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SOME QUALITATIVE PROPERTIES OF SOLUTIONS TO NAVIER-STOKES EQUATIONS

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SUMMARY

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Introduction

Navier-Stokes equations are useful because they describe the motion of fluids. They may be used to model the weather, ocean currents, the design of aircraft and cars, the study of blood flow, the analysis of pollution, and many other things. The Navier-Stokes equations are also of great interest in a purely mathematical sense. They have particular importance within the development of the modern mathematical theory of partial differential equations. Although the theory of partial differential equations has undergone a great development in the twentieth century, some fundamental questions remain unresolved. They are essentially concerned with the global existence and uniqueness of solutions, as well as their asymptotic behavior. More precisely, given a smooth datum at time zero, will the solution of the Navier-Stokes equations continue to be smooth and unique for all time? This question was posed in 1934 by J. Leray and is still without answer, neither in the positive nor in the negative. There is no uniqueness proof except for over small time intervals and it has been questioned whether the Navier-Stokes equations really describe general flows. But there is no proof for non-uniqueness either.
Uniqueness of the solutions of the equations of motion is the cornerstone of classical determinism (J. Earman 1986). If more than one solution were associated to the same initial data, the committed determinist will say that the space of the solutions is too large, beyond the real physical possibility, and that uniqueness can be restored if the unphysical solutions are excluded.
A question intimately related to the uniqueness problem is the regularity of the solution. Do the solutions to the Navier-Stokes equations blow-up in finite time? The solution is initially regular and unique, but at the instant T when it ceases to be unique (if such an instant exists), the regularity could also be lost.
One may imagine that blow-up of initially regular solutions never happens, or that it becomes more likely as the initial norm increases, or that there is blow-up, but only on a very thin set of probability zero. The best result in this direction concerning the possible loss of smoothness for the Navier-Stokes equations was obtained by L. Caffarelli (1982), R. Kohn and L. Nirenberg (1998), who proved that the one-dimensional Hausdorff measure of the singular set is zero.
We can say that if "some quantity" turns out to "be small", then the Navier-Stokes equations are well-posed in the sense of Hadamard (existence, uniqueness and stability of the corresponding solutions). For instance, the unique global solution exists when the initial value and the exterior force are small enough, and the solution is smooth depending on smoothness of these data. Another quantity that can be small is the dimension. If we are in dimension $n = 2$, the situation is easier than in dimension $n = 3$ and completely understood (P. Lions (1966), R. Temam

(1979)). Finally, if the domain $\Omega \subset \mathbb{R}^3$ is small, in the sense that $\Omega = \omega \times (0, \epsilon)$ is thin in one direction, say, then the question is also settled by M. Wiegner (1999). In this thesis, we study well-posedness for the Cauchy problem of incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \otimes u) - \nabla p, \\ \mathrm{div}(u) = 0, \\ u_0(0, x) = u_0, \end{cases} \tag{0.1}$$

where $t \in \mathbb{R}^+, x \in \mathbb{R}^d$ $(d \geq 2), u = (u_1, u_2, ..., u_d)$ denote the flow velocity vector and $p(t, x)$ describe the scalar pressure, $\nabla = (\partial_1, \partial_2, ..., \partial_d)$ is the gradient operator, $\Delta = \partial_1^2 + \partial_2^2 + ... + \partial_d^2$ is the Laplacian, $u_0(x) = (u_1^0, u_2^0, ..., u_d^0)$ is a given initial datum with $\mathrm{div}(u_0) = \partial_1 u_1^0 + \partial_2 u_2^0 + ... + \partial_d u_d^0 = 0$. For a tensor $F = (F_{ij})$ we define the vector $\nabla \cdot F$ by $(\nabla \cdot F)_i = \sum_{j=1}^d \partial_j F_{ij}$ and for two vectors u and v, we define their tensor product $(u \otimes v)_{ij} = u_i v_j$. It is to see that (0.1) can be rewritten in the following equivalent form:

$$\begin{cases} \partial_t u = \Delta u - \mathbb{P}\nabla \cdot (u \otimes u), \\ u_0(0, x) = u_0, \end{cases} \tag{0.2}$$

where the operator $\mathbb{P}$ is the Helmholtz-Leray projection onto the divergence-free fields

$$(\mathbb{P}f)_j = f_j + \sum_{1 \leq k \leq d} R_j R_k f_k,$$

here R_j are the Riesz transforms defined as

$$R_j = \frac{\partial_j}{\sqrt{-\Delta}} \quad \text{i.e.} \quad \widehat{R_j g}(\xi) = \frac{i\xi_j}{|\xi|}\hat{g}(\xi)$$

with $\hat{}$ denoting the Fourier transform. It is known that (0.2) is essentially equivalent to the following integral equation:

$$u = e^{t\Delta} u_0 - \int_0^t e^{(t-\tau)\Delta} \mathbb{P}\nabla \cdot (u \otimes u) \mathrm{d}\tau, \tag{0.3}$$

where the heat kernel $e^{t\Delta}$ is defined as

$$e^{t\Delta} u(x) = ((4\pi t)^{-d/2} e^{-|\cdot|^2/4t} * u)(x).$$

Note that (0.1) is scaling invariant in the following sense: if u solves (0.1), so does $u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$ and $p_\lambda(t, x) = \lambda^2 p(\lambda^2 t, \lambda x)$ with initial data $\lambda u_0(\lambda x)$. A function space X defined in $\mathbb{R}^d$ is said to be a critical space for (0.1) if its norm is invariant under the action of the scaling $f(x) \to \lambda f(\lambda x)$ for any $\lambda > 0$, i.e., $\|f(\cdot)\| = \|\lambda f(\lambda x)\|$. It is easy to see that the following spaces are critical spaces for NSE:

$$\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d) \hookrightarrow L^d(\mathbb{R}^d) \hookrightarrow \dot{B}_q^{\frac{d}{q}-1,\infty}(\mathbb{R}^d)_{(q<\infty)} \hookrightarrow BMO^{-1}(\mathbb{R}^d) \hookrightarrow \dot{B}_\infty^{-1,\infty}(\mathbb{R}^d). \tag{0.4}$$

It is remarkable feature that the Navier-Stokes equations are well-posed in the sense of Hadarmard (existence, uniqueness and continuous dependence on data) when the initial data are divergence-free and belong to the critical function spaces (except

$\dot{B}_{\infty}^{-1,\infty}$) listed in (0.4) (M. Cannone (1995) for $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$, $L^d(\mathbb{R}^d)$, and $\dot{B}_q^{\frac{d}{q}-1,\infty}(\mathbb{R}^d)$, see H. Koch (2001) for $BMO^{-1}(\mathbb{R}^d)$, and the recent ill-posedness result J. Bourgain (2008) for $\dot{B}_{\infty}^{-1,\infty}(\mathbb{R}^d)$). Very recently, ill-posedness of Navier-Stokes equations in critical Besov spaces $\dot{B}_{\infty,q}^{-1}$ was investigated B. Wang (2015) and finite time blowup for an averaged three-dimensional Navier-Stokes equation was investigated T. Tao. In the 1960s, mild solutions were first constructed by Kato and Fujita (1962) and Kato and Fujita (1964) that are continuous in time and take values in the Sobolev spaces $H^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$, say $u \in C([0,T]; H^s(\mathbb{R}^d))$. In 1992, a modern treatment for mild solutions in $H^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$ was given by Chemin (1992). In 1995, using the simplified version of the bilinear operator, Cannone proved the existence of mild solutions in $\dot{H}^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$, see M. Cannone (1995). Results on the existence of mild solutions with value in $L^q(\mathbb{R}^d), (q > d)$ were established in the papers of Fabes, Jones and Rivière (1972) and of Y. Giga (1986). Concerning the initial data in the space L^∞, the existence of a mild solution was obtained by Cannone and Meyer (1995). Moreover, in Cannone and Meyer (1995), they also obtained theorems on the existence of mild solutions with value in the Morrey-Campanato space $M_2^q(\mathbb{R}^d), (q > d)$ and the Sobolev space $H_q^s(\mathbb{R}^d), (q < d, \frac{1}{q} - \frac{s}{d} < \frac{1}{d})$, and in general in the so-called well-suited space $\mathcal{W}$ for the Navier-Stokes equations. The Navier-Stokes equations in the Morrey-Campanato spaces were also treated by T. Kato (1992) and Taylor M. Taylor (1992). In 1981, F. Weissler (1981) gave the first existence result for mild solutions in the half space $L^3(\mathbb{R}^3_+)$. Then Giga and Miyakawa (1985) generalized the result to $L^3(\Omega)$, where Ω is an open bounded domain in $\mathbb{R}^3$. Finally, in 1984, T. Kato (1984) obtained, by means of a purely analytical tool (involving only Hölder and Young inequalities and without using any estimate of fractional powers of the Stokes operator), an existence theorem in the whole space $L^3(\mathbb{R}^3)$. In (M. Cannone (1995), M. Cannone (1997), M. Cannone (1999)), Cannone showed how to simplify Kato's proof. The idea is to take advantage of the structure of the bilinear operator in its scalar form. In particular, the divergence ∇ and heat $e^{t\Delta}$ operators can be treated as a single convolution operator. In 1994, Kato and Ponce (1994) showed that NSE are well-posed when the initial data belong to the homogeneous Sobolev spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d), (d \leq q < \infty)$.

In this thesis, we use the progress achieved in the field of harmonic analysis for the last fifteen years to study the Navier-Stokes equations. We mean the use of the Fourier transform and its properties, better suited for the study of nonlinear problems.

Chapter 1 is devoted to the recalling of some well-known results of harmonic analysis.

In Chapter 2, we apply these tools to the study of the Cauchy problem for the Navier-Stokes equations.

Section 2.1 presents the general shift-invariant space of distributions and some Sobolev spaces over a shift-invariant Banach space of distributions.

From Sections 2.2 to Section 2.6, we construct mild solutions to (0.3), a natural approach is to iterate the transform $u \to e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla \cdot (u \otimes u)\mathrm{d}\tau$ and to find a fixed point u for this transform. This is the so-called Picard contraction method already in use by C. Oseen (1927) to establish the local existence of a classical solution to the Navier-Stokes equations for a regular initial value. By Theorem 1.3.1 (see Section 1.5 of Chapter 1), to find a fixed point u for the equation (0.3), we need to try to find a Banach space $\mathcal{E}_T$ of functions defined on $(0,T)\times\mathbb{R}^d$ so that the bilinear operator B defined by

$$B(u,v)(t)=\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes v)\mathrm{d}s \tag{0.5}$$

is bounded from $\mathcal{E}_T\times\mathcal{E}_T\to\mathcal{E}_T$. Section 2.2 to Section 2.6 are devoted to construct examples of such spaces $\mathcal{E}_T$. The obtained results have a standard relation between existence time and data size: large time with small data or large data with small time.

In Section 2.2, we study local and global well-posedness for the Navier-Stokes equations with initial data in homogeneous Sobolev spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ for $d\geq 2, 1<q\leq d$. The obtained result improves the known ones for $q=2$ and $q=d$. These cases were considered by many authors, see (M. Cannone (1995), J. Chemin (1992), H. Fujita and T. Kato (1964),T. Kato (1984), P. G. Lemarie-Rieusset (2002)).

