VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY
INSTITUTE OF MATHEMATICS

PHONG THI THU HUYEN

SHORTEST PATHS ALONG A SEQUENCE OF LINE SEGMENTS AND CONNECTED ORTHOGONAL CONVEX HULLS

DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY IN MATHEMATICS

HANOI - 2022

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Supervisor: Associate Professor PHAN THANH AN

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Confirmation

This dissertation was written on the basis of my research works carried out at Institute of Mathematics, Vietnam Academy of Science and Technology, under the supervision of Associate Professor Phan Thanh An. All the presented results have never been published by others.

January 10, 2022
The author

Phong Thi Thu Huyen

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Contents

Table of Notations

(X, ρ)	a metric space X with metric ρ
$[t_0, t_1], t_0, t_1 \in \mathbb{R}$	an interval in $\mathbb{R}$
γ	a path
$l(\gamma)$	the length of a path γ
$(\mathbb{E}, \\|.\\|)$	an Euclidean space $\mathbb{E}$ with norm $\\|.\\|$
$e_1, e_2, \ldots, e_k$	a sequence of line segments in $\mathbb{E}$
$a, b, c, p, q, \ldots$	some points in spaces
$[p, q], p, q \in \mathbb{E}$	a line segment between two points p and q
x_a, y_a	two coordinates of a point $a = (x_a, y_a)$
$P(a, b)_{(e_1,\ldots,e_k)}$	a path joining a and b with respect to the sequence $e_1, \ldots, e_k$
$SP(a, b)_{(e_1,\ldots,e_k)}$	a shortest path joining a and b with respect to the sequence $e_1, \ldots, e_k$
$\gamma_1 * \gamma_2$	the concatenation of γ_1 and γ_2
$\sigma : t_0 = \tau_0 < \tau_1 < \cdots < \tau_n = t_1$	a set of partitions of $[t_0, t_1]$
$\triangle abc$	a triangle with three vertexes a, b, and c
$\ell(a, b)$	an orthogonal line through a and b in the sense of orthogonal convexity
$s(a, b)$	an orthogonal line segment through a and b in the sense of orthogonal convexity
$\mathcal{F}(K)$	the family of all connected orthogonal convex hulls of the set K
P	a planar finite point set
COCH(P)	the connected orthogonal convex hull of P
P_{ah}	the set of points in P in the quadrant of $\ell(a, b)$
$\mathcal{P}_{ah}$	a staircase path joining a and h
$T(P)$	an orthogonal convex (x, y)-polygon

o-ext(COCH)(P)	the set of extreme points of COCH(P)

List of Figures

Introduction

This dissertation studies shortest paths and straightest paths along a sequence of line segments in Euclidean spaces and connected orthogonal convex hulls of a finite planar point set. They are meaningful problems in computational geometry.

Finding shortest paths (joining two given points, from a source point to many destinations, from a point to a line segment, ...) in a geometric domain (such as on surface of a polytope, a terrain, inside a simple polygon, ...) is a classical geometric optimization problem and has many applications in different areas such as robotics, geographic information systems and navigation (see, for example, Agarwal et al. [2], Sethian [50]). To date, many algorithms have been proposed to solve: touring polygons problems (see, for example, Dror et al. [27], Ahadi et al. [3]), the shortest path problem on polyhedral surfaces (see, for example, Mitchell et al. [40], [37], Chen and Han [22], Varadarajan and Agarwal [58]), the weighted region problem (see, for example, Aleksandrov et al. [8]), the shortest descending path problem (see, for example, Ahmed et al. [4], [5], Cheng and Jin [24]), and the shortest gentle descending path problem (see, for example, Ahmed et al. [6], Bose et al. [19], Mitchell et al. [38], [39]). However, the problem of finding the shortest path joining two points in three dimensions in the presence of general polyhedral obstacles is known to be computationally difficult (see, for example, Bajaj [16], Canny and Reif [21]).

Canny and Reif [21] studied the lower bounds of some robot motion planning problems and a specific problem is the shortest path problem. The problem is to find the shortest path between two points pass through some polyhedral obstacles, such that this path doesn't go through inside an obstacle (this is also called single-source single-destination problem). They showed that the number of shortest paths might grow exponentially, and the single-source single-destination problem is NP-hard. However, in some specific cases, the shortest path problem can be solved in polynomial time (see, for example, Sharir and Shorr [52], Mitchell et al., [40], Chen and Han [22]). In some shortest path problems, exact and approximate solutions are computed based on solving a subproblem of finding the shortest path joining two given points along a sequence of adjacent triangles (the adjacent triangles on a polyhedral surface). Several algorithms need to concatenate of a shortest path from a sequence of adjacent triangles with a line segment on a new adjacent triangle (see, for example, Chen and Han [22], Cook [26], Balasubramanian et al. [17], Cheng and Jin [23], Pham-Trong et al. [47], Xin and Wang [59], Tran et al. [55]). Not many properties of shortest paths on polyhedral surfaces have been shown (see, for example, Sharir and Shorr [52], Mitchell et al., [40], Bajaj [16], Canney and Reif [21]). They usually suppose paths as polylines. The properties of shortest paths along sequences of adjacent triangles are even less than that (see Mitchell et al. [40]). But the question is even more basic "Does the shortest path joining two points along a sequence of adjacent triangles exist uniquely?". The uniqueness of such a path is assumed even in geodesical spaces and generalized segment spaces (see Hai and An [31]). Thus, this question is important not only in computational respect but also in theoretical respect. In this dissertation, we will consider the existence and uniqueness of such shortest paths. We show how a shortest path bends at an edge, and moreover how two shortest paths can "glue" together to form one shortest path.

In the dissertation, we consider the problem of finding the shortest path between two points along a sequence of adjacent triangles in a general setting. The sequence of triangles is replaced by a sequence of ordered line segments. The 3D space is replaced by a Euclidean space. Let a, b be two points in Euclidean space $\mathbb{E}$ and $e_1, e_2, ..., e_k$ be line segments in $\mathbb{E}$, finding the shortest path joining a and b with respect to $e_1, e_2, ..., e_k$ in that order, our considered problem can be stated as follows

$$\begin{aligned} &\min_{(x_0, x_1, \ldots, x_{k+1})} \sum_{i=0}^{k} \|x_i - x_{i+1}\| \\ &\text{subject to}\, x_i \in e_i, \text{for}\, i = 1, 2, \ldots, k \\ &x_0 = a, \\ &x_{k+1} = b. \end{aligned}$$

The shortest path problems can be seen in many various ways. In the view of geometry, we have the description of the shortest path joining two points and going through a sequence of adjacent triangles. In the view of optimization, we have the formula of the above problem. A different point of view gives a new idea to solve these problems. For example, let consider a classical problem in computational geometry: the convex hull problem. The convex hull problem states as follows: given a finite planar point set, find the convex hull of the set. The boundary of the convex hull of a finite planar point set can be seen as the shortest path around the given set (see Li and Klette [36]). Then, the shortest path joining two points inside a simple polygon P can be constructed by the union of some parts of boundaries of the convex hull of some vertexes of P (see An and Hoai [14]). Vice versa, the shortest path can be used to find a part of the boundary of the convex hull of a polyline (see An [10]). The shortest path problems in different contexts have different uses and meanings. In computer graphics and image processing, the convex sets on a computer screen are orthogonal convex (see, for example, Hearn, Baker and Carithers [32]). The shortest path around one digital shape now is the connected orthogonal convex hull of the shape. We again can understand the connected orthogonal convex hull problem is the shortest path problem in a computer screen. This is how we come with the shortest path problem in Chapter 3: finding the connected orthogonal convex hull of a finite planar point set.

The computation of the convex hull of a finite planar point set has been studied extensively. It is natural to relate convex hulls to orthogonal convex hulls. Orthogonal convexity is also known as rectilinear, or (x, y)-convexity. It has found applications in several research fields, including illumination (see, for example, Abello et al. [1]), polyhedron reconstruction (see, for example, Biedl and Genc [18]), geometric search (see, for example, Son et al. [53]), and VLSI circuit layout design (see, for example, Uchoa et al. [56]), digital images processing (see, for example, Seo et al. [49]). For example, in images processing and pattern recognition, we have an image and want to find out which class of shapes will it belong. They usually find the contour (the bounded polygon) of the shape in the image. Then the orthogonal convex hull of the polygons is computed. From some characteristics of the orthogonal convex hull, they put the given shape in some given classes of shapes.

The notation of orthogonal convexity has been widely studied since early eighties (see Unger [57]) and some of its optimization properties are given in (see González-Aguilar et al. [28]). However, unlike the convex hulls, finding the orthogonal convex hull of a finite planar point set is fraught with difficulties. An orthogonal convex hull of a finite planar point set may be disconnected. Unfortunately the connected orthogonal convex hulls of a finite planar point set might be not unique, even countless. There exist several algorithms to find the connected orthogonal convex hulls of a finite planar point set [34], [41], [42], and [45]. In previous works, the definition of an orthogonal convex set was used to find a connected orthogonal convex hull of a finite planar point set, and no numerical result has been shown. The second question is "What is the explicit form of a connected orthogonal convex hull?". To answer this question, firstly, we consider some assumption when the connected orthogonal convex hull of a finite planar point set is unique. Secondly, we introduce the concept of extreme points of the smallest connected orthogonal convex hull of a finite planar point set, and show that this hull of a finite planar point set is totally determined by its extreme points and these points belong to the given finite planar point set. There arises the third question "How to detect these extreme points from the given finite planar point set?"

Let us consider Graham's convex hull algorithm (see, Graham [29]) that depends on an initial ordering of the given finite planar point set. The initial point with the other points in this order actually forms a star-convex set. Based on this shape, Graham constructed Graham's convex hull algorithm. Some advantages of Graham's convex hull algorithm can be found in Allison and Noga [9]. In case of connected orthogonal convex hulls, if we have a reasonably ordered points, we then can scan these ordered points to detect the connected orthogonal convex hull from these points. As a result, an efficient algorithm to find the connected orthogonal convex hull of a finite planar point set which is based on the idea of Graham's convex hull algorithm [29] will be presented in the dissertation. As can been seen later in the dissertation, our new algorithm takes only $O(n \log n)$ time, where n is the number of given points (Theorem 3.1).

The dissertation consists of three chapters about constrained optimization problems: Chapter 1 and Chapter 2 discuss the shortest paths joining two points in a region (a polygon or the surface of a polytope) in Euclidean spaces, and Chapter 3 studies the connected orthogonal convex hull of a finite planar point set.

In Chapters 1 and 2, the problem of shortest paths joining two points with respect to a sequence of line segments is examined. We present some analytical and geometric properties of shortest paths joining two given points with respect to a sequence of line segments in a Euclidean space, especially their existence, uniqueness, characteristics, and conditions for concatenation of two shortest paths to be a shortest path are presented. We then focus on straightest paths lying on a sequence of adjacent polygons in 2 or 3 dimensional spaces.

In Chapter 3, the problem of finding the connected orthogonal convex hull of a finite planar point set is considered. We present the concept of extreme points of a connected orthogonal convex hull of the set, and show that these points belong to the set. Then we prove that the connected orthogonal convex hull of a finite set of points is an orthogonal (x, y)-polygon where its convex vertexes are extreme points of its connected orthogonal convex hull. An efficient algorithm, which is based on the idea of Graham's convex hull algorithm, for finding the connected orthogonal convex hull of a finite planar point set is presented. We also show that the lower bound of computational complexity of such algorithms is $O(n \log n)$, where n is the number of given points. Some numerical results for finding the connected orthogonal convex hulls of random sets are given.

The dissertation is written on the basis of 2 papers in the List of Author's Related Papers on page 72: Paper [1] published in *Applied Mathematics and Computation* , and paper [2] published in *Journal of Convex Analysis.*

The results of the dissertation have been presented at

- The weekly seminar of the Department of Numerical Analysis and Scientific Computing, Institute of Mathematics, Vietnam Academy of Science and Technology;

- The 14th Workshop on "Optimization and Scientific Computing" (April 21-23, 2016, Ba Vi, Hanoi);

- The 18th Workshop on "Optimization and Scientific Computing" (August 20-22, 2020, Hoa Lac, Hanoi);

- The annual PhD Students Conferences, Institute of Mathematics, Vietnam Academy of Science and Technology (2016-2020, Hanoi).

Chapter 0

Preliminaries

This dissertation deals with shortest paths (joining two given points, from a source point to many destinations, from a point to a line segment, ...) in a geometric domain (such as on surface of a polytope, inside a simple polygon, a terrain, ...) in Euclidean spaces. In this chapter, we represent some preliminaries which are based on the books of O'Rourke [43] and Papadopoulos [46].

0.1 Paths

Given a metric space (X, ρ). A *path* in X is a continuous mapping γ from an interval $[t_0, t_1] \subset \mathbb{R}$ to X. We say that γ *joins* the point $\gamma(t_0)$ with the point $\gamma(t_1)$. The *length* of $\gamma : [t_0, t_1] \to X$ is the quantity

$$l(\gamma) = \sup_{\sigma} \sum_{i=1}^{k} \rho\big(\gamma(\tau_{i-1}), \gamma(\tau_i)\big),$$

where the supremum is taken over the set of partitions $\sigma : t_0 = \tau_0 < \tau_1 < \cdots < \tau_k = t_1$ of $[t_0, t_1]$.

The length of a path is additive, i.e., for any path $\gamma : [t_0, t_1] \to X$ and $t_* \in [t_0, t_1]$,

$$l(\gamma) = l\big(\gamma_{|_{[t_0,t_*]}}\big) + l\big(\gamma_{|_{[t_*,t_1]}}\big),$$

where $\gamma_{|_{[t_0,t_*]}}$ and $\gamma_{|_{[t_*,t_1]}}$ are restrictions of γ on $[t_0, t_*]$ and $[t_*, t_1]$, respectively (see [46], pp. 11–24).

For instance, let $(X, \|\cdot\|)$ be a normed space. A mapping $\gamma : [t_0, t_1] \to X$ is said to be an *affine path* if for any $\lambda \in [0, 1]$,

$$\gamma((1-\lambda)t_0 + \lambda t_1) = (1-\lambda)\gamma(t_0) + \lambda\gamma(t_1).$$

For $x, y \in X$, $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$, a path $\gamma : [t_0, t_1] \to X$ joining x with y is affine iff

$$\gamma(t) = (t_1 - t_0)^{-1}\big[(t_1 - t)x + (t - t_0)y\big].$$

In this case, γ has length $\|x - y\|$ and the image $\gamma([t_0, t_1])$ is the *line segment*

$$[x, y] := \{(1 - \lambda)x + \lambda y : 0 \leq \lambda \leq 1\}.$$

We assume that all paths in the dissertation have finite length.

Let t_0, t_1 and t_2 be real numbers satisfying $t_0 \leq t_1 \leq t_2$. If $\gamma_1 : [t_0, t_1] \to X$ and $\gamma_2 : [t_1, t_2] \to X$ are two paths satisfying $\gamma_1(t_1) = \gamma_2(t_1)$, then we can define the path $\gamma : [t_0, t_2] \to X$ by setting

$$\begin{aligned}\gamma(t) =& \gamma_1(t) \text{ if } t_0 \leq t \leq t_1,\\ \gamma(t) =& \gamma_2(t) \text{ if } t_1 \leq t \leq t_2.\end{aligned}$$

The path γ is called the *concatenation* of γ_1 and γ_2, denoted by $\gamma_1 * \gamma_2$, and we have $l(\gamma) = l(\gamma_1) + l(\gamma_2)$.

Let $\gamma : [t_0, t_1] \to X$ and $\eta : [\tau_0, \tau_1] \to X$ be two paths in X. We say that γ *is obtained from* η *by a change of parameter* if there exists a function $\psi : [t_0, t_1] \to [\tau_0, \tau_1]$ that is monotonic (non-decreasing function), surjective and that satisfies $\gamma = \eta \circ \psi$ (γ is the composite function of η and ψ).

The function ψ is called *the change of parameter*. It is proved that the equality $l(\gamma) = l(\eta)$ always holds.

We say that $\gamma : [t_0, t_1] \to X$ is *parametrized by arclength* if for all τ and τ' satisfying $t_0 \leq \tau \leq \tau' \leq t_1$, we have $l\big(\gamma_{|_{[\tau,\tau']}}\big) = \tau' - \tau$.

If $\gamma : [t_0, t_1] \to X$ is any path, then there always exists a path $\lambda : [0, l(\gamma)] \to X$ such that λ is parametrized by arclength and γ is obtained from λ by the change of the parameter $\psi : [t_0, t_1] \to [0, l(\gamma)]$ defined by $\psi(\tau) = l\big(\gamma_{|_{[t_0,\tau]}}\big)$. Most of definitions and results above can be found in [46], pp. 11–24.

In the dissertation, we do not consider a general metric space but examine in Euclidean spaces. We denote by $(\mathbb{E}, \|\cdot\|)$ a non-trivial Euclidean space and the induced distance is $\rho(x, y) = \|x - y\|$. For $x, y \in \mathbb{E}$, denote

$$\begin{aligned}]x, y] =& [x, y] \setminus \{x\},\\ [x, y[=& [x, y] \setminus \{y\}, \text{and}\\ ]x, y[=& [x, y] \setminus \{x, y\}.\end{aligned}$$

Note that when $x = y$,

$$[x, y] = \{x\},$$
$$[x, y[=]x, y] =]x, y[= \emptyset.$$

If $x \neq y$, each point $z \in]x, y[$ is called a (relative) *interior point* of $[x, y]$. By abuse of notation, sometimes we also call the image $\gamma([t_0, t_1])$ the path $\gamma : [t_0, t_1] \to \mathbb{E}$. For practical purposes, $\mathbb{E}$ is usually chosen to be the finite dimensional space $\mathbb{R}^d$ with $d = 2$ or $d = 3$.

In the problem of finding shortest paths on a polyhedral surface, a key geometric concept is the notion of *planar unfolding* around a sequence of adjacent convex polygons $\mathcal{S} = (f_1, f_2, \ldots, f_{k+1})$ (see [40], p. 649). Let $\mathcal{S} = (f_1, f_2, \ldots, f_{k+1})$ be a *sequence of (not necessary distinct) adjacent convex polygons* in $\mathbb{R}^2$ or $\mathbb{R}^3$, then $e_i := f_i \cap f_{i+1}$ is an edge for $1 \leq i \leq k$. We unfold the sequence $\mathcal{S}$ as follow: Rotate f_1 around e_1 until its plane coincides with that of f_2, rotate f_1 and f_2 around e_2 until their plane coincides with that of f_3, continue this way until all faces $f_1, f_2, \ldots, f_k$ lie on the plane of f_{k+1}.

0.2 Graham's Convex Hull Algorithm

First, we recall some useful definitions in computational geometry.

A *simple polygon* (see [35], p. 164) is an n-sided figure consisting of n segments

$$\overline{p_1p_2}, \overline{p_2p_3}, \overline{p_3p_4}, \ldots, \overline{p_{n-1}p_n}, \overline{p_np_1},$$

which intersect only at their endpoints and enclose a single region.

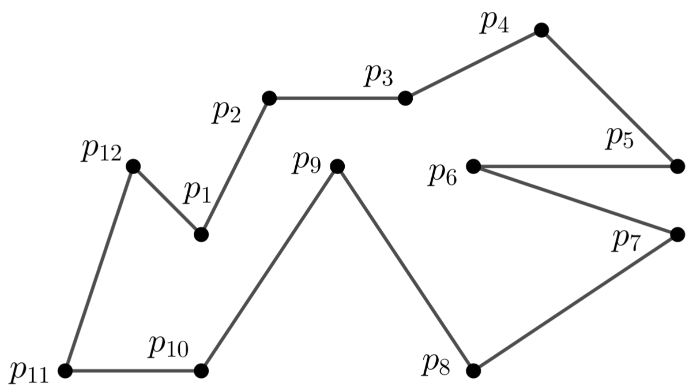


Figure 0.1: Illustration of a simple polygon $p_1p_2 \ldots p_{12}$

The *perimeter* of a (simple) polygon (see [35], p. 169) is defined to be the sum of the lengths of its sides.

A set $K \subset \mathbb{R}^d$ is *convex* (see [30], p. 8) if and only if for each pair of distinct points $a, b \in K$ the closed segment with endpoints a and b is contained in K. The *convex hull* of a set K is the smallest convex set that contains K.

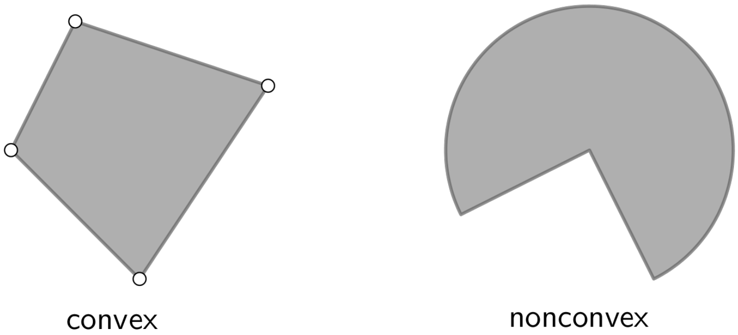


Figure 0.2: Illustration of a convex set, its extreme points (white spots), and a nonconvex set

Let K be a convex subset of $\mathbb{R}^d$. A point $x \in K$ is an *extreme point* (see [30], p. 17) of K provided $y, z \in K, 0 < \lambda < 1$, and $x = \lambda y + (1 - \lambda)z$ imply $x = y = z$. The set of all extreme points of K is denoted by ext$K$.

A compact convex set $K \subset \mathbb{R}^d$ is a *convex polytope* provided ext$K$ is a finite set.

A finite union of convex polytopes such that its space is connected is called a *polytope* (see [12], p.18). For a polytope (or polyhedral set) K, it is customary to call the points of ext$K$ *vertexes*.

Graham's convex hull algorithm is a sequential algorithm for finding the convex hull of a planar finite point set. This algorithm was presented in 1972 by R. Graham [29] in Bell Laboratories. It is easy to understand and to program on computer but not able to apply to finding convex hulls of finite point set in 3D. Here, we represent this algorithm which is based on [43].

Finding the convex hull of a finite point set in the plane is a fundamental problem in Computational Geometry. It can be used as an illustrated example for understanding the issues that arises in solving geometric problem by computer.

The convex hull problem in Computational Geometry is stated as follows:

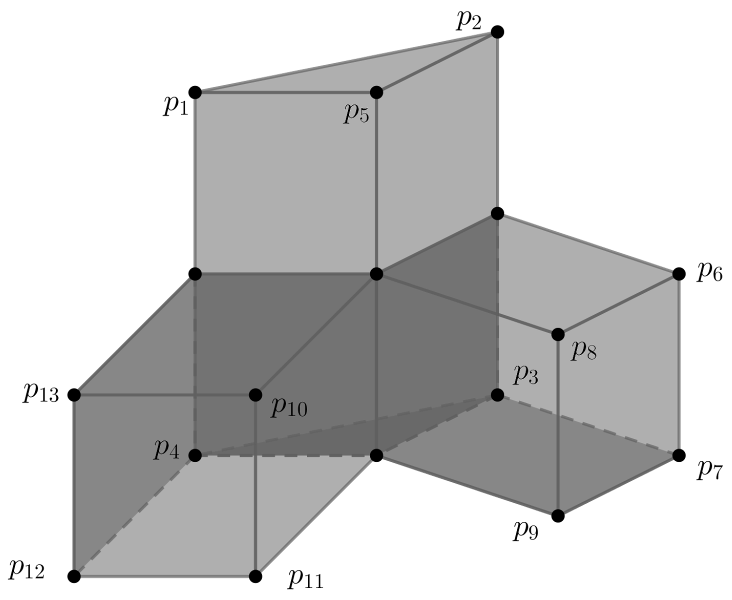


Figure 0.3: A polytope (in Computational Geometry) and its vertexes $\{p_1, p_2, \ldots, p_{13}\}$

given a finite point set P in the plane, find the convex hull of P. This problem can be seen as an optimization problem: given a finite point set P in the plane, find the shortest path around all points of P.

How to compute the convex hull? Or what does it mean to compute the convex hull? As we know, the convex hull of a finite point set in the plane is a convex polygon. A convex polygon can be represented by an ordered list of its vertexes, starting at arbitrary one. So the convex hull problem can be restated as follows: given a finite point set $P = \{p_1, p_2, \ldots, p_n\}$ in the plane, compute an ordered list of vertexes of the convex hull $\mathrm{conv}(P)$ of P, listed in counterclockwise order.

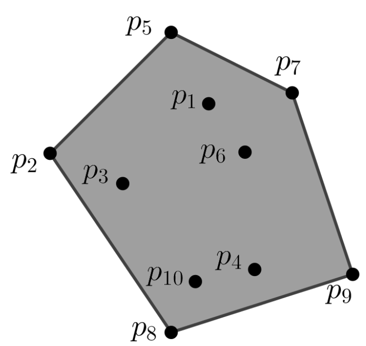


Figure 0.4: The convex hull of $\{p_1, p_2, \ldots, p_{10}\}$ is the convex polygon $p_2p_8p_9p_7p_5$.

