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Convergence Rates for the Tikhonov Regularization of Coefficient Identification Problems in Elliptic Equations

Speciality: Differential and Integral Equations

Speciality Code: 62 46 01 05

Dissertation submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY IN MATHEMATICS

Hanoi–2012

This work has been completed at Institute of Mathematics, Vietnam Academy of Science and Technology

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First referee: ....

Second referee: ....

Third referee: ....

To be defended at the Jury of Institute of Mathematics, Vietnam Academy of Science and Technology:

....

....

on .... 2012, at .... o'clock ....

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Introduction

Let Ω be an open bounded connected domain in $\mathbb{R}^d, d \geq 1$ with the Lipschitz boundary $\partial\Omega$, $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ be given. In this thesis we investigate the inverse problems of identifying the coefficient q in the Neumann problem for the elliptic equation

$$-\mathrm{div}(q\nabla u) = f \text{ in } \Omega, \tag{0.1}$$

$$q\frac{\partial u}{\partial n} = g \text{ on } \partial\Omega \tag{0.2}$$

and the coefficient a in the Neumann problem for the elliptic equation

$$-\Delta u + au = f \text{ in } \Omega, \tag{0.3}$$

$$\frac{\partial u}{\partial n} = g \text{ on } \partial\Omega \tag{0.4}$$

from imprecise values $z^\delta \in H^1(\Omega)$ of the exact solution $\overline{u}$ of (0.1)–(0.2) or (0.3)–(0.4) with

$$\|\overline{u} - z^\delta\|_{H^1(\Omega)} \leq \delta, \tag{0.5}$$

$\delta > 0$ being given. These problems are mathematical models in different topics of applied sciences, e.g. aquifer analysis. For practical models and surveys on these problems we refer the reader to our papers [1, 2, 3, 4] and the references therein. Physically, the state u in (0.1)–(0.2) or (0.3)–(0.4) can be interpreted as the piezometrical head of the ground water in Ω, the function f characterizes the sources and sinks in Ω and the function g characterizes the inflow and outflow through $\partial\Omega$, while the functionals q and a in these problems are called the *diffusion* (or *filtration* or *transmissivity*, or *conductivity*) and *reaction* coefficients, respectively. In the three-dimensional space the state u at point (x, y, z) of the flow region Ω is defined by

$$u = u(x, y, z) = \frac{p}{\rho g} + z,$$

where $p = p(x, y, z)$ is fluid pressure, $\rho = \rho(x, y, z)$ is density of the water and g is acceleration of gravity. For different kinds of the porous media, the diffusion coefficient varies in a large scale

Gravels	0.1 to 1 cm/sec
Sands	10^{-3} to 10^{-2} cm/sec
Silts	10^{-5} to 10^{-4} cm/sec
Clays	10^{-9} to 10^{-7} cm/sec
Limestone	10^{-4} to 10^{-2} cm/sec.

Suppose that the coefficient q in (0.1)–(0.2) is given so that we can determine the unique solution u and thus define a nonlinear coefficient-to-solution map from q to the solution $u = u(q) := U(q)$. Then the inverse problem has the form: solve the nonlinear equation

$$U(q) = \overline{u} \text{ for } q \text{ with } \overline{u} \text{ being given.}$$

Similarly, the identification problem (0.3)–(0.4) can be written as $U(a) = \overline{u}$ for a with $\overline{u}$ being given.

The above identification problems are well known to be ill-posed and there have been several stable methods for solving them such as stable numerical methods and regularization methods. Among these stable solving methods, the Tikhonov regularization seems to be most popular. However, although there have been many papers devoted to the subject, there have been very few ones devoted to the convergence rates of the methods. The authors of these works used the output least-squares method with the Tikhonov regularization of the nonlinear ill-posed problems and obtained some convergence rates under certain source conditions. However, working with *nonconvex functionals*, they are faced with difficulties in finding the global minimizers. Further, their source conditions are hard to check and require high regularity of the sought coefficient. To overcome the shortcomings of the above mentioned works, in this dissertation we do not use the output least-squares method, but use the *convex energy functionals* (see (0.6) and (0.7)) and then applying the Tikhonov regularization to these convex energy functionals. We obtain the convergence rates for three forms of regularization (L^2-regularization, total variation regularization and regularization of total variation combining with L^2-stabilization) of the inverse problems of identifying q in (0.1)–(0.2) and a in (0.3)–(0.4). Our source conditions are simple and much weaker than that by the other authors, since we remove the so-called "small enough condition" on the source functions which is popularized in the theory of regularization of nonlinear ill-posed problems but very hard to check. Furthermore, our results are valid for multi-dimensional identification problems. The crucial and new idea in the dissertation is that we use the convex energy functional

$$q \to J_{z^\delta}(q) := \frac{1}{2}\int_\Omega q|\nabla(U(q) - z^\delta)|^2 dx, \;\; q \in Q_{ad} \tag{0.6}$$

for identifying q in (0.1)–(0.2) and the convex energy functional

$$a \to G_{z^\delta}(a) := \frac{1}{2}\int_\Omega |\nabla(U(a) - z^\delta)|^2 dx + \frac{1}{2}\int_\Omega a(U(a) - z^\delta)^2 dx, \;\; a \in A_{ad} \tag{0.7}$$

for identifying a in (0.3)–(0.4) instead of the output least-squares ones. Here, $U(q)$ and $U(a)$ are the coefficient-to-solution maps for (0.1)–(0.2) and (0.3)–(0.4) with Q_{ad} and A_{ad} being the admissible sets, respectively.

The content of this dissertation is presented in four chapters. In Chapter 1, we will state the inverse problems of identifying the coefficient q in (0.1)–(0.2) and a in (0.3)–(0.4), and prove auxiliary results used in Chapters 2–4.

In Chapter 2, we apply L^2-regularization to these functionals. Namely, for identifying q in (0.1)–(0.2) we consider the *strictly convex* minimization problem

$$\min_{q \in Q_{ad}} J_{z^\delta}(q) + \rho\|q - q^*\|^2_{L^2(\Omega)}, \tag{0.8}$$

and for identifying a in (0.3)–(0.4) the *strictly convex* minimization problem

$$\min_{a \in A_{ad}} G_{z^\delta}(a) + \rho\|a - a^*\|^2_{L^2(\Omega)}, \tag{0.9}$$

where $\rho > 0$ is the regularization parameter, q^* and a^* respectively are a-priori estimates of sought coefficients q and a. Although these cost functions appear more complicated than that of the output least squares method, it is in fact much simpler because of its strict convexity, so there is no question on the uniqueness and localization of the minimizer. We will exploit this nice property to obtain convergence rates $\mathcal{O}(\sqrt{\delta})$, as $\delta \to 0$ and $\rho \sim \delta$, under simple and weak source conditions. Our main convergence results in Chapter 2 can now be stated as follows.

Let $q^\dagger$ be the q^*-minimum norm solution of the coefficient identification problem q in (0.1)–(0.2) (see § 2.1.1.) and q_ρ^δ be a solution of problem (0.8). Assume that there exists a functional $w^* \in H^1_\diamond(\Omega)^*$ (see § 1.1 for the definition of $H^1_\diamond(\Omega)$) such that

$$U'(q^\dagger)^* w^* = q^\dagger - q^*. \tag{0.10}$$

Here, $U'(q)^*$ is the adjoint of the Fréchet derivative of U at q. Then,

$$\|q_\rho^\delta - q^\dagger\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta}) \ \text{ and } \ \|U(q_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

The crucial assumption in our result on establishing the convergence rate of regularized solutions q_ρ^δ to the q^*-minimum norm solution $q^\dagger$ is the existence of a source element $w^* \in H^1_\diamond(\Omega)^*$ satisfying (0.10). This is a weak source condition and it does not require any smoothness of $q^\dagger$. Moreover, the smallness requirement on the source functions of the general convergence theory for nonlinear ill-posed problems, which is hard to check, is liberated in our source condition. In Theorem 2.1.6 we see that this condition is fulfilled for all the dimension d and hence a convergence rate $\mathcal{O}(\sqrt{\delta})$ of L^2-regularization is obtained under assumption that the sought coefficient $q^\dagger$ belongs to $H^1(\Omega)$ and the exact $U(q^\dagger) \in W^{2,\infty}(\Omega)$, $|\nabla U(q^\dagger)| \geq \gamma$ a.e. on Ω, where γ is a positive constant.

Similarly, let $a^\dagger$ be the a^*-minimum norm solution of the coefficient identification problem a in (0.3)–(0.4) (see § 2.2.1.) and a_ρ^δ be a solution of problem (0.9). Assume that there exists a functional $w^* \in H^1(\Omega)^*$ such that

$$U'(a^\dagger)^* w^* = a^\dagger - a^*. \tag{0.11}$$

Then,

$$\|a_\rho^\delta - a^\dagger\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta}) \ \text{ and } \ \|U(a_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$. Thus, in our source conditions the requirement on the smallness of the source functions is removed.

