SOME APPLICATIONS OF CHOI POLYNOMIALS OF LINEAR MAPS

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Abstract. This paper investigates the properties of Choi polynomials and their fundamental role in the theory of positive linear maps between matrix algebras. By focusing on Hermitian symmetric biquadratic forms, we establish a connection between the positivity of these forms and the structure of positive maps. We specifically explore the construction of indecomposable positive maps in matrix algebras, and their application as entanglement witnesses. Our analysis extends to the detection of Positive Partial Transpose (PPT) entangled states and the classification of edge PPT states in $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$. Our results provide a refined framework for identifying non-separable states that escape the standard PPT criterion, contributing to the broader understanding of entanglement distillation and quantum information theory.

1. Introduction

The classification of positive linear maps $\phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ remains a central challenge in operator theory and quantum information science. A map is called completely positive if its Choi matrix, defined as $C_\phi = \sum_{i,j} e_{ij} \otimes \phi(e_{ij})$, where $\{e_{ij}\}$ is the standard basis for M_m, is positive semi-definite. However, the structure of maps that are positive but not completely positive is significantly more intricate. These maps are essential for the detection of quantum entanglement, a task often formulated through the construction of entanglement witnesses (see [7, 8]).

A density operator ρ in a bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable if it can be expressed as a convex combination of product states. If a state is not separable, it is said to be entangled. The Positive Partial Transpose (PPT) criterion, introduced by Peres ([14]) and the Horodeckis ([6]), serves as a powerful necessary condition

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for separability: every separable state must remain positive under the partial transposition operator Γ. While this criterion is sufficient for dimensions 2×2 and 2×3, the existence of PPT entangled states (PPTES) in higher dimensions necessitates the use of indecomposable positive maps.

In this work, we focus on a specific class of homogeneous polynomials of degree four, known as Hermitian symmetric biquadratic forms (see [3]). As demonstrated in the seminal works of Choi and later refinements by Osaka, these forms are intrinsically linked to the range properties of positive maps. This class is a subclass of real-valued homogeneous polynomials of degree four. The properties of Hermitian symmetric biquadratic forms and establishes a rigorous mathematical framework linking polynomial algebra with linear operator theory. The content particularly emphasizes the role of biquadratic forms (hereafter called Choi polynomials) in classifying positive linear maps.

One of the foundational results presented is the existence of a one-to-one correspondence between a linear map $\phi : M_m \to M_n$ and its Choi polynomial, defined as $P_\phi(x, y) = y^*\phi(xx^*)y$. Through this correspondence, the properties of the map are fully reflected by the polynomial: The map ϕ is positive if and only if its corresponding Choi polynomial $P_\phi(x, y)$ is a positive semidefinite biquadratic form, meaning $P_\phi(x, y) \geq 0$ for all vectors $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$. The decomposability of a map is also completely characterized through the structure of its Choi polynomial. Specifically, a linear map ϕ is decomposable if and only if its Choi polynomial $P_\phi(x, y)$ can be expanded as a sum of squares of the moduli of bilinear forms and sesquilinear forms. In terms of matrices, this is equivalent to the Gram matrix W of the polynomial P_ϕ belonging to the decomposable Gram cone $\mathcal{D}(m, n) = \{W = Q + R^\Gamma : Q \geq 0, R \geq 0\}$, where Γ denotes the partial transpose. The most crucial objective is to provide a systematic method for constructing indecomposable biquadratic forms, which in turn generate indecomposable positive maps. Specifically, we construct the family of polynomials:

$$p_\epsilon(x, y) = p(x, y) - \epsilon||x \otimes y||^2$$

and show that there exists a threshold $\delta > 0$ such that for all $0 < \epsilon \leq \delta$, the polynomial $p_\epsilon(x, y)$ remains positive semidefinite but its decomposable structure is broken (it becomes an indecomposable form). This is particularly true when $Q + R^\Gamma$ and $Q^\Gamma + R$ are non-trivial projections. Due to the 1-1 correspondence, these newly constructed forms $p_\epsilon(x, y)$ are precisely the Choi polynomials of a new class of linear maps. These maps inherit the exact properties of their generating polynomials: they are positive maps but are indecomposable.

By applying the results established in Sections 2 and 3, we re-prove and extend several classes of decomposable and indecomposable maps in Section 4. This section demonstrates the practical application of biquadratic forms and Choi polynomials in characterizing map indecomposability, specifically through the analysis of weighted maps $\Phi_{a;m,n,\epsilon}$ and the construction of maps from edge PPT entangled states.

The paper is organized as follows. Section 2 introduces the notation for biquadratic forms and the Choi-Jamiołkowski isomorphism. Some sufficient and necessary conditions for a biquadratic form to be decomposable/indecomposable. Section 3 discusses the properties of Choi polynomials and their spectral characteristics. In Section 4, we apply these mathematical constructs to the problem of identifying PPTES and classifying edge states in $M_4(\mathbb{C}) \otimes M_4(\mathbb{C})$.

Notation: We denote by $\{e_i\}$ the standard basis for $\mathbb{C}^m$ (and similarly for $\mathbb{C}^n$) and set $e_{ij} = e_i e_j^*$. We define the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^n$ by $\langle x, y \rangle = y^* x$. Let M_n (resp., $M_n(\mathbb{R})$) denote the algebra of $n \times n$ matrices with complex (resp., real) entries. Let $\mathcal{S}_n$ be the space of all symmetric matrices of order n with real coefficients. Let $B(M_m, M_n)$ denote the space of all linear maps from M_m to M_n. The Choi matrix of a map $\phi : M_m \to M_n$, denoted by C_ϕ, is defined by $C_\phi = \sum_{i,j} e_{ij} \otimes \phi(e_{ij})$. The transpose of A is denoted by A^t. The partial transpose is a linear operator on $M_m \otimes M_n$ defined by $(A \otimes B)^\Gamma = A \otimes B^t$. Let $PPT[m, n]$ denote the set of all positive matrices in $M_m \otimes M_n$ with a positive partial transpose. Finally, let $\sum^2 K$ denote the set of all finite sums of squares of elements in K.

2. Biquadratic forms

In this section, we investigate a subclass of real-valued homogeneous polynomials of degree four, namely Hermitian symmetric biquadratic forms. Our aim is to characterize the positivity and decomposability of this class of forms.

2.1. **Real biquadratic forms.**

A real biquadratic form is a homogeneous polynomial of the form:

$$F(x, y) = \sum_{i,j,k,l} c_{ijkl} x_i x_j y_k y_l \quad (i \le j, k \le l, \; c_{ijkl} \in \mathbb{R})$$

with real indeterminates $x = (x_1, \ldots, x_m),\ y = (y_1, \ldots, y_n)$. Such an F is said to be positive semidefinite (or rnonnegative) if $F(x, y) \geq 0$ for all $x \in \mathbb{R}^m$,$y \in \mathbb{R}^n$.

An old question asks whether, if F is positive semidefinite, there must exist real bilinear forms f_i such that $F = \sum_i f_i^2$. Recall a real bilinear form f with real indeterminates x, y is a homogeneous polynomial which can be written as $f(x, y) = y^t Ax$, where A is an $n \times m$ matrix with real coefficients. The well-known fact that a positive semidefinite real biquadratic form must be a sum of squares of real bilinear forms provided that either $m = 2$ or $n = 2$ (see [1, Theorem 1]). However, if $m \geq 3$ and $n \geq 3$, Man D. Choi [2] gave an example to show that there exists a positive semidefinite real biquadratic form which is not a sum of square of real bilinear forms. As application, Choi also used this example to show that there exists a positive linear map from $\mathcal{S}_3$ to $\mathcal{S}_3$ which is not completely positive map. Follow this idea, we try to study the relation (and applications) of positive semidefinite biquadratic forms and positive maps.

Let $T : M_m(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ be a linear map. Then the Choi polynomial of T is determined by

$$P_T(x, y) = y^t T(xx^t)y, \qquad \forall x \in \mathbb{R}^m, y \in \mathbb{R}^n.$$

Then P_T is a real biquadratic form. If two linear maps ϕ and ψ from M_m to M_n have the same Choi polynomial, then $y^t\phi(xx^t)y = y^t\psi(xx^t)y$ for all $x \in \mathbb{R}^m, y \in \mathbb{R}^n$. Hence, $\phi(xx^t) = \psi(xx^t)$ for all x. As a consequence, $\phi(X) = \psi(X)$ for all symmetric $X \in M_m(\mathbb{R})$. However, since $\phi(e_{ij} + e_{ji}) = \psi(e_{ij} + e_{ji})$, we have $\phi(e_{ij}) + \phi(e_{ji}) = \psi(e_{ij}) + \psi(e_{ji})$.

Conversely, given a real biquaratic form $F(x, y) = \sum_{i \leq j, k \leq l} c_{ijkl} x_i x_j y_k y_l$ in m-variable x and n-variable y. We can write $F(x, y) = y^t B(x)y$, where $B(x)$ is a symmetric matrix whose coefficients are quadratic forms in x. Then we can get a linear operator T from $\mathcal{S}_m$ to $\mathcal{S}_n$ by $T(xx^t) = B(x)$. This implies that $F(x, y) = y^t T(xx^t)y$. Hence, we get the following remark (we can also see the proof of this remark in [2]).

Remark 1. The correspondence between operators $T : \mathcal{S}_m \to \mathcal{S}_n$ and real biquadratic forms P_T is one to one.

A linear map $T : M_m(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ is called a congruence map if there exist $m \times n$ matrices with real coefficients $V_i, i = 1, \dots, k$ such that

$$T(X) = \sum_{i=1}^{k} V_i^t X V_i \qquad \forall X \in M_m(\mathbb{R}).$$

Proposition 2.1 ([2])**.** *Let $T : \mathcal{S}_m \longrightarrow \mathcal{S}_n$ be a linear map.*

(a) *T is positive if and only if $P_T(x, y) \geq 0$ for all x, y.*
(b) *T is a congruence map if and only if P_T is a sum of square of real bilinear forms.*

Calderon [1] showed that every positive semidefinite real biquadratic form on $\mathbb{R}^m \times \mathbb{R}^n$ must be sum of square of real bilinear forms, provided that $m = 2$ or $n = 2$. Combine Proposition 2.1 and [1, Theorem 1], we get the following corollary.

Corollary 2.2. *A positive linear map $T : \mathcal{S}_2 \longrightarrow \mathcal{S}_n$ must be a congruence one.*

2.2. **Biquaratic forms.**

Definition 2.3. A complex polynomial $p(x, y)$ in $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$ is said to be *sesquilinear form* (linear in y and anti-linear in x) if it has the form:

$$p(x, y) = \sum_{i,j} p_{ij} \bar{x}_i y_j, \qquad p_{ij} \in \mathbb{C}.$$

A complex polynomial p is said to be *bilinear form* if it has the form:

$$p(x, y) = \sum_{i,j} p_{ij} x_i y_j, \qquad p_{ij} \in \mathbb{C}.$$

A complex polynomial p is called *biquadratic form* if it has the form:

$$p(x, y) = \sum_{i,j,k,l} p_{ijkl} x_i \bar{x}_j y_k \bar{y}_l, \qquad p_{ijkl} \in \mathbb{C}.$$

A biquadratic form p is said to be Hermitian symmetric (in the sense of [3]) if $p(x, y) \in \mathbb{R}$ for all $x \in \mathbb{C}^m$, $y \in \mathbb{C}^n$. Such a biquadratic form is also called a real-valued form.

Note that $p(x, y)$ is sesquilinear, meaning it is linear in y and anti-linear in x. A form p is called *positive semidefinite* (psd) if $p(x, y) \geq 0$ for all x, y. Sometimes, we use the term *nonnegative form* to refer to a psd form.

It is straightforward to see that p is a bilinear form if and only if there exists an $m \times n$ matrix A such that

$$p(x, y) = x^T Ay = \langle Ay, \bar{x} \rangle.$$

Analogously, p is a sesquilinear form if and only if there exists an $m \times n$ matrix A such that

$$p(x, y) = x^* Ay = \langle Ay, x \rangle.$$

Let $\mathrm{BLF}(m, n)$ (resp., $\mathrm{SLF}(m, n)$) denote the set of all bilinear forms (resp., sesquilinear forms) in $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$.

Remark 2. If p is a sesquilinear form and $p(x, y) \in \mathbb{R}$ for every real vectors x and y, then $p(x, y)$ is a real bilinear form.