In Section 2.3, we study local well-posedness for the Navier-Stokes equations with arbitrary initial data in homogeneous Sobolev spaces $\dot{H}_p^s(\mathbb{R}^d)$ for $d\geq 2, p>\frac{d}{2}$, and $\frac{d}{p}-1\leq s<\frac{d}{2p}$. The obtained result improves the known ones for $p>d$ and $s=0$ (see M. Cannone (1995), M. Cannone and Y. Meyer (1995)). In the case of critical indexes $s=\frac{d}{p}-1$, we prove global well-posedness for Navier-Stokes equations when the norm of the initial value is small enough. This result is a generalization of the ones in M. Cannone (1999) and P. G. Lemarie-Rieusset (2002) in which $(p=d, s=0)$ and $(p>d, s=\frac{d}{p}-1)$, respectively.

In Section 2.4, we introduce and study Sobolev-Fourier-Lorentz spaces $\dot{H}^s_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$. We then study local and global well-posedness for the Navier-Stokes equations with initial data in critical spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $d\geq 2, 1\leq p<\infty$, and $1\leq r<\infty$.

In Section 2.5, we study local well-posedness for the Navier-Stokes equations with the arbitrary initial value in homogeneous Sobolev-Lorentz spaces $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d):=(-\Delta)^{-s/2}L^{q,r}$ for $d\geq 2, q>1, s\geq 0$, $1\leq r\leq\infty$, and $\frac{d}{q}-1\leq s<\frac{d}{q}$. This result improves the known results for $q>d, r=q, s=0$, see M. Cannone (1995) and for $q=r=2, \frac{d}{2}-1<s<\frac{d}{2}$, see M. Cannone (1995) and J. Chemin 1992.

In the case of critical indexes $(s=\frac{d}{q}-1)$, we prove global well-posedness for NSE when the norm of the initial value is small enough. The result is a generalization of the result in M. Cannone (1997) for $q=r=d, s=0$.

In Section 2.6, for $0\leq m<\infty$ and index vectors $\mathbf{q}=(q_1,q_2,...,q_d), \mathbf{r}=(r_1,r_2,...,r_d)$, where $1<q_i<\infty, 1\leq r_i\leq\infty$, and $1\leq i\leq d$, we introduce and study mixed-norm Sobolev-Lorentz spaces $\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}$. Then we investigate the existence and uniqueness of solutions to the Navier-Stokes equations in the spaces $\mathcal{Q}:=\mathcal{Q}_T=L^p([0,T];\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}})$ where $p>2,\ T>0$, and initial data is taken in the class $\mathcal{I}=\{u_0\in(\mathcal{S}'(\mathbb{R}^d))^d, \mathrm{div}(u_0)=0: \|e^{\cdot\Delta}u_0\|_{\mathcal{Q}}<\infty\}$. In the case when $m=0,\ q_1=q_2=...=q_d=r_1=r_2=...=r_d$, our results recover those of Faber, Jones and Riviere (1972).

In Chapter 3, using the method of Foias-Temam, we show the vanishing of Hausdorff measure of the singular set in time of weak solutions to the Navier-Stokes equations in the 3D torus.

Chapter 1

Preliminaries

1.1 This section is devoted to the recalling of some well-known results of harmonic analysis.

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1.1.1. The Littlewood-Paley decomposition

We take an arbitrary function φ in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ and whose Fourier transform $\hat{\varphi}$ is such that $0 \leq \hat{\varphi}(\xi) \leq 1,\ \hat{\varphi}(\xi) = 1$ if $|\xi| \leq \frac{3}{4},\ \hat{\varphi}(\xi) = 0$ if $|\xi| \geq \frac{3}{2}$, and let $\psi(x) = 2^d\varphi(2x) - \varphi(x), \varphi_j(x) = 2^{dj}\varphi(2^j x),\ j \in \mathbb{Z}, \psi_j(x) = 2^{dj}\psi(2^j x),\ j \in \mathbb{Z}$. We denote by S_j and Δ_j , respectively, the convolution operators with φ_j and ψ_j. The set $\{S_j, \Delta_j\}_{j\in\mathbb{Z}}$ is the Littlewood-Paley decomposition.

1.1.2. The Besov spaces

The Littlewood-Paley decomposition is very useful because we can define (independently of the choice of the initial function φ) the following (inhomogeneous) Besov spaces.

Definition 1.1.1. *Let* $0 < p, q \leq \infty$ *and* $s \in \mathbb{R}$. *Then a tempered distribution* f *belongs to the (inhomogeneous) Besov space* $B_q^{s,p}$ *if and only if*

$$\|S_0 f\|_q + \Big(\sum_{j\geq 0} \big(2^{sj}\|\Delta_j f\|_q\big)^p\Big)^{\frac{1}{p}} < \infty.$$

For the sake of completeness, we also define the (inhomogeneous) Triebel-Lizorkin spaces, even if we will not make a great use of them in the study of the Navier-Stokes equations.

Definition 1.1.2. *Let* $0 < p \leq \infty, 0 < q < \infty$, *and* $s \in \mathbb{R}$. *Then a tempered distribution* f *belongs to the (inhomogeneous) Triebel-Lizorkin space* $F_q^{s,p}$ *if and only* *if*

$$\|S_0 f\|_q + \Big\| \Big(\sum_{j\geq 0} \big(2^{sj}|\Delta_j f|\big)^p \Big)^{\frac{1}{p}} \Big\|_q < \infty.$$

We are now ready to define the homogeneous version of the Besov and Triebel-Lizorkin spaces (G. Bourdaud (1993), G. Bourdaud (1988), M. Frazier (1991), J. Peetre (1976)). If $m \in \mathbb{Z}$, we denote by $\mathcal{P}_m$ the set of polynomials of degree $\leq m$ with the convention that $\mathcal{P}_m = \emptyset$ if $m < 0$. If $p = 1$ and $s - d/q \in \mathbb{Z}$, we put $m = s - d/q - 1$; if not, we put $m = [s - d/q]$, with the brackets denoting the integer part function.

Definition 1.1.3. *Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Then a tempered distribution f belongs to the (homogeneous) Besov space $\dot{B}^{s,p}_q$ if and only if*

$$\Big(\sum_{j\in\mathbb{Z}} \big(2^{sj}\|\Delta_j f\|_q\big)^p \Big)^{\frac{1}{p}} < \infty \text{ and } f = \sum_{j\in\mathbb{Z}} \Delta_j f \text{ in } \mathcal{S}'/\mathcal{P}_m.$$

Definition 1.1.4. *Let $0 < p \leq \infty, 0 < q < \infty$, and $s \in \mathbb{R}$. Then a tempered distribution f belongs to the (homogeneous) Triebel-Lizorkin space $\dot{F}^{s,p}_q$ if and only if*

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} \big(2^{sj}|\Delta_j f|\big)^p \Big)^{\frac{1}{p}} \Big\|_q < \infty \text{ and } f = \sum_{j\in\mathbb{Z}} \Delta_j f \text{ in } \mathcal{S}'/\mathcal{P}_m,$$

with an analogous modification as in the inhomogeneous case when $q = \infty$.

1.2 The Navier-Stokes equations

This thesis studies the Cauchy problem of the incompressible Navier-tokes equations (NSE) in the whole space $\mathbb{R}^d$ for $d \geq 2$,

$$\begin{cases} \partial_t u = \Delta u - \nabla\cdot(u\otimes u) - \nabla p, \\ \mathrm{div}(u) = 0, \\ u(0,x) = u_0, \end{cases} \tag{1.1}$$

which is a condensed writing for

$$\begin{cases} 1 \leq k \leq d, \quad \partial_t u_k = \Delta u_k - \sum_{l=1}^d \partial_l(u_l u_k) - \partial_k p, \\ \sum_{l=1}^d \partial_l u_l = 0, \\ 1 \leq k \leq d, \quad u_k(0,x) = u_{0k}. \end{cases}$$

The unknown quantities are the velocity $u(t,x) = (u_1(t,x), \ldots, u_d(t,x))$ of the fluid element at time t and position x and the pressure $p(t,x)$. Taking the divergence of (1.1), we obtain: $\Delta p = -\nabla\otimes\nabla\cdot(u\otimes u) = -\sum_{k=1}^d \sum_{l=1}^d \partial_k\partial_l(u_k u_l)$. Thus, we formally get the equations

$$\begin{cases} \partial_t u = \Delta u - \mathbb{P}\nabla\cdot(u\otimes u), \\ \mathrm{div}(u) = 0, \end{cases} \tag{1.2}$$

where $\mathbb{P}$ is the Helmholtz Leray projection operator defined as $\mathbb{P}f := f - \nabla\frac{1}{\Delta}(\nabla\cdot f)$ $= (I - \frac{\nabla\otimes\nabla}{\Delta})f$. We shall study the Cauchy problem for the equation (1.2) (looking for a solution on $(0,T)\times\mathbb{R}^d$ with the initial value u_0), and transform (1.2) into the integral equation

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)\mathrm{d}s. \tag{1.3}$$

1.3 Outline of the dissertation

For $T > 0$, we say that u is a mild solution of NSE on $[0,T]$ corresponding to a divergence-free initial datum u_0 when u satisfies the integral equation (1.3). We rewrite the equation (1.3) as following

$$u = U_0 - B(u,u), \tag{1.4}$$

where

$$B(u,v)(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes v)\mathrm{d}s \quad \text{and} \quad U_0 = e^{t\Delta}u_0. \tag{1.5}$$

Then we will find a fixed point u for the equation (1.4). This is the so-called Picard contraction method already in use by Oseen at the beginning of the 20th century to establish the (local) existence of a classical solution to the Navier-Stokes equations for a regular initial value, see C. Oseen (1927).

Theorem 1.3.1. *Let X be a Banach space, and let $B : X\times X \to X$ be a continuous bilinear form such that exists η so that $\|B(x,y)\| \le \eta\|x\|\|y\|$ for any x and $y\in X$. Then for any fixed $y\in X$ such that $\|y\| < 1/(4\eta)$, the equation $x = y - B(x,x)$ has a unique solution $\overline{x}\in X$ satisfying $\|\overline{x}\|\le R$, with $R = \frac{1-\sqrt{1-4\eta\|y\|}}{2\eta}$.*

By above Theorem, we need to try to find a Banach space $\mathcal{E}_T$ so that the bilinear operator B which is defined by (1.5) is bounded from $\mathcal{E}_T\times\mathcal{E}_T\to\mathcal{E}_T$.
Chapter 2 is devoted to construct examples of such spaces $\mathcal{E}_T$. The solutions we obtain through the Picard contraction principle are called mild solutions. We call a space $\mathcal{E}_T$ if we may apply the Picard contraction principle as an admissible path space for the Navier-Stokes equations, and the associated space E_T as an adapted value space.
Let us review some results. We will indicate what are the admissible path space $\mathcal{E}_T$ and the associated adapted space E_T.

• Classical admissible spaces are provided by the L^p theory of Kato (1984):
- For $d < p < \infty, C([0;T];L^p)$ is admissible with the associated adapted space $L^p(\mathbb{R}^d)$.
- For $p = d$, the space

$$\{f\in C([0;T];L^d) : \sup_{0<t<T}\sqrt{t}\|f\|_{L^\infty(\mathrm{d}x)} < \infty \text{ and } \lim_{t\to 0}\sqrt{t}\|f\|_{L^\infty(\mathrm{d}x)} = 0\}$$

is admissible with the associated adapted space $L^d(\mathbb{R}^d)$.

