Introduction

The Chip Firing Game (CFG) is a discrete dynamical model which was first defined by A. Björner, L. Lovász and W. Shor while studying the 'balancing game' [6, 7, 42]. The model has various applications in many fields of science such as physics [8, 16], computer science [6, 7, 23], social science [1, 2] and mathematics [2, 34, 35].

The model is a game which consists of a directed multi-graph G (also called *support graph*), the set of *configurations* on G and an *evolution rule* on this set of configurations. Here, a configuration c on G is a map from the set $V(G)$ of vertices of G to non-negative integers. For each vertex v, the integer $c(v)$ is regarded as the number of chips stored in v. In a configuration c, vertex v is *firable* (or *active*) if v has at least one outgoing arc and $c(v)$ is at least the out-degree of v. The *evolution rule* is defined as follows. When v is firable in c, c can be transformed into another configuration c' by moving one chip stored in v along each outgoing arc of v (Fig. 1).

We call this process *firing* v, and write $c \xrightarrow{v} c'$. An *execution* (or *legal firing sequence*) is a sequence of firing and is often written in the form $c_1 \xrightarrow{v_1} c_2 \xrightarrow{v_2} c_3 \cdots \rightarrow c_{k-1} \xrightarrow{v_{k-1}} c_k$, or $c_1 \xrightarrow{v_1, v_2, \ldots, v_{k-1}} c_k$. We write $c_1 \xrightarrow{*} c_k$ if we disregard which vertices are fired. The set of configurations which can be obtained from c

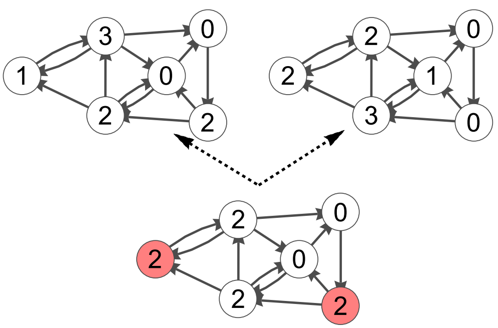

Figure 1 By firing firable vertices in the configuration at the bottom we obtain two new configurations that are presented at the top of the figure

by a sequence of firing is called *configuration space*, and denoted by $\mathrm{CFG(G, c)}$.

A CFG begins with an initial configuration c_0. It can be played forever or reaches a unique fixed point where no firing is possible [6, 7, 17, 23]. When the game reaches the unique fixed point, $\mathrm{CFG(G, c_0)}$ is an *upper locally distributive lattice* with the order defined by setting $c_1 \leq c_2$ if c_1 can be transformed into c_2 by a (possibly empty) sequence of firing [4, 22, 23, 31]. A CFG is *simple* if each vertex is fired at most once during any of its executions. Two CFGs are *equivalent* if their generated lattices are isomorphic. Let L(CFG) denote the class of lattices generated by CFGs. A well-known result is that $D \subsetneq \mathrm{L(CFG)} \subsetneq \mathrm{ULD}$ [38], where D and ULD denote the classes of distributive lattices and upper locally distributive lattices, respectively. Despite of the results on inclusion, one knows little about the structure of L(CFG), even an algorithm for determining whether a given ULD lattice is in L(CFG) is unknown so far.

The Chip Firing Game has many extended models. An important model is the Abelian Sandpile model (ASM), a restriction of CFGs on undirected graphs [6, 8, 33]. This model has been extensively studied in recent years. In [33], the author studied the class of lattices generated by ASMs, denoted by L(ASM), and showed that this class of lattices is strictly included in L(CFG) and strictly includes the class of distributive lattices. As L(CFG), the structure of L(ASM) is little known. An algorithm for determining whether a given ULD lattice is in L(ASM) is still open.

In Chapter 1, we will give criteria that completely characterize those classes of lattices. One of the most important discoveries in our study is pointing out a strong connection between the objects which do not seem to be closely related. These objects are meet-irreducible elements, simple CFGs, firing vertices of a CFG, and systems of linear inequalities. In particular, we establish a one-to-one correspondence between the firing vertices of a simple CFG and the meet-irreducible elements of the lattice generated by this CFG. Using this correspondence, we achieve a necessary and sufficient condition for L(CFG). By generalizing this correspondence to CFGs that are not necessarily simple, we also obtain a necessary and sufficient condition for L(ASM). Both conditions provide polynomial-time algorithms that address the above computational problems. As an application of these conditions, we present in this dissertation a lattice in $\mathrm{L(CFG)\backslash L(ASM)}$ that is smaller than the one shown in [33].

