VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY
INSTITUTE OF MATHEMATICS

NGUYEN THANH QUI

Coderivatives of Normal Cone Mappings and Applications

Speciality: Applied Mathematics
Speciality code: 62 46 01 12

SUMMARY

DOCTORAL DISSERTATION IN MATHEMATICS

HANOI - 2014

The dissertation was written on the basis of the author's research works carried at Institute of Mathematics, Vietnam Academy of Science and Technology

Supervisors:

1. **Prof. Dr. Hab. Nguyen Dong Yen**
2. **Dr. Bui Trong Kien**

First referee: ....
....

Second referee: ....
....

Third referee: ....
....

To be defended at the Jury of Institute of Mathematics, Vietnam Academy of Science and Technology:

....

....

on .... 2014, at .... o'clock ....

The dissertation is publicly available at:

- The National Library of Vietnam
- The Library of Institute of Mathematics

Introduction

Motivated by solving optimization problems, the concept of *derivative* was first introduced by Pierre de Fermat. It led to the *Fermat stationary principle*, which plays a crucial role in the development of *differential calculus* and serves as an effective tool in various applications. Nevertheless, many fundamental objects having no derivatives, no first-order approximations (defined by certain derivative mappings) occur naturally and frequently in mathematical models. The objects include nondifferentiable functions, sets with nonsmooth boundaries, and set-valued mappings. Since the classical differential calculus is inadequate for dealing with such functions, sets, and mappings, the appearance of generalized differentiation theories is an indispensable trend.

In the 1960s, differential properties of convex sets and convex functions have been studied. The fundamental contributions of J.-J. Moreau and R. T. Rockafellar have been widely recognized. Their results led to the beautiful theory of convex analysis. The derivative-like structure for convex functions, called *subdifferential*, is one of the main concepts in this theory. In contrast to the singleton of derivatives, subdifferential is a collection of subgradients. Convex programming which is based on convex analysis plays a fundamental role in Mathematics and in applied sciences.

In 1973, F. H. Clarke defined basic concepts of a generalized differentiation theory, which works for locally Lipschitz functions, in his doctoral dissertation under the supervision of R. T. Rockafellar. In Clarke's theory, convexity is a key point; for instance, subdifferential in the sense of Clarke is always a closed convex set. In the later 1970s, the concepts of Clarke have been developed for lower semicontinuous extended-real-valued functions in the works of R. T. Rockafellar, J.-B. Hiriart-Urruty, J.-P. Aubin, and others. Although the theory of Clarke is beautiful due to the convexity used, as well as to the elegant proofs of many fundamental results, the Clarke subdifferential and the Clarke normal cone face with the challenge of being too big, so too rough, in complicated practical problems where nonconvexity is an inherent property. Despite to this, Clarke's theory has opened a new chapter in the development of nonlinear analysis and optimization theory.

In the mid 1970s, to avoid the above-mentioned convexity limitations of the Clarke concepts, B. S. Mordukhovich introduced the notions of *limiting normal cone* and *limiting subdifferential* which are based entirely on dual-space constructions. His dual approach led to a modern theory of generalized differentiation with a variety of applications. Long before the publication of his books (2006), Mordukhovich's contributions to Variational Analysis had been presented in the well-known monograph of R. T. Rockafellar and R. J.-B. Wets (1998).

The limiting subdifferential is generally nonconvex and smaller than the Clarke subdifferential. Similarly, the limiting normal cone to a closed set in a Banach space is nonconvex in general and usually smaller than the Clarke normal cone. Therefore, necessary optimality conditions in nonlinear programming and optimal control in terms of the limiting subdifferential and limiting normal cone are much tighter than that given by the corresponding Clarke's concepts. Furthermore, the Mordukhovich criteria for the Lipchitz-like property (that is the pseudo-Lipschitz property in the original terminology of J.-P. Aubin, or the Aubin continuity as suggested by A. L. Dontchev and R. T. Rockafellar) and the metric regularity of multifunctions are remarkable tools to study stability of variational inequalities, generalized equations, and the Karush-Kuhn-Tucker point sets in parametric optimization problems. Note that if one uses Clarke's theory then only sufficient conditions for stability can be obtained. Meanwhile, Mordukhovich's theory provides one with both necessary and sufficient conditions for stability. Another advantage of the latter theory is that its system of calculus rules is much more developed than that of Clarke's theory. So, the wide range of applications and bright prospects of Mordukhovich's generalized differentiation theory are understandable.

As far as we understand, Variational Analysis is a new name of a mathematical discipline which unifies Nonsmooth Analysis, Set-Valued Analysis with applications to Optimization Theory and equilibrium problems.

Let X, W_1, W_2 are Banach spaces, $\varphi : X \times W_1 \to I\!R$ is a continuously Fréchet differentiable function, $\Theta : W_2 \rightrightarrows X$ is a multifunction (i.e., a set-valued map) with closed convex values. Consider the minimization problem

$$\min\{\varphi(x, w_1)|\ x \in \Theta(w_2)\} \tag{1}$$

depending on the parameters $w = (w_1, w_2)$, which is given by the data set $\{\varphi, \Theta\}$. According to the generalized Fermat rule, if $\bar{x}$ is a local solution of (1) then

$$0 \in f(\bar{x}, w_1) + N(\bar{x}; \Theta(w_2)),$$

where $f(\bar{x}, w_1) = \nabla_x \varphi(\bar{x}, w_1)$ denotes the partial derivative of φ with respect to x at $(\bar{x}, w_1)$ and

$$N(\bar{x}; \Theta(w_2)) = \{x^* \in X^*|\ \langle x^*, x - \bar{x}\rangle \leq 0,\ \forall x \in \Theta(w_2)\},$$

with X^* being the dual space of X, stands for the normal cone of $\Theta(w_2)$. This means that $\bar{x}$ is a solution of the following generalized equation

$$0 \in f(x, w_1) + \mathcal{F}(x, w_2), \tag{2}$$

where $\mathcal{F}(x, w_2) := N(x; \Theta(w_2))$ for every $x \in \Theta(w_2)$ and $\mathcal{F}(x, w_2) := \emptyset$ for every $x \notin \Theta(w_2)$, is the parametric normal cone mapping related to the multifunction $\Theta(\cdot)$. Equilibrium problems of the form (2) have been investigated intensively in the literature. Necessary and sufficient conditions for the Lipschitz-like property of the solution map $(w_1, w_2) \mapsto S(w_1, w_2)$ of (2) can be characterized by using the Mordukhovich criterion. According to the method proposed by Dontchev and Rockafellar (1996), which has been developed by A. B. Levy and B. S. Mordukhovich (2004) and by G. M. Lee and N. D. Yen (2011), one has to compute the Fréchet and the Mordukhovich coderivatives of $\mathcal{F} : X \times W_2 \rightrightarrows X^*$. Such a computation has been done by Dontchev and Rockafellar (1996) for the case $\Theta(w_2)$ is a fixed polyhedral convex set in $I\!R^n$, and by Yao and Yen (2010) for the case where $\Theta(w_2)$ is a fixed smooth-boundary convex set. The problem is rather difficult if $\Theta(w_2)$ depends on w_2.