• Prodi (1959) gave the following admissible spaces, plus the corresponding the associated adapted space

$$\mathcal{E}_T = L^q([0,T], L^p),\ E_T = \dot{B}_p^{\frac{d}{p}-1,q} \text{ with } \frac{d}{p} + \frac{2}{q} = 1 \text{ and } d < p < \infty.$$

• Cannone (1997) studied the space

$$\{f \in C([0;T]; L^d) : \sup_{0<t<T} t^{\frac{\alpha}{2}} \|f\|_{L^q(\mathrm{d}x)} < \infty \text{ and } \lim_{t\to 0} t^{\frac{\alpha}{2}} \|f\|_{L^q(\mathrm{d}x)} = 0\}, \tag{1.6}$$

$$\text{with } q > d \text{ and } \alpha = 1 - \frac{d}{q}, \tag{1.7}$$

which is admissible with the associated adapted space $L^d(\mathbb{R}^d)$.

• Gallagher and Planchon (2002) studied a Besov spaces scale

$$\mathcal{E}_T = L^q \dot{B}_p^{\frac{2}{q}+\frac{d}{p}-1,q},\ E_T = \dot{B}_p^{\frac{d}{p}-1,q} \text{ with } \frac{d}{p} + \frac{2}{q} > 1.$$

In Chapter 2 of this thesis we study some other admissible spaces with other associated adapted spaces.

In Section 2.2 of Chapter 2:
- For $2 < q \leq d$ and p be such that $q < p < \min\Big\{\frac{(d-2)q}{d-q},\ d+2\Big\}$, we consider the admissible space

$$L^p\big([0,T]; \dot{H}_p^{\frac{2+d-p}{p}}\big) \cap L^\infty\big([0,T]; \dot{H}_q^{d/q-1}\big)$$

which is admissible with the associated adapted space $\dot{H}_q^{d/q-1}(\mathbb{R}^d)$.
- For $1 < q \leq 2$ we consider the admissible space

$$L^{2q}\big([0,T]; \dot{H}_{\frac{dq}{d+1-q}}^{\frac{d+2-2q}{q}}\big) \cap L^\infty\big([0,T]; \dot{H}_q^{d/q-1}\big)$$

which is admissible with the associated adapted space $\dot{H}_q^{d/q-1}(\mathbb{R}^d)$.

In Section 2.3 of Chapter 2:
- For $p > \frac{d}{2}$, $\frac{d}{p} - 1 \leq s < \frac{d}{2p}$, $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$, and $r > \max\{p,q\}$, we consider the admissible space $\mathcal{K}_{q,T}^r \cap L^\infty([0,T]; \dot{H}_p^s)$ is admissible with the associated adapted space $\dot{H}_p^s(\mathbb{R}^d)$, where space $\mathcal{K}_{q,T}^r$ is made up of the functions $u(t,x)$ such that $\sup_{0<t<T} t^{\frac{\alpha}{2}} \big\|u(t,x)\big\|_{L^r} < \infty$ and $\lim_{t\to 0} t^{\frac{\alpha}{2}} \big\|u(t,x)\big\|_{L^r} = 0$ with $\alpha = d(\frac{1}{q} - \frac{1}{r})$.

In Section 2.4 of Chapter 2: For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$ we introduce and study the Sobolev-Fourier-Lorentz spaces $\dot{H}_{\mathcal{L}^{p,r}}^s(\mathbb{R}^d)$.
- For $1 < p \leq d, 1 \leq r < \infty$, and $\tilde{p}$ be such that $\frac{1}{2p} + \frac{[\frac{d}{p}]-1}{2d} < \frac{1}{\tilde{p}} < \min\Big\{\frac{[\frac{d}{p}]}{d}, \frac{1}{2} + \frac{[\frac{d}{p}]-1}{2d}\Big\}$, we consider the admissible space $\mathcal{K}_{\frac{d}{[\frac{d}{p}]},1,T}^{\tilde{p}} \cap L^\infty\big([0,T]; \dot{H}_{\mathcal{L}^{p,r}}^{\frac{d}{p}-1}\big)$ which is admissible with the associated adapted space $\dot{H}_{\mathcal{L}^{p,r}}^{\frac{d}{p}-1}(\mathbb{R}^d)$, where the space $\mathcal{K}_{p,r,T}^{\tilde{p}}$ is made up by the functions $u(t,x)$ such that $\sup_{0<t<T} t^{\frac{\alpha}{2}} \big\|u(t,x)\big\|_{\dot{H}_{\mathcal{L}^{\tilde{p},r}}^{\frac{d}{\tilde{p}}-1}} < \infty$ and $\lim_{t\to 0} t^{\frac{\alpha}{2}} \big\|u(t,x)\big\|_{\dot{H}_{\mathcal{L}^{\tilde{p},r}}^{\frac{d}{\tilde{p}}-1}} = 0$

with $\alpha = d\big(\frac{1}{p} - \frac{1}{\tilde{p}}\big)$.
- For $p \geq d, r \geq 1$, and $q > p$, we consider the admissible space $\mathcal{K}^q_{d,1,T} \cap L^\infty([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}})$ is admissible with the associated adapted space $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$.
- For $d - 1 < s < d$ and $r \geq 1$, we consider the admissible space $\mathcal{K}_{s,r,T} \cap L^\infty([0,T]; \dot{H}^{d-1}_{\mathcal{L}^{1,r}})$ which is admissible with the associated adapted space $\dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$, where the space $\mathcal{K}_{s,r,T}$ is made up by the functions $u(t,x)$ such that $\sup\limits_{0<t<T} t^{\frac{\alpha}{2}} \big\| u(t,x) \big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} < \infty$ and $\lim\limits_{t \to 0} t^{\frac{\alpha}{2}} \big\| u(t,x) \big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} = 0$ with $\alpha = s + 1 - d$.

In Section 2.5 of Chapter 2: For $q > 1, 1 \leq r \leq \infty$ and $0 \leq s < \frac{d}{q}$, we introduce and study the Sobolev-Lorentz spaces $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d)$, which are generalizations of the classical Sobolev spaces $\dot{H}^s_q(\mathbb{R}^d)$. For $s \geq 0, q > 1, r \geq 1, \frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}$, and $\tilde{q}$ be such that $\frac{1}{2}\Big(\frac{1}{q} + \frac{s}{d}\Big) < \frac{1}{\tilde{q}} < \min\Big\{\frac{1}{2} + \frac{s}{2d}, \frac{1}{q}\Big\}$, we consider the admissible space $\mathcal{K}^{s,\tilde{q}}_{q,1,T} \cap L^\infty([0,T]; \dot{H}^s_{L^{q,r}})$ which is admissible with the associated adapted space $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d)$, where space $\mathcal{K}^{s,\tilde{q}}_{q,r,T}$ is made up by the functions $u(t,x)$ such that $\sup\limits_{0<t<T} t^{\frac{\alpha}{2}} \big\| u(t,x) \big\|_{\dot{H}^s_{L^{\tilde{q},r}}} < \infty$ and $\lim\limits_{t \to 0} t^{\frac{\alpha}{2}} \big\| u(t,x) \big\|_{\dot{H}^s_{L^{\tilde{q},r}}} = 0$ with $\alpha = d(\frac{1}{q} - \frac{1}{\tilde{q}})$.

In Section 2.6 of Chapter 2: For $0 \leq m < \infty$ and index vectors $\mathbf{q} = (q_1, q_2, ..., q_d)$ and $\mathbf{r} = (r_1, r_2, ..., r_d)$, where $1 < q_i < \infty, 1 \leq r_i \leq \infty$ for $i = 1, 2, .., d$, we introduce and study mixed-norm Sobolev-Lorentz spaces $\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}$. For $\mathbf{q} > \mathbf{1},\ \mathbf{r} \geq \mathbf{1}, 2 < p < \infty$, and $m \geq 0$ be such that $m < \frac{1}{2}\sum_{i=1}^d \frac{1}{q_i},\ \frac{2}{p} - m + \sum_{i=1}^d \frac{1}{q_i} \leq 1,\ 2 < \frac{q_i}{\Big(1 - \frac{m}{\sum_{i=1}^d \frac{1}{q_i}}\Big)} < \infty,\ i = 1, 2, .., d$, we consider the admissible space $L^p([0,T]; \dot{H}^m_{L^{\mathbf{q},\mathbf{r}}})$ which is admissible with the associated adapted space $B^{m-\frac{2}{p},p}_{L^{\mathbf{q},\mathbf{r}}}$ (a Besov space).

In Chapter 3: Using the method of Foias-Temam, we investigate the Hausdorff dimension of the singular set in time of weak solutions to the Navier-Stokes equations.

Chapter 2

Mild solutions in some Sobolev spaces over a shift-invariant Banach space

In this chapter we investigate mild solutions to the Navier-Stokes equations in some Sobolev spaces over a shift-invariant Banach space of distributions.

2.1 The Sobolev spaces over a shift-invariant Banach space of distributions

We shall often use Banach spaces of distributions whose norms are invariant under translations $\|f\|_E = \|f(x-x_0)\|_E$ and on which dilations operate boundedly.

Definition 2.1.1. (Shift-invariant Banach spaces of distributions.)
A shift-invariant Banach space of test functions is a Banach space E such that we have the continuous embeddings $\mathcal{S}(\mathbb{R}^d) \hookrightarrow E \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ and so that:
(a) *for all $x_0 \in \mathbb{R}^d$ and for all $f \in E, f(x-x_0) \in E$ and $\|f\|_E = \|f(x-x_0)\|_E$,*
(b) *for all $\lambda > 0$ there exists $C_\lambda > 0$ so that for all $f \in E$ $f(\lambda x) \in E$ and $\|f(\lambda x)\|_E \le C_\lambda \|f\|_E$,*
(c) *$\mathcal{D}(\mathbb{R}^d)$ is dense in E.*

In the following definitions, we introduce the Sobolev spaces and their homogeneous spaces over a shift-invariant Banach space of distributions. Before proceeding to the definition of the Sobolev spaces, let us introduce several necessary notations. For a real number s, the operators $\dot{\Lambda}^s$ and $(Id - \Delta)^{s/2}$ are defined through the Fourier transform by

$$\big(\dot{\Lambda}^s f\big)^\wedge(\xi) = |\xi|^s \hat{f}(\xi) \text{ and } \big((Id-\Delta)^{s/2} f\big)^\wedge(\xi) = \big(1+|\xi|^2\big)^{s/2} \hat{f}(\xi).$$

Definition 2.1.2. (Sobolev spaces.)
Let E be a shift-invariant Banach space of distributions. Then, for $s \in \mathbb{R}$, the space H^s_E is defined as the space $(Id-\Delta)^{-s/2}E$, equipped with the norm $\big\|f\big\|_{H^s_E} = \big\|(Id-\Delta)^{s/2} f\big\|_E$.