In Chapter 1, we also give a necessary and sufficient condition for the class of lattices generated by the Chip-firing game defined on the class of acyclic digraphs. In [33], to prove $\mathrm{D} \subsetneq \mathrm{L(ASM)}$ the author studied simple CFGs on directed acyclic graphs (DAGs) and showed that such a CFG is equivalent to a CFG on an undirected graph. It is natural to study CFGs on DAGs which are not necessarily simple. Again our method is applicable to this model and we show that any CFG on a DAG is equivalent to a simple CFG on a DAG. As a corollary, the class of lattices generated by CFGs on DAGs is strictly included in L(ASM).

The lattice structure of a converging CFG on a digraph implies the strongly convergent property of the game. This property naturally leads to the definition of *recurrent configuration* from the viewpoint of Markov chain [30, 32]. The dollar game is an extended model of the Chip-firing game which is played on an undirected graph. The game has exactly one sink and the sink only can be fired if all other vertices are not firable [2]. In this model, the number of chips stored in the sink may be negative. The dollar game can be simulated easily by a CFG on a digraph with a global sink. By the viewpoint of Markov chain, the definition of recurrent configurations on a digraph with a global sink is not intuitive. However, in the case of the dollar game recurrent configurations have an alternative intuitive one. A configuration is called recurrent if it is stable and unchanged under firing at the sink and stablizing the resulting configuration. The dollar game has a natural generalization to the class of Eulerian digraphs as follows. An Eulerian digraph is a strongly connected digraph in which the indegree of each vertex is equal to its outdegree. An undirected graph can be regarded as an Eulerian graph by replacing each (undirected) edge e by two reverse arcs e' and e'' that have the same endpoints as e. The definition of the dollar game on Eulerian graphs is the same as of the one on undirected graphs, i.e. some vertex is chosen to be the sink that only can be fired if all other vertices are not firable [26].

The set of recurrent configurations of a dollar game on an undirected graph has many interesting properties such as it is an Abelian group with the addition defined by the stabilization, and the cardinality is equal to the number of spanning trees of the support graph, etc [2, 26, 45]. Remarkably N. Biggs defined the level of a recurrent configuration and made an intriguing conjecture about the relation between the generating function of recurrent configurations and the Tutte polynomial [1]. This conjecture later was proved by C. M. Lopéz [35]. An interesting consequence of this result is that Stanley's conjecture about pure $\mathcal{O}$-sequence holds for co-graphic matroids [36, 44]. Another direct consequence is that the generating function of recurrent configurations in a dollar game is independent of the sink. It only depends on the graph on which the game is defined. This fact is definitely not trivial. Currently, there is no proof for this fact without using the theorem of Merino Lopéz.

A lot of properties of recurrent configurations on undirected graphs can be extended to Eulerian digraphs without any difficulty [7, 26]. However, the situation is completely different when one tries to extend the sink-independent property of generating function to a larger class of graphs, in particular to Eulerian digraphs because a natural definition of the Tutte polynomial for digraphs is not known, even one for Eulerian digraphs. In Chapter 2, we show that this property holds not only for undirected graphs but also for Eulerian digraphs. Since the Tutte-polynomial approach does not work for Eulerian digraphs, we use another approach that is based on a level-preserved bijection between two sets of recurrent configurations with respect to two different sinks. The bijection also gives us some new insight into the groups of recurrent configurations.

There are a lot of polynomials that are defined on undirected graphs such as Tutte polynomial, chromatic polynomial, cover polynomial, etc. They count certain combinatorial objects. The Tutte polynomial is the most well-known one, it has many interesting properties and applications [9]. There is a number of articles that tried to give the polynomials as an attempt to define an analogue of Tutte polynomial for digraphs, or for some other objects [12, 20, 24]. They have some properties that are similar to those of the Tutte polynomial. Nevertheless, they are not natural analogues in the sense that one does not know a conversion between the properties of these polynomials to those of the Tutte polynomial, in particular how to obtain the Tutte polynomial on undirected graph from these polynomials [12]. The situation is not better for Eulerian digraphs, a natural analogue of the Tutte polynomial is unknown so far.

Also in Chapter 2, we show that the generating function of recurrent configurations on an Eulerian digraph can be a natural generalization of the Tutte polynomial in one variable to the class of Eulerian digraphs. It turns out from the sink-independent property of the generating function that the generating function is a characteristic of an Eulerian digraph, and we can denote it by $\mathcal{T}_G(y)$, regardless of the sink. By using this property, we derive a lot of properties that are generalizations of the usual those of $T(G;1,y)$ to Eulerian digraphs. These properties make us believe that the polynomial $\mathcal{T}_G(y)$ is quite a natural generalization of $T(G;1,y)$. By generalizing the result to strongly connected digraphs, we propose a conjecture that would be promising direction of looking for a natural generalization of $T(G;1,y)$ to strongly connected digraphs. In this chapter, we also propose another generalization of the Tutte polynomial in two variables to Eulerian digraphs.