We note that (see Theorem 2.2.6) the source condition (0.11) is fulfilled for the arbitrary dimension d and hence a convergence rate $\mathcal{O}(\sqrt{\delta})$ of L^2-regularization is obtained under hypothesis that the sought coefficient $a^\dagger$ is an element of $H^1(\Omega)$ and $|U(a^\dagger)| \geq \gamma$ a.e. on Ω, where γ is a positive constant.

To estimate a possible discontinuous or highly oscillating coefficient q, some authors used the output least-squares method with total variation regularization. Namely, they treated the *nonconvex* optimization problem

$$\min_{q \in \mathcal{Q}} \int_\Omega \left(U(q) - z^\delta\right)^2 dx + \rho \int_\Omega |\nabla q| \tag{0.12}$$

with $\int_\Omega |\nabla q|$ being the total variation of the function q. Total variation regularization originally introduced in image denoising by L. I. Rudin, S. J. Osher and E. Fatemi in the year 1992 has been used in several ill-posed inverse problems and analyzed by many authors over the last decades. This method is of particular interest for problems with possibility of discontinuity or high oscillation in the solution. Although there have been many papers using total variation regularization of ill-posed problems, there are very few ones devoted to the convergence rates. Only recently, in the year 2004 M. Burger and S. Osher investigated the convergence rates for convex variational regularization of linear ill-posed problems in the sense of the Bregman distance. This seminal paper has been intensively developed for several linear and nonlinear ill-posed problems.

We remark that the cost function appeared in (0.12) is not convex, it is difficult to find global minimizers and up to now there was no result on the convergence rates of the total regularization method for our inverse problems. To overcome this shortcoming, in Chapter 3, we do not use the output least-squares method, but apply the total variation regularization method to energy functionals $J_{z^\delta}(\cdot)$ and $G_{z^\delta}(\cdot)$, and obtain convergence rates for this approach. Namely, for identifying q, we consider the *convex* minimization problem

$$\min_{q \in Q_{ad}} J_{z^\delta}(q) + \rho \int_\Omega |\nabla q| \,, \tag{0.13}$$

and for identifying a the *convex* minimization problem

$$\min_{a \in A_{ad}} G_{z^\delta}(a) + \rho \int_\Omega |\nabla a| \,. \tag{0.14}$$

Our convergence results in Chapter 3 are as follows. Let $q^\dagger$ be a total variation-minimizing solution of the problem of identifying q in (0.1)–(0.2) (see § 3.1.1.) and q_ρ^δ be a solution of problem (0.13). Assume that there exists a functional $w^* \in H^1_\diamond(\Omega)^*$ such that

$$U'(q^\dagger)^* w^* \in \partial \left(\int_\Omega |\nabla(\cdot)| \right) (q^\dagger). \tag{0.15}$$

Then, $\|U(q_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$,

$$D_{TV}^{U'(q^\dagger)^* w^*}(q_\rho^\delta, q^\dagger) = \mathcal{O}(\delta) \text{ and } \left| \int_\Omega |\nabla q_\rho^\delta| - \int_\Omega |\nabla q^\dagger| \right| = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

Similarly, let $a^\dagger$ be a total variation-minimizing solution of the problem of identifying a in (0.3)–(0.4) (see § 3.2.1.) and a_ρ^δ be a solution of problem (0.14). Assume that there exists a functional $w^* \in H^1(\Omega)^*$ such that

$$U'(a^\dagger)^* w^* \in \partial \left(\int_\Omega |\nabla(\cdot)| \right) (a^\dagger). \tag{0.16}$$

Then, $\|U(a_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$,

$$D_{TV}^{U'(a^\dagger)^* w^*}(a_\rho^\delta, a^\dagger) = \mathcal{O}(\delta) \text{ and } \left| \int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a_\rho^\delta| \right| = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

However, our convergence rates in this approach are just in the sense of the Bregman distance which is in general not a metric. To enhance these results, in the last chapter we add an additional L^2-stabilization to the convex functionals (0.13) and (0.14) for respectively identifying q and a, and obtain convergence rates not only in the sense of the

Bregman distance but also in the $L^2(\Omega)$-norm. Namely, for identifying q in (0.1)–(0.2), we consider the *strictly convex* minimization problem

$$\min_{q\in Q_{ad}} J_{z^\delta}(q) + \rho\left(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q|\right), \tag{0.17}$$

and for identifying a in (0.3)–(0.4) the *strictly convex* minimization problem

$$\min_{a\in A_{ad}} G_{z^\delta}(a) + \rho\left(\frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a|\right). \tag{0.18}$$

We also note that, to our knowledge, up to now there is only the paper by G. Chavent and K. Kunisch in the year 1997 devoted to convergence rates for such a total variation regularization of a certain *linear* ill-posed problem.

Denote by q^δ_ρ the solution of (0.17), $q^\dagger$ the R-minimizing norm solution of the problem of identifying q in (0.1)–(0.2), where $R(\cdot) = \frac{1}{2}\|\cdot\|^2_{L^2(\Omega)} + \int_\Omega |\nabla\cdot|$. Assume that there exists a functional $w^* \in H^1_\diamond(\Omega)^*$ such that

$$U'(q^\dagger)^* w^* = q^\dagger + \ell \in \partial R(q^\dagger) \tag{0.19}$$

for some element ℓ in $\partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger)$. Then, we have the convergence rates

$$\|q^\delta_\rho - q^\dagger\|^2_{L^2(\Omega)} + D^\ell_{TV}(q^\delta_\rho, q^\dagger) = \mathcal{O}(\delta) \ \text{ and } \ \|U(q^\delta_\rho) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

Similarly, denote by a^δ_ρ the solution of (0.18), $a^\dagger$ the R-minimizing norm solution of the problem of identifying a in problem (0.3)–(0.4). Assume that there exists a function $w^* \in H^1(\Omega)^*$ such that

$$U'(a^\dagger)^* w^* = a^\dagger + \lambda \in \partial R(a^\dagger) \tag{0.20}$$

for some element λ in $\partial\left(\int_\Omega |\nabla(\cdot)|\right)(a^\dagger)$. Then, we have the convergence rates

$$\|a^\delta_\rho - a^\dagger\|^2_{L^2(\Omega)} + D^\lambda_{TV}(a^\delta_\rho, a^\dagger) = \mathcal{O}(\delta) \ \text{ and } \ \|U(a^\delta_\rho) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

We remark that (see Theorems 3.1.5, 3.2.5, 4.1.5 and 4.2.5) the source conditions (0.15), (0.16), (0.19) and (0.20) are valid for the dimension $d \le 4$ under some additional regularity assumptions on $q^\dagger$ and the exact $U(q^\dagger)$.

In the whole dissertation we assume that Ω is an open bounded connected domain in $\mathbb{R}^d, d \ge 1$ with the Lipschitz boundary $\partial\Omega$. The functions $f \in L^2(\Omega)$ in (0.1) or (0.3) and $g \in L^2(\partial\Omega)$ in (0.2) or (0.4) are given. The notation U is referred to the nonlinear coefficient-to-solution operators for the Neumann problems. We use the standard notion of Sobolev spaces $H^1(\Omega)$, $H^1_0(\Omega)$, $W^{1,\infty}(\Omega)$ and $W^{2,\infty}(\Omega)$ etc. For the simplicity of notation, as there will be no ambiguity, we write $\int_\Omega \cdots$ instead of $\int_\Omega \cdots dx$.

Chapter 1

Problem setting and auxiliary results

Let Ω be an open bounded connected domain in $\mathbb{R}^d, d \geq 1$ with the Lipschitz boundary $\partial\Omega$, $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ be given. In this work we investigate ill-posed nonlinear inverse problems of identifying the diffusion coefficient q in the Neumann problem for the elliptic equation (0.1)–(0.2) and the reaction coefficient a in the Neumann problem for the elliptic equation (0.3)–(0.4) from imprecise values z^δ of the exact solution $\overline{u}$ satisfying (0.5).