J.-C. Yao and N. D. Yen (2009a,b) first studied the case $\Theta(w_2) = \Theta(b) := \{x \in I\!R^n|\ Ax \leq b\}$ where A is an $m \times n$ matrix, b is a parameter. Some arguments from these papers have been used by R. Henrion, B. S. Mordukhovich and N. M. Nam (2010) to compute coderivatives of the normal cone mappings to a fixed polyhedral convex set in Banach space. Nam (2010) showed that the results of Yao and Yen on normal cone mappings to linearly perturbed polyhedra can be extended to an infinite dimensional setting. N. T. Q. Trang (2012) proposed some developments and refinements of the results of Nam.

Lee and Yen (2014) computed the Fréchet coderivatives of the normal cone mappings to a perturbed Euclidean balls and derived from the results a stability criterion for the Karush-Kuhn-Tucker point set mapping of parametric trust-region subproblems.

As concerning normal cone mappings to nonlinearly perturbed polyhedra, G. Colombo, R. Henrion, N. D. Hoang, and B. S. Mordukhovich (2012) have computed coderivatives of the normal cone to a rotating closed half-space.

The normal cone mapping considered by Lee and Yen (2014) is a special case of the normal cone mapping to the solution set $\Theta(w_2) = \Theta(p) := \{x \in X |\ \psi(x,p) \le 0\}$ where $\psi : X \times P \to I\!R$ is a $\mathcal{C}^2$-smooth function defined on the product space of Banach spaces X and P.

More generally, for the solution map

$$\Theta(w_2) = \Theta(p) := \{x \in X |\ \Psi(x,p) \in K\}$$

of a parametric generalized equality system with $\Psi : X \times P \to Y$ being a $\mathcal{C}^2$-smooth vector function which maps the product space $X \times P$ into a Banach space Y, $K \subset Y$ a closed convex cone, the problems of computing the Fréchet coderivative (respectively, the Mordukhovich coderivative) of the Fréchet normal cone mapping $(x, w_2) \mapsto \widehat{N}(x; \Theta(w_2))$ (respectively, of the limiting normal cone mapping $(x, w_2) \mapsto N(x; \Theta(w_2)))$, are interesting, but very difficult. All the above-mentioned normal cone mappings are special cases of the last two normal cone mappings. It will take some time before significant advances on these general problems can be done. Some aspects of this question have been investigated by R. Henrion, J. Outrata, and T. Surowiec (2009).

It is worthy to stress that coderivatives of normal cone mappings are nothing else as the second-order subdifferentials of the indicator functions of the set in question. The concepts of Fréchet and/or limiting second-order subdifferentials of extended-real-valued functions have been discussed by Mordukhovich (2006), R. A. Poliquin and R. T. Rockafellar (1998), Mordukhovich and Outrata (2001), N. H. Chieu, T. D. Chuong, J.-C. Yao, and N. D. Yen (2011), N. H. Chieu and N. Q. Huy (2011), Chieu and Trang (2012), Mordukhovich and Rockafellar (2012) from different points of views.

This dissertation studies some problems related to the generalized differentiation theory of Mordukhovich and its applications. Our main efforts concentrate on computing or estimating the Fréchet coderivative and the

Mordukhovich coderivative of the normal cone mappings to: a) linearly perturbed polyhedra in finite dimensional spaces, as well as in infinite dimensional reflexive Banach spaces; b) nonlinearly perturbed polyhedra in finite dimensional spaces; c) perturbed Euclidean balls.

Applications of the obtained results are used to study the metric regularity property and/or the Lipschitz-like property of the solution maps of some classes of parametric variational inequalities as well as parametric generalized equations.

Our results develop certain aspects of the preceding works Dontchev and Rockafellar (1996), Yao and Yen (2009a,b), Henrion, Mordukhovich and Nam (2010), Nam (2010), Lee and Yen (2014). The four open questions raised by Yao and Yen (2009a), Lee and Yen (2014) have been solved in this dissertation. Some of our techniques are new.

The dissertation has four chapters and a list of references.

Chapter 1 collects several basic concepts and facts on generalized differentiation, together with the well-known dual characterizations of the two fundamental properties of multifunctions: the local Lipschitz-like property defined by J.-P. Aubin and the metric regularity which has origin in Ljusternik's theorem.

Chapter 2 studies generalized differentiability properties of the normal cone mappings associated to perturbed polyhedral convex sets in reflexive Banach spaces. The obtained results lead to solution stability criteria for a class of variational inequalities in finite dimensional spaces under linear perturbations. This chapter answers the two open questions of Yao and Yen (2009a).

Chapter 3 computes the Fréchet and the Mordukhovich coderivatives of the normal cone mappings studied in the previous chapter with respect to total perturbations. As a consequence, solution stability of affine variational inequalities under nonlinear perturbations in finite dimensional spaces can be addressed by means of the Mordukhovich criterion and the coderivative formula for implicit multifunctions due to Levy and Mordukhovich (2004).

Based on a recent paper of Lee and Yen (2014), Chapter 4 presents a comprehensive study of the solution stability of a class of linear generalized equations connected with the parametric trust-region subproblems which are well-known in nonlinear programming. Exact formulas for the coderivatives of the normal cone mappings associated to perturbed Euclidean balls have been obtained. Combining the formulas with the necessary and the sufficient conditions for the local Lipschitz-like property of implicit multifunctions from a paper by Lee and Yen (2011), we get new results on stability of the Karush-Kuhn-Tucker point set maps of parametric trust-region subproblems. This chapter also solves the two open questions of Lee and Yen (2014).

Except for Chapter 1, each chapter has several illustrative examples.

The results of Chapter 2 and Chapter 3 were published on the journals *Nonlinear Analysis* [1], *Journal of Mathematics and Applications* [2], *Acta Mathematica Vietnamica* [3], *Journal of Optimization Theory and Applications* [4]. Chapter 4 is written on the basis of a joint paper by N. T. Qui and N. D. Yen, which has been accepted for publication on *SIAM Journal on Optimization* [5].

These results were reported by the author of this dissertation at Seminar of Department of Numerical Analysis and Scientific Computing of Institute of Mathematics (VAST, Hanoi), Workshops "Optimization and Scientific Computing" (Ba Vi, April 20-23, 2010; April 20-23, 2011), The 8^{th} Vietnam-Korea Workshop "Mathematical Optimization Theory and Applications" (University of Dalat, December 8-10, 2011), Summer Schools "Variational Analysis and Applications" (Institute of Mathematics (VAST, Hanoi), June 20-25, 2011; Institute of Mathematics (VAST, Hanoi) and Vietnam Institute for Advanced Study in Mathematics, May 28-June 03, 2012).

Chapter 1

Preliminary

This chapter reviews some background material of Variational Analysis. The basic concepts of generalized differentiation of multifunctions and extended-real-valued functions are taken from Mordukhovich (2006, Vols I and II).

1.1 Normal and Tangent Cones

Let $F : X \rightrightarrows X^*$ be a multifunction between a Banach space X and its dual X^*. The *sequential Painlevé-Kuratowski upper limit* of F as $x \to \bar{x}$ with respect to the norm topology of X and the weak* topology of X^* is given by

$$\operatorname*{Limsup}_{x\to\bar{x}} F(x) = \Big\{ x^* \in X^* \Big| \ \exists x_k \to \bar{x} \text{ and } x_k^* \overset{w^*}{\to} x^* \text{ with } x_k^* \in F(x_k), \forall k \in I\!N \Big\}.$$

Definition 1.1 Let Ω be a nonempty subset of a Banach space X.

