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ON MARTINGALES IN THE LIMIT AND CONVERGENCE OF THEIR SUBSEQUENCES

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Abstract. Martingales in the limit and mils were first introduced by Mucci (1976) and Talagrand (1985), respectively. They proved that every L^1-bounded mil converges a.s. Recently, Luu (1999) has extended this result to sequential mils. In this note we consider sequences of random variables which are not necessarily integrable. By using a stopping time method we shall give a convergence result for their subsequences.

1. Notations and definitions

Throughout this note, let $(\Omega, \mathcal{A}, P)$ be a complete probability space, N the set of all positive integers and $(\mathcal{A}_n,\ n \in N)$ an increasing sequence of complete sub-σ-fields of $\mathcal{A}$ with $\mathcal{A}_n \uparrow \mathcal{A}$. Let T denote the set of all bounded stopping times w.r.t. $(\mathcal{A}_n,\ n \in N)$. Then, endowed with the usual order "$\leq$" given by $\sigma \leq \tau$ iff $\sigma(\omega) \leq \tau(\omega)$ a.s., T becomes a directed set. Further, given $k \in N$ we denote by T^k the set of all bounded stopping times which take essentially at most k values. Then one can regard T^1 as N and each T^k as a cofinal subset of T.

For simplicity, given a cofinal subset Γ of T, $p \in N$ and $\tau \in T$ with $p \leq \tau$, we use the following notations:

$$\begin{aligned}
\Gamma(p) &= \{\gamma \in \Gamma,\ p \leq \gamma\};\\
\Gamma(p,\tau) &= \{\gamma \in \Gamma,\ p \leq \gamma \leq \tau\};\\
\tau^- &= \inf\{k \in N,\ P(\{\tau = k\}) > 0\},\\
\tau^+ &= \max\{k \in N,\ P(\{\tau = k\}) > 0\}.
\end{aligned}$$

We shall consider in this note only sequences $(\tau_n,\ n \in N)$ of T with $n \leq \tau_n \leq \tau_{n+1}^-$. Further for a sub-$\sigma$-field $\mathcal{B}$ of $\mathcal{A}$, we denote by $L^0(\mathcal{B})$ the set of all $\mathcal{B}$-measurable random variables and by $L^1(\mathcal{B})$ the Banach space of all (equivalence classes of) elements $X \in L^0(\mathcal{B})$ with

$$E(|X|) = \int_\Omega |X(\omega)| dP(\omega) < \infty.$$

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Unless otherwise stated we shall deal with only sequences $(X_n,\ n \in N)$ in $L^0(\mathcal{A})$ which are adapted to $(\mathcal{A}_n,\ n \in N)$.

Now, given a sequence $(X_n,\ n \in N)$ in $L^0(\mathcal{A})$ and $\tau \in T$, we define:

$$X_\tau(\omega) = X_{\tau(\omega)}(\omega), \quad \omega \in \Omega,$$
$$\mathcal{A}_\tau = \{A \in \mathcal{A},\ A \cap \{\tau = n\} \in \mathcal{A}_n,\ n \in N\}.$$

Then it is known (see [4]) that $(\mathcal{A}_\tau,\ \tau \in T)$ form an increasing family of complete sub-σ-fields of $\mathcal{A}$ and each X_τ is $\mathcal{A}_\tau$-measurable. Moreover, if $(X_n,\ n \in N)$ is integrable then so is $(X_\tau,\ \tau \in T)$.

For other related notions the reader is referred to [1]. In this note we start with the following definition.

Definition 1.1. A sequence $(X_n,\ n \in N)$ in $L^1(\mathcal{A})$ is said to be:

a) *a martingale in the limit* if

$$\lim_n \sup_{m \geq n} |E^n(X_m) - X_n| = 0 \quad \text{a.s.},$$

where given $\tau \in T$ and $X \in L^1(\mathcal{A})$, we mean by $E^\tau(X)$ the conditional expectation of X;

b) *a mil* if for every $\varepsilon > 0$ there exists $p \in N$ such that for each $n \in N(p)$ we have

$$P\Big(\sup_{q \in N(p,n)} |E^q(X_n) - X_q| > \varepsilon\Big) < \varepsilon. \tag{1.1}$$

Martingales in the limit were first introduced by Mucci [3] as the first important generalization of martingales. Later the notion is essentially extended to mils by Talagrand (1985), who proved that every L^1-bounded mil converges, a.s.