Proof. We can write

$$p(x, y) = \sum_{i,j} p_{ij} \bar{x}_i y_j, \qquad p_{ij} \in \mathbb{C}.$$

For all real vectors $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ we have $p(x, y) = \sum_{i,j} p_{ij} x_i y_j \in \mathbb{R}$. Thus, $p(x, y) = \overline{p(x, y)}$. Hence $p_{ij} = \bar{p}_{ij} \in \mathbb{R}$. This means that $p(x, y)$ is a real bilinear form when x, y are real. □

For $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$,

$$z(x, y) := x \otimes y = \big(x_1 y_1,\ x_1 y_2, \dots,\ x_1 y_n,\ x_2 y_1, \dots,\ x_m y_n\big)^T \in \mathbb{C}^{mn}.$$

Any finite sum of squares of moduli of *bilinear* forms can be written as

$$\sum_r \big|x^t A_r y\big|^2 \ = \ z(x, y)^* \, Q \, z(x, y) \qquad \text{for some } Q \succeq 0,$$

i.e., Q is the Gram matrix of the bilinear sum of squares.

For sesquilinear forms $x^* By$, use the monomial vector

$$w(x, y) := x \otimes \bar{y} = \big(x_1 \overline{y_1},\ \dots,\ x_m \overline{y_n}\big)^T \in \mathbb{C}^{mn},$$

and similarly

$$\sum_s \big|x^* B_s y\big|^2 \ = \ w(x, y)^* \, R \, w(x, y) \qquad \text{for some } R \succeq 0.$$

Observe that both $z(x, y)^* \, Q \, z(x, y)$ and $w(x, y)^* \, R \, w(x, y)$ are nonnegative Hermitian symmetric biquadratic forms.

Lemma 2.4. *Let p be a biquadratic form given by:*

$$p(x,y) = \sum_{i,j,k,l} p_{ijkl} x_i \bar{x}_j y_k \bar{y}_l, \qquad x \in \mathbb{C}^m, y \in \mathbb{C}^n.$$

There exist unique matrices Q and R such that:

$$p(x,y) = z(x,y)^* Q z(x,y) = w(x,y)^* R w(x,y),$$

where $z(x,y) = x \otimes y$ and $w(x,y) = \bar{x} \otimes y$. The relationship between Q and R is a partial transpose on the first subsystem. Furthermore, p is Hermitian symmetric if and only if $Q = Q^*$ (and a similar statement holds for R). In this case, the relationship between Q and R is also a partial transpose on the second subsystem.

Such a matrix Q in Lemma 2.4 is called a *Gram matrix* of p.

Proof. Let $z(x,y) = x \otimes y$. Its components are $z_{ik} = x_i y_k$, and $(z^*)_{jl} = \bar{x}_j \bar{y}_l$. We can rewrite the form as:

$$p(x,y) = \sum_{i,j,k,l} p_{ijkl} (\bar{x}_j \bar{y}_l)(x_i y_k)$$

Define $Q = \sum_{i,j=1}^m \sum_{k,l=1}^n Q_{(k,l)}^{(i,j)} e_i e_j^* \otimes f_k f_l^*$, where $Q_{(l,k)}^{(j,i)} = p_{ijkl}$. Then we obtain $p(x,y) = z^* Q z$. If $p(x,y) = z(x,y)^* P z(x,y)$, then we get $P_{(j,i)}^{(l,k)} = p_{ijkl} = Q_{(j,i)}^{(l,k)}$. Similarly, we can show the existence and uniqueness of R. Since $R_{(l,k)}^{(i,j)} = p_{ijkl} = Q_{(l,k)}^{(j,i)}$, we have $R = Q^{T_1}$, where T_1 denotes the transpose operation on the first Hilbert space $\mathbb{C}^m$.

If $p(x,y) \in \mathbb{R}$, then $z^* Q z = (z^* Q z)^* = z^* Q^* z$. Since $z(x,y)$ spans $\mathbb{C}^{mn}$, $Q = Q^*$. The converse follows from the property of Hermitian forms. In this case, $Q^{T_1} = Q^\Gamma$ and so R is the partial transpose on the second subsystem of Q. □

2.3. Decomposable biquadratic forms.

Definition 2.5. A biquadratic form p is said to be *decomposable* if it can be written as a finite sum of squares of the moduli of bilinear and sesquilinear forms. A positive semidefinite biquadratic form p is said to be *indecomposable* if it is not decomposable.

Let us denote by $\sum^2|\mathrm{BLF}|(m,n)$ (resp., $\sum^2|\mathrm{SLF}|(m,n)$) the set of all finite sums of the squared moduli of bilinear (resp., sesquilinear) forms in the variables $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$.

Denote

$$\mathcal{D}(m,n) \;=\; \left\{ W = Q + R^{\Gamma} \;:\; Q \succeq 0,\; R \succeq 0 \right\}.$$

Then $\mathcal{D}(m,n)$ is a cone and called the *decomposable Gram cone.*

Proposition 2.6. *Let $p(x,y) = (x \otimes y)^* \, W \, (x \otimes y)$ be a biquadratic form, where the Gram matrix $W = W^*$ is a square matrix of order mn and $x \in \mathbb{C}^m, y \in \mathbb{C}^n$. Then*

(i) *The polynomial $p \in \sum^2 |\mathrm{BLF}(m,n)|$ if and only if W is positive semidefinite.*
(ii) *The polynomial $p \in \sum^2 |\mathrm{SLF}(m,n)|$ if and only if W^{Γ} is positive semidefinite.*
(iii) *p is decomposable if and only if its Gram matrix W belongs to the decomposable Gram cone $\mathcal{D}(m,n)$.*

Proof. (i) If $f(x,y)$ is a bilinear form, then we can write

$$f(x,y) = \sum_{i,j} f_{ij} x_i y_j = \langle x \otimes y, \overline{\mathrm{vec}(f)} \rangle = (x \otimes y)^* \mathrm{vec}(f),$$

where $\mathrm{vec}(f)$ the vector of coefficients of f. Hence, if $p \in \sum^2 |\mathrm{BLF}(m,n)|$, then there are bilinear forms $f_1, \dots, f_k$ such that

$$p(x,y) = \sum_{i=1}^{k} |f_i(x,y)|^2 = \sum_{i=1}^{k} (x \otimes y)^* \mathrm{vec}(f_i) \mathrm{vec}(f_i)^* (x \otimes y).$$

By the uniqueness of the Gram matrix (Lemma 2.4), $W = \mathrm{vec}(f_1)\mathrm{vec}(f_1)^* + \cdots + \mathrm{vec}(f_k)\mathrm{vec}(f_k)^*$ is positive semidefinite. Conversely, if W is positive semidefinite, then there is a representation $W = \lambda_1 v_1 v_1^* + \cdots + \lambda_k v_k v_k^*$, where each v_i is a unit eigenvector corresponding to the eigenvalue λ_i of W. Hence,

$$p(x,y) = \sum_{i=1}^{k} \lambda_i (x \otimes y)^* \, v_i v_i^* \, (x \otimes y) = \sum_{i=1}^{k} \lambda_i |v_i^* (x \otimes y)|^2.$$

The proof of (ii) is the same as that of (i), and (iii) follows from (i), (ii) and Lemma 2.4. □

Let $W = W^* \in M_{mn}$. Define

$$V_W = \{ x \otimes y \mid (x \otimes y)^* W (x \otimes y) = 0 \}.$$

We say $W_1 \leq W_2$ iff $V_{W_1} \subset V_{W_2}$. This define a partial order on $\mathcal{D}(m,n)$.

Definition 2.7. A decomposable matrix $W \in \mathcal{D}(m,n)$ is said to be an *minimal* if $V_W = \{0\}$. That is $(x \otimes y)^* W x \otimes y > 0$ for every nonzero $x \otimes y$, $x \in \mathbb{C}^m, y \in \mathbb{C}^n$.

Lemma 2.8. *Let* $p(x,y) = (x\otimes y)^* W(x\otimes y)$ *be a decomposable form, where* $W = Q + R^\Gamma$ *for some positive semidefinite matrices* Q, R. *Then the following are equivalent.*

(1) W *is minimal.*

(2) $\min_{\|x\|=1,\|y\|=1} p(x,y) > 0$.

(3) $\ker Q \cap \ker R$ *has no nonzero product vectors.*

Proof. The equivalence of (1) and (2) follows from the fact that $p(tx, sy) = t^2 s^2 p(x,y)$ for any positive numbers s, t.

Now, we prove the equivalence of (1) and (3). We have

$$(x\otimes y)^* Q(x\otimes y) = \|Q^{\frac{1}{2}}(x\otimes y)\|^2.$$

Using $\mathrm{Tr}(X^\Gamma Y) = \mathrm{Tr}(XY^\Gamma)$, we also have

$$(x\otimes y)^* R(x\otimes y) = \mathrm{Tr}(R^\Gamma(x^*x\otimes(y^*y)^T)) = \mathrm{Tr}(R^\Gamma(x^*x\otimes(\bar{y}^*\bar{y}))) = \|(R)^{\frac{1}{2}}(x\otimes\bar{y})\|^2.$$

Hence,

$$p(x,y) = \|Q^{\frac{1}{2}}(x\otimes y)\|^2 + \|(R)^{\frac{1}{2}}(x\otimes\bar{y})\|^2 > 0,$$

for all nonzero $x\otimes y\in\mathbb{C}^m\otimes\mathbb{C}^n$ if and only if $\ker Q\cap\ker R$ has no nonzero product vectors $x\otimes y$. □

Denote $\mathbb{H}(M_m\otimes M_n)$ the space of Hermitian matrices in $(M_m\otimes M_n)$. We equip $\mathbb{H}(M_m\otimes M_n)$ with the Hilbert-Schmidt inner product

$$\langle X, Y\rangle := \mathrm{Tr}(XY),\ \ X = X^*, Y = Y^*\in M_m\otimes M_n.$$

Then partial transpose on the second subsystem $^\Gamma$ is self-adjoint, that is, for all $X = X^*, Y = Y^*\in M_m\otimes M_n$, we have

$$\langle X^\Gamma, Y\rangle = \langle X, Y^\Gamma\rangle.$$

Recall that for a cone $K\subset\mathbb{H}(M_m\otimes M_n)$,

$$K^* := \{\, X : \mathrm{Tr}(XY)\geq 0,\ \ \forall Y\in K\,\}$$

denotes its dual cone. Then it is well-known that the cone of positive semidefinite matrices is self-dual:

$$\{X\succeq 0\}^* = \{X\succeq 0\}.$$

In addition, the partial transpose on the second subsystem is self-adjoint and involutive, we have the following lemma (see [8] for more detail).

Lemma 2.9. $\mathcal{D}^* = \{ X \in M_m \otimes M_n : X \succeq 0 \ \&\ X^\Gamma \succeq 0\}$.

Clearly, a real valued polynomial can be written as a difference of sums of squares. On the other hand, given a real valued biquadratic form $f(x,y) = (x\otimes y)^* W x \otimes y$ with Gram matrix W, there exist a real number δ such that $f(x,y) + \delta\|x\otimes y\|^2$ is decomposable. Indeed, we choose δ to be at least the minimal eigenvalue of W. Then $W + \delta I$ is positive semidefinite.

In our attempt to construct positive indecomposable biquadratic forms, we usually pay attention to a class of the forms:

$$F(x,y) - \varepsilon||x\otimes y|| \qquad (x\in\mathbb{C}^m, y\in\mathbb{C}^n),$$

where $F(x,y)$ is a decomposable biquadratic form.

Theorem 2.10. *Let p be a decomposable biquadratic form*

$$p(x,y) = (x\otimes y)^* \left(Q + R^\Gamma\right)(x\otimes y) \qquad (x\in\mathbb{C}^m, y\in\mathbb{C}^n),$$

where Q and R are positive semidefinite square matrices in $M_m\otimes M_n$, Let $\varepsilon > 0$ and $p_\varepsilon(x,y) := p(x,y) - \varepsilon\|x\otimes y\|^2$. Then the followings hold.

(i) *$Q + R^\Gamma$ is minimal if and only if there exists a positive number δ such that the biquadratic form p_ε is positive semidefinite for every $\varepsilon \le \delta$.*

(ii) *p_ε is decomposable if and only if $Q + R^\Gamma - \varepsilon I$ is decomposable, where I is the identity matrix of order mn.*

Proof. Suppose that $W = Q + R^\Gamma$ is minimal. By Lemma 2.8, we have

$$\delta := \inf_{\|x\|=1,\|y\|=1} p(x,y) > 0.$$

For every positive $\varepsilon \le \delta$, and all nonzero product vector $x\otimes y$,

$$p_\varepsilon(x,y) = \|x\|^2\|y\|^2\left(p(x_1,y_1) - \varepsilon\right) \ge \|x\|^2\|y\|^2(\delta-\varepsilon) > 0,$$

where $x = \|x\|x_1$ and $y = \|y\|y_1$.