(i) Given $\bar{x} \in \Omega$ and $\varepsilon \geq 0$, we define the set of *ε-normals* to Ω at $\bar{x}$ by

$$\widehat{N}_\varepsilon(\bar{x}; \Omega) := \Big\{ x^* \in X^* \Big| \ \limsup_{x \overset{\Omega}{\to} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\| x - \bar{x} \|} \leq \varepsilon \Big\}.$$

When $\varepsilon = 0$, $\widehat{N}(\bar{x}; \Omega) := \widehat{N}_0(\bar{x}; \Omega)$ is the *Fréchet normal cone* to Ω at $\bar{x}$.

(ii) The *limiting normal cone* to Ω at $\bar{x} \in \Omega$ is the set

$$N(\bar{x}; \Omega) := \operatorname*{Limsup}_{x\to\bar{x},\ \varepsilon\downarrow 0} \widehat{N}_\varepsilon(x; \Omega).$$

If $\bar{x} \notin \Omega$, we put $\widehat{N}_\varepsilon(\bar{x}; \Omega) = \emptyset$ for all $\varepsilon \geq 0$, and put $N(\bar{x}; \Omega) = \emptyset$.

Let Ω be a subset of a Banach space X and $\bar{x} \in \Omega$. The *contingent cone* to Ω at $\bar{x}$ is the set

$$T(\bar{x};\Omega) := \operatorname*{Limsup}_{t\downarrow 0} \frac{\Omega - \bar{x}}{t}.$$

1.2 Coderivatives and Subdifferential

Definition 1.2 Let $F : X \rightrightarrows Y$ be a multifunction between Banach spaces X and Y.

(i) For any $(\bar{x},\bar{y}) \in X \times Y$ and $\varepsilon \geq 0$, *ε-coderivative* of F at $(\bar{x},\bar{y})$ is the multifunction $\widehat{D}^*_\varepsilon F(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$ defined by

$$\widehat{D}^*_\varepsilon F(\bar{x},\bar{y})(y^*) = \Big\{x^* \in X^* \big|\ (x^*,-y^*) \in \widehat{N}_\varepsilon\big((\bar{x},\bar{y}); \mathrm{gph}F\big)\Big\},\ \forall y^* \in Y^*.$$

The *Fréchet coderivative* of F at $(\bar{x},\bar{y})$ is the map $\widehat{D}^*F(\bar{x},\bar{y}) := \widehat{D}^*_0 F(\bar{x},\bar{y})$.

(ii) The *Mordukhovich coderivative* of F at $(\bar{x},\bar{y}) \in \mathrm{gph}F$ is the multifunction $D^*F(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$ given by

$$D^*F(\bar{x},\bar{y})(\bar{y}^*) = \operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y})\\ y^* \xrightarrow{w^*} \bar{y}^*,\ \varepsilon\downarrow 0}} \widehat{D}^*_\varepsilon F(x,y)(y^*).$$

If $(\bar{x},\bar{y}) \notin \mathrm{gph}F$, we put $D^*F(\bar{x},\bar{y})(y^*) = \emptyset$ for all $y^* \in Y^*$.

Let $\varphi : X \to \overline{I\!R}$ be an extended-real-valued function defined on a Banach space X. If $\varphi(x) > -\infty$ for all $x \in X$ and $\mathrm{dom}\varphi := \{x \in X|\ \varphi(x) < \infty\} \neq \emptyset$, then φ is said to be a *proper* function. To φ we associate the *epigraph* $\mathrm{epi}\varphi := \{(x,\alpha) \in X \times I\!R|\ \alpha \geq \varphi(x)\}$.

Definition 1.3 Let $\varphi : X \to \overline{I\!R}$ be finite at $\bar{x} \in X$.

(i) The *limiting subdifferential* of φ at $\bar{x}$ is the set

$$\partial\varphi(\bar{x}) := \big\{x^* \in X^*|\ (x^*,-1) \in N\big((\bar{x},\varphi(\bar{x})); \mathrm{epi}\varphi\big)\big\}.$$

When $\varphi(\bar{x}) = \infty$, one puts $\partial\varphi(\bar{x}) = \emptyset$.

(ii) For any $\bar{y} \in \partial\varphi(\bar{x})$, the mapping $\partial^2\varphi(\bar{x},\bar{y}) : X^{**} \rightrightarrows X^*$ with the values

$$\partial^2\varphi(\bar{x},\bar{y})(u) := (D^*\partial\varphi)(\bar{x},\bar{y})(u),\ \forall u \in X^{**},$$

is called the *limiting second-order subdifferential* of φ at $\bar{x}$ relative to $\bar{y}$.

The *indicator function* of Ω is the function $\delta(\cdot\,;\Omega) : X \to \overline{I\!R}$ defined by $\delta(x;\Omega) = 0$ if $x \in \Omega$ and $\delta(x;\Omega) = \infty$ if $x \notin \Omega$. If $F : X \rightrightarrows X^*$ given by $F(x) = N(x;\Omega)$ for all $x \in X$ and $(\bar{x}, \bar{x}^*) \in \mathrm{gph}F$, then we have

$$D^*F(\bar{x}, \bar{x}^*)(u) = \big(D^*\partial\delta(\cdot\,;\Omega)\big)(\bar{x}, \bar{x}^*)(u) = \partial^2\delta(\cdot\,;\Omega)(\bar{x}, \bar{x}^*)(u),\ \forall u \in X^{**}.$$

Thus the problem of computing the limiting second-order subdifferential of $\delta(\cdot\,;\Omega)$ reduces to that of computing coderivatives of $F(\cdot) = N(\cdot\,;\Omega)$.

1.3 Lipschitzian Properties and Metric Regularity

Let $F : X \rightrightarrows Y$ be a multifunction between Banach spaces and $(\bar{x}, \bar{y}) \in \mathrm{gph}F$.

Definition 1.4 F is *locally Lipschitz-like* around $(\bar{x}, \bar{y})$ with modulus $\ell \geq 0$ if there are neighborhoods U of $\bar{x}$, V of $\bar{y}$ such that

$$F(x) \cap V \subset F(u) + \ell\|x - u\|\bar{B}_Y,\ \ \forall x, u \in U.$$

Definition 1.5 F is *locally metrically regular* around $(\bar{x}, \bar{y})$ with modulus $\mu > 0$ if there are neighborhoods U of $\bar{x}$, V of $\bar{y}$, and $\gamma > 0$ such that

$$\mathrm{dist}(x; F^{-1}(y)) \leq \mu\,\mathrm{dist}(y; F(x))$$

for all $x \in U$ and $y \in V$ satisfying $\mathrm{dist}(y; F(x)) \leq \gamma$.

Theorem 1.1 (Modukhovich criterion for local Lipschitz-like property) *Let $F : X \rightrightarrows Y$ be a multifunction between finite dimensional spaces with its graph being locally closed around $(\bar{x}, \bar{y}) \in \mathrm{gph}F$. Then the following are equivalent:*

(i) *F is locally Lipschitz-like around $(\bar{x}, \bar{y})$.*

(ii) $D^*F(\bar{x}, \bar{y})(0) = \{0\}$.