Recently, the first author of this note has proved that the above convergence result still holds for the following much larger class of martingale-like sequences [2].

Definition 1.2. A sequence $(X_n,\ n \in N)$ in $L^1(\mathcal{A})$ is said to be *a sequential mil* if there exists a sequence $(\tau_n,\ n \in N)$ of T such that $(X_n,\ n \in N)$ is a $\{\tau_n\}$-mil, i.e., for every $\varepsilon > 0$, there exists $p \in N$ such that for each $n \in N(p)$ we have

$$P\Big(\sup_{q \in N(p,\tau_n)} |E^q(X_{\tau_n}) - X_q| > \varepsilon\Big) < \varepsilon, \tag{1.2}$$

where $n \leq \tau_n \leq \tau_{n+1}^-$ and $\{\tau_n\}$ is the set of all elements of $\tau_n,\ n \in N$.

It is clear that if (1.1) holds then so does (1.2) for $\tau_n = n,\ n \in N$. However in many applications, $(X_n,\ n \in N)$ is not always integrable and sometimes the convergence of a subsequence of $(X_n,\ n \in N)$ gives us enough information. This leads us to consider such sequences $(X_n,\ n \in N)$ in $L^0(\mathcal{A})$ for which there exists a sequence $(\tau_n,\ n \in N)$ of T such that the optional sampling sequence $(X_{\tau_n},\ n \in N)$ is integrable. The main aim of this note is to investigate the following class of martingale-like-sequences, where U is always a cofinal subset of N.

Definition 1.3. Given a sequence $(\tau_n,\ n \in N)$ of T and a cofinal subset U of N, we say that a sequence $(X_n,\ n \in N)$ in $L^0(\mathcal{A})$ is *a $\{\tau_n\}$-mil relative to U* if each $X_{\tau_n} \in L^1(\mathcal{A})$ and for every $\varepsilon > 0$ there exists $p \in U$ such that for every $n \in N(p)$ we have

$$P\Big(\sup_{q \in U(p,\tau_n)} |E^q(X_{\tau_n}) - X_q| > \varepsilon\Big) < \varepsilon. \tag{1.3}$$

Clearly, if a sequence $(X_n,\ n \in N)$ in $L^1(\mathcal{A})$ is a $\{\tau_n\}$-mil (in the sense of Definition 1.2) then by (1.2) it is a $\{\tau_n\}$-mil relative to N. More generally, if $V \subset U$ and $(\sigma_n,\ n \in N)$ is a subsequence of $(\tau_n,\ n \in N)$ then every $\{\tau_n\}$-mil relatively to U is a $\{\sigma_n\}$-mil relative to V.

For further related examples and the main results on the class we refer to the next section.

2. Main results

To explain the main sense of Definition 1.3, we start with the following partial characterization:

Theorem 2.1. *Let $(X_n,\ n \in N)$ be a sequence in $L^0(\mathcal{A})$ such that there exists a sequence $(\tau_n,\ n \in N)$ of T with $(X_{\tau_n},\ n \in N)$ converging in L^1 to some $X \in L^1(\mathcal{A})$. Then $(X_n,\ n \in N)$ is a $\{\tau_n\}$-mil relative to some cofinal subset U of N if and only if the subsequence $(X_u,\ u \in U)$ converges, a.s.*

Proof. Let $(X_n,\ n \in N)$, $(\tau_n,\ n \in N)$ and X be as in the theorem. Then for a given $\varepsilon > 0$ one can find $p_1 \in N$ such that for all $n \in N(p_1)$,

$$E\big(|X_{\tau_n} - X|\big) < \frac{\varepsilon^2}{9}\,\cdot \tag{2.1}$$

Consequently, by the maximal inequality, (2.1) implies

$$P\Big(\sup_{q \in N(1,\tau_n)} |E^q(X_{\tau_n}) - E^q(X)| > \frac{\varepsilon}{3}\Big) < \frac{\varepsilon}{3}\,\cdot \tag{2.2}$$

Further, by the martingale limit theorem, there exists $p_2 \in N(p_1)$ such that

$$P\Big(\sup_{q \in N(p_2)} |E^q(X) - X| > \frac{\varepsilon}{3}\Big) < \frac{\varepsilon}{3}\,\cdot \tag{2.3}$$