If a *stable configuration* (a configuration has no firable vertex) is componentwise greater than a recurrent configuration, then it is also a recurrent configuration [2, 26]. This is a typical property of recurrent configurations. This property implies that if we know the set of minimal recurrent configurations, then we know all recurrent configurations. For an undirected graph, all minimal recurrent configurations have the minimum number of chips. This fact implies that the problem of finding the minimum number of chips of a recurrent configuration on an undirected graph can be solved in polynomial time. In Chapter 3, we study the computational problem of finding the minimum number of chips of a recurrent configuration on a digraph with a global sink that we call *minimum recurrent problem* (MINREC problem). To study this computational problem, we give a connection to the classical computational problem *minimum feedback arc set* (MINFAS). A *feedback arc set* of a directed graph (digraph) G is a subset A of arcs of G such that removing A from G leaves an acyclic graph. The minimum feedback arc set problem is a classical combinatorial optimization on graphs in which one tries to minimize $|A|$. This problem has a long history and its decision version was one of Richard M. Karp's 21 NP-complete problems [29]. The problem is known to be still NP-hard for many smaller classes of digraphs such as tournaments, bipartite tournaments, and Eulerian multi-digraphs [13, 19, 21]. We prove in this dissertation that it is also NP-hard on Eulerian digraphs, a class in-between undirected and digraphs, in which the in-degree and the out-degree of each vertex are equal.

To give that connection, we study the properties of recurrent configurations on a digraph. In [26], the authors presented many properties of recurrent configurations on a digraph which are similar to those of recurrent configurations on undirected graphs. The authors also studied the Chip-firing game on Eulerian digraphs and presented many typical properties that can also be considered as natural generalizations of the undirected case. In this dissertation, we continue this work and present generalizations of more surprising properties. Since the minimal recurrent configurations are very important to understand the properties of recurrent configurations, it is worth studying properties of such recurrent configurations. It turns out from the study in [5, 6, 41] that we can associate a minimal recurrent configuration of an undirected graph G with an acyclic orientation of G. By giving the notion of maximal acyclic arc sets that can be regarded as a generalization of acyclic orientations of undirected graphs, we generalize the definitions and the results in [41] to the class of Eulerian digraphs. Although natural, these generalizations are not easy to see from the studies on undirected graphs. They allow us to derive a number of interesting properties of feedback arc sets and recurrent configurations of the Chip-firing game on Eulerian digraphs, and provide a polynomial reduction from the MINREC problem to the MINFAS problem on Eulerian digraphs. We extend a result of [19] and show that the MINFAS problem on Eulerian digraphs is also NP-hard, which implies the NP-hardness of the MINREC problem on general digraphs.

Chapter 1

CFG lattice

1.1 Preliminaries on lattice theory

In this section, we present some basic knowledges on the lattice theory that will play an important role for studying the class of lattices generated by the Chip firing game. Let $L = (X, \leq)$ be a partial order (X is equipped with a binary relation $\leq$ which is transitive, reflexive and antisymmetric). In this dissertation, we always work with a finite partial order, *i.e.* $|X| < \infty$. For $x, y \in X$, y is an *upper cover* of x if $x < y$ and for every $z \in X$, $x \leq z \leq y$ implies that $z = x$ or $z = y$. If y is an upper cover of x, then x is a *lower cover* of y, and then we write $x \prec y$. The partial order L can be presented by an acyclic digraph G=(X, E) that is defined by: $(x, y) \in E$ iff $x \prec y$ in L. Conversely, an acyclic digraph $G = (V, E)$ (simple digraph) defines a partial order $(V, \leq)$ by $v_1 \leq v_2$ if there is a directed path from v_1 to v_2 in G (the length of the path may be 0). A subset I of X is called an *ideal* of L if for every $x \in I$ and $y \in X$ such that $y \leq x$ we have $y \in I$.

The partial order L is a *lattice* if any two elements of L have a least upper bound (*join*) and a greatest lower bound (*meet*) . It follows immediately from the definition that every lattice has a unique minimum, denoted by $\mathbf{0}$, and a unique maximum, denoted by $\mathbf{1}$. When L is lattice, we have the following notations and definitions

- for every $x, y \in X$, $x \vee y$ and $x \wedge y$ denote the join and the meet of x, y, respectively.