The idea of Graham's convex hull algorithm is simple. First we sort all points of P in counterclockwise order by its angle from an arbitrary point of P. As we can see, on the boundary of the convex hull of P (the convex polygon) if we traverse through each vertexes (in counterclockwise order), we always turn left. Therefore, we scan the ordered list of points of P such that, each time turn right, we delete the processing point. The remaining points are the set of all extreme points of conv(P). We illustrate the above idea in the following pseudocode, which is based on [43], p. 74.

Algorithm 1 GRAHAM'S CONVEX HULL ALGORITHM

Input. A set of finite points P in the plane.

Output. List of extreme points of conv(P) in order.

Let p_0 be the point in P with the minimum y-coordinate, or the leftmost such point in case of a tie.

Let $< p_1, p_2, \dots, p_m >$ be the remaining points in P, sorted by polar angle in counterclockwise order around p_0 (if more than one point has the same angle, remove all but the one that is farthest from p_0).

Stack $S = \emptyset$

PUSH(p_0, S)

PUSH(p_1, S)

PUSH(p_2, S)

for $i \leftarrow 3$ **to** m

    while the angle formed by points NEXT-TO-TOP(S), TOP(S), and p_i makes nonleft turn

        POP(S)

    end while

    PUSH(p_i, S)

end for

return S

In Algorithm 1, we use "stack" to store the points in processing. In computer science, a *stack* is an abstract data type that is used as a set of elements, with two main principal operations:

- *push*, which adds an element to the set, and
- *pop*, which removes the most recently added element that was not yet removed.

Push and pop in stack can be understood as "last in, first out" (see Figure 0.5).

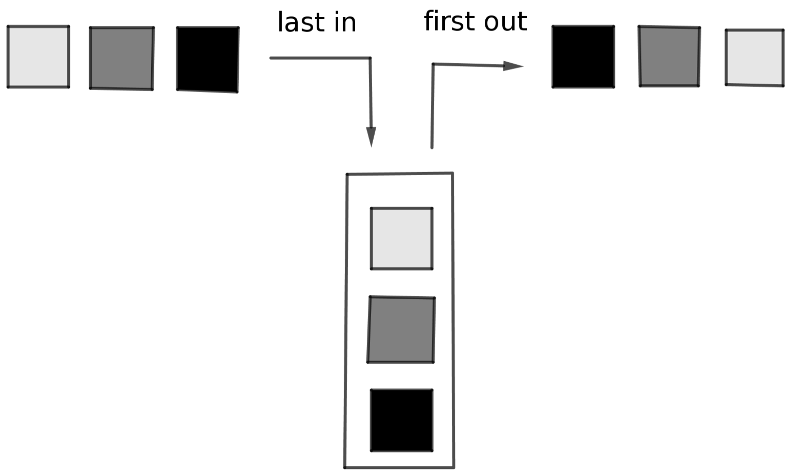


Figure 0.5: Illustration of a stack

We illustrate an example of Graham's convex hull algorithm here. List of given points is: $P = \{(32, 28), (34, 8), (26, 5), (15, 9), (34, 20), (13, 13), (30, 12), (29, 24), (19, 14), (20, 27)\}$. The first point of stack p_0 is picked by the leftmost lowest point of P. The ordered list of vertexes of $\text{conv}(P)$ is: $\{(26, 5), (34, 8), (34, 20), (32, 28), (20, 27), (13, 13), (15, 9)\}$. The convex hull of P is the convex polygon $abcdefg$ in Figure 0.6.

Theorem 0.1 (See [29], p. 132)**.** *The convex hull of n points in the plane can be found by Graham's convex hull algorithm in $O(n \log n)$ for sorting and no more than $2n$ iterations for the scan.*

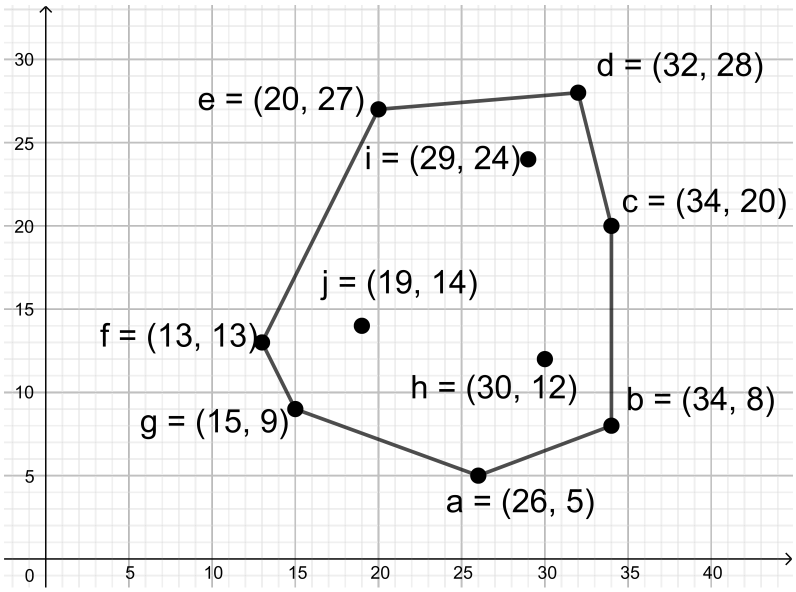


Figure 0.6: The convex hull of P is the convex polygon $abcdefg$.

Chapter 1

Shortest Paths with respect to a Sequence of Line Segments in Euclidean Spaces

In this chapter, we consider the following problem

$$\begin{aligned} &\min_{(x_0,x_1,\ldots,x_{k+1})} \sum_{i=0}^{i=k} \|x_i - x_{i+1}\| \\ &\text{such that}\, x_i \in e_i, \forall i = 1, 2, \ldots, k \\ &x_0 = a, \\ &x_{k+1} = b \end{aligned}$$

where a, b be two points in Euclidean space $\mathbb{E}$ and $e_1, e_2, \ldots, e_k$ be line segments in $\mathbb{E}$.

The present chapter is written on the basis of the paper [2] in the List of Author's Related Papers on page 72 of this dissertation.

1.1 Shortest Paths with respect to a Sequence of Ordered Line Segments

Definition 1.1. Let a, b be points in $\mathbb{E}$ and let $e_1, \ldots, e_k$ be a sequence of line segments (these line segments are not necessarily distinct and some of them may be singletons). A continuous map $\gamma : [t_0, t_1] \to \mathbb{E}$ that joins a with

b is called a *path* with respect to the sequence $e_1, \dots, e_k$ if there is a sequence of numbers $t_0 \le \bar{t}_1 \le \dots \le \bar{t}_k \le t_1$ such that $\gamma(\bar{t}_i) \in e_i$ for $i = 1, \dots, k$ (see Figure 1.1). The path γ is called a *shortest path* if its length does not exceed the length of any path.

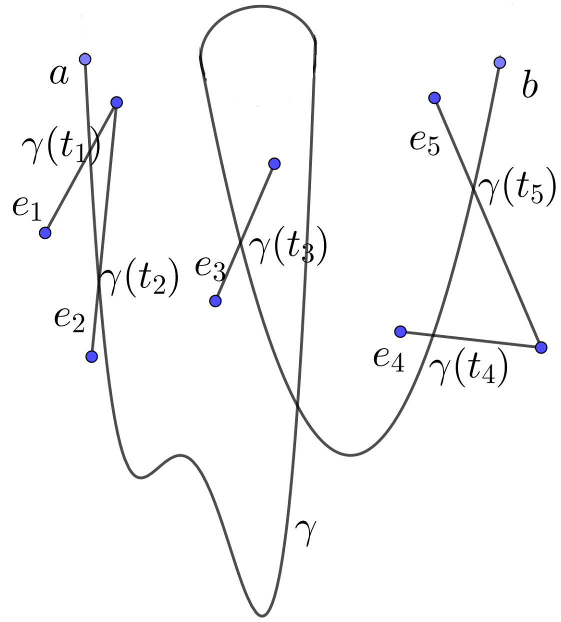


Figure 1.1: A path that joins a with b with respect to e_1, e_2, e_3, e_4 and e_5.

If $\gamma : [t_0, t_1] \to \mathbb{E}$ joins a with b and is a path with respect to the sequence $e_1, \dots, e_k$, we call it a $P(a, b)_{(e_1,\dots,e_k)}$. If we want to emphasize the intersection points $\gamma(\bar{t}_i)$ of γ and e_i, $t_0 \le \bar{t}_1 \le \dots \le \bar{t}_k \le t_1$, we say γ is a $P(a, b)_{(e_1,\dots,e_k)}$ corresponding to $\gamma(\bar{t}_i) \in e_i$, $1 \le i \le k$. If, in addition, γ is a shortest path, then we say that it is an $SP(a, b)_{(e_1,\dots,e_k)}$. In a concrete case, we only say shortest paths without specifying the starting and ending points, as well as the set of line segments.

We assume for convenience that $e_i \ne e_{i+1}$ for all $i = 1, \dots, k-1$. We also use the notation $SP(a, b)_\emptyset$ to denote any path that joins a with b, one-to-one, and has length $\|a-b\|$ and image $[a, b]$. Usually $SP(a, b)_\emptyset$ is assumed to be an affine path joining a with b. The aim of this section is to prove that shortest path joining two given points and with respect to a given sequence of line segments exists and in some sense is unique.

We start with an intuitive property of paths.

Lemma 1.1. *Let $t_0 < t_1$ and let $\gamma : [t_0, t_1] \to \mathbb{E}$ be any path.*

(a) $l(\gamma) \ge \|\gamma(t_0) - \gamma(t_1)\|$ and equality holds only if $\gamma([t_0, t_1]) = [\gamma(t_0), \gamma(t_1)]$.

(b) If γ is parametrized by arclength and $l(\gamma) = \|\gamma(t_0) - \gamma(t_1)\| > 0$, then γ is affine, that is,

$$\gamma(t) = \frac{t_1 - t}{t_1 - t_0}\gamma(t_0) + \frac{t - t_0}{t_1 - t_0}\gamma(t_1), \quad t \in [t_0, t_1].$$

Proof. Set $a = \gamma(t_0)$ and $b = \gamma(t_1)$.

(a) Choosing the partition $\{t_0, t_1\}$ of $[t_0, t_1]$ gives

$$l(\gamma) \geq \|\gamma(t_0) - \gamma(t_1)\| = \|a - b\|.$$

Suppose that $l(\gamma) = \|a - b\|$. If there were $x = \gamma(\tau) \in \gamma([t_0, t_1])$ such that $x \notin [a, b]$, then we would have

$$l(\gamma) \geq \|\gamma(t_0) - \gamma(\tau)\| + \|\gamma(\tau) - \gamma(t_1)\| = \|a - x\| + \|x - b\| > \|a - b\|,$$

a contradiction. Thus $\gamma([t_0, t_1]) \subset [a, b]$. Moreover, since $\gamma([t_0, t_1])$ is connected, $a = \gamma(t_0)$ and $b = \gamma(t_1)$, it follows that $\gamma([t_0, t_1]) = [a, b]$.

(b) First observe that $t_1 - t_0 = l(\gamma) = \|a - b\| > 0$ and by (a) $\gamma([t_0, t_1]) = [a, b]$. Fix $t_0 < t < t_1$ and suppose

$$t = (1 - \lambda)t_0 + \lambda t_1 \text{ and } \gamma(t) = (1 - \mu)a + \mu b, \lambda, \mu \in [0, 1].$$

Since γ is parametrized by arclength

$$l(\gamma_{|[t_0,t]}) = t - t_0 \geq \|\gamma(t) - a\| \text{ and } l(\gamma_{|[t,t_1]}) = t_1 - t \geq \|b - \gamma(t)\|.$$

Adding these inequalities gives

$$t_1 - t_0 \geq \|\gamma(t) - a\| + \|b - \gamma(t)\| \geq \|b - a\| = t_1 - t_0.$$

Hence $t - t_0 = \|\gamma(t) - a\|$, that is $\lambda(t_1 - t_0) = \mu\|b - a\| = \mu(t_1 - t_0)$, whence $\mu = \lambda = (t - t_0)/(t_1 - t_0)$. □

If $\gamma([t_0, t_1])$ is a path which joins $a = \gamma(t_0)$ with $b = \gamma(t_1)$, and $c \in \mathbb{E}$, for abbreviation, we denote by $\gamma * [b, c]$ the concatenation of γ and any one-to-one shortest path $\xi : [t_1, t_2] \to \mathbb{E}$ joining b and c. (If $b = c$, $\gamma * [b, c] = \gamma$.) The notation $[c, a] * \gamma$ is defined similarly. Clearly

$$l(\gamma * [b, c]) = l(\gamma) + \|c - b\|$$

and

$$l([c, a] * \gamma) = l(\gamma) + \|c - a\|.$$

Likewise, if $\gamma_1([t_0, t_1])$, $\gamma_2([t'_1, t_2])$ are paths which join a with b and c with d, respectively, and $t_1 < t'_1$, then $\gamma_1 * [b, c] * \gamma_2$ denotes the concatenation

$\gamma_1 * \xi * \gamma_2$, where ξ is defined on $[t_1, t_1']$, one-to-one, and is a shortest path joining b with c. In these notations, ξ is usually chosen to be affine.

In studying properties of shortest paths joining two points with respect to a sequence of line segments, the following simple result is essential for others.

Proposition 1.1. *Let $\gamma([t_0, t_1])$ be an $SP(a,b)_{(e_1,\ldots,e_k)}$ corresponding to $\bar{t}_0 = t_0, \bar{t}_{k+1} = t_1$, and $\gamma(\bar{t}_i) \in e_i$, $i = 1, \ldots, k$.*

(a) If $\bar{t}_{j-1} \le \alpha < \beta \le \bar{t}_j$, then $l(\gamma_{|[\alpha,\beta]}) = \|\gamma(\alpha) - \gamma(\beta)\|$ and $\gamma([\alpha, \beta]) = [\gamma(\alpha), \gamma(\beta)]$. If, in addition, γ is parametrized by arclength, then γ is affine on $[\alpha, \beta]$.

(b) $\gamma([t_0, t_1]) = \cup_{i=0}^{k} [\gamma(\bar{t}_i), \gamma(\bar{t}_{i+1})]$. Thus $\gamma([t_0, t_1])$ is a polyline.

Proof. Let $x := \gamma(\alpha)$, $y := \gamma(\beta)$. If $l(\gamma_{|[\alpha,\beta]}) > \|x - y\|$, then the path

$$\eta = \gamma_{|[t_0,\alpha]} * [x, y] * \gamma_{|[\beta,t_1]}$$

would be a $P(a,b)_{(e_1,\ldots,e_k)}$ and

$$l(\eta) = l(\gamma_{|[t_0,\alpha]}) + \|x - y\| + l(\gamma_{|[\beta,t_1]}) < l(\gamma),$$

a contradiction. Thus $l(\gamma_{|[\alpha,\beta]}) = \|x-y\|$ and hence, by Lemma 1.1, $\gamma([\alpha, \beta]) = [x, y]$. In particular $\gamma([\bar{t}_i, \bar{t}_{i+1}]) = [\gamma(\bar{t}_i), \gamma(\bar{t}_{i+1})]$ for $0 \le i \le k$. Therefore

$$\gamma([t_0, t_1]) = \cup_{i=0}^{k} \gamma([\bar{t}_i, \bar{t}_{i+1}]) = \cup_{i=0}^{k} [\gamma(\bar{t}_i), \gamma(\bar{t}_{i+1})],$$

which is a polyline. Finally, if γ is parametrized by arclength, Lemma 1.1 says that γ is affine on $[\alpha, \beta]$. □

Remark 1.1. Let $f_1, \ldots, f_{k+1}$ be a sequence of convex polygons (these polygons are not necessarily distinct and f_i and f_{i+1} may be identical), e_i a common edge of f_i and f_{i+1} and $a \in f_1$, $b \in f_{k+1}$ ($k \ge 1$). If $\gamma([t_0, t_1])$ is an $SP(a,b)_{(e_1,\ldots,e_k)}$, then by Proposition 1.1, $\gamma([t_0, t_1]) = \cup_{i=0}^{k} [\gamma(\bar{t}_i), \gamma(\bar{t}_{i+1})] \subset \cup_{i=1}^{k+1} f_i$, i.e., this shortest path lies on the polygons. Conversely, if $\gamma : [t_0, t_1] \to \cup_{i=1}^{k+1} f_i$ is a $P(a,b)_{(e_1,\ldots,e_k)}$ and has the minimum length in the family of all $P(a,b)_{(e_1,\ldots,e_k)}$ that lie totally on the polygons f_i, then $\gamma([t_0, t_1])$ is an $SP(a,b)_{(e_1,\ldots,e_k)}$.

Thus determining a shortest path on the sequence of convex polygons $f_1, \ldots, f_{k+1}$ in $\mathbb{R}^3$ (or in $\mathbb{R}^2$) is equivalent to determining an $SP(a,b)_{(e_1,\ldots,e_k)}$. This is a reason why we investigate shortest paths with respect to a sequence of line segments. The approach allows us to study paths even in cases where the unfolded images of a sequence of adjacent convex polygons on a plane overlap.

The following lemma says that restriction of a shortest path with respect to a sequence of line segments on each subinterval of its domain is again a shortest path with respect to a sequence of line segments. It can be seen as a "local property" of shortest paths with respect to a sequence of line segments.

Lemma 1.2. *Let $\gamma([t_0, t_1])$ be an $SP(a, b)_{(e_1,\dots,e_k)}$ and $t_0 =: \bar{t}_0 \le \bar{t}_1 \le \cdots \le \bar{t}_k \le \bar{t}_{k+1} := t_1$, $\gamma(\bar{t}_i) \in e_i$ for $i = 1, \dots, k$. If $1 \le m \le n \le k$ and $\bar{t}_{m-1} \le \alpha \le \bar{t}_m \le \bar{t}_n \le \beta \le \bar{t}_{n+1}$, then $\gamma_{|[\alpha,\beta]}$ is an $SP(\gamma(\alpha), \gamma(\beta))_{(e_m,\dots,e_n)}$.*

Proof. Let $x := \gamma(\alpha)$ and $y := \gamma(\beta)$. Clearly $\gamma_{|[\alpha,\beta]}$ is a $P(x, y)_{(e_m,\dots,e_n)}$. If $\gamma_{|[\alpha,\beta]}$ is not shortest, there exists a path $\eta : [\alpha, \beta] \to \mathbb{E}$ that is a $P(x, y)_{(e_m,\dots,e_n)}$ with $l(\eta) < l(\gamma_{|[\alpha,\beta]})$. Then $\xi := \gamma_{|[t_0,\alpha]} * \eta * \gamma_{|[\beta,t_1]}$ is a $P(a, b)_{(e_1,\dots,e_k)}$ but $l(\xi) < l(\gamma)$, a contradiction. □

Lemma 1.3. *Suppose $\gamma([t_0, t_1])$ is an $SP(a, b)_{(e_1,\dots,e_k)}$ and $t_0 =: \bar{t}_0 \le \bar{t}_1 \le \cdots \le \bar{t}_k \le \bar{t}_{k+1} := t_1$, $\gamma(\bar{t}_i) \in e_i$ for $i = 1, \dots, k$. If $l(\gamma_{|[t,t']}) > 0$ for all $t, t' \in [t_0, t_1]$ and $t < t'$, then γ is one-to-one on each subinterval $[\bar{t}_i, \bar{t}_{i+1}]$, $0 \le i \le k$.*

Proof. Suppose conversely that γ is not one-to-one on $[\bar{t}_j, \bar{t}_{j+1}]$, i.e., there exist

$$\bar{t}_j \le \alpha < \beta \le \bar{t}_{j+1} \text{ with } \gamma(\alpha) = \gamma(\beta).$$

Set $\delta := \beta - \alpha > 0$. Define $\eta : [t_0, t_1 - \delta] \to \mathbb{E}$ by

$$\eta(t) = \gamma(t) \text{ if } t_0 \le t \le \alpha$$
$$\text{and } \eta(t) = \gamma(t + \delta) \text{ if } \alpha < t \le t_1 - \delta.$$

Then η is a $P(a, b)_{(e_1,\dots,e_k)}$ and since $l(\gamma_{|[\alpha,\beta]}) > 0$,

$$l(\eta) = l(\eta_{|[t_0,\alpha]}) + l(\eta_{|[\alpha,t_1-\delta]}) = l(\gamma_{|[t_0,\alpha]}) + l(\gamma_{|[\beta,t_1]}) = l(\gamma) - l(\gamma_{|[\alpha,\beta]}) < l(\gamma).$$

This contradicts the fact that γ is an $SP(a, b)_{(e_1,\dots,e_k)}$. Thus

$$\gamma \text{ is one-to-one on each } [\bar{t}_i, \bar{t}_{i+1}].$$

□

Sometimes we make the following assumption.

(A) *The restriction of the path on each non-singleton interval of its domain has positive length.*

Assumption (A) means that the path is not constant on any interval with positive length in its domain. It is satisfied, for instance, if the path is parametrized by arclength or if it is one-to-one.

Example. Here we give an example of a function which does not satisfy the assumption (A). Let $\gamma : [0, 10\pi] \to \mathbb{R}^3$,

$$\gamma(t) = \begin{cases} (\cos t, \sin t, t) & \text{if } 0 \leq t \leq 2\pi, \\ (1, 0, 2\pi) & \text{if } 2\pi \leq t \leq 4\pi, \\ (\cos t, \sin t, t) & \text{if } 4\pi \leq t \leq 10\pi. \end{cases}$$

Then $l\big(\gamma([2\pi, 4\pi])\big) = 0$ since in this interval $\gamma =$ constant.

We now turn our attention to the existence and uniqueness of a shortest path joining two points with respect to a sequence of line segments.

Lemma 1.4. *Let $\mathcal{E} = (e_1, \ldots, e_k)$, $\gamma([t_0, t_1])$ and $\eta([\tau_0, \tau_1])$ be $P(a, b)_{\mathcal{E}}$ corresponding to $x_i = \gamma(\bar{t}_i)$ and $y_i = \eta(\bar{\tau}_i)$, $1 \leq i \leq k$, respectively.*

(a) If $a \neq x_1$, $a \neq y_1$ and the angle between $x_1 - a$ and $y_1 - a$ is not zero, γ and η are not both shortest.

(b) If $b \neq x_k$, $b \neq y_k$ and the angle between $x_k - b$ and $y_k - b$ is not zero, γ and η are not both shortest.

Proof. We prove for case (a), similar arguments apply to case (b). Suppose on the contrary that γ and η are both $SP(a, b)_{\mathcal{E}}$ and $l(\gamma) = l(\eta) = \sigma$. Let $x_0 = y_0 := a$, $x_{k+1} = y_{k+1} := b$ and

$$z_i := (x_i + y_i)/2, 0 \leq i \leq k + 1,$$

so $z_0 = a$ and $z_{k+1} = b$. Let $\bar{\alpha}_0 := 0$, $\bar{\alpha}_1 := \|z_1 - z_0\|$, $\bar{\alpha}_2 := \bar{\alpha}_1 + \|z_2 - z_1\|, \ldots,$ $\bar{\alpha}_{k+1} := \bar{\alpha}_k + \|z_{k+1} - z_k\|$. Define $\varphi : [0, \bar{\alpha}_{k+1}] \to \mathbb{E}$ by

$$\varphi(\bar{\alpha}_i) = z_i \text{ for } i = 0, \ldots, k + 1,$$

and φ is affine on each subinterval $[\bar{\alpha}_i, \bar{\alpha}_{i+1}]$. Then φ is a $P(a, b)_{\mathcal{E}}$ and

$$l(\varphi) = \sum_{i=0}^{k} l\big(\varphi_{|[\bar{\alpha}_i, \bar{\alpha}_{i+1}]}\big) = \sum_{i=0}^{k} \|z_i - z_{i+1}\|. \tag{1.1}$$

Likewise, according to linearity of length and Lemma 1.1, we have

$$l(\gamma) = \sum_{i=0}^{k} \|x_i - x_{i+1}\| \quad \text{and} \quad l(\eta) = \sum_{i=0}^{k} \|y_i - y_{i+1}\|. \tag{1.2}$$

Since the angle between $x_1 - a$ and $y_1 - a$ is not zero

$$2\|a - z_1\| < \|a - x_1\| + \|a - y_1\|. \tag{1.3}$$

Moreover, for $1 \leq i \leq k$, we have

$$2\|z_i - z_{i+1}\| = \|(x_i + y_i) - (x_{i+1} + y_{i+1})\| \leq \|x_i - x_{i+1}\| + \|y_i - y_{i+1}\|. \tag{1.4}$$

Using (1.1)–(1.4) we get

$$\begin{aligned} 2l(\varphi) &= 2\sum_{i=0}^{k} \|z_i - z_{i+1}\| < \|a - x_1\| + \|a - y_1\| + 2\sum_{i=1}^{k} \|z_i - z_{i+1}\| \\ &\leq \|a - x_1\| + \|a - y_1\| + \sum_{i=1}^{k} \|x_i - x_{i+1}\| + \sum_{i=1}^{k} \|y_i - y_{i+1}\| \\ &= \sum_{i=0}^{k} \|x_i - x_{i+1}\| + \sum_{i=0}^{k} \|y_i - y_{i+1}\| = l(\gamma) + l(\eta) = 2\sigma, \end{aligned}$$

implying $l(\varphi) < \sigma$. This is impossible. Therefore γ and η are not both shortest. □

Suppose that $\gamma([t_0, t_1])$ and $\eta([\tau_0, \tau_1])$ are $P(a, b)_{\mathcal{E}}$. We say that γ *equals* η if at least one path can be obtained from the other by a strictly increasing change of parameter. Clearly, if γ equals η, then $l(\gamma) = l(\eta)$. If $t_0 < t_1$, $\tau_0 < \tau_1$, and $\gamma([t_0, t_1])$ is a $P(a, b)_{\mathcal{E}}$, then there always exists a $P(a, b)_{\mathcal{E}}$ defined on $[\tau_0, \tau_1]$ that is equal to γ, say $\eta = \gamma \circ \varphi$, where

$$\varphi : [\tau_0, \tau_1] \to [t_0, t_1]$$

is defined by

$$\varphi(\tau) = (\tau_1 - \tau_0)^{-1}(\tau_1 - \tau)t_0 + (\tau_1 - \tau_0)^{-1}(\tau - \tau_0)t_1.$$

This equality relation is in fact an equivalence relation on the family of $P(a, b)_{\mathcal{E}}$. Whenever we identify ordered paths this way, the shortest path joining two points with respect to a given sequence of line segments is unique. This is stated in the following theorem which is also the main result in this section.

