VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY
INSTITUTE OF MATHEMATICS

TRẦN NHÂN TÂM QUYỀN

Convergence Rates for the Tikhonov Regularization of Coefficient Identification Problems in Elliptic Equations

Dissertation submitted in partial fulfillment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY IN MATHEMATICS

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Supervisor: Prof. Dr. habil. **ĐINH NHO HÀO**

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TỐC ĐỘ HỘI TỤ CỦA PHƯƠNG PHÁP CHỈNH TIKHONOV CHO CÁC BÀI TOÁN XÁC ĐỊNH HỆ SỐ TRONG PHƯƠNG TRÌNH ELLIPTIC

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This work has been completed at Institute of Mathematics, Vietnam Academy of Science and Technology under the supervision of Prof. Dr. habil. Đinh Nho Hào. I declare hereby that the results presented in it are new and have never been published elsewhere.

Author: Trần Nhân Tâm Quyền

Convergence Rates for the Tikhonov Regularization of Coefficient Identification Problems in Elliptic Equations

By
TRẦN NHÂN TÂM QUYỀN

ABSTRACT

Let Ω be an open bounded connected domain in $\mathbb{R}^d, d \geq 1$, with the Lipschitz boundary $\partial\Omega$, $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ be given. In this work we investigate convergence rates for the Tikhonov regularization of the ill-posed nonlinear inverse problems of identifying the diffusion coefficient q in the Neumann problem for the elliptic equation

$$-\mathrm{div}(q\nabla u) = f \text{ in } \Omega,$$
$$q\frac{\partial u}{\partial n} = g \text{ on } \partial\Omega,$$

and the reaction coefficient a in the Neumann problem for the elliptic equation

$$-\Delta u + au = f \text{ in } \Omega,$$
$$\frac{\partial u}{\partial n} = g \text{ on } \partial\Omega,$$

from imprecise values $z^\delta \in H^1(\Omega)$ of the exact solution u with $\|u - z^\delta\|_{H^1(\Omega)} \leq \delta$. The Tikhonov regularization is applied to convex energy functionals to stabilize these ill-posed nonlinear problems. Under weak source conditions without the smallness requirements on the source functions, we obtain convergence rates of the method.

Tốc độ hội tụ của phương pháp chỉnh Tikhonov cho các bài toán xác định hệ số trong phương trình elliptic

TÁC GIẢ
TRẦN NHÂN TÂM QUYỀN

TÓM TẮT

Giả sử Ω là một miền liên thông, mở và bị chặn trong $\mathbb{R}^d, d \geq 1$, với biên Lipschitz $\partial\Omega$ và các hàm $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ cho trước. Luận án nghiên cứu các bài toán ngược phi tuyến đặt không chỉnh xác định hệ số truyền tải q trong bài toán Neumann cho phương trình elliptic

$$-\mathrm{div}(q\nabla u) = f \text{ trong } \Omega,$$
$$q\frac{\partial u}{\partial n} = g \text{ trên } \partial\Omega$$

và hệ số phản ứng a trong bài toán Neumann cho phương trình elliptic

$$-\Delta u + au = f \text{ trong } \Omega,$$
$$\frac{\partial u}{\partial n} = g \text{ trên } \partial\Omega$$

khi nghiệm chính xác u được cho không chính xác bởi dữ kiện đo đạc $z^\delta \in H^1(\Omega)$ với $\|u - z^\delta\|_{H^1(\Omega)} \leq \delta$. Phương pháp chỉnh Tikhonov cho hai bài toán trên được áp dụng cho các phiếm hàm năng lượng lồi. Với điều kiện nguồn yếu không đòi hỏi tính đủ nhỏ của các hàm nguồn, ta thu được các đánh giá về tốc độ hội tụ của phương pháp chỉnh Tikhonov.

Contents

Function Spaces

$\mathbb{R}^d$	The d-dimensional Euclidean space
Ω	Open, bounded set with the Lipschitz boundary in $\mathbb{R}^d$
$C^k(\Omega)$	The set of k times continuously differential functions on Ω, $1 \le k \le \infty$
$C^\infty(\overline{\Omega})$	The set of infinitely differential functions on $\overline{\Omega}$
$C_c^k(\Omega)$	The set of functions in $C^k(\Omega)$ with compact support in Ω, $1 \le k \le \infty$
$L^p(\Omega)$	The Lebesgue space on Ω, $1 \le p \le \infty$
$W^{k,p}(\Omega)$	The Sobolev space of functions with k-th order weak derivatives in $L^p(\Omega)$
$W_0^{k,p}(\Omega)$	Closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$
$H^k(\Omega),\ H_0^k(\Omega)$	Abbreviations for the Hilbert spaces $W^{k,2}(\Omega)$, $W_0^{k,2}(\Omega)$
$H^{-k}(\Omega)$	Dual space of $H_0^k(\Omega)$
$BV(\Omega)$	Space of functions with bounded total variation on Ω, pp. 22

Notation

$\lvert x\rvert_{\ell^p}$	ℓ^p-norm of $x \in \mathbb{R}^d, 1 \le p \le \infty$
$\lVert u\rVert_X$	Norm of u in the normed space X
X^*	Dual space of the normed space X
$\langle u^*, u\rangle_{(X^*,X)}$	Duality product $u^*(u)$ of $u \in X$ and $u^* \in X^*$
$\langle u, v\rangle_H$	Inner product of u, v in the Hilbert space H
$\mathcal{L}(X,Y)$	Space of bounded linear operators between normed spaces X and Y
T^*	Adjoint in $\mathcal{L}(Y^*, X^*)$ of $T \in \mathcal{L}(X,Y)$
∇v	Gradient of the scalar function v
Δv	The Laplacian of the scalar function v
$\mathrm{div}\Upsilon$	Divergence of the vector-valued function Υ
$\partial R(q)$	Subdifferential of the proper convex functional R at $q \in \mathrm{Dom}R$, pp. 21
$D_R^\xi(p,q)$	The Bregman distance with respect to R and ξ of two elements p, q, pp. 21
$\int_\Omega \lvert\nabla v\rvert$	Total variation of the scalar function v, pp. 22 or seminorm in $W^{1,1}(\Omega)$
$\overline{u}$	Exact data, pp. 28, 29, 36
z^δ	Observed data, pp. 28, 29, 36
$Q,\ A,\ Q_{ad},\ A_{ad}$	Admissible sets of coefficients, 29, 35, 64, 78, 90, 101
$\underline{q},\ \overline{q},\ \underline{a},\ \overline{a}$	Positive constants, pp. 29, 35
$C_\Omega,\ \alpha,\ \beta,\ \Lambda_\alpha,\ \Lambda_\beta$	Positive constants, pp. 29, 36
$U(q),\ U(a)$	Coefficient-to-solution operators, pp. 29, 36
$J_{z^\delta}(q),\ G_{z^\delta}(a)$	Energy functionals, pp. 29, 36
ρ	Regularization parameter, pp. 42, 55, 64, 78, 90, 101
$q^*,\ a^*$	A-priori estimates of the true coefficients, pp. 42, 55
$q^\dagger,\ a^\dagger$	q^*-, a^*-solutions of the inverse problems, pp. 42, 56, 66, 80, 91, 101
$q_\rho^\delta,\ a_\rho^\delta$	Regularized solutions, pp. 43, 56, 66, 79, 91, 101
$\mathfrak{X}$	The space $L^\infty(\Omega) \cap BV(\Omega)$ with the norm $\lVert q\rVert_{L^\infty(\Omega)} + \lVert q\rVert_{BV(\Omega)}$, pp. 66
$\mathfrak{X}_{BV(\Omega)}$	The space $\mathfrak{X}$ with respect to the $BV(\Omega)$-norm, pp. 67
$\mathfrak{X}_{L^\infty(\Omega)}$	The space $\mathfrak{X}$ with respect to the $L^\infty(\Omega)$-norm, pp. 67
$H_\diamond^1(\Omega)$	Space of functions in $H^1(\Omega)$ with mean-zero, pp. 28

Introduction

The problem of identifying parameters in distributed parameter systems arising in groundwater hydrology, heat conduction, population models, seismic exploration and reservoir simulation attracted great attention from many scientists in the last 50 years or so. For surveys on the subject, we refer the reader to [4, 14, 19, 33, 34, 35, 37, 38, 40, 45, 46, 47, 49, 55, 58, 59, 60, 62, 61, 64, 68, 72, 76, 77, 89, 90, 91, 92, 95, 96, 97, 98, 105, 106, 110, 112, 113, 118, 119, 123, 126, 127, 128, 129, 130] and the references therein. The term "distributed parameter systems" means that the mathematical models in these situations are governed by partial differential equations. In this thesis we are interesting in the problem of identifying coefficients in groundwater hydrology, whose mathematical models contain function-coefficients which describe physical properties of the fluid flows or of the porous media. The identified coefficients appeared in the governing equations are not directly measurable from the physical point of view and have to be determined from historical observations. Such problems are called *inverse problems* which are in general very difficult to solve because of the nonuniqueness and instability (the ill-posedness) of the identified coefficients. The aim of this thesis is to study convergence rates for the Tikhonov regularization of these ill-posed nonlinear problems. Before presenting our results, to ease of reading we shortly describe the mathematical models on fluid flows and porous media.

0.1 Modelling

The governing equation of coefficients in an unsteady fluid flow can be represented by (see, for example, [14, 112])

$$s(t)\frac{\partial u}{\partial t} - \operatorname{div}(q(x)\nabla u) + a(x)u = f(x,t), \quad x \in \Omega \subset \mathbb{R}^d, t > 0 \tag{0.1}$$

accompanied by the initial and boundary conditions

$$\begin{aligned} u(x,0) &= u_0(x), x \in \Omega, \\ u(x,t) &= g_1(x,t), x \in \Gamma_1, t > 0, \\ q(x)\tfrac{\partial u(x,t)}{\partial n} &= g_2(x,t), x \in \Gamma_2, t > 0, \end{aligned} \tag{0.2}$$

where

Ω	flow region,
$\partial\Omega$	boundary of the flow region,
	$\partial\Omega = \Gamma_1 \cup \Gamma_2$, interior$(\Gamma_1) \cap$ interior$(\Gamma_2) = \emptyset$,
x	space variable,
t	time variable,
$u(x,t)$	head,

$q(x)$	diffusion coefficient,
$a(x)$	reaction coefficient,
$s(t)$	storage coefficient,
$f(x,t)$	source-sink term,
$u_0(x), g_1(x,t), g_2(x,t)$	specified functions,
$n = n(x)$	unit outer normal at $x \in \partial\Omega$,
$\frac{\partial}{\partial n}$	normal derivative,
$\frac{\partial}{\partial t}$	time derivative.

In the three-dimensional space the *hydraulic head* u at point (x, y, z) of the flow region Ω is defined by

$$u = u(x, y, z) = \frac{p}{\rho g} + z,$$

where $p = p(x, y, z)$ is fluid pressure, $\rho = \rho(x, y, z)$ is density of the fluid and g is acceleration of gravity.

The steady case of (0.1)–(0.2) is

$$-\mathrm{div}(q(x)\nabla u) + a(x)u = f(x), \quad x \in \Omega \subset \mathbb{R}^d, d \geq 1 \tag{0.3}$$

accompanied by the boundary condition

$$\begin{aligned} u(x) &= g_1(x), x \in \Gamma_1, \\ q(x)\frac{\partial u(x)}{\partial n} &= g_2(x), x \in \Gamma_2. \end{aligned} \tag{0.4}$$

Physically, u can be interpreted as the piezometrical head of the ground water in Ω, the function f characterizes the sources and sinks in Ω and the function g_2 characterizes the inflow and outflow through Γ_2 (see, for example [124]). We say that this boundary value problem is of the mixed type, if neither Γ_1 nor Γ_2 is empty, of the Dirichlet type if $\Gamma_2 = \emptyset$, and of the Neumann type if $\Gamma_1 = \emptyset$.

0.2 Inverse problems and ill-posedness

0.2.1. Inverse problems

The steady system (0.3)–(0.4) contains three known functions f, g_1 and g_2. When the coefficients q and a are given, the problem of uniquely solving u from the partial differential equation (0.3)–(0.4) is called the *forward problem*. Conversely, the problem of identifying the coefficients q and a from observed data of a solution u of (0.3)–(0.4) is called the *inverse problem*.

In this work we are working with the inverse problems for the steady cases of (0.1)–(0.2). Namely, we are concerned with the Neumann problem for (0.3). We investigate convergence rates for the Tikhonov regularization of the problems of identifying the coefficient q in the Neumann problem for the elliptic equation

$$-\mathrm{div}(q\nabla u) = f \text{ in } \Omega, \tag{0.5}$$

$$q\frac{\partial u}{\partial n} = g \text{ on } \partial\Omega \tag{0.6}$$

and the coefficient a in the Neumann problem for the elliptic equation

$$-\Delta u + au = f \text{ in } \Omega, \tag{0.7}$$

$$\frac{\partial u}{\partial n} = g \text{ on } \partial\Omega \tag{0.8}$$

from imprecise values $z^\delta \in H^1(\Omega)$ of a solution u with

$$\|u - z^\delta\|_{H^1(\Omega)} \leq \delta, \tag{0.9}$$

$\delta > 0$ being given, while f and g are prescribed. The problem of simultaneously estimating the coefficients q and a in (0.3) with either the Neumann, or the Dirichlet or mixed boundary condition has been studied in [15, 62, 64, 83, 84, 86, 87]. The functionals q and a in these problems are called the *diffusion* (or *filtration* or *transmissivity*, or *conductivity*) and *reaction* coefficients, respectively. For different kinds of the porous media, the diffusion coefficient varies in a large scale (see, for example, [123])

Gravels	0.1 to 1 cm/sec
Sands	10^{-3} to 10^{-2} cm/sec
Silts	10^{-5} to 10^{-4} cm/sec
Clays	10^{-9} to 10^{-7} cm/sec
Limestone	10^{-4} to 10^{-2} cm/sec.

These problems are mathematical models in different topics of applied sciences, e.g., from aquifer analysis. The coefficient identification problems in groundwater (or oil or other fluids) modeling have a history older than a half century, although we may encounter the first publication on hydraulic well with J. Dupuit in the year 1863. In 1950s and 1960s, analytical solutions were presented to identify the hydraulic conductivity and the storage coefficient around a well through fitting equipotential lines to aquifer testing data. In 1970s, numerical solutions were used to identify hydraulic coefficients by optimization of a "norm" in the state space. During the last four decades many techniques have been developed for solving inverse problems of parameter identification in distributed parameter systems such as hydraulic parameters, boundary conditions, pollution sources, dispersivities, adsorption kinetics, and filtration and reaction coefficients.