- for $x \in X$, x is a *meet-irreducible* if it has exactly one upper cover. The element x is a *join-irreducible* if x has exactly one lower cover. Let M and J denote the collections of the meet-irreducibles and the join-irreducibles of L, respectively. Let M_x, J_x be given by: $M_x = \{m \in M : x \leq m\}$ and $J_x = \{j \in J : j \leq x\}$. For $j \in J, m \in M$, if j is a minimal element in

$X \backslash \{x \in X : x \leq m\}$, then we write $j \downarrow m$. If m is a maximal element in $X \backslash \{x \in X : j \leq x\}$, then we write $j \uparrow m$, and $j \updownarrow m$ if $j \downarrow m$ and $j \uparrow m$.

- The lattice L is a *distributive lattice* if it satisfies one of the following equivalent conditions

  1. for every $x, y, z \in X$, we have $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
  2. for every $x, y, z \in X$, we have $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

  For a finite set A, $(2^A, \subseteq)$ is a distributive lattice. A lattice generated in this way is called *hypercube*.

- for $x, y \in X$ satisfying $x \leq y$, $[x, y]$ stands for set $\{z \in X : x \leq z \leq y\}$. If $x \neq \mathbf{1}$, x^+ denotes the join of all upper covers of x. Note that if x is a meet-irreducible, then x^+ is the unique upper cover of x. If $x \neq \mathbf{0}$, x^- denotes the meet of all lower covers of x. If x is a join-irreducible, then x^- is the unique lower cover of x. The lattice L is an *upper locally distributive (ULD) lattice* [37, 15] if for every $x \in X$, $x \neq \mathbf{1}$ implies the sublattice induced by $[x, x^+]$ is a hypercube. By dual notion, L is a *lower locally distributive (LLD) lattice* if for every $x \in X$, $x \neq \mathbf{0}$ implies that the sublattice induced by $[x^-, x]$ is a hypercube.

1.2 Lattices generated by CFGs

Let $G = (V, E)$ be a directed multi-graph. A vertex v of G is called *sink* if it has no outgoing edge. If a CFG, which is defined on a graph, reaches a fixed point, then its configuration space is a ULD lattice [7, 31]. If $\mathrm{CFG}(\mathrm{G}, \mathrm{c}_0)$ has a unique fixed point and $\mathrm{CFG}(\mathrm{G}, \mathrm{c}_0)$ is isomorphic to a ULD lattice L, we say $\mathrm{CFG}(\mathrm{G}, \mathrm{c}_0)$ generates L.

A lattice generated by a CFG is a ULD lattice. Conversely, given a ULD lattice L, is L in L(CFG)? This question was asked in [38]. Up to now, there exists no good criterion for L(CFG) that suggests a polynomial-time algorithm for this computational problem. We address this problem by giving a necessary and sufficient condition for L(CFG). From now until the end of this chapter, all CFGs are supposed to be simple since every CFG is equivalent to a simple CFG [38].

For each $m \in M$, $\mathfrak{U}_m$ denotes the collection of all minimal elements of $\{x \in X : \exists y \in X, x \prec y \text{ and } \mathfrak{m}(x, y) = m\}$ and $\mathfrak{L}_m$ denotes the collection of all maximal elements of $X \backslash \bigcup_{a \in \mathfrak{U}_m} \{x \in X : a \leq x\}$.

For each $m \in M$, the system of linear inequalities $\mathcal{E}(m)$ is given by:

$$\mathcal{E}(m) = \begin{cases} \{w - \sum_{x \in M \backslash M_a} e_x \geq 1 : a \in \mathfrak{L}_m\} \bigcup \{w \leq \sum_{x \in M \backslash M_a} e_x : a \in \mathfrak{U}_m\} & \text{if } \mathfrak{U}_m \neq \{\mathbf{0}\} \\ \{w \geq 1\} & \text{if } \mathfrak{U}_m = \{\mathbf{0}\}, \end{cases}$$

where w is an added variable.

By using the definition of systems of linear inequalities, we have the following necessary and sufficient condition for the class of lattices generated by CFGs.

Theorem 1.3 *L is in* L(CFG) *if and only if for each m in M, $\mathcal{E}(m)$ has non-negative integral solutions.*

The theorem implies a polynomial time algorithm for determining whether a given ULD lattice is in L(CFG). We can use the Karmarkar's algorithm [28] to find a non-negative integral solutions f_m of $\mathcal{E}(m)$. For each $m \in M$, the number of bits that are input to the algorithm is bounded by $O(|M| \times |X|)$. We have to run the Karmarkar's algorithm $|M|$ times. Hence the algorithm for determining whether a given ULD lattice is in L(CFG) can be implemented to run in $O(|M|^{6.5} \times |X|^2 \times log|X| \times log(log|X|))$ time.

