ACTA MATHEMATICA VIETNAMICA
Volume 19, Number 2, 1994, pp. 39–49

ESTIMATES OF THE SIZES OF RELATIONS IN RELATIONAL DATAMODEL

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1. Introduction

In relational datamodel a relation is a table matrix. The rows are the data records and the columns are the attributes. For the users the relatrional datamodel is a finite set of time-varying relations.

In this datamodel an important type of relationship between attributes is the functional dependency [2]. W.W. Armtrong [1] has presented a set of axioms (i.e. influence rules) for functional dependencies. Dual dependencies and strong dependencies were introduced and axiomatizied in [3, 4]. It has been shown that dual, strong dependencies have some practical importance (see [3]).

For a given family $F(D,S)$ of functional (dual, strong) dependencies and Sperner-system $\mathcal{K}$ there is a relation representing $F(D,S,\mathcal{K})$ [1, 3, 4]. In [7] J. Demetrovics, Z. Füredi and G.D.H. Kantona constructed the minimal relations representing some special Sperner-systems. However, the construction of minimal relation representing a vegin Sperner-system is very difficult in the general case. In [8] Demetrovics and this author proved that there is no algorithm for finding a minimal relation representing a given Sperner-system $\mathcal{K}$ such that its complexity is polynomial in the number of attributes or in the number of elements of $\mathcal{K}$. In [5, 6] Demetrovics and Gy. Gyepesi have estimated the sizes of minimal relations that represent the families of functional dependencies or Sperner-systems. In this paper we give lower and upper bounds for the sizes of minimal relations representing the families of dual or strong dependencies.

Received March 1st, 1991. Revised 25/1/1992 and 1/8/1994.

We start with some necessary definitions [1, 2, 3, 4]

DEFINITION 1.1. Let $\Omega = \{a_1, \ldots, a_n\}$ be a finite non-empty set of attributes. For each attribute a there is a non-empty set $D(a)$ of all possible values of that attribute. An arbitrary finite subset of Cartesian product $D(a_1) \times \cdots \times D(a_n)$ is called a relation over Ω.

It can be seen that a relation over Ω is a set of mappings $h : \Omega \longrightarrow \bigcup_{a \in \Omega} D(a)$, where $h(a) \in D(a)$ for all a.

DEFINITION 1.2. Let $R = \{h_1, \ldots h_m\}$ be a relation over the finite set of attributes Ω. Let $A, B \subseteq \Omega$. We say:

(1) B functionally depends on A in R (denoted as $A \xrightarrow{f} B$) if

$$(\forall h_i, h_j \in R)(\forall a \in A)\big(h_i(a) = h_j(a) \Rightarrow (\forall b \in B)(h_i(b) = h_j(b))\big);$$

(2) B dually dependes on A in R (denoted as $A \xrightarrow{d} B$) if

$$(\forall h_i, h_j \in R)\ \big((\exists a \in A)(h_i(a) = h_j(a) \Rightarrow (\exists b \in B)(h_i(b) = h_j(b))\big),$$

(3) B strongly depends on A in R (denoted as $A \xrightarrow{s} B$) if

$$(\forall h_i, h_j \in R)\ \big((\exists a \in A)(h_i(a) = h_j(a)) \Rightarrow (\forall b \in B)(h_i(b) = h_j(b))\big).$$

Let $F_R = \{(A, B) : A \xrightarrow{f} B\}$, $D_R = \{(A, B) : \xrightarrow{d} B\}$ and $S = \{(A, B) : A \xrightarrow{s} B\}$. $F_R(D_R, S_R)$ is called the family of functional (dual, strong) dependencies of R.

DEFINITION 1.3. Let Ω be a finite set, and denote by $P(\Omega)$ its power set. Let $Y \subseteq P(\Omega) \times P(\Omega)$. Then we say:

(1) Y satisfies the F-axioms if for all $A, B, C, D \subseteq \Omega$:

(F_1) $(A, A) \in Y$,

(F_2) $(A, B) \in Y, (B, C) \in Y \Longrightarrow (A, C) \in Y$,

(F_3) $(A, B) \in Y, A \subseteq C, D \subseteq B \Longrightarrow (C, D) \in Y$,

(F_4) $(A, B) \in Y, (C, D) \in Y \Longrightarrow (A \cup C, B \cup D) \in Y.$

(2) Y satisfies the D-axioms if for all $A, B, C, D \subseteq \Omega$,

(D_1) $(A, A) \in Y$,

(D_2) $(A, B) \in Y,\ (B, C) \in Y \Longrightarrow (A, C) \in Y$,

(D_3) $(A, B) \in Y,\ C \subseteq A,\ B = D \Longrightarrow (C, D) \in Y$,

(D_4) $(A, B) \in Y,\ (C, D) \in Y \Longrightarrow (A \cup C, B \cup D) \in Y$,

(D_5) $(A, \emptyset) \in Y \Longrightarrow A = \emptyset$.