Definition 2.1.3. (Homogeneous Sobolev spaces.) *Let E be a shift-invariant Banach space of distributions. Then, for $s \in \mathbb{R}$, the space $\dot{H}^s_E$ is defined as the closure* *of the space* $S_0 = \{f \in \mathcal{S} :\ 0 \notin \mathrm{Supp}\hat{f}\}$ *in the norm* $\|f\|_{\dot{H}^s_E} = \|\dot{\Lambda}^s f\|_E$.

In the rest of this chapter, we investigate mild solutions to NSE in the following Sobolev spaces over a shift-invariant Banach space of distributions:
- Sobolev spaces and homogeneous Sobolev spaces over the Lebesgue spaces, (Sections 2.2 and 2.3).
- Homogeneous Sobolev spaces over the Fourier-Lorentz spaces, (Section 2.4).
- Homogeneous Sobolev spaces over the Lorentz spaces, (Section 2.5).
- Homogeneous Sobolev over the mixed-norm Lorentz spaces, (Section 2.6).

2.2 Mild solutions in $\dot{H}_q^{\frac{d}{q}-1}$ and $H_q^{\frac{d}{q}-1} (1 < q \le d)$

In this section, we investigate mild solutions to NSE in the spaces $L^\infty\big([0,T]; \dot{H}_q^{\frac{d}{q}-1}\big)$ when the initial data belong to the homogeneous Sobolev spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d), (d \ge 2, 1 < q \le d)$. We obtain the existence of local mild solutions with arbitrary initial value and existence of global mild solutions when the norm of the initial value is small enough. The main results of this section are Theorems 2.2.3, 2.2.4, 2.2.5, and 2.2.6, the lemmas we need in order to prove these theorems are Lemmas 2.2.1 and 2.2.2 devoted to the study of the bilinear operator $B(u,v)(t)$ defined by (1.5). We prove these theorems by combining Lemmas 2.2.1 and 2.2.2 with fixed point algorithm Theorem 1.3.1.

Lemma 2.2.1. *Let* $d \ge 3,\ s \ge 0,\ p > 1,\ r > 2,$ *and* $T > 0$ *be such that* $\frac{s}{d} < \frac{1}{p} < \frac{1}{2} + \frac{s}{2d}$ *and* $\frac{2}{r} + \frac{d}{p} - s \le 1$. *Then the bilinear operator* $B(u,v)(t)$ *is continuous from* $L^r([0,T]; H_p^s) \times L^r([0,T]; H_p^s)$ *into* $L^r([0,T]; H_p^s)$, *and the following inequality holds* $\|B(u,v)\|_{L^r([0,T];H_p^s)} \le CT^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})}\|u\|_{L^r([0,T];H_p^s)}\|v\|_{L^r([0,T];H_p^s)}$, *where* C *is a positive constant independent of* T.

Lemma 2.2.2. *Let* $d \ge 3,\ 0 \le s < d,\ p > 1,\ r > 2,$ *and* $T > 0$ *be such that* $\frac{1}{p} < \frac{1}{2} + \frac{s}{2d}, \frac{2}{p} \ge \frac{s+1}{d}$, *and* $\frac{2}{r} + \frac{d}{p} - s = 1$. *Then the bilinear operator* $B(u,v)(t)$ *is continuous from* $L^r\big([0,T]; \dot{H}_p^s\big) \times L^r\big([0,T]; \dot{H}_p^s\big)$ *into* $L^\infty\Big([0,T]; \dot{B}_{\tilde{p}}^{\frac{d}{\tilde{p}}-1,\frac{r}{2}}\Big)$, *where* $\frac{1}{\tilde{p}} = \frac{2}{p} - \frac{s}{d}$, *and we have the inequality* $\|B(u,v)\|_{L^\infty\big([0,T];\dot{B}_{\tilde{p}}^{\frac{d}{\tilde{p}}-1,\frac{r}{2}}\big)} \le C\|u\|_{L^r([0,T];\dot{H}_p^s)}\|v\|_{L^r([0,T];\dot{H}_p^s)}$, *where* C *is a positive constant independent of* T.

Theorem 2.2.3. *Let* $3 \le d \le 4$ *and* $2 \le q \le d$. *There exists a positive constant* $\delta_{q,d}$ *such that for all* $T > 0$ *and for all* $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ *with* $\mathrm{div}(u_0) = 0$ *satisfying*

$$\|e^{\cdot\Delta}u_0\|_{L^4\big([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\big)} \le \delta_{q,d}, \tag{2.1}$$

NSE has a unique mild solution $u \in L^4\big([0,T]; \dot{H}^{d/q-1}_{2dq/(2d-q)}\big) \cap L^\infty\big([0,T]; \dot{H}_q^{d/q-1}\big)$. *Denoting* $w = u - e^{\cdot\Delta}u_0$, *then we have* $w \in L^4\big([0,T]; \dot{H}^{d/q-1}_{2dq/(2d-q)}\big) \cap L^\infty\big([0,T]; \dot{B}_q^{d/q-1,2}\big)$. *Finally, we have* $\|e^{\cdot\Delta}u_0\|_{L^4\big([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\big)} \lesssim \|u_0\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \lesssim \|u_0\|_{\dot{H}_q^{d/q-1}}$, *in particular, for arbitrary* $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ *the inequality* (2.1) *holds when* $T(u_0)$ *is small* *enough; and there exists a positive constant $\sigma_{q,d}$ such that for all $\|u_0\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \leq \sigma_{q,d}$ we can take $T = \infty$.*

Theorem 2.2.4. *Let $3 \leq d \leq 4$ and $2 \leq q \leq d$. There exists a positive constant $\delta_{q,d}$ such that for all $T > 0$ and for all $u_0 \in H_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ satisfying*

$$\|e^{\cdot\Delta}u_0\|_{L^4\left([0,T];H^{d/q-1}_{2dq/(2d-q)}\right)} \leq \delta_{q,d}, \tag{2.2}$$

NSE has a unique mild solution $u \in L^4\big([0,T];H^{d/q-1}_{2dq/(2d-q)}\big) \cap L^\infty\big([0,T];H_q^{d/q-1}\big)$. Finally, we have $\|e^{\cdot\Delta}u_0\|_{L^4\left([0,T];H^{d/q-1}_{2dq/(2d-q)}\right)} \leq \|e^{\cdot\Delta}u_0\|_{L^4\left([0,\infty);H^{d/q-1}_{2dq/(2d-q)}\right)} \lesssim \|u_0\|_{H_q^{d/q-1}}$, in particular, for arbitrary $u_0 \in H_q^{\frac{d}{q}-1}$ the inequality (2.2) holds when $T(u_0)$ is small enough.

Theorem 2.2.5. *Let $d \geq 3$ and $2 < q \leq d$. Then for any p be such that $q < p < \min\Big\{\frac{(d-2)q}{d-q},\ d+2\Big\}$, there exists a constant $\delta_{q,p,d} > 0$ such that for all $T > 0$ and for all $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ satisfying*

$$\Big\|e^{\cdot\Delta}u_0\Big\|_{L^p\left([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\right)} \leq \delta_{q,p,d}, \tag{2.3}$$

NSE has a unique mild solution $u \in L^p\big([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\big) \cap L^\infty\big([0,T];\dot{H}_q^{d/q-1}\big)$. Denoting $w = u - e^{\cdot\Delta}u_0$, then we have $w \in L^p\big([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\big) \cap L^\infty\Big([0,T];\dot{B}^{\frac{d+p-2}{p}-1,\frac{p}{2}}_{\frac{dp}{d+p-2}}\Big)$. Finally, we have $\Big\|e^{\cdot\Delta}u_0\Big\|_{L^p\left([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\right)} \leq \Big\|e^{\cdot\Delta}u_0\Big\|_{L^p\left([0,\infty);\dot{H}_p^{\frac{2+d-p}{p}}\right)} \simeq \|u_0\|_{\dot{B}_p^{\frac{d}{p}-1,p}} \lesssim \|u_0\|_{\dot{H}_q^{\frac{d}{q}-1}}$, in particular, for arbitrary $u_0 \in \dot{H}_q^{d/q-1}$ the inequality (2.3) holds when $T(u_0)$ is small enough; and there exists a positive constant $\sigma_{q,p,d}$ such that for all $\|u_0\|_{\dot{B}_p^{\frac{d}{p}-1,p}} \leq \sigma_{q,p,d}$ we can take $T = \infty$.

Remark 2.2.1. The case $q = d$ was treated by several authors, see for example Hongjie Dong (2007), T. Kato (1984), and M. Cannone (1995).

Theorem 2.2.6. *Let $d \geq 3$ and $1 < q \leq 2$. There exists a positive constant $\delta_{q,d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ satisfying*

$$\Big\|e^{\cdot\Delta}u_0\Big\|_{L^{2q}\left([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\right)} \leq \delta_{q,d}, \tag{2.4}$$

NSE has a unique mild solution $u \in L^{2q}\Big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\Big) \cap L^\infty\big([0,T];\dot{H}_q^{d/q-1}\big)$. Denoting $w = u - e^{\cdot\Delta}u_0$, then we have $w \in L^{2q}\Big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\Big) \cap L^\infty\big([0,T];\dot{B}_q^{\frac{d}{q}-1,q}\big)$. Finally, we have $\Big\|e^{\cdot\Delta}u_0\Big\|_{L^{2q}\left([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\right)} \leq \Big\|e^{\cdot\Delta}u_0\Big\|_{L^{2q}\left([0,\infty);\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\right)} \simeq \|u_0\|_{\dot{B}^{(d+1)/q-2,2q}_{dq/(d+1-q)}} \lesssim \|u_0\|_{\dot{H}_q^{d/q-1}}$, in particular, for arbitrary $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ the inequality (2.4) holds when $T(u_0)$ is small enough; and there exists a positive constant $\sigma_{q,d}$ such that for all $\|u_0\|_{\dot{B}^{(d+1)/q-2,2q}_{dq/(d+1-q)}} \leq \sigma_{q,d}$ we can take $T = \infty$.

Remark 2.2.2. The case $q = 2$ was treated by several authors, see for example P. G. Lemarie-Rieusset (2002), H. Fujita (1964), and M. Cannone (1995).

2.3 Mild solutions in Sobolev spaces of negative order

In this section, we present an different algorithm for constructing mild solutions in the spaces $L^\infty([0,T];\dot{H}^s_p(\mathbb{R}^d))$ to the Cauchy problem for NSE when the initial datum belongs to the Sobolev spaces $\dot{H}^s_p(\mathbb{R}^d)$, with $d \geq 2, p > \frac{d}{2}$, and $\frac{d}{p}-1 \leq s < \frac{d}{2p}$. This result improves the known results of Cannone (1997).