Theorem 1.1. *Let $a, b \in \mathbb{E}$ and let $e_1, \dots, e_k$ be a sequence of line segments (these line segments are not necessarily distinct and some of them may be singletons). There exists a shortest path joining a with b with respect to the sequence $e_1, \dots, e_k$. Moreover, this shortest path is unique in the family of paths with respect to a sequence of line segments satisfying assumption* (A).

Proof. *Existence.* Denote

$$\mathcal{E} = (e_1, \dots, e_k), K := e_1 \times \cdots \times e_k \subset \mathbb{E}^k, x_0 := a, \text{and}\, x_{k+1} := b.$$

Consider the function

$$\Phi : \mathbb{E}^k \to \mathbb{R},$$
$$\Phi(x_1, \dots, x_k) = \sum_{i=0}^{k} \|x_i - x_{i+1}\|.$$

As Φ is continuous and K is compact, there exists $(x_1^*, \dots, x_k^*) \in K$ such that $\Phi(x_1^*, \dots, x_k^*) = \sigma := \min_K \Phi$. Let

$$x_0^* := a, \quad x_{k+1}^* := b, \quad \bar{t}_0 := 0, \quad \bar{t}_{i+1} := \bar{t}_i + \|x_i^* - x_{i+1}^*\|, \quad 0 \le i \le k.$$

Clearly $\bar{t}_{k+1} = \sum_{i=0}^{k} \|x_i^* - x_{i+1}^*\| = \sigma$. Consider the mapping $\gamma_0 : [0, \sigma] \to \mathbb{E}$ defined by

$$\gamma_0(\bar{t}_i) = x_i^*, i = 0, \dots, k+1,$$

and γ_0 is affine on each subinterval $[\bar{t}_i, \bar{t}_{i+1}]$. Then γ_0 is a $P(a, b)_{\mathcal{E}}$ and

$$l(\gamma_0) = \sum_{i=0}^{k} l(\gamma_{0|[\bar{t}_i, \bar{t}_{i+1}]}) = \sum_{i=0}^{k} \|x_i^* - x_{i+1}^*\| = \Phi(x_1^*, \dots, x_k^*) = \sigma.$$

Suppose now that $\tilde{\gamma} : [t_0, t_1] \to \mathbb{E}$ is a $P(a, b)_{\mathcal{E}}$ corresponding to

$$\tilde{x}_i = \tilde{\gamma}(\tilde{t}_i) \in e_i, 1 \le i \le k.$$

Set $\tilde{x}_0 := a, \tilde{x}_{k+1} := b, \tilde{t}_0 := t_0, \text{and}\, \tilde{t}_{k+1} := t_1$. Then, by Lemma 1.1,

$$l(\tilde{\gamma}) = \sum_{i=0}^{k} l(\tilde{\gamma}_{|[\tilde{t}_i, \tilde{t}_{i+1}]}) \ge \sum_{i=0}^{k} \|\tilde{x}_i - \tilde{x}_{i+1}\| = \Phi(\tilde{x}_1, \dots, \tilde{x}_k) \ge \Phi(x_1^*, \dots, x_k^*) = l(\gamma_0).$$

Therefore γ_0 is an $SP(a, b)_{\mathcal{E}}$. Observe further that γ_0 is parametrized by arclength.

Uniqueness. Suppose that $\eta([\tau_0, \tau_1])$ is an $SP(a, b)_{\mathcal{E}}$ corresponding to

$$y_i = \eta(\bar{\tau}_i) \in e_i, 1 \le i \le k.$$

Set $\tau_0 =: \bar{\tau}_0, \bar{\tau}_{k+1} := \tau_1, y_0 := \eta(\bar{\tau}_0) = a,$ and $y_{k+1} := \eta(\bar{\tau}_{k+1}) = b$.

We first consider the case when η is parametrized by arclength and $\tau_0 = 0$. Then $\tau_1 = \tau_1 - \tau_0 = l(\eta) = \sigma$, so $[\tau_0, \tau_1] = [0, \sigma]$.

Suppose that the set $I = \{i : y_i \neq x_i^*\}$ is nonempty and $m := \min I$ (see Figure 1.2).

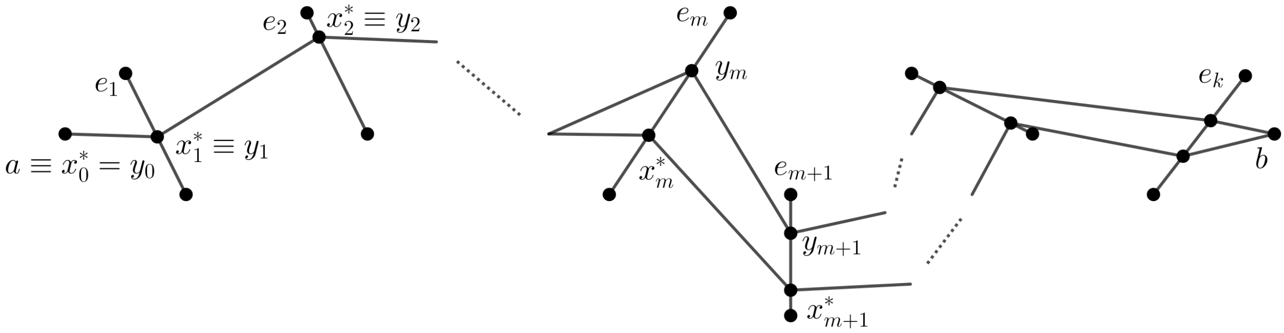


Figure 1.2: Illustration of Theorem 1.1

We have $y_0 = x_0^* = a$ and

$$y_{k+1} = x_{k+1}^* = b, 1 \leq m \leq k.$$

Furthermore, by definition $y_{m-1} = x_{m-1}^*$. If $x_m^* \notin [x_{m-1}^*, y_m]$ and $y_m \notin [x_{m-1}^*, x_m^*]$, then Lemma 1.4 says that

$$\gamma_{0|[\bar{t}_{m-1}, t_1]} \text{ and } \eta_{|[\bar{\tau}_{m-1}, \tau_1]}$$

are not both $SP(x_{m-1}^*, b)_{(e_m, \dots, e_k)}$. This and Lemma 1.2 imply that γ_0 and η are not both $SP(a, b)_{\mathcal{E}}$, a contradiction. Therefore

$$x_m^* \in [y_{m-1}, y_m[\text{ or } y_m \in [x_{m-1}^*, x_m^*[$$

(since $y_{m-1} = x_{m-1}^*$).

If $y_m \in [x_{m-1}^*, x_m^*[$, by Theorem 1.1, there exists

$$\bar{t}'_m \in [\bar{t}_{m-1}, \bar{t}_m[\text{ for which } \gamma_0(\bar{t}'_m) = y_m.$$

Since γ_0 is parametrized by arclength, Theorem 1.1 says that γ_0 is still affine on each subintervals, $[\bar{t}_{m-1}, \bar{t}'_m]$ and $[\bar{t}'_m, \bar{t}_{m+1}]$, and

$$\|x_{m-1}^* - y_m\| = l(\gamma_{0|[\bar{t}_{m-1}, \bar{t}'_m]}) = \bar{t}'_m - \bar{t}_{m-1}, \|y_m - x_{m+1}^*\| = \bar{t}_{m+1} - \bar{t}'_m.$$

Thus we can relabel

$$\bar{t}_m := \bar{t}'_m \text{ and } x_m^* := y_m.$$

Likewise, if $x_m^* \in [y_{m-1}, y_m[$, we can relabel $y_m := x_m^*$. If $J = I \setminus \{m\}$ is nonempty, set $n = \min J$. It follows that

$$x_n^* \in [y_{n-1}, y_n[\, \text{or} \, y_n \in [x_{n-1}^*, x_n^*[$$

and we continue to reduce the number of elements of J. After a finite number steps, we get

$$y_i = x_i^* \, \text{for all} \, i.$$

Thus we can assume

$$y_i = x_i^* \, \text{for all} \, i = 0, 1, \ldots, k+1$$

and will prove that $\eta = \gamma_0$ on $[0, \sigma]$.

Observe that

$$\begin{aligned} \bar{\tau}_{i+1} - \bar{\tau}_i &= l\big(\eta_{|[\bar{\tau}_i, \bar{\tau}_{i+1}]}\big) \\ &= \|\eta(\bar{\tau}_i) - \eta(\bar{\tau}_{i+1})\| \\ &= \|x_i^* - x_{i+1}^*\| \\ &= \bar{t}_{i+1} - \bar{t}_i \end{aligned}$$

for $i = 0, \ldots, k$. Condition $\bar{\tau}_0 = \bar{t}_0 = 0$ implies that

$$\bar{\tau}_1 = \bar{t}_1, \bar{\tau}_2 = \bar{t}_2, \ldots, \bar{\tau}_{k+1} = \bar{t}_{k+1} = \sigma.$$

Suppose $\bar{\tau}_{j+1} > \bar{\tau}_j$. Since η is parametrized by arclength and

$$l\big(\eta_{|[\bar{\tau}_j, \bar{\tau}_{j+1}]}\big) = \|\eta(\bar{\tau}_j) - \eta(\bar{\tau}_{j+1})\|,$$

Lemma 1.1 says that

$$\eta \, \text{is affine on} \, [\bar{\tau}_j, \bar{\tau}_{j+1}].$$

Hence $\eta(t) = \gamma_0(t)$ on $[\bar{\tau}_j, \bar{\tau}_{j+1}]$ and therefore

$$\eta(t) = \gamma_0(t) \, \text{on} \, [0, \sigma].$$

For the general case when $[\tau_0, \tau_1]$ is arbitrary and η is not necessarily parametrized by arclength but satisfies assumption (A), we express

$$\eta = \zeta \circ \psi,$$

where $\psi : [\tau_0, \tau_1] \to [0, \sigma]$ is defined by $\psi(\tau) = l\big(\eta_{|[\tau_0, \tau]}\big)$ and $\zeta : [0, \sigma] \to \mathbb{E}$ is parametrized by arclength (see [46]). Since ψ is strictly increasing and onto, we have

$$0 = \psi(\bar{\tau}_0) \leq \psi(\bar{\tau}_1) \leq \cdots \leq \psi(\bar{\tau}_{k+1}) = \sigma.$$

This and condition

$$\zeta\big(\psi(\bar{\tau}_i)\big) = \eta(\bar{\tau}_i) = y_i, \text{for } i = 0, 1, \ldots, k+1$$

imply that ζ is an $P(a,b)_{\mathcal{E}}$. Moreover, ζ is parametrized by arclength

$$l(\zeta) = \sigma - 0 = \sigma,$$

so that ζ is an $SP(a,b)_{\mathcal{E}}$. By the proof above $\zeta = \gamma_0$. Therefore $\eta = \gamma_0 \circ \psi$, i.e., the two paths η and γ_0 are equal. The proof is complete. □

Remark 1.2. Notice that the function Φ in the proof above is in general not strictly convex and hence the k-tuple $(x_1^*, \ldots, x_k^*) \in K$ is not unique. For instance, if $a, b \in \mathbb{E}$, $a \neq b$, and $\Phi : \mathbb{E} \to \mathbb{R}$ is defined by $\Phi(x) = \|x-a\| + \|x-b\|$, then Φ is not strictly convex, $\operatorname{argmin} \Phi = [a,b]$, and for each $x \in e := [a,b]$, $[a,x] * [x,b]$ is an $SP(a,b)_{(e)}$.

We can apply the arguments in the proof of the first part of Theorem 1.1 to prove the existence of solutions of problems with variable endpoints.

Corollary 1.1. *Let A, B be nonempty compact subsets of $\mathbb{E}$ and let $\mathcal{E} = (e_1, \ldots, e_k)$.*

(a) If $b \in \mathbb{E}$ is fixed, in the family of $P(a,b)_{\mathcal{E}}$, where $a \in A$, there exists a shortest path.

(b) If $a \in \mathbb{E}$ is fixed, in the family of $P(a,b)_{\mathcal{E}}$, where $b \in B$, there exists a shortest path.

(c) In the family of $P(a,b)_{\mathcal{E}}$, where $a \in A$ and $b \in B$, there exists a shortest path.

(d) In the family of $P(a,a)_{\mathcal{E}}$, where $a \in A$, there exists a shortest path.

Proof. (a) The proof is similar to the first part of that of Theorem 1.1, where Φ is replaced with

$$\tilde{\Phi} : \mathbb{E}^{k+1} \to \mathbb{R},$$
$$\tilde{\Phi}(x_0, x_1, \ldots, x_k) = \Big(\sum_{i=0}^{k-1} \|x_i - x_{i+1}\|\Big) + \|x_k - b\|$$

and K is replaced with $A \times e_1 \times \cdots \times e_k$. Proofs of parts (b)–(d) are similar. □

Observe that shortest paths with respect to a sequence of line segments in Corollary 1.1 may not be unique. Moreover, by Theorem 1.1, all shortest paths in this corollary are always polylines.

Applying Corollary 1.1(d) to the problem of finding an inscribed polygon in a given convex polygon $\mathcal{P} \subset \mathbb{R}^2$ with a minimum perimeter, we find that this problem has a solution. Some properties of angles of this inscribed polygon will be derived in the next section.

Corollary 1.2. *If $\gamma([t_0, t_1])$ and $\eta([\tau_0, \tau_1])$ are $SP(a, b)_{(e_1,\dots,e_k)}$ and parametrized by arclength, then $\eta(\tau) = \gamma(\tau - \tau_0 + t_0)$ and $\gamma(t) = \eta(t - t_0 + \tau_0)$. If, in addition, $\tau_0 = t_0$ (hence $\tau_1 = t_1$), then $\eta(t) = \gamma(t)$ for all $t \in [t_0, t_1]$.*

Proof. By Theorem 1.1, there is a strictly increasing and surjective function $\psi : [\tau_0, \tau_1] \to [t_0, t_1]$ such that $\eta = \gamma \circ \psi$. We have

$$\psi(\tau_0) = t_0 \text{ and } \psi(\tau_1) = t_1.$$

As γ and η are parametrized by arclength

$$\begin{aligned}\tau - \tau_0 &= l\big(\eta_{|[\tau_0,\tau]}\big)\\ &= l\big(\gamma_{|[\psi(\tau_0),\psi(\tau)]}\big)\\ &= \psi(\tau) - \psi(\tau_0) = \psi(\tau) - t_0 \quad \text{for every } \tau \in [\tau_0, \tau_1].\end{aligned}$$

It follows that $\psi(\tau) = \tau - \tau_0 + t_0$, giving

$$\eta(\tau) = \gamma\big(\psi(\tau)\big) = \gamma(\tau - \tau_0 + t_0).$$

Conversely, for each $t \in [t_0, t_1]$,

$$\tau := t - t_0 + \tau_0 \in [\tau_0, \tau_1]$$

and so $\eta(t - t_0 + \tau_0) = \eta(\tau) = \gamma(\tau - \tau_0 + t_0) = \gamma(t)$. □

Corollary 1.3. *Let $\mathcal{D}$ be a polytope in $\mathbb{R}^3$ whose faces are convex. Let $f_1, \dots, f_k$ be a sequence of faces of $\mathcal{D}$ such that $f_i \cap f_{i+1}$ is an edge for $i = 1, \dots, k-1$, $a \in f_1$, and $b \in f_k$. In the family of paths with respect to sequences of line segments satisfying assumption (A), there exists a unique shortest path with respect to a sequence of line segments lying on the surface of $\mathcal{D}$, joining a with b, and going orderly through $f_1, \dots, f_k$.*

As in Remark 1.1, this shortest path is an $SP(a, b)_{(e_1,\dots,e_k)}$, where $e_i = f_i \cap f_{i+1}$, $i = 1, \dots, k-1$.

Applying Theorem 1.1 to polytopes in $\mathbb{R}^3$ and simple polygons on the plane $\mathbb{R}^2$ we get the following well-known results that are shown in [40], [52].

Corollary 1.4. *(a) Let a and b be any two points on a domain $\mathcal{S}$ in $\mathbb{R}^3$ (or in $\mathbb{R}^2$) consisting of finite adjacent convex polygons. In the family of paths joining a with b and lying totally on $\mathcal{S}$, there exists a shortest path. This shortest path is a polyline.*

(b) Let a and b be any two points in a simple polygon $\mathcal{P} \subset \mathbb{R}^2$. There exists a shortest path joining a with b that lies totally in $\mathcal{P}$ and furthermore, it is a polyline.

Proof. (a) The case both a and b lie on the same polygon is trivial, so we assume that they belong to different polygons. Each path joining a with b and lying on $\mathcal{S}$ is a path with respect to some sequence $\mathcal{E}$ consisting of common edges of adjacent polygons in $\mathcal{S}$. Let $\gamma_{\mathcal{E}}$ be an $SP(a,b)_{(\mathcal{E})}$. Since the family of such sequences $\mathcal{E}$ is finite, the shortest path γ in the family $\{\gamma_{\mathcal{E}}\}$ is the required path. By Theorem 1.1, each $\gamma_{\mathcal{E}}$ is a polyline, so is γ.

(b) Partition $\mathcal{P}$ into nonoverlapping convex polygons and apply part (a) to obtain the required result. □

1.2 Concatenation of Two Shortest Paths

We have shown in Theorem 1.1 that every shortest path with respect to some sequence of line segments is a polyline. In this section we consider some geometric characteristics of shortest paths with respect to a sequence of line segments and represent conditions under which concatenation of two shortest paths with respect to two sequences of line segments is a shortest path with respect to a sequence of line segments. If $j < i$, then $SP(x,y)_{(e_i,\dots,e_j)}$ is understood to be $SP(x,y)_{\emptyset}$.

Lemma 1.5. *Suppose γ is a $P(a,b)_{(e_1,\dots,e_k)}$ and $b \in e_n \cap e_{n+1} \cap \dots \cap e_k$. The path γ is an $SP(a,b)_{(e_1,\dots,e_k)}$ iff it is an $SP(a,b)_{(e_1,\dots,e_{n-1})}$. Similarly, if $a \in e_1 \cap e_2 \cap \dots \cap e_m$, then γ is an $SP(a,b)_{(e_1,\dots,e_k)}$ iff it is an $SP(a,b)_{(e_{m+1},\dots,e_k)}$.*

This lemma is derived from the fact that under the condition $b \in e_n \cap \dots \cap e_k$, a path is a $P(a,b)_{(e_1,\dots,e_{n-1})}$ iff it is a $P(a,b)_{(e_1,\dots,e_k)}$. The other case is similar.

Lemma 1.6. *Let $\gamma([t_0,t_1])$ be an $SP(a,b)_{(e_1,\dots,e_k)}$ corresponding to $x_i = \gamma(\bar{t}_i) \in e_i$, $i = 1,\dots,k$.*

(a) If e_* is a line segment such that $e_* \cap [x_{j-1}, x_j] \neq \emptyset$, $2 \leq j \leq k$, and $e_* \neq e_{j-1}, e_* \neq e_j$, then γ is an $SP(a,b)_{(e_1,\ldots,e_{j-1},e_*,e_j,\ldots,e_k)}$. If $e_* \cap [a, x_1] \neq \emptyset$, and $e_* \neq e_1$, then γ is an $SP(a,b)_{(e_*,e_1,\ldots,e_k)}$. The case $e_* \cap [x_k, b] \neq \emptyset$ is similar.

(b) If $e_{*1}, \ldots, e_{*m}$ are line segments that contain the same point belonging to $[x_{j-1}, x_j]$ for some $j \in \{2, \ldots, k\}$, then γ is also an $SP(a,b)_{(e_1,\ldots,e_{j-1},e_{*1},\ldots,e_{*m},e_j,\ldots,e_k)}$. The cases $e_{*1}, \ldots, e_{*m}$ contain the same point of $[a, x_1]$ or $[x_k, b]$ are similar.

Proof. (a) Consider the case $2 \leq j \leq k$. Let

$$\mathcal{E}_* = (e_1, \ldots, e_{j-1}, e_*, e_j, \ldots, e_k)$$

and $\eta([\tau_0, \tau_1])$ an $P(a,b)_{\mathcal{E}_*}$ corresponding to

$$y_i = \eta(\bar{\tau}_i) \in e_i, y_* = \eta(\bar{\tau}_*) \in e_*(\bar{\tau}_{j-1} \leq \bar{\tau}_* \leq \bar{\tau}_j).$$

Clearly, η is also a $P(a,b)_{(e_1,\ldots,e_k)}$, so $l(\gamma) \leq l(\eta)$.
Now take an $x_* \in e_* \cap [x_{j-1}, x_j]$. Since $\gamma([\bar{t}_{j-1}, \bar{t}_j]) = [x_{j-1}, x_j]$, there is

$$\bar{t}_* \in [\bar{t}_{j-1}, \bar{t}_j] \,\text{with}\, \gamma(\bar{t}_*) = x_*.$$

Thus γ is also a $P(a,b)_{\mathcal{E}_*}$ and therefore γ is an $SP(a,b)_{\mathcal{E}_*}$. Other cases are proved similarly. (b) follows directly from (a). □

Next, we turn to our main problem of concatenation of two shortest paths with respect to two sequences of line segments. We first consider the simplest case: the concatenation of a path with respect to a sequence of line segments and a line segment. The following lemma says that if we elongate the first or last line segment of a shortest path with respect to a sequence of line segments, we get a new shortest path with respect to a sequence of line segments.

Lemma 1.7. *Let $\gamma([t_0, t_1])$ be an $SP(a,b)_{\mathcal{E}}$ corresponding to $x_i = \gamma(\bar{t}_i) \in e_i$, $e_i \in \mathcal{E} = (e_1, \ldots, e_n)$.*

(a) If $x_n \neq b$ and $q \in \mathbb{E}$ such that $b \in]x_n, q[$, then $\gamma * [b, q]$ is an $SP(a,q)_{\mathcal{E}}$.

(b) If $x_1 \neq a$ and $p \in \mathbb{E}$ such that $a \in]p, x_1[$, then $[p, a] * \gamma$ is an $SP(p,b)_{\mathcal{E}}$.

(c) Set $x_0 = a$. If $x_m \neq x_{m+1} = \cdots = x_n = b$, $(0 \leq m \leq n-1)$, and $b \in]x_m, q[$, then $\gamma * [b, q]$ is an $SP(a,q)_{\mathcal{E}}$.