Theorem 1.2 (Modukhovich criterion for metric regularity) *Let $F : X \rightrightarrows Y$ be a multifunction between finite dimensional spaces with its graph being locally closed around $(\bar{x}, \bar{y}) \in \mathrm{gph}F$. Then the following are equivalent:*

(i) *F is locally metrically regular around $(\bar{x}, \bar{y})$.*

(ii) $D^*F^{-1}(\bar{y}, \bar{x})(0) = \{0\}$.

Chapter 2

Linear Perturbations of Polyhedral Normal Cone Mappings

In this chapter, we differentiate the normal cone mappings to linearly perturbed polyhedral convex sets and apply the results to solution stability of affine variational inequalities. We will answer two open questions stated by Yao and Yen (2009a). This chapter is written on the basis of the results in [1], [2], and [3].

2.1 The Normal Cone Mapping $\mathcal{F}(x, b)$

Let X be a Banach space with its dual X^* and $T = \{1, 2, \ldots, m\}$ be an index set. Consider a vector system $\{\mathfrak{a}_i^* \in X^* |\ i \in T\}$, and a polyhedral convex set

$$\Theta(b) = \big\{x \in X |\ \langle \mathfrak{a}_i^*, x\rangle \le b_i,\ \forall i \in T\big\}$$

depending on $b = (b_1, \ldots, b_m) \in I\!\!R^m$. For every pair $(x, b) \in X \times I\!\!R^m$, we call

$$I(x, b) = \big\{i \in T |\ \langle \mathfrak{a}_i^*, x\rangle = b_i\big\}$$

the *active index set* of $\Theta(b)$ at x. For any $I \subset T$, put $\bar{I} = T \backslash I$. By b_I we denote the vector with the components b_i where $i \in I$. We will write $b_I \le 0$ (resp., $b_I \ge 0$, $b_I = 0$) if $b_i \le 0$ (resp., $b_i \ge 0$, $b_i = 0$) for all $i \in I$.

The multifunction $\mathcal{F} : X \times I\!\!R^m \rightrightarrows X^*$ defined by setting

$$\mathcal{F}(x, b) = N(x; \Theta(b)),\ \ \forall (x, b) \in X \times I\!\!R^m,$$

is said to be the *linearly perturbed polyhedral normal cone mapping* to the perturbed polyhedron $\Theta(b)$. Following Nam (2010), we have

$$\mathcal{F}(x,b) = \text{pos}\big\{\mathfrak{a}_i^*\big|\ i \in I(x,b)\big\},\ \ \forall (x,b) \in X \times I\!R^m.$$

2.2 The Fréchet Coderivative of $\mathcal{F}(x,b)$

Given $(x,b,x^*) \in \text{gph}\mathcal{F}$, we will write I for $I(x,b)$. We define

$$\Xi(x,b,x^*) = \Big\{(\lambda_i)_{i\in I} \in I\!R^{|I|}\Big|\ x^* = \sum_{i\in I}\lambda_i\mathfrak{a}_i^*,\ \ \lambda_i \geq 0\ \ \forall i \in I\Big\},$$

$$I_1(x,b,x^*) = \Big\{i \in I\Big|\ \lambda_i = 0 \text{ for some } (\lambda_j)_{j\in I} \in \Xi(x,b,x^*)\Big\},$$

$$H(x,b,x^*) = \Big\{(x^*,b^*,v)\Big|\,(x^*,v) \in \big(T(x;\Theta(b)) \cap \{x^*\}^\perp\big)^* \times T(x;\Theta(b)) \cap \{x^*\}^\perp,$$
$$x^* = -\textstyle\sum_{i\in I} b_i^*\mathfrak{a}_i^*,\ b_{\bar I}^* = 0,\ b_{I_1}^* \leq 0\Big\},$$

where $I_1 := I_1(x,b,x^*)$ and $b^* = (b_1^*,\ldots,b_m^*) \in I\!R^m$.

Theorem 2.1 *For any* $(\bar x,\bar b,\bar x^*) \in \text{gph}\mathcal{F}$, *we have*

$$\widehat{N}\big((\bar x,\bar b,\bar x^*);\text{gph}\mathcal{F}\big) = H(\bar x,\bar b,\bar x^*). \tag{2.1}$$

Theorem 2.2 *The Fréchet coderivative* $\widehat{D}^*\mathcal{F}(\bar x,\bar b,\bar x^*) : X^{**} \rightrightarrows X^* \times I\!R^m$ *of* $\mathcal{F}(\cdot)$ *at* $(\bar x,\bar b,\bar x^*) \in \text{gph}\mathcal{F}$ *is computed by*

$$\widehat{D}^*\mathcal{F}(\bar x,\bar b,\bar x^*)(v) = \Big\{(x^*,b^*) \in X^* \times I\!R^m\Big|\ x^* = -\textstyle\sum_{i\in I} b_i^*\mathfrak{a}_i^*,\ b_{\bar I}^* = 0,\ b_{I_1}^* \leq 0,$$
$$(x^*,-v) \in \big(T(\bar x;\Theta(\bar b)) \cap \{\bar x^*\}^\perp\big)^* \times \big(T(\bar x;\Theta(\bar b)) \cap \{\bar x^*\}^\perp\big)\Big\},\ \forall v \in X^{**},$$

where $I := I(\bar x,\bar b)$ *and* $I_1 := I_1(\bar x,\bar b,\bar x^*)$.

2.3 The Mordukhovich Coderivative of $\mathcal{F}(x,b)$

Following Henrion, Mordukhovich, and Nam (2010), for any sets P, Q with $P \subset Q \subset T$, we put

$$\mathcal{A}_{Q,P} = \text{span}\{\mathfrak{a}_i^*|\ i \in P\} + \text{pos}\{\mathfrak{a}_i^*|\ i \in Q\backslash P\},$$

$$\mathcal{B}_{Q,P} = \Big\{x \in X\Big|\ \langle\mathfrak{a}_i^*,x\rangle = 0\ \ \forall i \in P,\ \langle\mathfrak{a}_i^*,x\rangle \leq 0\ \ \forall i \in Q\backslash P\Big\}.$$

For each $(x,b,x^*)\in \mathrm{gph}\mathcal{F}$, we put

$$\mathcal{I}(x,b,x^*)=\Big\{P\subset I(x,b)\big|\ P\neq\emptyset,\ x^*\in\mathrm{pos}\{\mathfrak{a}_i^*|\ i\in P\}\Big\},$$

$$\mathcal{J}(x,b,x^*)=\Big\{P\in\mathcal{I}(x,b,x^*)\big|\ \mathfrak{a}_i^*,\ i\in P,\ \text{are linearly independent}\Big\},$$

$$\widehat{\mathcal{I}}(x,b,x^*)=\begin{cases}\mathcal{J}(x,b,x^*) & \text{if } x^*\neq 0\\ \mathcal{J}(x,b,x^*)\cup\{\emptyset\} & \text{if } x^*=0.\end{cases}$$