2.3.1. Solutions to the Navier-Stokes equations with the initial value in the Sobolev spaces $\dot{H}^s_p(\mathbb{R}^d)$ for $d \geq 2, p > \frac{d}{2}$, and $\frac{d}{p} - 1 \leq s < \frac{d}{2p}$

The main result of this subsection is Theorem 2.3.4, The lemmas we need in order to prove Theorem 2.3.4 are Lemmas 2.3.1, 2.3.2 and 2.3.3 devoted to the study of the bilinear operator $B(u,v)(t)$ defined by (1.5). We prove Theorem 2.3.4 by combining these lemmas with fixed point algorithm Theorem 1.3.1.
We define the space $\mathcal{N}^s_{p,T}$ which is made up of the functions $u(t,x)$ such that $\big\|u\big\|_{\mathcal{N}^s_{p,T}} := \sup_{0<t<T}\big\|u(t,x)\big\|_{\dot{H}^s_p} < \infty$, and $\lim_{t\to 0}\big\|u(t,x)\big\|_{\dot{H}^s_p} = 0$, with $p > 1$ and $s \geq \frac{d}{p}-1$. We now define the auxiliary space $\mathcal{K}^{\tilde{q}}_{q,T}$ which is made up of the functions $u(t,x)$ such that $\big\|u\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}} := \sup_{0<t<T} t^{\frac{\alpha}{2}}\big\|u(t,x)\big\|_{L^{\tilde{q}}} < \infty$, and $\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|u(t,x)\big\|_{L^{\tilde{q}}} = 0$, with $\tilde{q} \geq q \geq d$ and $\alpha = d(\frac{1}{q} - \frac{1}{\tilde{q}})$.

Lemma 2.3.1. *Suppose that $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ with $p > 1$ and $\frac{d}{p} - 1 \leq s < \frac{d}{p}$. Then for all $\tilde{q}$ satisfying $\tilde{q} > \max\{p,q\}$, where $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$, we have $e^{\cdot\Delta}u_0 \in \mathcal{K}^{\tilde{q}}_{q,\infty}$.*

Lemma 2.3.2. *Let $p > \frac{d}{2}$ and $\frac{d}{p} - 1 \leq s < \frac{d}{2p}$. Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{q}}_{q,T} \times \mathcal{K}^{\tilde{q}}_{q,T}$ into $\mathcal{N}^s_{p,T}$, where $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$ and $q < \tilde{q} \leq 2p$, and the following inequality holds $\big\|B(u,v)\big\|_{\mathcal{N}^s_{p,T}} \leq CT^{\frac{1}{2}(1+s-\frac{d}{p})}\big\|u\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}}\big\|v\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}}$, where C is a positive constant and independent of T.*

Lemma 2.3.3. *Let $d \leq q \leq \tilde{q}_2 < \infty$ and $q < \tilde{q}_1 < \infty$ be such that one of the following conditions $q < \tilde{q}_1 < 2d, q \leq \tilde{q}_2 < \frac{d\tilde{q}_1}{2d-\tilde{q}_1}$, or $2d \leq \tilde{q}_1 \leq 2q, q \leq \tilde{q}_2 < \infty$, or $2q < \tilde{q}_1 < \infty, \frac{\tilde{q}_1}{2} < \tilde{q}_2 < \infty$, is satisfied. Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{q}_1}_{q,T} \times \mathcal{K}^{\tilde{q}_1}_{q,T}$ into $\mathcal{K}^{\tilde{q}_2}_{q,T}$, and we have the inequality $\big\|B(u,v)\big\|_{\mathcal{K}^{\tilde{q}_2}_{q,T}} \leq CT^{\frac{1}{2}(1-\frac{d}{q})}\big\|u\big\|_{\mathcal{K}^{\tilde{q}_1}_{q,T}}\big\|v\big\|_{\mathcal{K}^{\tilde{q}_1}_{q,T}}$, where C is a positive constant and independent of T.*

Theorem 2.3.4. *Let $p > \frac{d}{2}$ and $\frac{d}{p} - 1 \leq s < \frac{d}{2p}$. Set $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$.*
(a) *For all $\tilde{q} > \max\{p,q\}$, there exists a positive constant $\delta_{q,\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ satisfying*

$$T^{\frac{1}{2}(1+s-\frac{d}{p})}\sup_{0<t<T} t^{\frac{d}{2}(\frac{1}{p}-\frac{s}{d}-\frac{1}{\tilde{q}})}\big\|e^{t\Delta}u_0\big\|_{L^{\tilde{q}}} \leq \delta_{q,\tilde{q},d}, \tag{2.5}$$

NSE has a unique mild solution $u \in \bigcap_{r>\max\{p,q\}} \mathcal{K}^r_{q,T} \cap L^\infty([0,T];\dot{H}^s_p)$*. In particular, the inequality* (2.5) *holds for arbitrary* $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ *when* $T(u_0)$ *is small enough.*
(b) *If* $s = \frac{d}{p} - 1$ *then for all* $\tilde{q} > \max\{p,d\}$ *there exists a constant* $\sigma_{\tilde{q},d} > 0$ *such that if* $\|u_0\|_{\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}} \leq \sigma_{\tilde{q},d}$ *and* $T = +\infty$ *then the inequality* (2.5) *holds.*

2.4 Mild solutions in the Sobolev-Fourier-Lorentz spaces

In this section, for $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, we introduce and study Sobolev-Fourier-Lorentz spaces $\dot{H}^s_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$. After that we show that the Navier-Stokes equations are well-posed when the initial datum belongs to the critical Sobolev-Fourier-Lorentz spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $d \geq 2, 1 \leq p < \infty$, and $1 \leq r < \infty$. The result that is a generalization of the known results for the cases $p = 1$ and $p = \infty$ studied by Le Jan (1997) and Zhen Lei Fang (2011), respectively.

2.4.1. The Sobolev-Fourier-Lozentz Space

Definition 2.4.1. (Fourier-Lebesgue spaces). (See Lars Hormander (1976).)
For $1 \leq p \leq \infty$*, the Fourier-Lebesgue spaces* $\mathcal{L}^p(\mathbb{R}^d)$ *are defined as the space* $\mathcal{F}^{-1}(L^{p'}(\mathbb{R}^d))$*,* $(\frac{1}{p'} + \frac{1}{p} = 1)$*, equipped with the norm* $\|f\|_{\mathcal{L}^p(\mathbb{R}^d)} := \|\mathcal{F}(f)\|_{L^{p'}(\mathbb{R}^d)}$*, where* $\mathcal{F}$ *and* $\mathcal{F}^{-1}$ *denote the Fourier transform and its inverse.*

Definition 2.4.2. (Sobolev-Fourier-Lebesgue spaces).
For $s \in \mathbb{R}$*, and* $1 \leq p \leq \infty$*, the Sobolev-Fourier-Lebesgue spaces* $\dot{H}^s_{\mathcal{L}^p}(\mathbb{R}^d)$ *are defined as the space* $\dot{\Lambda}^{-s}\mathcal{L}^p(\mathbb{R}^d)$*, equipped with the norm* $\|u\|_{\dot{H}^s_{\mathcal{L}^p}} := \|\dot{\Lambda}^s u\|_{\mathcal{L}^p}$.

Definition 2.4.3. (Fourier-Lorentz spaces). *For* $1 \leq p, r \leq \infty$*, the Fourier-Lorentz spaces* $\mathcal{L}^{p,r}(\mathbb{R}^d)$ *are defined as the space* $\mathcal{F}^{-1}(L^{p',r}(\mathbb{R}^d))$*,* $(\frac{1}{p'} + \frac{1}{p} = 1)$*, equipped with the norm* $\|f\|_{\mathcal{L}^{p,r}(\mathbb{R}^d)} := \|\mathcal{F}(f)\|_{L^{p',r}(\mathbb{R}^d)}$.

Definition 2.4.4. (Sobolev-Fourier-Lorentz spaces).
For $s \in \mathbb{R}$ *and* $1 \leq r, p \leq \infty$*, the Sobolev-Fourier-Lorentz spaces* $\dot{H}^s_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ *are defined as the space* $\dot{\Lambda}^{-s}\mathcal{L}^{p,r}(\mathbb{R}^d)$*, equipped with the norm* $\|u\|_{\dot{H}^s_{\mathcal{L}^{p,r}}} := \|\dot{\Lambda}^s u\|_{\mathcal{L}^{p,r}}$.

2.4.2. Solutions to the Navier-Stokes equations with the initial value in the critical spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $1 < p \leq d$ and $1 \leq r < \infty$

The main result of this subsection is Theorem 2.4.3, The lemmas we need in order to prove Theorem 2.4.3 are Lemmas 2.4.1 and 2.4.2 devoted to the study of the bilinear operator $B(u,v)(t)$ defined by (1.5). We prove Theorem 2.4.3 by combining these lemmas with fixed point algorithm Theorem 1.3.1.
We define an auxiliary space $\mathcal{K}^{\tilde{p}}_{p,r,T}$ which is made up by the functions $u(t,x)$ such that $\|u\|_{\mathcal{K}^{\tilde{p}}_{p,r,T}} := \sup_{0<t<T} t^{\frac{\alpha}{2}} \|u(t,x)\|_{\dot{H}^{\frac{d}{\tilde{p}}-1}_{\mathcal{L}^{\tilde{p},r}}} < \infty$, and $\lim_{t\to 0} t^{\frac{\alpha}{2}} \|u(t,x)\|_{\dot{H}^{\frac{d}{\tilde{p}}-1}_{\mathcal{L}^{\tilde{p},r}}} = 0$, with

$1 < p \le \tilde{p} < \infty, \frac{1}{p} - \frac{1}{d} < \frac{1}{\tilde{p}}, 1 \le r \le \infty, T > 0$, and $\alpha = \alpha(p, \tilde{p}) = d\Big(\frac{1}{p} - \frac{1}{\tilde{p}}\Big)$.
In the following lemmas, denote by $[x]$ the integer part of x and by $\{x\}$ the fraction part of x.

Lemma 2.4.1. *Let $1 < p \le d$. Then for all $\tilde{p}$ be such that $\frac{1}{2p} + \frac{[\frac{d}{p}]-1}{2d} < \frac{1}{\tilde{p}} < \min\Big\{\frac{[\frac{d}{p}]}{d}, \frac{1}{2} + \frac{[\frac{d}{p}]-1}{2d}\Big\}$, the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T} \times \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}$ into $\mathcal{K}^{p}_{p,1,T}$ and the following inequality holds $\big\|B(u,v)\big\|_{\mathcal{K}^{p}_{p,1,T}} \le C\Big\|u\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}}\Big\|v\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}}$, where C is a positive constant and independent of T.*

Lemma 2.4.2. *Let $1 < p \le d$. Then for all $\tilde{p}$ be such that $\frac{[\frac{d}{p}]-1}{d} < \frac{1}{\tilde{p}} < \min\Big\{\frac{[\frac{d}{p}]}{d}, \frac{1}{2} + \frac{[\frac{d}{p}]-1}{2d}\Big\}$, the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T} \times \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}$ into $\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T}$ and the following inequality holds $\big\|B(u,v)\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T}} \le C\Big\|u\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}}\Big\|v\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}}$, where C is a positive constant and independent of T.*

Theorem 2.4.3. *Let $1 < p \le d$ and $1 \le r < \infty$. Then for all $\tilde{p}$ be such that $\frac{1}{2p} + \frac{[\frac{d}{p}]-1}{2d} < \frac{1}{\tilde{p}} < \min\Big\{\frac{[\frac{d}{p}]}{d}, \frac{1}{2} + \frac{[\frac{d}{p}]-1}{2d}\Big\}$, there exists a positive constant $\delta_{p,\tilde{p},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ satisfying*

$$\sup_{0<t<T} t^{\frac{1}{2}([\frac{d}{p}]-\frac{d}{\tilde{p}})}\Big\|e^{t\Delta}u_0\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}} \le \delta_{p,\tilde{p},d}, \tag{2.6}$$

NSE has a unique mild solution $u \in \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T} \cap L^\infty\big([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}\big)$. In particular, the inequality (2.6) holds for arbitrary $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ when $T(u_0)$ is small enough, and there exists a positive constant $\sigma_{p,\tilde{p},d}$ such that we can take $T = \infty$ whenever $\Big\|u_0\Big\|_{\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}} \le \sigma_{p,\tilde{p},d}$.