In 1973, Neuman [103] classed the techniques for solving inverse problems of identifying coefficients in distributed parameter systems into either "direct" or "indirect". In the "direct method" the head variations and derivatives are assumed to be known or are estimated over the entire flow region and the original governing equations are transformed into linear first–order partial differential equations of the hyperbolic type for the unknown coefficients. With the combined knowledge of the heads and initial and boundary conditions, a direct solutions for the unknown coefficients may be possible. However, in practice, there is only a limited number of observations being available which are sparsely distributed in the flow region. The interpolation is then used to extend these data across the spatial domain, this process may contain serious errors that would cause errors in the results of identified coefficients. The output least-squares techniques have been used to the so–called "indirect methods", which try to minimize a "norm" in the state space of the difference between observed and calculated heads at specified observation points. The main advantage of the output least-squares methods are that the formulation of the inverse problem is applicable to the situation where the number of observations is limited and it does not require the derivatives of the measured data. However, these methods are several shortcomings. First, the object functional to be minimized is nonlinear and nonconvex. Second, the numerical implementation is often iterative that in order to obtain useful results a large number of repeated solution of the forward problem is required. Further, since the minimization being nonlinear and nonconvex, the third major shortcoming of these methods is that the critical dependence on the initial guesses of identified coefficients for rapid convergence of the iterative procedures. With large, poorly–conditioned functionals, the convergence may be slow, pick out only a local minimum, or even fail (see also [14, 45, 96, 113, 118, 127, 129]). One of the aims of this thesis is to overcome these serious shortcoming of the output least-squares method. We will use the energy functionals which are convex, rather than the output least-squares (see more details in § 0.4).

0.2.2. Ill-posedness

Suppose that the coefficient q in (0.5)–(0.6) is given so that we can determine the unique solution u and thus define a nonlinear coefficient-to-solution operator which maps from q to the solution $u = u(q) := U(q)$. Thus, the inverse problem in our setting is to solve the equation

$$U(q) = u$$

for q with u being given, which is the *nonlinear* and *ill-posed* inverse problem.

In 1923, Hadamard [57] introduced the notion of *well-posedness*. A problem is said to be well posed if the following conditions are satisfied

1. Existence: There is a solution of the problem.

2. Uniqueness: There is at most one solution of the problem.

3. Stability: The solution continuously depends on the data (in some appropriate topologies).

If at least one of the above conditions is not satisfied, the problem is said to be *ill-posed* (or *improperly posed*). He thought that such problems have no physical meaning. However, many important problems in practice and science are ill-posed (see [13, 26, 68, 112, 123]), in which the instability always causes serious problems. If a problem lacks the stability, a small error in observed data may lead to significant errors in the solution, that makes numerical solution extremely difficult.

Now we illustrate the ill-posedness of the problem of identifying the coefficient a in (0.7)–(0.8) from u by an example given by Baumeister and Kunisch [15]. Let

$$a(x) = 2, \;\; u(x) = 1$$

and

$$a_n(x) = \frac{2 - \cos(n+1)x}{1 + \frac{1}{(n+1)^2}\cos(n+1)x}, \;\; u_n(x) = \frac{1}{(n+1)^2}\cos(n+1)x + 1, \;\; n \in \mathbb{N}.$$

Then, for all $n \in \mathbb{N}$,

$$-u'' + au = -u''_n + a_n u_n \;\; \text{in } (0, \pi)$$

and

$$u'(0) = u'_n(0), \;\; u'(\pi) = u'_n(\pi).$$

One verifies that for all $n \in \mathbb{N}$,

$$\|u_n - u\|_{H^1(0,\pi)} \le \frac{\sqrt{2\pi}}{n+1},$$

while

$$\|a_n - a\|_{L^2(0,\pi)} \geq \sqrt{\frac{\pi}{2}}, \ \ \forall n \in \mathbb{N}.$$

Thus, the identification problem a in (0.7)–(0.8) is ill-posed in the $L^2(0,\pi)$ and $L^\infty(0,\pi)$-norms.

If we rewrite equation (0.5) as a first-order hyperbolic partial differential equation in the unknown q, which leads to

$$\nabla q \cdot \nabla u + q\Delta u = -f.$$

It turns out that if ∇u vanishes in subregions of Ω then it is impossible to determine q on the these subregions. This is one of the reasons why our coefficient identification problem is ill-posed (see, for example, [109]). This situation is possible when $f \neq 0$, although Alessandrini [5] has shown that if $f = 0$ and if $u_{|\partial\Omega}$ has a finite number of relative maxima and minima then ∇u only vanishes at a finite number of points in Ω, with finite multiplicity.

We note that in our setting we assume to have observations of $z^\delta \in L^2(\Omega)$, $\nabla z^\delta \in (L^2(\Omega))^d$ for the solution u and its gradient, respectively. Such assumptions have been used by many authors, e.g., Acar [1], Banks and Kunisch [13], Chan and Tai [24, 25], Chavent [26], Chavent and Kunisch [29], Chen and Zou [31], Ito and Kunisch [70, 71], Ito, Kroller and Kunisch [69], Keung and Zou [79], Knowles *et al* [84]–[86], Kohn and Lowe [88], Vainikko [121, 122], Vainikko and Kunisch [124], Zou [131]. In practice, the observation is measured at certain points and we need to interpolate the point observations to get distributed observations. The gradient may not be measurable directly, but there are several ways to approximate it. For example, Chan and Tai [24, 25] suggested to use the differentiation formulas by Anderssen and Hegland [8], Knowles *et al* [86] applied first a mollification process and then finite differences, whereas Kaltenbacher and Schröberl [73] used a Clément operator Π_{sm} for smoothing the data and obtained the gradient as a by-product (see also [32, pp. 154–157]). Kaltenbacher and Schröberl [73, pp. 679–680] showed that if $\|u - z^\delta\|_{L^2(\Omega)} \leq \epsilon$ and $u \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$, then one can explicitly choose smoothing parameters such that $\|u - \Pi_{sm} z^\delta\|_{L^2(\Omega)} \leq c\epsilon$ and $\|u - \Pi_{sm} z^\delta\|_{H^1(\Omega)} \leq c\sqrt{\epsilon}$ with c being the norm of Π_{sm} from $L^2(\Omega)$ to $L^2(\Omega)$. The question is if u and ∇u are given, whether our inverse problem is ill-posed? The case a is sought, the above explicit example by Baumeister and Kunisch [15] showed this fact in the L^2 and L^∞ norms. We could not find an explicit example showing that the problem of identifying q from the observations of u and ∇u is ill-posed in the same topologies. However, the ill-posedness of this problem was discussed by Kohn and Lowe [88, p. 123]. Furthermore, Vainikko [122] showed that this problem is ill-posed, even if $|\nabla u| \geq c > 0$ in Ω. Concerning this fact, see also the dissertation of Cherlenyak [32, pp. 147–154] where he gave a full proof of this fact along some private communications with Vainikko. Besides, Chavent and Kunisch [29, pp. 432–434] (see also [26, p. 28 and §4.9]) have also shown the ill-posedness of the problem. However, under certain additional assumption, the problem of identifying q in (0.5)–(0.6) is well-posed in the H^{-1}-norm as measurement error in the H^1-norm, where H^{-1} is the dual of H^1 (see also [85]). In fact, assume that the boundary $\partial\Omega$ is of class C^1 and $u \in W^{2,\infty}(\Omega)$ with $|\nabla u| \geq \gamma$ a.e. on Ω, where γ is a positive constant. Then, we can verify that if v is a solution of

$$\nabla \cdot (q\nabla u) = \nabla \cdot (p\nabla v) \ \text{ in } \Omega, \tag{0.10}$$

$$q\frac{\partial u}{\partial n} = p\frac{\partial v}{\partial n} \ \text{ on } \partial\Omega, \tag{0.11}$$

then

$$\|q - p\|_{H^{-1}(\Omega)} \le C\|u - v\|_{H^1(\Omega)}. \tag{0.12}$$

In fact, first we note for each element $\xi \in H^1(\Omega)$, there exists $\vartheta_\xi \in H^1(\Omega)$ satisfying

$$\nabla u \cdot \nabla \vartheta_\xi = \xi \tag{0.13}$$

(see Lemma 2.1.10 below). Further, there exists a positive constant C independent of ξ such that

$$\|\vartheta_\xi\|_{H^1(\Omega)} \le C\|\xi\|_{H^1(\Omega)}. \tag{0.14}$$

Since v solves (0.10)–(0.11), we have

$$\int_\Omega q\nabla u \nabla \vartheta_\xi = \int_\Omega p\nabla v \nabla \vartheta_\xi.$$

Then,

$$\int_\Omega (q - p)\nabla u \nabla \vartheta_\xi = \int_\Omega p\nabla(v - u)\nabla \vartheta_\xi.$$

By (0.13)–(0.14), we conclude that

$$\begin{aligned}\left|\int_\Omega (q - p)\xi\right| &= \left|\int_\Omega p\nabla(v - u)\nabla\vartheta_\xi\right| \\ &\le \|p\|_{L^\infty(\Omega)}\|\nabla(u - v)\|_{L^2(\Omega)}\|\nabla\vartheta_\xi\|_{L^2(\Omega)} \\ &\le C\|u - v\|_{H^1(\Omega)}\|\vartheta_\xi\|_{H^1(\Omega)} \\ &\le C\|u - v\|_{H^1(\Omega)}\|\xi\|_{H^1(\Omega)}\end{aligned}$$

for all $\xi \in H^1(\Omega)$, where the positive constant C is independent of ξ. This leads to estimate (0.12). However, the well-posedness of the identification problem in this way is not practicable since a good approximation in the H^{-1}-norm is physically useless. For the identification problem, it is interesting in special identification methods that are well-posed in the L^p-norm, $1 \le p \le \infty$.

Up to now, there have been many papers published devoted to the coefficient identification problems considered in this thesis. Different techniques and methods have been proposed for solving them such as output least squares methods [6, 26, 27, 29, 30, 36, 37, 46, 53, 99, 100], regularization methods [42, 54, 62, 60, 61, 59, 91, 102, 104, 116, 120], equation error methods [51, 76, 78], variational methods [88], integrating along characteristics [109], finite difference schemes [110], singular perturbation technique [5], the augmented Lagrangian technique [31, 52, 69, 70, 72], the long-time behavior of an associated dynamical system [65], the level set method [25, 114, 115], and iterative methods [11, 12, 20, 74, 75, 80]. In the next section we will describe with more details of these approaches to our inverse problems.

0.3 Review of Methods

We now discuss some of the techniques and methods that have been used for solving coefficient identification problems. Compared to the problem of identifying q in (0.5), the problem of identifying a in (0.7) has received less attention. However, there are some authors studied the problem such as Alt [6], Banks and Kunisch [13], Baumeister and

Kunisch [14], Chavent [26], Chavent, Kunisch and Roberts [30], Colonius and Kunisch [36, 37], Engl, Hanke and Neubauer [41], Engl, Kunisch and Neubauer [42], Hào and Quyen [59, 60, 61, 62], Hein and Meyer [64], Ito, Kroller and Kunisch [69], Ito and Kunisch [70, 72], Knowles [82, 83, 84], Neubauer [101], and Resmerita and Scherzer [108]. Thus, in the following we will describe some approaches introduced in [5, 42, 46, 65, 76, 88, 105, 108, 109, 110] for solving the coefficient identification problem q in (0.5).

0.3.1. Integrating along characteristics

In the article [109] Richter has written equation (0.5) as a first-order hyperbolic partial differential equation in the unknown $q = q(x)$, which leads to

$$\mathcal{L}(q,u) := \nabla q(x)\cdot\nabla u(x) + q(x)\Delta u(x) = -f(x), \ \ x\in\Omega\subset\mathbb{R}^2. \tag{0.15}$$

He assumes that

(H_1) $u\in C^2(\Omega)$, $f\in L^\infty(\Omega)$.

(H_2)

$$\inf_{\Omega}\max\{|\nabla u|,\Delta u\} > 0. \tag{0.16}$$

This condition is equivalent to the one that the domain Ω can be divided into subregions Ω_1 and Ω_2 in which $|\nabla u|$ and Δu are uniformly positive, respectively

$$\begin{aligned}&\Omega = \Omega_1\cup\Omega_2,\\ &|\nabla u| \ge k_1 > 0 \text{ in } \Omega_1,\\ &\Delta u \ge k_2 > 0 \text{ in } \Omega_2.\end{aligned} \tag{0.17}$$

(H_3) A "solution" to equation (0.15) means a function $q\in L^\infty(\Omega)$ which is continuous and differentiable along the characteristic curves of (0.15) and it holds the ordinary differential equation to which (0.15) reduces along such curves.

Then, the author concluded that for any f, the equation $\mathcal{L}(q,u) = -f$ has a unique solution $q = q(x)$ assuming prescribed values along the "inflow" boundary $\Gamma\subset\partial\Omega$ (essentially that portion of $\partial\Omega$ where the outer normal derivative of u is negative) and

$$\|q\|_{L^\infty(\Omega)} \le C(u)\left(\max\left\{\sup_{\Gamma}|q|, \frac{\|f\|_{L^\infty(\Omega)}}{k_2}\right\} + \frac{[u]}{k_1^2}\|f\|_{L^\infty(\Omega)}\right),$$

where

$$\begin{aligned}[u] &= \sup_{\Omega} u - \inf_{\Omega} u,\\ C(u) &= \max\left\{1, \exp\left(\frac{\xi[u]}{k_1}\right)\right\},\\ \xi &= \sup_{\Omega_1}\left\{-\frac{\Delta u}{|\nabla u|}\right\}.\end{aligned}$$

Now, suppose that $\mathcal{L}(p;v) = -g$, where p is the diffusion coefficient produced by a perturbed solution $v\approx u$ and forcing function $g\approx f$. Since

$$\mathcal{L}(q-p;u) = -\mathcal{L}(p;v-u) + (f-g),$$

we obtain the following continuous dependence

$$\|q-p\|_{L^\infty(\Omega)} \leq C(u)\left(\max\left\{\sup_{\Gamma}|q-p|, \frac{\widehat{C}}{k_2}\right\} + \frac{[u]}{k_1^2}\widehat{C}\right),$$

where

$$\widehat{C} = \|\nabla p\|_{L^\infty(\Omega)}\|\nabla(u-v)\|_{L^\infty(\Omega)} + \|p\|_{L^\infty(\Omega)}\|\Delta(u-v)\|_{L^\infty(\Omega)} + \|f-g\|_{L^\infty(\Omega)}.$$

0.3.2. Finite difference scheme

The author in the article [110] investigates a finite difference method for identifying of the coefficient in equation (0.5) under condition (0.17) as long as $q(x)$ is prescribed along the inflow portion of the boundary $\partial\Omega$ of Ω. For this scheme, equation (0.5) is viewed as a first order hyperbolic partial differential equation in the unknown $q(x)$, which reduces to (0.15). First we describe the numerical method in [110] on the unit square $(0,1)\times(0,1)$. We define a uniform grid as follows

$$(x_i, y_j) = (ih, jh),\ \ 0 \leq i, j \leq n+1,\ \ h = \frac{1}{n+1}.$$

Denote by Ω_h the set of interior grid points

$$\Omega_h = \{(x_i, y_j) \mid 1 \leq i, j \leq n\}$$

and Γ^h the discrete inflow boundary (a grid point in $\partial\Omega$ is in Γ^h if its nearest neighboring grid point in Ω_h has a higher u value; e.g., $(x_i, y_0) \in \Gamma^h$ for $i \in \{1, 2, ..., n\}$ if $u(x_i, y_1) > u(x_i, y_0)$). The grid values of $q(x,y), u(x,y)$ and $f(x,y)$ will be denoted by q_{ij}, u_{ij} and f_{ij}, respectively.