Conversely, there exists a positive number δ such that the biquadratic form p_ε is positive semidefinite for every $\varepsilon \le \delta$. Then

$$\min_{\|x\|=1,\|y\|=1} p(x,y) \ge \delta \min_{\|x\|=1,\|y\|=1} \|x\otimes y\|^2 = \delta > 0.$$

The proof of (ii) follows immediately from Proposition 2.6 and the fact that the Gram matrix of p_ε is $W - \varepsilon I$, where $W = Q + R^\Gamma$. □

Corollary 2.11. *Let p and p_ε be given as in Theorem 2.10(i). Suppose further that $\Pi := Q+R^\Gamma$ is a projection and Π^Γ a positive contraction. Then p_ε is indecomposable for every $0 < \varepsilon < \delta$.*

Proof. Let $M = I-\Pi$, where I is the identity matrix of order mn. By the hypothesis, M and M^Γ are positive semidefinite. Moreover,

$$\operatorname{Tr}((\Pi - \varepsilon I)M) = \operatorname{Tr}((\Pi - \varepsilon I)(I - \Pi)) = -\varepsilon \operatorname{Tr}(M) < 0.$$

By Lemma 2.9, $(\Pi - \varepsilon I)$ cannot lie in the decomposable Gram cone $\mathcal{D}(m, n)$. By Proposition 2.6, p_ε is indecomposable for every $0 < \varepsilon < \delta$. □

Example 2.12. Let us consider the real symmetric matrix $\Pi \in \mathbb{R}^{9\times 9}$ satisfying

$$18\Pi = \begin{pmatrix} 11 & -7 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ -7 & 11 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 11 & 2 & 2 & -7 & 2 & 2 & 2 \\ 2 & 2 & 2 & 11 & 2 & 2 & -7 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & -7 & 2 & 2 & 11 & 2 & 2 & 2 \\ 2 & 2 & 2 & -7 & 2 & 2 & 11 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 11 & -7 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & -7 & 11 \end{pmatrix}.$$

Then $\Pi^2 = \Pi = \Pi^*$ and $\operatorname{rank}(\Pi) = 5$ and $W = \Pi$ satisfies the conditions of Corollary 2.11. Hence, the biquadratic form

$$F_\varepsilon(x, y) = (x \otimes y)^*\Pi(x \otimes y) - \varepsilon\|x \otimes y\|^2 \qquad (x, y \in \mathbb{C}^3)$$

is indecomposable for every $0 < \varepsilon \leq \min_{\|x\|=1,\|y\|=1}(x \otimes y)^*\Pi(x \otimes y)$.

Next, we will show that $W = \Pi$ satisfies the hypothesis of Corollary 2.11.

Proof of Example 2.12. It is straightforward to check that $W = \Pi$ and W^Γ are projections and rank of Π is 5. Let $\mathcal{N} = \ker(\Pi) \subset \mathbb{C}^9$. A direct computation yields $\dim \mathcal{N} = 4$ and a basis of $\mathcal{N}$ consisting of the following four vectors (displayed as 3×3 matrices in the xy^t-reshape):

$$K_1 = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$K_3 = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad K_4 = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

In other word,

$$\ker(\Pi) = \{\mathrm{vec}(K) : K \in \mathrm{span}_{\mathbb{C}}\{K_1, K_2, K_3, K_4\}\}.$$

Let $z(x, y) = \mathrm{vec}(xy^t)$ corresponds to the rank-one matrix xy^t.

Claim 1. $\ker(\Pi) \cap \{\mathrm{vec}(xy^t) : x, y \neq 0\} = \{0\}$.

If $\mathrm{vec}(xy^t) \in \ker(\Pi)$, then $xy^t \in \mathrm{span}\{K_1, K_2, K_3, K_4\}$ and has rank one. By **Claim 1**, this forces $xy^t = 0$, hence $x = 0$ or $y = 0$.

Now, we prove **Claim 1.** Suppose $K = aK_1 + bK_2 + cK_3 + dK_4$ is a product vector, then K has rank ≤ 1. We will show that $a = b = c = d = 0$. Write

$$K = \begin{pmatrix} -\frac{a}{2} - b - c - d & -\frac{a}{2} - b - c - d & b \\ c & a & b \\ c & d & d \end{pmatrix}.$$

If $\mathrm{rank}(K) \leq 1$, then all 2×2 minors of K vanish. We get

$$c(d - a) = 0, \qquad c(d - b) = 0, \qquad d(a - b) = 0,$$

$$b(3a + 2b + 2c + 2d) = 0, \qquad b(a + 2b + 4c + 2d) = 0, \qquad d(a + 4b + 2c + 2d) = 0.$$

We now eliminate the following cases.

Case 1: $b \neq 0$. Then

$$3a + 2b + 2c + 2d = 0, \qquad a + 2b + 4c + 2d = 0.$$

Subtracting gives $c = a$. Substituting into $a + 2b + 4c + 2d = 0$ yields

$$5a + 2b + 2d = 0. \tag{1}$$

If $d = 0$, then $d(a - b) = 0$ is automatic, while $c(d - a) = a(0 - a) = -a^2 = 0$ forces $a = 0$ and then (1) gives $b = 0$, contradicting $b \neq 0$. Hence $d \neq 0$. Then $d(a + 4b + 2c + 2d) = 0$ implies

$$a + 4b + 2c + 2d = 0.$$

With $c = a$ this becomes $3a + 4b + 2d = 0$. Together with (1),

$$\begin{cases} 5a + 2b + 2d = 0, \\ 3a + 4b + 2d = 0. \end{cases} \qquad \text{We have } a = b.$$

Then $3a + 4b + 2d = 0$ gives $7b + 2d = 0$, so $d = -\frac{7}{2}b$. Now use $c(d - a) = 0$ with $c = a = b$:

$$0 = c(d-a) = b\Big(-\frac{7}{2}b - b\Big) = -\frac{9}{2}b^2,$$

so $b = 0$, contradiction. Therefore $b \neq 0$ is impossible.

Case 2: $b = 0$. Then K becomes

$$K = \begin{pmatrix} -\frac{a}{2} - c - d & -\frac{a}{2} - c - d & 0 \\ c & a & 0 \\ c & d & d \end{pmatrix}.$$

Similar arguments above, we imply that $d = 0$.

With $b = d = 0$, we have

$$K = \begin{pmatrix} -\frac{a}{2} - c & -\frac{a}{2} - c & 0 \\ c & a & 0 \\ c & 0 & 0 \end{pmatrix}.$$

Repeated the similar arguments above, we imply that $a = b = c = d = 0$.

□

Theorem 2.13 ([9]). *Let H_A, H_B be finite-dimensional complex Hilbert spaces and let $\Gamma = \mathrm{id} \otimes T$ denote the partial transpose with respect to a fixed product basis. Let $\rho \in \mathbb{H}(H_A \otimes H_B)$ satisfy the PPT conditions*

$$\rho \succeq 0, \qquad \rho^\Gamma \succeq 0.$$

For a product vector $x \otimes y$ we write $(x \otimes y)^\Gamma := x \otimes \overline{y}$.

Consider the following statements:

(A) *(*Edge (subtraction) definition*) There do not exist $\varepsilon > 0$ and a nonzero product vector $x \otimes y$ such that*

$$\rho - \varepsilon\,(x \otimes y)(x \otimes y)^* \succeq 0 \quad \textit{and} \quad \rho^\Gamma - \varepsilon\,(x \otimes \overline{y})(x \otimes \overline{y})^* \succeq 0.$$

(B) *(*Range-intersection condition*) There exists no nonzero product vector $x \otimes y$ such that $x \otimes y$ belongs to both the ranges of ρ and ρ^Γ.*

(C) *(*Kernel-orthogonality condition*) There exists no nonzero product vector $x \otimes y$ such that*

$$\langle u,\, x \otimes y\rangle = 0 \quad \forall u \in \ker(\rho), \qquad \langle v,\, x \otimes \overline{y}\rangle = 0 \quad \forall v \in \ker(\rho^{\Gamma}).$$

Then

$$(A) \iff (B) \iff (C).$$

Here, (A) in the theorem is the standard definition of an *edge PPT state* as introduced by Lewenstein–Kraus–Cirac–Horodecki [9, Sec. III].

Corollary 2.14. *Let ρ be a PPT entangled edge. Let $W = P + Q^{\Gamma}$, where P, Q be orthogonal projections on $\ker \rho, \ker \rho^{\Gamma}$, respectively. Then $\delta = \min_{|x|=1,|y|=1}(x \otimes y)^* W(x \otimes y) > 0$ and for every $0 < \varepsilon \leq \delta$, the biquadratic form $F_{\varepsilon}(x, y) = (x \otimes y)^* W(x \otimes y) - \varepsilon |x \otimes y|^2$ is positive semidefinite but not decomposable.*

Proof. Set

$$\delta := \min_{\|x\|=\|y\|=1} \langle x \otimes y, W(x \otimes y)\rangle.$$

For Horodecki PPT entangled states, by Theorem 2.13, this δ is strictly positive. For any $0 < \varepsilon \leq \delta$, the operator $A := W - \varepsilon I_8$ is block-positive (i.e., $P_{\varepsilon}(x, y) = (x \otimes y)^* A(x \otimes y) \geq 0$ for all $x \in \mathbb{C}^2, y \in \mathbb{C}^4$) but detects ρ in the sense that:

$$\mathrm{Tr}(A\rho) = -8\varepsilon < 0,$$

because $P\rho = 0$ and $Q\rho^{\Gamma} = 0$, hence $\mathrm{Tr}(P\rho) = 0$ and $\mathrm{Tr}(Q^{\Gamma}\rho) = \mathrm{Tr}(Q\rho^{\Gamma}) = 0$. Hence, $A \notin \mathcal{D}(m, n)$. Applying Proposition 2.6, we get the conclusion. □

3. Choi Polynomials

Let $\phi : M_m \to M_n$ be a linear map. Let P_{ϕ} denote the polynomial defined by:

$$P_{\phi}(x, y) = y^* \phi(xx^*) y \qquad (x \in \mathbb{C}^m, y \in \mathbb{C}^n).$$

We call P_{ϕ} *the Choi polynomial of ϕ*. (Note: M.D. Choi constructed a real biquadratic form on $\mathbb{R}^3 \times \mathbb{R}^3$ that is nonnegative everywhere but is not a sum of squares of real bilinear forms [2]. This result demonstrated the existence of a positive linear map on M_3 that is not decomposable.)

Example 3.1. Let F be a biquadratic form on $\mathbb{C}^2 \times \mathbb{C}^2$ defined by

$$F(x,y) = 2|x_1|^2|y_1|^2 - 2\sqrt{-1}\,x_2\bar{x}_1|y_1|^2 + 3\sqrt{-1}\,x_1\bar{x}_2|y_1|^2 + 3|x_2|^2|y_2|^2.$$

Then there is a linear map $\phi:\ M_2 \to M_2$ determined by

$$\phi(e_{11}) = \begin{pmatrix} 2 & 3\sqrt{-1} \\ -2\sqrt{-1} & 0 \end{pmatrix}, \quad \phi(e_{22}) = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\phi(e_{12}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \phi(e_{21}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

such that the Choi polynomial is $F(x,y)$. It is clear that ϕ is not self-adjoint and $F\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = 2 + \sqrt{-1}$ is not real.

Proposition 3.2. *Let ϕ, ψ be linear maps from M_m to M_n. Then the following statements hold true.*

(1) *$P_{\lambda\phi+\mu\psi}(x,y) = \lambda P_\phi(x,y) + \mu P_\psi(x,y)$ for all $x \in \mathbb{C}^m, y \in \mathbb{C}^n$, and any complex numbers λ, μ.*
(2) *$P_\phi(x,y) = 0$ for all $x \in \mathbb{C}^m, y \in \mathbb{C}^n$ if and only if $\phi(X) = 0$ for all $X \in M_m$.*
(3) *The correspondence that associates each linear map $\phi \in B[M_m, M_n]$ with its Choi polynomial P_ϕ is one-to-one between $B(M_m, M_n)$ and the set of all biquadratic forms in $x \in \mathbb{C}^m, y \in \mathbb{C}^n$.*

Proof. (1) is straightforward.

