*, then*

$$\mathrm{Sup}A + \mathrm{Sup}B = \mathrm{Sup}(A + B).$$

Proof. i) Let $a \in \mathrm{Sup}A$ be arbitrary. Then $ta \in \mathrm{Ub}(tA)$. Pick $b \in \mathrm{Ub}(tA)$ with $b \preceq ta$. Then $\dfrac{b}{t} \in \mathrm{Ub}A$ and $\dfrac{b}{t} \preceq a$. Since $a \in \mathrm{Sup}A$, $\dfrac{b}{t} = a$. Hence, $ta \in \mathrm{Sup}(tA)$. Thus, $t\mathrm{Sup}A \subseteq \mathrm{Sup}(tA)$. Replace A, t by $tA, \dfrac{1}{t}$, one has

$$\frac{1}{t}\mathrm{Sup}(tA) \subseteq \mathrm{Sup}A.$$

Hence, $\mathrm{Sup}(tA) \subseteq t\mathrm{Sup}A$.

ii) This is immediate from the fact $\mathrm{Sup}B \subseteq \mathrm{Ub}B \subseteq \mathrm{Ub}A$ and from the equality $\mathrm{Ub}A = \mathrm{Sup}A + C$.

iii) The inclusion follows from the facts $\mathrm{Sup}A + \mathrm{Sup}B \subseteq \mathrm{Ub}(A+B)$ and $\mathrm{Ub}(A+B) \subseteq \mathrm{Sup}(A+B) + C$. To prove the equality, without loss of generality we may assume that $\mathrm{ISup}A \neq \emptyset$. By Lemma 2.6, $\mathrm{Sup}A = \mathrm{ISup}A$ and it is a point. Let $b \in \mathrm{Sup}B$ be arbitrary. Then $\mathrm{Sup}A + b \in \mathrm{Ub}(A+B)$. Pick $x \in \mathrm{Ub}(A+B)$ such that

$$x \preceq \mathrm{Sup}A + b. \tag{5}$$

One has $x \succeq y + z$, for every $y \in A, z \in B$. Hence, $x - z \in \mathrm{Ub}A$, for every $z \in B$. Since $\mathrm{Ub}A = \mathrm{Sup}A + C$, $x - z \succeq \mathrm{Sup}A$, for every $z \in B$. Hence, we have $x - \mathrm{Sup}A \in \mathrm{Ub}B$. $b \in \mathrm{Sup}B$. This and (5) imply $x = \mathrm{Sup}A + b$. Therefore, $\mathrm{Sup}A + b \in \mathrm{Sup}(A+B)$.

Conversely, let $x \in \mathrm{Sup}(A+B)$ be arbitrary. By a similar proof, one has $x - \mathrm{Sup}A \in \mathrm{Ub}B$. Pick $b \in \mathrm{Ub}B$ such that

$$b \preceq x - \mathrm{Sup}A.$$

Then $\mathrm{Sup}A + b \in \mathrm{Ub}(A+B)$ and $\mathrm{Sup}A + b \preceq x$. Since $x \in \mathrm{Sup}(A+B)$ then $x = \mathrm{Sup}A + b$. Hence, $x - \mathrm{Sup}A \in \mathrm{Sup}B$. The proposition is proved. □

Proposition 2.23. *Let A be a nonempty subset of R^m and $a \in R^m$. If $\xi(a) = Sup\xi(A)$, for every $\xi \in C'$, then*

$$a = \mathrm{ISup}A.$$

Proof. For every $\xi \in C', x \in A$, one has $\xi(a) \geq \xi(x)$. Hence, $a \succeq x$. This means that $a \in \mathrm{Ub}A$. Now, let $b \in \mathrm{Ub}A$ be arbitrary. Then $b \succeq x$, for every $x \in A$. Hence, $\xi(b) \geq \mathrm{Sup}\xi(A) = \xi(a)$, for every $\xi \in C'$. This implies that $b \succeq a$. Hence, $a = \mathrm{ISup}A$. The proof is complete. □

Remark 2.24. We note that the duals to Proposition 2.20 to Proposition 2.23 for Infimum are also true.

3. Conjugate maps

Let $F : R^n \rightrightarrows R^m$ be a set-valued map. We recall that the C-epigraph of F is defined as the set

$$\mathrm{epi}F := \{(x,y) \in R^n \times R^m : y \in F(x) + C\}.$$

The domain of F is defined as the set

$$\mathrm{dom}F := \{x \in R^n : F(x) \neq \emptyset\}.$$

F is said to be convex (closed) with respect to C if epi$F$ is convex (closed).

It is easy to see that F is convex if and only if for every $x, y \in R^n, \lambda \in [0,1]$, one has

$$\lambda F(x) + (1-\lambda)F(y) \subseteq F(\lambda x + (1-\lambda)y) + C.$$

If f is a vector function from a subset $D \subseteq R^n$ to R^m then we shall identify f with the following set-valued map

$$F(x) = \begin{cases} \{f(x)\}, & x \in D, \\ \emptyset, & x \notin D. \end{cases}$$

Definition 3.1. Let $F : R^n \rightrightarrows R^m$ be a set-valued map with dom$F \neq \emptyset$. The conjugate map of F is a set-valued map from $L(R^n, R^m)$ to R^m defined by

$$F^*(A) := \mathrm{Sup} \bigcup_{x \in R^n} [A(x) - F(x)], \quad A \in L(R^n, R^m),$$

where $L(R^n, R^m)$ denotes the space of linear maps from R^n to R^m.

Definition 3.2. Let $F : R^n \rightrightarrows R^m$ be a set-valued map with dom$F^* \neq \emptyset$. The biconjugate map of F is a set-valued map from R^n to R^m defined by

$$F^{**}(x) := \mathrm{Sup} \bigcup_{A \in L(R^n, R^m)} [A(x) - F^*(A)], \;\; x \in R^n.$$

Example 3.3. i) Let $S \subseteq R^n$. The indicator map of S is a set-valued map from R^n to R^m defined by

$$I_S(x) := \begin{cases} \{0\}, & x \in S, \\ \emptyset, & x \notin S. \end{cases}$$

The support map of S is a set-valued map from $L(R^n, R^m)$ to R^m defined by

$$\mathrm{Supp}(S|A) := \mathrm{Sup}\{A(x) : x \in S\}, \quad A \in L(R^n, R^m).$$

One has

$$(I_S)^* = \text{Supp}(S|.).$$

ii) Let $B \in L(R^n, R^m)$, $a \in R^m$. Consider the affine function

$$F(x) = B(x) + a.$$

We have

$$F^*(A) := \begin{cases} \{-a\}, & A = B, \\ \emptyset, & A \neq B. \end{cases}$$

The following proposition presents some calculus rules for conjugate maps.