1.3 Lattices generated by Abelian Sandpile model

Abelian Sandpile model is the CFG model which is defined on connected undirected graphs. In this model, the support graph is undirected and it has a distinguished vertex which is called *sink* and never fires in the game even if it has enough chips. If we replace each undirected edge (v_1, v_2) in the support graph by two directed edges (v_1, v_2) and (v_2, v_1) and remove all out-edges of the sink, then we obtain an CFG on directed graph which has the same behavior as the old one. Thus a ASM can be regarded as a CFG on a directed multi-graph. We give an alternative definition of ASM on directed multi-graphs as follows. A $\mathrm{CFG}(\mathrm{G}, \mathrm{c}_0)$, where G is a directed multi-graph, is a ASM if G is connected, G has only one sink s and for any two distinct vertices v_1, v_2 of G, which are distinct from the sink, we have $E(v_1, v_2) = E(v_2, v_1)$. Therefore in this model we will continue to work on directed multi-graphs.

For each $\mathcal{E}(m)$, we define the system of linear inequalities $\mathfrak{E}(m)$ by replacing each variable e_x in $\mathcal{E}(m)$ by $e_{x,m}$ and w by w_m. Clearly, $\mathfrak{E}(m)$ is a system of linear inequalities whose variables are a subset of $\{e_{m_1,m_2} : m_1 \in M, m_2 \in M \text{ and } m_1 \neq m_2\} \cup \{w_m : m \in M\}$. Let U denote the set of all variables in $\bigcup\limits_{m \in M} \mathfrak{E}(m)$. The system Ω of linear inequalities is given by:

$$\Omega = \left(\bigcup_{m \in M} \mathfrak{E}(m) \right) \cup \{e_{m_1,m_2} = e_{m_2,m_1} : e_{m_1,m_2} \text{ and } e_{m_2,m_1} \text{ both are in } U\}.$$

The following theorem gives a necessary and sufficient condition for a lattice in L(ASM).

Theorem 1.4. $L \in \mathrm{L(ASM)}$ *if and only if* Ω *has non-negative integral solutions.*

This theorem implies a polynomial time algorithm for the problem of determining whether a given lattice is in L(ASM), and construct a corresponding CFG if there exists one. We again use the Karmarkar's algorithm for finding a non-negative integral solution of Ω. The number of variables of Ω is bounded by $O(|M|^2)$ and the number of bits, which are input to the algorithms for linear programming to find a non-negative integral solution of Ω, is bounded by $O\left(|M|^3 \times |X|\right)$. Therefore the algorithm can be implemented to run in $O(|M|^{13} \times |X|^2 \times log|X| \times log(log|X|))$ time.

1.4 Lattices generated by CFGs on acyclic graphs

In [33], the author gave a strong relation between ASM and the simple CFGs on acyclic graphs (directed acyclic graphs). The author pointed out that a simple CFG on an acyclic graph is equivalent to a ASM. In this subsection we study CFGs on acyclic graphs that are not necessarily simple. We show that each CFG on an acyclic graph is equivalent to a simple CFG on an acyclic graph. As a corollary, every lattice generated by a CFG on an acyclic graph is in L(ASM). We also give a necessary and sufficient criterion for lattices generated by CFGs on acyclic graphs. Firstly, we give a necessary condition for a lattice generated by a CFG on an acyclic graph.

Lemma 1.10. *If* L *is generated by a CFG on an acyclic graph, then* $\mathcal{G}$ *is acyclic, where* $\mathcal{G}$ *is the simple directed graph whose vertices are* M *and arcs are defined by:* $(m_1, m_2) \in E(\mathcal{G})$ *if and only if* $m_1 \in \bigcup\limits_{a \in \mathfrak{U}_{m_2}} (M \backslash M_a)$.

The following theorem is the main result of this subsection.

Theorem 1.5. *Any CFG on an acyclic graph is equivalent to a simple CFG on an acyclic graph, therefore equivalent to a ASM.*

Using Lemma 11 and a similar argument as in the proof of Theorem 5, we obtain a necessary and sufficient criterion for the class of lattices generated by CFGs on acyclic graphs

Corollary 1.3. *Let* $L \in \mathrm{L(CFG)}$. *Then* L *is generated by a CFG on an acyclic graph if and only if* $\mathcal{G}$ *is acyclic.*

Chapter 2

Generating function of recurrent configurations of an Eulerian digraph

2.1 Recurrent configurations on a digraph with global sink and recurrent configurations on an Eulerian digraph with a sink

All graphs in this section are assumed to be multi-digraphs without loops and an arc means an edge in a digraph. Graphs with loops will be considered in Section 2.3. We introduce in this section some notations and known results about recurrent configurations of CFG with a sink on general digraphs.