(3) Y satisfies the S-axioms if for all $A, B, C, D \subseteq \Omega$ and $a \in \Omega$,

(S_1) $(\{a\}, \{a\}) \in Y$,

(S_2) $(A, B) \in Y,\ (B, C) \in Y,\ B \neq \emptyset \Longrightarrow (A, C) \in Y$,

(S_3) $(A, B) \in Y, C \subseteq A, D \subseteq B \Longrightarrow (C, D) \in Y$,

(S_4) $(A, B) \in Y,\ (C, D) \in Y \Longrightarrow (A \cup C, B \cap D) \in Y$,

(S_5) $(A, B) \in Y,\ (C, D) \in Y \Longrightarrow (A \cap C, B \cup D) \in Y.$

DEFINITION 1.4. Let $Y \subseteq P(\Omega) \times P(\Omega)$. We say that Y is an f-(or d-, s-)family over Ω if Y satisfies the F-(resp., D-, S-)axioms.

It can be seen that F_R(resp., D_R, S_R) is an f-(resp., d-, s-)family over Ω.

DEFINITION 1.5. Let R be a relation over Ω, and $A \subseteq \Omega$. A is a key of R if $A \xrightarrow{f} \Omega$. A key A is a minimal key of R if for any $A' \subseteq A$, $A' \xrightarrow{f} \Omega$ implies $A' = A$.

Denote by $\mathcal{K}_R$ the set of all minimal keys of R. It is easy to see that $K_j, K_j \in \mathcal{K}_R$ implies $K_i \not\subseteq K_j$. Systems of subsets of Ω satisfying this condition are Sperner-systems. Consequently, $\mathcal{K}_R$ is a Sperner-system.

DEFINITION 1.6. Let R be a relation, $F(D, S)$ an f-(resp., d-, s-)family over Ω, $\mathcal{K}$ a Sperner-system over Ω. Then we say that R represents F(resp., $D, S, \mathcal{K}$) iff $F_R = F$ (resp., $D_R = D, S_R = S, \mathcal{K}_R = \mathcal{K}$).

REMARK. Let $F(D, S)$ be an f-(d-, s-)family, $\mathcal{K}$ a Sperner-system over Ω. Then there is a relation that represents $F(D, S, \mathcal{K})$ [1, 4].

DEFINITION 1.7. Let D be a d-family over Ω, and $(A, B) \in D$. We say that (A, B) is a maximal left side dependency of D if for all $A \subseteq A'$, $(A', B) \in D \Rightarrow A' = A$. Denote by $M(D)$ the set of all maximal left side dependencies of D. We say that A is a maximal left side of D if there is B such that $(A, B) \in M(D)$. Denote by $G(D)$ the set of all maximal left sides of D.

DEFINITION 1.8. Let $I \subseteq P(\Omega)$ be closed under intersection and $M \subseteq P(\Omega)$. Denote by M^+ the set $\{\cap M' \mid M' \subseteq M\}$. We say that M generates I if $M^+ = I$. With the convention $\cap \emptyset = \Omega$, if I is closed under intersection, then $\Omega \in I$.

2. Sizes of minimal relations

DEFINITION 2.1. Let $\mathcal{K}$ be a Sperner-system, F an f-family over Ω. Set

$$s(\mathcal{K}) = \min\{m \mid \mathcal{K}_R = \mathcal{K}, |R| = m, R \text{ is a relation over } \Omega\}.$$

$$s(n) = \max\{s(\mathcal{K}) \mid \mathcal{K} \text{ is a Sperner-system over } \Omega, |\Omega| = n\}.$$

$$S(F) = \min\{m \mid F_R = F, |R| = m, R \text{ is a relation over } \Omega\}.$$

$$S(n) = \max\{S(F) \mid F \text{ is an f-family over } \Omega, |\Omega| = n\}.$$

In [4] and [5] one can find the following fact:

Let $\Omega = \{a_1, \dots, a_n\}$ be a set of attributes. Then

$$\frac{1}{n^2}\binom{n}{[n/2]} \leq s(n) \leq \binom{n}{[n/2]} + 1,$$

$$\frac{1}{n^2}\binom{n}{[n/2]} \leq S(n) \leq (1 + \frac{c}{\sqrt{n}})\binom{n}{[n/2]})$$

for some constant c.