Proposition 3.4. *Let $F : R^n \rightrightarrows R^m$ be a set-valued map with $domF \neq \emptyset$. Then*

i) $(F + a)^*(A) = F^*(A) - a$, $(\forall a \in R^m, A \in L(R^n, R^m))$.

ii) $(tF)^*(A) = tF^*(\frac{A}{t})$, $(\forall t > 0, A \in L(R^n, R^m))$.

iii) $(F(. + b))^*(A) = F^*(A) - A(b)$, $(\forall b \in R^n, A \in L(R^n, R^m))$.

Proof. i) Let $a \in R^m, A \in L(R^n, R^m)$ be arbitrary. From the definition and Proposition 2.22, one has

$$\begin{aligned}
(F + a)^*(A) &= \text{Sup} \bigcup_{x \in R^n} [A(x) - F(x) - a] \\
&= \text{Sup}((\bigcup_{x \in R^n} [A(x) - F(x)]) - a) \\
&= (\text{Sup} \bigcup_{x \in R^n} [A(x) - F(x)]) - a \\
&= F^*(A) - a.
\end{aligned}$$

ii) Let $t > 0, A \in L(R^n, R^m)$ be arbitrary. From the definition and Proposition 2.22, one has

$$\begin{aligned}
(tF)^*(A) &= \text{Sup} \bigcup_{x \in R^n} [A(x) - tF(x)] \\
&= \text{Sup}t \bigcup_{x \in R^n} [\frac{A}{t}(x) - F(x)] \\
&= t\text{Sup} \bigcup_{x \in R^n} [\frac{A}{t}(x) - F(x)] \\
&= tF^*(\frac{A}{t}).
\end{aligned}$$

iii) Let $b \in R^n, A \in L(R^n, R^m)$ be arbitrary. From definition and Proposition 2.22, one has

$$\begin{aligned}(F(.+b))^*(A) &= \mathrm{Sup} \bigcup_{x\in R^n} [A(x) - F(x+b)] \\ &= \mathrm{Sup} \bigcup_{x\in R^n} [A(x+b) - F(x+b) - A(b)] \\ &= \Big(\mathrm{Sup} \bigcup_{x\in R^n} [A(x+b) - F(x+b)]\Big) - A(b) \\ &= F^*(A) - A(b).\end{aligned}$$

The proof is complete. $\square$

Now, we shall establish some basic properties of conjugate maps.

Proposition 3.5. *Let $F : R^n \rightrightarrows R^m$ be a set-valued map with $domF \neq \emptyset$. Then*

i) *F^* is convex and closed.*

ii) *If* $\mathrm{dom}F^* \neq \emptyset$ *then*

$$F(x) \subseteq F^{**}(x) + C, \quad (\forall x \in R^n).$$

Proof. i) For every $A, B \in L(R^n, R^m), \lambda \in [0, 1]$, from the definiton and Proposition 2.22, one has

$$\begin{aligned}&\lambda F^*(A) + (1-\lambda)F^*(B) \\ &= \lambda \mathrm{Sup} \bigcup_{x\in R^n} [A(x) - F(x)] + (1-\lambda)\mathrm{Sup} \bigcup_{x\in R^n} [B(x) - F(x)] \\ &= \mathrm{Sup} \bigcup_{x\in R^n} [\lambda A(x) - \lambda F(x)] + \mathrm{Sup} \bigcup_{x\in R^n} [(1-\lambda)B(x) - (1-\lambda)F(x)] \\ &\subseteq \mathrm{Sup}\Big(\bigcup_{x\in R^n} [\lambda A(x) - \lambda F(x)] + \bigcup_{x\in R^n} [(1-\lambda)B(x) - (1-\lambda)F(x)]\Big) + C.\end{aligned}$$

Since

$$\begin{aligned}&\bigcup_{x\in R^n} [(\lambda A + (1-\lambda)B)(x) - F(x)] \\ &\subseteq \bigcup_{x\in R^n} [\lambda A(x) - \lambda F(x)] + \bigcup_{x\in R^n} [(1-\lambda)B(x) - (1-\lambda)F(x)]\end{aligned}$$

then by Proposition 2.22, we have

$$\begin{aligned}
&\mathrm{Sup}\Big(\bigcup_{x\in R^n}[\lambda A(x)-\lambda F(x)]+\bigcup_{x\in R^n}[(1-\lambda)B(x)-(1-\lambda)F(x)]\Big)\ +C\\
&\subseteq \mathrm{Sup}\bigcup_{x\in R^n}[(\lambda A+(1-\lambda)B)(x)-F(x)]\ \ +C\\
&=F^*(\lambda A+(1-\lambda)B)+C.
\end{aligned}$$

Hence, F^* is convex.

Now, assume that a sequence $(A_k, y_k) \subseteq \mathrm{epi}F^*$ converges to some $(A, y) \in L(R^n, R^m) \times R^m$. For every $x \in \mathrm{dom}F, z \in F(x)$, from the definition of conjugates, one has

$$y_k \succeq A_k(x) - z, \ \ (\forall k \in N).$$

Let $k \to \infty$, since C is closed, $y \succeq A(x) - z$. Hence,

$$y \in \mathrm{Sup}\bigcup_{x\in R^n}[A(x)-F(x)]+C=F^*(A)+C.$$

This means that $(A, y) \in \mathrm{epi}F^*$.

ii) Let $x \in R^n$ be arbitrary. If $x \notin \mathrm{dom}F$ then the inclusion is obvious. Now, assume that $x \in \mathrm{dom}F$. Pick any $A \in \mathrm{dom}F^*$. From Lemma 2.5 and Remark 2.24, we have

$$\begin{aligned}
A(x)-F^*(A)&=A(x)-\mathrm{Sup}\bigcup_{z\in R^n}[A(z)-F(z)]\\
&=\mathrm{Inf}(A(x)-\bigcup_{z\in R^n}[A(z)-F(z)]).
\end{aligned}$$