For every $Q\subset T$, we define a *pseudo-face* of $\Theta(b)$ by putting

$$\mathfrak{F}_Q(b)=\Big\{x\in X\big|\ \langle\mathfrak{a}_i^*,x\rangle=b_i\ \ \forall i\in Q,\ \langle\mathfrak{a}_i^*,x\rangle<b_i\ \ \forall i\in T\backslash Q\Big\}.$$

Now, let $(x,b,x^*)\in\mathrm{gph}\mathcal{F}$, $I=I(x,b)$, $J=I\backslash I_1(x,b,x^*)$, $\mathcal{I}=\mathcal{I}(x,b,x^*)$, and $\widehat{\mathcal{I}}=\widehat{\mathcal{I}}(x,b,x^*)$. Define

$$\begin{aligned}\Sigma(x,b,x^*)=\bigcup_{P\subset Q\subset I,\,P\in\widehat{\mathcal{I}}}\Big\{(x^*,b^*,v)\big|\ &(x^*,v)\in\mathcal{A}_{Q,P}\times\mathcal{B}_{Q,P},\\ &x^*=-\sum_{i\in Q}b_i^*\mathfrak{a}_i^*,\ b^*_{\overline{Q}}=0,\ b^*_{Q\backslash P}\leq 0\Big\},\end{aligned}\tag{2.2}$$

$$\begin{aligned}\Sigma_0(x,b,x^*)=\bigcup_{\substack{P\subset Q\subset I,\,P\in\mathcal{I}\\ \mathfrak{F}_Q(b)\neq\emptyset}}\Big\{(x^*,b^*,v)\big|\ &(x^*,v)\in\mathcal{A}_{Q,P}\times\mathcal{B}_{Q,P},\\ &x^*=-\sum_{i\in Q}b_i^*\mathfrak{a}_i^*,\ b^*_{\overline{Q}}=0,\ b^*_{Q\backslash J}\leq 0\Big\}.\end{aligned}\tag{2.3}$$

Theorem 2.3 *For any* $(\bar{x},\bar{b},\bar{x}^*)\in\mathrm{gph}\mathcal{F}$, *the estimates*

$$\Sigma_0(\bar{x},\bar{b},\bar{x}^*)\subset N\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)\subset\Sigma(\bar{x},\bar{b},\bar{x}^*),\tag{2.4}$$

where $\Sigma(\bar{x},\bar{b},\bar{x}^*)$ *and* $\Sigma_0(\bar{x},\bar{b},\bar{x}^*)$ *are given respectively by* (2.2) *and* (2.3), *hold. Besides, if* $\bar{x}^*\neq 0$, *then*

$$\widehat{N}\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)\subset\Sigma_0(\bar{x},\bar{b},\bar{x}^*)\subset N\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big).\tag{2.5}$$

For $(x,b,x^*)\in\mathrm{gph}\mathcal{F}$, from Theorem 2.3 we infer that

$$\mathbf{\Omega}_0(x,b,x^*)(v)\subset D^*\mathcal{F}(x,b,x^*)(v)\subset\mathbf{\Omega}(x,b,x^*)(v),\ \forall v\in X^{**},$$

where

$$\mathbf{\Omega}(x,b,x^*)(v):=\Big\{(u^*,\eta^*)\in X^*\times I\!R^m\big|\ (u^*,\eta^*,-v)\in\Sigma(x,b,x^*)\Big\},$$

$$\mathbf{\Omega}_0(x,b,x^*)(v):=\Big\{(u^*,\eta^*)\in X^*\times I\!R^m\big|\ (u^*,\eta^*,-v)\in\Sigma_0(x,b,x^*)\Big\}.$$

2.4 AVIs under Linear Perturbations

Let $X = I\!R^n$ and consider $S : I\!R^m \times I\!R^n \rightrightarrows I\!R^n$ defined by

$$S(b, q) = \{x \in I\!R^n |\; q \in Mx + \mathcal{F}(x, b)\}, \tag{2.6}$$

where $M \in I\!R^{n\times n}$ is fixed, $(b, q) \in I\!R^m \times I\!R^n$ are parameters. Note that $S(b, q)$ can be rewritten as the solution set of a parametric *affine variational inequality* (AVI):

$$S(b, q) = \{x \in \Theta(b) |\; \langle Mx - q, y - x\rangle \geq 0,\ \forall y \in \Theta(b)\}.$$

Let $(\bar{b}, \bar{q}, \bar{x}) \in \mathrm{gph}S$. Clearly, $(\bar{x}, \bar{b}, \bar{x}^*) \in \mathrm{gph}\mathcal{F}$ with $\bar{x}^* := \bar{q} - M\bar{x}$. For every $x' \in I\!R^n$, we define the sets

$$\widehat{\mathbf{K}}_{M,\bar{q}}(x') = \bigcup_{v' \in I\!R^n} \Big\{(b', q') \in I\!R^{m+n} \Big|\; (-x', b', q') \in M^\top v' \times \{0_{I\!R^{m+n}}\} - \{0_{I\!R^{n+m}}\} \times \{v'\} + \widehat{D}^*\mathcal{F}(\bar{x}, \bar{b}, \bar{x}^*)(v') \times \{0_{I\!R^n}\}\Big\},$$

$$\mathbf{L}_{M,\bar{q}}(x') = \bigcup_{v' \in I\!R^n} \Big\{(b', q') \in I\!R^{m+n} \Big|\; (-x', b', q') \in M^\top v' \times \{0_{I\!R^{m+n}}\} - \{0_{I\!R^{n+m}}\} \times \{v'\} + \mathbf{\Omega}(\bar{x}, \bar{b}, \bar{x}^*)(v') \times \{0_{I\!R^n}\}\Big\}.$$

Theorem 2.4 *The following assertions hold*

(i) *If $S(\cdot)$ is locally metrically regular around $(\bar{b}, \bar{q}, \bar{x})$, then* $\ker\widehat{\mathbf{K}}_{M,\bar{q}} = \{0\}$.

(ii) *If* $\ker\mathbf{L}_{M,\bar{q}} = \{0\}$, *then $S(\cdot)$ is locally metrically regular around $(\bar{b}, \bar{q}, \bar{x})$.*

(iii) *If $\mathcal{F}(\cdot)$ is graphically regular at $(\bar{x}, \bar{b}, \bar{x}^*)$, then $S(\cdot)$ is locally metrically regular around $(\bar{b}, \bar{q}, \bar{x})$ if and only if* $\ker\widehat{\mathbf{K}}_{M,\bar{q}} = \{0\}$.

Theorem 2.5 *The following assertions are valid*

(i) *If $S(\cdot)$ is locally Lipschitz-like around $(\bar{b}, \bar{q}, \bar{x})$, then $\widehat{\mathbf{K}}_{M,\bar{q}}(0) = \{0\}$.*

(ii) *If $\mathbf{L}_{M,\bar{q}}(0) = \{0\}$, then $S(\cdot)$ is locally Lipschitz-like around $(\bar{b}, \bar{q}, \bar{x})$.*

(iii) *If $\mathcal{F}(\cdot)$ is graphically regular at $(\bar{x}, \bar{b}, \bar{x}^*)$, then $S(\cdot)$ is locally Lipschitz-like around $(\bar{b}, \bar{q}, \bar{x})$ if and only if $\widehat{\mathbf{K}}_{M,\bar{q}}(0) = \{0\}$.*

Chapter 3

Nonlinear Perturbations of Polyhedral Normal Cone Mappings

This chapter is devoted to the estimation of the Fréchet and the limiting normal cones to the graphs of the normal cone mappings to nonlinearly perturbed polyhedral convex sets in finite dimensional spaces. The obtained estimates are applied to solution stability of affine variational inequalities under nonlinear perturbations. The presentation given below comes from the results in [4].