(2) Suppose $P_\phi = 0$. Then $y^*\phi(xx^*)y = 0$ for all $x \in \mathbb{C}^m, y \in \mathbb{C}^n$. Hence, $\phi(xx^*) = 0$ for all $x \in \mathbb{C}^m$. As a consequence, $\phi(X) = 0$ for all Hermitian $X \in M_m$. Any matrix $X \in M_m$ can be decomposed as $X = X_1 + iX_2$, where the X_j are Hermitian. Hence $\phi(X) = 0$.

(3) Suppose p is a biquadratic forms. Then p can be written $p(x,y) = \sum c_{ijkl} x_i \bar{x}_j y_k \overline{y}_l$. We define a linear map $\phi:\ M_m \to M_n$ such that the (l,k) entry of $\phi(e_{ij})$ is c_{ijkl}.

Then, for $x \in \mathbb{C}^m, y \in \mathbb{C}^n$, we have

$$\begin{aligned} y^*\phi(xx^*)y &= \sum_{l=1}^{n} \overline{y_l} f_l^* \phi[(\sum_{i=1}^{m} x_i e_i)(\sum_{j=1}^{m} \overline{x}_j e_j^*)] \sum_{k=1}^{n} y_k f_k \\ &= \sum_{i,j=1}^{m} \sum_{k,l=1}^{n} x_i \overline{x}_j y_k \overline{y}_l [f_l^* \phi(e_i e_j^*) f_k] \\ &= \sum_{i,j=1}^{m} \sum_{k,l=1}^{n} c_{ijkl} x_i \overline{x}_j y_k \overline{y}_l = p(x,y). \end{aligned}$$

By (2), the corresponding between ϕ and P_ϕ is one to one and onto.

Indeed, if $\phi_1, \phi_2 \in B(M_m, M_n)$ satisfies $P_{\phi_1} = P_{\phi_2}$, then, $P_{\phi_1 - \phi_2} = 0$. Hence, $\phi_1 = \phi_2$. □

The Choi–Jamiołkowski isomorphism establishes a correspondence between a linear map $\phi : M_m \to M_n$ and its Choi matrix $C_\phi \in M_m \otimes M_n$, given by $C_\phi = \sum_{i,j=1}^m e_{ij} \otimes \phi(e_{ij})$. It is straightforward to compute that for all $x \in \mathbb{C}^m$,$y \in \mathbb{C}^n$

$$P_\phi(x,y) = \langle C_\phi(\overline{x} \otimes y), \overline{x} \otimes y \rangle,$$

where $\overline{x} = (\overline{x_1}, \ldots, \overline{x_m})^t \in \mathbb{C}^m$.

Let us recall (see, e.g., [8, 11]) that

$$B_1(M_m \otimes M_n) = \{X = X^* \in M_m \otimes M_n \mid (x \otimes y)^* X (x \otimes y) \geq 0, \quad \forall x \in \mathbb{C}^m, y \in \mathbb{C}^n\}.$$

Theorem 3.3. *Let ϕ be a linear map from M_m to M_n. Then the following statements hold.*

1. *ϕ is self-adjoint, if and only if $C_\phi = C_\phi^*$, if and only if P_ϕ is real-valued for all $x \in \mathbb{C}^m$, $y \in \mathbb{C}^n$ (in other word, P_ϕ is Hermitian symmetric).*
2. *ϕ is positive, if and only if C_ϕ belongs to $B_1(M_m \otimes M_n)$, if and only if $P_\phi(x,y) \geq 0$ for all x, y.*
3. *ϕ is completely positive, if and only if C_ϕ is positive, if and only if the Choi polynomial P_ϕ of ϕ is a sum of squares of sesquilinear forms (of the form $x^* A y$).*
4. *ϕ is completely copositive, if and only if the partial tranpose C_ϕ^Γ is positive, if and only if the Choi polynomial P_ϕ of ϕ is a sum of squares of bilinear forms (of the form $x^t B y$).*

(5) *ϕ is decomposable, if and only if $C_\phi \in PPT[m,n]$, if and only if the Choi polynomial P_ϕ of is a sum of squares of bilinear and sesquilinear forms.*

Proof. The equivalence between these properties of ϕ and of the Choi matrix C_ϕ are well-known (e.g., see [11, Section 3.3]). Hence, we need to prove the equivalence between ϕ and its Choi polynomial.

(1) If ϕ is self-adjoint (i.e., it maps a Hermitian matrix to a Hermitian one),then

$$\overline{P_\phi(x,y)} = (y^*\phi(xx^*)y)^* = y^*(\phi(xx^*))^*y = y^*\phi(xx^*)y \in \mathbb{R}.$$

Conversely, if $P_\phi(x,y)$ is real for all x, y then $\phi(xx^x)$ is Hermitian for every x. As a consequence, ϕ maps a Hermitian matrix to a Hermitian matrix. Hence, it is self-adjoint.

(2) This part follows from the definition of Choi polynomial and that ϕ is positive iff $\phi(xx^*) \geq 0$ for every $x \in \mathbb{C}^m$.

(3) If $\phi(X) = V^*XV$ then

$$P_\phi(x,y) = y^*V^*(xx^*)Vy = |x^*Vy|^2.$$

Conversely, if $P_\phi(x,y)$ is a sum of squares of sesquilinear forms, for simplicity, assume $P_\phi(x,y) = |f(x,y)|^2$, where f is a sesquilinear form. We can write $f(x,y) = x^*Ay$, where A is an $m \times n$-matrix. Hence,

$$P_\phi(x,y) = (x^*Ay)^*(x^*Ay) = y^*A^*xx^*Ay = y^*\phi(xx^*)y, \quad \phi(X) = A^*XA.$$

(4) If $\phi(X) = V^*X^tV$ then

$$P_\phi(x,y) = y^*V^*(xx^*)^tVy = |x^tVy|^2.$$

The converse statement is the same argument as the one in (3).

(5) This part follows from (3) and (4). □

In 1963, Størmer [15] showed that every positive linear map ϕ from M_2 to M_2 is decomposable. By [15, Theorem 8.2] every unital extreme linear map from M_2 to M_2 is unitarily equivalent to the one of the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & \alpha b + \beta c \\ \overline{\alpha} c + \overline{\beta} b & \gamma a + \epsilon b + \overline{\epsilon} c + \delta d \end{pmatrix},$$

where $|\epsilon|^2 = 2\gamma(\delta - |\alpha|^2 - |\beta|^2)$ in the case $\gamma \neq 0$ and $|\alpha| = 1$ or $|\beta| = 1$ in the case $\gamma = 0$. We will consider the case where $\gamma = 0$ (so $\epsilon = 0$) and $|\alpha| = 1$. Let us denote this map by ϕ_0 (including the non-unital case). Then, let $X = (x_{ij})$,

$$\phi_0(X) = \begin{pmatrix} x_{11} & \alpha x_{12} + \beta x_{21} \\ \overline{\alpha} x_{21} + \overline{\beta} x_{12} & \delta x_{22}, \end{pmatrix},$$

where $|\alpha| = 1$ and $\sqrt{\delta} = |\beta| + 1$.

Remark 3. ϕ_0 is unital if and only if $\beta = 0$; that is, ϕ_0 is extreme in the sense of [15] if and only if $\beta = 0$. In this case, ϕ_0 is completely positive.

Proof. $\phi_0(1) = 1$, if and only if, $\delta = 1$, if and only if, $\beta = 0$. The Choi matrix C_{ϕ_0} in this case is

$$C_{\phi_0} = \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{\alpha} & 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues are $\{0, 2\}$. □

In the case $\beta \neq 0$, the map ϕ_0 is not extreme. Using the Choi polynomial, we have the following decomposition of ϕ_0.

Corollary 3.4. *The positive linear map ϕ_0 is decomposable but is neither completely positive nor copositive, provided that $\beta \neq 0$. In addition, ϕ_0 can be decomposed uniquely as*

$$\phi_0 = \phi_1 + \phi_2,$$

where ϕ_1 is completely positive, ϕ_2 is completely copositive and

$$\phi_1(xx^*) = \begin{pmatrix} \delta^{-1/2}|x_1|^2 & x_1\overline{x}_2 \\ \overline{x}_1 x_2 & \sqrt{\delta}|x_2|^2, \end{pmatrix}, \quad \phi_2(xx^*) = |\beta| \begin{pmatrix} \delta^{-1/2}|x_1|^2 & x_2\overline{x}_1 \\ \overline{x}_2 x_1 & \sqrt{\delta}|x_2|^2, \end{pmatrix}.$$

Proof. We have

$$P_\phi(x, y) = |x_1|^2|y_1|^2 + \alpha x_1\overline{x}_2\overline{y}_1 y_2 + \overline{\alpha x}_1 x_2 y_1\overline{y}_2 + \beta\overline{x}_1 x_2\overline{y}_1 y_2 + \overline{\beta} x_1\overline{x}_2 y_1\overline{y}_2 + \delta|x_2|^2|y_2|^2.$$

Let $A = (a_{ij})$ and $B = (b_{kl})$ be two matrices in M_2. Let $F(x, y) := |x^*Ay|^2 + |x^tBy|^2$. Then

$$\begin{aligned} F(x,y) &= \left|\sum_{i,k} a_{ik}\overline{x}_i y_k\right|^2 + \left|\sum_{i,k} b_{ik} x_i y_k\right|^2 \\ &= \sum_{i,j,k,l} a_{ik}\overline{a}_{jl}\overline{x}_i x_j y_k \overline{y}_l + \sum_{i,j,k,l} b_{ik}\overline{b}_{jl} x_i \overline{x}_j y_k \overline{y}_l. \end{aligned}$$

Identify the coefficients of $F(x, y)$ and P_{ϕ_0} we get $a_{12} = a_{21} = b_{12} = b_{21} = 0$ and

$$\begin{aligned} &|a_{11}|^2 + |b_{11}|^2 = 1, \\ &|a_{22}|^2 + |b_{22}|^2 = \delta = (1 + |\beta|)^2, \\ &\overline{a}_{11}a_{22} = \alpha, \ \ \overline{b}_{11}b_{22} = \beta. \end{aligned}$$

Therefore,

$$A = e^{i\theta_1}\begin{pmatrix} \frac{\overline{\alpha}}{\sqrt{1+|\beta|}} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = e^{i\theta_2}\begin{pmatrix} \frac{\sqrt{|\beta|}}{\sqrt{1+|\beta|}} & 0 \\ 0 & \frac{\beta\sqrt{1+|\beta|}}{\sqrt{|\beta|}} \end{pmatrix},$$

for any θ_1, θ_2. However, the corresponding biquadratic forms of A and B:

$$\begin{aligned} F_1(x,y) &= |x^*Ay|^2 = y^*A^*xx^*Ay = y^*|A|xx^*|A|y, \\ F_2(x,y) &= |x^tBy|^2 = y^*B^*(x^t)^*x^tBy = y^*|B|(xx^*)^t|B|y \end{aligned}$$

are independent of θ_1 and θ_2. Hence, the resulting biquadratic forms F_1 and F_2 are uniquely determined. As a consequence, by Lemma 3.2, we can get the CP (ϕ_1) and coCP (ϕ_2) map uniquely. Recovery of the map ϕ_1, ϕ_2 from the biquadratic forms is followed from the identity $\phi_1(xx^*) = A^*xx^*A = |A|xx^*|A|$ and $\phi_2(xx^*) = |B|(xx^*)^t|B|$. □

Let $\phi : M_m \longrightarrow M_n$ be a linear map. Suppose that $\phi(M_m(\mathbb{R})) \subset M_n(\mathbb{R})$. Then there exists the restriction of ϕ on $M_m(\mathbb{R})$, denoted by $\phi_{\mathbb{R}}$ which is a linear map from $M_m(\mathbb{R})$ to $M_n(\mathbb{R})$ by $\phi_{\mathbb{R}}(X) = \phi(X)$ for $X \in M_m(\mathbb{R})$.