Proof. (a) Let $\eta([\tau_0, \tau_1])$ be any $P(a,q)_{\mathcal{E}}$ corresponding to

$$y_i = \eta(\bar{\tau}_i) \in e_i, i = 1, \dots, n.$$

Suppose

$$b = (1-\lambda)x_n + \lambda q \,\text{for some}\, \lambda \in]0,1[\,.$$

Set

$$\begin{aligned} z_i &:= (1-\lambda)x_i + \lambda y_i,\ 1 \le i \le n, \\ z_0 &:= (1-\lambda)x_1 + \lambda a, \\ z_{n+1} &:= (1-\lambda)x_n + \lambda q = b. \end{aligned}$$

Then for $i = 1, \dots, n-1$,

$$\begin{aligned} \|z_i - z_{i+1}\| &= \|(1-\lambda)(x_i - x_{i+1}) + \lambda(y_i - y_{i+1})\| \\ &\le (1-\lambda)\|x_i - x_{i+1}\| + \lambda\|y_i - y_{i+1}\|. \end{aligned} \tag{1.5}$$

Let ξ be the path going through $z_0, z_1, \dots, z_{n+1}$ such that

$$\xi \,\text{is an affine path joining}\, z_i \,\text{and}\, z_{i+1}, 0 \le i \le n.$$

Then ξ is a $P(z_0, b)_{\mathcal{E}}$. By Theorem 1.1, there exits

$$t^* \in [t_0, \bar{t}_1] \,\text{satisfying}\, \gamma(t^*) = z_0.$$

Lemma 1.2 states that $\gamma_{|[t^*,t_1]}$ is an $SP(z_0,b)_{\mathcal{E}}$. According to Theorem 1.1 and (1.5) we have

$$\begin{aligned} \|z_0 - x_1\| + \sum_{i=1}^{n-1} \|x_i - x_{i+1}\| + \|x_n - b\| &= l\big(\gamma_{|[t^*,t_1]}\big) \\ &\le l(\xi) \\ &= \sum_{i=0}^{n} \|z_i - z_{i+1}\| \\ &\le \|z_0 - z_1\| + \left[(1-\lambda)\sum_{i=1}^{n-1} \|x_i - x_{i+1}\| + \lambda \sum_{i=1}^{n-1} \|y_i - y_{i+1}\|\right] + \|z_n - b\|. \end{aligned} \tag{1.6}$$

Noting that

$$\begin{aligned} \|z_0 - x_1\| &= \lambda\|a - x_1\|, \quad \|x_n - b\| = \lambda\|x_n - q\|, \\ \|z_0 - z_1\| &= \lambda\|a - y_1\|, \quad \|z_n - b\| = \lambda\|y_n - q\|, \end{aligned}$$

we deduce from (1.6) that

$$\lambda\|a - x_1\| + \lambda\sum_{i=1}^{n-1} \|x_i - x_{i+1}\| + \lambda\|x_n - q\| \le \lambda\|a - y_1\| + \lambda\sum_{i=1}^{n-1} \|y_i - y_{i+1}\| + \lambda\|y_n - q\|.$$

Since

$$\|x_n - q\| = \|x_n - b\| + \|b - q\|,$$

the above inequality yields

$$\lambda\big(l(\gamma) + \|b - q\|\big) \leq \lambda l(\eta),$$

which implies

$$l(\gamma * [b, q]) = l(\gamma) + \|b - q\| \leq l(\eta).$$

Thus, $\gamma * [b, q]$ is an $SP(a, q)_{\mathcal{E}}$.

Part (b) is proved similarly. To prove (c) we observe that, by Lemma 1.5, γ is an $SP(a, b)_{(e_1, \ldots, e_m)}$. Applying part (a) we find that $\gamma * [b, q]$ is an $SP(a, q)_{(e_1, \ldots, e_m)}$. That $\gamma * [b, q]$ is an $SP(a, q)_{\mathcal{E}}$ follows from Lemma 1.6. □

We now study some characteristics of a shortest path with respect to a sequence of line segments basing on properties of angles between the path and line segments e_is. If u and v are nonzero vectors in $\mathbb{E}$, we denote by $\angle(u, v)$ the angle between u and v, which does not exceed π.

Theorem 1.2. *Let $\mathcal{E} = (e_1, \ldots, e_k)$ be a sequence of line segments and $\gamma([t_0, t_1])$ an $SP(a, b)_{(e_1, \ldots, e_{n-1})}$ corresponding to $x_i = \gamma(\bar{t}_i) \in e_i$ $(1 \leq i \leq n-1)$ and $n \leq k$. Suppose that $b \in e_n \cap \cdots \cap e_k$, each e_j is non-singleton for $j = n, \ldots, k$, and that $x_{n-1} \neq b$ (if $n = 1$, $x_0 := a$). Let $q \in \mathbb{E}$, $q \neq b$. Then $\gamma * [b, q]$ is an $SP(a, q)_{\mathcal{E}}$ iff for any $y_j \in e_j$ and $y_j \neq b$, $j = n, \ldots, k$, we have $\theta \geq \pi$, where*

$$\theta := \angle(x_{n-1} - b, y_n - b) + \sum_{j=n}^{k-1} \angle(y_j - b, y_{j+1} - b) + \angle(y_k - b, q - b).$$

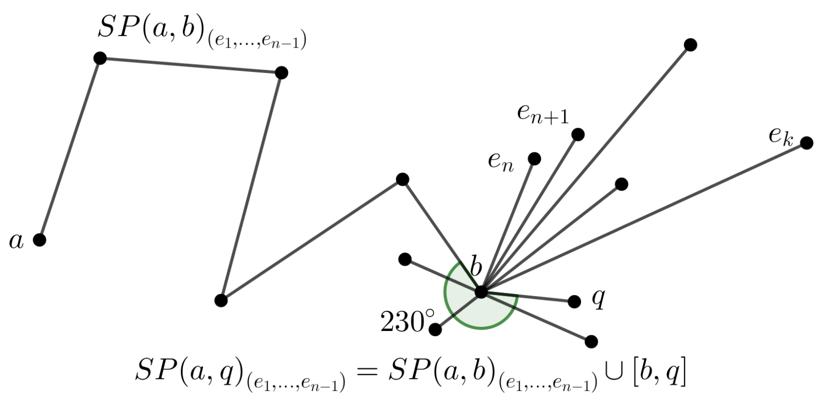


Figure 1.3: Illustration of Theorem 1.2

Proof. Suppose that $\gamma * [b,q]$ is an $SP(a,q)_{\mathcal{E}}$ and that

$$y_j \in e_j, y_j \neq b, j = n, \ldots, k.$$

We show that $\theta \geq \pi$. By Lemma 1.2

$$\xi := \gamma_{|[\bar{t}_{n-1},t_1]} * [b,q] \,\text{is an}\, SP(x_{n-1},q)_{(e_n,\ldots,e_k)}.$$

Setting

$$y_{k+1} := q, v_j := y_j - b, j = n, \ldots, k+1,$$

we have

$$\theta = \angle(x_{n-1} - b, v_n) + \sum_{j=n}^{k} \angle(v_j, v_{j+1}).$$

Let L_j be the line containing b and y_j, $n \leq j \leq k+1$. Let P be the plane containing L_n and x_{n-1} (if $x_{n-1} \in L_n$, P is any fixed plane containing L_n). Set $y'_n := y_n$. If $L_n \neq L_{n+1}$, denote by $y'_{n+1} \in P$ the point such that

$$\triangle x_{n-1}by'_n \,\text{and}\, \triangle y'_n b y'_{n+1} \,\text{do not overlap}$$

(i.e., x_{n-1} and y'_{n+1}, are on opposite sides of L_n) and

$$\triangle y'_n b y'_{n+1} = \triangle y_n b y_{n+1};$$

if $x_{n-1} \in L_n$, y'_{n+1} lies on any side of L_n. If $L_n = L_{n+1}$, i.e.,

$$y_{n+1} - b = \lambda(y_n - b) \,\text{for some}\, \lambda,$$

then we choose

$$y'_{n+1} \,\text{satisfying}\, y'_{n+1} - b = \lambda(y'_n - b).$$

Similarly $y'_{n+2} \in P$ is the point such that

$$\triangle y'_{n+1} b y'_{n+2} = \triangle y_{n+1} b y_{n+2}$$

and the triangles

$$\triangle y'_n b y'_{n+1} \,\text{and}\, \triangle y'_{n+1} b y'_{n+2} \,\text{do not overlap,}$$

and so on. We then obtain a sequence $y'_n, \ldots, y'_{k+1}$ in P such that

$$\|y'_j - y'_{j+1}\| = \|y_j - y_{j+1}\|,$$
$$\|y'_j - b\| = \|y_j - b\|,$$
$$\text{and}\, \angle(v'_j, v'_{j+1}) = \angle(v_j, v_{j+1}),$$

where $v'_j = y'_j - b$ for $j \geq n$. Then

$$\theta = \angle(x_{n-1} - b, v'_n) + \sum_{j=n}^{k} \angle(v'_j, v'_{j+1}).$$

If $\theta < \pi$, there exist

$$z_{n-1} \in \,]x_{n-1}, b[\,, z'_n \in \,]y'_n, b[\,, \ldots, z'_{k+1} \in \,]y'_{k+1}, b[$$

that are near b and such that they are collinear. Let $z_j \in \,]y_j, b[$ satisfy

$$\|z_j - b\| = \|z'_j - b\|, j \geq n.$$

We have

$$\begin{aligned}
\|z_{n-1} - z_n\| + \cdots + \|z_k - z_{k+1}\| &= \|z_{n-1} - z'_n\| + \cdots + \|z'_k - z'_{k+1}\| \\
&= \|z_{n-1} - z'_{k+1}\| \\
&< \|z_{n-1} - b\| + \|b - z'_{k+1}\| \\
&= \|z_{n-1} - b\| + \|b - z_{k+1}\|
\end{aligned}$$

and hence

$$[x_{n-1}, z_{n-1}] * [z_{n-1}, z_n] * \cdots * [z_k, z_{k+1}] * [z_{k+1}, q]$$

is a $P(x_{n-1}, q)_{(e_n,\ldots,e_k)}$ whose length is less than that of ξ. This is impossible. Therefore $\theta \geq \pi$.

Next, suppose conversely that for any

$$y_j \in e_j \text{ and } y_j \neq b, j = n, \ldots, k,$$

we have $\theta \geq \pi$. We first observe that $\gamma * [b, q]$ is an $P(a, q)_{\mathcal{E}}$. Let $\eta([\tau_0, \tau_1])$ be any $SP(a, q)_{\mathcal{E}}$ corresponding to

$$y_i = \eta(\bar{\tau}_i) \in e_i, i = 1, \ldots, k.$$

Let $\bar{\tau}_{k+1} := \tau_1$ and $y_{k+1} := \eta(\bar{\tau}_{k+1}) = q$. We need to show that

$$l(\gamma * [b, q]) \leq l(\eta).$$

If $b \in [y_j, y_{j+1}]$ for some $j \geq n$, then there is

$$\tau_b \in [\bar{\tau}_j, \bar{\tau}_{j+1}] \text{ with } \eta(\tau_b) = b.$$

Since $\eta_{|[\tau_0,\tau_b]}$ is a $P(a, b)_{(e_1,\ldots,e_{n-1})}$,

$$l(\eta) = l\big(\eta_{|[\tau_0,\tau_b]}\big) + l\big(\eta_{|[\tau_b,\tau_1]}\big) \geq l(\gamma) + \|b - q\| = l(\gamma * [b, q]).$$

Consider the case $b \notin [y_j, y_{j+1}]$ (and so $y_j \neq b$) for all $j = n, \ldots, k$. Let $y'_n, \ldots, y'_{k+1}$ be points defined as in the first part of the proof. Set

$$\theta_{n-1} := \angle(x_{n-1} - b, y'_n - b)$$

and for $n \leq j \leq k$,

$$\theta_j := \angle(x_{n-1} - b, y'_n - b) + \angle(y'_n - b, y'_{n+1} - b) + \cdots + \angle(y'_j - b, y'_{j+1} - b).$$

We have $\theta_k = \theta \geq \pi$. Let

$$r = \min\{j \geq n : \theta_j \geq \pi\}.$$

As $b \notin [y_r, y_{r+1}]$,

$$0 < \angle(y'_r - b, y'_{r+1} - b) = \angle(y_r - b, y_{r+1} - b) < \pi$$

and there exists

$$y^{*\prime} \in \,]y'_r, y'_{r+1}]$$

such that

$$\theta_{r-1} + \angle(y'_r - b, y^{*\prime} - b) = \pi.$$

Let $y^* \in \,]y_r, y_{r+1}]$ satisfy

$$\|y^* - y_r\| = \|y^{*\prime} - y'_r\|$$

and $\tau^* \in \,]\bar{\tau}_r, \bar{\tau}_{r+1}]$ satisfy

$$\eta(\tau^*) = y^*.$$

Since $b \in \,]x_{n-1}, y^{*\prime}[\,$, by Lemma 1.7(c)

$$\gamma * [b, y^{*\prime}] \text{ is an } SP(a, y^{*\prime})_{(e_1,\ldots,e_n)}.$$

Thus

$$\begin{aligned} l\big(\eta_{|[\tau_0,\tau^*]}\big) &= l\big(\eta_{|[\tau_0,\bar{\tau}_n]}\big) + \|y_n - y_{n+1}\| + \cdots + \|y_{r-1} - y_r\| + \|y_r - y^*\| \\ &= l\big(\eta_{|[\tau_0,\bar{\tau}_n]}\big) + \|y'_n - y'_{n+1}\| + \cdots + \|y'_{r-1} - y'_r\| + \|y'_r - y^{*\prime}\| \\ &\geq l\big(\gamma * [b, y^{*\prime}]\big) = l(\gamma) + \|b - y^{*\prime}\| = l(\gamma) + \|b - y^*\|. \end{aligned}$$

Hence

$$\begin{aligned} l(\eta) &\geq l\big(\eta_{|[\tau_0,\tau^*]}\big) + \|y^* - q\| \\ &\geq l(\gamma) + \|b - y^*\| + \|y^* - q\| \\ &\geq l(\gamma) + \|b - q\| \\ &= l(\gamma * [b, q]). \end{aligned}$$

□

Theorem 1.2 gives a step-by-step method to check whether a path with respect to a sequence of line segments is shortest: we just measure the angles between line segments of the path and e_is at points of intersection, from the starting point to the terminating point.

We now consider several consequences of Theorem 1.2. In computational geometry, edges of polygons intersect at their common endpoints. In such cases, Theorem 1.2 reduces to the following simple form.

Corollary 1.5. *Let $\mathcal{E} = (e_1, \dots, e_k)$ be a sequence of line segments and $\gamma([t_0, t_1])$ an $SP(a, b)_{(e_1,\dots,e_{n-1})}$ corresponding to $x_i = \gamma(\bar{t}_i) \in e_i$ $(1 \le i \le n-1)$ and $n \le k$. Suppose that b is the common endpoint of non-singleton line segments $e_n = [b, b_n], \dots,$ $e_k = [b, b_k]$, and that $x_{n-1} \neq b$ (if $n = 1$, $x_0 := a$). Let $q \in \mathbb{E}$, $q \neq b$. Then $\gamma * [b, q]$ is an $SP(a, q)_{\mathcal{E}}$ iff $\theta \ge \pi$, where*

$$\theta := \angle(x_{n-1} - b, b_n - b) + \sum_{j=n}^{k-1} \angle(b_j - b, b_{j+1} - b) + \angle(b_k - b, q - b).$$

Applying Corollary 1.5 we can prove the following result in [52]:

A shortest path joining two points on the surface of a convex polyhedron $\mathcal{D} \subset \mathbb{R}^3$ cannot pass through any vertex of $\mathcal{D}$.

Indeed, suppose a shortest path γ on the surface $\mathcal{S}$ of $\mathcal{D}$ joins a and b on $\mathcal{S}$ and goes through a vertex $v \in \mathcal{S}$. As γ can be seen as a shortest path with respect to a sequence of line segments, by Corollary 1.5, both the angle to the left and the angle to the right of γ at v are not less than π. Hence the angle of $\mathcal{D}$ at v is not less than 2π, a contradiction. Notice that the proof still holds for polyhedrons $\mathcal{D}$ whose angle at every vertex is strictly less than 2π.

Let us consider an example. Given $a, q \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, we show that

$$\gamma = [a, \mathbf{0}] * [\mathbf{0}, q]$$

is the shortest path that joins a with q and meets the x-, y-, and z-axes. Suppose η is any path joining a and q and meets x-, y-, and z-axes at u, v, and w, respectively. If one of the points u, v, w is coincident with $b := \mathbf{0} = (0, 0, 0)$, then clearly $l(\gamma) \le l(\eta)$. Assume that

$$b \notin \{u, v, w\}$$

and that

$$\eta \text{ is an } P(a, q)_{([b,v],[b,u],[b,w])}.$$

Since

$$\angle(a-b, v-b) + \angle(v-b, u-b) + \angle(u-b, w-b) + \angle(w-b, q-b) \geq \pi,$$

Corollary 1.5 says that, in the family of all $P(a,q)_{([b,v],[b,u],[b,w])}$, $l(\gamma) \leq l(\eta)$.

Corollary 1.6. *Suppose $\gamma([t_0, t_1])$ is an $SP(a,b)_{(e_1,\ldots,e_n)}$ corresponding to $x_i = \gamma(\bar{t}_i) \in e_i$, $1 \leq i \leq n$, $x_n \neq b$. Let $y \in e_n$, $y \neq x_n$, and let q be any point in $\mathbb{E}$ such that $q \neq x_n$ and $\angle(y-x_n, q-x_n) = \angle(y-x_n, b-x_n)$. Then $\gamma_{|[t_0,\bar{t}_n]} * [x_n, q]$ is an $SP(a,q)_{(e_1,\ldots,e_n)}$. In particular, if we rotate the last line segment $[x_n, b]$ around the line containing e_n by the angle $\angle(y-x_n, b-x_n)$, we then get a new shortest path with respect to a sequence of line segments.*

Proof. Set $x_0 := a$. If $x_0 = x_1 = \cdots = x_n$, then there is nothing to prove. So we assume that $m = \max\{i : x_i \neq b\}$ exists. By Theorem 1.2, for any $y_i \in e_i$, $y_i \neq x_n$, $m+1 \leq i \leq n$,

$$\angle(x_m - x_n, y_{m+1} - x_n) + \angle(y_{m+1} - x_n, y_{m+2} - x_n) + \cdots + \angle(y_n - x_n, b - x_n) \geq \pi.$$

Hence

$$\angle(x_m - x_n, y_{m+1} - x_n) + \cdots + \angle(y_n - x_n, q - x_n) \geq \pi$$

since

$$\angle(y_n - x_n, q - x_n) = \angle(y_n - x_n, b - x_n).$$

Applying Theorem 1.2 once more, we find that

$$\gamma_{|[t_0,\bar{t}_n]} * [x_n, q] \text{ is an } SP(a,q)_{(e_1,\ldots,e_n)}.$$

□

Corollary 1.7. *Let $\gamma([t_0, t_1])$ be an $SP(a,b)_{(e_1,\ldots,e_n)}$ corresponding to $x_i = \gamma(\bar{t}_i) \in e_i$, $1 \leq i \leq n$. Set $x_0 := a$, $x_{n+1} := b$. Suppose also that for some j, e_j is non-singleton, $x_{j-1} \neq x_j$, and $x_{j+1} \neq x_j$.*

(a) If $y_j \in e_j$, $y_j \neq x_j$, then $\theta := \angle(x_{j-1} - x_j, y_j - x_j) + \angle(y_j - x_j, x_{j+1} - x_j) \geq \pi$. In particular, if x_j is an interior point of e_j, then $\theta = \pi$.

(b) Let L be the line containing e_j. If $x_{j-1} \in L$ and $x_{j+1} \notin L$, then x_j is an end point of e_j with $\|x_{j-1} - x_j\| = \min_{x \in e_j} \|x_{j-1} - x\|$ and $\theta > \pi$. The case $x_{j+1} \in L$ and $x_{j-1} \notin L$ is similar.

(c) Let L be the line containing e_j. If $x_{j-1}, x_{j+1} \in L$, then either $\angle(x_{j-1} - x_j, x_{j+1} - x_j) = \pi$ or $\angle(x_{j-1} - x_j, y_j - x_j) = \angle(x_{j+1} - x_j, y_j - x_j) = \pi$, where $y_j \in e_j$, $y_j \neq x_j$. In the latter case x_j is an end point of e_j with $\|x_{j-1} - x_j\| = \min_{x \in e_j} \|x_{j-1} - x\|$.

Proof. (a) Applying Theorem 1.2 to $\gamma_{|[t_0,\bar{t}_{j+1}]}$ ($\bar{t}_{n+1} := t_1$) and the sequence $e_1, \ldots, e_j$, we get $\theta \geq \pi$. If x_j is an interior point of e_j, take $y'_j \in e_j$ such that

$$x_j \text{ is an interior point of } [y_j, y'_j].$$

We then also have

$$\theta' := \angle(x_{j-1} - x_j, y'_j - x_j) + \angle(y'_j - x_j, x_{j+1} - x_j) \geq \pi.$$

Since $\theta + \theta' = 2\pi$, $\theta = \theta' = \pi$.

(b) Suppose $x_{j-1} \in L$ and $x_{j+1} \notin L$. Since $\theta \neq \pi$, $\theta > \pi$ and according to part (a)

$$x_j \text{ must be an endpoint.}$$

If $\|x_{j-1} - x_j\| > \min_{x \in e_j} \|x_{j-1} - x\|$, there exists $\bar{x} \in e_j$ with

$$\|x_{j-1} - x_j\| > \|x_{j-1} - \bar{x}\|, \text{ i.e., } \bar{x} \in [x_{j-1}, x_j[\,.$$

Since $x_{j+1} \notin L$

$$\begin{aligned} \|x_{j-1} - x_j\| + \|x_j - x_{j+1}\| &= \|x_{j-1} - \bar{x}\| + \|\bar{x} - x_j\| + \|x_j - x_{j+1}\| \\ &> \|x_{j-1} - \bar{x}\| + \|\bar{x} - x_{j+1}\|, \end{aligned}$$

i.e.,

$$l(\gamma_{|[\bar{t}_{j-1},\bar{t}_{j+1}]}) > l([x_{j-1}, \bar{x}] * [\bar{x}, x_{j+1}]),$$

a contradiction. Thus

$$\|x_{j-1} - x_j\| = \min_{x \in e_j} \|x_{j-1} - x\|.$$

(c) If $x_j \in]x_{j-1}, x_{j+1}[$, then

$$\angle(x_{j-1} - x_j, x_{j+1} - x_j) = \pi.$$

Otherwise, say $x_{j-1} \in]x_j, x_{j+1}]$, an analysis which is similar to that in part (b) shows that

$$x_j \text{ is an end point of } e_j$$

with

$$\|x_{j-1} - x_j\| = \min_{x \in e_j} \|x_{j-1} - x\|$$

and

$$\angle(x_{j-1} - x_j, y_j - x_j) = \angle(x_{j+1} - x_j, y_j - x_j) = \pi, \text{for } y_j \in e_j, y_j \neq x_j.$$

□

The following result is a converse of Corollary 1.7 and it gives sufficient conditions for a path with respect to a sequence of line segments to be shortest.

Corollary 1.8. *Let* $\mathcal{E} = (e_1, \ldots, e_n)$ *be a sequence of line segments and* $\gamma([t_0, t_1])$ *an* $SP(a, b)_{(e_1,\ldots,e_{n-1})}$ *corresponding to* $x_i = \gamma(\bar{t}_i) \in e_i$, $1 \leq i \leq n-1$. *Let* $b \in e_n$ *and* $q \in \mathbb{E}$, $q \neq b$. *Set* $x_0 = a$ *and suppose that* $m = \max\{i : x_i \neq b\}$ *exits and that there is* $y \in e_n, y \neq b$.

(a) If $\theta := \angle(x_m - b, y - b) + \angle(y - b, q - b) = \pi$, *then* $\gamma * [b, q]$ *is an* $SP(a, q)_{\mathcal{E}}$.

(b) If b *is an end point of* e_n *and* $\theta > \pi$, *then* $\gamma * [b, q]$ *is an* $SP(a, q)_{\mathcal{E}}$.

Proof. (a) For any $y' \in e_n$, $y' \neq b$, we always have

$$\angle(x_m - b, y' - b) + \angle(y' - b, q - b) = \pi.$$

Thus by Theorem 1.2

$$\gamma * [b, q] \text{ is an } SP(a, q)_{(e_1,\ldots,e_m,e_n)}.$$

If $m < n - 1$, it follows from Lemma 1.6(b) that

$$\gamma * [b, q] \text{ is an } SP(a, q)_{\mathcal{E}}.$$

(b) follows directly from Corollary 1.5. □

To illustrate Corollaries 1.7 and 1.8 let us consider a triangle in $\mathbb{R}^2$ having acute angles (Figure 1.4). For $u \in e_1$ fixed, Corollary 1.8 states that the triangle uvw has minimum perimeter because angles θ at v and w are π. Figure 1.4 shows that

$$\angle(w - u, z_1 - u) + \angle(z_1 - u, v - u) \neq \pi.$$

Since the angles at u do not satisfy Corollary 1.7(a), uvw is not the inscribed triangle with minimum perimeter.

We hope that some results in this chapter could be used to study Steiner's problem (see [54]): Given a convex polygon in the plane, find an inscribed polygon of minimal perimeter.
We are now in a position to consider the general case of concatenation of two shortest paths with respect to two sequences of line segments. Roughly speaking, the following theorem says that if the last line segment of a shortest path with respect to a sequence of line segments and the first of the other overlap, the two paths can be joined to become a shortest path with respect to sequence of line segments.