Since $F(x) \subseteq A(x) - \bigcup_{z\in R^n}[A(z) - F(z)]$,

$$F(x) \subseteq \mathrm{Ub}(A(x) - F^*(A)).$$

Since this is true for every $A \in \mathrm{dom}F^*$ then

$$\begin{aligned}
F(x)&\subseteq \mathrm{Ub}\bigcup_{A\in\mathrm{dom}F^*}[A(x)-F^*(A)]\\
&=\mathrm{Sup}\bigcup_{A\in\mathrm{dom}F^*}[A(x)-F^*(A)]\ \ +C\\
&=F^{**}(x)+C.
\end{aligned}$$

The proposition is proved. □

Let f be a convex vector function from a nonempty subset $D \subseteq R^n$ to R^m. The subdifferential of f at $x \in D$ is defined as the set

$$\partial f(x) := \{A \in L(R^n, R^m) : f(y) - f(x) \succeq A(y - x), \text{ for every } y \in D\}.$$

Proposition 3.6. *Let f be a convex vector function from a nonempty subset $D \subseteq R^n$ to R^m and let $x \in D, A \in L(R^n, R^m)$. Then $A \in \partial f(x)$ if and only if $f^*(A) = A(x) - f(x)$.*

Proof. From the definition of subdifferential and Corollary 2.21, one has

$$\begin{aligned} A \in \partial f(x) &\Leftrightarrow A(x) - f(x) \succeq A(y) - f(y), \ \ (\forall y \in D) \\ &\Leftrightarrow A(x) - f(x) = \mathrm{ISup} \bigcup_{y \in D} \{A(y) - f(y)\} \\ &\Leftrightarrow A(x) - f(x) = f^*(A). \end{aligned}$$

The proof is complete. □

Lemma 3.7. *Let F be a set-valued map from R^n to R^m such that there exists* $\mathrm{IInf}F(x)$ *for every $x \in$* $\mathrm{dom}F$. *If F is convex then* $\mathrm{IInf}F$ *is convex on* $\mathrm{dom}F$.

Proof. For every $x, y \in \mathrm{dom}F, \lambda \in [0, 1]$, by Remark 2.24 and Corollary 2.19, one has

$$\begin{aligned} \lambda \mathrm{IInf}F(x) + (1 - \lambda)\mathrm{IInf}\, F(y) &= \mathrm{IInf}\, [\lambda F(x)] + \mathrm{IInf}\, [(1 - \lambda)F(y)] \\ &= \mathrm{IInf}\, [\lambda F(x) + (1 - \lambda)F(y)]. \end{aligned}$$

By the convexity of F, $\lambda F(x) + (1 - \lambda)F(y) \subseteq F(\lambda x + (1 - \lambda)y) + C$. Hence,

$$\begin{aligned} \mathrm{IInf}\, F(\lambda x + (1 - \lambda)y) &\in \mathrm{Lb}\, [\lambda F(x) + (1 - \lambda)F(y)] \\ &= \mathrm{IInf}\, [\lambda F(x) + (1 - \lambda)F(y)] - C. \end{aligned}$$

Therefore, $\lambda \mathrm{IInf}\, F(x) + (1-\lambda)\mathrm{IInf}\, F(y) \succeq \mathrm{IInf}\, F(\lambda x + (1-\lambda)y)$. The proof is complete. □

Lemma 3.8. *Let $F : R^n \rightrightarrows R^m$ be a set-valued map with* $\mathrm{dom}F \neq \emptyset$ *such that there exists* $\mathrm{IInf}F(x)$ *for every $x \in$* $\mathrm{dom}F$. *Then*

$$F^* = (\mathrm{IInf}F)^*.$$

Proof. Let $A \in L(R^n, R^m)$ be arbitrary. By Lemma 2.5, Lemma 2.6 and Proposition 2.22, for every $x \in \mathrm{dom}F$, one has

$$A(x) - \mathrm{IInf}F(x) = \mathrm{ISup}[A(x) - F(x)].$$

Hence,

$$\begin{aligned}
& y \in \mathrm{Ub}\Big(\bigcup_{x \in \mathrm{dom}F} [A(x) - \mathrm{IInf}F(x)]\Big) \\
& \quad \Leftrightarrow y \in \mathrm{Ub}(A(x) - \mathrm{IInf}F(x)), \quad (\forall x \in \mathrm{dom}F) \\
& \quad \Leftrightarrow y \in \mathrm{Ub}(\mathrm{ISup}[A(x) - F(x)]), \quad (\forall x \in \mathrm{dom}F) \\
& \quad \Leftrightarrow y \in \mathrm{Ub}(A(x) - F(x)), \quad (\forall x \in \mathrm{dom}F) \\
& \quad \Leftrightarrow y \in \mathrm{Ub}\Big(\bigcup_{x \in \mathrm{dom}F} [A(x) - F(x)]\Big).
\end{aligned}$$

Therefore

$$\mathrm{Ub}\Big(\bigcup_{x \in \mathrm{dom}F} [A(x) - \mathrm{IInf}F(x)]\Big) = \mathrm{Ub}\Big(\bigcup_{x \in \mathrm{dom}F} [A(x) - F(x)]\Big).$$

This implies that

$$\begin{aligned}
(\mathrm{IInf}F)^*(A) &= \mathrm{Sup}\Big(\bigcup_{x \in \mathrm{dom}F} [A(x) - \mathrm{IInf}F(x)]\Big) \\
&= \mathrm{Min}\left(\mathrm{Ub}\Big(\bigcup_{x \in \mathrm{dom}F} [A(x) - \mathrm{IInf}F(x)]\Big)\right) \\
&= \mathrm{Min}\left(\mathrm{Ub}\Big(\bigcup_{x \in \mathrm{dom}F} [A(x) - F(x)]\Big)\right) \\
&= \mathrm{Sup}\Big(\bigcup_{x \in \mathrm{dom}F} [A(x) - F(x)]\Big) \\
&= F^*(A).
\end{aligned}$$

The proof is complete. □

Corollary 3.9. *Let F be a convex set-valued map from R^n to R^m such that there exists $\mathrm{IInf}F(x)$ for every $x \in \mathrm{dom}F$ and let $x \in \mathrm{dom}F, A \in L(R^n, R^m)$. Then $A \in \partial\mathrm{IInf}F(x)$ if and only if $F^*(A) = A(x) - \mathrm{IInf}F(x)$.*

Proof. This is immediate from Proposition 3.6, Lemma 3.7 and Lemma 3.8. □

Proposition 3.10. *Let f be a vector function from a nonempty subset $D \subseteq R^n$ to R^m. Then $f^*(A)$ is bounded below for every $A \in \mathrm{dom} f^*$ and, if f is convex,* $\mathrm{dom} f^* \neq \emptyset$.