Proposition 3.5. *Let $\phi : M_m \longrightarrow M_n$ be a self-adjoint linear map with $\phi(M_m(\mathbb{R})) \subset M_n(\mathbb{R})$. If ϕ is decomposable, then the restriction $\phi_{\mathbb{R}} : \mathcal{S}_m \to \mathcal{S}_n$ is a congruence.*

Proof. For an $m \times n$ matrix $V = (v_{ij})$, we can write $V = Re(V) + iIm(V)$, where $Re(V) = (Re(v_{kl}))$ the real part and $Im(V) = (Im(v_{kl}))$ the imaginary part (which are matrices with real coefficients). Then for any $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, we have

$$y^*V^*xx^*Vy = |x^*Vy|^2 = |x^tRe(V)y|^2 + |x^tIm(V)y|^2.$$

Hence, $y^*V^*xx^*Vy$ is a sum of squares of real bilinear forms. Similarly, $y^*V^*(xx^*)^tVy$ is also a sum of squares of real bilinear forms. Hence, the Choi polynomial $P_{\phi_{\mathbb{R}}}(x,y)$ is the sum of squares of real bilinear forms and by Proposition 2.1, we get the conclusion. □

Proposition 3.6. *Let $\phi: M_2 \longrightarrow M_n$ be a linear map with $\phi(M_2(\mathbb{R})) \subset M_n(\mathbb{R})$. If ϕ is positive then the restriction $\phi_{\mathbb{R}}: \mathcal{S}_2 \to \mathcal{S}_n$ is a congruence map. In addition, there exists a completely positive map ϕ_1 from M_2 to M_n such that $\phi = \phi_1$ on $\mathcal{S}_2$ and $(\phi - \phi_1)(e_{12}) = -(\phi - \phi_1)(e_{21})$ is anti-symmetric.*

Proof. Since ϕ is positive, its restriction $\phi_{\mathbb{R}}$ on $M_m(\mathbb{R})$ is positive. By Proposition 2.1, the Choi polynomial of $\phi_{\mathbb{R}}$ is nonnegative, that is $P_{\phi_{\mathbb{R}}}(x,y) \geq 0$ for all x, y. By [1, Theorem 1], $P_{\phi_{\mathbb{R}}}$ is a sum of squares of real bilinear forms. By Proposition 2.1, the restriction $\phi_{\mathbb{R}}$ from $\mathcal{S}_2$ to $\mathcal{S}_n$ is a congruence map. That is, there are $m \times 2$ real matrices V_j such that

$$\phi_{\mathbb{R}}(Y) = \sum_j V_j^t Y V_j \qquad \forall Y = Y^t \in M_2(\mathbb{R}).$$

Define ϕ_1 from M_2 to M_n by

$$\phi_1(X) = \sum_j V_j^t X V_j \qquad \forall X \in M_2.$$

Then $\phi_1 = \phi$ on $\mathcal{S}_2$. Let $A = (\phi - \phi_1)(e_{12})$, then $0 = (\phi - \phi_1)(e_{12} + e_{21}) = A + (\phi - \phi_1)(e_{21})$. Hence $-A = (\phi - \phi_1)(e_{12}^*) = A^* = A^t$. □

Remark 4. Let $\phi_\varepsilon: M_2 \longrightarrow M_4$ be a linear map whose Choi polynomial

$$P_{\phi_\varepsilon}(x,y) = F_\varepsilon(x,y) = (x \otimes y)^*\Pi(x \otimes y) - \varepsilon\|x \otimes y\|^2 \qquad (x \in \mathbb{C}^2, y \in \mathbb{C}^4), \quad 0 < \varepsilon \leq \delta,$$

where Π, δ and $F_\varepsilon(x,y)$ are defined in Example 2.12. As in Example 2.12, the Choi polynomial is positive semidefinite and not decomposable. By Theorem 3.3, ϕ_ε is indecomposable. It is clear that $\phi_\varepsilon(M_2(\mathbb{R})) \subset M_4(\mathbb{R})$.

3.1. **Application of optimization to checking positivity.** A Choi polynomial of a self-adjoint linear map is a real-valued biquadratic form. As the same aregument in Lemma 2.8, we get the following corollary.

Corollary 3.7. *Let $\phi : M_m \longrightarrow M_n$ be a self-adjoint linear map. Then the following statements are equivalent.*

(i) *ϕ is positive.*

(ii) *For every positive real number r,*

$$\min_{||x||=||y||=r} P_\phi(x,y) \geq 0.$$

(iii) *There is a real number $r > 0$ such that*

$$\min_{||x||=||y||=r} P_\phi(x,y) \geq 0.$$

Proof. (i)⇒(ii). By Theorem 3.3(2), ϕ is positive if and only if $P_\phi(x,y)$ is nonnegative for all x, y. Hence,

$$\min_{||x||=r,||y||=r} P_\phi(x,y) \geq 0.$$

(iii)⇒(i) Suppose that there is a positive real number r such that

$$\min_{||x||=r,||y||=r} P_\phi(x,y) \geq 0.$$

There is a point $(x_0, y_0) \in \mathbb{C}^m \times \mathbb{C}^n$ such that $\|x_0\| = r = \|y_0\|$ and

$$\min_{\|x\|=r,\|y\|=r} P_\phi(x,y) = P_\phi(x_0,y_0) \geq 0.$$

Since $P_\phi(x,y)$ is homogeneous, we have

$$P_\phi(x,y) = \frac{||x||^2||y||^2}{r^4} P_\phi(\frac{r}{||x||}x, \frac{r}{||y||}y) \geq P_\phi(x_0,y_0) \geq 0.$$

The implication (ii) ⇒ (iii) is immediate. □

The characterization of positive linear maps are also related to the problem of determining whether a polynomial is bounded below or not.

Corollary 3.8. *Let $\phi : M_m \longrightarrow M_n$ be a self-adjoint linear map. Then ϕ is positive if and only if P_ϕ is bounded below, i.e,*

$$\inf\{P_\phi(x,y) \mid x \in \mathbb{C}^m, y \in \mathbb{C}^n\} > -\infty.$$

Note that if P_ϕ is Hermitian symmetric and if we replace $x_i = a_i + b_i\sqrt{-1}$, $y_j = c_j + d_j\sqrt{-1}$, where $a, b \in \mathbb{R}^m$ and $c, d \in \mathbb{R}^n$ then the obtained polynomial $F(a, b, c, d) = P_\phi(a + b\sqrt{-1}, c + d\sqrt{-1})$ is a real polynomial in a, b, c, d (with real coefficients). *There are some criteria on lower bounds* of real polynomials, see [4] and the references therein.

Proof of Corollary 3.8. If ϕ is positive, then $P_\phi(x, y) \geq 0$ for all x, y (by Theorem 3.3). Thus, P_ϕ is bounded below. Conversely, suppose that P_ϕ is bounded below, but assume for the sake of contradiction that there exists a point $(x_0, y_0) \in \mathbb{C}^m \times \mathbb{C}^n$ such that $P_\phi(x_0, y_0) < 0$. Consider the curve defined by $x(t) = tx_0$ and $y(t) = ty_0$ for $t \in \mathbb{R}$. Then, we have

$$P_\phi(tx_0, ty_0) = t^4 P_\phi(x_0, y_0) \to -\infty \quad \text{as} \quad t \to \infty.$$

This contradicts the assumption that P_ϕ is bounded below. □

Proposition 3.9. *Let $\phi : M_m \longrightarrow M_n$ be a linear map. Suppose that the Choi matrix C_ϕ has a negative eigenvalue $\lambda < 0$ with an eigenvector ω.*

(i) *If the solution set (the algebraic set)*

$$S := \{(x, y) \in \mathbb{C}^m \times \mathbb{C}^n \mid [x_1y_1 \cdots x_1y_n \; x_2y_1 \cdots x_my_1 \cdots x_my_n] = \omega^t\}$$

is non-empty, then ϕ is not positive.

(ii) *If ω is a product vector, then ϕ is not positive.*

Proof. (i) Suppose $(x_0, y_0) \in S$. That is, $x_0 \otimes y_0 = \omega$. Then

$$P_\phi(\overline{x}_0, y_0) = (x_0 \otimes y_0)^* C_\phi x_0 \otimes y_0 = \lambda||x_0 \otimes y_0||^2 < 0.$$

Now, by Theorem 3.3, ϕ is not positive.
(ii) If ω is a product vector, then ω belongs to S mentioned in (i). □

3.2. **Application of positive linear operators to sum of squares.** It is clear that a square of an element in $|\mathrm{SLF}(m, n)| \cup |\mathrm{BLF}(m, n)|$ is a Hermitian biquadratic form. On the other hand, if ϕ is self-adjoint, the Choi polynomial P_ϕ is Hermitian symmetric. Denote by $HBF(m, n)$ the set of all Hermitian symmetric biquadratic forms in $x \in \mathbb{C}^m, y \in \mathbb{C}^n$ and $HBF(m, n)_+$ the subset of all $F(x, y) \in HBF(m, n)$ satisfying $F(x, y) \geq 0$ for all x, y. It is clear that

$$\sum\nolimits^2 |\mathrm{SLF}(m, n)| + \sum\nolimits^2 |\mathrm{BLF}(m, n)| \subset HBF(m, n)_+. \tag{2}$$

An interesting problem is to classify the pairs (m,n) for which the reverse inclusion of (2) holds. It is well known that in the real case, this reverse inclusion holds if either $n=2$ or $m=2$, but fails if both m and n are at least 3 (see [2]). In the complex case, we can apply well-known results from the theory of decomposable maps.

Corollary 3.10. *The equality*

$$\sum^2 |SLF(m,n)| + \sum^2 |BLF(m,n)| = HBF(m,n)_+$$

holds if and only if $n+m \leq 5$.

Proof. This follows from Theorem 3.3 and the well-known fact that every positive linear map $\phi: M_m \to M_n$ is decomposable if and only if $m+n \leq 5$ (see [19]). □

We are interested in characterization of nonnegative biquadratic forms which do (or do not) belong to $\sum^2 |SLF(m,n)| + \sum^2 |BLF(m,n)|$. However, by Theorem 3.3, a real-valued biquadratic form p is decomposable if and only if its corresponding self-adjoint linear map ϕ is decomposable, where $P_\phi = p$. In Section 2, we study some classes of decomposable/indecomposable biquadratic forms, in particular the case where the Gram matrix of the biquadratic forms can be written as $W - \varepsilon I$, where W is minimal which is discussed before Lemma 2.8. The problem is still open in general .

4. Examples

In this section, we present several classes of examples (either new or extending previously well-known ones) of indecomposable linear maps. To achieve this, by Theorem 3.3, we construct indecomposable biquadratic forms $p(x,y)$. Then, the corresponding indecomposable map ϕ is determined as follows: the (k,l)-entry of $\phi(e_{ij})$ is the coefficient p_{ijkl} of the monomial $x_i\bar{x}_j y_l \bar{y}_k$ in the polynomial $p(x,y)$ (by Proposition 3.2). For example, as in Example 2.12, the polynomial F_ε is indecomposable, so by Theorem 3.3, its corresponding map $\Phi_\varepsilon: M_3 \to M_3$ is indecomposable for every $0 < \varepsilon \leq \delta \cong 0.0284$.

4.1. **Indecomposable maps from a given edge PPT entangled state.** We now illustrate the main results of the previous sections by means of a concrete and classical example, namely the $2 \otimes 4$ Horodecki family of PPT entangled states. The point of this example is that Theorem 2.13 and Corollary 2.14 provide a systematic way to construct positive semidefinite but indecomposable biquadratic forms from an edge PPT entangled state, while Theorem 3.3 then converts such forms into indecomposable linear maps. Thus the example below serves both as an application of the abstract theory and as an explicit model for the general mechanism. See Horodecki [6]. We consider the PPT entangled state $\rho_a \in M_2(\mathbb{C}) \otimes M_4(\mathbb{C}) \cong M_8(\mathbb{C})$ in the case of $a = 1/2$, as studied in [6], as follows:

$$\rho = \begin{pmatrix} \frac{1}{9} & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{9} & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 & \frac{1}{9} \\ 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{\sqrt{3}}{18} \\ \frac{1}{9} & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{9} & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{9} & 0 & \frac{\sqrt{3}}{18} & 0 & 0 & \frac{1}{6} \end{pmatrix}.$$

Proposition 4.1. *Let P and Q be the orthogonal projections onto $\ker \rho$ and $\ker \rho^\Gamma$, respectively, and define*

$$W := P + Q^\Gamma.$$

Then the following assertions hold.

(i) *ρ is an edge PPT entangled state.*

(ii) *The quantity*

$$\delta := \min_{\|x\|=\|y\|=1} (x \otimes y)^* W (x \otimes y)$$

is strictly positive.