Equation (0.15) is discretized as

$$\mathcal{L}^h(q_{ij}, u_{ij}) = -f_{ij},\ \ 1 \leq i, j \leq n \tag{0.18}$$

with

$$\mathcal{L}^h(q_{ij}, u_{ij}) = \frac{q_{ij}-q_{kj}}{h}\cdot\frac{u_{ij}-u_{kj}}{h} + \frac{q_{ij}-q_{il}}{h}\cdot\frac{u_{ij}-u_{il}}{h} + q_{ij}Hu_{ij},$$
$$Hu_{ij} = \frac{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{ij}}{h^2},$$

where

> k is the first index of the minimum of $\{u_{i-1,j}, u_{ij}, u_{i+1,j}\}$ and l is the second index of the minimum of $\{u_{i,j-1}, u_{ij}, u_{i,j+1}\}$.

Solving the equation $\mathcal{L}^h(q_{ij}, u_{ij}) = -f_{ij}$ for q_{ij}, we have

$$q_{ij} = \frac{q_{kj}\left(\frac{u_{ij}-u_{kj}}{h}\right) + q_{il}\left(\frac{u_{ij}-u_{il}}{h}\right) - hf_{ij}}{\frac{u_{ij}-u_{kj}}{h} + \frac{u_{ij}-u_{il}}{h} + hHu_{ij}}.$$

Under the assumption (0.16), Richter has showed that the discrete problem (0.18) has a unique solution q_{ij} assuming prescribed values on Γ^h. Further, if u and q are sufficiently regular in $\overline{\Omega}$, $u \in C^3(\overline{\Omega})$ and $q \in C^2(\overline{\Omega})$, then

$$\max_{0\leq i,j\leq n+1}|q_{ij} - q(x_i, y_j)| = \mathcal{O}(h)\quad \text{as } h \to 0$$

assuming $q_{ij} = q(x_i, y_j)$ on Γ^h. Finally, Richter has extended the applicability of this difference scheme to irregular domains and to problems in which the condition (0.16) does not hold but ∇u and Δu do not simultaneously vanish anywhere in Ω.

0.3.3. Output least-squares minimization

It seems that Frind and Pinder [48] were the first people who applied the output least-squares method to solve the problem of identifying the coefficient $q(x)$ in the Neumann problem for the elliptic equation (0.5)–(0.6). The least-squares approach says that if $u(p)$ is the solution of (0.5)–(0.6), where the coefficient q is replaced with p, then p is a good approximation of q if the difference of a measurement z of u and $u(p)$ is small in $L^2(\Omega)$. For practical purposes, we need to define finite dimensional spaces to implement this approach. Let $\{\Delta_h\}_{0<h<1}$ be a regular and quasi-uniform triangulation of $\overline{\Omega}$ with triangles T of diameter less than or equal to h. Given an L^2-measurement z of u, select finite dimensional subspaces A_h and V_h. To each coefficient $q_h \in A_h$, we associate with a $u_h(q_h) \in V_h$, where $u_h(q_h)$ solves (0.5)–(0.6) in a Galerkin approximation, i.e.

$$\int_\Omega q_h \nabla u_h(q_h) \cdot \nabla v_h dx = \int_\Omega f v_h dx + \int_{\partial\Omega} g v_h dS$$

for all $v_h \in V_h$, and

$$\int_\Omega u_h(q_h) dx = \int_\Omega z dx.$$

The least-squares approach to the approximate determination of q is to solve the following problem: Find $q_h \in A_h$ such that

$$\mathcal{J}(q_h) = \inf_{p_h \in Q_h} \mathcal{J}(p_h) \qquad (P_h)$$

with

$$\mathcal{J}(p_h) = \|u_h(p_h) - z\|^2_{L^2(\Omega)},$$

where

$$Q_h = \{p_h \in A_h \mid 0 < \underline{q} \le p_h \le \overline{q}\}$$

is the admissible set of coefficients and $\underline{q}, \overline{q}$ are given positive constants.

Falk in [46] has presented a very interesting error estimate for the approximation scheme (P_h). To this end, we need to formulate some hypotheses

(H_1) There are a constant unit vector $\vec{\nu}$ and a constant $\sigma > 0$ such that $\nabla u \cdot \vec{\nu} \ge \sigma$ for all $x \in \Omega$.

(H_2) $u \in W^{r+3,\infty}(\Omega)$ and $\Gamma = \left\{x \in \partial\Omega \mid \frac{\partial u}{\partial n} > 0\right\} \in C^{r+2}$ with $r \ge 1$.

(H_3) $q \in H^{r+1}(\Omega)$ and $0 < \underline{q} \le \min_{\overline{\Omega}} q(x) \le \max_{\overline{\Omega}} q(x) \le \overline{q}$. Further, $A_h = S_h^r$ and $V_h = S_h^{r+1}$, where

$$S_h^r = \{v \in C(\overline{\Omega}) \mid v_{|_T} \in P_r, \forall T \in \Delta_h\}$$

with P_r being the space of polynomials of degree less than or equal to r in the variables x_1 and x_2.

(H_4) The observation error is of the form

$$\|u - z\|_{L^2(\Omega)} \le \epsilon.$$

Then, for all h sufficiently small, we have

$$\|q - q_h\|_{L^2(\Omega)} \le C(h^r + h^{-2}\epsilon),$$

where q_h is any solution of problem (P_h) and C is a positive constant independent of h and ϵ. Therefore, if z is the continuous piecewise polynomial interpolation of degree $r+1$ of u, then

$$\|u - z\|_{L^2(\Omega)} = \mathcal{O}(h^{r+2}),$$

by the standard approximation result, and we have the estimate error

$$\|q - q_h\|_{L^2(\Omega)} = \mathcal{O}(h^r).$$

0.3.4. Equation error method

In this method we replace the exact solution u by the measurement data z in (0.5). With z and f being given, we consider the mapping $\psi(q) = \nabla\cdot(q\nabla z)+f$ and solve $\psi(q) = 0$ for the "true" coefficient $q = q(x)$ by solving the problem

$$\min_{q\in\mathcal{Q}_{ad}} \|\nabla\cdot(q\nabla z) + f\|_H^2, \tag{0.19}$$

where $\mathcal{Q}_{ad}$ is the admissible set of coefficients and H is an appropriately chosen Hilbert space in which the boundary conditions on z can be incorporated into (0.19). Differently from the output least-squares method, (0.19) is convex and hence the existence of a unique global minimizer follows.

Under an identifiability assumption the equation error method is realized with $H = L^2(\Omega)$. A multigrid algorithm is devised to solve the linear matrix equation which arises from discretization of (0.19) and application of a necessary optimality condition.

An alternative approach can be based on the weak formulation of (0.19). In the case of homogeneous Dirichlet boundary conditions $z_{|\partial\Omega} = 0$, it is given by

$$\min_{q\in\mathcal{Q}_{ad}} \|\nabla\cdot(q\nabla z) + f\|_{H^{-1}(\Omega)}^2, \tag{0.20}$$

where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. Note that (0.20) is equivalent to

$$\min_{q\in\mathcal{Q}_{ad}} \|\Delta^{-1}(\nabla\cdot(q\nabla z) + f)\|_{H_0^1(\Omega)}^2,$$

where Δ denotes the Laplacian from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ with homogeneous Dirichlet boundary conditions. On the other hand, since

$$\|\Delta^{-1}(\nabla\cdot(q\nabla z) + f)\|_{H_0^1(\Omega)}^2 = \sup_{v\in H_0^1(\Omega)} \left(-\int_\Omega q\nabla z\nabla v + \int_\Omega fv\right),$$

it is evident that the data are only differentiated once in the weak formulation of the equation error method (0.20) as opposed to two differentiations which are required in (0.19) with $H = L^2(\Omega)$. The analogue of the weak formulation with the Dirichlet boundary condition being replacing by the assumption of the availability of flux boundary data $q\frac{\partial z}{\partial n} = g$ on $\partial\Omega$ and its numerical treatment for smooth as well as for discontinuous coefficients q is given in [1].

0.3.5. Modified equation error and least-squares method

Kärkkäinen has introduced this method in [76]. Consider the problem of identifying the coefficient q in the homogeneous elliptic boundary value problem

$$\begin{aligned} -\mathrm{div}(q\nabla u) &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma_0, \\ \tfrac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_1, \end{aligned} \tag{0.21}$$

where Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with smooth boundary $\partial\Omega = \bar{\Gamma_0} \cup \bar{\Gamma_1}$ and Γ_0, Γ_1 are relative open disjoint subsets of $\partial\Omega$. The main idea of this method is to include an extra term to the least squares cost functional which takes into account the underlying equation (0.21), multipling a weight chosen according to the finite dimensional spaces, to balance the different amount of differentiation on both terms. This approach combines the output least-squares method with the equation error method to transform the identification problem into a minimization problem.

Let $\{\Delta_h\}_{0<h<1}$ be a triangulation of $\overline{\Omega}$ with triangles T of diameter less than or equal to h. If the boundary of $\partial\Omega$ is curved, we use triangles with one edge replaced by the curved segment of the boundary. Further, it is assumed that the family $\{\Delta_h\}_{0<h<1}$ is regular and quasi-uniform. Given an L^2-measurement z of u, select finite dimensional subspaces A_h and V_h for the coefficient and the state, respectively. To each coefficient $q_h \in A_h$, we associate with a $u_h(q_h) \in V_h$, where $u_h(q_h)$ solves (0.21) in a Galerkin approximation, i.e.

$$\int_\Omega q_h \nabla u_h(q_h) \cdot \nabla v_h dx = \int_\Omega f v_h dx$$

for all $v_h \in V_h \subset H_0^1(\Omega \cup \Gamma_1) = \{v \in H^1(\Omega) \mid v_{|\Gamma_0} = 0\}$. This approach to the approximate determination of q is to solve the following problem: Find $q_h \in A_h$ such that

$$\mathcal{J}(q_h) = \inf_{p_h \in Q_h} \mathcal{J}(p_h) \tag{P_h}$$

with

$$\mathcal{J}(p_h) = \|u_h(p_h) - z\|^2_{L^2(\Omega)} + \rho \cdot \|\nabla \cdot (p_h \nabla u_h(p_h)) + f\|^2_{L^2(\Omega)},$$

where

$$Q_h = \{p_h \in A_h \mid 0 < \underline{q} \leq p_h \leq \overline{q}\}$$

is the admissible set of coefficients and $\underline{q}, \overline{q}$ are given positive constants. Here $\rho > 0$ is the regularization parameter.

The advantage of this method over the others in the literature is that it does not substitute the observation z by u directly into the operator $\nabla \cdot (p_h \nabla u_h(p_h))$, which is a cause of a huge error in the numerical implementation. The error estimates in [76, 77] were derived under the following assumptions.

(H_1) Let z be a distributed L^2-observation of the state u with an observation error

$$\|u - z\|^2_{L^2(\Omega)} \leq \epsilon$$

(H_2) The functions in (0.21) have the following regularity

$$u \in H_0^1(\Omega \cup \Gamma_1) \cap H^{r+2}(\Omega) \cap W^{2,\infty}(\Omega), \;\; q \in H^{r+1}(\Omega) \cap W^{1,\infty}(\Omega), \;\; \text{and} \;\; f \in H^r(\Omega)$$

with $r \geq \frac{d}{2}$.

(H_3) We choose

$$V_h = S_{h,2}^{r+1,0} \text{ and } A_h = S_{h,1}^r,$$

where

$$S_{h,l}^r = \{v \in C^{l-1}(\overline{\Omega}) \mid v_{|T} \in P_r, \forall T \in \Delta_h\}$$

with P_r being the space of polynomials of degree less than or equal to r. We denote $S_{h,l}^{r,0} = S_{h,l}^r \cap H_0^1(\Omega \cup \Gamma_1)$, the subspace of $S_{h,l}^r$ of functions vanishing on Γ_0.

Then, for h small enough and the regularization parameter $\rho = h^4$, the minimizer q_h of (P_h) and the original coefficient q satisfy

$$\int_\Omega |q_n - q||\nabla u|^2 \leq C(h^r + h^{-2}\epsilon).$$

In dimension one, i.e. the domain Ω is an interval (a, b), a better estimate has been shown if we assume that at least on one end of the interval we have a Neumann condition $u'(a) = 0$ or $u'(b) = 0$. We change the cost functional in problem (P_h) to

$$\widetilde{\mathcal{J}}(p_h) = \|u_h(p_h) - z\|_{L^2(\Omega)}^2 + h^2\|(p_h u_h'(p_h))' + f\|_{-1}^2,$$

where $'$ denotes the differentiation with respect to the x-variable and the second norm is realized in the dual space $H_0^1(\Omega \cup \Gamma_1)^*$ of the test function space $H_0^1(\Omega \cup \Gamma_1)$, while Γ_1 is one end of the inteval (a, b). Then, for h small enough, we have the error estimate

$$\int_a^b |q_h - q||u'|dx \leq C(h^{r+1} + h^{-1}\epsilon),$$

where C is a positive constant independent of h.

0.3.6. Variational approach

We present here another numerical scheme for the reconstruction of the coefficient q in (0.5)–(0.6). This approach was developed in [88] and is motivated by the simple observation that for any positive weights γ_1 and γ_2

$$\int_\Omega |\sigma - q\nabla u|^2 dx + \gamma_1 \int_\Omega |\text{div}\sigma + f|^2 dx + \gamma_2 \int_{\partial\Omega} |\sigma \cdot n - g|^2 \geq 0, \tag{0.22}$$

for any choice of q and any vector field σ, and the minimum being achieved only when $\sigma = q\nabla u$ with q and u satisfying (0.5)–(0.6). This variational method for reconstructing the unknown coefficient involves minimizing (0.22) numerically over suitable finite-dimensional spaces of coefficients and vector fields, using the measured data u^m, f^m, and g^m.

Let $\{\Delta_h\}_{0<h<1}$ be a family of regular, quasi-uniform triangulations of $\overline{\Omega}$, an open bounded connected domain with a Lipschitz boundary. If the boundary of $\partial\Omega$ is curved, we use triangles with one edge replaced by the curved segment of the boundary. Given measurements u^m, f^m, and g^m of u, f, and g, respectively, and select finite dimensional subspaces A_h, K_h for the coefficient and the vector field variables. The variational method to identify the unknown coefficient q in (0.5)–(0.6) involves minimizing the functional

$$\mathcal{J}(q, \sigma) = \|\sigma - q \cdot \nabla u^m\|_{L^2(\Omega)}^2 + \gamma_1\|\text{div}\sigma + f^m\|_{L^2(\Omega)}^2 + \gamma_2\|\sigma \cdot n - g^m\|_{L^2(\partial\Omega)}^2, \tag{0.23}$$

over the finite-dimensional spaces of coefficients A_h and vector fields K_h. The advantage of this method is that we are dealing with a quadratic minimization problem which is extremely easy to implement. The disadvantage of this method is the large number of variables it uses, for example if σ and q are piecewise linear on a triangulation with N^2 nodes, then the functional to be minimized depends on $3N^2$ variables.

Variations of (0.23) are possible. For instance, one might consider using $\sigma \cdot n = g^m$ and div $\sigma = -f^m$ as constraints and minimizing $\|\sigma - q \cdot \nabla u^m\|^2_{L^2(\Omega)}$ or perhaps $\|\sigma - q \cdot \nabla u^m\|^2_{L^2(\Omega)} + \epsilon\|q\|^2_{H^1(\Omega)}$ for some small positive ϵ.

Now we make some assumptions.

(H_1) Let q and u satisfy equations (0.5)–(0.6) and let them have the following regularities: $q \in H^2(\Omega)$, $u \in H^3(\Omega)$, and $\Delta u \in C(\overline{\Omega})$.