Proof. Let $A \in \mathrm{dom} f^*$ be arbitrary. Pick any $x \in D$, one has

$$A(x) - f(x) \in \mathrm{Lb}\Big(\mathrm{Sup} \bigcup_{y \in D} \{A(y) - f(y)\}\Big) = \mathrm{Lb}(f^*(A)).$$

This means that $f^*(A)$ is bounded below.

Now, assume that f is convex. Let $x \in \mathrm{ri} D$. By Theorem 4.12 in [2], $\partial f(x) \neq \emptyset$. Pick any $A \in \partial f(x)$. By Proposition 3.6, $f^*(A) = A(x) - f(x)$. Hence, $\mathrm{dom} f^* \neq \emptyset$. The proof is complete. □

Corollary 3.11. *Let F be a set-valued map from R^n to R^m with* $\mathrm{dom} F \neq \emptyset$ *such that there exists* $\mathrm{IInf} F(x)$ *for every $x \in$* $\mathrm{dom} F$. *Then $F^*(A)$ is bounded below for every $A \in$* $\mathrm{dom} F^*$ *and* $\mathrm{dom} F^* \neq \emptyset$ *whenever F is convex.*

Proof. This is immediate from Proposition 3.10 , Lemma 3.7 and Lemma 3.8. □

Proposition 3.12. *Let f be a convex vector function from a nonempty subset $D \subseteq R^n$ to R^m. If D is closed then*

$$\mathrm{dom} f^{**} = D.$$

Proof. By Proposition 3.5, $D \subseteq \mathrm{dom} f^{**}$. Assume in the contrary that $\mathrm{dom} f^{**} \neq D$. Then there exists $x_0 \in \mathrm{dom} f^{**}$ such that $x_0 \notin D$. By the separation theorem, one can find a functional $\xi \in L(R^n, R)$ such that

$$\xi(x_0) > \mathrm{Sup}_{x \in D} \xi(x). \tag{6}$$

Let $y_0 \in riD$, by Theorem 4.12 in [2], $\partial f(y_0) \neq \emptyset$. Pick any point $A_0 \in \partial f(y_0)$. By Proposition 3.6, $f^*(A_0)$ reduces to a singleton. For every $c \in C$, define a linear map $\beta_c : R \to R^m$ as follows.

$$\beta_c(t) = tc, \quad (\forall t \in R).$$

From Corollary 2.19 and Proposition 2.22, one has

$$\begin{aligned} f^*(A_0) + \operatorname*{Sup}_{x\in D}(\beta_c\xi)(x) &= \operatorname{Sup}\bigcup_{x\in D}\{A_0(x) - f(x)\} + \operatorname*{Sup}_{x\in D}(\beta_c\xi)(x) \\ &= \operatorname{Sup}\Big(\bigcup_{x\in D}\{A_0(x) - f(x)\} + \bigcup_{x\in D}\{(\beta_c\xi)(x)\}\Big) \\ &\subseteq \operatorname{Sup}\bigcup_{x\in D}\{A_0(x) - f(x) + \beta_c\xi)(x)\} + C \\ &= f^*(A_0 + \beta_c\xi) + C. \end{aligned}$$

Observe that

$$\begin{aligned} \operatorname*{Sup}_{x\in D}(\beta_c\xi)(x) &= \beta_c(\operatorname*{Sup}_{x\in D}\xi(x)) \\ &= (\operatorname*{Sup}_{x\in D}\xi(x)).c. \end{aligned}$$

Hence, there exists $y_c \in f^*(A_0 + \beta_c\xi)$ such that

$$f^*(A_0) + (\operatorname*{Sup}_{x\in D}\xi(x)).c \succeq y_c.$$

Let $z \in f^{**}(x_0)$. From the definition of f^{**}, one has

$$\begin{aligned} z &\succeq (A_0 + \beta_c\xi)(x_0) - y_c \\ &\succeq [A_0(x_0) - f^*(A_0)] + [\xi(x_0) - \operatorname*{Sup}_{x\in D}\xi(x)].c, \end{aligned}$$

for all $c \in C$. By (6), this is impossible. Thus, $\mathrm{dom} f^{**} = D$. The proof is complete. □

Corollary 3.13. *Let F be a convex set-valued map from R^n to R^m with* $\mathrm{dom}F \neq \emptyset$ *such that there exists* $\mathrm{IInf}F(x)$ *for every* $x \in \mathrm{dom}F$*. If* $\mathrm{dom}F$ *is closed then*

$$\mathrm{dom}F^{**} = \mathrm{dom}F.$$

Proof. This is immediate from Proposition 3.12, Lemma 3.7 and Lemma 3.8. □

The rest of this section is intended to generalize the Fenchel-Moreau Theorem to the vector case.

Let f be a convex vector function from a nonempty subset $D \subseteq R^n$ to R^m. We say that f satisfies condition (H) at $x \in D$ if there is some $x_0 \in riD$ such that

$$f(x) = \lim_{\lambda \uparrow 1} f(\lambda x + (1-\lambda)x_0).$$

Obviously, if f is continuous at $x \in D$ then f satisfies the condition (H) at x. We note that the converse conclusion is not true in general. When $m = 1$ and f is closed then by [3, Chapter 2, Corollary 7.5.1], f satisfies the condition (H) at every $x \in D$. Furthermore, we have

Proposition 3.14. *Let f be a convex vector function from a nonempty subset $D \subseteq R^n$ to R^m. If f is subdifferentiable at $x \in D$, then f satisfies the condition* (H) *at x.*

We need the following lemma.