Suppose first that $(X_n,\ n \in N)$ is a $\{\tau_n\}$-mil relative to some cofinal subset U of N. Then by Definition 1.3, there exists $p \in U(p_2)$ such that for all $n \in N(p)$ we have:

$$P\Big(\sup_{q \in U(p,\tau_n)} |\ E^q(X_{\tau_n}) - X_q| > \frac{\varepsilon}{3}\Big) < \frac{\varepsilon}{3}\,\cdot$$

Thus, given $m \in N(p)$, and $n \in N(m)$ by (2.2) and (2.3) we have:

$$\begin{aligned}
P\big(\sup_{q\in U(p,m)} |X_q - X| > \varepsilon\big) &\le P\big(\sup_{q\in U(p,\tau_n)} |E^q(X_{\tau_n}) - X_q| > \frac{\varepsilon}{3}\big) \\
&\quad + P\big(\sup_{q\in U(p,\tau_n)} |E^q(X_{\tau_n}) - E^q(X)| > \frac{\varepsilon}{3}\big) \\
&\quad + P\big(\sup_{q\in U(p,\tau_n)} |E^q(X) - X| > \frac{\varepsilon}{3}\big) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This means that the subsequence $(X_u,\ u \in U)$ converges a.s. to X which proves the necessary condition.

Conversely, suppose that the last conclusion is true. Then for the same $\varepsilon > 0$ one can choose $k \in U(p_2)$ such that

$$P\big(\sup_{q\in U(k)} |X_q - X| > \frac{\varepsilon}{3}\big) < \frac{\varepsilon}{3}.$$

Thus, if $n \in N(k)$, by (2.2) and (2.3) we have

$$P\big(\sup_{q\in U(k,\tau_n)} |E^q(X_{\tau_n}) - X_q| > \varepsilon\big) < \varepsilon.$$

This proves (1.3) and the theorem. □

Example 2.1. There exists a $\{\tau_n\}$-mil relative to some cofinal subset U of N which converges in L^1. But it is not a $\{\tau_n\}$-mil.

Construction: Let $(X_n,\ n \in N)$ be a sequence in $L^1(\mathcal{A})$ which converges in L^1, but does not converge a.s. Let $k \in N$ and $(\tau_n,\ n \in N)$ a sequence in T^k. Then the sequence $(X_{\tau_n},\ n \in N)$ also converges in L^1. Thus, by the theorem $(X_n,\ n \in N)$ cannot be a $\{\tau_n\}$-mil since it does not converge a.s. On the other hand, $(X_n,\ n \in N)$ is a $\{\tau_n\}$-mil relative to some cofinal subset U of N. Hence, by the L^1-convergence of $(X_n,\ n \in N)$ it follows that there exists a cofinal subset U of N such that the subsequence $(X_u,\ u \in U)$ converges a.s. Thus, by the theorem (X_n) is a $\{\tau_n\}$-mil with respect to U. This completes the construction.

Example 2.2. There exists a $\{\tau_n\}$-mil relative to some cofinal subset U of N which does not converge in L^1 and it is not a $\{\tau_n\}$-mil.

Construction: Let $([0,1],\ \mathcal{B}_{[0,1]}, P)$ be the Lebesgue probability space on $[0,1]$. Given $k \in N$, let Q_k be the partition of $[0,1]$ in 2^k intervals of equal length and set $a_0 = 0,\ a_k = \sum_{j=1}^{k} 2^j$. Then for every $n \in N$ there exists a unique $k \in N$ such that $a_{k-1} < n \le a_k$. For this index n, we define $\mathcal{A}_n = \sigma - (Q_k)$ and the random variable X_n by $X_n = 2^k$ on the $(n - a_{k-1})$-th interval of Q_k and $X_n = 0$, elsewhere. Clearly,

a) $X_n,\ n \in N$, does not converge to zero in L^1;

b) $X_n,\ n \in N$, converges to zero, in probability.

Now, given $k \in N$, we define the stopping time τ_k by $\tau_k = a_{k-1} + 1$ on the last interval of Q_k, and $\tau_k = a_k$, elsewhere. It can be easily checked that

c) $X_{\tau_k} \equiv 0,\ k \in N$.

