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SHORTEST PATHS ALONG A SEQUENCE OF LINE SEGMENTS AND CONNECTED ORTHOGONAL CONVEX HULLS

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SUMMARY

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Introduction

This dissertation studies shortest paths and straightest paths along a sequence of line segments in Euclidean spaces and connected orthogonal convex hulls of a finite planar point set. They are meaningful problems in computational geometry.

Finding shortest paths (joining two given points, from a source point to many destinations, from a point to a line segment, ...) in a geometric domain (such as on the surface of a polytope, a terrain, inside a simple polygon, ...) is a classical geometric optimization problem and has many applications in different areas such as robotics, geographic information systems, and navigation (see, for example, Agarwal et al., 2002, Sethian, 1999). To date, many algorithms have been proposed: touring polygons problems (see, for example, Dror et al., 2003, Ahadi et al., 2014), the shortest path problem on polyhedral surfaces (see, for example, Mitchell et al., 1987, 1991, Chen and Han, 1996, Varadarajan and Agarwal, 2002), the weighted region problem (see, for example, Aleksandrov et al., 2005), the shortest descending path problem (see, for example, Ahmed et al., 2010, 2011, Cheng and Jin 2013), and the shortest gentle descending path problem (see, for example, Ahmed et al., 2009, Bose et al., 2011, Mitchell et al., 2000, 2015). However, the problem of finding the shortest path joining two points in three dimensions in the presence of general polyhedral obstacles is known to be computationally difficult (see, for example, Bajaj 1985, Canny and Reif, 1987).

In some shortest path problems, exact and approximate solutions are computed based on solving a subproblem of finding the shortest path joining two given points along a sequence of adjacent triangles (the adjacent triangles on a polyhedral surface). Several algorithms need to concatenate of a shortest path from a sequence of adjacent triangles with a line segment on a new adjacent triangle (see, for example, Chen and Han, 1996, Cook, 2014, Balasubramanian et al., 2009, Cheng and Jin, 2014, Pham-Trong et al., 2001, Xin and Wang, 2007, Tran et al., 2020). Not many properties of shortest paths on polyhedral surfaces have been shown (see, for example, Sharir and Shorr, 1986, Mitchell et al., 1987, Bajaj, 1985, Canney and Reif, 1987). They usually suppose paths as polylines. The properties of shortest paths along sequences of adjacent triangles are even less than that (see Mitchell et al., 1987). But the question is even more basic "Does the shortest path joining two points along a sequence of adjacent triangles exist uniquely?". The uniqueness of such a path is assumed even in geodesical spaces and generalized segment spaces (see Hai and An, 2013). Thus, this question is important not only in computational respect but also in theoretical respect. In this dissertation, we will consider the existence and uniqueness of such shortest paths. We show how a shortest path bends at an edge, and moreover how two shortest paths can "glue" together to form one shortest path.

In the dissertation, we consider the problem of finding the shortest path between two points along a sequence of adjacent triangles in a general setting. The sequence of triangles is replaced by a sequence of ordered line segments. The 3D space is replaced by a Euclidean space. Let a, b be two points in Euclidean space $\mathbb{E}$ and $e_1, e_2, ..., e_k$ be line segments in $\mathbb{E}$, finding the shortest path joining a and b with respect to $e_1, e_2, ..., e_k$ in that order, our considered problem can be stated as follows

$$\begin{aligned} \min_{(x_0,x_1,\ldots,x_{k+1})} & \sum_{i=0}^{k} \|x_i - x_{i+1}\| \\ \text{subject to} \, & x_i \in e_i, \text{for} \, i = 1, 2, \ldots, k \\ & x_0 = a, \\ & x_{k+1} = b. \end{aligned}$$

The shortest path problems can be seen in many various ways. In the view of geometry, we have the description of the shortest path joining two points and going through a sequence of adjacent triangles. In the view of optimization, we have the formula of the above problem. A different point of view gives a new idea to solve these problems. For example, let consider a classical problem in computational geometry: the convex hull problem. The convex hull problem states as follows: given a finite planar point set, find the convex hull of the set. The boundary of the convex hull of a finite planar point set can be seen as the shortest path around the given set (see Li and Klette, 2011). Then, the shortest path joining two points inside a simple polygon P can be constructed by the union of some parts of boundaries of the convex hull of some vertexes of P (see An and Hoai, 2011). Vice versa, the shortest path can be used to finding a part of the boundary of the convex hull of a polyline (see An, 2010). The shortest path problems in different contexts have different uses and meanings. In computer graphics and image processing, the convex sets on a computer screen are orthogonal convex (see, for example, Hearn, Baker and Carithers, 2014). The shortest path around one digital shape now is the connected orthogonal convex hull of the shape. We again can understand the connected orthogonal convex hull problem is the shortest path problem in a computer screen. This is how we come with the shortest path problem in Chapter 3: finding the connected orthogonal convex hull of a finite planar point set.

The computation of the convex hull of a finite planar point set has been studied extensively. It is natural to relate convex hulls to orthogonal convex hulls. Orthogonal convexity is also known as rectilinear, or (x, y)-convexity. It has found applications in several research fields, including illumination (see, for example, Abello et al., 1998), polyhedron reconstruction (see, for example, Biedl and Genc, 2011), geometric search (see, for example, Son et al., 2011), and VLSI circuit layout design (see, for example, Uchoa et al., 2002), digital images processing (see, for example, Seo et al., 2016).

The notation of orthogonal convexity has been widely studied since early eighties (see Unger, 1959) and some of its optimization properties are given in (see Gonza'lez-Aguilar et al., 2019). However, unlike the convex hulls, finding the orthogonal convex hull of a finite planar point set is fraught with difficulties. An orthogonal convex hull of a finite planar point set may be disconnected. Unfortunately the connected orthogonal convex hulls of a finite planar point set might be not unique, even countless. There exist several algorithms to find the connected orthogonal convex hulls of a finite planar point set (see Karlsson et al. 1998, Montuno and Fournier, 1982,...). In previous works, the definition of an orthogonal convex set was used to find a connected orthogonal convex hull of a finite planar point set, and no numerical result has been shown. The second question is "What is the explicit form of a connected orthogonal convex hull?". To answer this question, firstly, we consider some assumption when the connected orthogonal convex hull of a finite planar point set is unique. Secondly, we introduce the concept of extreme points of the smallest connected orthogonal convex hull of a finite planar point set, and show that this hull of a finite planar point set is totally determined by its extreme points and these points belong to the given finite planar point set. There arises the third question "How to detect these extreme points from the given finite planar point set?"

Let us consider Graham's convex hull algorithm (see, Graham, 1972) that depends on an initial ordering of the given finite planar point set. The initial point with the other points in this order actually forms a star-convex set. Based on this shape, Graham constructed Graham's convex hull algorithm. Some advantages of Graham's convex hull algorithm can be found in Allison and Noga, 1984. In case of connected orthogonal convex hulls, if we have a reasonably ordered points, we then can scan these ordered points to detect the connected orthogonal convex hull from these points. As a result, an efficient algorithm to find the connected orthogonal convex hull of a finite planar point set which is based on the idea of Graham's convex hull algorithm will be presented in the dissertation. As can been seen later in the dissertation, our new algorithm takes only $O(n \log n)$ time, where n is the number of given points (Theorem 3.1).

The dissertation consists of three chapters about constrained optimization problems: Chapter 1 and Chapter 2 discuss the shortest paths joining two points in a region (a polygon or the surface of a polytope) in Euclidean spaces, and Chapter 3 studies the connected orthogonal convex hull of a finite planar point set.

In Chapters 1 and 2, the problem of shortest paths joining two points with respect to a sequence of line segments is examined. We present some analytical and geometric properties of shortest paths joining two given points with respect to a sequence of line segments in a Euclidean space, especially their existence, uniqueness, characteristics, and conditions for concatenation of two shortest paths to be a shortest path are presented. We then focus on straightest paths lying on a sequence of adjacent polygons in 2 or 3 dimensional spaces.

In Chapter 3, the problem of finding the connected orthogonal convex hull of a finite planar point set is considered. We present the concept of extreme points of a connected orthogonal convex hull of the set, and show that these points belong to the set. Then we prove that the connected orthogonal convex hull of a finite set of points is an orthogonal (x, y)-polygon where its convex vertexes are extreme points of its connected orthogonal convex hull . An efficient algorithm, which is based on the idea of Graham's convex hull algorithm, for finding the connected orthogonal convex hull of a finite planar point set is presented. We also show that the lower bound of computational complexity of such algorithms is $O(n \log n)$, where n is the number of given points. Some numerical results for finding the connected orthogonal convex hulls of random sets are given.

