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REVERSES OF THE CAUCHY-BUNYAKOVSKY-SCHWARZ AND HEISENBERG INTEGRAL INEQUALITIES FOR VECTOR-VALUED FUNCTIONS IN HILBERT SPACES

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Abstract. Some reverses of the Cauchy-Bunyakovsky-Schwarz integral inequality for vector-valued functions in Hilbert spaces and applications for the Heisenberg inequality are provided.

1. Introduction

Assume that $(K;\langle\cdot,\cdot\rangle)$ is a Hilbert space over the real or complex number field $\mathbb{K}$. If $\rho:[a,b]\subset\mathbb{R}\to[0,\infty)$ is a Lebesgue integrable function with $\int_a^b\rho(t)\,dt=1$, then we may consider the space $L^2_\rho([a,b];K)$ of all functions $f:[a,b]\to K$, that are Bochner measurable and $\int_a^b\rho(t)\|f(t)\|^2\,dt<\infty$. It is well known that $L^2_\rho([a,b];K)$ endowed with the inner product $\langle\cdot,\cdot\rangle_\rho$ defined by

$$\langle f,g\rangle_\rho:=\int_a^b\rho(t)\langle f(t),g(t)\rangle\,dt$$

and generating the norm

$$\|f\|_\rho:=\left(\int_a^b\rho(t)\|f(t)\|^2\,dt\right)^{\frac{1}{2}},$$

is a Hilbert space over $\mathbb{K}$.

The following integral inequality is known in the literature as the Cauchy-Bunyakovsky-Schwarz (CBS) inequality

$$\int_a^b\rho(t)\|f(t)\|^2\,dt\int_a^b\rho(t)\|g(t)\|^2\,dt\geqslant\left|\int_a^b\rho(t)\langle f(t),g(t)\rangle\,dt\right|^2, \tag{1.1}$$

provided $f,\,g\in L^2_\rho([a,b];K)$.

Equality holds in (1.1) iff there exists a constant $\lambda\in\mathbb{K}$ such that $f(t)=\lambda g(t)$ for a.e. $t\in[a,b]$.

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Another version of the (CBS) inequality for one vector-valued and one scalar function is incorporated in:

$$\int_a^b \rho(t)|\alpha(t)|^2 dt \int_a^b \rho(t)\|f(t)\|^2 dt \geqslant \left\|\int_a^b \rho(t)\alpha(t)f(t)\,dt\right\|^2, \tag{1.2}$$

provided $\alpha \in L^2_\rho([a,b])$ and $f \in L^2_\rho([a,b];K)$, where $L^2_\rho([a,b])$ denotes the Hilbert space of scalar functions α for which $\int_a^b \rho(t)|\alpha(t)|^2 dt < \infty$. The equality holds in (1.2) iff there exists a vector $e \in K$ such that $f(t) = \overline{\alpha(t)}e$ for a.e. $t \in [a,b]$.

In this paper some reverses of the inequalities (1.1) and (1.2) are given under various assumptions for the functions involved. Natural applications for the Heisenberg inequality for vector-valued functions in Hilbert spaces are also provided.

2. Some reverse inequalities, the general case

The following result holds.

Theorem 1. *Let $f, g \in L^2_\rho([a,b];K)$ and $r > 0$ be such that*

$$\|f(t) - g(t)\| \leqslant r \leqslant \|g(t)\| \tag{2.1}$$

for a.e. $t \in [a,b]$. Then we have the inequalities:

$$\begin{aligned} 0 &\leqslant \int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt - \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right|^2 \\ &\leqslant \int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt - \left[\int_a^b \rho(t)\operatorname{Re}\langle f(t), g(t)\rangle dt\right]^2 \\ &\leqslant r^2 \int_a^b \rho(t)\|f(t)\|^2 dt. \end{aligned} \tag{2.2}$$

The constant $C = 1$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. We will use the following result obtained in [1].

In the inner product space $(H;\langle\cdot,\cdot\rangle)$, if $x, y \in H$ and $r > 0$ are such that $\|x - y\| \leqslant r \leqslant \|y\|$, then

$$\begin{aligned} 0 &\leqslant \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \\ &\leqslant \|x\|^2\|y\|^2 - [\operatorname{Re}\langle x, y\rangle]^2 \\ &\leqslant r^2\|x\|^2. \end{aligned} \tag{2.3}$$

The constant $c = 1$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller quantity.