Thus, by a), b), c) and the theorem the sequence $(X_n,\ n \in N)$ satisfies the first requirement of the example. Finally, since $(X_n,\ n \in N)$ does not converge a.s, by c) and the theorem it cannot be a $\{\tau_n\}$-mil. This completes the construction.

For other related examples, the interested reader is refered to Talagrand [5] and Luu [2].

Now we are going to prove the main result.

Theorem 2.2. *Let $(X_n,\ n \in N)$ be a $\{\tau_n\}$-mil relative to some cofinal subset U of N, with*

$$\liminf_{n \in N}\ E(|X_{\tau_n}|) < \infty.$$

Then the subsequence $(X_u,\ u \in U)$ converges a.s to some $X \in L^1(\mathcal{A})$.

Proof. Let $(X_n,\ n \in N)$, $(\tau_n,\ n \in N)$ and U be as in the theorem. Then, by passing to a subsequence of $(\tau_n,\ n \in N)$ if necessary, we can assume that

$$\sup_{n \in N}\ E(|X_{\tau_n}|) < \infty, \tag{2.4}$$

Assume on the contrary that $(X_u,\ u \in U)$ does not converge a.s. Then by adding to all $X_u,\ u \in U$, a suitable common constant if necessary, there exists a positive number $a > 0$ such that $p(A) > 0$, where

$$A = \Big\{ \limsup_{u \in U}\ X_u > \frac{5a}{4}, \quad \liminf_{u \in U}\ X_u < 0 \Big\}.$$

We shall show that this assumption implies that $\sup\limits_{n \in N}\ E(|X_{\tau_n}|) = \infty$ which contradicts (2.4), hence $(X_u,\ u \in U)$ converge a.s. To do this we make the following claim:

For every $n_1 \in N$ and $0 < \varepsilon < \dfrac{P(A)}{4}$ there exists $n_2 \in N(n_1)$ such that for each $E \in \mathcal{A}_{\tau_{n_1}}$ with $P(E) < \dfrac{P(A)}{2}$ and $n \in N(n_2)$ there exists $M \in \mathcal{A}_{\tau_{n_2}}$ with $M \cap E = \phi,\ P(M) < \varepsilon$ and such that

$$\int\limits_M X_{\tau_n} dP > \frac{aP(A)}{4}\,\cdot \tag{2.5}$$

To prove the claim, let $n_1 \in N$ and $0 < \varepsilon < \dfrac{P(A)}{4}$ be given. Then by the assumption on $(A_n,\ n \in N)$ and Definition 1.3 one can find a large enough

$k \in U(n_1)$ and $B \in \mathcal{A}_k$ with $P(A\Delta B) < \dfrac{\varepsilon}{4}$ such that for each $n \in N(k)$ we have:

$$P\Big(\sup_{q\in U(k,\tau_n)} |E^q(X_{\tau_n}) - X_q| > \frac{a}{8}\Big) < \frac{\varepsilon}{2}\cdot \tag{2.6}$$

Firstly, since

$$A \subset \Big\{\limsup_{u\in U} X_u > \frac{5a}{4}\Big\} \subset \Big\{\sup_{u\in U(k)} X_u > \frac{5a}{4}\Big\},$$

there exists $\sigma \in T$ with $P(\{\sigma \in U(k)\}) = 1$ such that

$$P\Big(A \cap \big\{X_\sigma > \frac{5a}{4}\big\}\Big) > P(A) - \frac{\varepsilon}{4}\cdot \tag{2.7}$$

Similarly, since

$$A \cap \Big\{X_\sigma > \frac{5a}{4}\Big\} \subset \Big\{\liminf_{u\in U} X_u < 0\Big\} \subset \Big\{\inf_{u\in U(\sigma^+)} X_u < 0\Big\},$$

there exists another $\gamma \in T$ with $P(\{\gamma \in U(\sigma^+)\}) = 1$ such that

$$P\Big(A \cap \big\{X_\sigma > \frac{5a}{4}\big\} \setminus \{X_\gamma < 0\}\Big) < \frac{\varepsilon}{4}\cdot$$