For a digraph $G = (V, E)$, a vertex s of G is called *global sink* if s does not have out-going edges, and for any vertex $v \in V$ there is a directed path from v to s (the length of the path may be 0). A configuration on G is a map from $V \backslash \{s\}$ to $\mathbb{N}$. When a chip goes into the sink, it vanishes. The interest is to assimilate two configurations that have the same number of chips on every vertices except on the sink. In the following definition, we assume that G has a global sink s.

Definition 2.1.[14, 26, 2] A stable configuration c is *recurrent* if and only if for any configuration d there is a configuration d' such that $c = (d + d')^{\circ}$.

There are several equivalent definitions of *recurrent* configurations. The one above says that c is recurrent if and only if it can be reached from any other configuration d by adding some chips (according to d') and then stabilize.

Definition 2.2.[14, 2]. Let $G = (V, E)$ be an Eulerian digraph with a distinguished vertex s of G which is called sink. A configuration c on G is a map from $V \backslash \{s\}$ to $\mathbb{N}$. The configuration c is recurrent on G if c is recurrent on the digraph H having a global sink s which is obtained from G by removing all arcs emanating from s.

2.2 Sink-independence of generating function of recurrent configurations on an Eulerian digraph

The following is the main result of this chapter.

Theorem 2.1. *Let G be an Eulerian digraph and s a vertex of G. For each recurrent configuration with respect to sink s, let $sum_{G,s}(c)$ denote $deg_G^+(s) + \sum_{v \neq s} c(v)$. The recurrent configurations with respect to sink s are denoted by $c_1, c_2, \ldots, c_p$ for some p. Then the sequence $(sum_{G,s}(c_i))_{1 \leq i \leq p}$ is independent of the choice of s up to a permutation of the entries.*

The result of Merino López [35] implies that Theorem 7 holds for undirected graphs. An undirected graph G can be considered as an Eulerian digraph by replacing each undirected edge e by two reverse directed arcs which have the same endpoints as e. With this conversion it makes sense to call an Eulerian digraph G *undirected* if for any two vertices v, v' of G we have $E_G(v, v') = E_G(v', v)$. The following known result is thus a particular case of Theorem 11, for the class of undirected graphs.

Theorem 2.2. *Let $\mathcal{C}$ be the set of all recurrent configurations with respect to some sink s. If G is an undirected graph (defined as a digraph), then $T_G(1, y) = \sum_{c \in \mathcal{C}} y^{level(c)}$, where $T_G(x, y)$ is the Tutte polynomial of G and $level(c) := -\frac{|A|}{2} + deg_G^+(s) + \sum_{v \neq s} c(v)$ for any $c \in \mathcal{C}$.*

2.3 Tutte-like properties of generating function of recurrent configurations

We present in this section a natural generalization of the partial Tutte polynomial in one variable, for the class of Eulerian digraphs. The Tutte polynomial has the recursive formula $T_G(1, y) = y\, T_{G_{\setminus e}}(1, y)$ if e is a loop. We have the following generalization.

Proposition 2.4. *If e is a loop, then $\mathcal{T}_G(y) = y\, \mathcal{T}_{G_{\setminus e}}(y)$.*

In order to generalize the recursive formula $T_G(1, y) = T_{G_{/e}}(1, y)$ if e is a bridge, we generalize the notion of bridge to directed graphs with the following.

Definition 2.2. *An arc b in G is called* bridge *if $G_{\setminus b}$ is not strongly connected.*

The second relation, extending the recursive formula on undirected graphs $T_G(1, y) = T_{G_{/e}}(1, y)$ if e is a bridge, is split into the two following propositions, depending on whether the bridge has a reverse arc.

Proposition 2.6. *Let e be a bridge of G such that it does not have a reverse arc. Then $\mathcal{T}_G(y) = \mathcal{T}_{G_{/e}}(y)$.*

The following can be considered as a generalization of the identity $T_G(1,y) = T_{G/e}(1,y)$ if e is a bridge of the Tutte polynomial $T_G(x,y)$ on undirected graphs.

Proposition 2.7. *Let e be a bridge of G such that it has a reverse arc e', and let H denote $G_{/e}$. Then $\mathcal{T}_G(y) = \frac{1}{y}\mathcal{T}_H(y)$ and $\mathcal{T}_G(y) = \mathcal{T}_{H_{\setminus e'}}(y)$.*

The recursive formula $T_G(1,y) = T_{G_{\setminus e}}(1,y) + T_{G_{/e}}(1,y)$ if e is neither a loop nor a bridge has the following generalization.