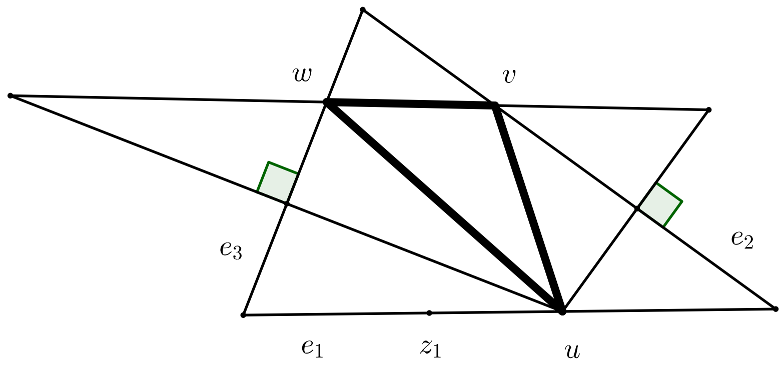


Figure 1.4: $SP(u,u)_{(e_1,e_2,e_3)} = [u,v] * [v,w] * [w,u]$

Theorem 1.3. *Let $\mathcal{E} = (e_1, \ldots, e_k)$ be a sequence of line segments. Suppose that $\gamma_1([t_0, t_1])$ is an $SP(a,b)_{(e_1,\ldots,e_{n-1})}$ and $\gamma_2([t^*, t_2])$ is an $SP(c,d)_{(e_n,\ldots,e_k)}$, where $t^* < t_1 < t_2$, and $\gamma_1(t_1) = \gamma_2(t_1) = b$. Suppose also that γ_1, γ_2 satisfy assumption (A). If $t_1 \leq \bar{t}_n \leq \cdots \leq \bar{t}_k \leq t_2$, $x_i := \gamma_2(\bar{t}_i) \in e_i$ for $n \leq i \leq k$, and if there exists $\epsilon > 0$ such that $\gamma_1([t_1 - \epsilon, t_1]) \subset \gamma_2([t^*, t_1])$, then the concatenation γ of γ_1 and $\gamma_{2|[t_1,t_2]}$ is an $SP(a,d)_{\mathcal{E}}$.*

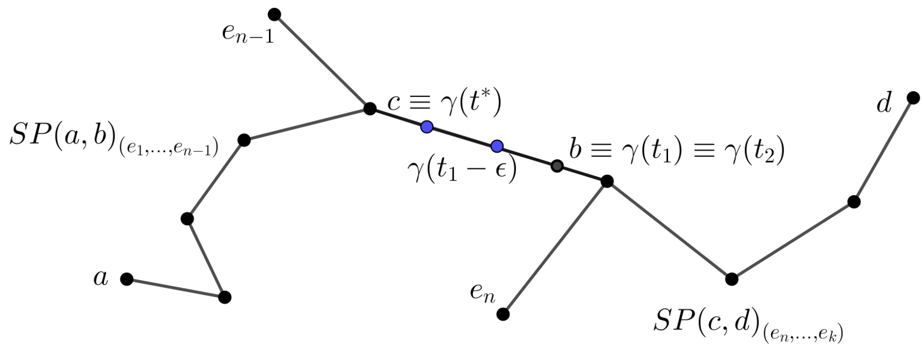


Figure 1.5: Illustration of Theorem 1.3 ($\gamma_1([t_1 - \epsilon, t_1]) \subset \gamma_2([t^*, t_1])$)

Proof. Let $t_0 =: \bar{t}_0 \leq \bar{t}_1 \leq \cdots \leq \bar{t}_{n-1} \leq t_1$,

$$x_i := \gamma_1(\bar{t}_i) \in e_i \,\text{for}\, i = 1, \ldots, n-1,$$
$$x_0 := \gamma_1(\bar{t}_0) = a.$$

We can also assume, by applying Lemma 1.7 if necessary, that $b = x_n \in e_n$, i.e., $\bar{t}_n = t_1$. Set $\bar{t}_{k+1} := t_2$ and note that γ is a $P(a,d)_{\mathcal{E}}$. The proof is divided into two cases.

Case 1: All e_j, $j \geq n$, are non-singleton. Set

$$x_{k+1} := \gamma_2(\bar{t}_{k+1}) = d,$$
$$m = \max\{i < n : x_i \neq b\},$$
$$r = \min\{j > n : x_j \neq b\}.$$

m and r exist due to assumption (A) and Theorem 1.1. Assume that

$$c \in [x_m, b[\,.$$

Then, by Lemma 1.2

$$\gamma_{2|[t^*,\bar{t}_r]} = [c, b] * [b, x_r]$$

is an $SP(c, x_r)_{(e_n,\ldots,e_{r-1})}$. Applying Theorem 1.2 to $\gamma_{2|[t^*,\bar{t}_r]}$ we find that

$$\text{all angles}\,\theta\,\text{at}\,b\,\text{are not less than}\,\pi.$$

Thus this theorem also says that

$$\gamma_1 * \gamma_{2|[t_1,\bar{t}_r]} = \gamma_1 * [b, x_r]$$

is an $SP(a, x_r)_{(e_1,\ldots,e_m,e_n,\ldots,e_{r-1})}$. If $m < n-1$, Lemma 1.6 again shows that

$$\gamma_1 * \gamma_{2|[t_1,\bar{t}_r]}$$

is an $SP(a, x_r)_{(e_1,\ldots,e_{r-1})}$. We continue in this fashion to obtain, after a finite number of times, that

$$\gamma\,\text{is an}\,SP(a, d)_{\mathcal{E}}.$$

Case 2: The line segment e_j is a singleton for some $j \geq n$. The particular case when e_n is a singleton, i.e., $e_n = \{b\}$, is derived immediately from the following lemma. □

Lemma 1.8. *Let $\zeta([\alpha_0, \alpha_1])$ be a $P(p, u)_{(e_1,\ldots,e_l)}$ and $\xi([\alpha_1, \alpha_2])$ a $P(u, q)_{(e_{l+1},\ldots,e_m)}$. Then $\psi = \zeta * \xi$ is an $SP(p, q)_{(e_1,\ldots,e_l,\{u\},e_{l+1},\ldots,e_m)}$ iff ζ is an $SP(p, u)_{(e_1,\ldots,e_l)}$ and ξ an $SP(u, q)_{(e_{l+1},\ldots,e_m)}$.*

Proof. Indeed, if ψ is a shortest path with respect to the sequence $e_1, \ldots, e_l$, $\{u\}, e_{l+1}$, then so are ζ and ξ (Lemma 1.2). Conversely, suppose ζ and ξ are shortest paths. Let $\eta([\tau_0, \tau_1])$ be any $P(p, q)_{(e_1,\ldots,e_l,\{u\},e_{l+1},\ldots,e_m)}$, corresponding to $y_i = \eta(\bar{\tau}_i) \in e_i$, $i = 1, \ldots, m$, and $\eta(\bar{\tau}_u) = u$ ($\bar{\tau}_l \leq \bar{\tau}_u \leq \bar{\tau}_{l+1}$). Since

$$\eta_{|[\tau_0,\bar{\tau}_u]}\,\text{is an}\,P(p, u)_{(e_1,\ldots,e_l)}$$

and

$$\eta_{|[\bar{\tau}_u,\tau_1]}\,\text{is an}\,P(u, q)_{(e_{l+1},\ldots,e_m)},$$

$$l(\eta) = l\big(\eta_{|[\tau_0,\bar{\tau}_u]}\big) + l\big(\eta_{|[\bar{\tau}_u,\tau_1]}\big) \geq l(\zeta) + l(\xi) = l(\psi).$$

Therefore ψ is an $SP(p,q)_{(e_1,\ldots,e_l,\{u\},e_{l+1},\ldots,e_m)}$.

We now prove the theorem for the case

$$s = \min\{j \geq n : e_j \text{ is a singleton}\} > n,$$

say $e_s = \{u\}$. If $t_1 = \bar{t}_{n+1} = \cdots = \bar{t}_s$, then

$$b = u \in e_{n+1} \cap \cdots \cap e_s$$

and by Lemma 1.5

$$\gamma_1 \text{ is also an } SP(a,b)_{(e_1,\ldots,e_{s-1})}$$

and

$$\gamma_{2|[t_1,t_2]} \text{ an } SP(b,d)_{(e_{s+1},\ldots,e_k)}.$$

Lemma 1.8 above states that

$$\gamma \text{ is an } SP(a,d)_{(e_1,\ldots,e_k)}.$$

If $\bar{t}_s > t_1$, by Case 1 we find that

$$\tilde{\gamma} = \gamma_1 * \gamma_{2|[t_1,\bar{t}_s]} \text{ is an } SP(a,u)_{(e_1,\ldots,e_{s-1})}.$$

We then apply Lemma 1.8 again to

$$\tilde{\gamma} \text{ and } \gamma_{2|[\bar{t}_s,t_2]}$$

to obtain the required result. The proof of the theorem is complete. □

If the last line segment of a shortest path with respect to a sequence of line segments and the first of the other do not overlap, we can use the following.

Corollary 1.9. *Suppose* $\gamma_1([t_0,t_1])$ *is an* $SP(a,b)_{(e_1,\ldots,e_n)}$ *and* $\gamma_2([t_1,t_2])$ *is an* $SP(b,c)_{(e_{n+1},\ldots,e_k)}$, $t_0 < t_1 < t_2$. *Suppose also that* γ_1,γ_2 *satisfy assumption (A). Then* $\gamma = \gamma_1 * \gamma_2$ *is an* $SP(a,c)_{(e_1,\ldots,e_k)}$ *iff there exists* $\epsilon > 0$ *such that* $\gamma_{1|[t_1-\epsilon,t_1]} * \gamma_{2|[t_1,t_1+\epsilon]}$ *is a shortest path with respect to the sequence* $e_1,e_2,\ldots,e_n$.

Loosely speaking, if the last segment of the first shortest path and the first segment of the second form a shortest path, then their concatenation is also a shortest path.

Proof. If $\gamma = \gamma_1 * \gamma_2$ is an $SP(a,c)_{(e_1,\ldots,e_k)}$, then

$$\zeta := \gamma_{1|[t_1-\epsilon,t_1]} * \gamma_{2|[t_1,t_1+\epsilon]} = \gamma_{|[t_1-\epsilon,t_1+\epsilon]}$$

is a shortest path with respect to some sequence of line segments for each $\epsilon \leq \min\{t_1 - t_0, t_2 - t_1\}$ (Lemma 1.2). Conversely suppose that ζ is a shortest path with respect to some sequence of line segments and satisfies assumption (A). Since

$$\gamma_{1|[t_1-\epsilon,t_1]} = \zeta_{|[t_1-\epsilon,t_1]},$$

applying Theorem 1.3 to γ_1 and ζ we find that $\xi = \gamma_1 * \zeta_{|[t_1,t_1+\epsilon]}$ is a shortest path with respect to some sequence of line segments. Applying Theorem 1.3 again to ξ and γ_2 we obtain that

$$\gamma = \xi * \gamma_{2|[t_1+\epsilon,t_2]}$$

is an $SP(a, c)_{(e_1,\ldots,e_k)}$. □

An et al. [13] and Hoai et al. [33] used a *direct multiple shooting method* to approximate shortest paths in a simple polygon in $\mathbb{R}^2$ and on surfaces of convex polytopes in $\mathbb{R}^3$. They divided the given region into subregions, found a shortest path in each subregion and then "glued" them. They applied one straightness condition, a version of Corollary 1.9, to adjust the paths to get better aproximations.

1.3 Conclusions

We have presented the existence and uniqueness of the shortest path (Theorem 1.1 and Corollary 1.2). We then have investigated some characteristics of angles between the shortest path and line segments e_is, especially when the path went through common points of e_is. Theorem 1.2 and Corollaries 1.7–1.8 have given a step-by-step method to determine whether a path is shortest. Sufficient conditions under which concatenation of two shortest paths is shortest have been also given (Theorem 1.3 and Corollary 1.9).

Chapter 2

Straightest Paths on a Sequence of Adjacent Polygons

Back to the shortest path problems, as we know, a line segment is the shortest path joining its two endpoints in 3D space. A very common technique is unfolding. Many researchers use it to solve the shortest path subproblem between two points on a sequence of triangles. If we can draw a line segment between two images inside the simple polygon after unfolding the sequence of triangles, then the inverse image of this line segment is the shortest path joining two given points. The question is how to find this shortest path without unfolding? Under the idea of the straightest geodesic of Polthier and Schmies [48] and the thought of answering the above question, we come to the definition of straightest paths.

The present chapter is written on the basis of the paper [2] in the List of Author's Related Papers on page 72 of this dissertation.

2.1 Straightest Paths

In [48] Polthier and Schmies presented a new concept of geodesics: straightest geodesics are paths that have equal path angle on both sides at each point. In this section we consider "straightest paths" which are slightly differ from the original and in fact are particular shortest paths with respect to some sequence of line segments. Let $\mathcal{S} = (f_1, f_2, \ldots, f_{k+1})$ be a sequence of (not necessary distinct) adjacent convex polygons in $\mathbb{R}^2$ or $\mathbb{R}^3$, then

$$e_i := f_i \cap f_{i+1} \text{ is an edge for } 1 \leq i \leq k.$$

Sometimes we make the following assumption.

(A) *The restriction of the path on each non-singleton interval of its domain has positive length.*

Definition 2.1. A *straightest path* $\gamma : [t_0, t_1] \to \cup_{i=1}^{k+1} f_i$ on the sequence $\mathcal{S}$ is a path that satisfies assumption (A) and the following conditions:

(a) there exist integers $1 \leq m \leq n \leq k$ and a sequence of numbers

$$t_0 =: \bar{t}_{m-1} < \bar{t}_m \leq \cdots \leq \bar{t}_n < \bar{t}_{n+1} := t_1$$

such that

$$x_i := \gamma(\bar{t}_i) \in e_i \,\text{for}\, m \leq i \leq n$$

and

$$x_{m-1} := \gamma(t_0) \in f_m, x_{n+1} := \gamma(t_1) \in f_{n+1};$$

(b) $l(\gamma_{|[\bar{t}_i, \bar{t}_{i+1}]}) = \|x_i - x_{i+1}\|$ for $m - 1 \leq i \leq n$;

(c) For $z_i \in e_i$, $z_i \neq x_i$, we have

$$\angle(x_{i-1} - x_i, z_i - x_i) + \angle(z_i - x_i, x_{i+1} - x_i) = \pi$$

if $x_i \in e_i$ is not a common vertex (a common endpoint of e_i and its adjacent line segments), and

$$\angle(x_{i^*-1} - x_i, z_{i^*} - x_i) + \sum_{j=i^*}^{r-1} \angle(z_j - x_i, z_{j+1} - x_i) + \angle(z_r - x_i, x_{r+1} - x_i) = \pi$$

if $x_{i^*-1} \neq x_{i^*} = x_{i^*+1} = \cdots = x_r \neq x_{r+1}$ and $i^* \leq i \leq r$.

It is conventional to define straightest path joining a with b on the same polygon (with respect to an empty sequence of common edges) to be any path that joins a with b, one-to-one, and has length $\|a - b\|$, i.e., a $SP(a, b)_\emptyset$.

Condition (a) states that a straightest path is a path with respect to the sequence $e_m, e_{m+1}, \ldots, e_n$. Condition (b) and Lemma 1.1 imply that $\gamma([\bar{t}_i, \bar{t}_{i+1}]) = [x_i, x_{i+1}] \subset f_{i+1}$ for $m - 1 \leq i \leq n$. We observe also that condition (c) does not depend on the choice of $z_i \in e_i$, $m \leq i \leq n$. Corollary 1.8 and Theorem 1.2 show that γ is an $SP(x_{m-1}, x_{n+1})_{(e_m,\ldots,e_n)}$. Note also that assumption (A), the conditions $t_0 < \bar{t}_m$, $\bar{t}_n < t_1$, and (b) imply that $x_0 \neq x_1$ and $x_n \neq x_{n+1}$. These conditions do not restrict the definition of straightest paths since if x_0 belongs to a common edge then we choose

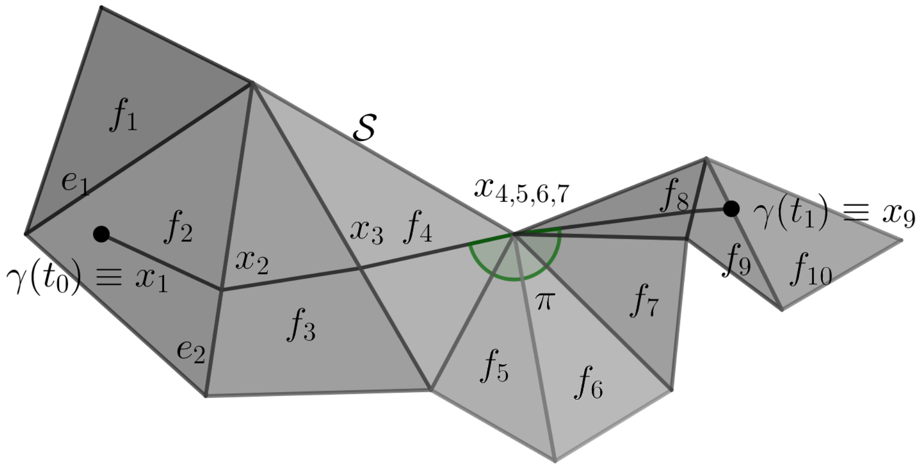


Figure 2.1: Straightest path γ from f_2 to f_9 on a sequence $\mathcal{S} = \bigcup_{i=1}^{i=10} f_i$

$m = \max\{j : x_0 \in f_j\}$. Similarly, we can assume that x_{n+1} does not belong to e_n.

There are reasons why we study straightest paths: First the shortest path joining two given points on the surface of a convex polyhedron $\mathcal{D} \subset \mathbb{R}^3$ does not go through any vertex of $\mathcal{D}$ (see [52], Lemma 4.1). Thus, by Corollary 1.7, the path is straightest. Second, every shortest path with respect to some sequence of line segments joining two points on a surface $\mathcal{S}$ consisting of convex polygons is composed of straightest paths joining vertexes of $\mathcal{S}$ (Proposition 2.1).

2.2 An Initial Value Problem on a Sequence of Adjacent Polygons

Here we recall the problem of an initial value problem in differential equations (see [25], p. 1-2).
Let D denote an open subset of $\mathbb{R} \times \mathbb{R}^n$, $f : D \to \mathbb{R}^n$ continuous. Then the *initial value problem* is a differential equation

$$x'(t) = f(t, x(t)), (' = \frac{d}{dt})$$

together with a point in the domain of f

$$(t_0, x_0) \in D,$$

called the *initial condition.*

With the straightest paths, we have also a similar initial value problem as follows.

Let $\gamma([t_0, t_1])$ be a straightest path on $\mathcal{S} = (f_1, f_2, \dots, f_{k+1})$, $\gamma(t_0) \in f_m$, and v a nonzero vector that is parallel to f_m. If there exists $t^* > t_0$ such that

$$\gamma([t_0, t^*]) \subset f_m$$

and

$$\gamma(t^*) - \gamma(t_0) = \lambda v \,\text{for some}\, \lambda > 0$$

we say that γ *starts* at $\gamma(t_0)$ in the direction of v.

Theorem 2.1. *(An initial value problem) Let $\mathcal{S} = (f_1, f_2, \dots, f_{k+1})$ be a sequence of adjacent convex polygons. Let $a, p \in f_m$, $a \neq p$, and $v = p - a$. Then there exists a unique longest straightest path $\gamma([t_0, t_1])$ on $\mathcal{S}$ starting at a and in the direction of v. Moreover, if $\eta([\tau_0, \tau_1])$ is any straightest path on $\mathcal{S}$ starting at a and in the direction of v, then η equals $\gamma_{|[t_0,t^*]}$ for some $t_0 \leq t^* \leq t_1$. Thus every straightest path on $\mathcal{S}$ can be extended to a longest straightest path.*

Proof. There is nothing to prove if $k = 0$, $\mathcal{S} = \{f_1\}$. Thus we assume, without loss of the generality, that $k \geq 1$, $a, p \in f_1$, and $a \notin f_2$.
First we construct γ. Denote $e_i = f_i \cap f_{i+1}$, $i = 1, \dots, k$. Set

$$t_0 = 0 \,\text{and}\, \bar{t}_1 = \max\{t : a + tv \in f_1\}.$$

Since $p = a + v \in f_1$, $\bar{t}_1 \geq 1$. Define

$$\gamma_1 : [0, \bar{t}_1] \to f_1$$

by

$$\gamma_1(0) = a \gamma_1(\bar{t}_1) = x_1 := a + \bar{t}_1 v \in f_1,$$

and

$$\gamma_1 \,\text{is affine on}\, [0, \bar{t}_1].$$

If $x_1 \notin e_1$ we put $t_1 := \bar{t}_1$ and $\gamma := \gamma_1$.
Suppose $x_1 \in e_1$. Choose any $z_1 \in e_1$, $z_1 \neq x_1$. If there exists $p_1 \in f_2$ such that

$$\angle(a - x_1, z_1 - x_1) + \angle(z_1 - x_1, p_1 - x_1) = \pi, \tag{2.1}$$

set

$$v_1 := p_1 - x_1, \sigma_1 := \max\{t : x_1 + tv_1 \in f_2\} \geq 1, \bar{t}_2 := \bar{t}_1 + \sigma_1,$$

and

$$x_2 := x_1 + \sigma_1 v_1 \in f_2.$$

We then define

$$\gamma_2 : [\bar{t}_1, \bar{t}_2] \to f_2$$

by

$$\gamma_2(\bar{t}_1) = x_1, \gamma_2(\bar{t}_2) = x_2,$$

and

$$\gamma_2 \text{ is affine on } [\bar{t}_1, \bar{t}_2].$$

Clearly $\gamma_1 * \gamma_2$ is a straightest path starting at a, in the direction of v, and joining a with x_2. Observe that such a p_1 always exists if x_1 is an interior point of e_1.

Suppose x_1 is an endpoint of e_1 and there is no $p_1 \in f_2$ satisfying (2.1). If x_1 is not a common vertex of e_1 and e_2, set

$$t_1 := \bar{t}_1 \text{ and } \gamma := \gamma_1.$$

Assume that x_1 is a common vertex of e_1 and e_2. Let

$$r := \max\{i : x_1 \in e_j,\ 1 \leq j \leq i\}.$$

Choose

$$z_j \in e_j, z_j \neq x_1, 1 \leq j \leq r,$$

and let z^*_{r+1} be a point on the second edge of f_{r+1} that has an endpoint x_1 and $z^*_{r+1} \neq x_1$. Let

$$\begin{aligned}\theta_1 :=& \angle(a - x_1, z_1 - x_1) + \angle(z_1 - x_1, z_2 - x_1) \\ &+ \cdots + \angle(z_{r-1} - x_1, z_r - x_1) + \angle(z_r - x_1, z^*_{r+1} - x_1).\end{aligned}$$

We call θ_1 the *angle of incidence at* x_1. If $\theta_1 < \pi$, set

$$t_1 := \bar{t}_1 \text{ and } \gamma := \gamma_1.$$

Assume $\theta_1 \geq \pi$. Put

$$s := \max\left\{i : \angle(a-x_1, z_1-x_1) + \angle(z_1-x_1, z_2-x_1) + \cdots + \angle(z_{i-1}-x_1, z_i-x_1) < \pi\right\}.$$

Since there is no $p_1 \in f_2$ satisfying (2.1), $2 \le s \le r$ and there exists $p_s \in f_{s+1} \setminus f_s$, such that

$$\angle(a - x_1, z_1 - x_1) + \angle(z_1 - x_1, z_2 - x_1) + \cdots + \angle(z_s - x_1, p_s - x_1) = \pi.$$

Set $\bar{t}_2 = \cdots = \bar{t}_s := \bar{t}_1$, $x_2 = \cdots = x_s := x_1$, and

$$v_s := p_s - x_1, \quad \sigma_s := \max\{t : x_s + tv_s \in f_{s+1}\}, \quad \bar{t}_{s+1} := \bar{t}_s + \sigma_s, \quad x_{s+1} := x_s + \sigma_s v_s.$$

As $\sigma_s \ge 1$, $\bar{t}_{s+1} > \bar{t}_s$. Define

$$\begin{aligned}
\gamma_i : [\bar{t}_{i-1}, \bar{t}_i] &\to f_i, \\
\gamma_i(t) &= x_1, 2 \le i \le s, \\
\gamma_{s+1} : [\bar{t}_s, \bar{t}_{s+1}] &\to f_{s+1}, \\
\gamma_{s+1}(\bar{t}_s) &= x_s, \\
\gamma_{s+1}(\bar{t}_{s+1}) &= x_{s+1},
\end{aligned}$$

and γ_{s+1} is affine.

From the construction we find that $\gamma_1 * \gamma_2 * \cdots * \gamma_{s+1}$ is a straightest path joining a with x_{s+1} and starting at a in the direction of v.

We continue in this fashion to finally obtain a sequence of points $x_1, \ldots, x_{n+1}$ satisfying the following conditions

i) $x_n \in e_n$, $x_{n+1} \in f_{n+1} \setminus f_n$, and $1 = \max\{t > 0 : x_n + t(x_{n+1} - x_n) \in f_{n+1}\}$;

ii) either $x_{n+1} \notin e_{n+1}$ (for instance, when $n = k$) or

iii) $x_{n+1} \in e_{n+1}$ and the angle of incidence at x_{n+1} is less than π.