Now we shall give lower and upper bounds for the sizes of minimal relations that represent a family of dual or strong dependencies. First we introduce the following notion.

DEFINITION 2.2. Let $R = \{h_1, \dots, h_m\}$ be a relation over Ω. Let $N_{ij} = \{a \in \Omega | h_i(a) \neq h_j(a), 1 \leq i < j \leq m\}$. We call N_{ij} the non-equality set of R. Denote by N_R the family of all non-equality sets of R. Set $T_R = \{A \in P(\Omega) \mid \exists N_{ij} \in N_R \text{ such that } A = N_{ij}\}$.

Based on the non-equality sets of R we are able to give a precise characterization for $D_R = D$.

LEMMA 2.1. *Let D be a d-family and R a relation over Ω. Then R represents D if and only if $G(D) = (T_R \setminus \{\Omega\})^+ \cup \{\emptyset\}$.*

PROOF. Let D_1, D_2 be the families of dual dependencies over Ω. Then $D_1 = D_2$ iff $G(D_1) = G(D_2)$. Consequently, we only have to prove that

$$G(D_R) = \left(T_R \setminus \{\Omega\}\right)^+ \cup \{\emptyset\}.$$

It is easy to see that $G(D_R)$ is closed under intersection and contains $\emptyset$, Ω. By the convention $\cap\emptyset = \Omega$ we obtain $\Omega \in (T_R \setminus \{\Omega\})^+$.

Suppose that $N_{ij} \neq \Omega$. It is obvious that $N_{ij} \neq \emptyset$. Clearly, for any $a \in \Omega \setminus N_{ij}$, we obtain $h(a_i) = h_j(a)$, but for all $b \in N_{ij}, h_i(b) \neq h_j(b)$, i.e. $a \cup N_{ij} \stackrel{d}{\nrightarrow} N_{ij}$. Hence $N_{ij} \in G(D_R)$. This implies $T_R \subseteq G(D_R)$. Thus, $(T_R \setminus \{\Omega\})^+ \cup \{\emptyset\} \subseteq G(D_R)$.

Conversely, if $A \in G(D_R) \setminus \{\emptyset, \Omega\}$, then if we assume that for all $h_j \in R$, there exists $a \in A$ such that $h_i(a) = h_j(a)$. So $\Omega \stackrel{d}{\longrightarrow} A$, which contradicts the definition of A. Consequently, there is an index pair (i,j) such that $A \subseteq N_{ij}$. Set $H = \{N_{ij} \mid A \subseteq N_{ij}\}$. If there is an N_{ij} such that $A = N_{ij}$, then it is obvious that $A \in T_R$.

If $A \subset \bigcap_{N_{ij} \notin H} N_{ij}$, then for all $N_{ij} \in H$, we obtain $A \not\subseteq N_{ij}$. So $\bigcap_{N_{ij} \in H} N_{ij} \stackrel{d}{\longrightarrow} A$ holds. This contradicts the assumption $A \in G(D_R) \setminus \{\emptyset, \Omega\}$. Consequently, we have $A = \bigcap_{N_{ij} \in H} N_{ij}$. According to the definition of T_R, $A \in T_R^+$. So $G(D_R) = (T_R \setminus \{\Omega\})^+ \cup \{\emptyset\}$. □

THEOREM 2.1. *Let $\Omega = \{a_1, \ldots, a_n\}$ be a finite set of attributes. Put*

$$U(D) = \min\{m \mid |R| = m, D_R = D, R \text{ is a relation over } \Omega\}.$$

$$U(n) = \max\{U(D) \mid D \text{ is a d-family over } \Omega\}.$$

Then there are two constants c_1, c_2 such that

$$\frac{1}{n(n - \frac{1}{2}log_2 n + c_1)} \binom{n}{[n/2]} \leq U(n) \leq 2(1 + \frac{c}{\sqrt{n}}) \binom{n}{[n/2]}.$$