Proposition 2.8. *Let e be an arc of G such that e is neither a loop nor a bridge, and e has a reverse arc e'. Then $\mathcal{T}_G(y) = y^{1+\lambda(\overline{G_{\setminus\{e,e'\}}})-\lambda(\overline{G})}\,\mathcal{T}_{G_{\setminus\{e,e'\}}}(y) + y^{\lambda(\overline{H})-\lambda(\overline{G})}\,\mathcal{T}_H(y)$, where H denotes $G_{/e}$. Moreover, if G is undirected, then $\mathcal{T}_G(y) = \mathcal{T}_{G_{\setminus\{e,e'\}}}(y) + y^{-E_G(e^-,e^+)+1}\,\mathcal{T}_{H_{\setminus e'}}(y)$.*

Let us present a new formula that does not exist for the Tutte polynomial on undirected graphs. If G is undirected, then it contains at least one arc that is a loop, or it satisfies the conditions of Proposition 10, Proposition 11 or Proposition 12. In every case, $\mathcal{T}_G(y)$ can be defined by a recursive formula on smaller graphs. However, there are Eulerian digraphs which do not contain any such arc, therefore no recursive formula generalizing those of the classical Tutte polynomial can be applied. Neither of the recursive formulas in Proposition 8, Proposition 10, Proposition 11 and Propostion 12 is useful in this case. The following new recursive formula handles this case, in order to complete the recursive definitions of $\mathcal{T}_G(y)$ on the class of general Eulerian digraphs.

Proposition 2.9. *Let G be an Eulerian digraph, s be a vertex of G, and N be the set of all out-neighbors of s. Then*

$$\mathcal{T}_G(y) = \sum_{\substack{W\subseteq N\\ W\neq\emptyset}} (-1)^{|W|+1} y^{\lambda(\overline{G_{/W\cup\{s\}}})-\lambda(\overline{G})-E_G(s,W)}$$

$$\frac{1}{(1-y)^{|W|}}\prod_{v\in W}\left(1-y^{E_G(s,v)}\right)\mathcal{T}_{G_{/W\cup\{s\}}}(y),$$

where $E_G(s,W)$ denotes the number of arcs e of G such that $e^- = s$ and $e^+ \in W$.

Note that the number of vertices of the digraph $G_{/W\cup\{s\}}$ is strictly smaller than G. Moreover, the digraph $G_{/W\cup\{s\}}$ is likely to have more loops than G, hence we could apply Proposition 8 to remove the loops in $G_{/W\cup\{s\}}$.

Chapter 3

NP-hardness of feedback arc set and minimum recurrent configuration problems of Chip-firing game on directed graphs

3.1 Acyclic arc sets on Eulerian digraphs

Throughout this chapter a graph always means a simple connected digraph. All results in this chapter can be generalized easily to the case of multi-graphs. An undirected graph is considered as a digraph in which for any edge linking u and v, we consider two arcs: one from u to v and another from v to u. With this convention an undirected graph is an Eulerian digraph.

Let $G = (V, E)$ be a digraph. For a subset A of E, let $G[A]$ denote the graph (V', E') with $V' = V$ and $E' = A$. A *feedback arc set* F of G is a subset of E such that removing the arcs in F from G leaves an acyclic graph. An *acyclic arc set* A of G is a subset of E such that the graph $G[A]$ is acyclic. Clearly, an acyclic arc set is the complement of a feedback arc set. A feedback arc set (resp. acyclic arc set) is *minimum* (resp. *maximum*) if it has minimum (resp. maximum) number of arcs over all feedback arc sets (resp. acyclic arc sets) of G. A feedback arc set A (resp. acyclic arc set A) is *minimal* (resp. *maximal*) if for any $e \in A$ (resp. $e \in E \backslash A$) we have $A \backslash \{e\}$ (resp. $A \cup \{e\}$) is not a feedback arc set (resp. acyclic arc set). An acyclic arc set A has a *sink* s if for any vertex v there is a path in A from s to v.

In the following important theorem, we assume $G = (V, E)$ to be an Eulerian digraph. The most important result is that finding a maximum acyclic arc set can be restricted to finding an acyclic arc set of the maximum size which has some special properties. This establishes a relation between the MINFAS problem and the MINREC problem on Eulerian digraphs via the following theorem.

Theorem 3.2. *Let N be the maximum number of arcs of an acyclic arc set of G. For every vertex s of G, there is an acyclic arc set of N arcs with sink s.*

We recall the definition of the MINFAS problem.

MINFAS Problem
Input: A digraph G.
Output: Minimum number of arcs of a feedback arc set of G.

When the input graphs are restricted to Eulerian digraphs, we call the above problem EMINFAS. Using the above theorem, we obtain the NP-hardness of MINFAS problem on Eulerian digraphs.