2.4.3. Solutions to the Navier-Stokes equations with the initial value in the critical spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $d \le p < \infty$ and $1 \le r < \infty$

The main result of this subsection is Theorem 2.4.7, The lemmas we need in order to prove Theorem 2.4.7 are Lemmas 2.4.4, 2.4.5, and 2.4.6 devoted to the study of the bilinear operator $B(u,v)(t)$ defined by (1.5). We prove Theorem 2.4.7 by combining these lemmas with fixed point algorithm Theorem 1.3.1.

Lemma 2.4.4. *Suppose that $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}$ with $d \le p < \infty$ and $1 \le r < \infty$. Then $e^{\cdot\Delta}u_0 \in \mathcal{K}^{\tilde{p}}_{d,1,\infty}$ for all $\tilde{p} > p$.*

Lemma 2.4.5. *Let $p \geq d$ and $d < \tilde{p} < 2p$. Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{p}}_{d,\infty,T} \times \mathcal{K}^{\tilde{p}}_{d,\infty,T}$ into $\mathcal{K}^{p}_{p,1,T}$, and we have the inequality $\big\|B(u,v)\big\|_{\mathcal{K}^{p}_{p,1,T}} \leq C\big\|u\big\|_{\mathcal{K}^{\tilde{p}}_{d,\infty,T}}\big\|v\big\|_{\mathcal{K}^{\tilde{p}}_{d,\infty,T}}$, where C is a positive constant and independent of T.*

Lemma 2.4.6. *Let $d < \tilde{p}_1 < \infty$ and $d \leq \tilde{p}_2 < \infty$ be such that one of the following conditions $d < \tilde{p}_1 < 2d, d \leq \tilde{p}_2 < \frac{d\tilde{p}_1}{2d-\tilde{p}_1}$, or $\tilde{p}_1 = 2d, d \leq \tilde{p}_2 < \infty$, or $2d < \tilde{p}_1 < \infty, \frac{\tilde{p}_1}{2} < \tilde{p}_2 < \infty$ is satisfied. Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{p}_1}_{d,\infty,T} \times \mathcal{K}^{\tilde{p}_1}_{d,\infty,T}$ into $\mathcal{K}^{\tilde{p}_2}_{d,1,T}$, and we have the inequality $\big\|B(u,v)\big\|_{\mathcal{K}^{\tilde{p}_2}_{d,1,T}} \leq C\big\|u\big\|_{\mathcal{K}^{\tilde{p}_1}_{d,\infty,T}}\big\|v\big\|_{\mathcal{K}^{\tilde{p}_1}_{d,\infty,T}}$, where C is a positive constant and independent of T.*

Theorem 2.4.7. *Let $p \geq d$ and $1 \leq r < \infty$. Then for any $\tilde{p}$ such that $\tilde{p} > p$, there exists a positive constant $\delta_{\tilde{p},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$, with $\mathrm{div}(u_0) = 0$ satisfying*

$$\sup_{0<t<T} t^{\frac{1}{2}(1-\frac{d}{\tilde{p}})}\big\|e^{t\Delta}u_0\big\|_{\mathcal{L}^{\tilde{p},\infty}} \leq \delta_{\tilde{p},d}, \tag{2.7}$$

NSE has a unique mild solution $u \in \bigcap_{q>p}\mathcal{K}^{q}_{d,1,T} \cap L^\infty([0,T];\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}})$. In particular, the inequality (2.7) holds for arbitrary $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $T(u_0)$ is small enough, and there exists a positive constant $\sigma_{\tilde{p},d}$ such that we can take $T = \infty$ whenever $\Big\|u_0\Big\|_{\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}} \leq \sigma_{\tilde{p},d}$.

2.4.4. Solutions to the Navier-Stokes equations with initial value in the critical spaces $\dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$ with $1 \leq r < \infty$

The main result of this subsection is Theorem 2.4.11, The lemmas we need in order to prove Theorem 2.4.11 are Lemmas 2.4.8, 2.4.9, and 2.4.10 devoted to the study of the bilinear operator $B(u,v)(t)$ defined by (1.5). We prove Theorem 2.4.11 by combining these lemmas with fixed point algorithm Theorem 1.3.1.
We define an auxiliary space $\mathcal{K}_{s,r,T}$ which is made up by the functions $u(t,x)$ such that $\big\|u\big\|_{\mathcal{K}_{s,r,T}} := \sup_{0<t<T} t^{\frac{\alpha}{2}}\Big\|u(t,x)\Big\|_{\dot{H}^{s}_{\mathcal{L}^{1,r}}} < \infty$, and $\lim_{t\to 0} t^{\frac{\alpha}{2}}\Big\|u(t,x)\Big\|_{\dot{H}^{s}_{\mathcal{L}^{1,r}}} = 0$, with $d-1 \leq s < d, 1 \leq r \leq \infty, T > 0$, and $\alpha = \alpha(s) = s + 1 - d$.

Lemma 2.4.8. *Suppose that $u_0 \in \dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$ with $1 \leq r < \infty$. Then $e^{\cdot\Delta}u_0 \in \mathcal{K}_{s,r,\infty}$ with $d-1 < s < d$.*

Lemma 2.4.9. *Let $d-1 < s < d$. Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}_{s,\infty,T} \times \mathcal{K}_{s,\infty,T}$ into $\mathcal{K}_{s,1,T}$ and we have the inequality $\big\|B(u,v)\big\|_{\mathcal{K}_{s,1,T}} \leq C\big\|u\big\|_{\mathcal{K}_{s,\infty,T}}\big\|v\big\|_{\mathcal{K}_{s,\infty,T}}$, where C is a positive constant and independent of T.*

Lemma 2.4.10. *Let $d-1 < s < d$. Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}_{s,\infty,T} \times \mathcal{K}_{s,\infty,T}$ into $\mathcal{K}_{d-1,1,T}$ and we have the inequality $\big\|B(u,v)\big\|_{\mathcal{K}_{d-1,1,T}} \leq C\big\|u\big\|_{\mathcal{K}_{s,\infty,T}}\big\|v\big\|_{\mathcal{K}_{s,\infty,T}}$, where C is a positive constant and independent of T.*

Theorem 2.4.11. *Let $d-1<s<d$ and $1\le r<\infty$. Then there exists a positive constant $\delta_{s,d}$ such that for all $T>0$ and for all $u_0\in \dot H^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$, with $\operatorname{div}(u_0)=0$ satisfying*

$$\sup_{0<t<T} t^{\frac{1}{2}(s+1-d)}\big\|e^{t\Delta}u_0\big\|_{\dot H^s_{\mathcal{L}^1}}\le \delta_{s,d}, \tag{2.8}$$

NSE has a unique mild solution $u\in \mathcal{K}_{s,r,T}\cap L^\infty([0,T];\dot H^{d-1}_{\mathcal{L}^{1,r}})$.
In particular, the inequality (2.8) holds for arbitrary $u_0\in \dot H^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$ when $T(u_0)$ is small enough, and there exists a positive constant $\sigma_{s,d}$ such that we can take $T=\infty$ whenever $\Big\|u_0\Big\|_{\dot H^{d-1}_{\mathcal{L}^1}}\le \sigma_{s,d}$.

2.5 Mild solutions in Sobolev-Lorentz spaces

In this section, we study local well-posedness for the Navier-Stokes equations (NSE) with the arbitrary initial value in homogeneous Sobolev-Lorentz spaces $\dot H^s_{L^{q,r}}(\mathbb{R}^d):=\dot\Lambda^{-s}L^{q,r}$ for $d\ge 2, q>1, s\ge 0,\ 1\le r\le\infty$, and $\frac{d}{q}-1\le s<\frac{d}{q}$, this result improves the known results for $q>d, r=q, s=0$ (see M. Cannone (1995)) and for $q=r=2, \frac{d}{2}-1<s<\frac{d}{2}$ (see Chemin (1992)).
In the case of critical indexes ($s=\frac{d}{q}-1$), we prove global well-posedness for NSE provided the norm of the initial value is small enough. The result that is a generalization of the result for $q=r=d, s=0$ (see M. Cannone (1997)).
The main result of this section is Theorem 2.5.4, The lemmas we need in order to prove Theorem 2.5.4 are Lemmas 2.5.1, 2.5.2, and 2.5.3 devoted to the study of the bilinear operator $B(u,v)(t)$ defined by (1.5). We prove Theorem 2.5.4 by combining these lemmas with fixed point algorithm Theorem 1.3.1.

2.5.1. The Sobolev-Lorentz spaces

Definition 2.5.1. (Sobolev-Lorentz spaces).
For $q>1, r\ge 1$, and $0\le s<\frac{d}{q}$, the Sobolev-Lorentz space $\dot H^s_{L^{q,r}}(\mathbb{R}^d)$ is defined as the space $I_s(L^{q,r}(\mathbb{R}^d))$, equipped with the norm $\big\|f\big\|_{\dot H^s_{L^{q,r}}}:=\big\|\dot\Lambda^s f\big\|_{L^{q,r}}$.

We define the auxiliary space $\mathcal{K}^{s,\tilde q}_{q,r,T}$ which is made up by the functions $u(t,x)$ such that $\big\|u\big\|_{\mathcal{K}^{s,\tilde q}_{q,r,T}}:=\sup\limits_{0<t<T} t^{\frac{\alpha}{2}}\big\|u(t,.)\big\|_{\dot H^s_{L^{\tilde q,r}}}<\infty$, and $\lim\limits_{t\to 0} t^{\frac{\alpha}{2}}\big\|u(t,.)\big\|_{\dot H^s_{L^{\tilde q,r}}}=0$, where $r,q,\tilde q,s$ being fixed constants satisfying $q,\tilde q\in(1,+\infty), r\ge 1, s\ge 0, \frac{s}{d}<\frac{1}{\tilde q}\le\frac{1}{q}\le\frac{s+1}{d}$, and $\alpha=\alpha(q,\tilde q)=d\Big(\frac{1}{q}-\frac{1}{\tilde q}\Big)$.