(iii) *For every $0 < \varepsilon \leq \delta$, the biquadratic form*

$$F_\varepsilon(x, y) := (x \otimes y)^* W (x \otimes y) - \varepsilon \|x \otimes y\|^2, \qquad x \in \mathbb{C}^2,\ y \in \mathbb{C}^4,$$

is positive semidefinite but not decomposable.

(iv) *Consequently, if $\Phi_\varepsilon : M_2(\mathbb{C}) \to M_4(\mathbb{C})$ is the linear map whose Choi polynomial is $P_{\Phi_\varepsilon} = F_\varepsilon$, then Φ_ε is indecomposable.*

Proof. It follows from [6] that there exists no product vector $x \otimes y \in \mathrm{Ran}(\rho_a)$ such that $x \otimes \overline{y} \in \mathrm{Ran}(\rho_a^\Gamma)$. Hence Theorem 2.13 applies, and therefore Corollary 2.14

yields ρ is an edge PPT entangled state,

$$\delta = \min_{\|x\|=\|y\|=1} (x \otimes y)^* W(x \otimes y) > 0,$$

and shows that for every $0 < \varepsilon \leq \delta$ the form

$$F_\varepsilon(x, y) = (x \otimes y)^* W(x \otimes y) - \varepsilon \|x \otimes y\|^2$$

is positive semidefinite but not decomposable.

Now let Φ_ε be the unique linear map associated with F_ε through the Choi-polynomial correspondence. Since

$$P_{\Phi_\varepsilon}(x, y) = F_\varepsilon(x, y) \geq 0 \text{ for all } x \in \mathbb{C}^2,\ y \in \mathbb{C}^4,$$

Theorem 3.3 (2) implies that Φ_ε is positive. Since F_ε is not decomposable, Theorem 3.3 (5) implies that Φ_ε is not decomposable. Therefore Φ_ε is an indecomposable positive map. □

4.2. **Indecomposable maps $\Phi_{a;m,n,\varepsilon}$.** Let $m, n \in \mathbb{N}$ with $n \geq m$, and set $r := n - m$. For

$$\varepsilon = (\varepsilon_0, \ldots, \varepsilon_r), \qquad 0 < \varepsilon_\alpha \leq 1,$$

define

$$\Phi_{a;m,n,\varepsilon} : M_m(\mathbb{C}) \to M_n(\mathbb{C}), \qquad \Phi_{a;m,n,\varepsilon}(X) = a\mathrm{Tr}(X) I_n - \sum_{\alpha=0}^{r} \varepsilon_\alpha V_\alpha X V_\alpha^*,$$

where the isometries $V_\alpha : \mathbb{C}^m \to \mathbb{C}^n$ are given by

$$V_\alpha e_p = f_{p+\alpha}, \qquad p = 1, \ldots, m, \quad \alpha = 0, \ldots, r,$$

and $\{e_p\}$ (resp. $\{f_q\}$) is the standard basis for $\mathbb{C}^m$ (resp. $\mathbb{C}^n$).

If $\varepsilon_\alpha = 1$ for every α, then the map $\Phi_{a;m,n,\mathbf{1}}$ is the same as $\Phi_{a;m,n}$ in [10]. As we can see below, using sum of squares arguments and Theorem 3.3, we can reprove some main results in [10] for the weighted maps $\Phi_{a;m,n,\varepsilon}$. In some cases, we even give necessary and sufficient condition when such a map is decomposable.

The Choi polynomial of $\Phi_{a;m,n,\varepsilon}$ is determined as

$$P_{\Phi_{a;m,n,\varepsilon}}(x, y) = y^* \Phi_{a;m,n,\varepsilon}(xx^*) y = a\|x\|^2 \|y\|^2 - \sum_{\alpha=0}^{r} \varepsilon_\alpha \left| \sum_{p=1}^{m} x_p \overline{y_{p+\alpha}} \right|^2, \quad \forall x \in \mathbb{C}^m, y \in \mathbb{C}^n.$$

Proposition 4.2. Let $\Phi_{a;m,n,\varepsilon} : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ as above. Its Choi polynomial is determined by

$$P_{\Phi_{a;m,n,\varepsilon}}(x,y) = \sum_{\alpha=0}^{r} \varepsilon_\alpha \sum_{1\le p<q\le m} |x_p\overline{y_{q+\alpha}} - x_q\overline{y_{p+\alpha}}|^2 + \sum_{p=1}^{m}\sum_{j=1}^{n}(a-s_j)|x_py_j|^2,$$

where

$$s_j := \sum_{\alpha=\max(0,j-m)}^{\min(r,j-1)} \varepsilon_\alpha, \qquad j=1,\dots,n.$$

In particular, if

$$a \ge \max_{1\le j\le n} s_j,$$

then $P_{\Phi_{a;m,n,\varepsilon}}$ is decomposable; hence $\Phi_{a;m,n,\varepsilon}$ is decomposable.

Proof. Let $x=(x_1,\dots,x_m)\in\mathbb{C}^m$ and $y=(y_1,\dots,y_n)\in\mathbb{C}^n$. We have

$$\begin{aligned} P_{\Phi_{a;m,n,\varepsilon}}(x,y) &= y^*\Big(a\,\mathrm{Tr}(xx^*)I_n - \sum_{\alpha=0}^{r}\varepsilon_\alpha V_\alpha xx^*V_\alpha^*\Big)y \\ &= a\|x\|^2\|y\|^2 - \sum_{\alpha=0}^{r}\varepsilon_\alpha\,|y^*V_\alpha x|^2. \end{aligned}$$

Now, by the definition of V_α, we have

$$y^*V_\alpha x = \sum_{p=1}^{m} x_p\overline{y_{p+\alpha}}.$$

Therefore

$$P_{\Phi_{a;m,n,\varepsilon}}(x,y) = a\|x\|^2\|y\|^2 - \sum_{\alpha=0}^{r}\varepsilon_\alpha\left|\sum_{p=1}^{m}x_p\overline{y_{p+\alpha}}\right|^2.$$

For each $\alpha=0,\dots,r$, define

$$u^{(\alpha)} := (y_{1+\alpha},\dots,y_{m+\alpha})\in\mathbb{C}^m.$$

Then

$$\left|\sum_{p=1}^{m}x_p\overline{y_{p+\alpha}}\right|^2 = |\langle x,u^{(\alpha)}\rangle|^2,$$

so

$$P_{\Phi_{a;m,n,\varepsilon}}(x,y) = \sum_{\alpha=0}^{r} \varepsilon_\alpha \Big(\|x\|^2 \|u^{(\alpha)}\|^2 - |\langle x, u^{(\alpha)} \rangle|^2 \Big) + \|x\|^2 \Big(a\|y\|^2 - \sum_{\alpha=0}^{r} \varepsilon_\alpha \|u^{(\alpha)}\|^2 \Big).$$

By the Lagrange identity,

$$\|x\|^2\|u\|^2 - |\langle x,u\rangle|^2 = \sum_{1\le p<q\le m} |x_p\overline{u_q} - x_q\overline{u_p}|^2 \qquad (u \in \mathbb{C}^m).$$

Applying this to $u = u^{(\alpha)}$ yields

$$\|x\|^2\|u^{(\alpha)}\|^2 - |\langle x,u^{(\alpha)}\rangle|^2 = \sum_{1\le p<q\le m} |x_p\overline{y_{q+\alpha}} - x_q\overline{y_{p+\alpha}}|^2 .$$

This gives the first sum in the asserted decomposition.

It remains to rewrite the second term. Observe that

$$\|u^{(\alpha)}\|^2 = \sum_{p=1}^{m} |y_{p+\alpha}|^2,$$

so

$$\sum_{\alpha=0}^{r} \varepsilon_\alpha \|u^{(\alpha)}\|^2 = \sum_{\alpha=0}^{r} \varepsilon_\alpha \sum_{p=1}^{m} |y_{p+\alpha}|^2.$$

Fix $j \in \{1,\dots,n\}$. The term $|y_j|^2$ appears in the inner sum exactly when

$$j = p+\alpha \quad \text{for some } p \in \{1,\dots,m\},$$

that is,

$$j-m \le \alpha \le j-1.$$

Since also $0 \le \alpha \le r$, the coefficient of $|y_j|^2$ is precisely

$$s_j = \sum_{\alpha=\max(0,j-m)}^{\min(r,j-1)} \varepsilon_\alpha.$$

Hence

$$\sum_{\alpha=0}^{r} \varepsilon_\alpha \|u^{(\alpha)}\|^2 = \sum_{j=1}^{n} s_j |y_j|^2,$$

and therefore

$$\|x\|^2 \Big(a\|y\|^2 - \sum_{\alpha=0}^{r} \varepsilon_\alpha \|u^{(\alpha)}\|^2 \Big) = \sum_{p=1}^{m}\sum_{j=1}^{n} (a-s_j)|x_p y_j|^2.$$

Combining the two parts proves the formula

$$P_{\Phi_{a;m,n,\varepsilon}}(x,y) = \sum_{\alpha=0}^{r} \varepsilon_\alpha \sum_{1\le p<q\le m} |x_p\overline{y_{q+\alpha}} - x_q\overline{y_{p+\alpha}}|^2 + \sum_{p=1}^{m}\sum_{j=1}^{n}(a-s_j)|x_p y_j|^2.$$

The first sum is a sum of squares of moduli of sesquilinear forms, while the second is a sum of squares of moduli of bilinear forms. Thus, if $a \ge \max_j s_j$, all coefficients $a - s_j$ are nonnegative and the form is decomposable. Consequently, $\Phi_{a;m,n,\varepsilon}$ is decomposable. □

Corollary 4.3 (The case $r = 1$)**.** Let $n = m+1$ and let

$$\Phi_{a;m,m+1,\varepsilon}(X) = a\,\mathrm{Tr}(X)I_{m+1} - \varepsilon_0 V_0 X V_0^* - \varepsilon_1 V_1 X V_1^*.$$

Then

$$\begin{aligned} P_{\Phi_{a;m,m+1,\varepsilon}}(x,y) = {} & \varepsilon_0 \sum_{1\le p<q\le m} |x_p\overline{y_q} - x_q\overline{y_p}|^2 + \varepsilon_1 \sum_{1\le p<q\le m} |x_p\overline{y_{q+1}} - x_q\overline{y_{p+1}}|^2 \\ & + \sum_{p=1}^{m}(a-\varepsilon_0)|x_p y_1|^2 + \sum_{p=1}^{m}\sum_{j=2}^{m}(a-\varepsilon_0-\varepsilon_1)|x_p y_j|^2 \\ & + \sum_{p=1}^{m}(a-\varepsilon_1)|x_p y_{m+1}|^2. \end{aligned}$$

In particular, if $a \ge \varepsilon_0 + \varepsilon_1$, then $\Phi_{a;m,m+1,\varepsilon}$ is decomposable.

Proof. This is a specialization of the proposition for $r = 1$. □

Corollary 4.4 (The case $r = 2$)**.** Let $n = m+2$ and

$$\Phi_{a;m,m+2,\varepsilon}(X) = a\,\mathrm{Tr}(X)I_{m+2} - \varepsilon_0 V_0 X V_0^* - \varepsilon_1 V_1 X V_1^* - \varepsilon_2 V_2 X V_2^*.$$

Then

$$\begin{aligned} P_{\Phi_{a;m,m+2,\varepsilon}}(x,y) = {} & \sum_{\alpha=0}^{2} \varepsilon_\alpha \sum_{1\le p<q\le m} |x_p\overline{y_{q+\alpha}} - x_q\overline{y_{p+\alpha}}|^2 + \sum_{p=1}^{m}(a-\varepsilon_0)|x_p y_1|^2 \\ & + \sum_{p=1}^{m}(a-\varepsilon_0-\varepsilon_1)|x_p y_2|^2 + \sum_{p=1}^{m}\sum_{j=3}^{m}(a-\varepsilon_0-\varepsilon_1-\varepsilon_2)|x_p y_j|^2 \\ & + \sum_{p=1}^{m}(a-\varepsilon_1-\varepsilon_2)|x_p y_{m+1}|^2 + \sum_{p=1}^{m}(a-\varepsilon_2)|x_p y_{m+2}|^2. \end{aligned}$$

Therefore, if $a \ge \varepsilon_0 + \varepsilon_1 + \varepsilon_2$, then $\Phi_{a;m,m+2,\varepsilon}$ is decomposable.

Proof. Again this follows from Proposition 4.2 when $r = 2$. □

Given a square matrix $X = X^*$, let $\lambda_{\max}(X)$ denote the largest eigenvalue of X.