Chapter 0

Preliminaries

This dissertation deals with shortest paths (joining two given points, from a source point to many destinations, from a point to a line segment, ...) in a geometric domain (such as on surface of a polytope, inside a simple polygon, a terrain, ...) in Euclidean spaces. In this chapter, we represent some preliminaries which are based on the books of O'Rourke, 1998 and Papadopoulos, 2014.

0.1 Paths

Given a metric space (X, ρ). A *path* in X is a continuous mapping γ from an interval $[t_0, t_1] \subset \mathbb{R}$ to X. We say that γ *joins* the point $\gamma(t_0)$ with the point $\gamma(t_1)$. The *length* of $\gamma : [t_0, t_1] \to X$ is the quantity

$$l(\gamma) = \sup_{\sigma} \sum_{i=1}^{k} \rho\big(\gamma(\tau_{i-1}), \gamma(\tau_i)\big),$$

where the supremum is taken over the set of partitions $\sigma : t_0 = \tau_0 < \tau_1 < \cdots < \tau_k = t_1$ of $[t_0, t_1]$.

The length of a path is additive, i.e., for any path $\gamma : [t_0, t_1] \to X$ and $t_* \in [t_0, t_1]$,

$$l(\gamma) = l\big(\gamma_{|_{[t_0,t_*]}}\big) + l\big(\gamma_{|_{[t_*,t_1]}}\big),$$

where $\gamma_{|_{[t_0,t_*]}}$ and $\gamma_{|_{[t_*,t_1]}}$ are restrictions of γ on $[t_0, t_*]$ and $[t_*, t_1]$, respectively.

For instance, let $(X, \|\cdot\|)$ be a normed space. A mapping $\gamma : [t_0, t_1] \to X$ is said to be an *affine path* if for any $\lambda \in [0, 1]$,

$$\gamma((1-\lambda)t_0 + \lambda t_1) = (1-\lambda)\gamma(t_0) + \lambda\gamma(t_1).$$

For $x, y \in X$, $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$, a path $\gamma : [t_0, t_1] \to X$ joining x with y is affine iff

$$\gamma(t) = (t_1 - t_0)^{-1}\big[(t_1 - t)x + (t - t_0)y\big].$$

In this case, γ has length $\|x - y\|$ and the image $\gamma([t_0, t_1])$ is the *line segment*

$$[x, y] := \{(1-\lambda)x + \lambda y : 0 \le \lambda \le 1\}.$$

We assume that all paths in the dissertation have finite length.

Let t_0, t_1 and t_2 be real numbers satisfying $t_0 \le t_1 \le t_2$. If $\gamma_1 : [t_0, t_1] \to X$ and $\gamma_2 : [t_1, t_2] \to X$ are two paths satisfying $\gamma_1(t_1) = \gamma_2(t_1)$, then we can define the path

$\gamma : [t_0, t_2] \to X$ by setting

$$\gamma(t) = \gamma_1(t) \text{ if } t_0 \le t \le t_1,$$
$$\gamma(t) = \gamma_2(t) \text{ if } t_1 \le t \le t_2.$$

The path γ is called the *concatenation* of γ_1 and γ_2, denoted by $\gamma_1 * \gamma_2$, and we have $l(\gamma) = l(\gamma_1) + l(\gamma_2)$.

Let $\gamma : [t_0, t_1] \to X$ and $\eta : [\tau_0, \tau_1] \to X$ be two paths in X. We say that γ *is obtained from* η *by a change of parameter* if there exists a function $\psi : [t_0, t_1] \to [\tau_0, \tau_1]$ that is monotonic (in the weak sense), surjective and that satisfies $\gamma = \eta \circ \psi$ (the composite function of η and ψ).

The function ψ is called *the change of parameter*. It is proved that the equality $l(\gamma) = l(\eta)$ always holds.

We say that $\gamma : [t_0, t_1] \to X$ is *parametrized by arclength* if for all τ and τ' satisfying $t_0 \le \tau \le \tau' \le t_1$, we have $l\big(\gamma_{|_{[\tau,\tau']}}\big) = \tau' - \tau$.

If $\gamma : [t_0, t_1] \to X$ is any path, then there always exists a path $\lambda : [0, l(\gamma)] \to X$ such that λ is parametrized by arclength and γ is obtained from λ by the change of the parameter $\psi : [t_0, t_1] \to [0, l(\gamma)]$ defined by $\psi(\tau) = l\big(\gamma_{|_{[t_0,\tau]}}\big)$.

In the dissertation, we do not consider a general metric space but examine in Euclidean spaces. We denote by $(\mathbb{E}, \|\cdot\|)$ a non-trivial Euclidean space and the induced distance is $\rho(x, y) = \|x - y\|$. For $x, y \in \mathbb{E}$, denote

$$]x, y] = [x, y] \setminus \{x\},$$
$$[x, y[= [x, y] \setminus \{y\}, \text{ and}$$
$$]x, y[= [x, y] \setminus \{x, y\}.$$

Note that when $x = y$,

$$[x, y] = \{x\},$$
$$[x, y[=]x, y] =]x, y[= \emptyset.$$

If $x \ne y$, each point $z \in]x, y[$ is called a (relative) *interior point* of $[x, y]$. By abuse of notation, sometimes we also call the image $\gamma([t_0, t_1])$ the path $\gamma : [t_0, t_1] \to \mathbb{E}$. For practical purposes, $\mathbb{E}$ is usually chosen to be the finite dimensional space $\mathbb{R}^d$ with $d = 2$ or $d = 3$.

In the problem of finding shortest paths on a polyhedral surface, a key geometric concept is the notion of *planar unfolding* around a sequence of adjacent convex polygons $\mathcal{S} = (f_1, f_2, \ldots, f_{k+1})$ (see Mitchel et al. 1987). Let $\mathcal{S} = (f_1, f_2, \ldots, f_{k+1})$ be a *sequence of (not necessary distinct) adjacent convex polygons* in $\mathbb{R}^2$ or $\mathbb{R}^3$, then $e_i := f_i \cap f_{i+1}$ is an edge for $1 \le i \le k$. We unfold the sequence $\mathcal{S}$ as follow: Rotate f_1 around e_1 until its plane coincides with that of f_2, rotate f_1 and f_2 around e_2 until their plane coincides with that of f_3, continue this way until all faces $f_1, f_2, \ldots, f_k$ lie on the plane of f_{k+1}.

0.2 Graham's Convex Hull Algorithm

First, we recall some useful definitions in computational geometry.

A *simple polygon* (see Lang and Murrow, 1988) is an n-sided figure consisting of n segments $\overline{p_1p_2}, \overline{p_2p_3}, \overline{p_3p_4}, \ldots, \overline{p_{n-1}p_n}, \overline{p_np_1}$, which intersect only at their endpoints and enclose a single region.

The *perimeter* of a (simple) polygon (see Lang and Murrow, 1988) is defined to be the sum of the lengths of its sides.

A set $K \subset \mathbb{R}^d$ is *convex* (see Grunbaum, 2003) if and only if for each pair of distinct points $a, b \in K$ the closed segment with endpoints a and b is contained in K. The *convex hull* of a set K is the smallest convex set that contains K.

Let K be a convex subset of $\mathbb{R}^d$. A point $x \in K$ is an *extreme point* (see Grunbaum, 2003) of K provided $y, z \in K, 0 < \lambda < 1$, and $x = \lambda y + (l - \lambda)z$ imply $x = y = z$. The set of all extreme points of K is denoted by ext$K$.

A compact convex set $K \subset \mathbb{R}^d$ is a *convex polytope* provided ext$K$ is a finite set.

A finite union of convex polytopes such that its space is connected is called a *polytope* (see An, 2017). For a polytope (or polyhedral set) K , it is customary to call the points of ext$K$ *vertexes*.

Graham's convex hull algorithm is a sequential algorithm for finding the convex hull of a planar finite point set. This algorithm was presented in 1972 by R. Graham in Bell Laboratories. It is easy to understand and to program on computer but not able to apply to finding convex hulls of finite point set in 3D. Here, we represent this algorithm which is based on Orourke, 1998.

Finding the convex hull of a finite point set in the plane is a fundamental problem in Computational Geometry. It can be used as an illustrated example for understanding the issues that arises in solving geometric problem by computer.

The convex hull problem in Computational Geometry is stated as follows: given a finite point set P in the plane, find the convex hull of P. This problem can be seen as an optimization problem: given a finite point set P in the plane, find the shortest path around all points of P.

How to compute the convex hull? Or what does it mean to compute the convex hull? As we know, the convex hull of a finite point set in the plane is a convex polygon. A convex polygon can be represented by an ordered list of its extreme vertexes, starting at arbitrary one. So the convex hull problem can be restated as follows: given a finite point set $P = \{p_1, p_2, \ldots, p_n\}$ in the plane, compute an ordered list of extreme vertexes of the convex hull conv(P) of P, listed in counterclockwise order.