We then set $t_1 := \bar{t}_{n+1} = \bar{t}_n + \sigma_n$ and define

$$\begin{aligned}
&\gamma_{n+1} : [\bar{t}_n, t_1] \to f_{n+1} \text{ by} \\
&\gamma_{n+1}(\bar{t}_n) = x_n, \\
&\gamma_{n+1}(t_1) = x_{n+1} = x_n + \sigma_n v_n,
\end{aligned}$$

and γ_{n+1} is affine.

Let

$$\gamma = \gamma_1 * \gamma_2 * \cdots * \gamma_{n+1}.$$

We find that γ satisfies assumption (A) and is an $P(a, x_{n+1})_{(e_1,\ldots,e_n)}$. Moreover, γ also satisfies conditions (b) and (c) of Definition 2.1. Thus γ is a straightest path starting at a and in the direction of v.

Suppose now that $\eta([\tau_0, \tau_1])$ is a straightest path on $\mathcal{S}$ starting at a and in the direction of v. Let $c := \eta(\tau_1)$. The step-by-step construction of γ shows that there exists $t_c \in]t_0, t_1]$ such that $c = \gamma(t_c)$ and η and $\gamma_{|[t_0,t_c]}$ are shortest paths joining a with c with respect to the same sequence of line segments. Thus by the uniqueness of shortest path (Theorem 1.1), η equals $\gamma_{|[t_1,t_c]}$. This implies also that η can be extended to γ and γ is the unique longest straightest path. □

Remark 2.1. The shortest path joining two given points on a sequence of adjacent convex polygons always exists but this may not be true for the straightest path (for example, see Figure 2.2).

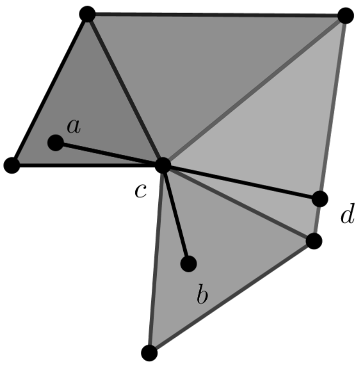


Figure 2.2: The shortest path from a to b along this sequence of triangles is $[a, c] \cup [c, b]$. The straightest path from a to b on this sequence doesn't exist.

We now investigate the image of a straightest path under a planar unfolding.

Suppose that $\gamma([t_0, t_1])$ is a straightest path joining

$$a \in f_1 \text{ and } b \in f_{k+1}, t_0 < \bar{t}_1 \leq \cdots \leq \bar{t}_k < t_1, x_i := \gamma(\bar{t}_i) \in e_i, \text{ for } 1 \leq i \leq k.$$

For each i, choose

$$z_i \in e_i, z_i \neq x_i.$$

Let a_1 be the image of a under the planar unfolding around e_1. Since $a \neq x_1$, $a_1 \neq x_1$. If $x_1 \neq x_2$, we have

$$\angle(a_1 - x_1, z_1 - x_1) = \angle(a - x_1, z_1 - x_1)$$

so that

$$\angle(a_1 - x_1, z_1 - x_1) + \angle(z_1 - x_1, x_2 - x_1) = \pi.$$

Hence

$$x_1 \in]a_1, x_2[\,,$$

i.e., a_1, x_1, x_2 are collinear. Similarly, let a_2 and $x_{1,2}$ be the images of a_1 and x_1 under the planar unfolding around e_2, respectively. If $x_2 \neq x_3$, then

$$\angle(x_{1,2} - x_2, z_2 - x_2) = \angle(x_1 - x_2, z_2 - x_2)$$

whence

$$\angle(x_{1,2} - x_2, z_2 - x_2) + \angle(z_2 - x_2, x_3 - x_2) = \pi,$$

i.e., $a_2, x_{1,2}, x_2, x_3$ are collinear. If $x_1 = x_2 = \cdots = x_r \neq x_{r+1}$, then by condition (c)

$$a_r, x_1, x_{r+1} \text{ are collinear},$$

where

$$a_r \text{ is the image of } a \text{ under the planar unfolding around } e_1, e_2, \ldots, e_r.$$

Repeating this argument we finally find that the image of γ under the planar unfolding around $e_1, \ldots, e_k$ is a line segment. Conversely, if the image of γ under the planar unfolding around $e_1, \ldots, e_k$ is a line segment, then the angles of γ at edges are π, so γ is a straightest path. We thus arrive at the following result.

Lemma 2.1. *Let γ be a $P(a,b)_{(e_1,\ldots,e_k)}$ ($a \in f_1$, $b \in f_{k+1}$) on the sequence $\mathcal{S}$ in $\mathbb{R}^3$. Then γ is straightest iff its planar unfolding around $e_1, \ldots, e_k$ is a line segment.*

Thus straightest paths are paths whose images under planar unfoldings are line segments. Therefore if there exists a straightest path joining two points $a \in f_m$ and $b \in f_{n+1}$ on the sequence $\mathcal{S}$, ($m \leq n$), then it is the shortest path joining these points that lies entirely in the polygons and passes through $e_m, \ldots, e_n$. Conversely, we have the following result presented by O'Rourke et al. (see [44], Lemma 1).

Proposition 2.1. *Every shortest path joining two vertexes on $\mathcal{S} = (f_1, f_2, \ldots, f_{k+1})$ is composed of straightest paths joining vertexes of $\mathcal{S}$.*

Proof. Let $\gamma([t_0, t_1])$ be a shortest path joining two vertexes $a \in f_1$ and $b \in f_{k+1}$ on $\mathcal{S}$

$$t_0 < \bar{t}_1 \leq \cdots \leq \bar{t}_k < t_1, x_i = \gamma(\bar{t}_i) \in e_i, 1 \leq i \leq k.$$

If γ does not pass through any vertexes of $\mathcal{S}$ except a and b, then each x_i is an interior point of e_i. By Corollary 1.7, for

$$z_i \in e_i, z_i \neq x_i, i = 1, \ldots, k,$$

$$\angle(x_{i-1} - x_i, z_i - x_i) + \angle(z_i - x_i, x_{i+1} - x_i) = \pi,$$

where $x_0 := a$ and $x_{k+1} := b$. Thus γ is straightest.

If γ passes through vertexes

$$v_0 := a, v_1 = \gamma(t_1^*), \ldots, v_l = \gamma(t_l^*), v_{l+1} := b$$

(in this order) and there are no vertexes belonging to

$$\gamma(\,]t_0, t_1^*[\,), \gamma(\,]t_1^*, t_2^*[\,), \ldots, \gamma(\,]t_l^*, t_1[\,),$$

then by Lemma 1.2, restrictions of γ on $[t_0, t_1^*], \ldots, [t_l^*, t_1]$ are shortest paths joining vertexes v_j and v_{j+1} of $\mathcal{S}$. By the proof above, each restriction is a straightest path. Thus γ is composed of straightest paths joining vertexes of $\mathcal{S}$. □

Theorem 2.2. *Let $\mathcal{S} = (f_1, f_2, \ldots, f_{k+1})$ be a sequence of adjacent convex polygons and let a, q_1, q_2, q_3 be points in f_m such that $q_2 \in\,]q_1, q_3[$ and a, q_1, q_3 are not collinear. Let $v_i = q_i - a$, $i = 1, 2, 3$. Assume that γ_1, γ_2 and γ_3 are straightest paths starting at a and in the directions of v_1, v_2, v_3, respectively, and γ_1, γ_3 cut a line segment $e \subset f_n$ $(n \geq m)$ at y_1 and y_3, respectively. If γ_2 is the longest straightest path, then it meets e at some point $y_2 \in\,]y_1, y_3[\,$.*

Proof. We prove for the case $\mathcal{S}$ is in $\mathbb{R}^3$, the other case is similar. Without restriction of generality we assume that $m = 1$ and $n = k + 1$. Let $\mathcal{U}$ be the planar unfolding around the sequence $e_1, \ldots, e_k$. By Lemma 2.1, the images of $\gamma_1, \gamma_2, \gamma_3$ under $\mathcal{U}$ are line segments $[a', y_1^*]$, $[a', y_2^*]$, $[a', y_3^*]$, respectively and we have

$$y_1 \in [a', y_1^*] \,\text{and}\, y_3 \in [a', y_3^*].$$

Let q_1', q_2', q_3' be the images of q_1, q_2, q_3 under $\mathcal{U}$. Since q_2 is between q_1 and q_3, q_2' is between q_1' and q_3'. Assume

$$q_2' = \alpha q_1' + \beta q_3' \,\text{where}\, \alpha, \beta > 0, \alpha + \beta = 1.$$

Letting $v_i' := q_i' - a', i = 1, 2, 3,$ we get

$$v_2' = \alpha v_1' + \beta v_3'.$$

Since q_i lies on the path γ_i, and

$$q_i \neq a, q_2' \in \,]a', y_2^*], q_1' \in \,]a', y_1],$$

and $q_3' \in \,]a', y_3]$.
Assume

$$y_1 - a' = \lambda v_1' \text{ and } y_3 - a' = \nu v_3', \lambda, \nu > 0.$$

Set

$$\mu := \lambda\nu/(\alpha\nu + \beta\lambda) > 0.$$

We have

$$y_2 := a' + \mu v_2' = a' + \mu\alpha v_1' + \mu\beta v_3' = (\alpha\nu + \beta\lambda)^{-1}(\alpha\nu y_1 + \beta\lambda y_3) \in \,]y_1, y_3[\,.$$

Let $\mathcal{S}'$ be the sequence of images of $f_1, \ldots, f_{k+1}$ under the planar unfolding $\mathcal{U}$. $\mathcal{S}'$ is the sequence of convex polygons with adjacent edges being images of $e_1, \ldots, e_k$ under $\mathcal{U}$ and the triangle $a'y_1y_3$ is contained in the union of polygons in $\mathcal{S}'$. As the image of γ_2 is $[a', y_2^*]$, the longest line segment starting at a' in the direction of v_2', we have

$$[a', y_2] \subset [a', y_2^*].$$

This means that the line segment $[a', y_2^*]$ meets $[y_1, y_3]$ at $y_2 \in \,]y_1, y_3[\,$, i.e., the longest straightest path γ_2 meets e at y_2. □

Theorem 2.1 can be used to find a straightest path on a sequence of convex polygons from a given point and a given direction. Moreover, using Proposition 2.1 one can construct the shortest path joining two points and passing through a sequence of convex polygons from its finite straightest paths (see An [11], Theorem 5.1).

2.3 Conclusions

We have introduced the concept of straightest paths on a sequence S of adjacent convex polygons in $\mathbb{R}^2$ or $\mathbb{R}^3$ established a relation between shortest paths and straightest paths joining two points on S and considered a discrete initial value problem of straightest paths.

Chapter 3

Finding the Connected Orthogonal Convex Hull of a Finite Planar Point Set

In this chapter, we consider the third shortest path problem in the context of computer graphics and image processing: finding the connected orthogonal convex hull of a finite planar point set P. The solution should be an orthogonal convex polygon bounding P, with minimum perimeter and minimum area. For an arbitrary given finite point set, the connected orthogonal convex hulls of this set maybe not unique. Therefore, we consider some assumptions under that the uniqueness of the connected orthogonal convex hull of the given set is preserved. Then we study the construction of the connected orthogonal convex hull of a finite planar point set and build an efficient algorithm for finding it. In Chapter 3, we only consider the subsets of $\mathbb{R}^2$.

The present chapter is written on the basis of the paper [1] in the List of Author's Related Papers on page 72 of this dissertation.

3.1 Orthogonal Convex Sets and their Properties

Definition 3.1. (See [57], p. 1747) A set $K \subset \mathbb{R}^2$ is said to be *orthogonal convex* if its intersection with any horizontal or vertical line is convex.

You can see that a set $K \subset \mathbb{R}^2$ is orthogonal convex if its intersection with any horizontal or vertical line is empty, a point, or a line segment.

In some previous papers (see [45], Definition 2.1 and [57], p. 1747), a slightly different definition of orthogonal convexity were given. Here, we use the term "convex" to cover line segments with or without its endpoints. Furthermore, our definition can be extended for $\mathbb{R}^n$. Observe that any convex set is orthogonal convex as seen in Figure 3.1 (a), but the reverse may be not true as seen in Figure 3.1 (b).

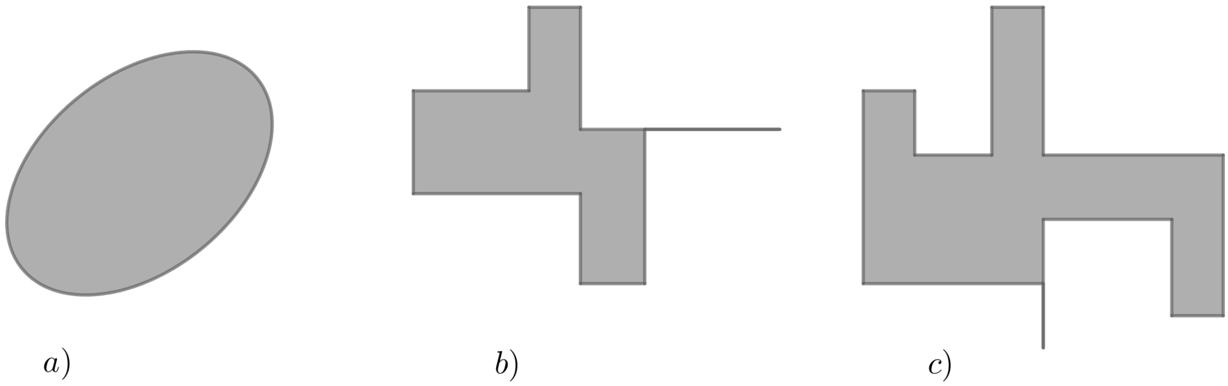


Figure 3.1: a) and b) two orthogonal convex sets; c) not an orthogonal convex set

K is said to be *connected orthogonal convex* if it is orthogonal convex and connected.

It is obvious that the intersection of any family (finite or infinite) of orthogonal convex sets is orthogonal convex. An *orthogonal convex hull* (see [41], Definition 5) of a set $K \subset \mathbb{R}^2$ is the smallest orthogonal convex set which contains K. Thus, the orthogonal convex hull of K is the intersection of all orthogonal convex sets containing K and therefore, the orthogonal convex hull of a set is unique. But it may be not connected.

Definition 3.2. (See [45], Definition 2.3) A *connected orthogonal convex hull* of K is a minimal connected orthogonal convex set containing K.

We use the term "minimal" here since all connected orthogonal sets containing the given set form a partially ordered set. As we know a partially ordered set can have at most one maximum and at most one minimum and may have multiple maximal or minimal elements.

In Figure 3.2 we display a set of three distinct points in the plane. Observe that the orthogonal convex hull of the set is itself, and it is disconnected as in Figure 3.2 (a). The connected orthogonal convex hulls of the set are not unique, as in Figure 3.2 (b, c).

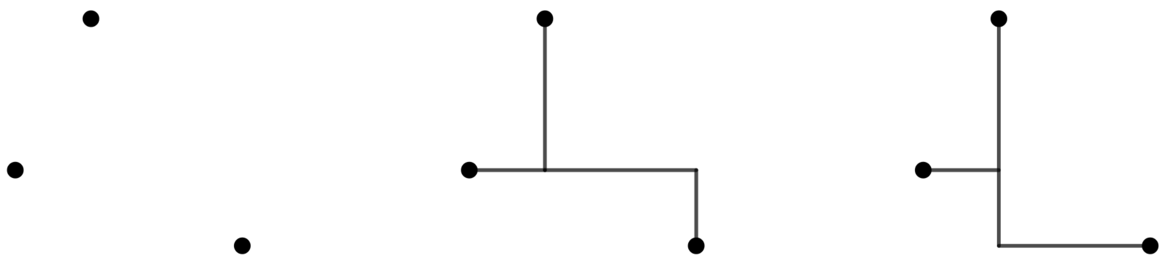


Figure 3.2: a) the set of three points in the plane and its orthogonal convex hull consists of these points; b), c) connected orthogonal convex hulls of these points

Note that a connected orthogonal convex hull of a finite planar point set is compact. We define a line to be *rectilinear* if the line is parallel to either x-axe or y-axe. A half line or a line segment are *rectilinear* if the lines on which they lie are rectilinear.

Let $a \neq b$ be two given points in the plane. We define an *orthogonal line* $\ell(a, b)$ through a, b as follows: if $x_a \neq x_b$ and $y_a \neq y_b$, $\ell(a, b)$ is the union of two rectilinear half lines having the same starting point, and if $x_a = x_b$ or $y_a = y_b$, $\ell(a, b)$ is the line through a and b.

Let $\ell(a, b)$ be an orthogonal line segment through a and b, and c be the common point of two half-lines of $\ell(a, b)$. We define the *orthogonal line segment* $s(a, b)$ be two line segments $[a, c] \cup [c, b]$.

Thus, an orthogonal line $\ell(a, b), (x_a \neq x_b, y_a \neq y_b)$ separates the plane into two regions, as shown in Figure 3.3. The quadrant region together with the orthogonal line $\ell(a, b)$ will be called a *quadrant* determined by the orthogonal line. In Figure 3.3, the quadrant regions are shaded.

We now define support lines for connected orthogonal convex hulls that are similar to tangent lines in the construction of convex hulls.

Definition 3.3. Given a set $K \subset \mathbb{R}^2$. An $\ell(a, b)$ is an orthogonal supporting line (O-support, for brevity) of a set K (a and b might not belong to K) if the intersection of $\ell(a, b)$ with K is non-empty and either all points of

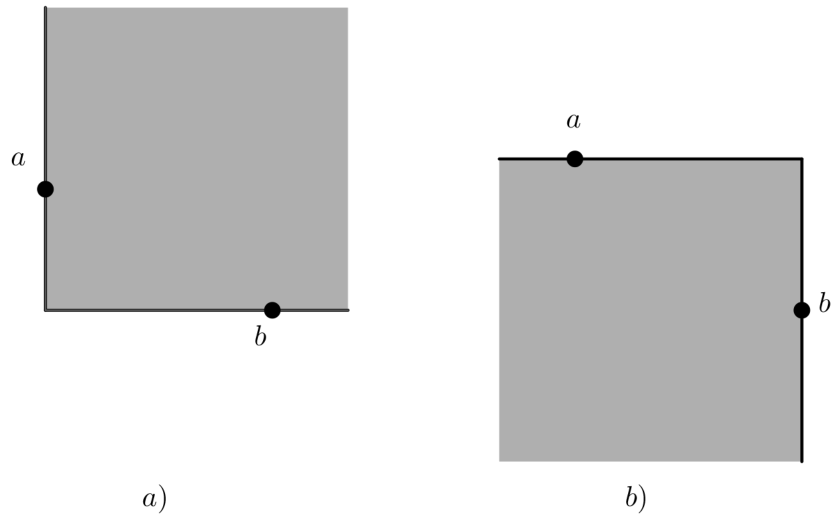


Figure 3.3: Orthogonal lines through the points a and b with $x_a \neq x_b$ and $y_a \neq y_b$ and their quadrants (shadow regions)

$K \setminus (K \cap \ell(a,b))$ are not on the quadrant of $\ell(a,b)(x_a \neq x_b, y_a \neq y_b)$, or all points of $K \setminus (K \cap \ell(a,b))$ are on one open half plane which is determined by the line $\ell(a,b)(x_a = x_b, \text{or}\, y_a = y_b)$.

We also call $\ell(a,b)$ be the *orthogonal supporting line segments*. In a concrete case, we only say O-supports without specific it is an orthogonal line or orthogonal line segments.

Two O-supports $\ell(a,b)$ and $\ell(c,d)$ of a set K are said to be *opposite* if their half lines meet in exactly two points. Such O-supports are indicated in Figure 3.4.

As we know the connected orthogonal convex hulls of a finite planar point set maybe not unique. We now define a new kind of a point of a connected orthogonal convex hull, then we will prove that when these points exist, the connected orthogonal convex hulls are not unique.

Let λ be a simple polyline in $\mathbb{R}^2$ whose edges $[v_{i-1}, v_i], 1 \leq i \leq n$, are parallel to the coordinate axes. Such a path λ is called a *staircase path* (see [20], p. 22) if and only if the associated vectors alternate in direction. That is, for i odd, the vectors $\overrightarrow{v_{i-1}v_i}$ have the same direction, and for i even, the vectors $\overrightarrow{v_{i-1}v_i}$ have the same direction, $1 \leq i \leq n$.

We denote by $\mathcal{F}(K)$ the family of all connected orthogonal convex hulls of K. For $E \in \mathcal{F}(K)$, if there exist two opposite O-supports H and L

of K intersecting in only two points, say p and q, with $x_p \neq x_q, y_p \neq y_q$, then there exists a staircase path connecting p and q in E (for example, the boundary from p to q of the rectangle having rectilinear edges and diagonal $[p, q]$). We define all points on such path (not including p and q) to be *semi-isolated* points of E. Thus, if E has a semi-isolated point, then E has infinity number of semi-isolated points. Therefore, the set of all semi-isolated points of elements of $\mathcal{F}(K)$ is the rectangle with the diagonal $[p, q]$ excepting $\{p, q\}$. Some semi-isolated points are illustrated in Figure 3.4.

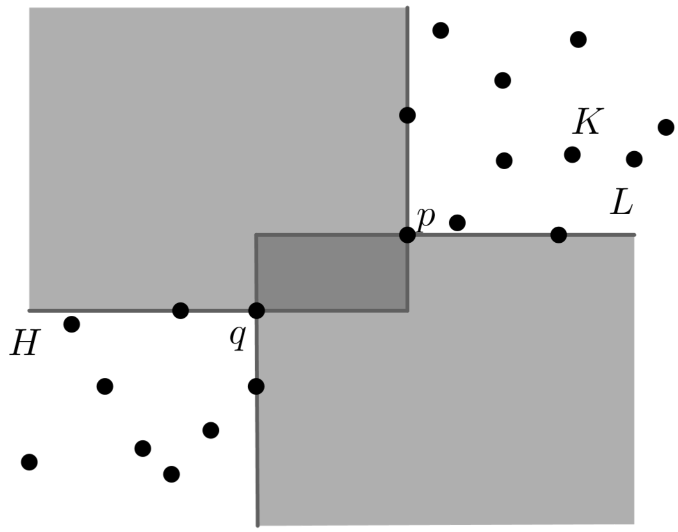


Figure 3.4: Two opposite O-supports H and L of the set K and the set of semi-isolated points (the rectangle with the diagonal $[p, q]$ excepting $\{p, q\}$)

It follows immediately from the definition of semi-isolated points the following result.

Proposition 3.1. *If $E \in \mathcal{F}(K)$ has no semi-isolated point, then every $F \in \mathcal{F}(K)$ also has no semi-isolated point. Therefore, the existence of semi-isolated points does not depend on each element $E \in \mathcal{F}(K)$.*

Proof. Indeed, on the contrary, suppose that E has no semi-isolated point and there exists $F \in \mathcal{F}(K)$, F has semi-isolated points. Since E has no semi-isolated point, there do not exist two O-supports of K such that they meet each other in exactly two points. By the definition of semi-isolated points, F cannot have semi-isolated point. Therefore, our contrary assumption is not true. By the above argument, the existence of semi-isolated points does not depend on each element $E \in \mathcal{F}(K)$. □

Note that the intersection of all connected orthogonal convex hulls of a set may be a disconnected set, and it also might not be the orthogonal convex hull of the set. As shown in Figure 3.2, connected orthogonal convex hulls of a finite planar point set may be countless. The following proposition provides some conditions such that the family of all connected orthogonal convex hulls of a given set has only one element.

Proposition 3.2. *Let P be a finite planar point set, and $\mathcal{F}(P)$ the family of all connected orthogonal convex hulls of P. If there exists an element of $\mathcal{F}(P)$ that has no semi-isolated point, then $\bigcap_{E\in\mathcal{F}(P)} E$ is a connected orthogonal convex hull of P. Therefore, $\mathcal{F}(P)$ has only one element.*

Proof. We will prove that $S = \bigcap_{E\in\mathcal{F}(P)} E$ is orthogonal convex and connected.

i) We first prove that S is orthogonal convex. Suppose that the intersection of S and a vertical (horizontal, respectively) line contains two points e and f, where e is the highest point and f is the lowest point (e is the leftmost point and f is the rightmost point, respectively). Then $e, f \in E$, for all $E \in \mathcal{F}(P)$. Since E is orthogonal convex, we get

$$[e, f] \subset E, \text{for all } E \in \mathcal{F}(P).$$

Therefore $[e, f] \subset S$. This implies that S is orthogonal convex.

ii) We now prove that S is connected. Suppose that S is disconnected. Then it has at least two connected closed components. As S is orthogonal convex, follow the result of two forms of an orthogonal convex set (see Ottmann et al. [45], Proposition 2.4), we assume that the first case occurs and that there are only two connected components T_1 "under" T_2 (case (a) in Figure 3.5), the second case can be proved similarly (case (b) in Figure 3.5). Therefore, we have $x_p < x_q, y_p < y_q$ for all $p \in T_1$ and $q \in T_2$. Let

a be the highest rightmost of T_1,

b be the rightmost highest point of T_1,

c be the lowest leftmost of T_2,

d be the leftmost lowest point of T_2.