If (2.1) holds true, then

$$\|f-g\|_\rho^2 = \int_a^b \rho(t)\|f(t)-g(t)\|^2 dt \leqslant r^2 \int_a^b \rho(t)\,dt = r^2$$

and

$$\|g\|_\rho^2 = \int_a^b \rho(t)\|g(t)\|^2 dt \geqslant r^2 \int_a^b \rho(t)\,dt = r^2$$

and thus $\|f-g\|_\rho \leqslant r \leqslant \|g\|_\rho$. Applying the inequality (2.3) for $\left(L_\rho^2([a,b];K),\right.$ $\left.\langle\cdot,\cdot\rangle_p\right)$, we deduce the desired inequality (2.2).

If we choose $\rho(t) = \dfrac{1}{b-a}$, $f(t) = x$, $g(t) = y$, $x,y \in K$, $t \in [a,b]$, then from (2.2) we recapture (2.3) for which the constant $c = 1$ in front of r^2 is best possible. □

We next point out some general reverse inequalities for the second (CBS) inequality (1.2).

Theorem 2. *Let $\alpha \in L_\rho^2([a,b])$, $g \in L_\rho^2([a,b];K)$ and $a \in K$, $r > 0$ such that $\|a\| > r$. If the following condition holds*

$$\|g(t) - \bar{\alpha}(t)\,a\| \leqslant r\,|\alpha(t)| \tag{2.4}$$

for a.e. $t \in [a,b]$, (note that, if $\alpha(t) \neq 0$ for a.e. $t \in [a,b]$, then the condition (2.4) is equivalent to

$$\left\|\frac{g(t)}{\bar{\alpha}(t)} - a\right\| \leqslant r \tag{2.5}$$

for a.e. $t \in [a,b]$), then we have the following inequality

$$\begin{aligned}
&\left(\int_a^b \rho(t)\,|\alpha(t)|^2 dt \int_a^b \rho(t)\,\|g(t)\|^2 dt\right)^{\frac{1}{2}} \\
&\leqslant \frac{1}{\sqrt{\|a\|^2 - r^2}} \mathrm{Re}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, a\right\rangle \\
&\leqslant \frac{\|a\|}{\sqrt{\|a\|^2 - r^2}} \left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|;
\end{aligned} \tag{2.6}$$

$$
\begin{aligned}
(2.7)\qquad 0 &\leqslant \left(\int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt\right)^{\frac{1}{2}} \\
&\quad - \left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\| \\
&\leqslant \left(\int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt\right)^{\frac{1}{2}} \\
&\quad - \operatorname{Re}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, \frac{a}{\|a\|}\right\rangle \\
&\leqslant \frac{r^2}{\sqrt{\|a\|^2 - r^2}\left(\|a\| + \sqrt{\|a\|^2 - r^2}\right)} \left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, \frac{a}{\|a\|}\right\rangle \\
&\leqslant \frac{r^2}{\sqrt{\|a\|^2 - r^2}\left(\|a\| + \sqrt{\|a\|^2 - r^2}\right)} \left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|;
\end{aligned}
$$

$$
\begin{aligned}
(2.8)\qquad &\int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt \\
&\leqslant \frac{1}{\|a\|^2 - r^2}\left[\operatorname{Re}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, a\right\rangle\right]^2 \\
&\leqslant \frac{\|a\|^2}{\|a\|^2 - r^2}\left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
(2.9)\qquad 0 &\leqslant \int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt - \left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|^2 \\
&\leqslant \int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt \\
&\quad - \left[\operatorname{Re}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, \frac{a}{\|a\|}\right\rangle\right]^2 \\
&\leqslant \frac{r^2}{\|a\|^2\left(\|a\|^2 - r^2\right)}\left[\operatorname{Re}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, a\right\rangle\right]^2 \\
&\leqslant \frac{r^2}{\|a\|^2 - r^2}\left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|^2.
\end{aligned}
$$

*All the inequalities (*2.6*) - (*2.9*) are sharp.*

Proof. From (2.4) we deduce

$$\|g(t)\|^2 - 2\operatorname{Re}\langle g(t), \bar{\alpha}(t)\,a\rangle + |\alpha(t)|^2\,\|a\|^2 \leqslant |\alpha(t)|^2\,r^2$$

for a.e. $t \in [a, b]$, which is clearly equivalent to:

$$\|g(t)\|^2 + \left(\|a\|^2 - r^2\right) |\alpha(t)|^2 \leqslant 2\mathrm{Re} \langle \alpha(t) g(t), a \rangle \tag{2.10}$$

for a.e. $t \in [a, b]$.