1.1 Diffusion coefficient identification problem

1.1.1. Problem setting

We consider problem (0.1)–(0.2). Assume that the functions f and g satisfy the compatibility condition $\int_\Omega f + \int_{\partial\Omega} g = 0$. Then, a function $u \in H^1_\diamond(\Omega) := \left\{u \in H^1(\Omega) \middle| \int_\Omega u dx = 0\right\}$ is said to be a weak solution of problem (0.1)–(0.2), if $\int_\Omega q\nabla u \nabla v = \int_\Omega fv + \int_{\partial\Omega} gv$ for all $v \in H^1_\diamond(\Omega)$. We assume that the coefficient q belongs to the set

$$Q := \{q \in L^\infty(\Omega) \mid 0 < \underline{q} \leq q(x) \leq \overline{q} \text{ a.e. on } \Omega\} \tag{1.1}$$

with $\underline{q}$ and $\overline{q}$ being given positive constants. Then, by the aid of the Poincaré-Friedrichs inequality in $H^1_\diamond(\Omega)$, we obtain that there exists a positive constant α depending only on $\underline{q}$ and the domain Ω such that the following coercivity condition is fulfilled

$$\int_\Omega q|\nabla u|^2 \geq \alpha \|u\|^2_{H^1(\Omega)} \text{ for all } u \in H^1_\diamond(\Omega) \text{ and } q \in Q. \tag{1.2}$$

Here,

$$\alpha := \frac{\underline{q}C_\Omega}{1 + C_\Omega} > 0 \tag{1.3}$$

with C_Ω being the positive constant, depending only on Ω, appeared in the Poincaré-Friedrichs inequality: $C_\Omega \int_\Omega v^2 \leq \int_\Omega |\nabla v|^2$ for all $v \in H^1_\diamond(\Omega)$. It follows from inequality (1.2) and the Lax-Milgram lemma that for all $q \in Q$, there is a unique weak solution in $H^1_\diamond(\Omega)$ of (0.1)–(0.2) which satisfies the inequality $\|u\|_{H^1(\Omega)} \leq \Lambda_\alpha \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right)$, where Λ_α is a positive constant depending only on α.

Thus, in the direct problem we defined the nonlinear coefficient-to-solution operator $U : Q \subset L^\infty(\Omega) \to H^1_\diamond(\Omega)$ which maps the coefficient $q \in Q$ to the solution $U(q) \in H^1_\diamond(\Omega)$

of problem (0.1)–(0.2). The inverse problem is stated as follows:

$$\text{Given } \overline{u} := U(q) \in H^1_\diamond(\Omega) \text{ find } q \in Q.$$

We assume that instead of the exact $\overline{u}$ we have only its observations $z^\delta \in H^1_\diamond(\Omega)$ such that (0.5) satisfies. Our problem is to reconstruct q from z^δ. For solving this problem we minimize the *convex functional* $J_{z^\delta}(q)$ defined by (0.6) over Q. Since the problem is ill-posed, we shall use the Tikhonov regularization to solve it in a stable way and establish convergence rates for the method.

1.1.2. Some preliminary results

Lemma 1.1.1. *The coefficient-to-solution operator $U : Q \subset L^\infty(\Omega) \to H^1_\diamond(\Omega)$ is continuously Fréchet differentiable on the set Q. For each $q \in Q$, the Fréchet derivative $U'(q)$ of $U(q)$ has the property that the differential $\eta := U'(q)h$ with $h \in L^\infty(\Omega)$ is the unique weak solution in $H^1_\diamond(\Omega)$ of the Neumann problem*

$$-div\,(q\nabla\eta) = div\,(h\nabla U(q)) \;\; in\ \Omega,$$
$$q\frac{\partial\eta}{\partial n} = -h\frac{\partial U(q)}{\partial n} \;\; on\ \partial\Omega$$

in the sense that it satisfies the equation $\int_\Omega q\nabla\eta\nabla v = -\int_\Omega h\nabla U(q)\nabla v$ for all $v \in H^1_\diamond(\Omega)$. Moreover, $\|\eta\|_{H^1(\Omega)} \le \frac{\Lambda_\alpha}{\alpha}\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right)\|h\|_{L^\infty(\Omega)}$ for all $h \in L^\infty(\Omega)$.

We note that $U : Q \subset L^\infty(\Omega) \to H^1_\diamond(\Omega)$ is in fact infinitely Fréchet differentiable.

Lemma 1.1.2. *The functional $J_{z^\delta}(\cdot)$ is convex on the convex set Q.*

1.2 Reaction coefficient identification problem

1.2.1. Problem setting

Recall that a function $u \in H^1(\Omega)$ is said to be a weak solution of (0.3)–(0.4), if it satisfies the equality $\int_\Omega \nabla u\nabla v + \int_\Omega auv = \int_\Omega fv + \int_{\partial\Omega} gv$ for all $v \in H^1(\Omega)$. For all $u \in H^1(\Omega)$ and $a \in A$, where

$$A := \{a \in L^\infty(\Omega) \mid 0 < \underline{a} \le a(x) \le \overline{a} \text{ a.e. on } \Omega\} \tag{1.4}$$

with $\underline{a}$ and $\overline{a}$ being given positive constants, the following coercivity condition

$$\int_\Omega |\nabla u|^2 + \int_\Omega au^2 \ge \beta\,\|u\|^2_{H^1(\Omega)} \tag{1.5}$$

holds. Here,

$$\beta := \min\{1, \underline{a}\} > 0. \tag{1.6}$$

In virtue of the Lax-Milgram lemma for each $a \in A$, there exists a unique weak solution of (0.3)–(0.4) which satisfies inequality $\|u\|_{H^1(\Omega)} \le \Lambda_\beta\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right)$, where Λ_β is a positive constant depending only on β.

Therefore, we can define the nonlinear coefficient-to-solution mapping $U : A \subset L^\infty(\Omega) \to H^1(\Omega)$ which maps each $a \in A$ to the unique solution $U(a) \in H^1(\Omega)$ of (0.3)–(0.4). Our inverse problem is formulated as:

$$\text{Given } \overline{u} = U(a) \in H^1(\Omega) \text{ find } a \in A.$$

Assume that instead of the exact $\overline{u}$ we have only its observations $z^\delta \in H^1(\Omega)$ such that (0.5) satisfies. Our problem is to reconstruct a from z^δ. For this purpose we minimize the *convex functional* $G_{z^\delta}(a)$ defined by (0.7) over A. Since the problem is ill-posed, we shall use the Tikhonov regularization to solve it in a stable way and establish the convergence rates for method.

1.2.2. Some preliminary results

Lemma 1.2.1. *The mapping $U : A \subset L^\infty(\Omega) \to H^1(\Omega)$ is continuously Fréchet differentiable with the derivative $U'(a)$. For each h in $L^\infty(\Omega)$, the differential $\eta := U'(a)h \in H^1(\Omega)$ is the unique solution of the problem*

$$-\Delta\eta + a\eta = -hU(a) \text{ in } \Omega,$$
$$\frac{\partial \eta}{\partial n} = 0 \text{ on } \partial\Omega,$$

in the sense that it satisfies the equation $\int_\Omega \nabla\eta\nabla v + \int_\Omega a\eta v = -\int_\Omega hU(a)v$ for all $v \in H^1(\Omega)$. Furthermore, the estimate $\|\eta\|_{H^1(\Omega)} \le \frac{\Lambda_\beta}{\beta}\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right)\|h\|_{L^\infty(\Omega)}$ holds for all $h \in L^\infty(\Omega)$.

As in the previous paragraph, we note that the mapping $U : A \subset L^\infty(\Omega) \to H^1(\Omega)$ is infinitely Fréchet differentiable.

Lemma 1.2.2. *The functional $G_{z^\delta}(\cdot)$ is convex on the convex set A.*

This chapter was written on the basis of the papers

[1] Dinh Nho Hào and Tran Nhan Tam Quyen (2010), Convergence rates for Tikhonov regularization of coefficient identification problems in Laplace-type equations, *Inverse Problems* **26**, 125014 (23pp).

[4] Dinh Nho Hào and Tran Nhan Tam Quyen (2012), Convergence rates for Tikhonov regularization of a two-coefficient identification problem in an elliptic boundary value problem, *Numerische Mathematik* **120**, pp. 45–77.

Chapter 2

L^2-regularization

In this chapter the convex functionals $J_{z^\delta}(\cdot)$ and $G_{z^\delta}(\cdot)$ defined by (0.6) and (0.7) are used for identifying the coefficient q and a in (0.1)–(0.2) and (0.3)–(0.4), respectively. We apply L^2-regularization to these functionals and obtain convergence rates $\mathcal{O}(\sqrt{\delta})$ of regularized solutions in the $L^2(\Omega)$-norm as the error level $\delta \to 0$ and the regularization parameter $\rho \sim \delta$.

2.1 Convergence rates for L^2-regularization of the diffusion coefficient identification problem

2.1.1. L^2-regularization

For solving the problem of identifying the coefficient q in (0.1)–(0.2) we solve the minimization problem

$$\min_{q\in Q} J_{z^\delta}(q) + \rho\|q - q^*\|^2_{L^2(\Omega)}, \qquad (\mathbf{P}^q_{\rho,\delta})$$

where $\rho > 0$ is the regularization parameter and q^* is an a-priori estimate of the true coefficient which is identified. The cost functional of problem $(\mathbf{P}^q_{\rho,\delta})$ is weakly lower semi-continuous in the $L^2(\Omega)$-norm and strictly convex, it attains a *unique solution* q^δ_ρ on the nonempty, convex, bounded and closed in the $L^2(\Omega)$-norm and hence weakly compact set Q which we consider as the regularized solution of our identification problem.