PROOF. We assume that D is a d-family over Ω and $G(D)$ is a set of all maximal left sides of D. Set

$$Q = \{A \in G(D) \mid A \neq \Omega, A \neq \emptyset, (\forall B, C \in G(D))(A = B \cap C) \Rightarrow A = B \text{ or } A = C\}.$$

Then if $|Q| = 0$, we construct the relation $R = \{h_1, h_2\}$, where $h_1(a) = 0$, $h_2(a) = 1$ for all $a \in \Omega$. If $Q = \{A_1, \ldots, A_2 \mid k \geq 1\}$, then we set $R = \{h_1, h_2, \ldots, h_{2k-1}, h_{2k}\}$ where for $i = 1, \ldots, k$ and $a \in \Omega$, $h_{2i-1}(a) = 2i - 1$,

$$h_{2i}(a) = \begin{cases} 2i - 1 & \text{if } a \in \Omega \setminus A_i, \\ 2i & \text{otherwise.} \end{cases}$$

It is obvious that if $|Q| = 0$, then $D_R = D$ by Lemma 2.1. If $|Q| \neq 0$, then it is easy to see that $Q \cup \{\Omega\} = N_R$, where N_R is a family of non-equality sets of R. On the other hand, $(N_R \setminus \{\Omega\})^+ \cup \{\emptyset\} = G(D)$. By Lemma 2.1 we see that R represents D.

It is known [9] that there is a constant c_2 such that

$$|Q| \leq (1 + \frac{c_2}{\sqrt{n}} \binom{n}{[n/2]}.$$

Consequently,

$$U(n) \leq 2\left(1 + \frac{c_2}{\sqrt{n}}\right) \binom{n}{[n/2]}.$$

We now prove the lower bound of $U(n)$. First let us make the following trivial observations:

(i) Let R be a relation over Ω with m rows. Then there exists a relation R' over Ω such that R' uses no more than m symbols and $D_{R'} = D_R$.

(ii) Let R be a relation over Ω with rows and $m' > m$. Then there is a relation R' over Ω with m' rows such that $D_{R'} = D_R$. From (i) and (ii) it can be seen that the number of d-families that are represented by a relation with $U(n)$ rows is not greater than $U(n)^{n.U(n)}$. Thus, the number of d-families over Ω is not greater than $U(n)^{n.U(n)}$. On the other hand, let $\mathcal{K}$ be a Sperner-system such that $|\mathcal{K}| = \binom{n}{[n/2]}$. Clearly, for all $H \subset \mathcal{K}$, $H^+ \cup \{\emptyset\}$ is a set of maximal left sides of some d-families over Ω. It can be seen that there are $2^{\binom{n}{[n/2]}}$ such d-families. Consequently, $U(n)^{n.U(n)} \geq 2^{\binom{n}{[n/2]}}$.

It is easily to check that

$$n.U(n)\log_2 U(n) \geq \binom{n}{[n/2]}.$$

Since

$$U(n) \leq 2(1+\frac{c_2}{\sqrt{n}})\binom{n}{[n/2]},$$

from Stirling's formula it follows that $\binom{n}{[n/2]} \leq \frac{2}{\sqrt{n}}$. So we obtain

$$U(n)\left(\log_2(1+\frac{c_2}{\sqrt{n}}) - \frac{1}{2}\log_2 n + n + 1\right) \geq \binom{n}{[n/2]}.$$

Hence

$$\frac{\binom{n}{[n/2]}}{n.\left(n + c_1 - \frac{1}{2}\log_2 n\right)} \leq U(n),$$

where $c_1 = 1 + \log_2(1 + c_2)$. □

We now estimate the size of minimal relations representing a family of strong dependencies.

DEFINITION 2.3. Let R be a finite set. The mapping $E : P(\Omega) \to P(\Omega)$ is called a strong operation over Ω if for every $a, b \in \Omega$ and $A \subseteq \Omega$, the following properties hold:

(i) $a \in E_{(\{a\})}$,

(ii) $b \in E_{(\{a\})} \Longrightarrow E_{(\{b\})} \subseteq E_{(\{a\})}$.