Lemma 3.15. *Let $g : [0,1] \to R$ be convex. If g is subdifferentiable at 1 then*

$$\lim_{t \uparrow 1} g(t) = g(1).$$

Proof. By the convexity of g, the limit $\lim\limits_{t \uparrow 1} g(t)$ exists. Pick any $A \in \partial g(1)$. Then for every $t \in [0,1]$, one has $g(t) - g(1) \geq A(t-1)$. Hence,

$$\lim_{t \uparrow 1} g(t) \geq g(1).$$

By the convexity of g, this implies that

$$\lim_{t \uparrow 1} g(t) = g(1).$$

The proof is complete. □

Proof of Proposition 3.14. Let $x_0 \in riD$. For every $\xi \in C'$, we define a function $f_\xi : [0,1] \to R$ as follows.

$$f_\xi(t) = \xi f(tx + (1-t)x_0).$$

Obviously, f_ξ is convex on $[0,1]$. Pick any $A \in \partial f(x)$ and put

$$A_\xi(t) = t\xi A(x - x_0), \ \ (\forall t \in R).$$

A direct verification shows that $A_\xi \in \partial f_\xi(1)$, i.e. f_ξ is subdifferentiable at 1. By Lemma 3.15, one has

$$\begin{aligned}\lim_{t\uparrow 1}\xi f(tx+(1-t)x_0) &= \lim_{t\uparrow 1} f_\xi(t)\\ &= f_\xi(1)\\ &= \xi f(x).\end{aligned}$$

Since C is closed, convex and pointed, $\mathrm{int}C' \neq \emptyset$. Hence,

$$\lim_{t\uparrow 1}\xi f(tx+(1-t)x_0) = \xi f(x), \quad \text{for all } \xi \in L(R^m, R).$$

This implies that

$$\lim_{t\uparrow 1} f(tx+(1-t)x_0) = f(x).$$

The proof is complete. $\square$

Lemma 3.16. *Let f be a convex vector function from a subset $D \subseteq R^n$ to R^m and let $x \in D$. Assume that there exists $x_0 \in riD$ such that $f(x) = \lim_{t\uparrow 1} f(tx+(1-t)x_0)$, (i.e., f satisfies condition* (H) *at x.). Then for every sequence $((A_k, x_k))_k \subseteq L(R^n, R^m) \times [x_0, x]$ such that $x_k \to x$ and $A_k \in \partial f(x_k)$, one has*

$$\lim_{k\to\infty} A_k(x-x_k) = 0.$$

Proof. For every $\xi \in C'$, we define a function f_ξ from [0,1] to R as follows.

$$f_\xi(t) = \xi f(tx+(1-t)x_0).$$

Obviously, f_ξ is convex on [0,1].

For every $k \geq 1$, represent x_k as $x_k = \lambda_k x + (1-\lambda_k)x_0$, for some $\lambda_k \in [0,1]$. Since $x_k \to x$, $\lambda_k \to 1$. Without loss of generality, we may assume $\lambda_k \uparrow 1$. Put

$$A_{k,\xi}(t) = t\xi A_k(x-x_0), \ \ (\forall t \in R).$$

A direct verification shows that $A_{k,\xi} \in \partial f_\xi(\lambda_k)$. We claim that $A_{k,\xi}(1-\lambda_k) \to 0$. Indeed, since f_ξ is convex on [0,1] then one of two following cases holds.

i) There exists $t_0 \in [0,1)$ such that f_ξ increases on $[t_0, 1)$. Without loss of generality, we may assume $t_0 = 0$. From the definition of subdifferential, for every $k \geq 1$, one has

$$\begin{aligned} A_{k,\xi}(1-\lambda_k) &\leq f_\xi(1) - f_\xi(\lambda_k), \\ A_{k,\xi}(-\lambda_k) &\leq f_\xi(0) - f_\xi(\lambda_k) \leq 0. \end{aligned}$$

Hence,

$$0 \leq A_{k,\xi}(1-\lambda_k) \leq f_\xi(1) - f_\xi(\lambda_k).$$

Since $f_\xi(1) - f_\xi(\lambda_k) = \xi[f(x) - f(\lambda_k x + (1-\lambda_k)x_0)] \to 0$ then

$$A_{k,\xi}(1-\lambda_k) \to 0 \text{ as } k \to \infty.$$

ii) f_ξ decreases on [0,1). Since $\lim\limits_{\lambda \uparrow 1} f_\xi(\lambda) = f_\xi(1)$ then f_ξ decreases on [0,1]. From the definition of subdifferential and the monotonicity of subdifferential, one has

$$\begin{cases} (A_{k+1,\xi} - A_{k,\xi})(\lambda_{k+1} - \lambda_k) \geq 0, \\ A_{k,\xi}(1-\lambda_k) \leq f_\xi(1) - f_\xi(\lambda_k) \leq 0. \end{cases}$$

Hence, $(A_{k,\xi})_k$ is an increasing and bounded from above sequence. Therefore $\lim\limits_{k\to\infty} A_{k,\xi} \in R$. Consequently,

$$\lim_{k\to\infty} A_{k,\xi}(1-\lambda_k) = 0.$$

We have seen that, in both cases, our claim is true. Thus, for every $\xi \in C'$, we have

$$\begin{aligned} \lim_{k\to\infty} \xi A_k(x - x_k) &= \lim_{k\to\infty} \xi A_k((1-\lambda_k)(x - x_0)) \\ &= \lim_{k\to\infty} A_{k,\xi}(1-\lambda_k) \\ &= 0. \end{aligned}$$

Since C is closed, convex and pointed, $\mathrm{int}C' \neq \emptyset$. Hence, $\lim\limits_{k\to\infty} \xi A_k(x - x_k) = 0$, for every $\xi \in L(R^m, R)$. This implies that

$$\lim_{k\to\infty} A_k(x - x_k) = 0.$$

The proof is complete. □

The following theorem is the main result of this section.

Theorem 3.17. *Let f be a convex vector function from a nonempty subset $D \subseteq R^n$ to R^m. If f satisfies the condition* (H) *at $x \in D$ then*

$$f^{**}(x) = f(x).$$

Proof. First, we consider the case $x \in riD$. By Proposition 4.12 in [2], $\partial f(x) \neq \emptyset$. Pick any $A_0 \in \partial f(x)$. By Proposition 3.6, $f(x) = A_0(x) - f^*(A_0)$. This and Proposition 3.5 implies

$$f(x) \in \mathrm{Ub}\Big(\bigcup_{A\in L(R^n,R^m)} [A(x) - f^*(A)]\Big)\cap$$
$$\cap\ cl\Big(co\Big(\bigcup_{A\in L(R^n,R^m)} [A(x) - f^*(A)]\Big)\Big).$$

Hence, by Corollary 2.21,

$$\begin{aligned} f(x) &= \mathrm{ISup} \bigcup_{A\in L(R^n,R^m)} [A(x) - f^*(A)] \\ &= f^{**}(x). \end{aligned}$$