Lemma 2.5.1. *If $u_0\in \dot H^s_{L^{q,r}}(\mathbb{R}^d)$ with $q>1, r\ge 1, s\ge 0$, and $\frac{s}{d}<\frac{1}{q}\le\frac{s+1}{d}$ then for all $\tilde q$ satisfying $\frac{s}{d}<\frac{1}{\tilde q}<\frac{1}{q}$, we have $e^{\cdot\Delta}u_0\in \mathcal{K}^{s,\tilde q}_{q,1,\infty}$, and the following imbedding map $\dot H^s_{L^{q,r}}(\mathbb{R}^d)\hookrightarrow \dot B^{s-(\frac{d}{q}-\frac{d}{\tilde q}),\infty}_{\tilde q}(\mathbb{R}^d)$.*

Lemma 2.5.2. *Let $s, q \in \mathbb{R}$ be such that $s \geq 0, q > 1$, and $\frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}$. Then for all $\tilde{q}$ satisfying $\frac{s}{d} < \frac{1}{\tilde{q}} < \min\Big\{\frac{1}{2} + \frac{s}{2d}, \frac{1}{q}\Big\}$, the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T} \times \mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T}$ into $\mathcal{K}^{s,\tilde{q}}_{q,1,T}$ and the following inequality holds $\big\|B(u,v)\big\|_{\mathcal{K}^{s,\tilde{q}}_{q,1,T}} \leq C.T^{\frac{1}{2}(1+s-\frac{d}{q})}\big\|u\big\|_{\mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T}}\big\|v\big\|_{\mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T}}$, where C is a positive constant independent of T.*

Lemma 2.5.3. *Let $s, q \in \mathbb{R}$ be such that $s \geq 0, q > 1$, and $\frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}$. Then for all $\tilde{q}$ satisfying $\frac{1}{2}\Big(\frac{1}{q} + \frac{s}{d}\Big) < \frac{1}{\tilde{q}} < \min\Big\{\frac{1}{2} + \frac{s}{2d}, \frac{1}{q}\Big\}$, the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T} \times \mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T}$ into $\mathcal{K}^{s,q}_{q,1,T}$ and the following inequality holds $\big\|B(u,v)\big\|_{\mathcal{K}^{s,q}_{q,1,T}} \leq C.T^{\frac{1}{2}(1+s-\frac{d}{q})}\big\|u\big\|_{\mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T}}\big\|v\big\|_{\mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T}}$, where C is a positive constant independent of T.*

Theorem 2.5.4. *Let $s \geq 0, q > 1, r \geq 1$ be such that $\frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}$.*
(a) *For all $\tilde{q}$ satisfying $\frac{1}{2}\Big(\frac{1}{q} + \frac{s}{d}\Big) < \frac{1}{\tilde{q}} < \min\Big\{\frac{1}{2} + \frac{s}{2d}, \frac{1}{q}\Big\}$, there exists a positive constant $\delta_{s,q,\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^s_{L^{q,r}}(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ satisfying*

$$T^{\frac{1}{2}(1+s-\frac{d}{q})}\sup_{0<t<T} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})}\big\|e^{t\Delta}u_0\big\|_{\dot{H}^s_{\tilde{q}}} \leq \delta_{s,q,\tilde{q},d}, \tag{2.9}$$

NSE has a unique mild solution $u \in \mathcal{K}^{s,\tilde{q}}_{q,1,T} \cap L^\infty\big([0,T];\dot{H}^s_{L^{q,r}}\big)$. In particular, for arbitrary $u_0 \in \dot{H}^s_{L^{q,r}}$ with $\mathrm{div}(u_0) = 0$, there exists $T(u_0)$ small enough such that the inequality (2.9) holds.
(b) *If $1 < q \leq d$, and $s = \frac{d}{q} - 1$ then for any $\tilde{q}$ be such that $\frac{1}{q} - \frac{1}{2d} < \frac{1}{\tilde{q}} < \min\Big\{\frac{1}{2} + \frac{1}{2q} - \frac{1}{2d}, \frac{1}{q}\Big\}$, there exists a positive constant $\sigma_{q,\tilde{q},d}$ such that if $\big\|u_0\big\|_{\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}} \leq \sigma_{q,\tilde{q},d}$ and $T = \infty$ then the inequality (2.9) holds.*

2.6 Mild solutions in mixed-norm Sobolev-Lorentz spaces

In this section, for $0 \leq m < \infty$ and index vectors $\mathbf{q} = (q_1, q_2, ..., q_d), \mathbf{r} = (r_1, r_2, ..., r_d)$, where $1 < q_i < \infty, 1 \leq r_i \leq \infty$, and $1 \leq i \leq d$, we introduce and study mixed-norm Sobolev-Lorentz spaces $\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}$. Then we investigate the existence and uniqueness of solutions to the NSE in the spaces $L^p([0,T];\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}})$. In the case when $m = 0$, $q_1 = q_2 = ... = q_d = r_1 = r_2 = ... = r_d$, our results recover those of Faber, Jones and Riviere (1972).

2.6.1. Mixed-norm Lorentz spaces

Given a measurable function u on $\mathbb{R}^d$ and index vectors $\mathbf{q} = (q_1, q_2, ..., q_d)$, $\mathbf{r} = (r_1, r_2, ..., r_d)$, where $1 < q_i < \infty, 1 \leq r_i \leq \infty, 1 \leq i \leq d$, we can define the norm $\|u\|_{L^{\mathbf{q},\mathbf{r}}}$ by calculating first the L^{q_1,r_1} - Lorentz norm of $u(x_1, x_2, ..., x_d)$ with respect to the variable x_1, and then L^{q_2,r_2} - Lorentz norm of the resulting quantity with respect to the variable x_2, and so on, finishing with the L^{q_d,r_d} - Lorentz norm with respect to the variable x_d: $\|u\|_{L^{\mathbf{q},\mathbf{r}}} = \left\| ... \left\| \|u\|_{L^{q_1,r_1}_{x_1}} \right\|_{L^{q_2,r_2}_{x_2}} ... \right\|_{L^{q_d,r_d}_{x_d}}$. For a short notation, for a vector $\mathbf{q} = (q_1, q_2, ..., q_d)$ we will write $\frac{1}{\mathbf{q}}$ for the vector $\left(\frac{1}{q_1}, \dots, \frac{1}{q_d}\right)$.

2.6.2. $L^p L^{\mathbf{q},\mathbf{r}}$ solutions of the Navier-Stokes equations

Definition 2.6.1. *For $m \in \mathbb{R}$ and $\boldsymbol{q}, \boldsymbol{r} \in \mathbb{R}^d$, $\boldsymbol{1} < \boldsymbol{q} < \infty, \boldsymbol{1} \leq \boldsymbol{r} \leq \infty$, the space $\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}}$ is defined as the space $\dot{\Lambda}^{-m} L^{\boldsymbol{q},\boldsymbol{r}}$, equipped with the norm $\|u\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}} = \left\|\dot{\Lambda}^m u\right\|_{L^{\mathbf{q},\mathbf{r}}}$.*

Lemma 2.6.1. *Let $\boldsymbol{q} = (q_1, q_2, ..., q_d)$, $\boldsymbol{r} = (r_1, r_2, ..., r_d), 2 < p < \infty$, $m \geq 0$, and $0 < T < \infty$, be such that $m < \frac{1}{2}\sum_{i=1}^d \frac{1}{q_i}$, $\frac{2}{p} - m + \sum_{i=1}^d \frac{1}{q_i} \leq 1$, $1 \leq r_i \leq \infty$, $2 < \frac{q_i}{\left(1 - \frac{m}{\sum_{i=1}^d \frac{1}{q_i}}\right)} < \infty$, $i = 1, 2, .., d$. Then the bilinear operator B is continuous from $L^p([0,T]; \dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}}) \times L^p([0,T]; \dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})$ to $L^p([0,T]; \dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})$ and we have the inequality*

$$\left\|B(u,v)\right\|_{L^p([0,T];\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})} \lesssim T^{\frac{1}{2}(1+m-\frac{2}{p}-\sum_{i=1}^d \frac{1}{q_i})} \left\|u\right\|_{L^p([0,T];\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})} \cdot \left\|v\right\|_{L^p([0,T];\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})}. \quad (2.10)$$

Combining Lemma 2.6.1 with Theorem 1.3.1 we obtain the following existence result

Theorem 2.6.2. *Let $\boldsymbol{q} = (q_1, q_2, ..., q_d)$, $\boldsymbol{r} = (r_1, r_2, ..., r_d), 2 < p < \infty$, and $m \geq 0$ be such that $m < \frac{1}{2}\sum_{i=1}^d \frac{1}{q_i}$, $\frac{2}{p} - m + \sum_{i=1}^d \frac{1}{q_i} \leq 1, 1 \leq r_i \leq \infty$, $2 < \frac{q_i}{\left(1 - \frac{m}{\sum_{i=1}^d \frac{1}{q_i}}\right)} < \infty$, $i = 1, 2, .., d$.*

(a) *There exists a positive constant $\delta_{(m,\boldsymbol{q},\boldsymbol{r},p)} > 0$ such that for all $T > 0$ and for all $u_0 \in \mathcal{S}'(\mathbb{R}^d)$ with $\operatorname{div}(u_0) = 0$, satisfying*

$$T^{\frac{1}{2}(1+m-\frac{2}{p}-\sum_{i=1}^d \frac{1}{q_i})} \left\|e^{\cdot\Delta} u_0\right\|_{L^p([0,T];\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})} \leq \delta_{(m,\boldsymbol{q},\boldsymbol{r},p)}, \quad (2.11)$$

there is a unique mild solution $u \in L^p([0,T]; \dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})$ for NSE.
If $e^{\cdot\Delta} u_0 \in L^p([0,1]; \dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})$, then the inequality (2.11) holds when $T(u_0)$ is small enough.
(b) *If $\frac{2}{p} + \sum_{i=1}^d \frac{1}{q_i} - m = 1$ then there exists a positive $\delta_{(m,\boldsymbol{q},\boldsymbol{r},p)} > 0$ such that we can take $T = \infty$ whenever $\left\|e^{\cdot\Delta} u_0\right\|_{L^p([0,\infty];\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})} \leq \delta_{(m,\boldsymbol{q},\boldsymbol{r},p)}$.*

2.6.3. Uniqueness theorems

In this subsection, we give a theorem on the uniqueness of solutions.

Theorem 2.6.3. *If $u \in L^p\big((0,T), (L^{\boldsymbol{q},\infty}(\mathbb{R}^d))^d\big)$ is a Leray weak solution associated with u_0, where $p \in \mathbb{R}, \boldsymbol{q} \in \mathbb{R}^d$, $\boldsymbol{2} < \boldsymbol{q} < \infty$, $2 < p < \infty$ and $\frac{2}{p} + \sum_{i=1}^d \frac{1}{q_i} = 1$, then the condition iii) of Theorem in the book of P. G. Lemarie-Rieusset (2002) is satisfied, and $u \in L^p\big((0,T), (X_r)^d\big)$ where $r = \sum_{i=1}^d \frac{1}{q_i} \in (0,1), \frac{2}{p} + r = 1$, and u is the unique Leray solution associated with u_0 on $(0,T)$.*

Chapter 3

Hausdorff dimension of the set of singularities for a weak solutions

In this chapter we investigate the Hausdorff dimension of the possible singular set in time of weak solutions to the Navier-Stokes equation on the three dimensional torus under some regularity conditions of Serrin's type. The results in the paper relate the regularity conditions of Serrin's type to the Hausdorff dimension of the singular set set in time. The result that is a generalization of the results of V. Scheffer (1977) and J. Leray (1934).