Now we introduce the notion of *q^*-minimum norm solution.*

Lemma 2.1.1. *The set $\Pi_Q(\overline{u}) := \{q \in Q \mid U(q) = \overline{u}\}$ is nonempty, convex, bounded and closed in the $L^2(\Omega)$-norm. Hence there is a unique solution $q^\dagger$ of problem*

$$\min_{q\in\Pi_Q(\overline{u})} \|q - q^*\|^2_{L^2(\Omega)} \qquad (\boldsymbol{K}^q)$$

which is called by the q^*-minimum norm solution of the identification problem.

Our goal is to investigate the convergence rate of regularized solutions q^δ_ρ to the q^*-minimum norm solution $q^\dagger$ of the equation $U(q) = \overline{u}$.

Theorem 2.1.2. *There exists a unique solution q^δ_ρ of problem $(\boldsymbol{P}^q_{\rho,\delta})$.*

Theorem 2.1.3. *For a fixed regularization parameter $\rho > 0$, let (z^{δ_n}) be a sequence which converges to z^δ in the $H^1(\Omega)$ and $(q_\rho^{\delta_n})$ be unique minimizers of problems*

$$\min_{q\in Q} J_{z^{\delta_n}}(q) + \rho\|q - q^*\|^2_{L^2(\Omega)}.$$

Then, $(q_\rho^{\delta_n})$ converges to the unique solution q_ρ^δ of $(\boldsymbol{P}^q_{\rho,\delta})$ in the $L^2(\Omega)$-norm.

Theorem 2.1.4. *For any positive sequence $(\delta_n) \to 0$, let $\rho_n := \rho(\delta_n)$ be such that $\rho_n \to 0$ and $\frac{\delta_n^2}{\rho_n} \to 0$ as $n \to \infty$. Moreover, let (z^{δ_n}) be a sequence satisfying $\|\overline{u} - z^{\delta_n}\|_{H^1(\Omega)} \le \delta_n$ and $\left(q_{\rho_n}^{\delta_n}\right)$ be the unique minimizers of the problems*

$$\min_{q\in Q} J_{z^{\delta_n}}(q) + \rho_n\|q - q^*\|^2_{L^2(\Omega)}.$$

Then, $\left(q_{\rho_n}^{\delta_n}\right)$ converges to the unique solution $q^\dagger$ of problem $(\boldsymbol{K}^q)$ in the $L^2(\Omega)$-norm.

2.1.2. Convergence rates

Now we state our main result on convergence rates for L^2-regularization of the problem of estimating the coefficient q in the Neumann problem (0.1)–(0.2).

We remark that since $L^\infty(\Omega) = L^1(\Omega)^* \subset L^\infty(\Omega)^*$, any $q \in L^\infty(\Omega)$ can be considered as an element in $L^\infty(\Omega)^*$, the dual space of $L^\infty(\Omega)$, by $\langle q, h\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} = \int_\Omega qh$ for all $h \in L^\infty(\Omega)$ and $\|q\|_{(L^\infty(\Omega))^*} \le \mathrm{mes}(\Omega)\|q\|_{L^\infty(\Omega)}$. Besides, for $q \in Q$, the mapping $U'(q) : L^\infty(\Omega) \to H^1_\diamond(\Omega)$ is a continuous linear operator. Denote by $U'(q)^* : H^1_\diamond(\Omega)^* \to L^\infty(\Omega)^*$ the dual operator of $U'(q)$. Then, $\langle U'(q)^* w^*, h\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} = \langle w^*, U'(q)h\rangle_{\left(H^1_\diamond(\Omega)^*, H^1_\diamond(\Omega)\right)}$ for all $w^* \in H^1_\diamond(\Omega)^*$ and $h \in L^\infty(\Omega)$.

The main result of this section is the following.

Theorem 2.1.5. *Assume that there exists a function $w^* \in H^1_\diamond(\Omega)^*$ such that*

$$q^\dagger - q^* = U'(q^\dagger)^* w^*. \tag{2.1}$$

Then,

$$\|q_\rho^\delta - q^\dagger\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta}) \text{ and } \|U(q_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

Remark 2.1.1. In our condition the source function is in $H^1_\diamond(\Omega)^*$, but not in the Hilbert space. Moreover, we do not require the “small enough condition” on the source function which seems to be extremely restrictive of the theory of regularization for nonlinear ill-posed problems.

2.1.3. Discussion of the source condition

The condition (2.1) is a weak source condition and does not require any smoothness of $q^\dagger$. Moreover, the smallness requirement on source functions of the general convergence theory for nonlinear ill-posed problems, which is hard to check, is liberated in our source condition. We note that the source condition (2.1) is fulfilled if and only if there exists a function $w \in H^1_\diamond(\Omega)$ such that

$$\langle q^\dagger - q^*, h\rangle_{L^2(\Omega)} = \langle w, U'(q^\dagger)h\rangle_{H^1(\Omega)} \tag{2.2}$$

for all h belonging to $L^\infty(\Omega)$.

In the following, as q^* is only an a-priori estimate of $q^\dagger$, for simplicity, we assume that $q^* \in H^1(\Omega)$. The following result gives a sufficient condition for (2.2) with a quite weak hypothesis about the regularity of the sought coefficient.

Theorem 2.1.6. *Assume that the boundary $\partial\Omega$ is of class C^1 and $q^\dagger$ belongs to $H^1(\Omega)$. Moreover, suppose that the exact $\overline{u} \in W^{2,\infty}(\Omega)$ and $|\nabla\overline{u}| \geq \gamma$ a.e. on Ω, where γ is a positive constant. Then, the condition* (2.2) *is fulfilled and hence a convergence rate $\mathcal{O}(\sqrt{\delta})$ of L^2-regularization is obtained.*

We remark that the hypothesis $|\nabla\overline{u}| \geq \gamma$ on Ω is quite natural, as if $|\nabla\overline{u}|$ vanishes in a subregion of Ω, then it is impossible to determine q on it. This is one of the reasons why our coefficient identification problem is ill-posed.

The proof of this theorem is based on the following auxiliary result.

Lemma 2.1.7. *Assume that the boundary $\partial\Omega$ is of class C^1 and $u \in W^{2,\infty}(\Omega)$ and $|\nabla u| \geq \gamma$ a.e. on Ω, where γ is a positive constant. Then, for any element $\widetilde{q} \in H^1(\Omega)$, there exists $v \in H^1(\Omega)$ satisfying*

$$\nabla u \cdot \nabla v = \widetilde{q}.$$

Further, there exists a positive constant C independent of $\widetilde{q}$ such that $\|v\|_{H^1(\Omega)} \leq C\|\widetilde{q}\|_{H^1(\Omega)}$.

2.2 Convergence rates for L^2-regularization of the reaction coefficient identification problem

2.2.1. L^2-regularization

Now we use the functional $G_{z^\delta}(a)$ with L^2-regularization to solve the problem of identifying the coefficient a in (0.3)–(0.4). Namely, we solve the strictly convex minimization problem

$$\min_{a\in A} G_{z^\delta}(a) + \rho\|a - a^*\|^2_{L^2(\Omega)} \tag{$\mathbf{P}^a_{\rho,\delta}$}$$

with $\rho > 0$ being the regularization parameter, a^* an a-priori estimate of the true coefficient.

Lemma 2.2.1. *The set $\Pi_A(\overline{u}) := \{a \in A \mid U(a) = \overline{u}\}$ is nonempty, convex, bounded and closed in the $L^2(\Omega)$-norm. Hence there is a unique solution $a^\dagger$ of problem*

$$\min_{a\in\Pi_A(\overline{u})} \|a - a^*\|^2_{L^2(\Omega)} \tag{$\boldsymbol{K}^a$}$$

which is called by the a^*-minimum norm solution of the identification problem.

Theorem 2.2.2. *There exists a unique solution a^δ_ρ of problem $(\boldsymbol{P}^a_{\rho,\delta})$.*

Theorem 2.2.3. *For a fixed regularization parameter $\rho > 0$, let (z^{δ_n}) be a sequence converging to z^δ in $H^1(\Omega)$ and $(a^{\delta_n}_\rho)$ be unique minimizers of problems*

$$\min_{a\in A} G_{z^{\delta_n}}(a) + \rho\|a - a^*\|^2_{L^2}.$$

Then, $(a^{\delta_n}_\rho)$ converges to the unique solution a^δ_ρ of $(\boldsymbol{P}^a_{\rho,\delta})$ in the $L^2(\Omega)$-norm.

Theorem 2.2.4. *For any positive sequence* $(\delta_n) \to 0$, *let* $\rho_n := \rho(\delta_n)$ *be such that* $\rho_n \to 0$ *and* $\frac{\delta_n^2}{\rho_n} \to 0$ *as* $n \to \infty$. *Moreover, let* (z^{δ_n}) *be a sequence satisfying* $\|\overline{u} - z^{\delta_n}\|_{H^1(\Omega)} \le \delta_n$ *and* $\left(a_{\rho_n}^{\delta_n}\right)$ *be the unique minimizers of the problems*

$$\min_{a\in A} G_{z^{\delta_n}}(a) + \rho_n \|a - a^*\|^2_{L^2(\Omega)}.$$

Then, $\left(a_{\rho_n}^{\delta_n}\right)$ *converges to the unique solution* $a^\dagger$ *of problem* $(\boldsymbol{K}^a)$ *in the* $L^2(\Omega)$*-norm.*

2.2.2. Convergence rates

We now state the convergence rate of regularized solutions a_ρ^δ to the a^*-minimum norm solution $a^\dagger$ of the equation $U(a) = \overline{u}$.