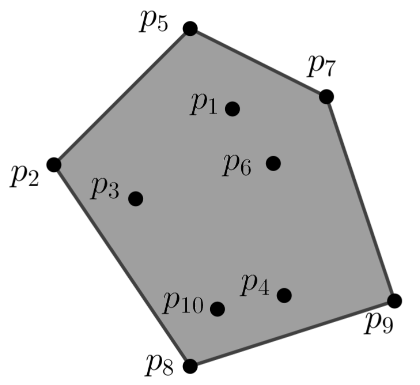

Figure 0.1: The convex hull of $\{p_1, p_2, \ldots, p_{10}\}$ is the convex polygon $p_2p_8p_9p_7p_5$.

The idea of Graham's convex hull algorithm is simple. First we sort all points of P in counterclockwise order by its angle from an arbitrary point of P. As we can see, on the boundary of the convex hull of P (the convex polygon) if we traverse through each vertexes (in counterclockwise order), we always turn left. Therefore, we scan the ordered list of points of P such that, each time turn right, we delete the processing point. The remaining points are the set of all extreme points of $\mathrm{conv}(P)$. We illustrate the above idea in the following pseudocode, which is based on Orourke, 1998.

Algorithm 1 GRAHAM'S CONVEX HULL ALGORITHM

Input. A set of finite points P in the plane.
Output. List of extreme points of $\mathrm{conv}(P)$ in order.
Let p_0 be the point in P with the minimum y-coordinate, or the leftmost such point in case of a tie.
Let $< p_1, p_2, \ldots, p_m >$ be the remaining points in P, sorted by polar angle in counterclockwise order around p_0 (if more than one point has the same angle, remove all but the one that is farthest from p_0).
PUSH(p_0, S)
PUSH(p_1, S)
PUSH(p_2, S)
for $i \leftarrow 3$ **to** m
  while the angle formed by points NEXT-TO-TOP(S), TOP(S), and p_i makes nonleft turn
    POP(S)
  end while
  PUSH(p_i, S)
end for
return S

In Algorithm 1, we use "stack" to store the points in processing. In computer science, a *stack* is an abstract data type that is used as a set of elements, with two main principal operations:

- *push*, which adds an element to the set, and

- *pop*, which removes the most recently added element that was not yet removed.

Push and pop in stack can be understood as "last in, first out" (see Figure 0.2).

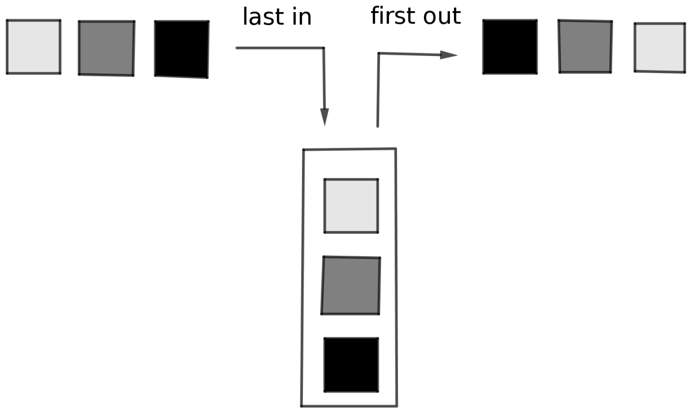

Figure 0.2: Illustration of a stack

In the dissertation, an illustrative example for Graham's convex hull algorithm has been given.

Theorem 0.1 (See Graham, 1972) *The convex hull of n points in the plane can be found by Graham's convex hull algorithm in $O(n \log n)$ for sorting and no more than $2n$ iterations for the scan.*

Chapter 1

Shortest Paths with respect to a Sequence of Line Segments in Euclidean Spaces

In this chapter, we consider the following problem

$$\begin{aligned} \min_{(x_0, x_1, \ldots, x_{k+1})} & \sum_{i=0}^{i=k} \|x_i - x_{i+1}\| \\ \text{such that}\ & x_i \in e_i, \forall i = 1, 2, \ldots, k \\ & x_0 = a, \\ & x_{k+1} = b \end{aligned}$$

where a, b be two points in Euclidean space $\mathbb{E}$ and $e_1, e_2, \ldots, e_k$ be line segments in $\mathbb{E}$.

1.1 Shortest Paths with respect to a Sequence of Ordered Line Segments

Definition 1.1 Let a, b be points in $\mathbb{E}$ and let $e_1, \ldots, e_k$ be a sequence of line segments (these line segments are not necessarily distinct and some of them may be singletons). A continuous map $\gamma : [t_0, t_1] \to \mathbb{E}$ that joins a with b is called a *path* with respect to the sequence $e_1, \ldots, e_k$ if there is a sequence of numbers $t_0 \le \bar{t}_1 \le \cdots \le \bar{t}_k \le t_1$ such that $\gamma(\bar{t}_i) \in e_i$ for $i = 1, \ldots, k$ (see Figure 1.1). The path γ is called a *shortest path* if its length does not exceed the length of any path.

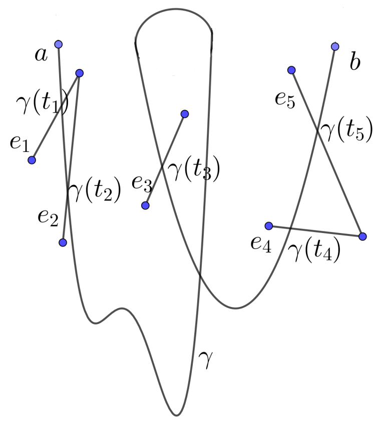

Figure 1.1: A path that joins a with b with respect to e_1, e_2, e_3, e_4 and e_5.

If $\gamma : [t_0, t_1] \to \mathbb{E}$ joins a with b and is a path with respect to the sequence $e_1, \ldots, e_k$, we call it a $P(a, b)_{(e_1,\ldots,e_k)}$. If we want to emphasize the intersection points $\gamma(\bar{t}_i)$ of γ and e_i, $t_0 \le \bar{t}_1 \le \cdots \le \bar{t}_k \le t_1$, we say γ is a $P(a, b)_{(e_1,\ldots,e_k)}$ corresponding to $\gamma(\bar{t}_i) \in e_i$, $1 \le i \le k$. If, in addition, γ is a shortest path, then we say that it is an $SP(a, b)_{(e_1,\ldots,e_k)}$. In a concrete case, we only say shortest paths without specific they from where to where and go through any line segments.

We assume for convenience that $e_i \ne e_{i+1}$ for all $i = 1, \ldots, k-1$. We also use the notation $SP(a, b)_\emptyset$ to denote any path that joins a with b, one-to-one, and has length $\|a - b\|$ and image $[a, b]$. Usually $SP(a, b)_\emptyset$ is assumed to be an affine path joining a with b. The aim of this section is to prove that shortest path joining two given points and with respect to a given sequence of line segments exists and in some sense is unique.

We start with an intuitive property of paths.

Lemma 1.1 *Let $t_0 < t_1$ and let $\gamma : [t_0, t_1] \to \mathbb{E}$ be any path.*

(a) $l(\gamma) \ge \|\gamma(t_0) - \gamma(t_1)\|$ and equality holds only if $\gamma([t_0, t_1]) = [\gamma(t_0), \gamma(t_1)]$.

(b) If γ is parametrized by arclength and $l(\gamma) = \|\gamma(t_0) - \gamma(t_1)\| > 0$, then γ is affine, that is,

$$\gamma(t) = \frac{t_1 - t}{t_1 - t_0}\gamma(t_0) + \frac{t - t_0}{t_1 - t_0}\gamma(t_1), \quad t \in [t_0, t_1].$$

If $\gamma\big([t_0,t_1]\big)$ is a path which joins $a=\gamma(t_0)$ with $b=\gamma(t_1)$, and $c\in\mathbb{E}$, for abbreviation, we denote by $\gamma*[b,c]$ the concatenation of γ and any one-to-one shortest path $\xi:[t_1,t_2]\to\mathbb{E}$ joining b and c. (If $b=c$, $\gamma*[b,c]=\gamma$.) The notation $[c,a]*\gamma$ is defined similarly. Clearly $l\big(\gamma*[b,c]\big)=l(\gamma)+\|c-b\|$ and $l\big([c,a]*\gamma\big)=l(\gamma)+\|c-a\|$. Likewise, if $\gamma_1\big([t_0,t_1]\big)$, $\gamma_2\big([t_1',t_2]\big)$ are paths which join a with b and c with d, respectively, and $t_1<t_1'$, then $\gamma_1*[b,c]*\gamma_2$ denotes the concatenation $\gamma_1*\xi*\gamma_2$, where ξ is defined on $[t_1,t_1']$, one-to-one, and is a shortest path joining b with c. In these notations, ξ is usually chosen to be affine.