If we multiply (2.10) by $\rho(t) \geqslant 0$ and integrate over $t \in [a, b]$, then we deduce

$$\int_a^b \rho(t) \|g(t)\|^2 dt + \left(\|a\|^2 - r^2\right) \int_a^b \rho(t) |\alpha(t)|^2 dt \\ \leqslant 2\mathrm{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, a \right\rangle . \tag{2.11}$$

Now, dividing (2.11) by $\sqrt{\|a\|^2 - r^2} > 0$, we get

$$\frac{1}{\sqrt{\|a\|^2 - r^2}} \int_a^b \rho(t) \|g(t)\|^2 dt + \sqrt{\|a\|^2 - r^2} \int_a^b \rho(t) |\alpha(t)|^2 dt \\ \leqslant \frac{2}{\sqrt{\|a\|^2 - r^2}} \mathrm{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, a \right\rangle . \tag{2.12}$$

On the other hand, by the elementary inequality

$$\frac{1}{\alpha} p + \alpha q \geqslant 2\sqrt{pq}, \qquad \alpha > 0, \ p, q \geqslant 0,$$

we can state that

$$2\sqrt{\int_a^b \rho(t) |\alpha(t)|^2 dt} \cdot \sqrt{\int_a^b \rho(t) \|g(t)\|^2 dt} \\ \leqslant \frac{1}{\sqrt{\|a\|^2 - r^2}} \int_a^b \rho(t) \|g(t)\|^2 dt + \sqrt{\|a\|^2 - r^2} \int_a^b \rho(t) |\alpha(t)|^2 dt. \tag{2.13}$$

Making use of (2.12) and (2.13), we deduce the first part of (2.6).

The second part of (2.6) is obvious by Schwarz's inequality

$$\mathrm{Re} \left\langle \int_a^b \rho(t) \alpha(t) g(t) dt, a \right\rangle \leqslant \left\| \int_a^b \rho(t) \alpha(t) g(t) dt \right\| \|a\| .$$

If $\rho(t) = \frac{1}{b-a}$, $\alpha(t) = 1$, $g(t) = x \in K$, then from (2.6) we get

$$\|x\| \leqslant \frac{1}{\sqrt{\|a\|^2 - r^2}} \mathrm{Re} \langle x, a \rangle \leqslant \frac{\|x\| \|a\|}{\sqrt{\|a\|^2 - r^2}},$$

provided $\|x - a\| \leqslant r < \|a\|$, $x, a \in K$. The sharpness of this inequality has been shown in [1], and we omit the details.

The other inequalities are obvious consequences of (2.6) and we omit the details. □

3. Some particular cases

It has been shown in [1] that for $A, a \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) and $x, y \in H$, where $(H; \langle \cdot, \cdot \rangle)$ is an inner product over the real or complex number field $\mathbb{K}$, the following inequality holds

$$\begin{aligned} \|x\| \|y\| &\leqslant \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A}+\bar{a}\right)\langle x, y\rangle\right]}{\left[\operatorname{Re}\left(A\bar{a}\right)\right]^{\frac{1}{2}}} \\ &\leqslant \frac{1}{2} \cdot \frac{|A+a|}{\left[\operatorname{Re}\left(A\bar{a}\right)\right]^{\frac{1}{2}}} \left|\langle x, y\rangle\right| \end{aligned} \tag{3.1}$$

provided $\operatorname{Re}(A\bar{a}) > 0$ and

$$\operatorname{Re}\langle Ay - x, x - ay\rangle \geqslant 0, \tag{3.2}$$

or, equivalently,

$$\left\| x - \frac{a+A}{2} \cdot y \right\| \leqslant \frac{1}{2} |A - a| \|y\|, \tag{3.3}$$

holds. The constant $\frac{1}{2}$ is best possible in (3.1).

From (3.1), we can deduce the following results

$$\begin{aligned} 0 &\leqslant \|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \\ &\leqslant \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A}+\bar{a}-2\left[\operatorname{Re}\left(A\bar{a}\right)\right]^{\frac{1}{2}}\right)\langle x, y\rangle\right]}{\left[\operatorname{Re}\left(A\bar{a}\right)\right]^{\frac{1}{2}}} \\ &\leqslant \frac{1}{2} \cdot \frac{\left|\bar{A}+\bar{a}-2\left[\operatorname{Re}\left(A\bar{a}\right)\right]^{\frac{1}{2}}\right|}{\left[\operatorname{Re}\left(A\bar{a}\right)\right]^{\frac{1}{2}}} \left|\langle x, y\rangle\right| \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} 0 &\leqslant \|x\| \|y\| - \left|\langle x, y\rangle\right| \\ &\leqslant \frac{1}{2} \cdot \frac{|A+a| - 2\left[\operatorname{Re}\left(A\bar{a}\right)\right]^{\frac{1}{2}}}{\left[\operatorname{Re}\left(A\bar{a}\right)\right]^{\frac{1}{2}}} \left|\langle x, y\rangle\right|. \end{aligned} \tag{3.5}$$