3.1 The Normal Cone Mapping $\mathcal{F}(x, A, b)$

For every $(A, b) \in I\!R^{m\times n} \times I\!R^m$, consider a polyhedral convex set

$$\Theta(A, b) = \big\{x \in I\!R^n |\ Ax \le b\big\},$$

where A and b are parameters. Let $T = \{1, 2, \ldots, m\}$ be a fixed index set. For $(x, A, b) \in I\!R^n \times I\!R^{m\times n} \times I\!R^m$ with $A = (a_{ij})_{m\times n}$, $b = (b_1, \ldots, b_m)$, we call

$$I(x, A, b) = \big\{i \in T |\ A_i x = b_i\big\}.$$

the *active index set* of $\Theta(A, b)$ at x. For any $\Gamma := \{i_1, \ldots, i_r\} \subset T$, we denote the column vector

$$a_{\Gamma,j} = \begin{pmatrix} a_{i_1 j} \\ \vdots \\ a_{i_r j} \end{pmatrix} \quad \text{for every } j \in \{1, \ldots, n\}.$$

The multifunction $\mathcal{F} : I\!R^n \times I\!R^{m\times n} \times I\!R^m \rightrightarrows I\!R^n$ given by

$$\mathcal{F}(x, A, b) = N(x; \Theta(A, b)), \ \ \forall (x, A, b) \in I\!R^n \times I\!R^{m\times n} \times I\!R^m, \tag{3.1}$$

is said to be the *nonlinearly perturbed polyhedral normal cone mapping* to the perturbed polyhedron $\Theta(A,b)$.

We discuss solution stability of the parametric affine variational inequality (AVI) problem

$$\text{Find } x \in \Theta(A,b) \quad \text{s.t. } \langle Mx - q, u - x\rangle \geq 0,\ \forall u \in \Theta(A,b), \tag{3.2}$$

where $M \in I\!R^{n\times n}$ is fixed, and $A \in I\!R^{m\times n}$, $b \in I\!R^m$, $q \in I\!R^n$ are subject to change. Let $S(A,b,q)$ be the solution set of (3.2). Then we have

$$S(A,b,q) = \{x \in I\!R^n |\ 0 \in Mx - q + \mathcal{F}(x,A,b)\}, \tag{3.3}$$

where $\mathcal{F}(x,A,b)$ is given by (3.1).

3.2 Estimation of the Fréchet Normal Cone to gph$\mathcal{F}$

For any $(x,A,b,\xi^*) \in \mathrm{gph}\mathcal{F}$, we put

$$\Xi(x,A,b,\xi^*) := \Big\{(\lambda_i)_{i\in I}\Big|\ \xi^* = \sum_{i\in I}\lambda_i A_i^\top,\ \lambda_i \geq 0\ \forall i \in I\Big\},$$

and

$$I_1(x,A,b,\xi^*) := \Big\{i \in I\Big|\ \lambda_i = 0 \text{ for some } (\lambda_j)_{j\in I} \in \Xi(x,A,b,\xi^*)\Big\},$$

where $I := I(x,A,b)$. Using $I_1 := I_1(x,A,b,\xi^*)$, we construct the set

$$\begin{aligned}
&\mathcal{H}(x,A,b,\xi^*)\\
&\quad = \Big\{(x^*,A^*,b^*,\xi)\Big|\ x^* \in \big(T(x;\Theta(A,b)) \cap \{\xi^*\}^\perp\big)^*,\\
&\qquad\qquad x^* = -\textstyle\sum_{i\in I} b_i^* A_i^\top,\ \xi \in T(x;\Theta(A,b)) \cap \{\xi^*\}^\perp,\\
&\qquad\qquad a^*_{I_1,j} \leq 0 \text{ if } x_j < 0,\ a^*_{I_1,j} \geq 0 \text{ if } x_j > 0,\\
&\qquad\qquad a^*_{I_1,j} = 0 \text{ if } x_j = 0,\ A^*_{\bar{I}} = 0,\ b^*_{\bar{I}} = 0,\ b^*_{I_1} \leq 0\Big\},
\end{aligned}$$

where $A^* = (a^*_{ij})_{m\times n} \in I\!R^{m\times n}$ and $b^* = (b_1^* \ldots b_m^*)^\top \in I\!R^m$.

Theorem 3.1 *For any* $(\bar{x},\bar{A},\bar{b},\bar{\xi}^*) \in \mathrm{gph}\mathcal{F}$*, we have*

$$\widehat{N}\big((\bar{x},\bar{A},\bar{b},\bar{\xi}^*);\mathrm{gph}\mathcal{F}\big) \subset \mathcal{H}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*). \tag{3.4}$$

3.3 Estimation of the Limiting Normal Cone to $\mathrm{gph}\mathcal{F}$

Given a matrix $A \in I\!R^{m\times n}$ and subsets P, Q of T satisfying $P \subset Q$, following Henrion, Mordukhovich, and Nam (2010) we put

$$\mathcal{A}_{Q,P}(A) = \mathrm{span}\{A_i^\top \mid i \in P\} + \mathrm{pos}\{A_i^\top \mid i \in Q \setminus P\},$$

$$\mathcal{B}_{Q,P}(A) = \Big\{v \in I\!R^n \Big| \ \langle A_i^\top, v\rangle = 0 \ \forall i \in P,\ \langle A_i^\top, v\rangle \le 0 \ \forall i \in Q \setminus P\Big\}.$$

For each $(x, A, b, \xi^*) \in \mathrm{gph}\mathcal{F}$, we put

$$\mathcal{I}(x, A, b, \xi^*) := \Big\{P \subset I(x, A, b) \Big| \ P \ne \emptyset,\ \xi^* \in \mathrm{pos}\{A_i^\top \mid i \in P\}\Big\},$$

$$\mathcal{J}(x, A, b, \xi^*) := \Big\{P \in \mathcal{I} \Big| \ A_i^\top,\ i \in P,\ \text{are linearly independent}\Big\}$$

with $\mathcal{I} = \mathcal{I}(x, A, b, \xi^*)$, and

$$\widehat{\mathcal{I}}(x, A, b, \xi^*) := \begin{cases} \mathcal{J}(x, A, b, \xi^*), & \text{if } \xi^* \ne 0,\\ \mathcal{J}(x, A, b, \xi^*) \cup \{\emptyset\}, & \text{if } \xi^* = 0. \end{cases}$$

Using the abbreviations $I := I(x, A, b)$ and $\widehat{\mathcal{I}} := \widehat{\mathcal{I}}(x, A, b, \xi^*)$, we define

$$\begin{aligned}\mathbf{\Sigma}(x, A, b, \xi^*) := \bigcup_{P\subset Q\subset I,\, P\in\widehat{\mathcal{I}}} \Big\{(x^*, A^*, b^*, \xi) \Big| \ & (x^*, \xi) \in \mathcal{A}_{Q,P}(A) \times \mathcal{B}_{Q,P}(A),\\ & x^* = -\textstyle\sum_{i\in Q} b_i^* A_i^\top,\\ & b^*_{\overline{Q}} = 0,\ \ b^*_{Q\setminus P} \le 0,\ \ A^*_{\overline{Q}} = 0,\\ & a^*_{Q\setminus P,j} \le 0 \text{ if } x_j < 0,\ \ a^*_{Q\setminus P,j} \ge 0 \text{ if } x_j > 0\Big\}.\end{aligned}$$

Vectors $\{v_j\}_{j\in J}$ are called *positively linearly independent* if from conditions $\sum_{j\in J} \lambda_j v_j = 0$ and $\lambda_j \ge 0$ for all $j \in J$ it follows that $\lambda_j = 0$ for all $j \in J$.