(H_2) Set $Q_h^{(k)} = \{w \in C(\overline{\Omega}) \mid w_{|_T} \in P_k, \forall T \in \Delta_h\}$ with P_k being the set of polynomials of degree less than or equal to k and

$$A_h = \{w \in Q_h^{(1)} \mid 0 < \underline{q} \leq w \leq \overline{q}\},$$
$$K_h = Q_h^{(1)} \times Q_h^{(1)},$$

where $\underline{q}$ and $\overline{q}$ are positive constants.

(H_3) Let u^m, f^m, and g^m be measurements of u, f and g correspondingly, where $\|u - u^m\|_{H^1(\Omega)} < \epsilon$, $\|f - f^m\|_{L^2(\Omega)} < \lambda_1$, and $\|g - g^m\|_{L^2(\partial\Omega)} < \lambda_2$.

(H_4) Assume that $u^m \in Q_h^{(k)}$ for some fixed k.

Then, the authors in [88] conclude that:

□ Let $(q_{p,h}, \sigma_{p,h}) \in A_h \times K_h$ solve the problem

$$\min_{(q,\sigma)\in A_h\times K_h} \{\|\sigma - q \cdot \nabla u^m\|^2_{L^2(\Omega)} + h^2\|\text{div}\sigma + f^m\|^2_{L^2(\Omega)} + h\|\sigma \cdot n - g^m\|^2_{L^2(\partial\Omega)}\}.$$

If (0.16) holds, then

$$\|q_{p,h} - q\|_{L^2(\Omega)} \leq C(h + \epsilon h^{-1} + \lambda_1 + h^{-1/2}\lambda_2).$$

Further, if $u^m = u_{h,I}^{(2)}$, the piecewise quadratic interpolation of u on Δ_h, then

$$\|q_{p,h} - q\|_{L^2(\Omega)} \leq C(h + \lambda_1 + h^{-1/2}\lambda_2).$$

□ Let $(q_{p,h}, \sigma_{p,h}) \in A_h \times K_h$ solve the problem

$$\min_{(q,\sigma)\in A_h\times K_h} \{\|\sigma - q \cdot \nabla u^m\|^2_{L^2(\Omega)} + h^2\|\text{div}\sigma + f^m\|^2_{L^2(\Omega)} + h\|\sigma \cdot n - g^m\|^2_{L^2(\partial\Omega)} + \rho\|\nabla q\|_{L^2(\Omega)}\},$$

where $\rho \sim (h^2 + \epsilon + h\lambda_1 + h^{1/2}\lambda_2)^2$. Then,

$$\int_\Omega |q_{p,h} - q||\nabla u|^2 \leq C(h + \epsilon h^{-1} + \lambda_1 + h^{-1/2}\lambda_2).$$

If $u^m = u_{h,I}^{(2)}$, then

$$\int_\Omega |q_{p,h} - q||\nabla u|^2 \leq C(h + \lambda_1 + h^{-1/2}\lambda_2).$$

□ Let $(q_{p,h}, \sigma_{p,h}) \in A_h \times K_h$ solve the problem

$$\min_{(q,\sigma)\in A_h\times K_h} \{\|\sigma - q \cdot \nabla u^m\|^2_{L^2(\Omega)} + \|\text{div}\sigma + f^m\|^2_{L^2(\Omega)} + h^{-1}\|\sigma \cdot n - g^m\|^2_{L^2(\partial\Omega)} + \rho\|\nabla q\|_{L^2(\Omega)}\},$$

where $\rho \sim (h + \epsilon + \lambda_1 + h^{-1/2}\lambda_2)^2$. Then, if $u \in C^2(\overline{\Omega})$ and $|\nabla u| \neq 0$ on $\overline{\Omega}$, one has

$$\|q_{p,h} - q\|_{L^2(\Omega)} \leq C(h + \epsilon + \lambda_1 + h^{-1/2}\lambda_2)^{1/2}.$$

Further, if $f^m = f^{(1)}_{h,I}$, $g^m = g^{(1)}_{h,I}$ and $(q_{p,h}, \sigma_{p,h}) \in A_h \times K_h$ solve

$$\min_{(q,\sigma)\in A_h\times K_h} \{\|\sigma - q \cdot \nabla u^m\|^2_{L^2(\Omega)} + \|\mathrm{div}\sigma + f^m\|^2_{L^2(\Omega)} + \rho\|\nabla q\|_{L^2(\Omega)}\},$$

then

$$\|q_{p,h} - q\|_{L^2(\Omega)} \leq C(h + \epsilon)^{1/2}.$$

In addition, if $u^m = u^{(1)}_{h,I}$, then

$$\|q_{p,h} - q\|_{L^2(\Omega)} \leq Ch^{1/2}.$$

All the positive constants C in these error estimates are independent of h, ϵ, λ_1 and λ_2.

0.3.7. Singular perturbation

In the article [5] Alessandrini has proposed a singular perturbation technique to determine the spatially varying coefficient in the special case $f = 0$ in the partial differential equation (0.5) when the Dirichlet boundary condition is $u = g$ on $\partial\Omega$, where Ω is a connected, C^2-smooth, bounded domain in $\mathbb{R}^2$ and g is a smooth function which is precisely known. Moreover, it is assumed that q satisfies the ellipticity condition

$$0 < \underline{q} \leq q(x) \leq \overline{q}, \quad x \in \Omega$$

along with the following regularity hypothesis

$$|\nabla q| \leq E, \quad x \in \Omega,$$

where $\underline{q}$, $\overline{q}$ and E are fixed positive constants.

Alessandrini has proved that if g has a finite number N of relative maxima and minima on $\partial\Omega$, then the gradient of u vanishes only at a finite number of interior points, and only with a finite multiplicity. Moreover, the number of interior critical points and their multiplicities are controlled in terms of N. Alessandrini's algorithm consists of an approximation procedure. It has been shown that as $\epsilon \to 0$, the solution q_ϵ of the elliptic boundary value problem

$$\begin{aligned} \epsilon\Delta q_\epsilon + \mathrm{div}(q\nabla u) &= 0 \text{ in } \Omega, \\ q_\epsilon &= q \text{ on } \partial\Omega \end{aligned} \tag{0.24}$$

converges to q in $L^p_{loc(\Omega)}$ for every $1 \leq p < \infty$. Hence an approximate identification is performed solving the problem (0.24) with a suitably chosen value of ϵ. It is worth mentioning that under a very smooth hypothesis for q, q_ϵ, u, g, Ω and the boundary values $q_{|\partial\Omega}$ of q the following estimate holds

$$\int_\Omega |q - q_\epsilon||\nabla u|^2 dx \leq C\epsilon^{\frac{1}{2}}.$$

0.3.8. Long-time behavior of an associated dynamical system

Hoffmann and Sprekels in [65] have proposed a new and ingenious technique to reconstruct coefficients in elliptic equations. An algorithm is developed to identify the unknown coefficients without a minimization technique. This method is based on the construction of certain time-dependent problems which contain the original equation as asymptotic steady state. The specific equation they considered is

$$-\mathrm{div}(A^*\nabla u^*) = f^* \text{ in } \Omega \tag{0.25}$$

where Ω is an open and bounded set in $\mathbb{R}^d$, $u^* \in H_0^1(\Omega)$ and $f^* \in H^{-1}(\Omega)$. The algorithm seeks to determine a pointwise symmetric matrix function $A^* \in L^\infty(\Omega)$ for solving (0.25). Here $A^* \in L^\infty(\Omega)$ means that $a_{ij}^* \in L^\infty(\Omega)$ for all entries of A^*.

The main idea of this method is to regard (0.25) as an asymptotic for $t \to \infty$ steady state of the following system of parabolic equations

$$\begin{aligned} \frac{\partial u}{\partial t} - \mathrm{div}(A^*\nabla u^*) = f^* \text{ a.e. on } (0,T), \\ u(0) = u^0 \in H_0^1(\Omega), \\ \frac{\partial A}{\partial t} = (\nabla u(t) \otimes \nabla(u(t) - u^*)) \text{ a.e. on } (0,T), \\ A(0) = A^0 \in L^\infty(\Omega), \text{ symmetric.} \end{aligned} \tag{0.26}$$

Here for $u, v \in \mathbb{R}^d$, the $(d \times d)$-matrix $u \otimes v$ is defined by

$$(u \otimes v)_{i,j} = \frac{1}{2}(u_i v_j + u_j v_i), \ \ \forall i, j = 1, ..., d.$$

In this method, equation (0.26) is replaced by a regularizing equation with the hope that $A(t)$ converges in some sense to a solution of (0.25) as $t \to \infty$.

For $\epsilon > 0$, we consider the following dynamical system

$$\begin{aligned} -\epsilon\tfrac{\partial}{\partial t}\Delta u - \mathrm{div}(A^*\nabla u^*) &= f^* \text{ on } (0,T), \\ u(0) &= u^0 \in H_0^1(\Omega), \\ \tfrac{\partial A}{\partial t} &= (\nabla u(t) \otimes \nabla(u(t) - u^*)) \text{ on } (0,T), \\ A(0) &= A^0 \in L^\infty(\Omega), \text{ symmetric.} \end{aligned} \tag{0.27}$$

The system (0.27) has a unique solution $(u(t), A(t))$ for all t. They show that for each sequence $t_n \to \infty$, there exists a subsequence (t_{k_n}) such that $A(t_{k_n}) \to A_\infty$ weakly in $L^2(\Omega)$ where A_∞ satisfies (0.25), under the hypothesis that (0.25) has at least one positive definite solution $A^* \in L^\infty(\Omega)$.

The key tool in this result is the a-priori estimate:

$$\sup_{t\geq 0}\{\|\nabla u(t) - \nabla u^*\|^2_{L^2(\Omega)} + \|A(t) - A^*\|^2_{L^2(\Omega)}\} \\ + \int_0^\infty \|\nabla u(t) - \nabla u^*\|^2_{L^2(\Omega)} dt \leq C < \infty, \tag{0.28}$$

where $C = C(u^0, A^0, A^*)$ is a positive constant. For practical purposes, it is necessary to replace the system (0.27) by a finite-dimensional scheme. To this end, a Galerkin approximation is proposed. Under additional assumptions, it can be shown that the a-priori estimation (0.28) holds in the finite dimensional case. This estimate is used again to show that if $A^n(t)$ is the n-dimensional Galerkin solution of the system (0.27), then $\lim_{t\to\infty} A^n(t) = A^n_\infty \in L^\infty(\Omega)$ and $A^n_\infty \to A_\infty$ weakly in $L^2(\Omega)$, where A_∞ satisfies (0.25). It is worth noticing that this method gives a matrix coefficient, not a scalar one. Besides, in this context the solution of (0.25) is not unique. The method of [65] presumably chooses a particular solution, but it is not clear which one.

0.3.9. Regularization

The problems of identifying the coefficient q in (0.5)–(0.6) and the coefficient a in (0.7)–(0.8) are well known to be ill-posed, and there have been several stable methods for solving them such as numerical methods presented in subsections §§ 0.3.1.–0.3.8. (see also [13, 51, 56, 69, 73, 79, 87, 104, 125, 128, 131]) and regularization methods [11, 12, 20, 58, 74, 75, 80, 122, 123]. Among these stable solving methods, the Tikhonov regularization seems to be most popular. Therefore, we will explain this technique applied to our inverse problems in more details.

For solving the nonlinear ill-posed inverse problems of identifying the coefficient q in (0.5)–(0.6) and the coefficient a in (0.7)–(0.8), Engl, Kunisch and Neubauer in [42] consider the general nonlinear ill-posed equation

$$U(q) = u, \tag{0.29}$$

for q with u being given, where $U : \mathcal{Q} \subset \mathcal{H} \to \mathcal{U}$ is a nonlinear mapping between the Hilbert spaces $\mathcal{H}$ and $\mathcal{U}$, and $\mathcal{Q}$ is some admissible set of the coefficients. The output least-squares method with the Tikhonov regularization is then formulated as follows

$$\min_{q\in\mathcal{Q}} \|U(q) - z^\delta\|^2_{\mathcal{U}} + \rho\|q - q^*\|^2_{\mathcal{H}}. \tag{0.30}$$

Here, $\rho > 0$ is a regularization parameter, q^* is an a-priori estimate of the true coefficient, and z^δ is the observed data of the exact data u with a measurement error of level $\delta > 0$, that is, $\|u - z^\delta\|_{\mathcal{U}} \le \delta$.

They assume that:

(A_1) There exists a solution q^δ_ρ of the problem (0.30).

(A_2) The mapping U is Fréchet differentiable and its Fréchet derivative U' is Lipschitz continuous with Lipschitz constant L.

(A_3) There exists a source element $w \in \mathcal{U}$ such that

$$q^\dagger - q^* = U'(q^\dagger)^* w$$

and the "small enough condition" holds

$$L\|w\|_{\mathcal{U}} < 1, \tag{0.31}$$

where $q^\dagger$ is a q^*-minimum-norm solution of equation (0.29) defined by

$$U(q^\dagger) = u \text{ and } \|q^\dagger - q^*\|_{\mathcal{H}} = \min\{\|q - q^*\|_{\mathcal{H}} \mid U(q) = u\}.$$

Then, they conclude that the regularized minimizers q^δ_ρ of the problem (0.30) converge to $q^\dagger$ with the rate $\delta^{1/2}$ as $\delta \to 0$ and $\rho \sim \delta$. Namely,

$$\|q^\delta_\rho - q^\dagger\|_{\mathcal{H}} = \mathcal{O}(\sqrt{\delta}) \text{ as } \delta \to 0 \text{ and } \rho \sim \delta.$$

However, in their work these authors could apply their theory only to the one-dimensional cases of the above coefficient identification problems with the requirement H^3-regularity and H^2-regularity on sought coefficients q and a in (0.5)–(0.6) and (0.7)–(0.8), respectively.

To find more features (e.g. points of discontinuity) of the sought coefficients, recently many authors considered the nonlinear ill-posed problems (0.29) in Banach spaces, especially in the space of functions with bounded total variation. This approach is promising. However, it is difficult, since the Tikhonov functionals are not differentiable and regularization theory in Banach spaces is not so developed. To summary some ideas in this direction, we introduce the notion of the Bregman distance. Let $\mathcal{L}$ be a Banach space with $\mathcal{L}^*$ being its dual space. Suppose that $R : \mathcal{L} \to (-\infty, +\infty]$ is a proper convex functional and $\partial R(q)$ stands for the subdifferential of R at $q \in \text{Dom}R := \{q \in \mathcal{L} \mid R(q) < +\infty\} \neq \emptyset$ defined by

$$\partial R(q) := \{q^* \in \mathcal{L}^* \mid R(p) \geq R(q) + \langle q^*, p - q\rangle_{(\mathcal{L}^*,\mathcal{L})} \text{ for all } p \in \mathcal{L}\}.$$

The set $\partial R(q)$ may be empty; however, if R is continuous at q, then it is nonempty. Further, $\partial R(q)$ is convex and weak* compact (see, [39], Propositions 5.1, 5.2, p. 21–22). In case $\partial R(q) \neq \emptyset$, for any fixed $p \in \mathcal{L}$ we denote by

$$D_R(p, q) := \{R(p) - R(q) + \langle q^*, p - q\rangle_{(\mathcal{L}^*,\mathcal{L})} \mid q^* \in \partial R(q)\}.$$

Then, for a fixed element $q^* \in \partial R(q)$,

$$D_R^{q^*}(p, q) := R(p) - R(q) + \langle q^*, p - q\rangle_{(\mathcal{L}^*,\mathcal{L})}$$

is called the *Bregman distance with respect to* R *and* q^* *of two elements* $p, q \in \mathcal{L}$.