If one assumes that $A = M$, $a = m$, $M \geqslant m > 0$, then from (3.1), (3.4) and (3.5) we deduce the much simpler and more useful results:

$$\|x\| \|y\| \leqslant \frac{1}{2} \cdot \frac{M+m}{\sqrt{Mm}} \operatorname{Re}\langle x, y\rangle, \tag{3.6}$$

$$0 \leqslant \|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \leqslant \frac{1}{2} \cdot \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{\sqrt{Mm}} \operatorname{Re}\langle x, y\rangle \tag{3.7}$$

and

$$0 \leqslant \|x\| \|y\| - |\langle x, y\rangle| \leqslant \frac{1}{2} \cdot \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{\sqrt{Mm}} |\langle x, y\rangle|, \tag{3.8}$$

provided

$$\operatorname{Re}\langle My - x, x - my\rangle \geqslant 0$$

or, equivalently

$$\left\| x - \frac{M+m}{2} y \right\| \leqslant \frac{1}{2}(M - m)\|y\|. \tag{3.9}$$

Squaring the second inequality in (3.1), we can get the following results as well:

$$0 \leqslant \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leqslant \frac{1}{4} \cdot \frac{|A - a|^2}{\operatorname{Re}(A\bar{a})} |\langle x, y\rangle|^2, \tag{3.10}$$

provided (3.2) or (3.1) holds. Here the constant $\frac{1}{4}$ is also best possible.

Using the above inequalities for vectors in inner product spaces, we are able to state the following theorem concerning reverses of the (CBS) integral inequality for vector-valued functions in Hilbert spaces.

Theorem 3. *Let $f, g \in L^2_\rho([a, b]; K)$ and $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$. If*

$$\operatorname{Re}\langle \Gamma g(t) - f(t), f(t) - \gamma g(t)\rangle \geqslant 0 \tag{3.11}$$

for a.e. $t \in [a, b]$, or, equivalently,

$$\left\| f(t) - \frac{\gamma + \Gamma}{2} \cdot g(t) \right\| \leqslant \frac{1}{2} |\Gamma - \gamma| \|g(t)\| \tag{3.12}$$

for a.e. $t \in [a, b]$, then we have the inequalities

$$\begin{aligned} &\left(\int_a^b \rho(t) \|f(t)\|^2 dt \int_a^b \rho(t) \|g(t)\|^2 dt\right)^{\frac{1}{2}} \\ &\leqslant \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{\Gamma} + \bar{\gamma}\right) \int_a^b \rho(t) \langle f(t), g(t)\rangle dt\right]}{[\operatorname{Re}(\Gamma\bar{\gamma})]^{\frac{1}{2}}} \\ &\leqslant \frac{1}{2} \cdot \frac{|\Gamma + \gamma|}{[\operatorname{Re}(\Gamma\bar{\gamma})]^{\frac{1}{2}}} \left| \int_a^b \rho(t) \langle f(t), g(t)\rangle dt \right|, \end{aligned} \tag{3.13}$$

$$0 \leqslant \left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \int_a^b \rho(t)\operatorname{Re}\langle f(t), g(t)\rangle dt \leqslant \frac{1}{2}\cdot\frac{\operatorname{Re}\left[\left\{\bar{\Gamma}+\bar{\gamma}-2\left[\operatorname{Re}(\Gamma\bar{\gamma})\right]^{\frac{1}{2}}\right\}\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right]}{\left[\operatorname{Re}(\Gamma\bar{\gamma})\right]^{\frac{1}{2}}} \leqslant \frac{1}{2}\cdot\frac{\left|\bar{\Gamma}+\bar{\gamma}-2\left[\operatorname{Re}(\Gamma\bar{\gamma})\right]^{\frac{1}{2}}\right|}{\left[\operatorname{Re}(\Gamma\bar{\gamma})\right]^{\frac{1}{2}}}\left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right|, \tag{3.14}$$

$$0 \leqslant \left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right| \leqslant \frac{1}{2}\cdot\frac{|\Gamma+\gamma|-2\left[\operatorname{Re}(\Gamma\bar{\gamma})\right]^{\frac{1}{2}}}{\left[\operatorname{Re}(\Gamma\bar{\gamma})\right]^{\frac{1}{2}}}\left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right|, \tag{3.15}$$

and

$$0 \leqslant \int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt - \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right|^2 \leqslant \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}\left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right|^2. \tag{3.16}$$

The constants $\frac{1}{2}$ *and* $\frac{1}{4}$ *above are sharp.*

In the case where Γ, γ are positive real numbers, the following corollary incorporating more convenient reverses for the (CBS) integral inequality, may be stated.