Now, assume that x is a relative boundary point of D. From the hypothesis, there exists $x_0 \in riD$ such that

$$f(x) = \lim_{t\uparrow 1} f(tx + (1-t)x_0). \tag{7}$$

Let $(t_k) \subseteq (0,1)$ be an increasing sequence converging to 1. Put $x_k = t_k x + (1-t_k)x_0$. Obviously, $(x_k) \subseteq \mathrm{ri}D \cap [x_0, x]$ and $x_k \to x$. For every k, pick any $A_k \in \partial f(x_k)$. By Proposition 3.6, $f(x_k) = A_k(x_k) - f^*(A_k)$. Hence,

$$\|f(x) - [A_k(x) - f^*(A_k)]\| \leq \|f(x) - f(x_k)\| + \|A_k(x - x_k)\|.$$

Letting $k \to \infty$, by (7) and Lemma 3.16, we obtain

$$\|f(x) - [A_k(x) - f^*(A_k)]\| \to 0.$$

By this and by Proposition 3.5, one has

$$f(x) \in \mathrm{Ub}\Big(\bigcup_{A \in L(R^n, R^m)} [A(x) - f^*(A)] \Big) \cap$$
$$\cap \ cl\Big(co\Big(\bigcup_{A \in L(R^n, R^m)} [A(x) - f^*(A)] \Big)\Big).$$

Hence, by Corollary 2.21,

$$\begin{aligned} f(x) &= \mathrm{ISup} \bigcup_{A \in L(R^n, R^m)} [A(x) - f^*(A)] \\ &= f^{**}(x). \end{aligned}$$

The proof is complete. □

Corollary 3.18. *Let $F : R^n \rightrightarrows R^m$ be a convex set-valued map such that there exists* $\mathrm{IInf}F(x)$*, for every $x \in \mathrm{dom}F$. If* $\mathrm{IInf}F$ *satisfies the condition* (H) *at $x \in \mathrm{dom}\ F$ then*

$$F^{**}(x) = \mathrm{IInf}F(x).$$

Proof. This is immediate from Theorem 3.17, Lemma 3.7 and Lemma 3.8. □

4. Directional derivative

Let D be a nonempty convex subset of R^n and let $x \in D$. The direction $v \in R^n$ is said to be a feasible direction of D at x if there exists $t > 0$ such that $x + tv \in D$. The set of the feasible directions of D at x is denoted by $\mathrm{T}(D; x)$.

We recall that the contingent cone of D at $x \in R^n$ is defined as the set

$$\mathrm{K}(D; x) := \{v \in R^n : \exists v_i \to v, \exists t_i \downarrow 0 \text{ such that } x + t_i v_i \in D\}.$$

The following results describe the structure of the set $\mathrm{T}(D; x)$.

Proposition 4.1. *For every $x \in D$, one has*

i) *$T(D; x)$ is a convex cone.*

ii) *$T(D; x) \subseteq K(D; x)$.*

The equality holds if $x \in \mathrm{ri}D$. *In this case,*

$$T(D;x) = K(D;x) = \mathrm{span}D,$$

where span$D$ *denotes the subspace which is parallel to the affine hull of* D.

Proof. i) It is not difficult to see that $\mathrm{T}(D;x)$ is a cone. Let $u, v \in \mathrm{T}(D;x), \lambda \in [0,1]$ be arbitrary. Then there exist $s, t > 0$ such that $x + su, x + tv \in \mathrm{T}(D;x)$. Since D is convex then $x + \mu su + (1-\mu)tv \in D$, for every $\mu \in [0,1]$. If we choose

$$\mu = \frac{\lambda t}{\lambda t + (1-\lambda)s},$$
$$t_1 = \frac{st}{\lambda t + (1-\lambda)s},$$

then

$$x + t_1(\lambda u + (1-\lambda)v) = x + \mu su + (1-\mu)tv.$$

Hence, $\lambda u + (1-\lambda)v \in \mathrm{T}(D;x)$.

ii) The inclusion is obvious from definition. To prove the last assertion, without loss of generality, we assume that $x = 0$. Then we can find $\varepsilon > 0$ such that $\bar{B}_{spanD}(0,\epsilon) \subseteq D$. For every $v \in \mathrm{K}(D;0) \setminus \{0\}$, there exists $v_i \to v, t_i \downarrow 0$, such that $t_i v_i \in D$. Then $\varepsilon \dfrac{v_i}{\|v_i\|} \in \bar{B}_{spanD}(0,\varepsilon)$. Hence,

$$\varepsilon \frac{v}{\|v\|} = \lim \varepsilon \frac{v_i}{\|v_i\|} \in \bar{B}_{spanD}(0,\varepsilon) \subseteq D.$$

This implies that $v \in \mathrm{T}(D;x)$. The equality $\mathrm{T}(D;x) = \mathrm{span}D$ is trivial. The proof is complete. □

We note that, the inclusion in Proposition 4.1 is strict in general. For instance, let $D = \bar{B}((0,1),1) \subseteq R^2$, $x = (0,0)$. Then

$$\mathrm{K}(D;x) = \{(y,z) \in R^2 : z \geq 0\},$$

and

$$\mathrm{T}(D;x) = \{(y,z) \in R^2 : z > 0\}.$$

Definition 4.2. Let f be a convex vector function from a nonempty subset $D \subseteq R^n$ to R^m. The directional derivative of f at $x \in D$ in the direction $v \in \mathrm{T}(D;x)$ is the following limit if such exists

$$f'(x;v) = \lim_{t \downarrow 0} \frac{f(x+tv) - f(x)}{t}.$$

Proposition 4.3. *Let f be a convex vector function from a nonempty subset $D \subseteq R^n$ to R^m and let $x \in D, v \in T(D;x)$. If the set*

$$\Big\{\frac{f(x+tv)-f(x)}{t} : t > 0, x + tv \in D\Big\}$$

is bounded below then there exists the directional derivative of f at x in the direction v and

$$f'(x;v) = \mathrm{IInf}\Big\{\frac{f(x+tv)-f(x)}{t} : t > 0, x + tv \in D\Big\}.$$

Furthermore, $f'(x;.)$ is positively homogeneous and if dom$f'(x;.)$ is convex then $f'(x;.)$ is convex.