Theorem 2.2.5. *Assume that there exists a function* $w^* \in H^1(\Omega)^*$ *such that*

$$a^\dagger - a^* = U'(a^\dagger)^* w^*. \tag{2.3}$$

Then,

$$\|a_\rho^\delta - a^\dagger\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta}) \text{ and } \|U(a_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ *and* $\rho \sim \delta$.

2.2.3. Discussion of the source condition

The source condition (2.3) is equivalent to the following one: there exists a function $w \in H^1(\Omega)$ such that

$$\langle a^\dagger - a^*, h\rangle_{L^2(\Omega)} = \langle w, U'(a^\dagger)h\rangle_{H^1(\Omega)} \tag{2.4}$$

for all $h \in L^\infty(\Omega)$. We see that this condition is satisfied under a weak hypothesis about the regularity of the sought coefficient. Further, the smallness requirement on the source functions of the general convergence theory for nonlinear ill-posed problems is liberated in our source condition.

Theorem 2.2.6. *Assume that* $\frac{a^\dagger - a^*}{U(a^\dagger)}$ *is an element of* $H^1(\Omega)$. *Then, the condition* (2.4) *is fulfilled and hence a convergence rate* $\mathcal{O}(\sqrt{\delta})$ *of* L^2*-regularization is obtained.*

We close this section by the following note.

Remark 2.2.1. The hypothesis that $\frac{a^\dagger - a^*}{U(a^\dagger)}$ belongs to $H^1(\Omega)$ is satisfied if there exists a positive constant γ such that $|U(a^\dagger)| \ge \gamma$ a.e. on Ω and $a^\dagger - a^*$ is an element of $H^1(\Omega)$.

This chapter was written on the basis of the papers

[1] Dinh Nho Hào and Tran Nhan Tam Quyen (2010), Convergence rates for Tikhonov regularization of coefficient identification problems in Laplace-type equations, *Inverse Problems* **26**, 125014 (23pp).

[4] Dinh Nho Hào and Tran Nhan Tam Quyen (2012), Convergence rates for Tikhonov regularization of a two-coefficient identification problem in an elliptic boundary value problem, *Numerische Mathematik* **120**, pp. 45–77.

Chapter 3

Total variation regularization

In this chapter we apply total variation regularization to the convex functionals $J_{z^\delta}(\cdot)$ and $G_{z^\delta}(\cdot)$ respectively defined by (0.6) and (0.7) and obtain convergence rates $\mathcal{O}(\delta)$ of regularized solutions to solutions of our identification problems in the sense of the Bregman distance.

3.1 Convergence rates for total variation regularization of the diffusion coefficient identification problem

3.1.1. Regularization by the total variation

To estimate coefficients that may discontinuous or highly oscillating, we apply the total variation regularization method and arrive at the convex minimization problem

$$\min_{q \in Q_{ad}} J_{z^\delta}(q) + \rho \int_\Omega |\nabla q| \tag{$\mathcal{P}^q_{\rho,\delta}$}$$

for identifying the coefficient q in (0.1)–(0.2), where

$$Q_{ad} := Q \cap BV(\Omega) \tag{3.1}$$

is the admissible set of coefficients, $BV(\Omega)$ is the space of functions with bounded total variation and $\rho > 0$ is the regularization parameter. Theorem 3.1.1 shows that problem $(\mathcal{P}^q_{\rho,\delta})$ has a solution q^δ_ρ. Further, the problem

$$\min_{q \in \Pi_{Q_{ad}}(\overline{u})} \int_\Omega |\nabla q| \tag{$\mathcal{K}^q$}$$

also has a solution which is called the *total variation-minimizing solution* of the equation $U(q) = \overline{u}$, where

$$\Pi_{Q_{ad}}(\overline{u}) := \{q \in Q_{ad} \mid U(q) = \overline{u}\}. \tag{3.2}$$

Our aim in this section is to investigate the convergence rates of regularized solutions q^δ_ρ to the total variation-minimizing solution $q^\dagger$ of equation $U(q) = \overline{u}$.

Theorem 3.1.1. *(i) There exists a solution q^δ_ρ of problem $(\mathcal{P}^q_{\rho,\delta})$.*

(ii) There exists a solution $q^\dagger$ of problem $(\mathcal{K}^q)$.

In the following we denote by $\mathfrak{X} := L^\infty(\Omega) \cap BV(\Omega)$. Then, $\mathfrak{X}$ is a Banach space with the norm $\|q\|_{\mathfrak{X}} := \|q\|_{L^\infty(\Omega)} + \|q\|_{BV(\Omega)}$. Further, $L^\infty(\Omega)^* \subset \mathfrak{X}^*$ and $BV(\Omega)^* \subset \mathfrak{X}^*$. In addition, we will write $\mathfrak{X}_{BV(\Omega)} := \big(\mathfrak{X}, \|\cdot\|_{BV(\Omega)}\big)$ (respectively, $\mathfrak{X}_{L^\infty(\Omega)} := (\mathfrak{X}, \|\cdot\|_{L^\infty(\Omega)})$) to denote the space $\mathfrak{X}$ with respect to the $BV(\Omega)$-norm (respectively, the $L^\infty(\Omega)$-norm).

Theorem 3.1.2. *For a fixed regularization parameter $\rho > 0$, let (z^{δ_n}) be a sequence which converges to z^δ in the $H^1(\Omega)$-norm and $(q_\rho^{\delta_n})$ be minimizers of the problems*

$$\min_{q \in Q_{ad}} J_{z^{\delta_n}}(q) + \rho \int_\Omega |\nabla q|.$$

Then, there exist a subsequence $(q_\rho^{\delta_{1_n}})$ of $(q_\rho^{\delta_n})$ and $\widetilde{q} \in Q_{ad}$ such that $(q_\rho^{\delta_{1_n}})$ converges to $\widetilde{q}$ in the $L^1(\Omega)$-norm and $\lim_n \int_\Omega |\nabla q_\rho^{\delta_{1_n}}| = \int_\Omega |\nabla \widetilde{q}|$. Further, $\widetilde{q}$ is a solution to $(\mathcal{P}^q_{\rho,\delta})$.

Theorem 3.1.3. *For any positive sequence $(\delta_n) \to 0$, let $\rho_n := \rho(\delta_n)$ be such that $\rho_n \to 0$ and $\frac{\delta_n^2}{\rho_n} \to 0$ as $n \to \infty$. Moreover, let (z^{δ_n}) be a sequence satisfying $\|\overline{u} - z^{\delta_n}\|_{H^1(\Omega)} \le \delta_n$ and $(q_{\rho_n}^{\delta_n})$ be minimizers of problems*

$$\min_{q \in Q_{ad}} J_{z^{\delta_n}}(q) + \rho_n \int_\Omega |\nabla q|.$$

Then, there exist a subsequence $(q_{\rho_{1_n}}^{\delta_{1_n}})$ of $(q_{\rho_n}^{\delta_n})$ and an element $\widehat{q} \in \Pi_{Q_{ad}}(\overline{u})$ such that $\lim_n \|q_{\rho_{1_n}}^{\delta_{1_n}} - \widehat{q}\|_{L^1(\Omega)} = 0$ and $\lim_n \int_\Omega |\nabla q_{\rho_{1_n}}^{\delta_{1_n}}| = \int_\Omega |\nabla \widehat{q}|$. Further, $\widehat{q}$ is a solution to problem $(\mathcal{K}^q)$ and $\lim_n D_{TV}^\ell(q_{\rho_{1_n}}^{\delta_{1_n}}, \widehat{q}) = 0$ for all $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(\widehat{q})$.

3.1.2. Convergence rates

Now we state our main result on convergence rates of regularized solutions q_ρ^δ to the total variation-minimizing solution $q^\dagger$.

Theorem 3.1.4. *Let $q^\dagger$ be a solution of $(\mathcal{K}^q)$. Assume that there exists a functional $w^* \in H^1_\diamond(\Omega)^*$ such that*

$$U'(q^\dagger)^* w^* \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger). \tag{3.3}$$

Then,

$$D_{TV}^{U'(q^\dagger)^* w^*}(q_\rho^\delta, q^\dagger) = \mathcal{O}(\delta) \quad \textit{and} \quad \left|\int_\Omega |\nabla q_\rho^\delta| - \int_\Omega |\nabla q^\dagger|\right| = \mathcal{O}(\delta), \tag{3.4}$$

and $\|U(q_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$ as $\delta \to 0$ and $\rho \sim \delta$.