Corollary 1. *Let* $f, g \in L_\rho^2([a,b]; K)$ *and* $M \geqslant m > 0$. *If*

$$\operatorname{Re}\langle Mg(t) - f(t), f(t) - mg(t)\rangle \geqslant 0 \tag{3.17}$$

for a.e. $t \in [a,b]$, *or, equivalently,*

$$\left\| f(t) - \frac{m+M}{2}\cdot g(t)\right\| \leqslant \frac{1}{2}(M-m)\|g(t)\| \tag{3.18}$$

for a.e. $t \in [a,b]$, *then we have the inequalities*

$$\left(\int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} \leqslant \frac{1}{2}\cdot\frac{M+m}{\sqrt{mM}}\int_a^b \rho(t)\operatorname{Re}\langle f(t), g(t)\rangle dt, \tag{3.19}$$

$$0 \leqslant \left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \int_a^b \rho(t) \operatorname{Re}\langle f(t), g(t)\rangle dt \leqslant \frac{1}{2} \cdot \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{\sqrt{mM}} \int_a^b \rho(t) \operatorname{Re}\langle f(t), g(t)\rangle dt, \tag{3.20}$$

$$0 \leqslant \left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \left|\int_a^b \rho(t) \langle f(t), g(t)\rangle dt\right| \leqslant \frac{1}{2} \cdot \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{\sqrt{mM}} \left|\int_a^b \rho(t) \langle f(t), g(t)\rangle dt\right|, \tag{3.21}$$

and

$$0 \leqslant \int_a^b \rho(t)\|f(t)\|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt - \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right|^2 \leqslant \frac{1}{4} \cdot \frac{(M-m)^2}{mM} \left|\int_a^b \rho(t)\langle f(t), g(t)\rangle dt\right|^2. \tag{3.22}$$

The constants $\frac{1}{2}$ *and* $\frac{1}{4}$ *above are best possible.*

On utilising the general result of Theorem 2, we are able to state a number of interesting reverses for the (CBS) inequality in the case when one function takes vector-values while the other is a scalar function.

Theorem 4. *Let* $\alpha \in L_\rho^2([a,b])$, $g \in L_\rho^2([a,b];K)$, $e \in K$, $\|e\| = 1$, $\gamma, \Gamma \in \mathbb{K}$ *with* $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$. *If*

$$\left\|g(t) - \bar{\alpha}(t) \cdot \frac{\Gamma+\gamma}{2} e\right\| \leqslant \frac{1}{2}|\Gamma-\gamma|\,|\alpha(t)| \tag{3.23}$$

for a.e. $t \in [a,b]$, *or, equivalently*

$$\operatorname{Re}\langle \Gamma\bar{\alpha}(t) e - g(t), g(t) - \gamma\bar{\alpha}(t) e\rangle \geqslant 0 \tag{3.24}$$

for a.e. $t \in [a,b]$, *(note that, if* $\alpha(t) \neq 0$ *for a.e.* $t \in [a,b]$, *then* (3.23) *is equivalent to*

$$\left\|\frac{g(t)}{\overline{\alpha(t)}} - \frac{\Gamma+\gamma}{2} e\right\| \leqslant \frac{1}{2}|\Gamma-\gamma| \tag{3.25}$$

for a.e. $t \in [a,b]$*, and* (3.24) *is equivalent to*

$$\operatorname{Re}\left\langle \Gamma e - \frac{g(t)}{\alpha(t)}, \frac{g(t)}{\alpha(t)} - \gamma e \right\rangle \geqslant 0 \tag{3.26}$$

for a.e. $t \in [a,b]$*), then the following reverse inequalities are valid:*

$$\begin{aligned} &\left(\int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt\right)^{\frac{1}{2}} \\ &\leqslant \frac{\operatorname{Re}\left[\left(\bar{\Gamma}+\bar{\gamma}\right)\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle\right]}{2\left[\operatorname{Re}\left(\Gamma\bar{\gamma}\right)\right]^{\frac{1}{2}}} \\ &\leqslant \frac{1}{2}\cdot\frac{|\Gamma+\gamma|}{\left[\operatorname{Re}\left(\Gamma\bar{\gamma}\right)\right]^{\frac{1}{2}}}\left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|; \end{aligned} \tag{3.27}$$