Proof. Put $t_0 = \mathrm{Sup}\{t > 0 : x + tv \in D\}$. It is clear $t_0 > 0$. Then the function

$$g(t) := \frac{f(x+tv)-f(x)}{t}$$

is defined on $(0, t_0)$ and takes value in R^m. From the hypothesis, g is bounded below on $(0, t_0)$. Let $t, t' \in (0, t_0), t \leq t'$. By [1, Chapter 1, Proposition 6.2], ξf is convex, for every $\xi \in C'$, hence

$$\frac{\xi f(x+tv) - \xi f(x)}{t} \leq \frac{\xi f(x+t'v) - \xi f(x)}{t'}.$$

i.e.

$$\xi g(t) \leq \xi g(t'), \quad (\forall \xi \in C').$$

This implies $g(t) \leq g(t')$, i.e. g is increasing on $(0, t_0)$. Then by Corollary 2.15, one has

$$\begin{aligned} f'(x;v) &= \lim_{t\downarrow 0} g(t) \\ &= \mathop{\mathrm{IInf}}_{t\in(0,t_0)} g(t) \\ &= \mathrm{IInf}\{\frac{f(x+tv)-f(x)}{t} : t > 0, x + tv \in D\}. \end{aligned}$$

From the definition it follows that $f'(x;.)$ is positively homogeneous.

Now, assume that dom$f'(x;.)$ is convex. Let $u, v \in \mathrm{dom}f'(x;.), \lambda \in (0,1)$ be arbitrary. Then for every $t > 0$, by the convexity of f, one has

$$\begin{aligned} \frac{f(x+t(\lambda u + (1-\lambda)v)) - f(x)}{t} \preceq \lambda &\frac{f(x+tu)-f(x)}{t} \\ &+ (1-\lambda)\frac{f(x+tv)-f(x)}{t}. \end{aligned}$$

Letting $t \downarrow 0$, by the closedness of C, we have

$$f'(x; \lambda u + (1-\lambda)v) \preceq \lambda f'(x;u) + (1-\lambda) f'(x;v).$$

The proof is complete. $\square$

Proposition 4.4. *Let f be a convex vector function from a nonempty subset $D \subseteq R^n$ to R^m. If f is subdifferentiable at $x \in D$ then*

$$\mathrm{dom} f'(x;.) = \mathrm{T}(D;x).$$

Proof. Obviously, $\mathrm{dom} f'(x;.) \subseteq \mathrm{T}(D;x)$. Let $v \in \mathrm{T}(D;x)$ be arbitrary. Pick any $A \in \partial f(x)$. For every $t > 0$ such that $x + tv \in D$, one has

$$f(x+tv) - f(x) \succeq tA(v).$$

Hence,

$$\Big\{ \frac{f(x+tv) - f(x)}{t} : t > 0, x + tv \in D \Big\}$$

is bounded below by $A(v)$. Then by Proposition 4.3, $v \in \mathrm{dom} f'(x;.)$. The proof is complete. $\square$

We note that if $x \in riD$ then by Proposition 4.1, $\mathrm{T}(D;x) = spanD$ and by Theorem 4.12 in [2], f is subdifferentiable at x. Hence, $\mathrm{dom} f'(x;.) = spanD$.

The following proposition is the most remarkable result of this section.

Proposition 4.5. *Let f be a convex vector function from a nonempty subset $D \subseteq R^n$ to R^m and let $x \in riD$. Then for every $v \in T(D;x)$, one has*

$$f'(x;v) = \mathrm{ISup}\{A(v) : A \in \partial f(x)\}.$$

Proof. For every $\xi \in C' \setminus \{0\}$, by [1, Chapter 1, Proposition 6.2], ξf is convex and by Theorem 4.6 in [2],

$$\partial(\xi f)(x) = \xi \partial f(x).$$

Then by [3, Chapter 5, Theorem 23.4], for every $v \in \mathrm{T}(D;x)$, we have

$$\begin{aligned}(\xi f)'(x;v) &= \mathrm{Sup}\{B(v) : B \in \partial(\xi f)(x)\} \\ &= \mathrm{Sup}\{\xi A(v) : A \in \partial f(x)\} \\ &= \mathrm{Sup} \xi \{A(v) : A \in \partial f(x)\}.\end{aligned}$$

On the other hand, one has

$$\begin{aligned}(\xi f)'(x;v) &= \lim_{t\downarrow 0}\frac{\xi f(x+tv)-\xi f(x)}{t}\\ &= \xi\lim_{t\downarrow 0}\frac{f(x+tv)-f(x)}{t}\\ &= \xi f'(x;v).\end{aligned}$$

Hence,

$$\xi f'(x;v) = \mathrm{Sup}\xi\{A(v) : A \in \partial f(x)\}.$$

Then by Proposition 2.23,

$$f'(x;v) = \mathrm{ISup}\{A(v) : A \in \partial f(x)\}.$$

The proof is complete. □

Proposition 4.6. *Let f be a convex vector function from a nonempty subset $D \subseteq R^n$ to R^m and let $x \in D, A \in L(R^n.R^m)$. Then $A \in \partial f(x)$ if and only if $A(v) \preceq f'(x;v)$, for every $v \in T(D;x)$.*

Proof. The 'only if' part is immediate from Proposition 4.3 and from the proof of Proposition 4.4.

For the 'if' part, let $y \in D$ be arbitrary. Set $v := y - x$. Then $v \in \mathrm{T}(D;x)$. From the hypothesis and from Proposition 4.3, one has

$$A(v) \preceq f'(x;v) \preceq f(x+v) - f(x).$$

Hence, $A \in \partial f(x)$. The proof is complete. □

Lemma 4.7. *Let $a \in R^m$. If $a \notin -C$ then $\{ta : t \geq 0\}$ is unbounded from above.*

Proof. Since C is convex and closed then, by the separation theorem, there exists $\xi \in C' \setminus \{0\}$ such that

$$\xi(a) > \mathrm{Sup}\{\xi(x) : x \in -C\} = 0.$$

Assume the contrary that $\{ta : t > 0\}$ is bounded from above by some $b \in R^m$. Then $-b + ta \in -C$, for every $t \geq 0$. Hence,

$$\xi(a) > -\xi(b) + t\xi(a),$$

for every $t \geq 0$. Letting $t \to \infty$, we arrive at a contradiction. The proof is complete. $\square$

Corollary 4.8. *Let f be a convex vector function from a nonempty subset $D \subseteq R^n$ to R^m and let $x \in D$ with* $\mathrm{dom} f'(x;.) = T(D;x)$. *Then*

$$\partial f(x) = \mathrm{dom}[f'(x;.)]^*.$$

Proof. Let $A \in \partial f(x)$ be arbitrary. By Proposition 4.6, $A(v) \preceq f'(x;v)$, for every $v \in \mathrm{T}(D;x)$. Hence,

$$\bigcup_{v \in \mathrm{dom} f'(x;.)} \{A(v) - f'(x;v)\}$$

is bounded from above by 0. This implies that $A \in \mathrm{dom}[f'(x;.)]^*$.