Theorem 3.2 *Let $(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) \in \mathrm{gph}\mathcal{F}$ and let $I = I(\bar{x}, \bar{A}, \bar{b})$. If the vectors $\{\bar{A}_i^\top \mid i \in I\}$ are positively linearly independent, then*

$$N\big((\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*); \mathrm{gph}\mathcal{F}\big) \subset \mathbf{\Sigma}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*). \tag{3.5}$$

By Theorem 3.2, on setting

$$\mathbf{\Lambda}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi) = \Big\{(x^*, A^*, b^*) \Big| \ (x^*, A^*, b^*, -\xi) \in \mathbf{\Sigma}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)\Big\} \tag{3.6}$$

for every $\xi \in I\!R^n$ and recalling that

$$D^*\mathcal{F}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi) = \Big\{(x^*, A^*, b^*) \Big| \ (x^*, A^*, b^*, -\xi) \in N\big((\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*); \mathrm{gph}\mathcal{F}\big)\Big\},$$

we have

$$D^*\mathcal{F}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi) \subset \mathbf{\Lambda}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi),\ \forall \xi \in I\!R^n.$$

3.4 AVIs under Nonlinear Perturbations

Consider $S(\cdot)$ given by (3.3). Let $\bar{w} = (\bar{A}, \bar{b}, \bar{q}, \bar{x}) \in \mathrm{gph}S$. It is clear that $(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) \in \mathrm{gph}\mathcal{F}$ with $\bar{\xi}^* := \bar{q} - M\bar{x}$. For each $x^* \in I\!R^n$, we put

$$\begin{aligned}\mathbf{K}(\bar{w})(x^*) = \bigcup_{\xi \in I\!R^n} \Big\{(A^*, b^*, q^*)\Big|\ & (-x^*, A^*, b^*, q^*) \in M^\top \xi \times \{(0_{I\!R^{m\times n}}, 0_{I\!R^m}, 0_{I\!R^n})\} \\ & +\{(0_{I\!R^n}, 0_{I\!R^{m\times n}}, 0_{I\!R^m})\} \times \{-\xi\} \\ & +D^*\mathcal{F}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi) \times \{0_{I\!R^n}\}\Big\},\end{aligned}$$

$$\begin{aligned}\mathbf{L}(\bar{w})(x^*) = \bigcup_{\xi \in I\!R^n} \Big\{(A^*, b^*, q^*)\Big|\ & (-x^*, A^*, b^*, q^*) \in M^\top \xi \times \{(0_{I\!R^{m\times n}}, 0_{I\!R^m}, 0_{I\!R^n})\} \\ & +\{(0_{I\!R^n}, 0_{I\!R^{m\times n}}, 0_{I\!R^m})\} \times \{-\xi\} \\ & +\mathbf{\Lambda}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi) \times \{0_{I\!R^n}\}\Big\},\end{aligned}$$

where $\mathbf{\Lambda}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi)$ is given by (3.6).

Theorem 3.3 *Let $\bar{w} = (\bar{A}, \bar{b}, \bar{q}, \bar{x}) \in \mathrm{gph}S$. If the vectors $\{\bar{A}_i^\top |\ i \in I(\bar{x}, \bar{A}, \bar{b})\}$ are positively linearly independent, then the following assertions are valid:*

(i) *If* $\mathrm{ker}\mathbf{L}(\bar{w}) = \{0\}$*, then $S(\cdot)$ is locally metrically regular around $\bar{w}$.*

(ii) *If $\mathbf{L}(\bar{w})(0) = \{0\}$, then $S(\cdot)$ is locally Lipschitz-like around $\bar{w}$.*

Theorem 3.4 *Let $\bar{w} = (\bar{A}, \bar{b}, \bar{q}, \bar{x}) \in \mathrm{gph}S$. If the vectors $\{\bar{A}_i^\top |\ i \in I(\bar{x}, \bar{A}, \bar{b})\}$ are positively linearly independent, then there exists $\delta > 0$ such that $S(\cdot)$ is locally metrically regular around any point $w \in \bar{B}(\bar{w}, \delta) \cap \mathrm{gph}S$.*

Chapter 4

A Class of Linear Generalized Equations

Solution stability of a class of linear generalized equations in finite dimensional Euclidean spaces is investigated in this chapter by means of generalized differentiation. Since the trust-region subproblems can be regarded as linear generalized equations, the obtained results on stability of linear generalized equations lead to new results on stability of the parametric trust-region subproblems. The two open problems stated by Lee and Yen (2014) are solved. The results presented below are taken from [5].

4.1 Linear Generalized Equations

The concept of *generalized equation* introduced in 1979 by Robinson has been recognized as an efficient tool for dealing with various questions in optimization theory. We consider the linear generalized equations of the form

$$0 \in Ax + b + N(x; E(\alpha)), \tag{4.1}$$

where symmetric $n \times n$ matrix $A \in I\!R^{n\times n}$, vector $b \in I\!R^m$, and real number $\alpha > 0$ are parameters, $E(\alpha) := \{x \in I\!R^n \mid \|x\| \leq \alpha\}$, and $N(x; E(\alpha))$ is the normal cone to $E(\alpha)$ at x. The solution set of (4.1) is denoted by $S(A, b, \alpha)$.

Let $\mathcal{N} : I\!R^n \times I\!R \rightrightarrows I\!R^n$ be defined by $\mathcal{N}(x, \alpha) = N(x; E(\alpha))$ if $\alpha > 0$, and $\mathcal{N}(x, \alpha) = \emptyset$ if $\alpha \leq 0$. Thus $\mathcal{N}(\cdot)$ is a multifunction with closed convex values.

4.2 Formulas for Coderivatives

This section provides exact formulas for the Fréchet and the Mordukhovich coderivatives of $\mathcal{N}(\cdot)$ at every point belonging to gph$\mathcal{N}$ in various cases.

Fix any point $(x, \alpha, v) \in \text{gph}\mathcal{N}$.

Theorem 4.1 *If $\|x\| = \alpha$ and $v \neq 0$, then $v = \mu x$ with $\mu = \|v\| \cdot \|x\|^{-1}$ and*

$$\widehat{D}^*\mathcal{N}(x, \alpha, v)(v') = \begin{cases} \left\{(x', \alpha') \in I\!R^n \times I\!R \,\middle|\, x' = -\frac{\alpha'}{\alpha}x + \mu v'\right\}, & \text{if } \langle v', x\rangle = 0 \\ \emptyset, & \text{if } \langle v', x\rangle \neq 0 \end{cases}$$

for every $v' \in I\!R^n$.