Then there exist two O-supports of P one connects a and d and the other connects b and c. These O-supports intersect in each other.

This contradicts the fact that every element of $\mathcal{F}(P)$ has no semi-isolated point. Thus, S is connected.

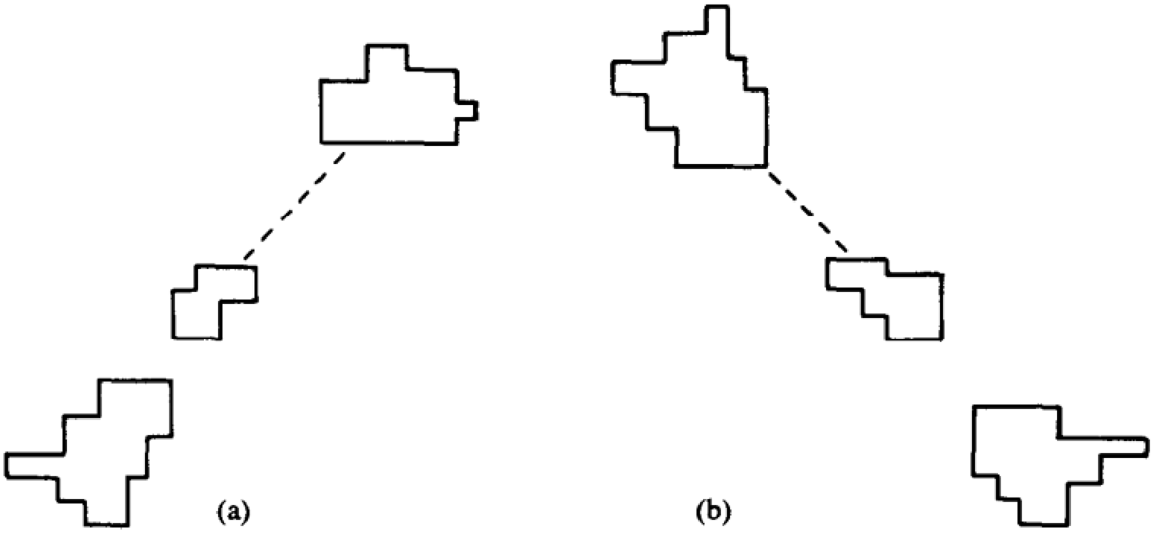


Figure 3.5: Two forms of an orthogonal convex set

Thus, S is the minimal connected orthogonal convex set containing P. Hence, $S \in \mathcal{F}(P)$ and therefore $\mathcal{F}(P)$ has only one element. The proof is complete. □

By Proposition 3.2, when no semi-isolated point can arise, $\mathcal{F}(P)$ has only one element, we then denoted it by COCH(P). From now on, when the notation COCH(P) is used, we understand that elements of $\mathcal{F}(P)$ have no semi-isolated point.

Corollary 3.1. *Let P be a finite set of points in the plane. Then* COCH(P) *is the intersection of all connected orthogonal convex sets containing P.*

Proof. Let T be the intersection of all connected orthogonal convex sets containing P. We now prove that COCH(P) $= T$.
Since COCH(P) is a connected orthogonal convex set, $T \subset$ COCH(P).
For each F is a connected orthogonal convex set, we have COCH(P) $\subset F$.
Thus COCH(P) $\subset T$. □

The intersection of all connected orthogonal convex sets which contains P might not be the orthogonal convex hull of P, even when we assume that every connected orthogonal convex hull of P has no semi-isolated point (see Figure 3.6).

The converse of Proposition 3.2 is also true by the following.

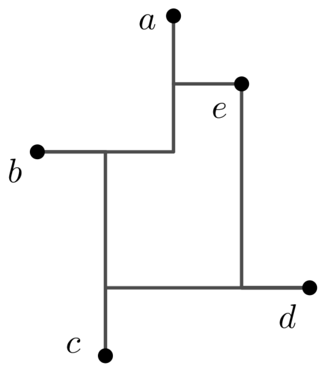


Figure 3.6: Every connected orthogonal convex hull of $\{a, b, c, d, e\}$ has no semi-isolated point.

Remark 3.1. Let P be a finite set of points in the plane and $\mathcal{F}(P)$ be the family of all connected orthogonal convex hulls of P . If $\cap\{E : E \in \mathcal{F}(P)\}$ is a connected orthogonal convex hull of P then all elements of $\mathcal{F}(P)$ has no semi-isolated points. Indeed, let $S = \cap\{E : E \in \mathcal{F}(P)\}$. As S is a connected orthogonal convex hull of $P, S \subset E$, for all $E \in \mathcal{F}(P)$. By the definition of connected orthogonal convex hull of P, E is a minimal connected orthogonal set which contains P. Thus $E \subset S$. Hence $S = E$, for all $E \in \mathcal{F}(P)$. It means that $\mathcal{F}(P)$ has only one element which is S. We next prove that S has no semi-isolated point. Indeed, on the contrary, suppose that S has semi-isolated points, there exist two opposite $O-$supports H and L of P intersecting in only two points, say p and q, with $x_p \neq x_q; y_p \neq y_q$. There exists a staircase path γ connecting p and q. Let $\mathcal{R}$ be the minimum rectilinear bounded rectangle of P. Let A be the set which is the union of γ and two closed rectangles made by H, L and $\mathcal{R}$. Two closed rectangles are not shaded as in Figure 3.7. Then A is connected orthogonal convex set which contains P. By the definition of connected orthogonal convex hull, there exists a connected orthogonal convex hull, say S_0, which is a minimal connected orthogonal convex set such that $S_0 \subset A$. Clearly, $\gamma \subset S_0$. Since there are many staircase paths as γ, there exist several distinct connected orthogonal convex hulls as S_0. This contradicts the fact that $\mathcal{F}(P)$ has only one element which is S.

Remark 3.2. Let P be a finite set of points in the plane. If the orthogonal convex hull of P is connected, then it is also the connected orthogonal convex hull of P. Indeed, let P_1, P_2 be two finite point sets in the plane and $P_1 \subset P_2$.

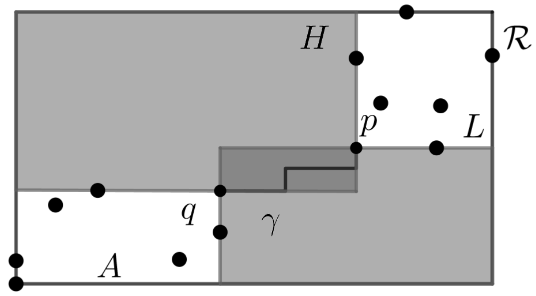


Figure 3.7: Illustration of Remark 3.1

Then, the convex hull of P_1 is also a subset of the convex hull of P_2. Here, $\mathrm{COCH}(P_1) \subset \mathrm{COCH}(P_2)$ holds under the assumption of no semi-isolated point.

Remark 3.3. Let P_1, P_2 be two finite point sets in the plane and $P_1 \subset P_2$. Suppose that there exist $E \in \mathcal{F}(P_1), F \in \mathcal{F}(P_2)$ such that E and F have no semi-isolated point. Then $\mathrm{COCH}(P_1) \subset \mathrm{COCH}(P_2)$. Note that $P_1 \subset P_2$ yields $P_1 \subset P_2 \subset \mathrm{COCH}(P_2)$ and $\mathrm{COCH}(P_2)$ is a connected orthogonal convex set containing P_1. By Proposition 3.2, $\mathrm{COCH}(P_1)$ is the connected orthogonal convex hull of P_1. Therefore, $\mathrm{COCH}(P_1) \subset \mathrm{COCH}(P_2)$.

In the plane, two perpendicular rectilinear lines that define an orthogonal line $\ell(a, b)$ divide the plane into four open *corners*: I, II, III and IV, and two rectilinear lines, as shown in Figure 3.8. Two corners I and III (or II and IV) are opposite. Thus, II and two half lines through a, b form the quadrant of $\ell(a, b)$.

Proposition 3.3. *Let P be a finite set of points in the plane and the following condition hold*

(B) [1] *There exists an O-support $\ell(a, b)$ of P $(a, b \in P)$ such that all points of $P \setminus \{a, b\}$ lie only in both two opposite corners of $\ell(a, b)$.*

Then there exists a connected orthogonal convex hull of P that has semi-isolated points and vice versa.

Proof. Suppose that all points of $P \setminus \{a, b\}$ lie only in two opposite corners

[1] (A) condition in paper [15].

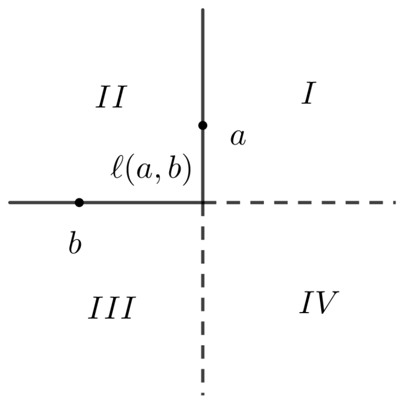


Figure 3.8: Four corners I, II, III and IV of an orthogonal line $\ell(a,b)$. Corner II belongs to the quadrant of $\ell(a,b)$.

I and III of $\ell(a,b)$. Let

c be a point of corner I in that set having the smallest y-coordinate,

d be a point of corner III in that set having the greatest x-coordinate

(because P is finte, such c and d exist). Then an orthogonal line $\ell(c,d)$ is an O-support and the intersection of $\ell(c,d)$ and $\ell(a,b)$ is exact two distinct points. Therefore, connected orthogonal convex hulls of P have semi-isolated points. the reverse is obtained from the definition of semi-isolated points. Thus the proof is complete. □

Let P be a finite planar point set. By identifying four points of P with minimum and maximum x and y coordinates: one adds these points to the convex hull and discards all points falling inside the quadrilateral form. We call this the "throw-away" principle (see [7], p. 219).

Let us describe COCH(P) when P is a finite planar point set. Based on the "throw-away" principle, we take the points a, b, c, d, e, f, g, and h belonging to P such that

(C) a (b, respectively) is the leftmost (rightmost, respectively) of highest points of P,

- e (f, respectively) is the rightmost (leftmost, respectively) of lowest points of P,
- c (d, respectively) is the highest (lowest, respectively) of rightmost points of P,

- g (h, respectively) is the lowest (highest, respectively) of leftmost points of P.

Then the rectangle $pquv$ formed by a, b, c, d, e, f, g, h is the minimum rectangle containing P and its edges are rectilinear (p is the intersection of the lines through ab and gh, q is the intersection of the lines through ab and cd, u is the intersection of the lines through cd and ef, v is the intersection of the lines through ef and gh). Assume without loss of generality that p, q, u, v are pairwise different. This rectangle $pquv$ is connected orthogonal convex and therefore contains COCH(P). In addition, the rectilinear line segments $[a, b], [c, d], [e, f], [g, h]$ belong the boundary of COCH(P) as (C) holds and COCH(P) is orthogonal convex. Assume without loss of generality that p, q, u, v are pairwise different and $a \neq p$.

Suppose that $E \in \mathcal{F}(P)$ has semi-isolated points. According to Proposition 3.3, there are two opposite O-supports of P such that one lies in $\ell(a, h)$ and the other lies in $\ell(d, e)$, or one lies in $\ell(b, c)$ and the other lies in $\ell(f, g)$. Based on Proposition 3.3, procedure Semi-Isolated_Point($P, flag$) finds out whether a connected orthogonal convex hull of a set of points has semi-isolated points or not.

From now on, we only consider the finite set of points P in the plane such that $|P| > 1$ and P does not satisfy the condition (B).

Proposition 3.4. *Let P be a finite planar point set. For two distinct points a and b in P there exists at most one O-support of P through a and b.*

Proof. Suppose $\ell_1(a, b)$ through a and b having starting point at (x_a, y_b) is an O-support of P and $\ell_2(a, b)$ is another orthogonal line through a and b having starting point (x_b, y_a). Then $\ell_1(a, b)$ and $\ell_2(a, b)$ are opposite O-supports of P. Since P does not satisfy the condition (B), there is no semi-isolated point. Hence, there exists a point of P that is in the rectangle determined by $\ell_1(a, b)$ and $\ell_2(a, b)$. Therefore, $\ell_2(a, b)$, from the definition of O-support, cannot be an O-support. □

3.2 Construction of the Connected Orthogonal Convex Hull of a Finite Planar Point Set

We need some definitions.

A *rectilinear polygon* (see [41], Definition 1) is a simple polygon whose edges are rectilinear (i.e., they are parallel to either x or y axis). The polygon has therefore only 90 and 270 degree internal angles.

Definition 3.4. (See [41], Definition 2) An $(x, y)-$*polygon* is one of the following: a) a point; b) connected rectilinear line segments; c) a rectilinear polygon; and d) a connected union of type b) and/or type c) rectilinear polygons.

Let us describe COCH(P) when P is a finite planar point set. Take the points a, b, c, d, e, f, g, and h belonging to P satisfying (C).

Consider the case $a \neq h, g \neq f, e \neq d$ and $c \neq b$. Take the orthogonal lines $\ell(a, h)$ through a and h, $\ell(g, f)$ through g and f , $\ell(e, d)$ through e and d, and $\ell(c, b)$ through c and b such that $p \notin \ell(a, h)$, $v \notin \ell(g, f)$, $u \notin \ell(e, d)$, and $q \notin \ell(c, b)$ (see Figure 3.9). We define

(D) If $a \neq h$, P_{ah} is the set of points of P in the quadrant of $\ell(a, h)$. Otherwise, $P_{ah} := \{a\}$.

- If $b \neq c$, P_{cb} is the set of points of P in the quadrant of $\ell(c, b)$. Otherwise, $P_{cb} := \{b\}$.
- If $g \neq f$, P_{gf} is the set of points of P in the quadrant of $\ell(g, f)$. Otherwise, $P_{gf} := \{f\}$.
- If $e \neq d$, P_{ed} is the set of points of P in the quadrant of $\ell(e, d)$. Otherwise, $P_{ed} := \{e\}$.

We use the scanline technique given in ([34], p. 231–234) to find the successive layers of maximal elements of P_{ah}, where a point $w \in P_{ah}$ is maximal if there are no other points z of P_{ah} such that $x_w \geq x_z$ and $y_w \leq y_z$. Assume that $p_0 = a, p_1, \ldots, p_{k-1}, p_k = h$ are such maximal elements. They form a staircase path, say, $\mathcal{P}_{ah}$ (i.e., a union of parts of O-supports through p_i and p_{i+1}, $i = 0, \ldots, k-1$) joining a and h such that the region formed by this path and $\ell(a, h)$ consists of P_{ah} and the area of this region is minimum in the set of all orthogonal polygons containing $P \cap P_{ah}$.

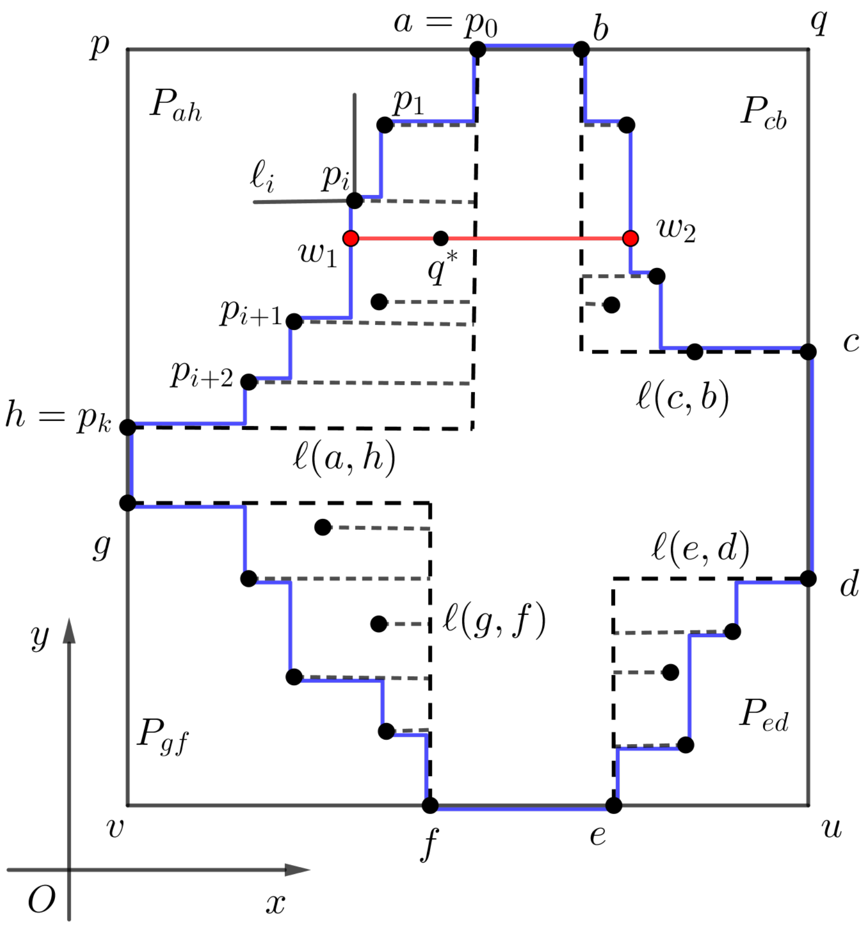


Figure 3.9: The case $a \neq h$. A staircase path (coloured in blue) joining a and h is formed by maximal elements $p_0 = a, p_1, \ldots, p_{k-1}, p_k = h$ of the set P_{ah}.

Thus there is an orthogonal convex (x, y)-polygon, say $T(P)$, formed by the rectilinear line segments $[a, b], [c, d], [e, f], [g, h]$ and staircase paths $\mathcal{P}_{cb}$, $\mathcal{P}_{ah}$, $\mathcal{P}_{gf}$, $\mathcal{P}_{ed}$, joining c and b, a and h, g and f, e and d, respectively.

Proposition 3.5. *Let P be a finite planar point set and $\mathcal{F}(P)$ be the family of all connected orthogonal convex hulls of P. Then the intersection* COCH(P) *of all connected orthogonal convex sets is an orthogonal convex (x, y)-polygon formed by the rectilinear line segments $[a, b], [c, d], [e, f], [g, h]$ and staircase paths $\mathcal{P}_{cb}$, $\mathcal{P}_{ah}$, $\mathcal{P}_{gf}$, $\mathcal{P}_{ed}$.*

Proof. Suppose that $T(P)$ is the region formed by $[a, b]$, $[c, d]$, $[e, f]$, $[g, h]$, $\mathcal{P}_{cb}$, $\mathcal{P}_{ah}$, $\mathcal{P}_{gf}$, $\mathcal{P}_{ed}$. By the construction of $T(P)$, we have $P \subset T(P)$ and $T(P)$ has no semi-isolated point. It follows from Remark 3.3 that COCH(P) $\subset$ T(P). We are in position to prove that $T(P) \subset$ COCH(P). Assume the contrary that there exists $q^* \in T(P)$. Because COCH(P) has no semi-isolated point, a

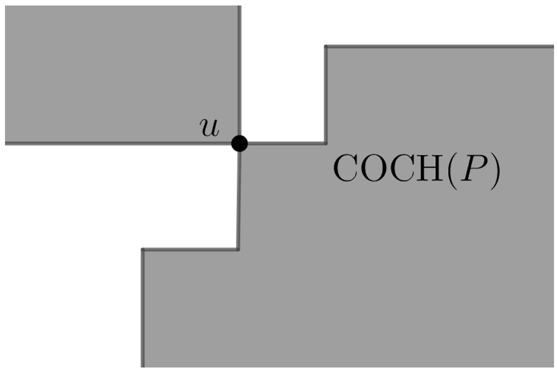


Figure 3.10: u is an extreme point of COCH(P).

horizonal line through q^* intersects two of staircase paths $\mathcal{P}_{cb}$, $\mathcal{P}_{ah}$, $\mathcal{P}_{gf}$, $\mathcal{P}_{ed}$, say at $w_1 \in \mathcal{P}_{ah}$ and $w_2 \in \mathcal{P}_{cb}$ (see Figure 3.9). Assume that w_1 belongs to the O-support L of P through p_i, p_{i+1} and between these points. Because P does not satisfy (B), the part of L between p_i and p_{i+1} belongs to COCH(P). It implies that $w_1 \in$ COCH(P). Similarly, $w_2 \in$ COCH(P). Therefore, $q^* \in [w_1, w_2] \subset$ COCH(P). Hence, $T(P) \subset$ COCH(P). □

Definition 3.5. Let P be a finite planar point set. We define a point $u \in$ COCH(P) to be an *extreme* point of COCH(P) if there exists an orthogonal line L (u is the starting point of two half lines of L) whose intersection with COCH(P) is only u and there is no point of COCH(P) $\setminus \{u\}$ which lies in the quadrant determined by L. We denote all extreme points of COCH(P) briefly by o-ext(COCH(P)).

In Figure 3.10 we display a point $u \in$ COCH(P) that is an extreme point of COCH(P). Consider the boundary of COCH(P) between a and h. It is a staircase path formed by the maximal elements $p_0 = a, p_1, \ldots, p_{k-1}, p_k = h$ (Figure 3.9). We claim that p_i is an extreme point of COCH(P). Indeed, if it was not an extreme point of COCH(P), there was some point of P_{ah} in the quadrant of an orthogonal line, say ℓ_i at p_i. It follows that p_i was not a maximal element, a contradition. Thus, all points $p_0 = a, p_1, \ldots, p_{k-1}, p_k = h$ are extreme points of COCH(P). Therefore, we obtain the following.

Proposition 3.6. *Let P be a finite planar point set. All extreme points of* COCH(P) *in P_{ah} are maximal elements of in P_{ah}. They are formed by the staircase paths $\mathcal{P}_{ah}$. Consequently, all extreme points of* COCH(P) *are formed by the staircase paths $\mathcal{P}_{cb}$, $\mathcal{P}_{ah}$, $\mathcal{P}_{gf}$, $\mathcal{P}_{ed}$ and therefore they belong to P.*

Proposition 3.7. *Let P be a finite planar point set. Then,*

$$\text{COCH(P)} = \text{COCH(o-ext(COCH(P)))}.$$

Proof. Let $T = \text{COCH}(\text{o-ext}(\text{COCH}(P)))$. By Proposition 3.6, $\text{o-ext}(\text{COCH}(P)) \subseteq P$. Then, by Remark 3.3, $T = \text{COCH}(\text{o-ext}(\text{COCH}(P))) \subseteq \text{COCH}(P)$.

On the other hand, by Proposition 3.5, the COCH(P) is formed by the rectilinear line segments determined by extreme points of COCH(P). It implies that $\text{COCH}(P) \subseteq T$. The proof is complete. □

Obviously, the staircase path $\mathcal{P}_{cb}$ ($\mathcal{P}_{ah}$, $\mathcal{P}_{gf}$, $\mathcal{P}_{ed}$, respectively) can be seen as an union of finite set of O-supports, and each O-support goes through two extreme points of P_{cb} P_{ah}, P_{gf}, P_{ed}, respectively). It follows directly from Proposition 3.6 the following.

Corollary 3.2. *The connected orthogonal convex hull of a finite planar point set P is an orthogonal convex (x, y)-polygon whose boundary is union of finite set of O-supports, and each O-support goes through two extreme points of P.*

3.3 Algorithm, Implementation and Complexity

3.3.1 New Algorithm Based on Graham's Convex Hull Algorithm

Let P be a finite planar point set and the condition (B) for P does not hold. In case of connected orthogonal convex hulls, if we have a reasonably ordered points, we then can scan these ordered points to get candidates for extreme points of COCH(P).

Assume without loss of generality that $p \neq q \neq u \neq v$ and $a \neq p$. First of all, for each set P_{cb} P_{ah}, P_{gf}, P_{ed}, we reorder points due to their y-coordinates only. Then we use Graham's convex hull algorithm. More details can be seen in Algorithm 2.

Let a and b be two distinct points. We call $\ell(a, b)$ *parallel* to $[q, u] \cup [u, v]$ if the first half-line through a parallel to $[q, u]$ and the second half-line through b parallel to $[u, v]$. We determine an orthogonal line $\ell(p_t, p_{t-1})$ through two points p_t, p_{t-1} in P_{ah}, P_{gf}, P_{ed}, P_{cb} as follows

(E) In P_{ah}: If $x_{p_t} \leq x_{p_{t-1}}$, $\ell(p_{t-1}, p_t)$ is parallel to $[q, u] \cup [u, v]$. Otherwise, $\ell(p_t, p_{t-1})$ is parallel to $[p, q] \cup [q, u]$.