In studying properties of shortest paths joining two points with respect to a sequence of line segments, the following simple result is essential for others.

Proposition 1.1 *Let $\gamma\big([t_0,t_1]\big)$ be an $SP(a,b)_{(e_1,\ldots,e_k)}$ corresponding to $\gamma(\bar{t}_i)\in e_i$, $i=1,\ldots,k$.*

(a) If $\bar{t}_{j-1}\le\alpha<\beta\le\bar{t}_j$, then $l\big(\gamma_{|[\alpha,\beta]}\big)=\|\gamma(\alpha)-\gamma(\beta)\|$ and $\gamma([\alpha,\beta])=[\gamma(\alpha),\gamma(\beta)]$. If, in addition, γ is parametrized by arclength, then γ is affine on $[\alpha,\beta]$.

(b) $\gamma\big([t_0,t_1]\big)=\cup_{i=0}^{k}[\gamma(\bar{t}_i),\gamma(\bar{t}_{i+1})]$. Thus $\gamma\big([t_0,t_1]\big)$ is a polyline.

Remark 1.1 Let $f_1,\ldots,f_{k+1}$ be a sequence of convex polygons (these polygons are not necessarily distinct and f_i and f_{i+1} may be identical), let e_i be a common edge of f_i and f_{i+1} and let $a\in f_1$, $b\in f_{k+1}$ ($k\ge1$). If $\gamma([t_0,t_1])$ is an $SP(a,b)_{(e_1,\ldots,e_k)}$, then by Proposition 1.1, $\gamma([t_0,t_1])=\cup_{i=0}^{k}[\gamma(\bar{t}_i),\gamma(\bar{t}_{i+1})]\subset\cup_{i=1}^{k+1}f_i$, i.e., this shortest path lies on the polygons. Conversely, if $\gamma:[t_0,t_1]\to\cup_{i=1}^{k+1}f_i$ is a $P(a,b)_{(e_1,\ldots,e_k)}$ and has the minimum length in the family of all $P(a,b)_{(e_1,\ldots,e_k)}$ that lie totally on the polygons f_i, then $\gamma([t_0,t_1])$ is an
$SP(a,b)_{(e_1,\ldots,e_k)}$.

Sometimes we make the following assumption.

(A) *The restriction of the path on each non-singleton interval of its domain has positive length.*

Assumption (A) means that the path is not constant on any interval with positive length in its domain. It is satisfied, for instance, if the path is parametrized by arclength or if it is one-to-one.

Theorem 1.1 *Let $a,b\in\mathbb{E}$ and let $e_1,\ldots,e_k$ be a sequence of line segments (these line segments are not necessarily distinct and some of them may be singletons). There exists a shortest path joining a with b with respect to the sequence $e_1,\ldots,e_k$. Moreover, this shortest path is unique in the family of paths with respect to a sequence of line segments satisfying assumption* (A).

We can apply the arguments in the proof of the first part of Theorem 1.1 to prove the existence of solutions of problems with variable endpoints.

Corollary 1.1 *Let A,B be nonempty compact subsets of $\mathbb{E}$ and let $\mathcal{E}=(e_1,\ldots,e_k)$.*

(a) If $b\in\mathbb{E}$ is fixed, in the family of $P(a,b)_{\mathcal{E}}$, where $a\in A$, there exists a shortest path.

(b) If $a \in \mathbb{E}$ is fixed, in the family of $P(a,b)_{\mathcal{E}}$, where $b \in B$, there exists a shortest path.

(c) In the family of $P(a,b)_{\mathcal{E}}$, where $a \in A$ and $b \in B$, there exists a shortest path.

(d) In the family of $P(a,a)_{\mathcal{E}}$, where $a \in A$, there exists a shortest path.

Observe that shortest paths with respect to a sequence of line segments in Corollary 1.1 may not be unique. Moreover, by Theorem 1.1, all shortest paths in this corollary are always polylines.

Applying Corollary 1.1(d) to the problem of finding an inscribed polygon in a given convex polygon $\mathcal{P} \subset \mathbb{R}^2$ with a minimum perimeter, we find that this problem has a solution. Some properties of angles of this inscribed polygon will be derived in the next section.

Corollary 1.2 *If $\gamma([t_0, t_1])$ and $\eta([\tau_0, \tau_1])$ are $SP(a,b)_{(e_1,\dots,e_k)}$ and parametrized by arclength, then $\eta(\tau) = \gamma(\tau - \tau_0 + t_0)$ and $\gamma(t) = \eta(t - t_0 + \tau_0)$. If, in addition, $\tau_0 = t_0$ (hence $\tau_1 = t_1$), then $\eta(t) = \gamma(t)$ for all $t \in [t_0, t_1]$.*

1.2 Concatenation of Two Shortest Paths

We have shown in Theorem 1.1 that every shortest path with respect to some sequence of line segments is a polyline. In this section we consider some geometric characteristics of shortest paths with respect to a sequence of line segments and represent conditions under which concatenation of two shortest paths with respect to two sequences of line segments is a shortest path with respect to a sequence of line segments.

We first consider the simplest case: the concatenation of a path with respect to a sequence of line segments and a line segment. If u and v are nonzero vectors in $\mathbb{E}$, we denote by $\angle(u,v)$ the angle between u and v, which does not exceed π.

Theorem 1.2 *Let $\mathcal{E} = (e_1, \dots, e_k)$ be a sequence of line segments and $\gamma([t_0, t_1])$ an $SP(a,b)_{(e_1,\dots,e_{n-1})}$ corresponding to $x_i = \gamma(\bar{t}_i) \in e_i$ $(1 \le i \le n-1)$ and $n \le k$. Suppose that $b \in e_n \cap \dots \cap e_k$, each e_j is non-singleton for $j = n, \dots, k$, and that $x_{n-1} \ne b$ (if $n = 1$, $x_0 := a$). Let $q \in \mathbb{E}$, $q \ne b$. Then $\gamma * [b,q]$ is an $SP(a,q)_{\mathcal{E}}$ iff for any $y_j \in e_j$ and $y_j \ne b$, $j = n, \dots, k$, we have $\theta \ge \pi$, where*

$$\theta := \angle(x_{n-1} - b, y_n - b) + \sum_{j=n}^{k-1} \angle(y_j - b, y_{j+1} - b) + \angle(y_k - b, q - b).$$

Theorem 1.2 gives a step-by-step method to check whether a path with respect to a sequence of line segments is shortest: we just measure the angles between line segments of the path and e_is at points of intersection, from the starting point to the terminating point.

Corollary 1.3 *Let $\gamma([t_0, t_1])$ be an $SP(a,b)_{(e_1,\dots,e_n)}$ corresponding to $x_i = \gamma(\bar{t}_i) \in e_i$, $1 \le i \le n$. Set $x_0 := a$, $x_{n+1} := b$. Suppose also that for some j, e_j is non-singleton, $x_{j-1} \ne x_j$, and $x_{j+1} \ne x_j$.*

(a) *If* $y_j \in e_j$, $y_j \neq x_j$, *then* $\theta := \angle(x_{j-1} - x_j, y_j - x_j) + \angle(y_j - x_j, x_{j+1} - x_j) \geq \pi$. *In particular, if* x_j *is an interior point of* e_j, *then* $\theta = \pi$.

(b) *Let* L *be the line containing* e_j. *If* $x_{j-1} \in L$ *and* $x_{j+1} \notin L$, *then* x_j *is an end point of* e_j *with* $\|x_{j-1} - x_j\| = \min_{x \in e_j} \|x_{j-1} - x\|$ *and* $\theta > \pi$. *The case* $x_{j+1} \in L$ *and* $x_{j-1} \notin L$ *is similar.*

(c) *If* $x_{j-1}, x_{j+1} \in L$, *then either* $\angle(x_{j-1} - x_j, x_{j+1} - x_j) = \pi$ *or* $\angle(x_{j-1} - x_j, y_j - x_j) = \angle(x_{j+1} - x_j, y_j - x_j) = \pi$, *where* $y_j \in e_j$, $y_j \neq x_j$. *In the latter case* x_j *is an end point of* e_j *with* $\|x_{j-1} - x_j\| = \min_{x \in e_j} \|x_{j-1} - x\|$.

The following result is a converse of Corollary 1.3 and it gives sufficient conditions for a path with respect to a sequence of line segments to be shortest.

Corollary 1.4 *Let* $\mathcal{E} = (e_1, \ldots, e_k)$ *be a sequence of line segments and* $\gamma([t_0, t_1])$ *an* $SP(a,b)_{(e_1,\ldots,e_{n-1})}$ *corresponding to* $x_i = \gamma(\bar{t}_i) \in e_i$, $1 \leq i \leq n-1$. *Let* $b \in e_n$ *and* $q \in \mathbb{E}$, $q \neq b$. *Set* $x_0 = a$ *and suppose that* $m = \max\{i : x_i \neq b\}$ *exits and that there is* $y \in e_n, y \neq b$.