Proposition 4.5. $\Phi_{a;m,n,\varepsilon}$ is positive if and only if

$$a \geq \sup_{\|x\|=1} \lambda_{\max}\left(\sum_{\alpha=0}^{r} \varepsilon_\alpha V_\alpha x x^* V_\alpha^*\right), \qquad \forall x \in \mathbb{C}^m, \|x\| = 1.$$

Proof. $\Phi_{a;m,n,\varepsilon}$ is positive if and only if, for every unit vector $x \in \mathbb{C}^m$,

$$\Phi_{a;m,n,\varepsilon}(xx^*) \geq 0.$$

By the definition of $\Phi_{a;m,n,\varepsilon}$, we have

$$\Phi_{a;m,n,\varepsilon}(xx^*) = aI_n - \sum_{\alpha=0}^{r} \varepsilon_\alpha V_\alpha x x^* V_\alpha^*.$$

Hence,

$$aI_n - \sum_{\alpha=0}^{r} \varepsilon_\alpha V_\alpha x x^* V_\alpha^* \geq 0 \Longleftrightarrow a \geq \lambda_{\max}\left(\sum_{\alpha=0}^{r} \varepsilon_\alpha V_\alpha x x^* V_\alpha^*\right), \qquad \text{for all } \|x\| = 1.$$

□

Corollary 4.6 (The case $r = 0$)**.** Suppose $n = m$, so that

$$\Phi_{a;m,m,\varepsilon}(X) = a \operatorname{Tr}(X) I_m - \varepsilon_0 X.$$

Then

$$\Phi_{a;m,m,\varepsilon} \text{ is decomposable } \iff \Phi_{a;m,m,\varepsilon} \text{ is positive } \iff a \geq \varepsilon_0.$$

Proof. By Proposition 4.2, $a \geq \varepsilon_0$ implies decomposability. Conversely, suppose that $\Phi_{a;m,m,\varepsilon}$ is decomposable, then the map is positive and by Proposition 4.5, we have $a \geq \varepsilon_0$. □

Corollary 4.7. Assume $m = 2$ and $n = 2 + r$. Then $\Phi_{a;2,2+r,\varepsilon}$ is positive if and only if $a \geq \lambda_{\max}(J_\varepsilon)$, where

$$J_\varepsilon = \begin{pmatrix} \varepsilon_0 & \frac{1}{2}\sqrt{\varepsilon_0\varepsilon_1} & 0 & \cdots & 0 \\ \frac{1}{2}\sqrt{\varepsilon_0\varepsilon_1} & \varepsilon_1 & \frac{1}{2}\sqrt{\varepsilon_1\varepsilon_2} & \ddots & \vdots \\ 0 & \frac{1}{2}\sqrt{\varepsilon_1\varepsilon_2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \varepsilon_{r-1} & \frac{1}{2}\sqrt{\varepsilon_{r-1}\varepsilon_r} \\ 0 & \cdots & 0 & \frac{1}{2}\sqrt{\varepsilon_{r-1}\varepsilon_r} & \varepsilon_r \end{pmatrix}.$$

Proof. By the previous proposition, positivity is equivalent to

$$a \geq \sup_{\|x\|=1} \lambda_{\max}\left(\sum_{\alpha=0}^{r} \varepsilon_\alpha V_\alpha x x^* V_\alpha^*\right).$$

Let $x = (x_1, x_2) \in \mathbb{C}^2$ with $\|x\| = 1$. Define

$$v_\alpha := \sqrt{\varepsilon_\alpha}\, V_\alpha x, \qquad \alpha = 0, \dots, r.$$

Then

$$\sum_{\alpha=0}^{r} \varepsilon_\alpha V_\alpha x x^* V_\alpha^* = \sum_{\alpha=0}^{r} v_\alpha v_\alpha^*.$$

The nonzero eigenvalues of this matrix coincide with the nonzero eigenvalues of the Gram matrix

$$G(x) := \big(\langle v_\beta, v_\alpha\rangle\big)_{\alpha,\beta=0}^{r}.$$

Since

$$V_\alpha x = x_1 f_{\alpha+1} + x_2 f_{\alpha+2},$$

the supports of $V_\alpha x$ and $V_\beta x$ are disjoint unless $|\alpha - \beta| \leq 1$. More precisely,

$$\langle V_\beta x, V_\alpha x\rangle = \begin{cases} 1, & \alpha = \beta, \\ \overline{x_1} x_2, & \beta = \alpha + 1, \\ x_1 \overline{x_2}, & \alpha = \beta + 1, \\ 0, & |\alpha - \beta| \geq 2. \end{cases}$$

Therefore

$$G(x) = \begin{pmatrix} \varepsilon_0 & \sqrt{\varepsilon_0\varepsilon_1}\,\overline{x_1}x_2 & 0 & \cdots & 0 \\ \sqrt{\varepsilon_0\varepsilon_1}\,x_1\overline{x_2} & \varepsilon_1 & \sqrt{\varepsilon_1\varepsilon_2}\,\overline{x_1}x_2 & \ddots & \vdots \\ 0 & \sqrt{\varepsilon_1\varepsilon_2}\,x_1\overline{x_2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \varepsilon_{r-1} & \sqrt{\varepsilon_{r-1}\varepsilon_r}\,\overline{x_1}x_2 \\ 0 & \cdots & 0 & \sqrt{\varepsilon_{r-1}\varepsilon_r}\,x_1\overline{x_2} & \varepsilon_r \end{pmatrix}.$$

Now $|x_1 x_2| \leq \frac{1}{2}$ because $\|x\| = 1$. Thus the maximal possible largest eigenvalue is attained when $|x_1 x_2| = \frac{1}{2}$, that is, when $|x_1| = |x_2| = 1/\sqrt{2}$. For such a choice, $G(x)$

is J_ε:

$$J_\varepsilon = \begin{pmatrix} \varepsilon_0 & \frac{1}{2}\sqrt{\varepsilon_0\varepsilon_1} & 0 & \cdots & 0 \\ \frac{1}{2}\sqrt{\varepsilon_0\varepsilon_1} & \varepsilon_1 & \frac{1}{2}\sqrt{\varepsilon_1\varepsilon_2} & \ddots & \vdots \\ 0 & \frac{1}{2}\sqrt{\varepsilon_1\varepsilon_2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \varepsilon_{r-1} & \frac{1}{2}\sqrt{\varepsilon_{r-1}\varepsilon_r} \\ 0 & \cdots & 0 & \frac{1}{2}\sqrt{\varepsilon_{r-1}\varepsilon_r} & \varepsilon_r \end{pmatrix}.$$

Hence

$$\sup_{\|x\|=1} \lambda_{\max}\left(\sum_{\alpha=0}^{r} \varepsilon_\alpha V_\alpha x x^* V_\alpha^*\right) = \lambda_{\max}(J_\varepsilon).$$

Thus positivity, and therefore decomposability, is equivalent to $a \geq \lambda_{\max}(J_\varepsilon)$. □

Corollary 4.8 (The case $m = 2$, $r = 1$)**.** For

$$\Phi_{a;2,3,\varepsilon}(X) = a\,\mathrm{Tr}(X) I_3 - \varepsilon_0 V_0 X V_0^* - \varepsilon_1 V_1 X V_1^*,$$

the following are equivalent:

(1) $\Phi_{a;2,3,\varepsilon}$ is positive;
(2) $a \geq \dfrac{\varepsilon_0 + \varepsilon_1 + \sqrt{\varepsilon_0^2 - \varepsilon_0\varepsilon_1 + \varepsilon_1^2}}{2}$.

Proof. By the preceding proposition, it suffices to compute the largest eigenvalue of

$$J_\varepsilon = \begin{pmatrix} \varepsilon_0 & \frac{1}{2}\sqrt{\varepsilon_0\varepsilon_1} \\ \frac{1}{2}\sqrt{\varepsilon_0\varepsilon_1} & \varepsilon_1 \end{pmatrix}.$$

The largest eigenvalue is

$$\lambda_{\max}(J_\varepsilon) = \frac{\varepsilon_0 + \varepsilon_1 + \sqrt{\varepsilon_0^2 - \varepsilon_0\varepsilon_1 + \varepsilon_1^2}}{2}.$$

□

Remark 5. The inequality $a \geq \max_{1 \leq j \leq n} s_j$ is an explicit sufficient condition for decomposability for arbitrary m and r. In general it need not be necessary. However, it becomes exact in some situations, notably when $r = 0$.

4.2.1. *Unweighted decomposability.* In this subsection, we consider the unweighted case $\Phi_{a;m,n,\varepsilon}$ (in the previous subsection) when $\varepsilon_\alpha = 1$ for all α. In this case, we write $\Phi_{a;m,n} := \Phi_{a;m,n,\mathbf{1}}$. Let $m, n \in \mathbb{N}$ with $n \geq m$, and put $r := n - m$. Consider

$$\Phi_{a;m,n}(X) = a\,\mathrm{Tr}(X) I_n - \sum_{\alpha=0}^{r} V_\alpha X V_\alpha^*, \qquad X \in M_m(\mathbb{C}).$$

where

$$V_\alpha e_p = f_{p+\alpha}, \qquad p = 1, \dots, m,\ \alpha = 0, \dots, r.$$

Then

$$P_{\Phi_{a;m,n}}(x, y) = \sum_{\alpha=0}^{r} \sum_{1\leq p<q\leq m} |x_p \overline{y_{q+\alpha}} - x_q \overline{y_{p+\alpha}}|^2 + \sum_{p=1}^{m}\sum_{j=1}^{n} (a - c_j)|x_p y_j|^2,$$

where

$$c_j := \#\{\alpha \in \{0, \dots, r\} : 1 + \alpha \leq j \leq m + \alpha\}, \qquad j = 1, \dots, n.$$

Equivalently,

$$c_j = \min(r, j-1) - \max(0, j-m) + 1, \qquad j = 1, \dots, n.$$

In particular,

$$\max_{1\leq j\leq n} c_j = r + 1,$$

and therefore, if

$$a \geq r + 1 = n - m + 1,$$

then $P_{\Phi_{a;m,n}}$ is a sum of squares of moduli of sesquilinear forms and bilinear forms. Hence $\Phi_{a;m,n}$ is decomposable. Hence, we get the following corollary.

Corollary 4.9. Let $m, n \in \mathbb{N}$ with $n \geq m$, and put $r := n - m$. Consider

$$\Phi_{a;m,n} : M_m(\mathbb{C}) \to M_n(\mathbb{C}), \qquad \Phi_{a;m,n}(X) = a\,\mathrm{Tr}(X) I_n - \sum_{\alpha=0}^{r} V_\alpha X V_\alpha^*$$

as above. Then the following hold.

(i) If $a \geq r + 1 = n - m + 1$, then $\Phi_{a;m,n}$ is decomposable.

(ii) The case $r = 0$. The following are equivalent:
  - (a) $\Phi_{a;m,m}$ is decomposable;
  - (b) $\Phi_{a;m,m}$ is positive;
  - (c) $a \geq 1$.

(iii) The case $m = 2$. The following are equivalent:

(a) $\Phi_{a;2,2+r}$ is positive;
(b) $a \geq 1 + \cos\Big(\dfrac{\pi}{r+2}\Big)$.

Proof. Part (i) is exactly the unweighted decomposability criterion (Proposition 4.2)).

(ii) follows from Corollary 4.6.

(iii) (m=2) By Corollary 4.7, we have

$$\Phi_{a;2,2+r} \text{ is positive } \iff \lambda_{\max}(J)),$$

where

$$J = \begin{pmatrix} 1 & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & \ddots & \vdots \\ 0 & \frac{1}{2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & \frac{1}{2} \\ 0 & \cdots & 0 & \frac{1}{2} & 1 \end{pmatrix} \in M_{r+1}(\mathbb{C}).$$

The eigenvalues of this Toeplitz tridiagonal matrix are well known:

$$\lambda_j(J) = 1 + \cos\Big(\frac{j\pi}{r+2}\Big), \qquad j = 1, \ldots, r+1.$$

Hence

$$\lambda_{\max}(J) = 1 + \cos\Big(\frac{\pi}{r+2}\Big).$$

Therefore

$$\Phi_{a;2,2+r} \text{ is positive } \iff a \geq 1 + \cos\Big(\frac{\pi}{r+2}\Big).$$

□

4.3. **Indecomposable maps based on unextendible product bases.** We next show how the results of the previous sections apply to orthonormal unextendible families of product vectors. This yields a concrete class of positive semidefinite but indecomposable biquadratic forms, and therefore, via the Choi polynomial correspondence, a class of indecomposable positive maps. The argument is an immediate consequence of Corollary 2.11 together with the UPB construction in [17, Theorem 3].