(iii) $E(A) = \bigcap_{a \in A} E(a)$.

It is easy to see that by the convention $\cap\emptyset = \Omega$ we obtain $E(\emptyset) = \Omega$. For $A, B \in P(\Omega)$ we have $E(A \cup B) = E(A) \cap E(B)$ and if $A \subseteq B$, then $E(B) \subseteq E(A)$.

LEMMA 2.2. [10] *Let S be an s-family over Ω. We define the mapping $E_S : P(\Omega) \longrightarrow P(\Omega)$ as follows: $E_S(A) = \{a \in \Omega \mid (A, \{a\}) \in S\}$ for all A. Then E_S is a strong operation. Conversely, if E is a strong operation over Ω, then there is exactly one s-family*

$$S = \{(A, B) | A, B \in P(\Omega) \text{ and } B \subseteq E(A)\}$$

such that $E_S = E$.

LEMMA 2.3. *Let S be an s-family and $R = \{h_1, \ldots, h_m\}$ a relation over Ω. Denote by T_{ij} the set $\{a \in \Omega \mid h_i(a) = h_j(a), 1 \leq i < j \leq m\}$. Then R represents S iff for each $a \in \Omega$:*

$$E_S(\{a\}) = \begin{cases} \bigcap\limits_{a \in T_{ij}} T_{ij} & \text{if } \{T_{ij} | a \in T_{ij}\} \neq \emptyset, \\ \Omega & \text{otherwise.} \end{cases}$$

PROOF. By Lemma 2.2, $S_R = S$ if and only if $E_{S_R} = E_S$. Consequently, we have to show that $E_{S_R}(\{a\}) = \bigcap\limits_{a \in T_{ij}} T_{ij}$ if there exists T_{ij} such that $a \in T_{ij}$, and $E_{S_R}(\{a\}) = \Omega$ otherwise. It is easy to see that $E_{S_R}(\{a\}) = \{b \in \Omega \mid \{a\} \xrightarrow{s} \{b\}\}$. According to the strong dependency we know that for any $a \in \Omega, \{a\} \xrightarrow{s} B$ iff $\{a\} \xrightarrow{f} B$. Let us denote by Q the set $\{T_{ij} \mid a \in T_{ij}\}$. It is obvious that if $Q = \emptyset$, then $\{a\} \xrightarrow{f} \Omega$. Thus, $E_{S_R}(\{a\}) = \Omega$. If $Q \neq 0$, then we set $A = \bigcap\limits_{a \in T_{ij}} T_{ij}$. If $Q = \{T_{ij} \mid 1 \leq i < j \leq m\}$, it is obvious that $\{a\} \xrightarrow{f} A$. If $Q \subset \{T_{ij} | 1 \leq i < j \leq m\}$, then $h_i(a) \neq h_j(a)$ for $T_{ij} \notin Q$. Consequently, we also have $\{a\} \xrightarrow{f} A$. Denote by A' the subset of Ω such that $\{a\} \xrightarrow{f} A'$

implies $\{a\} \xrightarrow{f} A''$ for any $A'' \subset A$. It can be seen that $A' = A$. According to the definition of E_{S_R} we obtain $E_{S_R}(\{a\}) = \bigcap_{a \in T_{ij}} T_{ij}$. Thus, if $S = S_R$, then

$$E_S(\{a\}) = \begin{cases} \bigcap_{a \in T_{ij}} T_{ij} & \text{if } \{T_{ij} \mid a \in T_{ij}\} \neq \emptyset, \\ \Omega & \text{otherwise.} \end{cases}$$

Conversely, if E_S satisfies (1), then we obtain $E_S(\{a\}) = E_{S_R}(\{a\})$ for any $a \in \Omega$. By Lemma 2.2 and since E_S, E_{S_R} are strong operations, we obtain $E_S(A) = E_{S_R}(A)$ for all $A \subseteq \Omega$. Hence, $E_S = E_{S_R}$. The proof is complete. □

THEOREM 2.4. *Let $\Omega = \{a_1, \ldots, a_n\}$ be a finite set of attributes. Put*

$$V(S) = \min\{m \mid S_R = S,\ R \text{ is a relation over } \Omega,\ |R| = m\}$$

$$V(n) = \max\{V(S) \mid S \text{ is an s-family over } \Omega\}.$$

Then $\sqrt{2\log_2 n} \leq V(n) \leq n + 1$.