(a) *If* $\theta := \angle(x_m - b, y - b) + \angle(y - b, q - b) = \pi$, *then* $\gamma * [b, q]$ *is an* $SP(a,q)_{\mathcal{E}}$.

(b) *If* b *is an end point of* e_n *and* $\theta > \pi$, *then* $\gamma * [b, q]$ *is an* $SP(a,q)_{\mathcal{E}}$.

In the dissertation, an illustrative example for Corollaries 1.3 and 1.4 has been given. We are now in a position to consider the general case of concatenation of two shortest paths with respect to two sequences of line segments. Roughly speaking, the following theorem says that if the last line segment of a shortest path with respect to a sequence of line segments and the first of the other overlap, the two paths can be joined to become a shortest path with respect to sequence of line segments.

Theorem 1.3 *Let* $\mathcal{E} = (e_1, \ldots, e_k)$ *be a sequence of line segments. Suppose that* $\gamma_1([t_0, t_1])$ *is an* $SP(a,b)_{(e_1,\ldots,e_{n-1})}$ *and* $\gamma_2([t^*, t_2])$ *is an* $SP(c,d)_{(e_n,\ldots,e_k)}$, *where* $t^* < t_1 < t_2$, *and* $\gamma_1(t_1) = \gamma_2(t_1) = b$. *Suppose also that* γ_1, γ_2 *satisfy assumption (A). If* $t_1 \leq \bar{t}_n \leq \cdots \leq \bar{t}_k \leq t_2$, $x_i := \gamma_2(\bar{t}_i) \in e_i$ *for* $n \leq i \leq k$, *and if there exists* $\epsilon > 0$ *such that* $\gamma_1([t_1 - \epsilon, t_1]) \subset \gamma_2([t^*, t_1])$, *then the concatenation* γ *of* γ_1 *and* $\gamma_{2|[t_1,t_2]}$ *is an* $SP(a,d)_{\mathcal{E}}$.

If the last line segment of a shortest path with respect to a sequence of line segments and the first of the other do not overlap, we can use the following.

Corollary 1.5 *Suppose* $\gamma_1([t_0, t_1])$ *is an* $SP(a,b)_{(e_1,\ldots,e_n)}$ *and* $\gamma_2([t_1, t_2])$ *is an* $SP(b,c)_{(e_{n+1},\ldots,e_k)}$, $t_0 < t_1 < t_2$. *Suppose also that* γ_1, γ_2 *satisfy assumption (A). Then* $\gamma = \gamma_1 * \gamma_2$ *is an* $SP(a,c)_{(e_1,\ldots,e_k)}$ *iff there exists* $\epsilon > 0$ *such that* $\gamma_{1|[t_1-\epsilon,t_1]} * \gamma_{2|[t_1,t_1+\epsilon]}$ *is a shortest path with respect to the sequence* $e_1, e_2, \ldots, e_n$.

Loosely speaking, if the last segment of the first shortest path and the first segment of the second form a shortest path, then their concatenation is also a shortest path.

Chapter 2

Straightest Paths on a Sequence of Adjacent Polygons

Back to the shortest path problems, as we know, a line segment is the shortest path joining its two endpoints in 3D space. A very common technique is unfolding. Many researchers use it to solve the shortest path subproblem between two points on a sequence of triangles. If we can draw a line segment between two images inside the simple polygon after unfolding the sequence of triangles, then the inverse image of this line segment is the shortest path joining two given points. The question is how to find this shortest path without unfolding? Under the idea of the straightest geodesic of Polthier and Schmies, 1998 and the thought of answering the above question, we come to the definition of straightest paths.

2.1 Straightest Paths

In 1998, Polthier and Schimies presented a new concept of geodesics: straightest geodesics are paths that have equal path angle on both sides at each point. In this section we consider "straightest paths" which are lightly differ from the original and in fact are particular shortest paths with respect to some sequence of line segments. Let $\mathcal{S} = (f_1, f_2, \ldots, f_{k+1})$ be a sequence of (not necessary distinct) adjacent convex polygons in $\mathbb{R}^2$ or $\mathbb{R}^3$, i.e. $e_i := f_i \cap f_{i+1}$ is an edge for $1 \le i \le k$.

Sometimes we make the following assumption.

(A) *The restriction of the path on each non-singleton interval of its domain has positive length.*

Definition 2.1 A *straightest path* $\gamma : [t_0, t_1] \to \cup_{i=1}^{k+1} f_i$ on the sequence $\mathcal{S}$ is a path that satisfies assumption (A) and the following conditions:

(a) there exist integers $1 \le m \le n \le k$ and a sequence of numbers

$$t_0 =: \bar{t}_{m-1} < \bar{t}_m \le \cdots \le \bar{t}_n < \bar{t}_{n+1} := t_1$$

such that

$$x_i := \gamma(\bar{t}_i) \in e_i \text{ for } m \le i \le n$$

and

$$x_{m-1} := \gamma(t_0) \in f_m,$$

$$x_{n+1} := \gamma(t_1) \in f_{n+1};$$

(b) $l\big(\gamma_{|[\bar{t}_i,\bar{t}_{i+1}]}\big) = \|x_i - x_{i+1}\|$ for $m-1 \le i \le n$;

(c) For $z_i \in e_i$, $z_i \neq x_i$, we have

$$\angle(x_{i-1}-x_i, z_i - x_i) + \angle(z_i - x_i, x_{i+1}-x_i) = \pi$$

if $x_i \in e_i$ is not a common vertex, and

$$\angle(x_{i^*-1}-x_i, z_{i^*}-x_i) + \sum_{j=i^*}^{r-1} \angle(z_j - x_i, z_{j+1}-x_i) + \angle(z_r - x_i, x_{r+1}-x_i) = \pi$$

if $x_{i^*-1} \neq x_{i^*} = x_{i^*+1} = \cdots = x_r \neq x_{r+1}$ and $i^* \le i \le r$.

It is conventional to define straightest path joining a to b on the same polygon (with respect to an empty sequence of common edges) to be any path that joins a to b, one-to-one, and has length $\|a-b\|$, i.e., a $SP(a,b)_\emptyset$.

Condition (a) states that a straightest path is a path with respect to the sequence $e_m, e_{m+1}, \dots, e_n$. Condition (b) and Lemma 1.1 imply that $\gamma([\bar{t}_i, \bar{t}_{i+1}]) = [x_i, x_{i+1}] \subset f_{i+1}$ for $m-1 \le i \le n$. We observe also that condition (c) does not depend on the choice of $z_i \in e_i$, $m \le i \le n$. Corollary 1.4 and Theorem 1.2 show that γ is an $SP(x_{m-1}, x_{n+1})_{(e_m,\dots,e_n)}$. Note also that assumption (A), the conditions $t_0 < \bar{t}_m$, $\bar{t}_n < t_1$, and (b) imply that $x_0 \neq x_1$ and $x_n \neq x_{n+1}$. These conditions do not restrict the definition of straightest paths since if x_0 belongs to a common edge then we choose $m = \max\{j : x_0 \in f_j\}$. Similarly, we can assume that x_{n+1} does not belong to e_n.

2.2 An Initial Value Problem on a Sequence of Adjacent Polygons

Let $\gamma([t_0,t_1])$ be a straightest path on $\mathcal{S} = (f_1, f_2, \dots, f_{k+1})$, $\gamma(t_0) \in f_m$, and v a nonzero vector that is parallel to f_m. If there exists $t^* > t_0$ such that $\gamma([t_0,t^*]) \subset f_m$ and $\gamma(t^*) - \gamma(t_0) = \lambda v$ for some $\lambda > 0$ we say that γ *starts* at $\gamma(t_0)$ in the direction of v.

As usual, we first consider the problem of existence of straightest paths.

Theorem 2.1 *Let $\mathcal{S} = (f_1, f_2, \dots, f_{k+1})$ be a sequence of adjacent convex polygons. Let $a, p \in f_m$, $a \neq p$, and $v = p - a$. Then there exists a unique longest straightest path $\gamma([t_0,t_1])$ on $\mathcal{S}$ starting at a and in the direction of v. Moreover, if $\eta([\tau_0,\tau_1])$ is any straightest path on $\mathcal{S}$ starting at a and in the direction of v, then η equals $\gamma_{|[t_0,t^*]}$ for some $t_0 \le t^* \le t_1$. Thus every straightest path on $\mathcal{S}$ can be extended to a longest straightest path.*

Remark 2.1 The shortest path joining two given points on a sequence of adjacent convex polygons always exists but this may not be true for straightest path.