Theorem 3.1. *The EMINFAS problem is NP-hard.*

3.2 NP-hardness of minimum recurrent configuration problem

3.2.1 Chip-firing game on Eulerian digraphs with sink and firing graph

Let $G = (V, E)$ be an Eulerian digraph (connected) and a distinguished vertex s of G that is called *sink*. Let $G_{\setminus s}$ be the graph G in which the out-going arcs of s have been deleted. Clearly $G_{\setminus s}$ has a global sink s. The Chip-firing game on G with sink s is the ordinary Chip-firing game that is defined on the graph $G_{\setminus s}$. Let β be the configuration defined by: for every $v \in V\backslash\{s\}$, $\beta(v) = 1$ if $(s, v) \in E$ and $\beta(v) = 0$ otherwise. Since G is Eulerian, $\beta \sim \mathbf{0}$ (after firing -1 time every vertex, except the sink). The following introduces the notion of firing graph which will play a central role in studying the minimum recurrent configuration problem [41].

Definition 3.6. *Let c be a recurrent configuration and $c + \beta = d_0 \overset{w_1}{\to} d_1 \overset{w_2}{\to} d_2 \overset{w_3}{\to} d_3 \ldots \overset{w_k}{\to} d_k$ a legal firing sequence of c such that $d_k = c$. This sequence of legal firings can be presented by $(w_1, w_2, \ldots, w_k)$ since d_i is completely defined by $w_1, w_2, \ldots, w_i$ for $i \geq 1$. The graph $\mathcal{F} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = V$ and $\mathcal{E} = \{(s, w_i) : (s, w_i) \in E\} \cup \{(w_i, w_j) : i < j$ and $(w_i, w_j) \in E\}$ is called a* firing graph *of c.*

3.2.2 Minimal recurrent configurations and maximal acyclic arc sets

In this subsection, we work with the Chip-firing game on an Eulerian digraph $G = (V, E)$ with sink s. For two configurations c' and c, we write $c' \leq c$ if $c'(v) \leq c(v)$ for every $v \in V \backslash \{s\}$. A recurrent configuration c is *minimal* if whenever $c' \neq c$ and $c' \leq c$, c' is not recurrent. When c has the minimum total number of chips over all recurrent configurations, we say that c is *minimum*. Let $\mathcal{M}$ be the set of all minimal recurrent configurations of the game.

Let $\mathcal{A}$ be the set of all maximal acyclic arc sets A of G such that s is a unique sink of A. The following is the main result of this subsection which gives a connection between the MINFAS problem and the EMINFAS problem.

Theorem 3.4. *Let $\mathcal{F}_c$ denote the firing graph of c, the map from $\mathcal{M}$ to $\mathcal{A}$, defined by: $c \mapsto \mathcal{F}_c$, is bijective. Moreover, this map gives a minimum recurrent configuration to a maximum acyclic arc set.*

3.2.3 NP-hardness of minimum recurrent configuration problem

In this subsection, we study the computational complexity of the following main problem.

MINREC problem

Input: A graph G with a global sink.
Output: Minimum total number of chips of a recurrent configuration of G.

If the input graphs are restricted to undirected graphs G with a sink s, the problem can be solved in polynomial time since all minimal recurrent configurations have the same total number of chips, namely $\frac{E(G)}{2}$. Nevertheless, the problem is NP-hard for general digraphs. In particular, we show that the problem is NP-hard when the input graphs are restricted to Eulerian digraphs.

Theorem 3.3. *The $EMINREC$ problem is NP-hard, so is the MINREC problem.*

Author's papers used in dissertation

1. Lattices generated by Chip Firing Game models: Criteria and recognition algorithms (with Thi Ha Duong Phan), *European Journal of Combinatorics 34 (2013) pp. 812-832.*

2. Feedback arc set problem and NP-hardness of minimum recurrent configuration problem of Chip-firing game on directed graphs (with Kevin Perrot). Accepted for publication in *Annals of Combinatorics*.

3. Chip-firing game and partial Tutte polynomial for Eulerian digraphs (with Kevin Perrot), preprint.

Author's other relevant papers

4. Fixed-point forms of the parallel symmetric sandpile model (with Enrico Formenti, Tran Thi Thu Huong and Thi Ha Duong Phan), *Theoretical Computer Science 533 (2014), pp. 1-14.*

5. On the set of Fixed Points of the Parallel Symmetric Sand Pile Model (with Thi Ha Duong Phan and Kevin Perrot), *Automata 2011, DMTCS : Automata 2011 - 17th International Workshop on Cellular Automata and Discrete Complex Systems, pages 17-28.*

6. A polynomial-time algorithm for reachability problem of a subclass of Petri net and Chip Firing Games (with Manh Ha Le and Thi Ha Duong Phan), *IEEE-RIVF International Conference on Computing and Communication Technologies (2012), pages 189-194, ISBN: 978-1-4244-8072-2.*

7. Orbits of rotor-router operation and stationary distribution of random walks on directed graphs, preprint.

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