3.1.3. Discussion of the source condition

We note that the source condition (3.3) is fulfilled if and only if there exists a functional $w^* \in H^1_\diamond(\Omega)^*$ such that

$$\int_\Omega |\nabla q| - \int_\Omega |\nabla q^\dagger| - \langle U'(q^\dagger)^* w^*, q - q^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \ge 0 \tag{3.5}$$

for all $q \in \mathfrak{X}$. To further analyze our source condition, we assume that the sought coefficient belongs to $H^1(\Omega)$. Therefore, the admissible set of coefficients is restricted to $\widehat{Q_{ad}} = Q \cap H^1(\Omega) \subset Q \cap BV(\Omega)$.

Theorem 3.1.5. *Let the boundary $\partial\Omega$ be of class C^1 and the dimension $d \le 4$. Suppose that a solution $q^\dagger$ to $(\mathcal{K}^q)$ has the property that there is an element $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger)$ such that $\langle \ell, q\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)} = \langle \widehat{\ell}, q\rangle_{L^2(\Omega)}$ for all $q \in L^\infty(\Omega)\cap H^1(\Omega)$, where $\widehat{\ell}$ is some element of $H^1(\Omega)$. Further, assume that the exact $\overline{u} \in W^{2,\infty}(\Omega)$, $|\nabla \overline{u}| \ge \gamma$ a.e. on Ω with γ being a positive constant. Then, the condition (3.5) is fulfilled and hence convergence rates of pure total variation regularization in (3.4) are obtained.*

We remark that the requirement on $q^\dagger$ of the theorem is fulfilled at least on a set which is everywhere dense on $H^1(\Omega)$ as the boundary $\partial\Omega$ is of class C^1 and the dimension $d \le 4$.

3.2 Convergence rates for total variation regularization of the reaction coefficient identification problem

3.2.1. Regularization by the total variation

For solving the problem of identifying the coefficient a in (0.3)–(0.4), in this section we solve the convex minimization problem

$$\min_{a\in A_{ad}} G_{z^\delta}(a) + \rho\int_\Omega |\nabla a|, \tag{$\mathcal{P}^a_{\rho,\delta}$}$$

where

$$A_{ad} := A \cap BV(\Omega) \tag{3.6}$$

is the admissible set of coefficients and $\rho > 0$ is the regularization parameter. Problem $(\mathcal{P}^a_{\rho,\delta})$ has a solution a^δ_ρ which takes as the regularized solution to the inverse problem. On the other hand, the problem

$$\min_{a\in \Pi_{A_{ad}}(\overline{u})} \int_\Omega |\nabla a| \tag{$\mathcal{K}^a$}$$

also has a solution, which is called the *total variation-minimizing solution* of equation $U(a) = \overline{u}$, where

$$\Pi_{A_{ad}}(\overline{u}) := \{a \in A_{ad} \mid U(a) = \overline{u}\}. \tag{3.7}$$

Theorem 3.2.1. *(i) There exists a solution of problem $(\mathcal{P}^a_{\rho,\delta})$.*

(ii) There exists a solution of problem $(\mathcal{K}^a)$.

Theorem 3.2.2. *For a fixed regularization parameter $\rho > 0$, let (z^{δ_n}) be a sequence which converges to z^δ in the $H^1(\Omega)$-norm and $(a^{\delta_n}_\rho)$ be minimizers of the problems*

$$\min_{a\in A_{ad}} G_{z^{\delta_n}}(a) + \rho\int_\Omega |\nabla a|.$$

Then, there exist a subsequence $(a^{\delta_{1n}}_\rho)$ of $(a^{\delta_n}_\rho)$ and $\widetilde{a} \in A_{ad}$ such that $(a^{\delta_{1n}}_\rho)$ converges to $\widetilde{a}$ in the $L^1(\Omega)$-norm and $\lim_n \int_\Omega |\nabla a^{\delta_{1n}}_\rho| = \int_\Omega |\nabla \widetilde{a}|$. Further, $\widetilde{a}$ is a solution to $(\mathcal{P}^a_{\rho,\delta})$.

Theorem 3.2.3. *For any positive sequence $(\delta_n) \to 0$, let $\rho_n := \rho(\delta_n)$ be such that $\rho_n \to 0$ and $\frac{\delta_n^2}{\rho_n} \to 0$ as $n \to \infty$. Moreover, let (z^{δ_n}) be a sequence satisfying $\|\overline{u} - z^{\delta_n}\|_{H^1(\Omega)} \le \delta_n$ and $(a^{\delta_n}_{\rho_n})$ be minimizers of problems*

$$\min_{a\in A_{ad}} G_{z^{\delta_n}}(a) + \rho_n\int_\Omega |\nabla a|.$$

Then, there exists a subsequence $(a_{\rho_{1n}}^{\delta_{1n}})$ *of* $(a_{\rho_n}^{\delta_n})$ *and an element* $\widehat{a} \in \Pi_{A_{ad}}(\overline{u})$ *such that* $\lim_n \|a_{\rho_{1n}}^{\delta_{1n}} - \widehat{a}\|_{L^1(\Omega)} = 0$ *and* $\lim_n \int_\Omega |\nabla a_{\rho_{1n}}^{\delta_{1n}}| = \int_\Omega |\nabla \widehat{a}|$*. Further,* $\widehat{a}$ *is a solution to problem* $(\mathcal{K}^a)$ *and* $\lim_n D_{TV}^{\ell}(a_{\rho_{1n}}^{\delta_{1n}}, \widehat{a}) = 0$ *for all* $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(\widehat{a})$*.*

3.2.2. Convergence rates

Now we state the result on convergence rates of regularized solutions a_ρ^δ to the total variation-minimizing solution $a^\dagger$ of equation $U(a) = \overline{u}$.

Theorem 3.2.4. *Let* $a^\dagger$ *be a solution of* $(\mathcal{K}^a)$*. Assume that there exists a functional* $w^* \in H^1(\Omega)^*$ *such that*

$$U'(a^\dagger)^* w^* \in \partial \left(\int_\Omega |\nabla(\cdot)| \right) (a^\dagger). \tag{3.8}$$

Then,

$$D_{TV}^{U'(a^\dagger)^* w^*}(a_\rho^\delta, a^\dagger) = \mathcal{O}(\delta) \ \ \text{and} \ \ \left| \int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a_\rho^\delta| \right| = \mathcal{O}(\delta), \tag{3.9}$$

and $\|U(a_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$ *as* $\delta \to 0$ *and* $\rho \sim \delta$*.*

3.2.3. Discussion of the source condition

The source condition (3.8) is equivalent to the following one: there exists a function $w^* \in H^1(\Omega)^*$ such that

$$\int_\Omega |\nabla a| - \int_\Omega |\nabla a^\dagger| - \langle U'(a^\dagger)^* w^*, a - a^\dagger \rangle_{\left(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)}\right)} \geq 0 \tag{3.10}$$

for all $a \in \mathfrak{X}$. To further analyze this condition we assume that the admissible set of coefficients is restricted to $\widehat{A_{ad}} = A \cap H^1(\Omega) \subset A \cap BV(\Omega)$.

Theorem 3.2.5. *Let the boundary* $\partial\Omega$ *be of class* C^1 *and the dimension* $d \leq 4$*. Suppose that a solution* $a^\dagger$ *to* $(\mathcal{K}^a)$ *has the property that there is an element* $\lambda \in \partial \left(\int_\Omega |\nabla(\cdot)| \right) (a^\dagger)$ *such that* $\langle \lambda, a \rangle_{\left(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)}\right)} = \langle \widehat{\lambda}, a \rangle_{L^2(\Omega)}$ *for all* $a \in L^\infty(\Omega) \cap H^1(\Omega)$*, where* $\widehat{\lambda}$ *is some element of* $H^1(\Omega)$*. Furthermore, assume that there exists a positive constant* γ *such that* $|\overline{u}| \geq \gamma$ *a.e. on* Ω*. Then, the condition* (3.10) *is fulfilled and hence convergence rates of pure total variation regularization* (3.9) *are obtained.*

This chapter was written on the basis of the paper

[2] Dinh Nho Hào and Tran Nhan Tam Quyen (2011), Convergence rates for total variation regularization of coefficient identification problems in elliptic equations I, *Inverse Problems* **27**, 075008 (28pp).

Chapter 4

Regularization of total variation combining with L^2-stabilization

In this chapter total variation regularization combining with L^2-stabilization is applied to the convex functionals $J_{z^\delta}(\cdot)$ and $G_{z^\delta}(\cdot)$ respectively defined by (0.6) and (0.7). We obtain convergence rates of regularized solutions to solutions of the identification problems in the sense of the Bregman distance and in the $L^2(\Omega)$-norm.

4.1 Convergence rates for total variation regularization combining with L^2-stabilization of the diffusion coefficient identification problem

4.1.1. Regularization by total variation combining with L^2-stabilization

For identifying the coefficient q in (0.1)–(0.2), in this section we solve the strictly convex minimization problem

$$\min_{q \in Q_{ad}} J_{z^\delta}(q) + \rho \left(\frac{1}{2} \|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q| \right), \tag{$\mathbb{P}^q_{\rho,\delta}$}$$

where Q_{ad} defined by (3.1) is the admissible set of coefficients, $\rho > 0$ is the regularization parameter.