Conversely, let $A \in \mathrm{dom}[f'(x;.)]^*$ be arbitrary. From the definition of conjugate functions, the set

$$\bigcup_{v \in \mathrm{dom} f'(x;.)} \{A(v) - f'(x;v)\}$$

is bounded from above. We shall show that

$$A(v) - f'(x;v) \preceq 0,$$

for every $v \in \mathrm{dom} f'(x;.)$. Then by Proposition 4.6, $A \in \partial f(x)$, which completes the proof. Indeed, assume in the contrary that there is a point $v \in \mathrm{dom} f'(x;.)$ such that $A(v) - f'(x;v) \notin -C$. Since $f'(x;.)$ is positively homogeneous, by Lemma 4.7, the set $\{A(tv) - f'(x;tv) : t \geq 0\}$ is unbounded from above. We arrive at a contradiction. $\square$

5. Conjugate duality

We conclude this paper by giving an application of conjugate maps to the dual problems of vector optimization problems.

Let $C \subseteq R^m$ and $K \subseteq R^k$ be closed, convex and pointed cones. Let F and G be set-valued maps from R^n to R^m and R^k respectively, and let X be a nonempty subset of R^n such that

$$X \subseteq \mathrm{dom} F \cap \mathrm{dom} G.$$

Let us consider the vector optimization problem (VP) with set-valued data:

$$\begin{aligned}&\min F(x)\\&s.t.\ x\in X,\ G(x)\cap -K\neq\emptyset.\end{aligned}$$

Put

$$X_0 := \{x\in X|\ G(x)\cap -K\neq\emptyset\}.$$

Assume that $X_0\neq\emptyset$. We recall that $x_0\in X_0$ is said to be an optimal solution of (VP) if $F(x_0)\cap \mathrm{Min}(F(X_0)|C)\neq\emptyset$. $x_0\in X_0$ is said to be an ideal optimal solution of (VP) if $F(x_0)\cap \mathrm{IMin}(F(X_0)|C)\neq\emptyset$.

Define a perturbation for (VP) as a map from $R^n\times R^k$ to R^m by the rule

$$\begin{aligned}&\Phi(x,b)=F(x),\ x\in X,\ G(x)\cap -(K+b)\neq\emptyset;\\&\Phi(x,b)=\emptyset,\ \text{otherwise.}\end{aligned}$$

Observe that $\mathrm{dom}\Phi$ is nonempty since X_0 is nonempty. The perturbed problem corresponding to a vector $b\in R^k$ will be of the form

$$(Pb)\qquad\qquad \begin{aligned}&\min \Phi(x,b)\\&\text{s.t. } x\in R^n.\end{aligned}$$

It is clear that problem (VP) is the same as (P_0).

The conjugate map Φ^* of Φ is a set-valued map from $L(R^n\times R^k,R^m)$ to R^m defined by

$$\Phi^*(z,y)=\mathrm{Sup}\Big(\bigcup_{(x,b)\in R^n\times R^k}[z(x)+y(b)-\Phi(x,b)]|\ C\Big),$$
$$y\in L(R^k,R^m),\ z\in L(R^n,R^m).$$

For every fixed $z\in L(R^n,R^m)$, we then have a vector problem

$$\begin{aligned}&\max -\Phi^*(z,y)\\&\text{s.t. } y\in L(R^k,R^m).\end{aligned}$$

Let us denote this latter problem by (D^*) in the special case where $z=0$ and call it the conjugate dual of (VP).

We note that by [1, Chapter 5, Proposition 2.1], Φ is C-convex in both variables on $R^n \times R^k$ if X is convex, F is C-convex, G is K-convex on X. By Proposition 3.5 above, Φ^* is C-convex.

Recall from [1] that a triple $(x, a, b) \in R^n \times R^m \times R^k$ is said to be feasible if $x \in X,\ a \in F(x)$ and $b \in G(x) \cap -K$. For the dual problem a feasible couple $(y, a') \in L(R^k, R^m) \times R^m$ means that $a' \in -\Phi^*(0, y)$.

Theorem 5.1 (Weak duality). *For every feasible triple (x_0, a_0, b_0) of (VP) and feasible couple (y_0, a_0') of (D^*), one has*

$$a_0' \preceq_C a_0.$$

Proof. We have

$$\begin{aligned} a_0' &\in -\Phi^*(0, y_0) \\ &= -\mathrm{Sup}\Big(\bigcup_{(x,b)\in R^n\times R^k} [y_0(b) - \Phi(x, b)]\Big) \\ &= \mathrm{Inf}\Big(\bigcup_{(x,b)\in R^n\times R^k} [\Phi(x, b) - y_0(b)]\Big). \end{aligned}$$

Hence, $a_0' \in \mathrm{Lb}\Phi(x_0, 0) = \mathrm{Lb}F(x_0)$. Thus, $a_0' \preceq_C a_0$. The proof is complete. □

Theorem 5.2. *If there are $x_0 \in X,\ y_0 \in L(R^k, R^m)$ such that*

$$0 \in \Phi(x_0, 0) + \Phi^*(0, y_0)$$

then x_0 is an ideal optimal solution of (VP), y_0 is an ideal optimal solution of (D^*) and the ideal optimal values of these problems are equal.

Proof. Let $a_0 \in \Phi(x_0, 0) \cap -\Phi^*(0, y_0)$. Then (y_0, a_0) is a feasible couple of (D^*). For every feasible triple (x, a, b) of (VP), by Theorem 5.1, one has

$$a_0 \preceq_C a.$$

Then $a_0 = \mathrm{IMin}\,(F(X_0)|\ C)$ and x_0 is an ideal optimal solution of (VP).

Now, let $b_0 \in G(x_0) \cap -K$. Then (x_0, a_0, b_0) is a feasible triple of (VP). For every feasible couple (y, a') of (D^*), by Theorem 5.1, one has

$$a' \preceq_C a_0.$$

Then $a_0 = \mathrm{IMax}\left(-\Phi^*(0, L(R^k, R^m))|\ C\right)$ and y_0 is an ideal optimal solution of (D^*). The proof is complete. $\square$

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