Theorem 4.2 *If $\|x\| = \alpha$ and $v = 0$, then*

$$\widehat{D}^*\mathcal{N}(x, \alpha, v)(v') = \begin{cases} \left\{(x', \alpha') \in I\!R^n \times I\!R \,\middle|\, x' = -\frac{\alpha'}{\alpha}x,\ \alpha' \leq 0\right\}, & \text{if } \langle v', x\rangle \geq 0 \\ \emptyset, & \text{if } \langle v', x\rangle < 0, \end{cases}$$

for every $v' \in I\!R^n$.

Lemma 4.1 (Lee and Yen (2014)) *If $\|x\| < \alpha$, then $v = 0$ and*

$$D^*\mathcal{N}(x, \alpha, v)(v') = \widehat{D}^*\mathcal{N}(x, \alpha, v)(v') = \{(0_{I\!R^n}, 0_{I\!R})\},\ \forall v' \in I\!R^n.$$

Theorem 4.3 *If $\|x\| = \alpha$ and if $v \neq 0$, then we have*

$$\begin{aligned} D^*\mathcal{N}(x, \alpha, v)(v') &= \widehat{D}^*\mathcal{N}(x, \alpha, v)(v') \\ &= \begin{cases} \left\{(x', \alpha') \in I\!R^n \times I\!R \,\middle|\, x' = -\frac{\alpha'}{\alpha}x + \mu v'\right\}, & \text{if } \langle v', x\rangle = 0 \\ \emptyset, & \text{if } \langle v', x\rangle \neq 0 \end{cases} \end{aligned}$$

for every $v' \in I\!R^n$, where $\mu := \|v\| \cdot \|x\|^{-1}$.

Theorem 4.4 *Suppose that $\|x\| = \alpha$ and $v = 0$. For every $v' \in I\!R^n$, the following hold*

(i) *If $\langle v', x\rangle \neq 0$, then*

$$D^*\mathcal{N}(x, \alpha, v)(v') = \begin{cases} \left\{(x', \alpha') \in I\!R^{n+1} \,\middle|\, x' = -\frac{\alpha'}{\alpha}x,\ \alpha' \leq 0\right\}, & \text{if } \langle v', x\rangle > 0 \\ \{(0_{I\!R^n}, 0_{I\!R})\}, & \text{if } \langle v', x\rangle < 0. \end{cases}$$

(ii) *If $\langle v', x\rangle = 0$, then*

$$D^*\mathcal{N}(x, \alpha, v)(v') = \left\{(x', \alpha') \in I\!R^n \times I\!R \,\middle|\, x' = -\frac{\alpha'}{\alpha}x,\ \alpha' \in I\!R\right\}.$$

4.3 Necessary and Sufficient Conditions for Stability

Using the coderivative formulas of $\mathcal{N}(\cdot)$, conditions for stability of the solution map $(A, b, \alpha) \mapsto S(A, b, \alpha)$ of (4.1) are obtained in this section.

By Martinez (1994), if $x \in E(\alpha)$ is a local minimum of the problem

$$\min \left\{ f(x) = \frac{1}{2} x^\top A x + b^\top x \,\middle|\, x \in E(\alpha) \right\},$$

which is called the *trust-region subproblem*, then there exists a Lagrange multiplier $\lambda \geq 0$ such that

$$(A + \lambda I)x = -b, \quad \lambda(\|x\| - \alpha) = 0,$$

where I denotes the $n \times n$ unit matrix.

Theorem 4.5 *For any $(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) \in \mathrm{gph} S$, the following assertions hold:*

(i) *If $\|\bar{x}\| < \bar{\alpha}$, then the map $S(\cdot)$ is locally Lipschitz-like around $(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x})$ if and only if $\det \bar{A} \neq 0$.*

(ii) *If $\|\bar{x}\| = \bar{\alpha}$ and $\bar{A}\bar{x} + \bar{b} \neq 0$, then $S(\cdot)$ is locally Lipschitz-like around $(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x})$ if and only if $\det Q(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) \neq 0$, where*

$$Q(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) := \begin{pmatrix} \bar{A} + \mu I & -\frac{1}{\bar{\alpha}}\bar{x} \\ \bar{x}^\top & 0 \end{pmatrix} \tag{4.2}$$

with μ being the unique Lagrange multiplier associated to $\bar{x}$.

Theorem 4.6 *Let $(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) \in \mathrm{gph} S$ be such that $\|\bar{x}\| = \bar{\alpha}$ and $\bar{A}\bar{x} + \bar{b} = 0$. Then, the following hold*

(i) *If $S(\cdot)$ is locally Lipschitz-like around $(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x})$, then the constraint qualification below is satisfied*

$$\begin{cases} \bar{A}v' - \frac{\alpha'}{\bar{\alpha}}\bar{x} = 0 \\ \langle v', \bar{x} \rangle \geq 0 \\ v' \in I\!R^n, \ \alpha' \leq 0 \end{cases} \implies \begin{cases} v' = 0 \\ \alpha' = 0. \end{cases} \tag{4.3}$$

(ii) *If $\det \bar{A} \neq 0$, $\det Q_1(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) \neq 0$, where*

$$Q_1(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) := \begin{pmatrix} \bar{A} & -\frac{1}{\bar{\alpha}}\bar{x} \\ \bar{x}^\top & 0 \end{pmatrix}, \tag{4.4}$$

and (4.3) is satisfied, then $S(\cdot)$ is locally Lipschitz-like around $(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x})$.

General Conclusions

The main results of this dissertation include:

1. An exact formula for the Fréchet coderivative and some upper and lower estimates for the Mordukhovich coderivative of the normal cone mappings to linearly perturbed polyhedral convex sets in reflexive Banach spaces.
2. Upper estimates for the Fréchet and the limiting normal cone to the graphs of the normal cone mappings to nonlinearly perturbed polyhedral convex sets in finite dimensional spaces.
3. Exact formulas for the Fréchet and the Mordukhovich coderivatives of the normal cone mappings to perturbed Euclidean balls.
4. Conditions for the local Lipschitz-like property and local metric regularity of the solution maps of parametric affine variational inequalities under linear/nonlinear perturbations, and conditions for the local Lipschitz-like property of the solution maps of a class of linear generalized equations in finite dimensional spaces.

References

[1] N. T. Qui, *Linearly perturbed polyhedral normal cone mappings and applications*, Nonlinear Anal., **74** (2011), pp. 1676–1689.

[2] N. T. Qui, *New results on linearly perturbed polyhedral normal cone mappings*, J. Math. Anal. Appl., **381** (2011), pp. 352–364.

[3] N. T. Qui, *Upper and lower estimates for a Fréchet normal cone*, Acta Math. Vietnam., **36** (2011), pp. 601–610.

[4] N. T. Qui, *Nonlinear perturbations of polyhedral normal cone mappings and affine variational inequalities*, J. Optim. Theory Appl., **153** (2012), pp. 98–122.

[5] N. T. Qui and N. D. Yen, *A class of linear generalized equations*, SIAM J. Optim., **24** (2014), pp. 210–231.