Theorem 4.1.1 shows that problem ($\mathbb{P}^q_{\rho,\delta}$) has a *unique solution* q^δ_ρ. Further, the problem

$$\min_{q \in \Pi_{Q_{ad}}(\overline{u})} \left(\frac{1}{2} \|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q| \right), \tag{$\mathbb{K}^q$}$$

also has a *unique solution* $q^\dagger$, which we call *R-minimizing solution* to our inverse problem, where

$$R(\cdot) := \frac{1}{2} \| \cdot \|^2_{L^2(\Omega)} + \int_\Omega |\nabla (\cdot)| \,. \tag{4.1}$$

Our aim in this section is to investigate convergence rates of q^δ_ρ to the R-minimizing solution $q^\dagger$ of the equation $U(q) = \overline{u}$.

Theorem 4.1.1. *(i) There exists a unique solution* q_ρ^δ *of problem* $(\mathbb{P}^q_{\rho,\delta})$.

(ii) There exists a unique solution $q^\dagger$ *of problem* $(\mathbb{K}^q)$.

Theorem 4.1.2. *For a fixed regularization parameter* $\rho > 0$, *let* (z^{δ_n}) *be a sequence which converges to* z^δ *in the* $H^1(\Omega)$*-norm and* $(q_\rho^{\delta_n})$ *be the unique minimizers of problems*

$$\min_{q\in Q_{ad}} J_{z^{\delta_n}}(q) + \rho\left(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q|\right).$$

Then, $(q_\rho^{\delta_n})$ *converges to the unique solution* q_ρ^δ *of* $(\mathbb{P}^q_{\rho,\delta})$ *in the* $L^2(\Omega)$*-norm. Further,* $\lim_n \int_\Omega |\nabla q_\rho^{\delta_n}| = \int_\Omega |\nabla q_\rho^\delta|$.

Theorem 4.1.3. *For any positive sequence* $(\delta_n) \to 0$, *let* $\rho_n := \rho(\delta_n)$ *be such that* $\rho_n \to 0$ *and* $\frac{\delta_n^2}{\rho_n} \to 0$ *as* $n \to \infty$. *Moreover, let* (z^{δ_n}) *be a sequence satisfying* $\|\overline{u} - z^{\delta_n}\|_{H^1(\Omega)} \le \delta_n$ *and* $(q_{\rho_n}^{\delta_n})$ *be the unique minimizers of problems*

$$\min_{q\in Q_{ad}} J_{z^{\delta_n}}(q) + \rho_n\left(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q|\right).$$

Then, $\left(q_{\rho_n}^{\delta_n}\right)$ *converges to the unique solution* $q^\dagger$ *of problem* $(\mathbb{K}^q)$ *in the* $L^2(\Omega)$*-norm. Further,* $\lim_n \int_\Omega |\nabla q_{\rho_n}^{\delta_n}| = \int_\Omega |\nabla q^\dagger|$ *and* $\lim_n D^\ell_{TV}(q_{\rho_n}^{\delta_n}, q^\dagger) = 0$ *for all* $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger)$.

4.1.2. Convergence rates

We recall that

$$\partial R(q^\dagger) = q^\dagger + \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger) \subset \mathfrak{X}^*,$$

where the functional $R(\cdot)$ is defined by (4.1).

Theorem 4.1.4. *Assume that there exists a functional* $w^* \in H^1_\diamond(\Omega)^*$ *such that*

$$U'(q^\dagger)^* w^* = q^\dagger + \ell \in \partial R(q^\dagger) \tag{4.2}$$

for some element ℓ *in* $\partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger)$. *Then,*

$$\|q_\rho^\delta - q^\dagger\|^2_{L^2(\Omega)} + D^\ell_{TV}(q_\rho^\delta, q^\dagger) = \mathcal{O}(\delta) \tag{4.3}$$

and $\|U(q_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$ *as* $\delta \to 0$ *and* $\rho \sim \delta$. *Moreover, if* $\ell \in \mathfrak{X}^*$ *can be identified with an element of* $L^2(\Omega)$, *then the following convergence rate is obtained*

$$\left|\int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q_\rho^\delta|\right| = \mathcal{O}(\sqrt{\delta}) \text{ as } \delta \to 0 \text{ and } \rho \sim \delta. \tag{4.4}$$

4.1.3. Discussion of the source condition

Condition (4.2) is fulfilled if and only if there exists a function $w^* \in H^1_\diamond(\Omega)^*$ such that

$$\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q| - \frac{1}{2}\|q^\dagger\|^2_{L^2(\Omega)} - \int_\Omega |\nabla q^\dagger| - \langle U'(q^\dagger)^* w^*, q - q^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \ge 0$$

for all $q \in \mathfrak{X}$. To further analyze this condition we assume that the sought coefficient belongs to $H^1(\Omega)$. Therefore, the admissible set of sought coefficients is restricted to $\widehat{Q_{ad}} = Q \cap H^1(\Omega) \subset Q \cap BV(\Omega)$.

Moreover, if ℓ can be identified with an element of $L^2(\Omega)$, i.e., there exists an element $\widetilde{\ell}$ in $L^2(\Omega)$ such that $\langle \ell, q\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} = \langle \widetilde{\ell}, q\rangle_{L^2(\Omega)}$ for all q in $\mathfrak{X}$, then the convergence rate (4.4) is also established.

Theorem 4.1.5. *Let the boundary $\partial\Omega$ be of class C^1 and the dimension $d \le 4$. Assume that $q^\dagger$ has the property that there is an element $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger)$ such that $\langle \ell, q\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} = \langle \widehat{\ell}, q\rangle_{L^2(\Omega)}$ for all $q \in L^\infty(\Omega) \cap H^1(\Omega)$, where $\widehat{\ell}$ is some element of $H^1(\Omega)$. Further, suppose that the exact $\overline{u} \in W^{2,\infty}(\Omega)$, $|\nabla \overline{u}| \ge \gamma$ a.e. on Ω with γ being a positive constant. Then, the convergence rates* (4.3) *and* (4.4) *are obtained.*

We note that as the boundary $\partial\Omega$ is of class C^1 and the dimension $d \le 4$, the requirement on $q^\dagger$ of the theorem is fulfilled at least on a set which is everywhere dense on $H^1(\Omega)$.

4.2 Convergence rates for total variation regularization combining with L^2-stabilization of the reaction coefficient identification problem

4.2.1. Regularization by the total variation combining with L^2-stabilization

For identifying the coefficient a in (0.3)–(0.4), we solve the strictly convex minimization problem

$$\min_{a \in A_{ad}} G_{z^\delta}(a) + \rho\left(\frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a|\right), \tag{$\mathbb{P}^a_{\rho,\delta}$}$$

where A_{ad} defined by (3.6) and $\rho > 0$ is the regularization parameter.

Theorem 4.2.1 shows that problem ($\mathbb{P}^a_{\rho,\delta}$) has a *unique solution* a^δ_ρ. On the other hand, the problem

$$\min_{a \in \Pi_{A_{ad}}(\overline{u})} \frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a|, \tag{$\mathbb{K}^a$}$$

also has a *unique solution* $a^\dagger$, which we also call *R-minimizing solution* to our inverse problem, where the functional $R(\cdot)$ is defined by (4.1).

In this section we investigate the convergence rates of a^δ_ρ to the solution $a^\dagger$ of the equation $U(a) = \overline{u}$.

Theorem 4.2.1. *(i) There exists a unique solution of problem* ($\mathbb{P}^a_{\rho,\delta}$).

(ii) There exists a unique solution of problem ($\mathbb{K}^a$).

Theorem 4.2.2. *For a fixed regularization parameter $\rho > 0$, let (z^{δ_n}) be a sequence which converges to z^δ in $H^1(\Omega)$ and $(a^{\delta_n}_\rho)$ be the unique minimizers of problems*

$$\min_{a \in A_{ad}} G_{z^{\delta_n}}(a) + \rho\left(\frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a|\right).$$

Then, $(a^{\delta_n}_\rho)$ converges to the unique solution a^δ_ρ of ($\mathbb{P}^a_{\rho,\delta}$) *in the $L^2(\Omega)$-norm. Further,* $\lim_n \int_\Omega |\nabla a^{\delta_n}_\rho| = \int_\Omega |\nabla a^\delta_\rho|$.