3.1 Functional setting of the equations

In this chapter, we consider the initial value problem for the non stationary Navier-Stokes equations on the torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, or in other words in $\mathbb{R}^3$ with periodic boundary conditions

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} - \Delta u_i + \frac{\partial p}{\partial x_i} - f_i = 0 \;\text{ on }\; \mathbb{T}^3_T := \mathbb{T}^3 \times (0,T), i = \overline{1,3} \tag{3.1}$$

$$\operatorname{div}(u) = \sum_{i=0}^{3} \frac{\partial u_i}{\partial x_i} = 0 \;\text{ on }\; \mathbb{T}^3_T, \tag{3.2}$$

$$u(x,0) = u_0(x) \;\text{ in }\; \mathbb{T}^3 \times \{0\}, \tag{3.3}$$

where $f_i(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)), u_0(x)$ are given functions with $u_0(x)$ satisfying the condition $\operatorname{div}(u_0) = 0$. Denote by $\dot{\mathcal{V}}(\mathbb{T}^3)$ the space of all infinitely differentiable solenoidal vector fields with zero averaging on $\mathbb{T}^3$; by $\dot{\mathcal{V}}(\mathbb{T}^3_T)$ the space of all compactly supported in $\mathbb{T}^3_T$ infinitely differentiable solenoidal vector fields with zero averaging on $\mathbb{T}^3$ for each $t \in [0,T]$; H, V are the closures of the set $\dot{\mathcal{V}}(\mathbb{T}^3)$ in the spaces $L^2(\mathbb{T}^3), H^1(\mathbb{T}^3)$, respectively. Assume that $f \in L^\infty(0,T;V'), u_0 \in H$, where V' is the dual space of V. A weak solution of the problem (3.1) - (3.3) in $\mathbb{T}^3_T$

is a vector field such that

$$u \in L^2((0,T);V) \cap L^\infty((0,T);H) \cap C([0,T];L^2_{\mathrm{w}});$$

$$\int_{\mathbb{T}^3_T} \Big(-\sum_{i=1}^{3} u_i \frac{\partial v_i}{\partial t} + \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} + u_i u_j \frac{\partial v_i}{\partial x_j} \Big) \mathrm{d}x\mathrm{d}t =< f, v >, \forall v \in \dot{C}^\infty(\mathbb{T}^3_T);$$

$$\frac{1}{2}\int_{\mathbb{T}^3} \sum_{i=1}^{3} |u_i(x,t_1)|^2 \mathrm{d}x + \int_{\mathbb{T}^3 \times (t_0,t_1)} \sum_{i,j=1}^{3} \Big|\frac{\partial u_i}{\partial x_j}\Big|^2 \mathrm{d}x\mathrm{d}t \le \frac{1}{2}\int_{\mathbb{T}^3} \sum_{i=1}^{3} |u_i(x,t_0)|^2 \mathrm{d}x;$$

$$\forall t_0 \in [0,T]\backslash\Sigma,\ t_1 \in [t_0,T],\ \text{where } \Sigma \text{ has Lebesgue measure zero and } 0 \notin \Sigma;$$

$$\|u(x,t) - u_0(x)\|_{L^2(\mathbb{T}^3)} \to 0 \text{ as } t \to 0,$$

where $< .,. >$ is the pairing between V and V'. It was proved by Leray that there exists at least one weak solution of the problem (3.1) - (3.3).

3.2 Weak solutions in $L^r H^\alpha$

Lemma 3.2.1. *Assume that* $f \in L^\infty(0,T;V_{\alpha-1}),\ u_0 \in V_\alpha,\ \frac{1}{2} < \alpha < \frac{3}{2}$. *Then there exists a unique strong solution to the Navier-Stokes equations, satisfying*

$$u \in L^2(0,T^{**};V_{\alpha+1}) \cap C(0,T^{**};V_\alpha)$$

where $T^{**} = \min(T,T^*)$, *with* $T^* = \frac{C(\alpha,N)}{(|u(0)|^2_\alpha+1)^{\frac{2}{2\alpha-1}}}$.

Let $\alpha \in (\frac{1}{2},\frac{3}{2})$. We say that a weak solution u is $H^\alpha(\mathbb{T}^3)$-regular on (t_1,t_2) if $u \in C((t_1,t_2),H^\alpha(\mathbb{T}^3))$. We obtaine the following theorem by using Lemma 3.2.1.

Theorem 3.2.2. *Assume that* $\alpha \in (\frac{1}{2},\frac{3}{2}), u_0 \in H, f \in L^\infty(0,T;V_{\alpha-1})$ *and* u *is a weak solution of the Navier-Stokes equations and satisfies the condition* $u \in L^r(0,T;V_\alpha)$*, with* $r > 0$ *and* $r(2\alpha-1) < 4$. *Then there exists a closed set* $S_\alpha \subset [0,T]$ *such that* $u \in C([0,T] \setminus S_\alpha;V_\alpha)$ *and* $\mu_{1-\frac{r(2\alpha-1)}{4}}(S_\alpha) = 0$.

3.3 Weak solutions in $L^r W^{1,q}$

Lemma 3.3.1. *Assume that* $f \in L^\infty(0,T;L^q(\mathbb{T}^3)),\ u_0 \in W^{1,q}(\mathbb{T}^3),\ q \in [2,3)$. *Then there exists a constant* T^{**} *depending only on* $\|\nabla u(0)\|^q_{L^q(\mathbb{T}^3)}, q, \sup_{0\le t\le T}\|f(t)\|_{L^q(\mathbb{T}^3)}$ *and a unique strong solution to the Navier-Stokes equations, satisfying*

$$u \in L^2(0,T^{**};\tilde{W}^{2,q}) \cap C(0,T^{**};W^{1,q}).$$

Let $q \in [2,3)$. We say that a weak solution u is $W^{1,q}(\mathbb{T}^3)$ - regular on (t_1,t_2) if $u \in C((t_1,t_2),W^{1,q}(\mathbb{T}^3))$. We obtaine the following theorem by using Lemma 3.3.1

Theorem 3.3.2. *Assume that* $q \in [2,3), u_0 \in H,\ f \in L^\infty(0,T;L^p(\mathbb{T}^3))$ *and* u *is a weak solution of the Navier-Stokes equations and satisfies the following condition* $u \in L^r(0,T;W^{1,q}),\ \frac{r(2q-3)}{2q} < 1$. *Then there exists a closed set* $S_q \subset [0,T]$ *such that* $u \in C([0,T] \setminus S_q;W^{1,q})$ *and* $\mu_{1-\frac{r(2q-3)}{2q}}(S_q) = 0$.

Conclusions

In this thesis, we construct mild solutions to the Navier-Stokes equations by applying the Picard contraction principle. For the Sobolev spaces $\dot{H}^s_q$ $(q > 1, \frac{d}{q} - 1 \leq s < \frac{d}{q})$, we obtain the local existence of mild solutions in the spaces $L^\infty\big([0,T]; \dot{H}^s_q(\mathbb{R}^d)\big)$ with arbitrary initial value in $\dot{H}^s_q(\mathbb{R}^d)$, in the case of critical indexes $(q > 1, s = \frac{d}{q} - 1)$ we get the existence of global mild solutions in the spaces $L^\infty([0,\infty); \dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d))$ when the norm of the initial value is small enough. The same argument is applied to following spaces:

- Critical Sobolev-Fourier-Lorentz spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d), (r \geq 1, 1 \leq p < \infty)$.

- Sobolev-Lorentz spaces $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d), (s \geq 0, q > 1, r \geq 1, \frac{d}{q} - 1 \leq s < \frac{d}{q})$ with critical indexes $s = \frac{d}{q} - 1$.

- For $0 \leq m < \infty$ and index vectors $\mathbf{q} = (q_1, q_2, ..., q_d), \mathbf{r} = (r_1, r_2, ..., r_d)$, where $1 < q_i < \infty, 1 \leq r_i \leq \infty$, and $1 \leq i \leq d$, we introduce and study mixed-norm Sobolev-Lorentz spaces $\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}$. Then we investigate the existence and uniqueness of solutions to the Navier-Stokes equations in the spaces $\mathcal{Q} := \mathcal{Q}_T = L^p([0,T]; \dot{H}^m_{L^{\mathbf{q},\mathbf{r}}})$ where $p > 2,\ T > 0$, and initial data is taken in the class $\mathcal{I} = \{u_0 \in (\mathcal{S}'(\mathbb{R}^d))^d, \operatorname{div}(u_0) = 0 : \|e^{\cdot\Delta}u_0\|_{\mathcal{Q}} < \infty\}$. The results have a standard relation between existence time and data size: large time with small data or large data with small time. In the case with $T = \infty$ and critical indexes $\frac{2}{p} + \sum_{i=1}^d \frac{1}{q_i} - m = 1$, the space $\mathcal{I}$ coincides with the homogeneous Besov space $\dot{B}^{m-\frac{2}{p},p}_{L^{\mathbf{q},\mathbf{r}}}$.

Finally, we investigate the Hausdorff dimension of the possible singular set in time of weak solutions to the Navier-Stokes equations on the three dimensional torus under some regularity conditions of Serrin's type. The results in the chapter relate the regularity conditions of Serrin's type to the Hausdorff dimension of the singular set in time.

List of the author's publications related to the dissertation

[1] D. Q. Khai and N. M. Tri, *Well-posedness for the Navier-Stokes equations with data in Sobolev-Lorentz spaces*, Nonlinear Analysis, **149** (2017), 130-145.

[2] D. Q. Khai, *Well-posedness for the Navier-Stokes equations with datum in the Sobolev spaces*, Acta Math Vietnam (2016). doi:10.1007/s40306-016-0192-x.

[3] D. Q. Khai and N. M. Tri, *Well-posedness for the Navier-Stokes equations with datum in Sobolev-Fourier-Lorentz spaces*, Journal of Mathematical Analysis and Applications, **437** (2016), 854-781.

[4] D. Q. Khai and N. M. Tri, *On the initial value problem for the Navier-Stokes equations with the initial datum in critical Sobolev and Besov spaces*, Journal of Mathematical Sciences the University of Tokyo, **23** (2016), 499-528.

[5] D. Q. Khai and N. M. Tri, *On the Hausdorff dimension of the singular set in time for weak solutions to the nonstationary Navier-Stokes equations on torus*,Vietnam Journal of Mathematics, **43** (2015), 283-295.

[6] D. Q. Khai and N. M. Tri, *Solutions in mixed-norm Sobolev-Lorentz spaces to the initial value problem for the Navier-Stokes equations*, Journal of Mathematical Analysis and Applications, **417** (2014), 819-833.

Author's other relevant papers

[7] D. Q.Khai, N.M. Tri, *On general axisymmetric explicit solutions for the Navier-Stokes equations*, International Journal of Evolution Equations, **6** (2013), 325-336..

[8] D. Q. Khai and V. T. T. Duong,*On the initial value problem for the Navier-Stokes equations with the initial datum in the Sobolev spaces*,
preprint arXiv:1603.04219.

[9] D. Q. Khai and N. M. Tri, *The existence and decay rates of strong solutions for Navier-Stokes Equations in Bessel-potential spaces*, preprint, arXiv:1603.01896.

[10] D. Q. Khai and N. M. Tri *The existence and space-time decay rates of strong solutions to Navier-Stokes Equations in weighed* $L^\infty(|x|^\gamma \mathrm{dx}) \cap L^\infty(|x|^\beta \mathrm{dx})$ *spaces*, preprint, arXiv:1601.01441.

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