In general, the Bregman distance is not a metric on $\mathcal{L}$. However, for each $q^* \in \partial R(q)$ the $D_R^{q^*}(p, q) \geq 0$ for any $p \in \mathcal{L}$ and $D_R^{q^*}(q, q) = 0$. Further, in case R is a strictly convex function, $D_R^{q^*}(p, q) = 0$ if and only if $p = q$. The notion of Bregman distance was first given by Bregman [18] for Fréchet differentiable R and it was generalized by Kiwiel [81] to nonsmooth but strictly convex R. Burger and Osher [21] further generalized this notion for R being neither smooth nor strictly convex.

Resmerita and Scherzer in [108] considered the general nonlinear ill-posed equation (0.29), where $U : \mathcal{D}(U) \subset \mathcal{L} \to \mathcal{F}$ is a nonlinear mapping between the Banach spaces $\mathcal{L}$ and $\mathcal{F}$, and $\mathcal{D}(U)$ is the domain of the operator U. The output least-squares method with regularization by the proper convex functional $R(\cdot)$ is then formulated as follows

$$\min_{q \in \mathcal{D}(U)} \frac{1}{2}\|U(q) - z^\delta\|_{\mathcal{F}}^2 + \rho R(q). \tag{0.32}$$

Here, $\rho > 0$ is the regularization parameter and z^δ is the observed data of the exact one $\overline{u}$ with $\|\overline{u} - z^\delta\|_{\mathcal{F}} \leq \delta$, $\delta > 0$.

They assume that:

(H_1) There exists an R-minimizing solution $q^\dagger$ of equation (0.29) defined by

$$U(q^\dagger) = \overline{u} \text{ and } R(q^\dagger) = \min\{R(q) \mid U(q) = \overline{u}\}.$$

(H_2) There exists a solution q_ρ^δ of problem (0.32).

(H_3) The mapping U is Gâteaux differentiable and there exists a positive constant γ such that for any $q \in \mathcal{D}(U) \cap B_r(q^\dagger)$

$$\|U(q) - U(q^\dagger) - U'(q^\dagger)(q - q^\dagger)\|_{\mathcal{F}} \leq \gamma D_R^\xi(q, q^\dagger) \tag{0.33}$$

for all $\xi \in \partial R(q^\dagger)$, where $B_r(q^\dagger) := \{q \in \mathcal{L} \mid \|q - q^\dagger\|_{\mathcal{L}} < r, r > 0\}$.

(H_4) There exists a source element $w \in \mathcal{F}^*$ such that

$$\xi := U'(q^\dagger)^* w \in \partial R(q^\dagger)$$

and the "small enough condition" holds

$$\gamma \|w\|_{\mathcal{F}^*} < 1. \tag{0.34}$$

Then, they conclude that the minimizers q_ρ^δ of problem (0.32) converge to $q^\dagger$ with the rate δ as $\delta \to 0$ and $\rho \sim \delta$ in the sense of the Bregman distance, i.e.,

$$D_R^\xi(q_\rho^\delta, q^\dagger) = \mathcal{O}(\delta) \text{ as } \delta \to 0 \text{ and } \rho \sim \delta.$$

To apply Resmerita and Scherzer's theory to identification problems, we briefly introduce the notion of the space of functions with bounded total variation; for more details, the reader may consult Ambrosio, Fusco, and Pallara [7], Attouch, Buttazzo and Michaille [9], Evans and Gariepy [44], and Guisti [50]. A function $q \in L^1(\Omega)$ is said to be of bounded total variation if

$$TV(q) := \int_\Omega |\nabla q| := \sup \left\{ \int_\Omega q \text{ div } g dx \mid g \in C_c^1(\Omega)^d, \ |g(x)|_{\ell^\infty} \le 1, \ x \in \Omega \right\} < \infty. \tag{0.35}$$

Here $|\cdot|_{\ell^\infty}$ denotes the ℓ^∞-norm on $\mathbb{R}^d$ defined by $|x|_{\ell^\infty} = \max_{1\le i\le d} |x_i|$. The space of all functions in $L^1(\Omega)$ with bounded total variation is denoted by

$$BV(\Omega) = \left\{ q \in L^1(\Omega) \;\middle|\; \int_\Omega |\nabla q| < \infty \right\}.$$

It is the Banach space under the norm

$$\|q\|_{BV(\Omega)} := \|q\|_{L^1(\Omega)} + \int_\Omega |\nabla q|.$$

Further, if Ω is an open bounded set in $\mathbb{R}^d, d \ge 1$ with Lipschitz boundary, then $W^{1,1}(\Omega) \subsetneq BV(\Omega)$ (Giusti [50], pp. 3–4).

In their work these authors could apply their theory only to the problem of identifying the coefficient a in (1.3)–(1.4). They take $\mathcal{L} = BV(\Omega)$, the space of functions with bounded total variation, and

$$R(\cdot) = \frac{1}{2}\|\cdot\|_{L^2(\Omega)}^2 + \int_\Omega |\nabla(\cdot)| \tag{0.36}$$

with $\int_\Omega |\nabla(\cdot)|$ being the total variation defined. However, in this particular case the small enough condition (0.34) has been uncontrollable.

In this work, by our new approach based on *convex energy functionals* (see more details in § 0.4), we will see that our source conditions are easy to check and much weaker than that by Engl, Kunisch and Neubauer [42] and Resmerita and Scherzer [108], since we remove the so-called *small enough condition* (0.31) and (0.34) on the source functions which is popularized and very hard to check in the theory of regularization of nonlinear ill-posed problems (see [41, 42, 108]).

0.4 Summary of the Dissertation

In this dissertation we investigate convergence rates for the Tikhonov regularization of the problems of identifying the coefficient q in the Neumann problem for the elliptic equation (0.5)–(0.6) and the coefficient a in the Neumann problem for the elliptic equation (0.7)–(0.8) as the error level δ in (0.9) tends to zero.

Although there have been many papers devoted to the subject, there have been very few ones devoted to the convergence rates for the methods. Earlier, there was only the paper by Engl, Kunisch and Neubauer [42] devoted to convergence rates for the Tikhonov regularization of the above mentioned problems. Recently, as a generalization of [42], Resmerita and Scherzer [108] investigated convergence rates for convex variational regularization. These authors use the output least-squares method with the Tikhonov regularization of the nonlinear ill-posed problems and obtain some convergence rates under certain source conditions. However, working with *nonconvex functions*, they are faced with difficulties in finding the global minimizers. Further, their source conditions are hard to check and require high regularity of the sought coefficient. To overcome the shortcomings of the above mentioned works, in this thesis we do not use the output least-squares method but follow Knowles [82, 83, 87] and Zou [131] in using the *convex energy functionals* (see (0.37) and (0.38)) and then applying the Tikhonov regularization to these convex energy functionals. We obtain the convergence rates for three forms of regularization (L^2-regularization, total variation regularization and regularization of total variation combining with L^2-stabilization) of the inverse problems of identifying q in (0.5)–(0.6) and a in (0.7)–(0.8). Our source conditions are simple and much weaker than that by Engl, Kunisch and Neubauer [42] and Resmerita and Scherzer [108], since we remove the small enough condition (0.31) and (0.34) on the source functions in the theory of regularization for nonlinear ill-posed problems which is rather restrictive. Furthermore, our results are applicable to multi-dimensional identification problems. The crucial and new idea in the dissertation is that we use the convex energy functional

$$q \to J_{z^\delta}(q) := \frac{1}{2}\int_\Omega q|\nabla(U(q) - z^\delta)|^2 dx, \;\; q \in Q_{ad} \tag{0.37}$$

for identifying q in (0.5)–(0.6) and the convex energy functional

$$a \to G_{z^\delta}(a) := \frac{1}{2}\int_\Omega |\nabla(U(a) - z^\delta)|^2 dx + \frac{1}{2}\int_\Omega a(U(a) - z^\delta)^2 dx, \;\; a \in A_{ad} \tag{0.38}$$

for identifying a in (0.7)–(0.8) (see Lemmas 1.1.5 and 1.2.4) instead of the output least-squares ones. Here, $U(q)$ and $U(a)$ are the coefficient-to-solution maps for (0.5)–(0.6) and (0.7)–(0.8) with Q_{ad} and A_{ad} being the admissible sets, respectively. Then, we apply the Tikhonov regularization to these functionals and obtain convergence rates of the method.

The content of this dissertation is presented in four chapters. In Chapter 1, we will state the inverse problems of identifying the coefficient q in (0.5)–(0.6) and a in (0.7)–(0.8), and prove auxiliary results used in Chapters 2–4.

In Chapter 2, we apply L^2-regularization to these convex energy functionals and investigate convergence rates of the method. Namely, for identifying q in (0.5)–(0.6) we consider the *strictly convex* minimization problem

$$\min_{q \in Q_{ad}} J_{z^\delta}(q) + \rho\|q - q^*\|^2_{L^2(\Omega)}, \tag{0.39}$$

and for identifying a in (0.7)–(0.8) the *strictly convex* minimization problem

$$\min_{a\in A_{ad}} G_{z^\delta}(a) + \rho\|a - a^*\|^2_{L^2(\Omega)}, \tag{0.40}$$

where $\rho > 0$ is the regularization parameter, q^* and a^* respectively are a-priori estimates of sought coefficients q and a. Although these cost functions appear more complicated than that of the output least squares method, it is in fact much simpler because of its strictly convexity, so there is no question on the uniqueness and localization of the minimizer. We will exploit this nice property to obtain convergence rates $\mathcal{O}(\sqrt{\delta})$, as $\delta \to 0$ and $\rho \sim \delta$, under simple and weak source conditions. Our main convergence results in Chapter 2 can now be stated as follows.

Let $q^\dagger$ be the q^*-minimum norm solution of the coefficient identification problem q in (0.5)–(0.6) (see § 2.1.1.) and q^δ_ρ be a solution of problem (0.39). Assume that there exists a functional $w^* \in H^1_\diamond(\Omega)^*$ (see § 1.1.2. for the definition of $H^1_\diamond(\Omega)$) such that

$$U'(q^\dagger)^* w^* = q^\dagger - q^*. \tag{0.41}$$

Here, $U'(q)^*$ is the adjoint of the Fréchet derivative of U at q. Then,

$$\|q^\delta_\rho - q^\dagger\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta}) \text{ and } \|U(q^\delta_\rho) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

The crucial assumption in our result on establishing the convergence rate of regularized solutions q^δ_ρ to the q^*-minimum norm solution $q^\dagger$ is the existence of a source element $w^* \in H^1_\diamond(\Omega)^*$ satisfying (0.41). This is a weak source condition and it does not require any the smoothness of $q^\dagger$. Moreover, the smallness requirement on the source functions of the general convergence theory for nonlinear ill-posed problems in [41, 42], which is hard to check, is liberated in our source condition. In Theorem 2.1.8 we see that this condition is fulfilled for all the dimension d and hence a convergence rate $\mathcal{O}(\sqrt{\delta})$ of L^2-regularization is obtained under assumption that the sought coefficient $q^\dagger$ belongs to $H^1(\Omega)$ and the exact $U(q^\dagger) \in W^{2,\infty}(\Omega)$, $|\nabla U(q^\dagger)| \geq \gamma$ a.e. on Ω, where γ is a positive constant.

Similarly, let $a^\dagger$ be the a^*-minimum norm solution of the coefficient identification problem a in (0.7)–(0.8) (see § 2.2.1.) and a^δ_ρ be a solution of problem (0.40). Assume that there exists a functional $w^* \in H^1(\Omega)^*$ such that

$$U'(a^\dagger)^* w^* = a^\dagger - a^*. \tag{0.42}$$

Then,

$$\|a^\delta_\rho - a^\dagger\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta}) \text{ and } \|U(a^\delta_\rho) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$. Thus, in our source conditions the requirement on the smallness of the source functions is removed.

We note that (see Theorem 2.2.7) the source condition (0.42) is fulfilled for the arbitrary dimension d and hence a convergence rate $\mathcal{O}(\sqrt{\delta})$ of L^2-regularization is obtained under hypothesis that the sought coefficient $a^\dagger$ is an element of $H^1(\Omega)$ and $|U(a^\dagger)| \geq \gamma$ a.e. on Ω, where γ is a positive constant.

To estimate a possible discontinuous or highly oscillating coefficient q, some authors used the output least-squares method with total variation regularization (see, e.g. [2, 11, 22, 24, 25, 125] and part 2 of § 0.3.9.). Namely, they treated the *nonconvex* optimization problem

$$\min_{q\in\mathcal{Q}} \int_\Omega \left(U(q) - z^\delta\right)^2 dx + \rho \int_\Omega |\nabla q| \tag{0.43}$$

with $\int_\Omega |\nabla q|$ being the total variation of the function q defined by (0.35).

Total variation regularization originally introduced in image denoising by Rudin, Osher and Fatemi [111] has been used in several ill-posed inverse problems and analyzed by many authors over the last decades. This method is of particular interest for problems with possibility of discontinuity or high oscillation in the solution (see, for example, [2, 10, 23, 25, 28, 31, 56, 79, 115] and the references therein). Although there have been many papers using total variation regularization of ill-posed problems, there are very few ones devoted to the convergence rates. Only recently, Burger and Osher [21] investigated the convergence rates for convex variational regularization of linear ill-posed problems in the sense of the Bregman distance. This seminal paper has been intensively developed for several linear and nonlinear ill-posed problems [66, 107, 108], etc.

We remark that the cost function appeared in (0.43) is not convex, it is difficult to find global minimizers. To overcome this shortcoming, in Chapter 3, we do not use the output least-squares method, but apply the total variation regularization method to energy functionals $J_{z^\delta}(\cdot)$ and $G_{z^\delta}(\cdot)$, and obtain convergence rates for this approach. Namely, for identifying q, we consider the *convex* minimization problem

$$\min_{q \in Q_{ad}} J_{z^\delta}(q) + \rho \int_\Omega |\nabla q| \,, \tag{0.44}$$

and for identifying a the *convex* minimization problem

$$\min_{a \in A_{ad}} G_{z^\delta}(a) + \rho \int_\Omega |\nabla a| \,. \tag{0.45}$$

Our convergence results in Chapter 3 are as follows. Let $q^\dagger$ be a total variation-minimizing solution of the problem of identifying q in (0.5)–(0.6) (see § 3.1.1.) and q_ρ^δ be a solution of problem (0.44). Assume that there exists a functional $w^* \in H^1_\diamond(\Omega)^*$ such that

$$U'(q^\dagger)^* w^* \in \partial \left(\int_\Omega |\nabla(\cdot)| \right) (q^\dagger). \tag{0.46}$$

Then, $\|U(q_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$,

$$D_{TV}^{U'(q^\dagger)^* w^*}(q_\rho^\delta, q^\dagger) = \mathcal{O}(\delta) \ \text{ and } \ \left| \int_\Omega |\nabla q_\rho^\delta| - \int_\Omega |\nabla q^\dagger| \right| = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

Similarly, let $a^\dagger$ be a total variation-minimizing solution of the problem of identifying a in (0.7)–(0.8) (see § 3.2.1.) and a_ρ^δ be a solution of problem (0.45). Assume that there exists a functional $w^* \in H^1(\Omega)^*$ such that

$$U'(a^\dagger)^* w^* \in \partial \left(\int_\Omega |\nabla(\cdot)| \right) (a^\dagger). \tag{0.47}$$

Then, $\|U(a_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$,

$$D_{TV}^{U'(a^\dagger)^* w^*}(a_\rho^\delta, a^\dagger) = \mathcal{O}(\delta) \ \text{ and } \ \left| \int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a_\rho^\delta| \right| = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

However, our convergence rates in this approach are just in the sense of the Bregman distance which is in general not a metric. To enhance these results, in the last chapter we add an additional L^2-stabilization to the convex energy functionals (0.44) and (0.45) for respectively identifying q and a, and obtain convergence rates not only in the sense of the Bregman distance but also in the $L^2(\Omega)$-norm. Namely, for identifying q in (0.5)–(0.6), we consider the *strictly convex* minimization problem

$$\min_{q \in Q_{ad}} J_{z^\delta}(q) + \rho \left(\frac{1}{2} \|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q| \right), \tag{0.48}$$

and for identifying a in (0.7)–(0.8) the *strictly convex* minimization problem

$$\min_{a \in A_{ad}} G_{z^\delta}(a) + \rho \left(\frac{1}{2} \|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a| \right). \tag{0.49}$$

We also note that, to our knowledge, up to now there is only the paper by Chavent and Kunisch [28] devoted to convergence rates for such a total variation regularization of a certain *linear* ill-posed problem.