Theorem 4.2.3. *For any positive sequence* $(\delta_n) \to 0$*, let* $\rho_n := \rho(\delta_n)$ *be such that* $\rho_n \to 0$ *and* $\frac{\delta_n^2}{\rho_n} \to 0$ *as* $n \to \infty$*. Moreover, let* (z^{δ_n}) *be a sequence satisfying* $\|\overline{u} - z^{\delta_n}\|_{H^1(\Omega)} \le \delta_n$ *and* $(a_{\rho_n}^{\delta_n})$ *be the unique minimizers of problems*

$$\min_{a \in A_{ad}} G_{z^{\delta_n}}(a) + \rho_n \left(\frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a| \right).$$

Then, $(a_{\rho_n}^{\delta_n})$ *converges to the unique solution* $a^\dagger$ *of problem* $(\mathbb{K}^a)$ *in the* $L^2(\Omega)$*-norm. Further,* $\lim_n \int_\Omega |\nabla a_{\rho_n}^{\delta_n}| = \int_\Omega |\nabla a^\dagger|$ *and* $\lim_n D_{TV}^\ell(a_{\rho_n}^{\delta_n}, a^\dagger) = 0$ *for all* $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(a^\dagger)$*.*

4.2.2. Convergence rates

Theorem 4.2.4. *Assume that there exists a function* $w^* \in H^1(\Omega)^*$ *such that*

$$U'(a^\dagger)^* w^* = a^\dagger + \lambda \in \partial R(a^\dagger) \tag{4.5}$$

for some element λ *in* $\partial\left(\int_\Omega |\nabla(\cdot)|\right)(a^\dagger)$*. Then,*

$$\|a_\rho^\delta - a^\dagger\|^2_{L^2(\Omega)} + D_{TV}^\lambda(a_\rho^\delta, a^\dagger) = \mathcal{O}(\delta) \tag{4.6}$$

and $\|U(a_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$ *as* $\delta \to 0$ *and* $\rho \sim \delta$*. Further, if* $\lambda \in \mathfrak{X}^*$ *can be identified with an element of* $L^2(\Omega)$*, then the convergence rate*

$$\left| \int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a_\rho^\delta| \right| = \mathcal{O}(\sqrt{\delta}) \text{ as } \delta \to 0 \text{ and } \rho \sim \delta, \tag{4.7}$$

is also established.

4.2.3. Discussion of the source condition

The source condition (4.5) is equivalent to the following one: there exists a function $w^* \in H^1(\Omega)^*$ such that

$$\frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a| - \frac{1}{2}\|a^\dagger\|^2_{L^2(\Omega)} - \int_\Omega |\nabla a^\dagger| - \langle U'(a^\dagger)^* w^*, a - a^\dagger \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \ge 0 \tag{4.8}$$

for all $a \in \mathfrak{X}$. To further analyze our condition we assume that the admissible set of coefficients is restricted to $\widehat{A_{ad}} = A \cap H^1(\Omega) \subset A \cap BV(\Omega)$.

Theorem 4.2.5. *Let the boundary* $\partial\Omega$ *be of class* C^1 *and the dimension* $d \le 4$*. Suppose that* $a^\dagger$ *has the property that there is an element* $\lambda \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(a^\dagger)$ *such that* $\langle \lambda, a \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} = \langle \widehat{\lambda}, a \rangle_{L^2(\Omega)}$ *for all* $a \in L^\infty(\Omega) \cap H^1(\Omega)$*, where* $\widehat{\lambda}$ *is some element of* $H^1(\Omega)$*. Further, assume that there exists a positive constant* γ *such that* $|\overline{u}| \ge \gamma$ *a.e. on* Ω*. Then, the condition* (4.8) *is fulfilled and hence convergence rates* (4.6) *and* (4.7) *are obtained.*

THIS CHAPTER WAS WRITTEN ON THE BASIS OF THE PAPER

[3] Dinh Nho Hào and Tran Nhan Tam Quyen (2012), Convergence rates for total variation regularization of coefficient identification problems in elliptic equations II, *Journal of Mathematical Analysis and Applications* **388**, pp. 593–616.

General Conclusions

Let Ω be an open bounded connected domain in $\mathbb{R}^d, d \geq 1$ with the Lipschitz boundary $\partial\Omega$, $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ be given. In this work we investigate ill-posed nonlinear inverse problems of identifying the diffusion coefficient q in the Neumann problem for the elliptic equation

$$
\begin{aligned}
-\operatorname{div}(q\nabla u) &= f \text{ in } \Omega,\\
q\frac{\partial u}{\partial n} &= g \text{ on } \partial\Omega
\end{aligned}
$$

and the reaction coefficient a in the Neumann problem for the elliptic equation

$$
\begin{aligned}
-\Delta u + au &= f \text{ in } \Omega,\\
\frac{\partial u}{\partial n} &= g \text{ on } \partial\Omega,
\end{aligned}
$$

when u is imprecisely given by z^δ with $\|u - z^\delta\|_{H^1(\Omega)} \leq \delta$ and $\delta > 0$. These problems frequently accounter in practice and attracted great attention from many researchers during the last 50 years or so. They are difficult due to their nonlinearity and ill-posedness, therefore regularization methods for them are required. However, up to now there have been very few results on convergence rates of suggested regularization methods. The famous one (and the only one) by Engl, Kunisch and Neubauer required the small enough condition of the source functions which is very difficult to check and applicable only to one-dimensional above identification problems. The drawback of this work and many related ones is that the authors follow the least-squares approach and thus they are faced with nonconvex minimization problems, the global minima of which are impossible to find. In this dissertation, we do not follow this approach, but regularize the above identification problems by correspondingly minimizing the (strictly) convex functionals

$$
\frac{1}{2}\int_\Omega q|\nabla(U(q) - z^\delta)|^2 + \rho\mathcal{R}(q)
$$

and

$$
\frac{1}{2}\int_\Omega |\nabla(U(a) - z^\delta)|^2 + \frac{1}{2}\int_\Omega a(U(a) - z^\delta)^2 + \rho\mathcal{R}(a)
$$

over the admissible sets of coefficients, where $U(q)$ ($U(a)$) is the solution of the first (second) Neumann boundary value problem, $\rho > 0$ is the regularization parameter and either

$$
\mathcal{R}(\cdot) = \|\cdot\|^2_{L^2(\Omega)}
$$

or

$$
\mathcal{R}(\cdot) = \int_\Omega |\nabla(\cdot)|
$$

or

$$\mathcal{R}(\cdot) = \frac{1}{2}\|\cdot\|^2_{L^2(\Omega)} + \int_\Omega |\nabla(\cdot)|.$$

One of the advantage of our approach is that the above minimization problems are convex and we can find their global minima. Furthermore, taking their solutions as the regularized solutions to the corresponding identification problems, we obtain the convergence rates of them to solutions of our inverse problems under weak source conditions

$$\text{there exists a function } w^* \in H^1_\diamond(\Omega)^* \text{ such that } U'(q^\dagger)^* w^* \in \partial\mathcal{R}(q^\dagger)$$

for the first problem and

$$\text{there exists a function } w^* \in H^1(\Omega)^* \text{ such that } U'(a^\dagger)^* w^* \in \partial\mathcal{R}(a^\dagger)$$

for the second problem with $q^\dagger$ and $a^\dagger$ respectively being the total $\mathcal{R}$-minimizing solutions of the coefficient identification problems. Our source conditions are simple and weak, since we remove the so-called "small enough condition" on the source functions that is standard in the theory of regularization of nonlinear ill-posed problems but very hard to check. Furthermore, our results are valid for multi-dimensional identification problems. They are the first results affirmatively answering the question whether total variation regularization can provide convergence rates for coefficient identification problems in partial differential equations.

List of the author's publications related to the dissertation

[1] Dinh Nho Hào and Tran Nhan Tam Quyen (2010), Convergence rates for Tikhonov regularization of coefficient identification problems in Laplace-type equations, *Inverse Problems* **26**, 125014 (23pp).

[2] Dinh Nho Hào and Tran Nhan Tam Quyen (2011), Convergence rates for total variation regularization of coefficient identification problems in elliptic equations I, *Inverse Problems* **27**, 075008 (28pp).

[3] Dinh Nho Hào and Tran Nhan Tam Quyen (2012), Convergence rates for total variation regularization of coefficient identification problems in elliptic equations II, *Journal of Mathematical Analysis and Applications* **388**, pp. 593–616.

[4] Dinh Nho Hào and Tran Nhan Tam Quyen (2012), Convergence rates for Tikhonov regularization of a two-coefficient identification problem in an elliptic boundary value problem, *Numerische Mathematik* **120**, pp. 45–77.

The results of the dissertation have been presented

by Prof. Dr. habil. Dinh Nho Hào

1) Mini Special Semester on Inverse Problems, Johann Radom Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Jun 29th – Jul 2nd, 2010.

2) Scientific Conference in Celebration of the 35-Year History of Vietnam Academy of Science and Technology, Oct 25, 2010.

3) International Conference on Analysis and Applied Mathematics, Sai Gon University, Viet Nam, Mar 14, 2011.

and by Tran Nhan Tam Quyen

4) PhD Students Conference, Hanoi Institute of Mathematics, Oct 30, 2009.

5) PhD Students Conference, Hanoi Institute of Mathematics, Oct 29, 2010.