$$\begin{aligned} 0 &\leqslant \left(\int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt\right)^{\frac{1}{2}} - \left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\| \\ &\leqslant \left(\int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt\right)^{\frac{1}{2}} \\ &\qquad - \operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle\right] \\ &\leqslant \frac{|\Gamma-\gamma|^2}{2\sqrt{\operatorname{Re}\left(\Gamma\bar{\gamma}\right)}\left(|\Gamma+\gamma|+2\sqrt{\operatorname{Re}\left(\Gamma\bar{\gamma}\right)}\right)} \\ &\qquad \times \operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle\right] \\ &\leqslant \frac{|\Gamma-\gamma|^2}{2\sqrt{\operatorname{Re}\left(\Gamma\bar{\gamma}\right)}\left(|\Gamma+\gamma|+2\sqrt{\operatorname{Re}\left(\Gamma\bar{\gamma}\right)}\right)}\left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|; \end{aligned} \tag{3.28}$$

$$\begin{aligned} &\int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt \\ &\leqslant \frac{1}{4}\cdot\frac{1}{\operatorname{Re}\left(\Gamma\bar{\gamma}\right)}\left[\operatorname{Re}\left(\left(\overline{\Gamma}+\overline{\gamma}\right)\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle\right)\right]^2 \\ &\leqslant \frac{1}{4}\cdot\frac{|\Gamma+\gamma|^2}{\operatorname{Re}\left(\Gamma\bar{\gamma}\right)}\left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|^2 \end{aligned} \tag{3.29}$$

and

$$0 \leqslant \int_a^b \rho(t)|\alpha(t)|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt - \left\|\int_a^b \rho(t)\alpha(t)g(t)\,dt\right\|^2 \tag{3.30}$$
$$\leqslant \int_a^b \rho(t)|\alpha(t)|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt$$
$$- \left[\operatorname{Re}\left(\frac{\overline{\Gamma}+\overline{\gamma}}{|\Gamma+\gamma|}\left\langle \int_a^b \rho(t)\alpha(t)g(t)\,dt, e\right\rangle\right)\right]^2$$
$$\leqslant \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|^2\operatorname{Re}(\Gamma\bar{\gamma})}\left[\left(\operatorname{Re}\left(\overline{\Gamma}+\overline{\gamma}\right)\left\langle \int_a^b \rho(t)\alpha(t)g(t)\,dt, e\right\rangle\right)\right]^2$$
$$\leqslant \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}\left\|\int_a^b \rho(t)\alpha(t)g(t)\,dt\right\|^2.$$

The constants $\frac{1}{2}$ *and* $\frac{1}{4}$ *above are sharp.*

In the particular case of positive constants, the following simpler version of the above inequalities may be stated.

Corollary 2. *Let* $\alpha \in L^2_\rho([a,b]) \setminus \{0\}$, $g \in L^2_\rho([a,b];K)$, $e \in K$, $\|e\| = 1$ *and* $M, m \in \mathbb{R}$ *with* $M \geqslant m > 0$. *If*

$$\left\|\frac{g(t)}{\bar{\alpha}(t)} - \frac{M+m}{2}\cdot e\right\| \leqslant \frac{1}{2}(M-m) \tag{3.31}$$

for a.e. $t \in [a,b]$, *or, equivalently,*

$$\operatorname{Re}\left\langle Me - \frac{g(t)}{\bar{\alpha}(t)}, \frac{g(t)}{\bar{\alpha}(t)} - me\right\rangle \geqslant 0 \tag{3.32}$$

for a.e. $t \in [a,b]$, *then we have*

$$\left(\int_a^b \rho(t)|\alpha(t)|^2 dt \int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} \tag{3.33}$$
$$\leqslant \frac{1}{2}\cdot\frac{M+m}{\sqrt{Mm}}\operatorname{Re}\left\langle \int_a^b \rho(t)\alpha(t)g(t)\,dt, e\right\rangle$$
$$\leqslant \frac{1}{2}\cdot\frac{M+m}{\sqrt{Mm}}\left\|\int_a^b \rho(t)\alpha(t)g(t)\,dt\right\|;$$