Denote by q^δ_ρ the solution of (0.48), $q^\dagger$ the R-minimizing norm solution of the problem of identifying q in (0.5)–(0.6), where $R(\cdot)$ is defined by (0.36). Assume that there exists a functional $w^* \in H^1_\diamond(\Omega)^*$ such that

$$U'(q^\dagger)^* w^* = q^\dagger + \ell \in \partial R(q^\dagger) \tag{0.50}$$

for some element ℓ in $\partial \left(\int_\Omega |\nabla(\cdot)| \right)(q^\dagger)$. Then, we have the convergence rates

$$\|q^\delta_\rho - q^\dagger\|^2_{L^2(\Omega)} + D^\ell_{TV}(q^\delta_\rho, q^\dagger) = \mathcal{O}(\delta) \text{ and } \|U(q^\delta_\rho) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

Similarly, denote by a^δ_ρ the solution of (0.49), $a^\dagger$ the R-minimizing norm solution of the problem of identifying a in problem (0.7)–(0.8). Assume that there exists a function $w^* \in H^1(\Omega)^*$ such that

$$U'(a^\dagger)^* w^* = a^\dagger + \lambda \in \partial R(a^\dagger) \tag{0.51}$$

for some element λ in $\partial \left(\int_\Omega |\nabla(\cdot)| \right)(a^\dagger)$. Then, we have the convergence rates

$$\|a^\delta_\rho - a^\dagger\|^2_{L^2(\Omega)} + D^\lambda_{TV}(a^\delta_\rho, a^\dagger) = \mathcal{O}(\delta) \text{ and } \|U(a^\delta_\rho) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

We remark that (see Theorems 3.1.11, 3.2.7, 4.1.9 and 4.2.6) the source conditions (0.46), (0.47), (0.50) and (0.51) are valid for the dimension $d \le 4$ under some additional regularity assumptions on $q^\dagger$ and the exact $U(q^\dagger)$.

Thus, our source conditions are much weaker than that by Engl, Kunisch and Neubauer [42] and Resmerita and Scherzer [108].

Some authors have analyzed the convergence rates for the Tikhonov regularization of nonlinear ill-posed problems without the smallness conditions. For example, in [101] Neubauer considered the nonlinear ill-posed problem (0.29), where $U : \mathcal{D}(U) \subset \mathcal{H} \to \mathcal{U}$ is a nonlinear mapping between the Hilbert spaces $\mathcal{H}$ and $\mathcal{U}$, and $\mathcal{D}(U)$ is the domain of the operator U. He approached the output least-squares method with the Tikhonov regularization and used Hilbert scales to remove the smallness conditions. However, Neubauer could only apply his theory to the one-dimensional case of the coefficient identification problem a in (0.7)–(0.8) with requiring a in $H^6(\Omega)$, some restrictions on the exact solution and some unpleasant boundary conditions. Further, how to solve his nonconvex minimization problem in Hilbert scales is not clear. On the other hand, in a very interesting series of papers [17, 63, 67], Hofmann and coworkers investigated the interplay of source conditions and structural nonlinearity conditions in establishing convergence rates for the Tikhonov regularization of the nonlinear ill-posed problem (0.29) with $U : \mathcal{D}(U) \subset \mathcal{L} \to \mathcal{F}$ being a nonlinear mapping between the Banach spaces $\mathcal{L}$ and $\mathcal{F}$. In these papers, they replaced the condition (0.33) by the following stronger ones

$$\|U(q) - U(q^\dagger) - U'(q^\dagger)(q - q^\dagger)\|_{\mathcal{F}} \le \gamma \|U(q) - U(q^\dagger)\|_{\mathcal{F}}^{c_1} D_R^{\xi}(q, q^\dagger)^{c_2} \tag{0.52}$$

for q in a weakly sequentially pre-compact of the Banach space $\mathcal{U}$ and $0 \le c_1, c_2$, $0 < c_1 + c_2 \le 1$. In cases $c_1 = 1, c_2 = 0$ and $0 < c_1 < 1, c_1 + c_2 = 1$, they proved some convergence rates for the Tikhonov regularization without any additional smallness condition. However, similarly to [108], these authors aimed at minimizing the discrepancy $\|U(q) - z^\delta\|_{\mathcal{F}}$ and thus arrived at the Tikhonov functional

$$\psi(\|U(q) - z^\delta\|_{\mathcal{F}}) + \rho R(q) \tag{0.53}$$

with a misfit function ψ which is nonconvex. Our approach is completely different. It is not aiming at minimizing the discrepancy $\|U(q) - z^\delta\|_{\mathcal{F}}$, but an energy functional which is convex. Therefore, the theory of Hofmann *et al* is in a different framework and not comparable with our approach to the inverse problems considered in this work.

In the whole thesis we assume that Ω is an open bounded connected domain in $\mathbb{R}^d, d \ge 1$ with the Lipschitz boundary $\partial\Omega$. The functions $f \in L^2(\Omega)$ in (0.5) or (0.7) and $g \in L^2(\partial\Omega)$ in (0.6) or (0.8) are given. The notation U is referred to the nonlinear coefficient-to-solution operators for the Neumann problems. We use the standard notion of Sobolev spaces $H^1(\Omega)$, $H_0^1(\Omega)$, $W^{1,\infty}(\Omega)$ and $W^{2,\infty}(\Omega)$ etc from the books [3, 43, 93, 117]. For the simplicity of notation, as there will be no ambiguity, we write $\int_\Omega \cdots$ instead of $\int_\Omega \cdots dx$.

Chapter 1

Problem setting and auxiliary results

Let Ω be an open bounded connected domain in $\mathbb{R}^d, d \geq 1$, with the Lipschitz boundary $\partial\Omega$, $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ be given. In this work we investigate ill-posed nonlinear inverse problems of identifying the diffusion coefficient q in the Neumann problem for the elliptic equation

$$-\mathrm{div}(q\nabla u) = f \text{ in } \Omega, \tag{1.1}$$

$$q\frac{\partial u}{\partial n} = g \text{ on } \partial\Omega, \tag{1.2}$$

and the reaction coefficient a in the Neumann problem for the elliptic equation

$$-\Delta u + au = f \text{ in } \Omega, \tag{1.3}$$

$$\frac{\partial u}{\partial n} = g \text{ on } \partial\Omega, \tag{1.4}$$

from imprecise values $z^\delta \in H^1(\Omega)$ of the exact solution $\overline{u}$ of (1.1)–(1.2) or (1.3)–(1.4) with

$$\|\overline{u} - z^\delta\|_{H^1(\Omega)} \leq \delta, \tag{1.5}$$

$\delta > 0$ being given.

1.1 Diffusion coefficient identification problem

1.1.1. Problem setting

We consider problem (1.1)–(1.2). Assume that the functions f and g satisfy the compatibility condition

$$\int_\Omega f + \int_{\partial\Omega} g = 0.$$

Then, a function u in $H^1_\diamond(\Omega)$, the close subspace of $H^1(\Omega)$ consisting all the functions $u \in H^1(\Omega)$ with mean-zero:

$$H^1_\diamond(\Omega) := \left\{ u \in H^1(\Omega) \mid \int_\Omega u dx = 0 \right\},$$

is said to be a weak solution of problem (1.1)–(1.2), if

$$\int_\Omega q\nabla u \nabla v = \int_\Omega fv + \int_{\partial\Omega} gv, \ \ \forall v \in H^1_\diamond(\Omega). \tag{1.6}$$

We assume that the coefficient q belongs to the set

$$Q := \{q \in L^\infty(\Omega) \mid 0 < \underline{q} \le q(x) \le \overline{q} \text{ a.e. on } \Omega\} \tag{1.7}$$

with $\underline{q}$ and $\overline{q}$ being given positive constants. Then, by the aid of the Poincaré-Friedrichs inequality in $H^1_\diamond(\Omega)$, we obtain that there exists a positive constant α depending only on $\underline{q}$ and the domain Ω such that the following coercivity condition is fulfilled

$$\int_\Omega q|\nabla u|^2 \ge \alpha \left\|u\right\|^2_{H^1(\Omega)} \text{ for all } u \in H^1_\diamond(\Omega) \text{ and } q \in Q. \tag{1.8}$$

Here,

$$\alpha := \frac{\underline{q}C_\Omega}{1 + C_\Omega} > 0 \tag{1.9}$$

with C_Ω being the positive constant, depending only on Ω, appeared in the Poincaré-Friedrichs inequality:

$$C_\Omega \int_\Omega v^2 \le \int_\Omega |\nabla v|^2 \quad \text{for all } v \in H^1_\diamond(\Omega). \tag{1.10}$$

It follows from inequality (1.8) and the Lax-Milgram lemma (see, e.g. [94]) that for all $q \in Q$, there is a unique weak solution in $H^1_\diamond(\Omega)$ of (1.1)–(1.2) which satisfies the inequality

$$\|u\|_{H^1(\Omega)} \le \Lambda_\alpha \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right), \tag{1.11}$$

where Λ_α is a positive constant depending only on α.

Thus, in the direct problem we defined the nonlinear coefficient-to-solution operator

$$U : Q \subset L^\infty(\Omega) \to H^1_\diamond(\Omega)$$

which maps the coefficient $q \in Q$ to the solution $U(q) \in H^1_\diamond(\Omega)$ of problem (1.1)–(1.2).

The inverse problem is stated as follows:

$$\text{Given } \overline{u} := U(q) \in H^1_\diamond(\Omega) \text{ find } q \in Q.$$

We assume that instead of the exact $\overline{u}$ we have only its observations $z^\delta \in H^1_\diamond(\Omega)$ such that (1.5) satisfies. Our problem is to reconstruct q from z^δ. For solving this problem we minimize the *convex functional*

$$J_{z^\delta}(q) := \frac{1}{2}\int_\Omega q|\nabla(U(q) - z^\delta)|^2 \tag{1.12}$$

over Q (see Lemma 1.1.5). However, since the problem is ill-posed, in the next chapters we shall use the Tikhonov regularization to solve it in a stable way.

1.1.2. Differentiability of the coefficient-to-solution operator

Lemma 1.1.1. *The coefficient-to-solution operator $U : Q \subset L^\infty(\Omega) \to H^1_\diamond(\Omega)$ is continuously Fréchet differentiable on the set Q. For each $q \in Q$, the Fréchet derivative $U'(q)$ of*

$U(q)$ has the property that the differential $\eta := U'(q)h$ with $h \in L^\infty(\Omega)$ is the unique weak solution in $H^1_\diamond(\Omega)$ of the Neumann problem

$$-div\,(q\nabla\eta) = div\,(h\nabla U(q)) \ \ in\ \Omega,$$
$$q\frac{\partial\eta}{\partial n} = -h\frac{\partial U(q)}{\partial n} \ \ on\ \partial\Omega$$

in the sense that it satisfies the equation

$$\int_\Omega q\nabla\eta\nabla v = -\int_\Omega h\nabla U(q)\nabla v \tag{1.13}$$

for all $v \in H^1_\diamond(\Omega)$. Moreover,

$$\|\eta\|_{H^1(\Omega)} \le \frac{\Lambda_\alpha}{\alpha}\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right)\|h\|_{L^\infty(\Omega)} \tag{1.14}$$

for all $h \in L^\infty(\Omega)$.

Proof. For any $q \in Q$ and $h \in L^\infty(\Omega)$, in virtue of the coercivity condition (1.8) and due to the Lax-Milgram lemma, we conclude that the variational equation (1.13) defines a unique solution $\eta := \eta(h) \in H^1_\diamond(\Omega)$. We will see that for q fixed in Q, the $\eta = \eta(\cdot)$ defines a bounded linear operator from $L^\infty(\Omega)$ to $H^1_\diamond\Omega)$. In fact, for any $h_1, h_2 \in L^\infty(\Omega)$ and $\xi_1, \xi_2 \in \mathbb{R}$, and $v \in H^1_\diamond(\Omega)$, we have

$$\begin{aligned}\int_\Omega q\nabla\eta(\xi_1h_1+\xi_2h_2)\nabla v &= -\int_\Omega(\xi_1h_1+\xi_2h_2)\nabla U(q)\nabla v\\ &= -\xi_1\int_\Omega h_1\nabla U(q)\nabla v - \xi_2\int_\Omega h_2\nabla U(q)\nabla v\\ &= \xi_1\int_\Omega q\nabla\eta(h_1)\nabla v + \xi_2\int_\Omega q\nabla\eta(h_2)\nabla v\\ &= \int_\Omega q\nabla\left(\xi_1\eta(h_1)+\xi_2\eta(h_2)\right)\nabla v.\end{aligned}$$

Choose $v = \eta(\xi_1h_1+\xi_2h_2) - \xi_1\eta(h_1) - \xi_2\eta(h_2)$, it follows from the last equality and (1.8) that

$$\begin{aligned}0 &= \int_\Omega q|\nabla(\eta(\xi_1h_1+\xi_2h_2) - \xi_1\eta(h_1) - \xi_2\eta(h_2))|^2\\ &\ge \alpha\left\|\eta(\xi_1h_1+\xi_2h_2) - \xi_1\eta(h_1) - \xi_2\eta(h_2)\right\|^2_{H^1(\Omega)}.\end{aligned}$$

Thus, $\eta = \eta(\cdot)$ is a linear operator of h. Besides, from (1.8) and (1.13) we get

$$\begin{aligned}\alpha\|\eta\|^2_{H^1(\Omega)} &\le -\int_\Omega h\nabla U(q)\nabla\eta\\ &\le \|h\|_{L^\infty(\Omega)}\|\eta\|_{H^1(\Omega)}\|U(q)\|_{H^1(\Omega)}.\end{aligned}$$

From the last inequality and (1.11) we arrive at (1.14). Hence $\eta = \eta(\cdot)$ is bounded from $L^\infty(\Omega)$ to $H^1_\diamond(\Omega)$.