- In P_{gf}: If $x_{p_t} \leq x_{p_{t-1}}$, $\ell(p_t, p_{t-1})$ is parallel to $[q, u] \cup [u, v]$. Otherwise, $\ell(p_t, p_{t-1})$ is parallel to $[p, q] \cup [q, u]$.
- In P_{ed}: If $x_{p_t} \leq x_{p_{t-1}}$, $\ell(p_t, p_{t-1})$ is parallel to $[u, v] \cup [v, p]$. Otherwise, $\ell(p_t, p_{t-1})$ is parallel to $[v, p] \cup [p, q]$.
- In P_{cb}: If $x_{p_t} \leq x_{p_{t-1}}$, $\ell(p_t, p_{t-1})$ is parallel to $[u, v] \cup [v, p]$. Otherwise, $\ell(p_t, p_{t-1})$ is parallel to $[v, p] \cup [p, q]$.

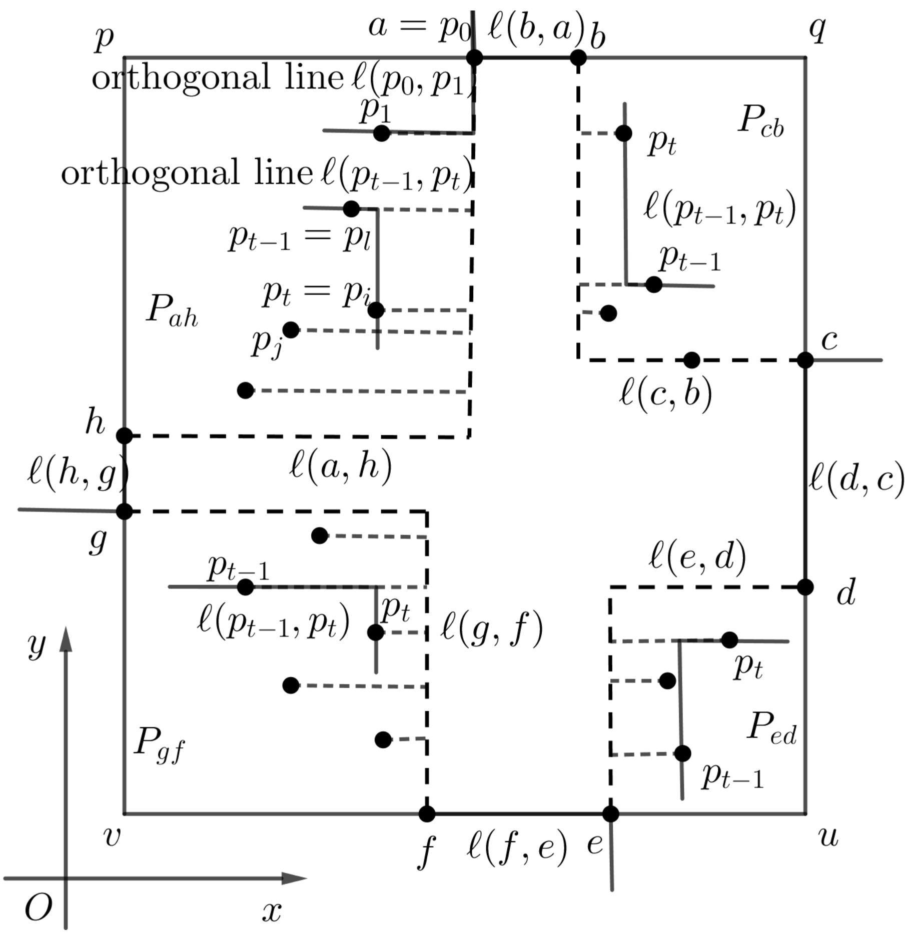


Figure 3.11: The orthogonal line $\ell(p_t, p_{t-1})$ is defined by the relation between x_{p_t} and $x_{p_{t-1}}$ and by the location of p_t, p_{t-1} in P_{ah}, P_{gf}, P_{ed} and P_{cb}.

Let $\ell(q_1, q_2)$ be an orthogonal line through two points q_1, q_2 and its rectilinear half lines have starting point q_3. If the triple (q_1, q_3, q_2) forms a clockwise circuit, and a point q_4 is not in the quadrant determined by ℓ, then q_4 is to the *left* of ℓ (see Figure 3.12 (a)). In other words, a point q_4 is to the left of

ℓ if q_4 is to the left of the directed line q_1q_3 or q_4 is to the left of the directed line q_3q_2[2]. From the construction of the COCH(P), we develop an algorithm, which uses only this kind of orthogonal lines through two given points (see Figure 3.12 (b)). Before starting the algorithm we present a procedure to

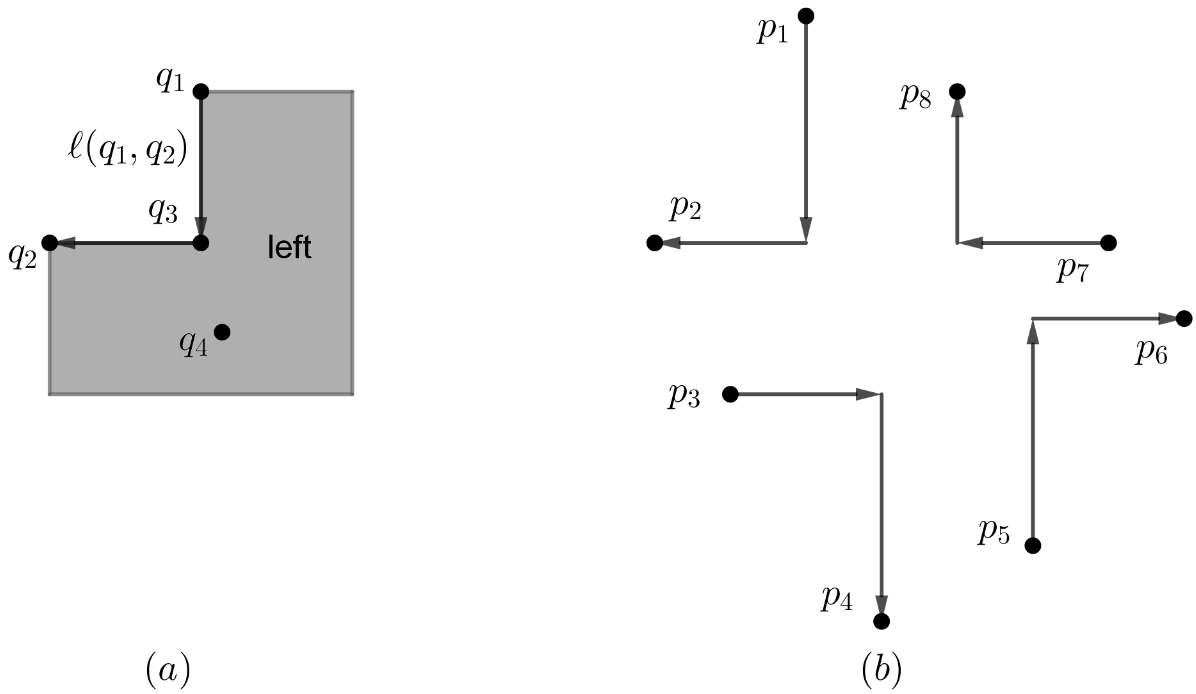


Figure 3.12: (a) The point q_4 is left to $\ell(q_1, q_2)$; (b) Four cases of orthogonal lines.

determine a given point set satisfying condition (B) or not. Under the assumptions of Proposition 3.4, two given points of a finite planar point set P determine at most one O-support of P. Therefore, we need only to check points of P in the $\ell(a, h)$'s quadrant and in the $\ell(b, c)$'s quadrant. If all points of P in $\ell(a, h)$'s quadrant do not satisfy the condition (B), then we continue to check the points of P in $\ell(b, c)$'s quadrant.

Example 1. A demonstration of the procedure is shown in Figure 3.13.

The input is $P = \{(1, 10), (2, 12), (3, 8), (4, 4), (6, 6), (7, 2)\}$. The highest points is $a = (2, 12)$, the leftmost point is $h = (1, 10)$. $S := \{\ell(a, h)\}$. Consider $(3, 8), (4, 4), (6, 6), (7, 2) \in P$. When $S := \{\ell(a, h)\}$, flag $= 0$. Therefore, we continue to check points of P in $\ell(b, c)$'s quadrant. $b \equiv a \equiv (2, 12)$, $c = (7, 2)$. $S = \{\ell(b, i), \ell(i, m), \ell(m, c)\}$. Similarly, the procedure gives flag $= 1$. Therefore, all elements of $\mathcal{F}(P)$ have semi-isolated points.

[2] c is left of a directed line ab iff $(x_b - x_a)(y_c - y_a) - (x_c - x_a)(y_b - y_a) > 0$.

Procedure 1 DETERMINE WHETHER A CONNECTED ORTHOGONAL CONVEX HULL OF P HAS SEMI-ISOLATED POINTS.

procedure SEMI-ISOLATED_POINT(P, flag)

    Input. A set of finite points P in the plane.

    Output. Determine whether a connected orthogonal convex hull of P has semi-isolated points.

    Find a and h (or c and b) satisfying (C).

    $A = \{$all points of P in P_{ah} (or in P_{cb})$\}$;

    $S = \{$all O-supports of P in P_{ah} (or in P_{cb}) of two points of $A\}$;

    flag $= 0$;

    while ℓ in S **do**

        if all points of P on II and IV (or I and III) of ℓ **then**

            flag $= 1$; **break**;

        end if

    end while

return flag;

end procedure

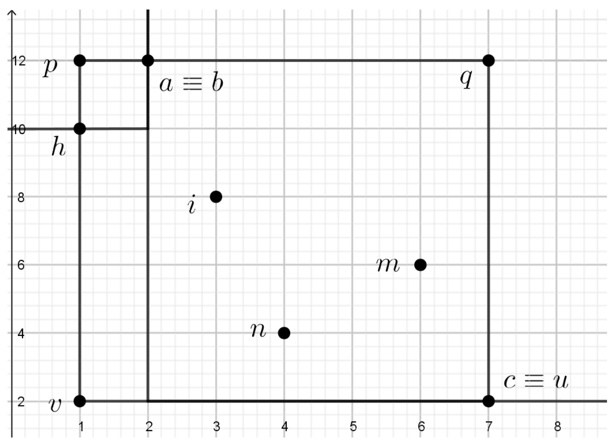


Figure 3.13: $P = \{a, b, c, h, i, m, n\}$ and a connected orthogonal hull of P has semi-isolated points.

We now present an efficient algorithm based on the idea of Graham's convex hull algorithm.

Algorithm 2 FINDING THE CONNECTED ORTHOGONAL CONVEX HULL

Input. A set of finite distinct points P in the plane.

Output. List of extreme points of COCH(P) in order.

Find $a, b, c, d, e, f, g, h \in P$ satisfying (C) and P_{cb}, P_{ah}, P_{gf}, and P_{ed} satisfying (D).

Sort all points of $P_{ah} \cup P_{gf}$ in decreasing their y-coordinates. If two points have the same y-coordinate, the one having smaller x-coordinate is chosen. Suppose that after sorting, the points are $p_0, p_1, \ldots, p_k$.

Sort all points of $P_{ed} \cup P_{cb}$ in ascending their y-coordinates. If two points have the same y-coordinate, the one having bigger x-coordinate is chosen. Suppose that after sorting, the points are $p_{k+1}, p_{k+2}, \ldots, p_m$.

Stack $S = \emptyset$

PUSH(p_0, S)

PUSH(p_1, S)

PUSH(p_2, S)

for $i \leftarrow 3$ **to** m

    while p_i is not to the left of the orthogonal line ℓ(NEXT-TO-TOP(S), TOP(S))

        POP(S)

    end while

    PUSH(p_i, S)

end for

return S

Theorem 3.1. *Algorithm 2 determines* COCH(P). *The time complexity is* $O(n \log n)$, *where* n *is the number of points of* P.

Proof. As we have seen that a, b, c, d, e, f, g, h are extreme point of COCH(P), we can assume without loss of generality, that $P = P_{ah}$ ($b = c, g = f, e = d$).

Firstly, we claim that each point which is poped from the stack S is impossible to be an extreme point of COCH(P). Indeed, suppose that

$$\text{point } p_i \text{ is poped from the stack}$$

i.e., there is

$$\text{some } p_j \text{ such that } p_j \text{ is not on the left of } \ell(p_{t-1}, p_t) \text{ and } p_t = p_i.$$

Assume that $p_l = p_{t-1}$. Then $l < i < j$ as we order points of P_{cb} in decreasing their y-coordinates. Hence

$$p_i \in P_{ah}, x_{p_i} > x_{p_j} \text{ and } y_{p_i} > y_{p_j}$$

(see Figure 3.11). Thus

$$p_i \text{ is not a maximal element of } P_{ah}.$$

According to Proposition 3.6

$$p_j \text{ is not an extreme point of COCH}(P).$$

Next, we claim that when the algorithm stops, the points on stack always are extreme points of COCH(P). Indeed, by Proposition 3.6

$$a = p_0, p_1 \text{ are extreme points of COCH}(P).$$

We now prove by induction. Assume that

$$p_{t-1} \in S \text{ with } t \geq 2 \text{ is an extreme point of COCH}(P)$$

we prove that

$$p_t \text{ is an extreme point of COCH}(P), \text{ too.}$$

Indeed, we have

$$p_j \text{ is left of } \ell(p_{t-1}, p_t)$$

for all $j > i$, where

$$p_t = p_i, p_{t-1} = i', i' < i.$$

As $j > i$ and the sort of points of P_{ah} is in decreasing their y-coordinates we have $y_{p_j} < y_{p_i}$. On the other hand, as

$$p_{t-1}, p_t \text{ are consecutive points in } S,$$

we get that

$$\text{all points } p_m \in P_{ah} (i' < m < i) \text{ are left of } \ell(p_{t-1}, p_t).$$

It follows from Proposition 3.6 and the fact that p_{t-1} is a maximal element of P_{ah} that $x_{p_t} > x_{p_{t-1}}$. Thus

$$p_t \text{ is a maximal element of } P_{ah}.$$

It follows from Proposition 3.6 that

$$p_t \text{ is an extreme point of } \mathrm{COCH}(P).$$

We now turn to analysis the complexity of the algorithm. Step 3 needs $O(n)$ time. Steps 4 and 5 need $O(n \log n)$ time. Steps 10-13 take $O(n)$ time. Therefore, Algorithm 2 takes $O(n \log n)$ time. □

Example 2. A demonstration of the algorithm is shown in Figure 3.14. The input is $P = \{(0,1), (10,0), (7,3), (5,4), (1,6), (10,4), (7,8), (6,9), (8,10)\}$ that does not satisfy (B). A highest point is $a = b = (8,10)$, a leftmost point is $g = h = (0,1)$, a lowest point is $e = f = (10,0)$, the rightmost point is $c = d = (10,4)$.

After sorting via y-coordinates, we have the list of points

$$P = \{(8,10), (6,9), (7,8), (1,6), (5,4), (7,3), (0,1), (10,0), (10,4)\}.$$

Below is shown the stack S and the value of i at the for loop:

$$\begin{aligned}
i = 2 :& (8,10), (6,9) \\
i = 3 :& (8,10), (6,9), (7,8) \\
i = 4 :& (8,10), (6,9), (1,6) \\
i = 5 :& (8,10), (6,9), (1,6), (5,4) \\
i = 6 :& (8,10), (6,9), (1,6), (5,4), (7,3) \\
i = 7 :& (8,10), (6,9), (1,6), (0,1) \\
i = 8 :& (8,10), (6,9), (1,6), (0,1), (10,0) \\
i = 9 :& (8,10), (6,9), (1,6), (0,1), (10,0), (10,4).
\end{aligned}$$

Hence, COCH(P) is determined by the extreme points $(8,10)$, $(6,9)$, $(1,6)$, $(0,1)$, $(10,0)$, $(10,4)$ and their order.

3.3.2 Complexity

The lower bound of algorithms for finding the connected orthogonal convex hull can be proved similarly to lower bound of finding convex hulls (see [51],

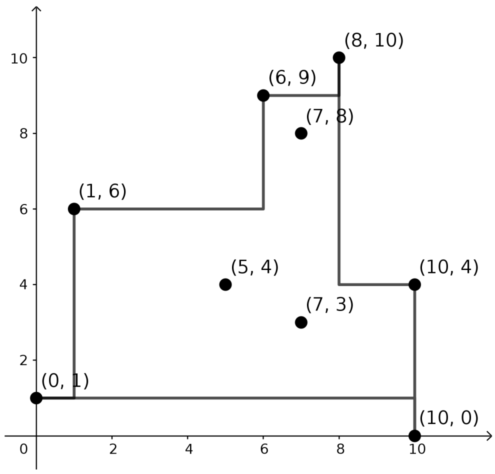


Figure 3.14: The connected orthogonal convex hull of $P = \{(0,1),(10,0),(7,3),(5,4),(1,6),(10,4),(7,8),(6,9),(8,10)\}$

Section 3.4).

Proposition 3.8. *Lower bound on computational complexity of algorithms for finding the connected orthogonal convex hull of a finite planar point set is the same as for sorting, it means $O(n \log n)$.*

Proof. We have presented Algorithm 2 that runs in $O(n \log n)$ time to find the connected convex hull of a finite set of points. We will prove that any algorithm for finding the connected convex hull of a finite set of points cannot run faster than sorted algorithms (Hence, since the lower bound of sorted algorithms is $O(n \log n)$, this implies the required proof).

Suppose that problem A is an unsorted list P_1 of numbers to be sorted, $x_1, x_2, \ldots, x_n$ and we have some algorithm B that constructs the connected orthogonal convex hull as a (x, y)-polygon of n points in $T(n)$ time. Now we will use B to solve A in time $T(n) + O(n)$, where the additional $O(n)$ represents the time to convert the solution of B to a solution of A.

Let $P_1 \subset [w_1, w_2]$. Take $w = (w_1 + w_2)/2$. Now we have the set

$$P := \{(x_i \in P_1, |x_i - w|), i = 1, \ldots, n\} \cup \{(w_1, w - w_1), (w_2, w_2 - w)\}$$

in the plane, as shown in Figure 3.15, where $\{(w_1, w - w_1), (w_2, w_2 - w)\}$ are artificial points. P lies on the graph of the function $y = |x - w|$ and does not satisfy (B). We use algorithm B to construct the connected orthogonal convex hull COCH(P) of these points. It follows from Proposition 3.6 that every point of P is extreme point of COCH(P). The order in which the points

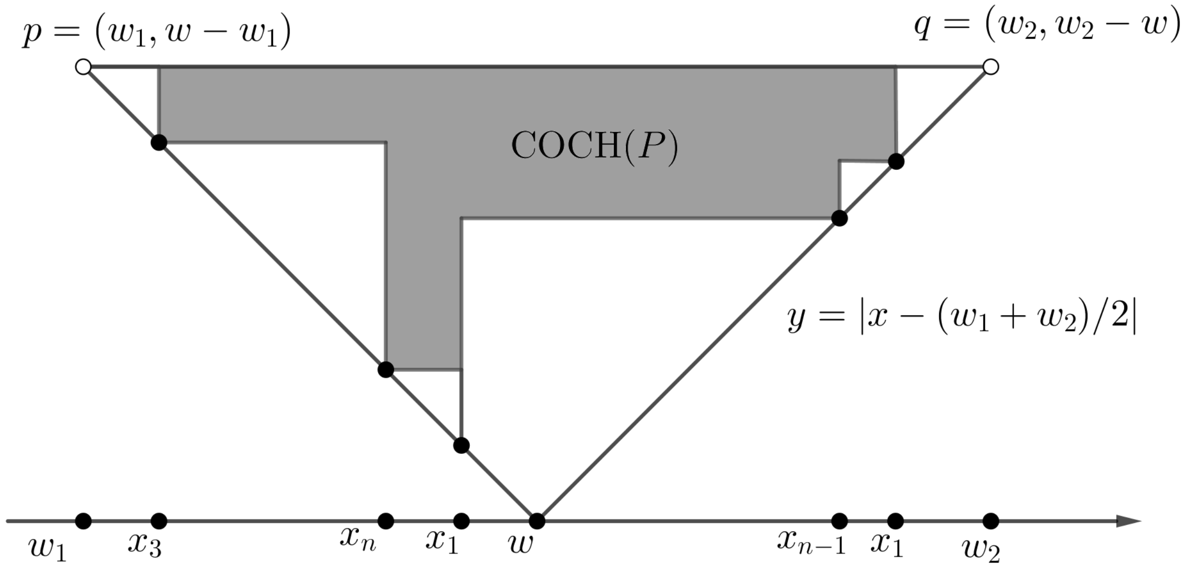


Figure 3.15: The order in which the points of P (black spots and two white spots) occur on the hull COCH(P) in counterclockwise from p is the sorted order of $x_1, x_2, \ldots, x_n$.

of P occur on the hull in counterclockwise from p is the sorted order for P_1. Thus we can use any algorithm for finding the connected orthogonal convex hull to sort the list P_1, but it cannot run faster than sorted algorithms. □

3.3.3 Implementation

Our algorithm was implemented in Python. Tests were run on a PC 3.20GHz with an intel Core i5 and 8 GB of memory. The actual run times of our algorithm on the set of a finite number of points which is uniformly random positioned in the interior of a square of size 10000000 having sides parallel to the coordinate lines are given in Table 3.1. In our experiments, the more points we add the less cases of semi-isolated point happen. In the code we consider the case $a = h = p$ as follows: we take the last point in the ordered points in Step 4 of Algorithm 2 to be the starting point p_0 and $a = h$ to be the second point p_1. Following the way of sorting all points of P we obtain p_0 as an extreme point of COCH(P).

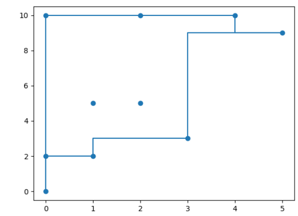
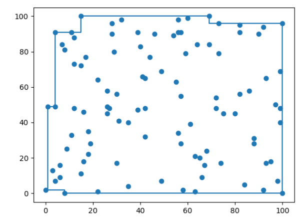
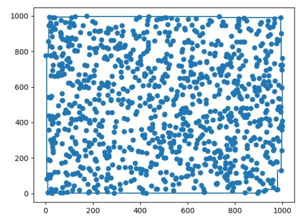
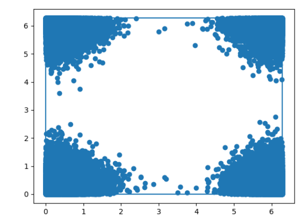
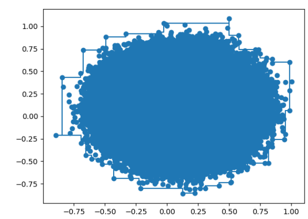
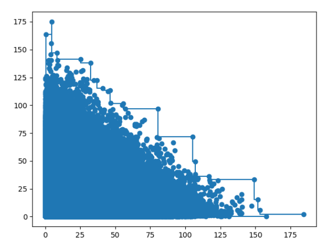
Number of points	Time (s)	Illustrations of connected orthogonal convex hulls of some planar point sets
10	0.001	
100	0.008	
1000	0.08	
10000	0.1	
100000	38.91	
1000000	8919.93	

Table 3.1: Time (an average of 100 runs) required to compute the connected orthogonal convex hull of the set of n points with integer coordinates randomly positioned in the interior of a square of size 10000000 having sides parallel to the coordinate lines.

3.4 Conclusions

In Sections 3.1 and 3.2, we have detected in what circumstances, there exists the connected orthogonal convex hull of a planar points set (Propositions 3.2). Following the uniqueness of the connected orthogonal convex hull, we have provided the construction of the hull which is an (x, y)-orthogonal polygon (Proposition 3.5 and Corollary 3.2), and its extreme vertexes belong to the given points (Proposition 3.6). We have presented a procedure to determine if a given finite planar point set has the connected orthogonal convex hull. Section 3.3 has contained the main algorithm, which has been based on the idea of Graham's convex hull algorithm, for finding the connected orthogonal convex hull of a finite planar point set (Algorithm 2) and it has stated that the lower bound of such algorithm is $O(n \log n)$ (Proposition 3.8). Some numerical results have shown the connected orthogonal convex hulls of some sets of a finite number of points which has been randomly positioned in the interior of a given square (Table 3.1).

General Conclusions

This dissertation has applied different tools from convex analysis, optimization theory, and computational geometry to study some constrained optimization problems in computational geometry.

The main results of the dissertation have included:

- the existence and uniquesness of shortest paths along a sequence of line segments;
- conditions for concatenation of two shortest paths to be a shortest paths;
- straightest paths and longest straightest paths on a sequence of adjacent triangles;
- a property of connected orthogonal convex hulls; the construction of the connected orthgonal convex hull via extreme points;
- an efficient algorithm to finding the connected orthogonal convex hull of a finite planar point set and an evaluating the time complexity for all algorithms which find the connected orthogonal convex hull of a finite planar point set.

List of Author's Related Papers

1. An, P.T., Huyen, P.T.T., Le, N.T.: A modified Graham's scan algorithm for finding the connected orthogonal convex hull of a finite planar point set. Applied Mathematics and Computation **397**, (2021). Article ID 125889

2. Hai, N.N., An, P.T., Huyen, P.T.T.: Shortest paths along a sequence of line segments in Euclidean spaces. Journal of Convex Analysis **26**(4), 1089–1112 (2019)

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