In the problem of finding shortest paths on a polyhedral surface, a key geometric concept is the notion of *planar unfolding* around a sequence of adjacent convex polygons $\mathcal{S} = (f_1, f_2, \dots, f_{k+1})$ (see Mitchell et al., 1987). We unfold the sequence $\mathcal{S}$ as follow: Rotate f_1 around e_1 until its plane coincides with that of f_2, rotate f_1 and f_2 around e_2 until their plane coincides with that of f_3, continue this way until all faces $f_1, f_2, \ldots, f_k$ lie on the plane of f_{k+1}. The idea of planar unfolding was used to prove Theorem 1.2. We now investigate the image of a straightest path under a planar unfolding.

Lemma 2.1 *Let γ be a $P(a,b)_{(e_1,\ldots,e_k)}$ ($a \in f_1$, $b \in f_{k+1}$) on the sequence $\mathcal{S}$ in $\mathbb{R}^3$. Then γ is straightest iff its planar unfolding around $e_1, \ldots, e_k$ is a line segment.*

Theorem 2.2 *Let $\mathcal{S} = (f_1, f_2, \ldots, f_{k+1})$ be a sequence of adjacent convex polygons and let a, q_1, q_2, q_3 be points in f_m such that $q_2 \in]q_1, q_3[$ and a, q_1, q_3 are not collinear. Let $v_i = q_i - a$, $i = 1, 2, 3$. Assume that γ_1, γ_2 and γ_3 are straightest paths starting at a and in the directions of v_1, v_2, v_3, respectively, and γ_1, γ_3 cut a line segment $e \subset f_n$ ($n \geq m$) at y_1 and y_3, respectively. If γ_2 is the longest straightest path, then it meets e at some point $y_2 \in]y_1, y_3[$.*

Chapter 3

Finding the Connected Orthogonal Convex Hull of a Finite Planar Point Set

In this chapter, we consider the third shortest path problem in the context of computer graphics and image processing: finding the connected orthogonal convex hull of a finite planar point set P. The solution should be an orthogonal convex polygon bounding P, with minimum perimeter and minimum area. For an arbitrary given finite point set, the connected orthogonal convex hulls of this set maybe not unique. Therefore, we consider some assumptions under that the uniqueness of the connected orthogonal convex hull of the given set is preserved. Then we study the construction of the connected orthogonal convex hull of a finite planar point set and build an efficient algorithm for finding it. In Chapter 3, we only consider the subsets of $\mathbb{R}^2$.

3.1 Orthogonal Convex Sets and their Properties

Definition 3.1 (See Unger, 1959) A set $K \subset \mathbb{R}^2$ is said to be *orthogonal convex* if its intersection with any horizontal or vertical line is convex.

In some previous papers (see Karlsson et al., 1988 and Unger, 1959), a slightly different definition of orthogonal convexity were given. Here, we use the term "convex" to cover line segments with or without its endpoints. Furthermore, our definition can be extended for $\mathbb{R}^n$. Observe that any convex set is orthogonal convex as seen in Figure 3.1 (a), but the reverse may be not true as seen in Figure 3.1 (b).

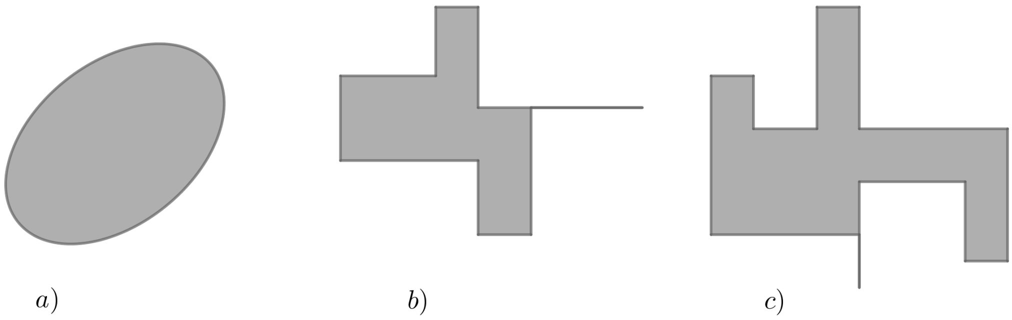

Figure 3.1: a) and b) two orthogonal convex sets; c) not an orthogonal convex set

K is said to be *connected orthogonal convex* if it is orthogonal convex and connected.

It is obvious that the intersection of any family (finite or infinite) of orthogonal convex sets is orthogonal convex. An *orthogonal convex hull* (see Montuno and Fournier, 1982) of a set $K \subset \mathbb{R}^2$ is the smallest orthogonal convex set which contains K. Thus, the orthogonal convex hull of K is the intersection of all orthogonal convex sets containing K and therefore, the orthogonal convex hull of a set is unique. But it may be not connected.

Definition 3.2 (See Ottmann et al., 1984) A *connected orthogonal convex hull* of K, is a minimal connected orthogonal convex set containing K.

In Figure 3.2 we display a set of three distinct points in the plane. Observe that the orthogonal convex hull of the set is itself, and it is disconnected as in Figure 3.2 (a). The connected orthogonal convex hulls of the set are not unique, as in Figure 3.2 (b, c).

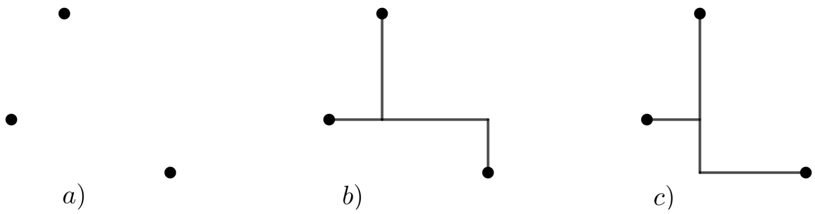

Figure 3.2: a) the set of three points in the plane and its orthogonal convex hull consists of these points; b), c) connected orthogonal convex hulls of these points

Note that a connected orthogonal convex hull of a finite planar point set is compact. We define a line to be *rectilinear* if the line is parallel to either x-axe or y-axe. A half line or a line segment are *rectilinear* if the lines on which they lie are rectilinear.

Let $a \neq b$ be two given points in the plane. We define an *orthogonal line* $\ell(a,b)$ through a, b as follows: if $x_a \neq x_b$ and $y_a \neq y_b$, $\ell(a,b)$ is the union of two rectilinear half lines having the same starting point, and if $x_a = x_b$ or $y_a = y_b$, $\ell(a,b)$ is the line through a and b.

Let $\ell(a, b)$ be an orthogonal line segment through a and b and c be the common point of two half-lines of $\ell(a, b)$. We define the *orthogonal line segment* $s(a, b)$ be two line segments $[a, c] \cup [c, b]$.

Thus, an orthogonal line $\ell(a, b), (x_a \neq x_b, y_a \neq y_b)$ separates the plane into two regions, as shown in Figure 3.3. The quadrant region together with the orthogonal line $\ell(a, b)$ will be called a *quadrant* determined by the orthogonal line. In Figure 3.3, the quadrant regions are shaded.

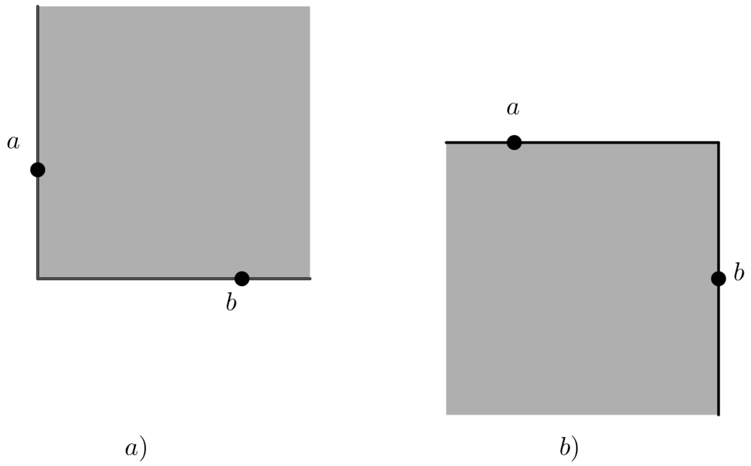

Figure 3.3: Orthogonal lines through the points a and b with $x_a \neq x_b$ and $y_a \neq y_b$ and their quadrants (shadow regions)

We now define support lines for connected orthogonal convex hulls that are similar to tangent lines in the construction of convex hulls.

Definition 3.3 Given a set $K \subset \mathbb{R}^2$. An $\ell(a, b)$ is an orthogonal supporting line (O-support, for brevity) of a set K (a and b might not belong to K) if the intersection of $\ell(a, b)$ with K is non-empty and either all points of $K \setminus (K \cap \ell(a, b))$ are not on the quadrant of $\ell(a, b)(x_a \neq x_b, y_a \neq y_b)$, or all points of $K \setminus (K \cap \ell(a, b))$ are on one open half plane which is determined by the line $\ell(a, b)(x_a = x_b, \text{or}\, y_a = y_b)$.