(3.34)

$$\begin{aligned}
0 &\leqslant \left(\int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt\right)^{\frac{1}{2}} - \left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\| \\
&\leqslant \left(\int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt\right)^{\frac{1}{2}} - \operatorname{Re}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle \\
&\leqslant \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{Mm}} \operatorname{Re}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle \\
&\leqslant \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{Mm}} \left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|
\end{aligned}$$

$$\begin{aligned}
0 &\leqslant \int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt \\
&\leqslant \frac{1}{4}\cdot\frac{(M+m)^2}{Mm}\left[\operatorname{Re}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle\right]^2 \\
&\leqslant \frac{1}{4}\cdot\frac{(M+m)^2}{Mm}\left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|^2
\end{aligned} \tag{3.35}$$

and

(3.36)

$$\begin{aligned}
0 &\leqslant \int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt - \left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|^2 \\
&\leqslant \int_a^b \rho(t)\,|\alpha(t)|^2\,dt \int_a^b \rho(t)\,\|g(t)\|^2\,dt - \left[\operatorname{Re}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle\right]^2 \\
&\leqslant \frac{1}{4}\cdot\frac{(M-m)^2}{Mm}\left[\operatorname{Re}\left\langle \int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt, e\right\rangle\right]^2 \\
&\leqslant \frac{1}{4}\cdot\frac{(M-m)^2}{Mm}\left\|\int_a^b \rho(t)\,\alpha(t)\,g(t)\,dt\right\|^2.
\end{aligned}$$

The constants $\frac{1}{2}$ *and* $\frac{1}{4}$ *above are sharp.*

4. Reverses of the Heisenberg inequality

It is well known that if $(H;\langle\cdot,\cdot\rangle)$ is a real or complex Hilbert space and $f:[a,b]\subset\mathbb{R}\to H$ is an *absolutely continuous vector-valued* function, then f is differentiable almost everywhere on $[a,b]$, the derivative $f':[a,b]\to H$ is Bochner integrable on $[a,b]$ and

$$f(t)=\int_a^t f'(s)\,ds \quad \text{for any } \; t\in[a,b]. \tag{4.1}$$

The following theorem provides a version of the Heisenberg inequalities in the general setting of Hilbert spaces.

Theorem 5. *Let $\varphi : [a,b] \to H$ be an absolutely continuous function with the property that $b \|\varphi(b)\|^2 = a \|\varphi(a)\|^2$. Then we have the inequality:*

$$\left(\int_a^b \|\varphi(t)\|^2 \, dt \right)^2 \leqslant 4 \int_a^b t^2 \|\varphi(t)\|^2 \, dt \cdot \int_a^b \|\varphi'(t)\|^2 \, dt. \tag{4.2}$$

The constant 4 is best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. Integrating by parts, we have successively

$$\begin{aligned} \int_a^b \|\varphi(t)\|^2 \, dt &= t \|\varphi(t)\|^2 \Big|_a^b - \int_a^b t \frac{d}{dt} \left(\|\varphi(t)\|^2 \right) dt \\ &= b \|\varphi(b)\|^2 - a \|\varphi(a)\|^2 - \int_a^b t \frac{d}{dt} \langle \varphi(t), \varphi(t) \rangle \, dt \\ &= - \int_a^b t \left[\langle \varphi'(t), \varphi(t) \rangle + \langle \varphi(t), \varphi'(t) \rangle \right] dt \\ &= -2 \int_a^b t \operatorname{Re} \langle \varphi'(t), \varphi(t) \rangle \, dt \\ &= 2 \int_a^b \operatorname{Re} \langle \varphi'(t), (-t) \varphi(t) \rangle \, dt. \end{aligned} \tag{4.3}$$

If we apply the (CBS) integral inequality

$$\int_a^b \operatorname{Re} \langle g(t), h(t) \rangle \, dt \leqslant \left(\int_a^b \|g(t)\|^2 \, dt \int_a^b \|h(t)\|^2 \, dt \right)^{\frac{1}{2}}$$

for $g(t) = \varphi'(t)$, $h(t) = -t\varphi(t)$, $t \in [a,b]$, then we deduce the desired inequality (4.2).

The fact that 4 is the best possible constant in (4.2) follows from the fact that in the (CBS) inequality, the case of equality holds iff $g(t) = \lambda h(t)$ for a.e. $t \in [a,b]$ and λ a given scalar in $\mathbb{K}$. We omit the details. $\square$

For details on the classical Heisenberg inequality see, for instance, [2].

The following reverse of the Heisenberg type inequality (4.2) holds.