By a simple estimation we have

(2.8)

$$\begin{aligned}
P\Big(B \cap \big\{X_\sigma > \frac{5a}{4}\big\} \setminus \{X_\gamma < 0\}\Big) &\le P\Big(A \cap \big\{X_\sigma > \frac{5a}{4}\big\} \setminus \{X_\gamma < 0\}\Big)\\
&\quad + P\Big(\Big[A \cap \big\{X_\sigma > \frac{5a}{4}\big\}\Big]\Delta\Big[B \cap \big\{X_\gamma > \frac{5a}{4}\big\}\Big]\Big)\\
&< \frac{\varepsilon}{4} + P(A\Delta B)\\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}\cdot
\end{aligned}$$

Now set $n_2 = \gamma^+$ and let $n \in N(n_2),\ E \in \mathcal{A}_{\tau_{n_1}}$ with $P(E) < \dfrac{P(A)}{2}$ be given. We define

$$\begin{aligned}
C &= \big\{|E^\sigma(X_{\tau_n}) - X_\sigma| > \frac{a}{8}\big\},\\
D &= \big\{|E^\gamma(X_{\tau_n}) - X_\gamma| > \frac{a}{8}\big\}.
\end{aligned}$$

Then by (2.6) we have

$$\max\{P(C), P(D)\} \le P\Big(\sup_{q\in U(k,\tau_n)} |E^q(X_{\tau_n}) - X_q| > \frac{a}{8}\Big) < \frac{\varepsilon}{2}\cdot \tag{2.9}$$

Thus, if we put

$$C_1 = B \cap \big\{X_\sigma > \frac{5a}{4}\big\} \setminus (C \cup E)$$

then by (2.7) and (2.8) we obtain

$$\begin{aligned}
P(C_1) &\geq P\Big(B \cap \{X_\sigma > \frac{5a}{4}\}\Big) - P(C) - P(E) \\
&\geq P\Big(A \cap \{X_\sigma > \frac{5a}{4}\}\Big) - P(A \Delta B) - \frac{\varepsilon}{2} - \frac{P(A)}{2} \\
&> P(A) - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} - \frac{\varepsilon}{2} - \frac{P(A)}{2} = \frac{P(A)}{2} - \varepsilon.
\end{aligned}$$

This implies

$$P(C_1) > \frac{P(A)}{4} \cdot \tag{2.10}$$

Note that $C_1 \in \mathcal{A}_\sigma$ and on C_1 we have

$$E^\sigma(X_{\tau_n}) > X_\sigma - \frac{a}{8} > \frac{5a}{8} - \frac{a}{8} = \frac{9a}{8} \cdot$$

Hence

$$\int\limits_{C_1} X_{\tau_n} dP = \int\limits_{C_1} E^\sigma(X_{\tau_n}) dP > \frac{9aP(C_1)}{8} \cdot \tag{2.11}$$

Similarly, set

$$D_1 = C_1 \cap \{X_\gamma < 0\} \setminus D.$$

Then $D_1 \in \mathcal{A}_\gamma$ and on D_1 we have

$$E^\sigma(X_{\tau_n}) < X_{\tau_n} - \frac{a}{8} < \frac{a}{8} \cdot$$

Hence

$$\int\limits_{D_1} X_{\tau_n} dP = \int\limits_{D_1} E^\gamma(X_{\tau_n}) dP < \frac{aP(D_1)}{8} \leq \frac{aP(C_1)}{8} \cdot$$

This together with (2.10) and (2.11) yield

$$\int\limits_{C_1 \setminus D_1} X_{\tau_n} dP > \frac{9aP(C_1)}{8} - \frac{aP(C_1)}{8} = aP(C_1) > \frac{aP(A)}{4} \cdot \tag{2.12}$$

Thus, if we take $M = C_1 \setminus D_1$ then $M \in \mathcal{A}_{\tau_{n_2}}$, $M \cap E \subset C_1 \cap E = \phi$ and by (2.8) we have

$$\begin{aligned}
P(M) &= P(C_1 \setminus [C_1 \cap \{X_j < 0\} \setminus D]) \\
&\leq P([C_1 \setminus \{X_\gamma < 0\}]) + P(D) \\
&< P\Big(B \cap \{X_\sigma > \frac{5a}{4}\} \setminus \{X_\gamma < 0\}\Big) + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This together with (2.12) prove (2.5) and the claim.