Proposition 4.10. *Let*

$$E = \{z_1, \ldots, z_k\} \subseteq \mathbb{C}^m \otimes \mathbb{C}^n$$

be an orthonormal unextendible family of product vectors, and let

$$P_E := \sum_{i=1}^{k} z_i z_i^*$$

be the orthogonal projection onto $H_E := \operatorname{span} E$. *Define*

$$\delta_E := \min_{\|x\|=\|y\|=1} \langle x \otimes y,\, P_E(x \otimes y)\rangle.$$

Then $\delta_E > 0$. *Moreover, for every* $0 < \varepsilon \le \delta_E$, *the biquadratic form*

$$P_\varepsilon(x, y) := (x \otimes y)^*(P_E - \varepsilon I)(x \otimes y), \qquad x \in \mathbb{C}^m,\ y \in \mathbb{C}^n,$$

is positive semidefinite and indecomposable.

Consequently, if $\Phi_\varepsilon : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ *denotes the unique linear map whose Choi polynomial is* $P_{\Phi_\varepsilon} = P_\varepsilon$, *then* Φ_ε *indecomposable.*

Proof. The result follows directly from [17, Theorem 3], Corollary 2.11, and Theorem 3.3. □

4.4. **Indecomposability of the Tanahashi-Tomiyama's map** $\tau_{4,1}$**.** In the following we reprove the indecomposability of the Tanahashi-Tomiyama's map $\tau_{4,1}$ by sums of squares. The indecomposability of $\tau_{n,k}$ was well-known for $1 \le k \le n-2$ (see, e.g. [16, 12, 13, 18, 5]).

Recall that $\tau_{4,1} : M_4(\mathbb{C}) \to M_4(\mathbb{C})$ be the linear map

$$\tau_{4,1}(X) = 3\,\varepsilon(X) + \varepsilon(SXS^*) - X,$$

where $\varepsilon(X)$ denotes the diagonal part of X, and $S = [\delta_{i,j+1}]$ is the cyclic shift matrix. More explicitly,

$$\tau_{4,1}(X) = \begin{pmatrix} 2x_{11} + x_{44} & -x_{12} & -x_{13} & -x_{14} \\ -x_{21} & 2x_{22} + x_{11} & -x_{23} & -x_{24} \\ -x_{31} & -x_{32} & 2x_{33} + x_{22} & -x_{34} \\ -x_{41} & -x_{42} & -x_{43} & 2x_{44} + x_{33} \end{pmatrix}, \qquad X = (x_{ij}) \in M_4(\mathbb{C}).$$

The Choi polynomial of $\tau_{4,1}$ is determined by

$$P_{\tau_{4,1}}(x, y) = y^* \tau_{4,1}(xx^*) y, \qquad x, y \in \mathbb{C}^4.$$

Firstly, we show the positivity of $P_{\tau_{4,1}}$.

Proposition 4.11. $P_{\tau_{4,1}}(x, y) \ge 0$ *for all* $x, y \in \mathbb{C}^4$.

Proof. Let

$$x = (x_1, x_2, x_3, x_4)^T, \qquad y = (y_1, y_2, y_3, y_4)^T,$$

and set

$$a_i := |x_i|^2, \qquad b_i := |y_i|^2 \qquad (i = 1, 2, 3, 4).$$

Indices are taken cyclically modulo 4, so $a_0 = a_4$.

From the explicit formula for $\tau_{4,1}(xx^*)$, we have

$$P_{\tau_{4,1}}(x, y) = \sum_{i=1}^{4} (3a_i + a_{i-1})\, b_i - \left| \sum_{i=1}^{4} \overline{x_i} y_i \right|^2.$$

Thus it suffices to prove

$$\left| \sum_{i=1}^{4} \overline{x_i} y_i \right|^2 \le \sum_{i=1}^{4} (3a_i + a_{i-1})\, b_i.$$

Assume first that

$$3a_i + a_{i-1} > 0 \qquad (i = 1, 2, 3, 4).$$

Then

$$\sum_{i=1}^{4} \overline{x_i} y_i = \sum_{i=1}^{4} \frac{\overline{x_i}}{\sqrt{3a_i + a_{i-1}}} \sqrt{3a_i + a_{i-1}}\, y_i.$$

By the Cauchy–Schwarz inequality,

$$\left| \sum_{i=1}^{4} \overline{x_i} y_i \right|^2 \le \left(\sum_{i=1}^{4} \frac{a_i}{3a_i + a_{i-1}} \right) \left(\sum_{i=1}^{4} (3a_i + a_{i-1})\, b_i \right).$$

Therefore it remains to prove that

$$\sum_{i=1}^{4} \frac{a_i}{3a_i + a_{i-1}} \le 1.$$

Assume now that all $a_i > 0$, and define $u_i := \dfrac{a_{i-1}}{a_i} > 0 \qquad (i = 1, 2, 3, 4)$. Then $u_1 u_2 u_3 u_4 = 1$, and $\dfrac{a_i}{3a_i + a_{i-1}} = \dfrac{1}{3 + u_i}$. Hence it is enough to show that

$$\sum_{i=1}^{4} \frac{1}{3 + u_i} \le 1, \qquad \text{whenever } u_1 u_2 u_3 u_4 = 1.$$

Multiplying both sides by $\prod_{i=1}^{4}(3+u_i)$, this is equivalent to

$$\prod_{i=1}^{4}(3+u_i) - \sum_{i=1}^{4}\prod_{j\neq i}(3+u_j) \geq 0.$$

Since $u_1u_2u_3u_4 = 1$, the left-hand side becomes

$$2\sum_{1\leq i<j<k\leq 4} u_iu_ju_k + 3\sum_{1\leq i<j\leq 4} u_iu_j - 26.$$

Applying the arithmetic–geometric mean inequality, we obtain

$$\frac{1}{6}\sum_{1\leq i<j\leq 4} u_iu_j \geq \left(\prod_{1\leq i<j\leq 4} u_iu_j\right)^{1/6} = (u_1u_2u_3u_4)^{1/2} = 1,$$

and

$$\frac{1}{4}\sum_{1\leq i<j<k\leq 4} u_iu_ju_k \geq \left(\prod_{1\leq i<j<k\leq 4} u_iu_ju_k\right)^{1/4} = (u_1u_2u_3u_4)^{3/4} = 1.$$

Therefore,

$$2\sum_{1\leq i<j<k\leq 4} u_iu_ju_k + 3\sum_{1\leq i<j\leq 4} u_iu_j \geq 26,$$

and hence $\sum_{i=1}^{4}\frac{1}{3+u_i} \leq 1$. It follows that $\sum_{i=1}^{4}\frac{a_i}{3a_i+a_{i-1}} \leq 1$. Therefore

$$P_{\tau_{4,1}}(x,y) = \sum_{i=1}^{4}(3a_i + a_{i-1})\, b_i - \left|\sum_{i=1}^{4}\overline{x_i}y_i\right|^2 \geq 0.$$

This proves the claim when all $a_i > 0$. If some $a_i = 0$, the conclusion follows by continuity. $\square$

Proposition 4.12. *The Choi polynomial $P_{\tau_{4,1}}(x,y) := y^*\tau_{4,1}(xx^*)y$, where x and y are vectors in $\mathbb{C}^4$, is indecomposable.*

Proof. Let $x = (x_1,x_2,x_3,x_4)$, $y = (y_1,y_2,y_3,y_4)$. We now restrict to real variables $x,y \in \mathbb{R}^4$, and consider the Choi polynomial (the real case):

$$\begin{aligned} p(x,y) &= (2x_1^2 + x_4^2)y_1^2 + (2x_2^2 + x_1^2)y_2^2 + (2x_3^2 + x_2^2)y_3^2 \\ &\quad + (2x_4^2 + x_3^2)y_4^2 - 2\sum_{1\leq i<j\leq 4} x_ix_jy_iy_j. \end{aligned}$$

Equivalently,

$$p(x,y) = 2\sum_{i=1}^{4} x_i^2 y_i^2 - 2\sum_{1\le i<j\le 4} x_i x_j y_i y_j + x_4^2 y_1^2 + x_1^2 y_2^2 + x_2^2 y_3^2 + x_3^2 y_4^2. \tag{3}$$

Assume, for contradiction, that p is a sum of square of real bilinear forms

$$p(x,y) = \sum_{r=1}^{L} |F_r(x,y)|^2 \tag{4}$$

for some real bilinear forms

$$F_r(x,y) = \sum_{i,j=1}^{4} a_{ij}^{(r)} x_i y_j \qquad a_{ij}^{(r)} \in \mathbb{R}.$$

Since each summand in (4) is nonnegative, every zero of p is a common zero of all F_r.

We first use eight sparse zeros. From (3), one checks directly that

$$p(e_1,e_3) = p(e_1,e_4) = p(e_2,e_1) = p(e_2,e_4) = 0,$$

$$p(e_3,e_1) = p(e_3,e_2) = p(e_4,e_2) = p(e_4,e_3) = 0,$$

where e_1, e_2, e_3, e_4 are the standard basis vectors of $\mathbb{R}^4$. Therefore every F_r vanishes at these eight points. Writing

$$F_r(x,y) = \sum_{i,j=1}^{4} a_{ij}^{(r)} x_i y_j,$$

we obtain

$$\begin{aligned} F_r(x,y) = {} & a_{11}^{(r)} x_1 y_1 + a_{12}^{(r)} x_1 y_2 + a_{22}^{(r)} x_2 y_2 + a_{23}^{(r)} x_2 y_3 \\ & + a_{33}^{(r)} x_3 y_3 + a_{34}^{(r)} x_3 y_4 + a_{41}^{(r)} x_4 y_1 + a_{44}^{(r)} x_4 y_4. \end{aligned} \tag{5}$$

Observe from (3), that for every sign vector

$$s = (s_1, s_2, s_3, s_4) \in \{\pm 1\}^4$$

we have $p(s,s) = 0$. Substituting $x = y = s$ into (5), we obtain

$$F_r(s,s) = c_r + a_{12}^{(r)} s_1 s_2 + a_{23}^{(r)} s_2 s_3 + a_{34}^{(r)} s_3 s_4 + a_{41}^{(r)} s_4 s_1,$$

where

$$c_r := a_{11}^{(r)} + a_{22}^{(r)} + a_{33}^{(r)} + a_{44}^{(r)}.$$

Since $F_r(s,s) = 0$ for all $s \in \{\pm 1\}^4$, and the five functions

$$1, \qquad s_1 s_2, \qquad s_2 s_3, \qquad s_3 s_4, \qquad s_4 s_1$$

are linearly independent on $\{\pm 1\}^4$, it follows that

$$a_{12}^{(r)} = a_{23}^{(r)} = a_{34}^{(r)} = a_{41}^{(r)} = 0, \qquad a_{11}^{(r)} + a_{22}^{(r)} + a_{33}^{(r)} + a_{44}^{(r)} = 0.$$

Therefore every F_r is of the form

$$F_r(x,y) = \alpha_r x_1 y_1 + \beta_r x_2 y_2 + \gamma_r x_3 y_3 + \delta_r x_4 y_4, \qquad \alpha_r + \beta_r + \gamma_r + \delta_r = 0. \tag{6}$$

Thus each F_r is a linear combination of the diagonal-difference forms

$$x_2 y_2 - x_1 y_1, \qquad x_3 y_3 - x_1 y_1, \qquad x_4 y_4 - x_1 y_1.$$

Now every square $|F_r(x,y)|^2$ with F_r of the form (6) expands only into monomials

$$x_i^2 y_i^2, \qquad x_i x_j y_i y_j \quad (i \neq j),$$

that is, only monomials involving matched pairs (i,i). In particular, no such square can produce any of the shifted monomials

$$x_4^2 y_1^2, \qquad x_1^2 y_2^2, \qquad x_2^2 y_3^2, \qquad x_3^2 y_4^2.$$

Hence no sum of such squares can contain those monomials. But the explicit formula (3) shows that p contains all four of them, each with coefficient $+1$:

$$x_4^2 y_1^2 + x_1^2 y_2^2 + x_2^2 y_3^2 + x_3^2 y_4^2.$$

This contradicts (4). Therefore p is not a sum of squares of real bilinear forms. By Propositions 3.5 and 2.1, $P_{\tau_{4,1}}$ is indecomposable. $\square$

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