We also call $s(a, b)$ be the *orthogonal supporting line segments*. In a concrete case, we only say O-supports without specific it is an orthogonal line or orthogonal line segments.

Two O-supports $\ell(a, b)$ and $\ell(c, d)$ of a set K are said to be *opposite* if their half lines meet in exactly two points. Such O-supports are indicated in Figure 3.4.

As we know the connected orthogonal convex hulls of a finite planar point set maybe not unique. We now define a new kind of a point of a connected orthogonal convex hull, then we will prove that when these points exist, the connected orthogonal convex hulls are not unique.

Let λ be a simple polyline in $\mathbb{R}^2$ whose edges $[v_{i-1}, v_i], 1 \leq i \leq n$, are parallel to the coordinate axes. Such a path λ is called a *staircase path* (see Breen, 1984) if and only if the associated vectors alternate in direction. That is, for i odd, the vectors $\overrightarrow{v_{i-1}v_i}$ have the same direction, and for i even, the vectors $\overrightarrow{v_{i-1}v_i}$ have the same direction, $1 \leq i \leq n$.

We denote by $\mathcal{F}(K)$ the family of all connected orthogonal convex hulls of K. For $E \in \mathcal{F}(K)$, if there exist two opposite O-supports H and L of K intersecting in only two points, say p and q, with $x_p \neq x_q, y_p \neq y_q$, then there exists a staircase path connecting p and q in E (for example, the boundary from p to q of the rectangle having rectilinear edges and diagonal $[p, q]$). We define all points on such path (not including p and q) to be *semi-isolated* points of E. Thus, if E has a semi-isolated point, then E has infinity number of semi-isolated points. Therefore, the set of all semi-isolated points of elements of $\mathcal{F}(K)$ is the rectangle with the diagonal $[p, q]$ excepting $\{p, q\}$. Some semi-isolated points are illustrated in Figure 3.4.

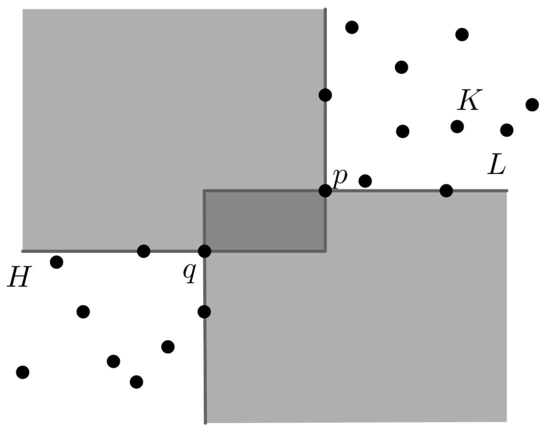

Figure 3.4: Two opposite O-supports H and L of the set K and the set of semi-isolated points (the rectangle with the diagonal $[p, q]$ excepting $\{p, q\}$)

It follows immediately from the definition of semi-isolated points the following result.

Remark 3.1 If $E \in \mathcal{F}(K)$ has no semi-isolated point, then every $F \in \mathcal{F}(K)$ also has no semi-isolated point. Therefore, the existence of semi-isolated points does not depend on each element $E \in \mathcal{F}(K)$.

The following proposition provides some conditions such that the family of all connected orthogonal convex hulls of a given set has only one element.

Proposition 3.1 *Let P be a finite planar point set and $\mathcal{F}(P)$ the family of all connected orthogonal convex hulls of P. If there exists an element of $\mathcal{F}(P)$ that has no semi-isolated point, then $\bigcap_{E \in \mathcal{F}(P)} E$ is a connected orthogonal convex hull of P. Therefore, $\mathcal{F}(P)$ has only one element.*

By Proposition 3.1, when no semi-isolated point can arise, $\mathcal{F}(P)$ has only one element, we then denoted it by COCH(P). From now on, when the notation COCH(P) is used, we understand that elements of $\mathcal{F}(P)$ have no semi-isolated point.

Corollary 3.1 *Let P be a finite set of points in the plane. Then* COCH(P) *is the intersection of all connected orthogonal convex sets containing P.*

The intersection of all connected orthogonal convex sets which contains P might not be the orthogonal convex hull of P, even when we assume that every connected orthogonal convex hull of P has no semi-isolated point.

Remark 3.2 Let P be a finite set of points in the plane. If the orthogonal convex hull of P is connected, then it is also the connected orthogonal convex hull of P.

Let P_1, P_2 be two finite point sets in the plane and $P_1 \subset P_2$. Then, the convex hull of P_1 is also a subset of the convex hull of P_2. Here, $\mathrm{COCH}(P_1) \subset \mathrm{COCH}(P_2)$ holds under the assumption of no semi-isolated point.

Remark 3.3 Let P_1, P_2 be two finite point sets in the plane and $P_1 \subset P_2$. Suppose that there exist $E \in \mathcal{F}(P_1), F \in \mathcal{F}(P_2)$ such that E and F have no semi-isolated point. Then $\mathrm{COCH}(P_1) \subset \mathrm{COCH}(P_2)$.

Proposition 3.2 *Let P be a finite set of points in the plane and the following condition hold*

(B) There exists an O-support $\ell(a, b)$ of P $(a, b \in P)$ such that all points of $P \setminus \{a, b\}$ lie only in both two opposite corners of $\ell(a, b)$.

Then there exists a connected orthogonal convex hull of P that has semi-isolated points and vice versa.

Suppose that $E \in \mathcal{F}(P)$ has semi-isolated points. According to Proposition 3.2, there are two opposite O-supports of P such that one lies in $\ell(a, h)$ and the other lies in $\mathrm{l}(d, e)$, or one lies in $\mathrm{l}(b, c)$ and the other lies in $\ell(f, g)$. In the dissertation, a procedure to determine a given point set satisfying condition (B) or not has been given.

From now on, we only consider the finite set of points P in the plane such that $|P| > 1$ and P does not satisfy the condition (B).

3.2 Construction of the Connected Orthogonal Convex Hull of a Finite Planar Point Set

We need some definitions.

A *rectilinear polygon* (see Montuno and Fournier, 1982) is a simple polygon whose edges are rectilinear (i.e., they are parallel to either x or y axis). The polygon has therefore only 90 and 270 degree internal angles.

Definition 3.4 (See Montuno and Fournier, 1982) An $(x, y)-$*polygon* is one of the following: a) a point; b) connected rectilinear line segments; c) a rectilinear polygon; and d) a connected union of type b) and/or type c) (x, y)-polygons.

Definition 3.5 Let P be a finite planar point set. We define a point $u \in \mathrm{COCH}(P)$ to be an *extreme* point of $\mathrm{COCH}(P)$ if there exists an orthogonal line L (u is the starting point of two half lines of L) whose intersection with $\mathrm{COCH}(P)$ is only u and there is no point of $\mathrm{COCH}(P) \setminus \{u\}$ which lies in the quadrant determined by L. We denote all extreme points of $\mathrm{COCH}(P)$ briefly by o-ext($\mathrm{COCH}(P)$).

Similar to convex hulls, we present some properties for the connected convex hull of a finite planar point set.

Proposition 3.3 *Let P be a finite planar point set. Then,*

$$\mathrm{COCH(P)} = \mathrm{COCH(o-ext(COCH(P)))}.$$

Proposition 3.4 *The connected orthogonal convex hull of a finite planar point set P is an orthogonal convex (x, y)-polygon whose boundary is union of finite set of O-supports, and each O-support goes through two extreme points of P.*

3.3 Algorithm, Implementation and Complexity

3.3.1 New Algorithm Based on Graham's Convex Hull Algorithm

Let P be a finite planar point set and the condition (B) for P does not hold In case of connected orthogonal convex hulls, if we have a reasonably ordered points, we then can scan these ordered points to get candidates for extreme points of COCH(P).

We take the points a, b, c, d, e, f, g, and h belonging to P such that

(C) a (b, respectively) is the leftmost (rightmost, respectively) of highest points of P,

- e (f, respectively) is the rightmost (leftmost, respectively) of lowest points of P,
- c (d, respectively) is the highest (lowest, respectively) of rightmost points of P,
- g (h, respectively) is the lowest (highest, respectively) of leftmost points of P.

Consider the case $a \neq h, g \neq f, e \neq d$ and $c \neq b$. Take the orthogonal lines $\ell(a, h)$ through a and h, $\ell(g, f)$ through g and f , $\ell(e, d)$ through e and d, and $\ell(c, b)$ through c and b such that $p \notin \ell(a, h)$, $v \notin \ell(g, f)$, $u \notin \ell(e, d)$, and $q \notin \ell(c, b)$. We define

(D) If $a \neq h$, P_{ah} is the set of points of P in the quadrant of $\ell(a, h)$. Otherwise, $P_{ah} := \{a\}$.