Using the claim we can construct by induction an increasing sequence $(n_p,\ p \in N)$ of N with the following property: For $E \in \mathcal{A}_{\tau_{n_p}}$ with $P(E) < \dfrac{P(A)}{2}$ and $n \in N(n_{p+1})$ there exists $M \in \mathcal{A}_{\tau_{n_{p+1}}}$ with $M \cap E = \phi,\ P(M) < 2^{-(p+1)}P(A)$ such that $\displaystyle\int_M X_{\tau_n} dP > \frac{aP(A)}{4}$. Thus, given $p \in N(2)$, we can construct by finite induction disjoint sets $(M_j)_{j \le p}$ with $M_1 = \phi$, $M_j \in \mathcal{A}_{\tau_{n_j}}$ $\ P(M_j) < 2^{-j}P(A)$ and

$$\int_{M_j} X_{\tau_n} dP > \frac{aP(A)}{4}, \quad 2 \le j \le p.$$

This implies that

$$E(|X_{\tau_n}|) \ge \int_M X_{\tau_n} dP > \frac{(p-1)aP(A)}{4},$$

where

$$M = \sum_{j=1}^{p} M_j.$$

Therefore,

$$\sup_{n \in N} E(|X_{\tau_n}|) = \infty$$

which contradicts (2.4). Thus, $(X_u,\ u \in U)$ converge a.s.

Finally, to complete the proof it remains to show that if $(X_u,\ u \in U)$ converges a.s. to some $X \in L^0(\mathcal{A})$ then $X \in L^1(\mathcal{A})$. This fact is not trivial at all since one cannot apply Fatou's lemma directly neither to the subsequence $(X_u,\ u \in U)$ nor to $(X_{\tau_n},\ n \in N)$. However, since U is cofinal, by passing to a subsequence of $(\tau_n,\ n \in N)$ if necessary, we can suppose also that the sequence $(u(n),\ n \in N)$ with

$$u(n) = \max\{q \in U : q \le \tau_n\}, \quad n \in N,$$

is a well-defined strictly increasing subsequence of U. Moreover, by Definition 1.3 the sequence $(Y_n,\ n \in N)$ converges to zero in probability, where

$$Y_n = E^{u(n)}(X_{\tau_n}) - X_{u(n)}, \quad n \in N.$$

Therefore, there exists a cofinal subset V of N such that $(Y_v,\ v \in V)$ converges to zero a.s. This fact together with Fatou's lemma show that

$$\begin{aligned} E(|X|) &= E\big(\lim_{u \in U} |X_u|\big) = E\big(\lim_{v \in V} |X_{u(v)}|\big) = E\big(\lim_{v \in V} |E^{u(v)}(X_{\tau_v})|\big) \\ &\le \liminf_{v \in V}\ E\big(|E^{u(v)}(X_{\tau_v})|\big) \le \liminf_{v \in V}\ E\big(|X_{\tau_v}|\big) \\ &\le \sup_{n \in N}\ E\big(|X_{\tau_v}|\big) < \infty. \end{aligned}$$

This proves the integrability of X and hence the proof is complete. $\square$

In conclusion, it is worth noting that the proof of the theorem is based on a stopping time technique and follows from the pattern of the proof of Theorem 2 of [2]. However, the construction is completely different. Further, the proof of Theorem 4 of [2] could not be applied to prove the theorem since in that proof the author essentially used the fact that if $(X_n,\ n \in N)$ is a $\{\tau_n\}$-mil then the sequence $(X_{\tau_n},\ n \in N)$ is a mil w.r.t. $(\mathcal{A}_{\tau_n},\ n \in N)$. For more information the reader is refered again to the previous examples.

References

[1] G. A. Edgar and L. Sucheston, *Stopping times and directed processes*, Encyclopedia of Math. and its Applications 47, Cambridge University Press, 1992.

[2] D. Q. Luu, *On further classes of martingale-like-sequences and some decomposition and convergence theorems*, Glasgow Math. J. **41** (1999), 313-322.

[3] A. Mucci, *Another martingale convergence theorem*, Pacific J. Math. **64** (1976), 539-541.

[4] J. Neveu, *Discrete parameter martingales*, North-Holland Math. Library, 1975.

[5] M. Talagrand, *Some structure results for martingales in the limit and pramarts*, Ann. Probab. **13** (1985), 1192-1203.

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