Now we show that $U(q)$ is Fréchet differentiable. In fact, taking h in $L^\infty(\Omega)$ such that $q + h \in Q$, we have

$$\begin{aligned}\int_\Omega q\nabla U(q)\nabla v &= \int_\Omega fv + \int_{\partial\Omega} gv\\ &= \int_\Omega (q+h)\nabla U(q+h)\nabla v\end{aligned}$$

for all $v \in H^1_\diamond(\Omega)$. Therefore,

$$\begin{aligned}\int_\Omega (q+h)\nabla(U(q+h)-U(q))\nabla v &= \int_\Omega q\nabla U(q)\nabla v - \int_\Omega (q+h)\nabla U(q)\nabla v\\ &= -\int_\Omega h\nabla U(q)\nabla v\\ &= \int_\Omega q\nabla\eta\nabla v,\ \ \forall v \in H^1_\diamond(\Omega),\end{aligned}$$

with η defined by equality (1.13). Then,

$$\begin{aligned}\int_\Omega (q+h)\nabla(U(q+h)-U(q)-\eta)\nabla v &= \int_\Omega q\nabla\eta\nabla v - \int_\Omega (q+h)\nabla\eta\nabla v\\ &= -\int_\Omega h\nabla\eta\nabla v\end{aligned}$$

for all $v \in H^1_\diamond(\Omega)$. Choose $v = U(q+h) - U(q) - \eta$, by inequality (1.8), we get

$$\alpha\|U(q+h)-U(q)-\eta\|_{H^1(\Omega)} \le \|h\|_{L^\infty(\Omega)}\|\eta\|_{H^1(\Omega)}.$$

From the last inequality and (1.24) we obtain

$$\begin{aligned}\|U(q+h)-U(q)-\eta\|_{H^1(\Omega)} &\le \frac{\Lambda_\alpha}{\alpha^2}\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right)\|h\|^2_{L^\infty(\Omega)}\\ &= o\left(\|h\|_{L^\infty(\Omega)}\right).\end{aligned}$$

Since η is a bounded linear operator of h from $L^\infty(\Omega)$ to $H^1_\diamond\Omega)$, we conclude that $U : Q \subset L^\infty(\Omega) \to H^1_\diamond(\Omega)$ is continuously Fréchet differentiable and its differential $U'(q)h$ is η. The lemma is proved. $\square$

We note that the mapping $U : Q \subset L^\infty(\Omega) \to H^1_\diamond(\Omega)$ is in fact infinitely Fréchet differentiable, but we do not use this fact so omit its proof here.

1.1.3. Some preliminary results

The following results are needed to prove the main results of the next chapters.

Lemma 1.1.2. *Let (q_n) be a bounded sequence in the $L^\infty(\Omega)$-norm and $q \in L^\infty(\Omega)$. Assume that (q_n) converges to q in the $L^1(\Omega)$-norm. Then,*

$$\lim_n \int_\Omega |q_n - q|v^2 = 0$$

for each v in $L^2(\Omega)$.

Proof. By the assumption of the lemma, we have

$$\sup_{n\in\mathbb{N}}\left(\int_\Omega |q_n-q|v^2\right) \le \sup_{n\in\mathbb{N}}\left(\|q_n\|_{L^\infty(\Omega)} + \|q\|_{L^\infty(\Omega)}\right)\int_\Omega v^2 < \infty.$$

Hence

$$\limsup_n \int_\Omega |q_n-q|v^2 \le \sup_{n\in\mathbb{N}}\left(\int_\Omega |q_n-q|v^2\right) < \infty.$$

By the definition of limsup, there is a subsequence (q_{1_n}) of (q_n) such that

$$\limsup_n \int_\Omega |q_n - q| v^2 = \lim_n \int_\Omega |q_{1_n} - q| v^2.$$

On the other hand, since the sequence (q_{1_n}) converges to q in the $L^1(\Omega)$-norm, it has a subsequence denoted by the same symbol which converges to q a.e. on Ω. Applying the Lebesgue dominated convergence theorem, we get

$$\lim_n \int_\Omega |q_{1_n} - q| v^2 = \int_\Omega \lim_n |q_{1_n} - q| v^2 = 0.$$

Therefore, $\limsup_n \int_\Omega |q_n - q| v^2 = 0$. The lemma is proved. □

Remark 1.1.1. It is useful to note that the set Q defined by (1.7) is closed in the $L^p(\Omega)$-norm for all $p \in [1, +\infty]$.

Lemma 1.1.3. *Assume that the sequence $(q_n) \subset Q$ converges to q in the $L^1(\Omega)$-norm. Then, the sequence $(U(q_n))$ weakly converges to $U(q)$ in $H^1(\Omega)$.*

To prove this result we make the following note.

Remark 1.1.2. If $\mathfrak{F}$ is a normed linear space and (f_n) is a sequence in $\mathfrak{F}$ such that for each subsequence (f_{1_n}) of (f_n), there exists a subsequence (f_{2_n}) of (f_{1_n}) which converges (weakly converges, resp.) to a fixed element $f_0 \in \mathfrak{F}$, then the whole sequence (f_n) also converges (weakly converges, resp.) to f_0.

Proof of Lemma 1.1.3. Since Q is closed in the $L^1(\Omega)$-norm, $q \in Q$ and $U(q)$ is well-defined. For all $v \in H^1_\diamond(\Omega)$ and $n \in \mathbb{N}$, we have

$$\int_\Omega q_n \nabla U(q_n) \nabla v = \int_\Omega fv + \int_{\partial\Omega} gv. \tag{1.15}$$

and, due to (1.11),

$$\|U(q_n)\|_{H^1(\Omega)} \leq \Lambda_\alpha \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right).$$

It follows from the last inequality that there exists a subsequence of $(U(q_n))$ denoted by the same symbol and some element $\Theta \in H^1_\diamond(\Omega)$ such that $(U(q_n))$ weakly converges to Θ in $H^1(\Omega)$. We rewrite, for all $v \in H^1_\diamond(\Omega)$,

$$\begin{aligned} &\int_\Omega q_n \nabla U(q_n) \nabla v - \int_\Omega q \nabla \Theta \nabla v \\ &\qquad = \int_\Omega (q_n - q) \nabla U(q_n) \nabla v + \int_\Omega q \nabla (U(q_n) - \Theta) \nabla v. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left| \int_\Omega (q_n - q) \nabla U(q_n) \nabla v \right| &\leq \int_\Omega |q_n - q| |\nabla U(q_n)| |\nabla v| \\ &\leq \left(\int_\Omega |q_n - q| |\nabla U(q_n)|^2 \right)^{1/2} \left(\int_\Omega |q_n - q| |\nabla v|^2 \right)^{1/2} \\ &\leq \sqrt{2\overline{q}} \|U(q_n)\|_{H^1(\Omega)} \left(\int_\Omega |q_n - q| |\nabla v|^2 \right)^{1/2} \\ &\leq \sqrt{2\overline{q}} \Lambda_\alpha \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right) \left(\int_\Omega |q_n - q| |\nabla v|^2 \right)^{1/2}. \end{aligned}$$

Thus, applying Lemma 1.1.2, we obtain

$$\lim_n \int_\Omega (q_n - q)\nabla U(q_n)\nabla v = 0.$$

On the other hand, since $(U(q_n))$ weakly converges to Θ in $H^1(\Omega)$, we get

$$\int_\Omega q\nabla(U(q_n) - \Theta)\nabla v \to 0$$

as n goes to ∞. Therefore,

$$\int_\Omega q_n \nabla U(q_n)\nabla v \to \int_\Omega q\nabla\Theta\nabla v \tag{1.16}$$

for all $v \in H^1_\diamond(\Omega)$, when n tends to ∞. Combining (1.15) and (1.16) we conclude that $\Theta = U(q)$. Since the element $U(q)$ is unique, we conclude that the whole sequence $(U(q_n))$ also converges to $U(q)$. The lemma is proved. □

Now, we outline some properties of the functional $J_{z^\delta}(\cdot)$ defined by (1.12).

Lemma 1.1.4. *The functional $J_{z^\delta}(\cdot)$ is continuous on the set Q with respect to the $L^2(\Omega)$-norm.*

Proof. For any fixed $q \in Q$ take a sequence $(q_n) \subset Q$ convergent to q in the $L^2(\Omega)$-norm. By the Cauchy-Schwarz inequality, the estimate

$$\left(\int_\Omega |q_n - q|\right)^2 \le \operatorname{mes}(\Omega)\int_\Omega |q_n - q|^2$$

holds for all $n \in \mathbb{N}$. Since $\operatorname{mes}(\Omega) < \infty$, it follows that (q_n) converges to q in the $L^1(\Omega)$-norm, too. Thus, in virtue of Lemma 1.1.3 we get that the sequence $(U(q_n))$ weakly converges to $U(q)$ in $H^1(\Omega)$. Due to (1.6), we have

$$\begin{aligned} J_{z^\delta}(q_n) = \frac{1}{2}\int_\Omega f(U(q_n) - z^\delta) + \frac{1}{2}\int_{\partial\Omega} g(U(q_n) - z^\delta) \\ - \frac{1}{2}\int_\Omega (q_n - q)\nabla(U(q_n) - z^\delta)\nabla z^\delta - \frac{1}{2}\int_\Omega q\nabla(U(q_n) - z^\delta)\nabla z^\delta. \end{aligned}$$

Since $U(q_n)$ weakly converges to $U(q)$ in $H^1(\Omega)$, we get

$$\begin{aligned} &\int_\Omega f(U(q_n) - z^\delta) + \int_{\partial\Omega} g(U(q_n) - z^\delta) \\ &\qquad \to \int_\Omega f(U(q) - z^\delta) + \int_{\partial\Omega} g(U(q) - z^\delta) \end{aligned}$$

and

$$\int_\Omega q\nabla(U(q_n) - z^\delta)\nabla z^\delta \to \int_\Omega q\nabla(U(q) - z^\delta)\nabla z^\delta$$

as n goes to ∞.

On the other hand, applying the Cauchy-Schwarz inequality, the estimate (1.11) and the definition of the set Q, we have

$$\begin{aligned}
\left|\int_\Omega (q_n - q)\nabla z^\delta \nabla(U(q_n) - z^\delta)\right|^2 &\le \int_\Omega |q_n - q||\nabla(U(q_n) - z^\delta)|^2 \int_\Omega |q_n - q||\nabla z^\delta|^2 \\
&\le 2\overline{q}\|U(q_n) - z^\delta\|^2_{H^1(\Omega)} \int_\Omega |q_n - q||\nabla z^\delta|^2 \\
&\le 4\overline{q}\left(\|U(q_n)\|^2_{H^1(\Omega)} + \|z^\delta\|^2_{H^1(\Omega)}\right)\int_\Omega |q_n - q||\nabla z^\delta|^2 \\
&\le C\int_\Omega |q_n - q||\nabla z^\delta|^2
\end{aligned}$$

for all $n \in \mathbb{N}$, where $C = 4\overline{q}\left(\Lambda_\alpha^2(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)})^2 + \|z^\delta\|^2_{H^1(\Omega)}\right)$ is a constant independent of n. Then, applying Lemma 1.1.2, we have

$$\int_\Omega (q_n - q)\nabla(U(q_n) - z^\delta)\nabla z^\delta \to 0$$

as n tends to ∞. Thus,

$$\begin{aligned}
\lim_n J_{z^\delta}(q_n) &= \frac{1}{2}\int_\Omega f(U(q) - z^\delta) + \frac{1}{2}\int_{\partial\Omega} g(U(q) - z^\delta) - \frac{1}{2}\int_\Omega q\nabla(U(q) - z^\delta)\nabla z^\delta \\
&= J_{z^\delta}(q)
\end{aligned}$$

The lemma is proved. □

Lemma 1.1.5. *The functional $J_{z^\delta}(\cdot)$ is convex on the convex set Q.*

Proof. For all $q \in Q$ and $h \in L^\infty(\Omega)$, we have

$$J'_{z^\delta}(q)h = \frac{1}{2}\int_\Omega h|\nabla(U(q) - z^\delta)|^2 + \int_\Omega q\nabla(U(q) - z^\delta)\nabla U'(q)h.$$

Since $U(q) - z^\delta \in H^1_\diamond(\Omega)$ and (1.13), the last equation yields

$$\begin{aligned}
J'_{z^\delta}(q)h &= \frac{1}{2}\int_\Omega h|\nabla(U(q) - z^\delta)|^2 - \int_\Omega h\nabla U(q)\nabla(U(q) - z^\delta) \\
&= -\frac{1}{2}\int_\Omega h|\nabla U(q)|^2 + \frac{1}{2}\int_\Omega h|\nabla z^\delta|^2.
\end{aligned}$$

Then, for all $q \in Q$ and $h, k \in L^\infty(\Omega)$, the second Fréchet derivative of $J_{z^\delta}(\cdot)$ is given by

$$\begin{aligned}
J''_{z^\delta}(q)(h,k) &= -\int_\Omega h\nabla U(q)\cdot\nabla U'(q)k \\
&= \int_\Omega q\nabla U'(q)h\cdot\nabla U'(q)k.
\end{aligned}$$

Therefore,

$$J''_{z^\delta}(q)(h,h) = \int_\Omega q\nabla U'(q)h\cdot\nabla U'(q)h \ge 0$$

for all $q \in Q$ and $h \in L^\infty(\Omega)$. This completes the proof. □

Lemma 1.1.6. *Let (z^{δ_n}) be a sequence which converges to z^δ in the $H^1(\Omega)$-norm and (q_n) be some sequence in the set Q. Then,*

$$\liminf_n J_{z^{\delta_n}}(q_n) = \liminf_n J_{z^\delta}(q_n).$$

Proof. We have

$$\begin{aligned} J_{z^\delta}(q_n) &= \frac{1}{2}\int_\Omega q_n |\nabla(U(q_n) - z^{\delta_n} + z^{\delta_n} - z^\delta)|^2 \\ &= J_{z^{\delta_n}}(q_n) + \int_\Omega q_n \nabla(U(q_n) - z^{\delta_n})\nabla(z^{\delta_n} - z^\delta) + \frac{1}{2}\int_\Omega q_n |\nabla(z^{\delta_n} - z^\delta)|^2. \end{aligned}$$

The last two terms in the right hand side of the last equality go to zero when n tends to ∞ since (z^{δ_n}) converges to z^δ in $H^1(\Omega)$. The lemma is proved. □

Lemma 1.1.7. *The estimate*

$$\|U(q) - z^\delta\|^2_{H^1(\Omega)} \le \frac{2}{\alpha} J_{z^\delta}(q)$$

holds for all q belonging to Q with the positive constant α defined by (1.9).

Proof. By the definition of the set Q in (1.7), of the functional $J_{z^\delta}(\cdot)$ in (1.12) and the Poincaré-Friedrichs inequality (1.10), we have

$$\begin{aligned} \|U(q) - z^\delta\|^2_{H^1(\Omega)} &\le \int_\Omega |\nabla(U(q) - z^\delta)|^2 + \frac{1}{C_\Omega}\int_\Omega |\nabla(U(q) - z^\delta)|^2 \\ &\le \frac{1 + C_\Omega}{\underline{q} C_\Omega}\int_\Omega q |\nabla(U(q) - z^\delta)|^2 \\ &= \frac{2}{\alpha} J_{z^\delta}(q). \end{aligned}$$

The lemma is proved. □

1.2 Reaction coefficient identification problem

1.2.1. Problem setting

Recall that a function $u \in H^1(\Omega)$ is said to be a weak solution of (1.3)–(1.4), if it satisfies the equality

$$\int_\Omega \nabla u \nabla v + \int_\Omega auv = \int_\Omega fv + \int_{\partial\Omega} gv, \ \ \forall v \in H^1(\Omega). \tag{1.17}$$

For all $u \in H^1(\Omega)$ and $a \in A$, where

$$A := \{a \in L^\infty(\Omega) \mid 0 < \underline{a} \le a(x) \le \overline{a} \text{ a.e. on } \Omega\} \tag{1.18}$$

with $\underline{a}$ and $\overline{a}$ being given positive constants, the following coercivity condition

$$\int_\Omega |\nabla u|^2 + \int_\Omega au^2 \ge \beta \|u\|^2_{H^1(\Omega)} \tag{1.19}$$

holds. Here,

$$\beta := \min\{1, \underline{a}\} > 0. \tag{1.20}$$

In virtue of the Lax-Milgram lemma for each $a \in A$, there exists a unique weak solution of (1.3)–(1.4) which satisfies inequality

$$\|u\|_{H^1(\Omega)} \leq \Lambda_\beta \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right), \tag{1.21}$$

where Λ_β is a positive constant depending only on β.