PROOF. Assume that S is an s-family over $\Omega = \{a_1, \ldots, a_n\}$. Let $T = \{A \in P(\Omega) \mid \exists a \in \Omega \text{ such that } E_S(\{a\}) = A\}$. Suppose that $T = \{A_1, \ldots, A_k\}$. We set

$$N = \{A \in T \mid A \neq \Omega, (\forall B, C \in T)\ (A = B \cap C \Rightarrow A = B \text{ or } A = C)\}.$$

If $|N| = 0$, then we set $R = \{h_1, h_2\}$, where $h_1(a) = 0$ and $h_2(a) = 1$ for all $a \in \Omega$. If $|N| \neq 0$, we suppose that $N = \{B_1, \ldots, B_t\}$ $(t \leq k)$. Set $R = \{h_0, h_1, \ldots, h_t\}$ as follows:

$$h_i(a) = \begin{cases} 0 & \text{if } a \in B_i, \\ i & \text{otherwise,} \end{cases}$$

Clearly, if $|N| = 0$ for all $a \in \Omega$ with $h_0(a) = 0$, then $S_R = S$. In case $|N| \neq 0$, we shall show that relation R represents S. By Lemma 2.3 we prove that for each $a \in \Omega$,

$$E(S\{a\}) = \begin{cases} \bigcap_{a \in T_{ij}} T_{ij} & \text{if } \{T_{ij} \mid a \in T_{ij}\} = \emptyset, \\ \Omega & \text{otherwise} \end{cases}$$

where $T_{ij} = \{a \in \Omega \mid h_i(a) = h_j(a),\ 1 \leq i < j \leq t\}$ and $E_S(A) = \{a \in \Omega \mid (A, \{a\}) \in S\}$. It is easy to see that if $E_S(\{a\}) = \Omega$, then there is no T_{ij} such that $a \in T_{ij}$ by the construction of R. If $E_S(\{a_{i_t}\}) = B_t$, then from the condition (ii) in the definition of strong operation we know that $a_{i_t} \in B_k$ implies $E_S(\{a_{i_t}\}) = B_t \subseteq B_k$. Consequently, $E_S(\{a_{i_t}\}) = \bigcap_{a \in T_{ij}} T_{ij} = T_{0t} = B_t$. Hence $E_S(\{a_{i_t}\}) = B_{j_1} \cap \cdots \cap B_{j_t}$. Clearly, for any B_k $(a_{i_t} \in B_k)$ we have $E_S(\{a_{i_t}\}) \subseteq B_k$ by (ii). Therefore, $E_S(\{a_{i_t}\}) = \bigcap_{a \in T_{ij}} T_{ij} = \bigcap_{a=1} T_{0j_q}$. By (ii), for any $a \in \Omega$ with $E_S(\{a\}) = E_S(\{a_{i_t}\})$ we obtain $E_S(\{a\}) = \bigcap_{a \in T_{ij}} T_{ij} = T_{0t}$ or $E_S(\{a\}) = \bigcap_{a \in T_{ij}} T_{ij} = \bigcap_{q=1}^{t} T_{0j_q}$. Consequently, $S_R = S$. It can be seen that $|N| \leq n$. Hence $V(n) \leq n+1$.

For the lower bound of $V(n)$ we suppose that R is a relation that represents S. For all $B_s, B_t \in N$ $(B_s \neq B_t)$ we have

$$\big\{(i,j) \mid 1 \leq i < j \leq m, |R| = m, B_s \subseteq T_{ij}\big\} \neq \big\{(i,j) \mid 1 \leq i < j \leq m, |R| = m, B_t \subseteq T_{ij}\big\}.$$

Consequently, $|N| \leq 2$. Let $S = \{(\{a\},\{a\}) \mid a \in \Omega\} \cup \{(\emptyset, A), (A, \emptyset) | A \subseteq \Omega\}$. It is easy to see that $N = \{a \mid a \in \Omega\}$. Hence $|N| = n$ and $\sqrt{2.\log_2 n} \leq V(n)$, as required. □

It is interesting to note that by virtue of Theorem 2.4, the number of rows of minimal relation that represents an s-family linearly depends on the number of attributes. However, the number of rows of minimal relation which represents an f-family, or d-family, or a Sperner-system exponentially depends on the number of attributes.

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