Theorem 6. *Assume that $\varphi : [a,b] \to H$ is as in the hypothesis of Theorem 5. In addition, if there exists a $r > 0$ such that*

$$\left\| \varphi'(t) - t\varphi(t) \right\| \leqslant r \leqslant \left\| \varphi'(t) \right\| \tag{4.4}$$

for a.e. $t \in [a,b]$*, then we have the inequalities*

$$0 \leqslant \int_a^b t^2 \|\varphi(t)\|^2 dt \int_a^b \|\varphi'(t)\|^2 dt - \frac{1}{4}\left(\int_a^b \|\varphi(t)\|^2 dt\right)^2 \leqslant r^2 \int_a^b t^2 \|\varphi(t)\|^2 dt. \tag{4.5}$$

Proof. We observe, by the identity (4.3), that

$$\frac{1}{4}\left(\int_a^b \|\varphi(t)\|^2 dt\right)^2 = \left(\int_a^b \operatorname{Re}\langle\varphi'(t), t\varphi(t)\rangle dt\right)^2. \tag{4.6}$$

Now, if we apply Theorem 1 for the choices $f(t) = t\varphi(t)$, $g(t) = \varphi'(t)$, and $\rho(t) = \frac{1}{b-a}$, then by (2.2) and (4.6) we deduce the desired inequality (4.5). □

Remark 1. Interchanging the place of $t\varphi(t)$ with $\varphi'(t)$ in Theorem 6, we also have

$$0 \leqslant \int_a^b t^2 \|\varphi(t)\|^2 dt \int_a^b \|\varphi'(t)\|^2 dt - \frac{1}{4}\left(\int_a^b \|\varphi(t)\|^2 dt\right)^2 \leqslant \rho^2 \int_a^b \|\varphi'(t)\|^2 dt, \tag{4.7}$$

provided

$$\|\varphi'(t) - t\varphi(t)\| \leqslant \rho \leqslant |t| \|\varphi(t)\|$$

for a.e. $t \in [a,b]$, where $\rho > 0$ is a given positive number.

The following result also holds.

Theorem 7. *Assume that* $\varphi : [a,b] \to H$ *is as in the hypothesis of Theorem* 5. *In addition, if there exists* $M \geqslant m > 0$ *such that*

$$\operatorname{Re}\langle Mt\varphi(t) - \varphi'(t), \varphi'(t) - mt\varphi(t)\rangle \geqslant 0 \tag{4.8}$$

for a.e. $t \in [a,b]$*, or, equivalently,*

$$\left\|\varphi'(t) - \frac{M+m}{2} t\varphi(t)\right\| \leqslant \frac{1}{2}(M-m)|t| \|\varphi(t)\| \tag{4.9}$$

for a.e. $t \in [a,b]$*, then we have the inequalities*

$$\int_a^b t^2 \|\varphi(t)\|^2 dt \int_a^b \|\varphi'(t)\|^2 dt \leqslant \frac{1}{16} \cdot \frac{(M+m)^2}{Mm}\left(\int_a^b \|\varphi(t)\|^2 dt\right)^2 \tag{4.10}$$

and

$$\int_a^b t^2 \|\varphi(t)\|^2 dt \int_a^b \|\varphi'(t)\|^2 dt - \frac{1}{4}\left(\int_a^b \|\varphi(t)\|^2 dt\right)^2 \leqslant \frac{1}{16} \cdot \frac{(M-m)^2}{Mm}\left(\int_a^b \|\varphi(t)\|^2 dt\right)^2 \tag{4.11}$$

respectively.

Proof. We use Corollary 1 for the choices $f(t) = \varphi'(t)$, $g(t) = t\varphi(t)$, $\rho(t) = \frac{1}{b-a}$ to get

$$\int_a^b \left\|\varphi'(t)\right\|^2 dt \int_a^b t^2 \left\|\varphi(t)\right\|^2 dt \leqslant \frac{(M+m)^2}{4Mm}\left(\int_a^b \operatorname{Re}\left\langle \varphi'(t), t\varphi(t)\right\rangle dt\right)^2.$$

Since by (4.6)

$$\left(\int_a^b \operatorname{Re}\left\langle \varphi'(t), t\varphi(t)\right\rangle dt\right)^2 = \frac{1}{4}\left(\int_a^b \left\|\varphi(t)\right\|^2 dt\right)^2,$$

we deduce the desired result (4.10).

The inequality (4.11) follows from (4.10), and we omit the details. □

Remark 2. If one is interested in reverses for the Heisenberg inequality for scalar valued functions, then all the other inequalities obtained above for one scalar function may be applied as well. For the sake of brevity, we do not list them here.

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