- If $b \neq c$, P_{cb} is the set of points of P in the quadrant of $\ell(c, b)$. Otherwise, $P_{cb} := \{b\}$.
- If $g \neq f$, P_{gf} is the set of points of P in the quadrant of $\ell(g, f)$. Otherwise, $P_{gf} := \{f\}$.
- If $e \neq d$, P_{ed} is the set of points of P in the quadrant of $\ell(e, d)$. Otherwise, $P_{ed} := \{e\}$.

Let a and b be two distinct points. We call $\ell(a, b)$ *parallel* to $[q, u] \cup [u, v]$ if the first half-line through a parallel to $[q, u]$ and the second half-line through b parallel to $[u, v]$. We determine an orthogonal line $\ell(p_t, p_{t-1})$ through two points p_t, p_{t-1} in P_{ha}, P_{gf}, P_{ed}, P_{cb} as follows

(E) In P_{ha}: If $x_{p_t} \leq x_{p_{t-1}}$, $\ell(p_{t-1}, p_t)$ is parallel to $[q, u] \cup [u, v]$. Otherwise, $\ell(p_t, p_{t-1})$ is parallel to $[p, q] \cup [q, u]$.

- In P_{gf}: If $x_{p_t} \leq x_{p_{t-1}}$, $\ell(p_t, p_{t-1})$ is parallel to $[q, u] \cup [u, v]$. Otherwise, $\ell(p_t, p_{t-1})$ is parallel to $[p, q] \cup [q, u]$.

- In P_{ed}: If $x_{p_t} \leq x_{p_{t-1}}$, $\ell(p_t, p_{t-1})$ is parallel to $[u, v] \cup [v, p]$. Otherwise, $\ell(p_t, p_{t-1})$ is parallel to $[v, p] \cup [p, q]$.

- In P_{cb}: If $x_{p_t} \leq x_{p_{t-1}}$, $\ell(p_t, p_{t-1})$ is parallel to $[u, v] \cup [v, p]$. Otherwise, $\ell(p_t, p_{t-1})$ is parallel to $[v, p] \cup [p, q]$.

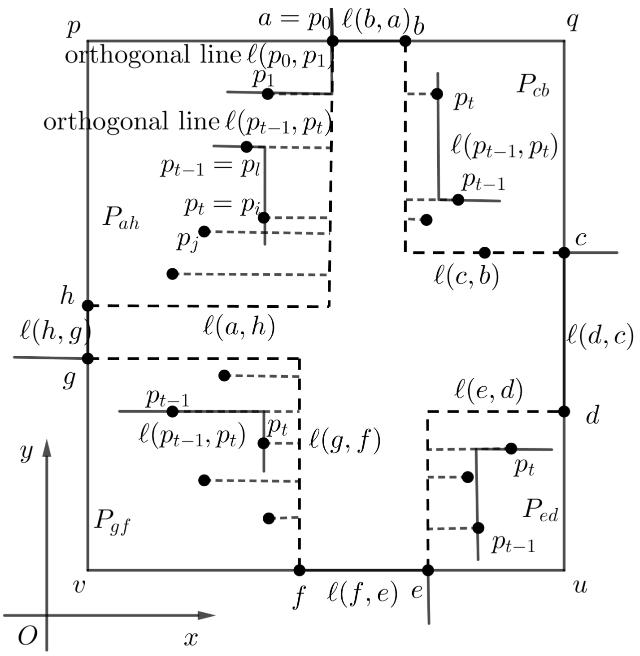

Figure 3.5: The orthogonal line $\ell(p_t, p_{t-1})$ is defined by the relation between x_{p_t} and $x_{p_{t-1}}$ and by the location of p_t, p_{t-1} in P_{ha}, P_{gf}, P_{ed} and P_{cb}.

Let $\ell(q_1, q_2)$ be an orthogonal line through two points q_1, q_2 and its rectilinear half lines have starting point q_3. If the triple (q_1, q_3, q_2) forms a clockwise circuit, and a point q_4 is not in the quadrant determined by ℓ, then q_4 is to the *left* of ℓ. In other words, a point q_4 is to the left of ℓ if q_4 is to the left of the directed line q_1q_3 or q_4 is to the left of the directed line q_3q_2[1]. From the construction of the COCH(P), we develop an algorithm, which uses only this kind of orthogonal lines through two given points.

[1] c is left of a directed line ab iff $(x_b - x_a)(y_c - y_a) - (x_c - x_a)(y_b - y_a) > 0$.

Algorithm 2 FINDING THE CONNECTED ORTHOGONAL CONVEX HULL

Input. A set of finite distinct points P in the plane.
Output. List of extreme points of COCH(P) in order.
Find $a, b, c, d, e, f, g, h \in P$ satisfying (C) and P_{cb}, P_{ah}, P_{gf}, and P_{ed} satisfying (D).
Sort all points of $P_{ah} \cup P_{gf}$ in decreasing their y-coordinates. If two points have the same y-coordinate, the one having smaller x-coordinate is chosen.
Sort all points of $P_{ed} \cup P_{cb}$ in ascending their y-coordinates. If two points have the same y-coordinate, the one having bigger x-coordinate is chosen.
Label these points to $p_0, p_1, \ldots, p_m$.
PUSH(p_0, S)
PUSH(p_1, S)
PUSH(p_2, S)
for $i \leftarrow 3$ **to** m
    while p_i is left of the orthogonal line ℓ(NEXT-TO-TOP(S), TOP(S))
        POP(S)
    end while
    PUSH(p_i, S)
end for
return S

Theorem 3.1 *Algorithm 2 determines* COCH(P). *The time complexity is* $O(n \log n)$, *where* n *is the number of points of* P.

In the dissertation, an illustrative example for Algorithm 2 has been given.

3.3.2 Complexity

The lower bound of algorithms for finding the connected orthogonal convex hull can be proved similar to the lower bound of finding convex hulls (see Shamos, 1978).

Proposition 3.5 *Lower bound on computational complexity of algorithms for finding the connected orthogonal convex hull of a finite planar point set is the same as for sorting, it means* $O(n \log n)$.

Number of points	Time (s)
10	0.001
100	0.008
1000	0.08
10000	0.1
100000	38.91
1000000	8919.93

Table 3.1: Time (an average of 100 runs) required to compute the connected orthogonal convex hull of the set of n points with integer coordinates randomly positioned in the interior of a square of size 10000000 having sides parallel to the coordinate lines.

3.3.3 Implementation

Our algorithm was implemented in Python. Tests were run on a PC 3.20GHz with an intel Core i5 and 8 GB of memory. The actual run times of our algorithm on the set of a finite number of points which is uniformly random positioned in the interior of a square of size 10000000 having sides parallel to the coordinate lines are given in Table 3.1. In our experiments, the more points we add the less cases of semi-isolated point happen. In the code we consider the case $a = h = p$ as follows: we take the last point in the ordered points in Step 4 of Algorithm 2 to be the starting point p_0 and $a = h$ to be the second point p_1. Following the way of sorting all points of P we obtain p_0 as an extreme point of COCH(P).

General Conclusions

This dissertation has applied different tools from convex analysis, optimization theory, and computational geometry to study some constrained optimization problems in computational geometry.

The main results of the dissertation include:

- the existence and uniqueness of shortest paths along a sequence of line segments;
- conditions for concatenation of two shortest paths to be a shortest paths;
- straightest paths and longest straightest paths on a sequence of adjacent triangles;
- a property of connected orthogonal convex hulls; the construction of the connected orthogonal convex hull via extreme points;
- an efficient algorithm to find the connected orthogonal convex hull of a finite planar point set and an evaluating the time complexity for all algorithms which find the connected orthogonal convex hull of a finite planar point set.

List of Author's Related Papers

1. N. N. Hai, P. T. An and P. T. T. Huyen.: Shortest paths along a sequence of line segments in Euclidean spaces, *Journal of Convex Analysis*, vol. 26 (4), pp. 1089-1112 (2019)

2. P. T. An, P. T. T. Huyen and N. T. Le.: A modified Graham's convex hull algorithm for finding the connected orthogonal convex hull of a finite planar point set, *Applied Mathematics and Computation*, vol. 397, (2021). Article ID 125889

The results of this dissertation have been presented at

- The weekly seminar of the Department of Numerical Analysis and Scientific Computing, Institute of Mathematics, Vietnam Academy of Science and Technology;
- The 14th Workshop on "Optimization and Scientific Computing" (April 21-23, 2016, Ba Vi, Hanoi);
- The 18th Workshop on "Optimization and Scientific Computing" (August 20-22, 2020, Hoa Lac, Hanoi);
- The annual PhD Students Conferences, Institute of Mathematics, Vietnam Academy of Science and Technology (2016-2020, Hanoi).