Therefore, we can define the nonlinear coefficient-to-solution mapping

$$U : A \subset L^\infty(\Omega) \to H^1(\Omega)$$

which maps each $a \in A$ to the unique solution $U(a) \in H^1(\Omega)$ of (1.3)–(1.4). Our inverse problem is formulated as:

$$\text{Given } \overline{u} = U(a) \in H^1(\Omega) \text{ find } a \in A.$$

Now, we assume that instead of the exact $\overline{u}$ we have only its observations $z^\delta \in H^1(\Omega)$ such that (1.5) is satisfied. Our problem is to reconstruct a from z^δ. For solving this problem we minimize the *convex functional*

$$G_{z^\delta}(a) := \frac{1}{2}\int_\Omega |\nabla(U(a) - z^\delta)|^2 + \frac{1}{2}\int_\Omega a(U(a) - z^\delta)^2 \tag{1.22}$$

over A (see Lemma 1.2.4 below). However, since the problem is ill-posed, in the next chapters we shall use the Tikhonov regularization to solve it in a stable way.

1.2.2. Differentiability of the coefficient-to-solution operator

Lemma 1.2.1. *The mapping $U : A \subset L^\infty(\Omega) \to H^1(\Omega)$ is continuously Fréchet differentiable with the derivative $U'(a)$. For each h in $L^\infty(\Omega)$, the differential $\eta := U'(a)h \in H^1(\Omega)$ is the unique solution of the problem*

$$-\Delta\eta + a\eta = -hU(a) \text{ in } \Omega,$$
$$\frac{\partial\eta}{\partial n} = 0 \text{ on } \partial\Omega,$$

in the sense that it satisfies the equation

$$\int_\Omega \nabla\eta\nabla v + \int_\Omega a\eta v = -\int_\Omega hU(a)v \tag{1.23}$$

for all $v \in H^1(\Omega)$. Furthermore, the estimate

$$\|\eta\|_{H^1(\Omega)} \leq \frac{\Lambda_\beta}{\beta}\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right) \|h\|_{L^\infty(\Omega)} \tag{1.24}$$

holds for all $h \in L^\infty(\Omega)$.

Proof. For any $a \in A$ and $h \in L^\infty(\Omega)$, by the coercivity condition (1.19) and an application of the Lax-Milgram lemma, we conclude that the variational equation (1.23) defines a unique solution $\eta := \eta(h) \in H^1(\Omega)$. Further, it directly follows from (1.23) that η is a linear operator of h. On the other hand, from (1.19) and (1.23), we get

$$\begin{aligned}\beta \|\eta\|^2_{H^1(\Omega)} &\le -\int_\Omega hU(a)\eta \\ &\le \|h\|_{L^\infty(\Omega)}\|\eta\|_{H^1(\Omega)}\|U(a)\|_{H^1(\Omega)}.\end{aligned}$$

From the last inequality and (1.21) we arrive at (1.24). Thus, $\eta = \eta(h)$ is a continuous linear operator from $L^\infty(\Omega)$ to $H^1(\Omega)$.

Now we show that the mapping U is Fréchet differentiable on A. In fact, for any fixed $a \in A$, taking h in $L^\infty(\Omega)$ such that $a + h \in A$, we have

$$\begin{aligned}\int_\Omega \nabla U(a)\nabla v + \int_\Omega aU(a)v &= \int_\Omega fv + \int_{\partial\Omega} gv \\ &= \int_\Omega \nabla U(a+h)\nabla v + \int_\Omega (a+h)U(a+h)v\end{aligned}$$

for all $v \in H^1(\Omega)$. Therefore,

$$\begin{aligned}\int_\Omega \nabla(U(a+h) - U(a))\nabla v + \int_\Omega (a+h)(U(a+h) - U(a))v \\ = -\int_\Omega hU(a)v \\ = \int_\Omega \nabla\eta\nabla v + \int_\Omega a\eta v, \ \ \forall v \in H^1(\Omega),\end{aligned}$$

with η being defined by equality (1.23). Then,

$$\int_\Omega \nabla(U(a+h) - U(a) - \eta)\nabla v + \int_\Omega (a+h)(U(a+h) - U(a) - \eta)v = -\int_\Omega h\eta v$$

for all $v \in H^1(\Omega)$. Choose $v = U(a+h) - U(a) - \eta$, by inequality (1.19), we get that

$$\beta\|U(a+h) - U(a) - \eta\|_{H^1(\Omega)} \le \|h\|_{L^\infty(\Omega)}\|\eta\|_{H^1(\Omega)}.$$

From the last inequality and (1.24), we obtain

$$\begin{aligned}\|U(a+h) - U(a) - \eta\|_{H^1(\Omega)} &\le \frac{\Lambda_\beta}{\beta^2}\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right)\|h\|^2_{L^\infty(\Omega)} \\ &= o\left(\|h\|_{L^\infty(\Omega)}\right).\end{aligned}$$

Since η is a bounded linear operator of h from $L^\infty(\Omega)$ to $H^1_0\Omega)$, we conclude that $U : A \subset L^\infty(\Omega) \to H^1(\Omega)$ is continuously Fréchet differentiable and its differential $U'(a)h$ is η. The lemma is proved. □

As in the previous paragraph, we note that the mapping $U : A \subset L^\infty(\Omega) \to H^1(\Omega)$ is infinitely Fréchet differentiable. Besides, Colonius and Kunisch [37] considered the mapping $U : A \subset L^2(\Omega) \to H^1(\Omega)$ and stated that U is Fréchet differentiable, but they did not prove this assertion.

1.2.3. Some preliminary results

Lemma 1.2.2. *Assume that the sequence $(a_n) \subset A$ converges to a in the $L^1(\Omega)$-norm. Then, the sequence $(U(a_n))$ weakly converges to $U(a)$ in $H^1(\Omega)$.*

Proof. Since A is closed in the $L^1(\Omega)$-norm, we have $a \in A$ and so $U(a)$ is well-defined. For all $v \in H^1(\Omega)$ and $n \in \mathbb{N}$, we have

$$\int_\Omega \nabla U(a_n)\nabla v + \int_\Omega a_n U(a_n)v = \int_\Omega fv + \int_{\partial\Omega} gv \tag{1.25}$$

and, due to (1.21),

$$\|U(a_n)\|_{H^1(\Omega)} \le \Lambda_\beta \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right).$$

It follows from the last estimate that there exists a subsequence of $(U(a_n))$ denoted by the same symbol and some element $\Theta \in H^1(\Omega)$ such that $(U(a_n))$ weakly converges to Θ in $H^1(\Omega)$. For all $v \in H^1(\Omega)$ and $n \in \mathbb{N}$, we get

$$\begin{aligned}\int_\Omega \nabla U(a_n)\nabla v + \int_\Omega a_n U(a_n)v - \int_\Omega \nabla\Theta\nabla v - \int_\Omega a\Theta v \\ = \int_\Omega \nabla(U(a_n)-\Theta)\nabla v + \int_\Omega a(U(a_n)-\Theta)v \\ + \int_\Omega (a_n - a)U(a_n)v.\end{aligned}$$

In virtue of Lemma 1.1.2, the last term in the right hand side of the last equality tends to zero when n goes to ∞. On the other hand, since $(U(a_n))$ weakly converges to Θ in $H^1(\Omega)$, we get

$$\int_\Omega \nabla(U(a_n)-\Theta)\nabla v + \int_\Omega a(U(a_n)-\Theta)v \to 0$$

as n goes to ∞. Thus,

$$\int_\Omega \nabla U(a_n)\nabla v + \int_\Omega a_n U(a_n)v \to \int_\Omega \nabla\Theta\nabla v + \int_\Omega a\Theta v \tag{1.26}$$

for all $v \in H^1(\Omega)$, when n tends to ∞. Hence it follows from (1.26) and (1.25) that $\Theta = U(a)$. The lemma is proved. □

Lemma 1.2.3. *The functional $G_{z^\delta}(\cdot)$ defined by* (1.22) *is continuous on the set A with respect to the $L^2(\Omega)$-norm.*

Proof. For any fixed $a \in A$, let $(a_n) \subset A$ be a sequence which converges to a in the $L^2(\Omega)$-norm. Due to Lemma 1.2.2, the sequence $(U(a_n))$ weakly converges to $U(a)$ in $H^1(\Omega)$. By (1.17), we have

$$\begin{aligned}2G_{z^\delta}(a_n) &= \int_\Omega f(U(a_n)-z^\delta) + \int_{\partial\Omega} g(U(a_n)-z^\delta) \\ &\quad - \int_\Omega \nabla(U(a_n)-z^\delta)\nabla z^\delta - \int_\Omega a(U(a_n)-z^\delta)z^\delta \\ &\quad - \int_\Omega (a_n-a)(U(a_n)-z^\delta)z^\delta.\end{aligned}$$

Since $U(a_n)$ weakly converges to $U(a)$ in $H^1(\Omega)$, we get

$$\begin{aligned}\int_\Omega f(U(a_n)-z^\delta) + \int_{\partial\Omega} g(U(a_n)-z^\delta) \\ \to \int_\Omega f(U(a)-z^\delta) + \int_{\partial\Omega} g(U(a)-z^\delta)\end{aligned}$$

and

$$\int_\Omega \nabla(U(a_n) - z^\delta)\nabla z^\delta + \int_\Omega a(U(a_n) - z^\delta)z^\delta \\ \to \int_\Omega \nabla(U(a) - z^\delta)\nabla z^\delta + \int_\Omega a(U(a) - z^\delta)z^\delta$$

as n goes to ∞. On the other hand, applying Lemma 1.1.2, we have

$$\int_\Omega (a_n - a)(U(a_n) - z^\delta)z^\delta \to 0$$

as n tends to ∞. Thus,

$$\begin{aligned}\lim_n 2G_{z^\delta}(a_n) &= \int_\Omega f(U(a) - z^\delta) + \int_{\partial\Omega} g(U(a) - z^\delta) \\ &- \int_\Omega \nabla(U(a) - z^\delta)\nabla z^\delta - \int_\Omega a(U(a) - z^\delta)z^\delta \\ &= 2G_{z^\delta}(a)\end{aligned}$$

The lemma is proved. □

Lemma 1.2.4. *The functional $G_{z^\delta}(\cdot)$ is convex on the convex set A.*

Proof. In fact, for all $a \in A$ and $h \in L^\infty(\Omega)$, we have

$$\begin{aligned}G'_{z^\delta}(a)(h) &= \int_\Omega \nabla(U(a) - z^\delta)\nabla U'(a)h \\ &+ \frac{1}{2}\int_\Omega h(U(a) - z^\delta)^2 + \int_\Omega a(U(a) - z^\delta)U'(a)h.\end{aligned}$$

It follows from equality (1.23) that

$$\begin{aligned}G'_{z^\delta}(a)(h) &= -\int_\Omega h(U(a) - z^\delta)U(a) + \frac{1}{2}\int_\Omega h(U(a) - z^\delta)^2 \\ &= -\frac{1}{2}\int_\Omega hU^2(a) + \frac{1}{2}\int_\Omega h(z^\delta)^2.\end{aligned}$$

Then, for all $a \in A$ and $h, k \in L^\infty(\Omega)$, the second Fréchet derivative of $G_{z^\delta}(\cdot)$ is given by

$$G''_{z^\delta}(a)(h,k) = -\int_\Omega hU(a)U'(a)k.$$

Again using (1.23), from the last equality we have

$$G''_{z^\delta}(a)(h,k) = \int_\Omega \nabla U'(a)h\nabla U'(a)k + \int_\Omega aU'(a)hU'(a)k.$$

Therefore,

$$G''_{z^\delta}(a)(h,h) = \int_\Omega |\nabla U'(a)h|^2 + \int_\Omega a|U'(a)h|^2 \geq 0$$

for all $a \in A$ and $h \in L^\infty(\Omega)$. Hence the functional $G_{z^\delta}(\cdot)$ is convex on A. The lemma is proved. □

Lemma 1.2.5. *Let (z^{δ_n}) be a sequence which converges to z^δ in the $H^1(\Omega)$-norm and (a_n) be some sequence in the set A. Then,*

$$\liminf_n G_{z^\delta}(a_n) = \liminf_n G_{z^{\delta_n}}(a_n).$$

Proof. We rewrite

$$\begin{aligned} G_{z^\delta}(a_n) &= \frac{1}{2}\int_\Omega |\nabla(U(a_n) - z^{\delta_n} + z^{\delta_n} - z^\delta)|^2 + \frac{1}{2}\int_\Omega a_n(U(a_n) - z^{\delta_n} + z^{\delta_n} - z^\delta)^2 \\ &= G_{z^{\delta_n}}(a_n) + \int_\Omega \nabla(U(a_n) - z^{\delta_n})\nabla(z^{\delta_n} - z^\delta) + \frac{1}{2}\int_\Omega |\nabla(z^{\delta_n} - z^\delta)|^2 \\ &+ \int_\Omega a_n(U(a_n) - z^{\delta_n})(z^{\delta_n} - z^\delta) + \frac{1}{2}\int_\Omega a_n(z^{\delta_n} - z^\delta)^2. \end{aligned}$$

By the assumption, we get

$$\int_\Omega \nabla(U(a_n) - z^{\delta_n})\nabla(z^{\delta_n} - z^\delta) + \frac{1}{2}\int_\Omega |\nabla(z^{\delta_n} - z^\delta)|^2 \to 0$$

and

$$\int_\Omega a_n(U(a_n) - z^{\delta_n})(z^{\delta_n} - z^\delta) + \frac{1}{2}\int_\Omega a_n(z^{\delta_n} - z^\delta)^2 \to 0$$

as n tends to ∞. Hence we arrive at the desired equality. The lemma is proved. □

Lemma 1.2.6. *The estimate*

$$\|U(a) - z^\delta\|^2_{H^1(\Omega)} \le \frac{2}{\beta} G_{z^\delta}(a)$$

holds for all a belonging to A with the positive constant β defined by (1.20).

Proof. For all $a \in A$, we have

$$\begin{aligned} 2G_{z^\delta}(a) &\ge \int_\Omega |\nabla(U(q) - z^\delta)|^2 + \underline{a}\int_\Omega (U(q) - z^\delta)^2 \\ &\ge \min\{1, \underline{a}\}\left(\int_\Omega |\nabla(U(q) - z^\delta)|^2 + \int_\Omega (U(q) - z^\delta)^2\right). \end{aligned}$$

The lemma is proved. □

In the next chapters, the convex functionals $J_{z^\delta}(q)$ and $G_{z^\delta}(a)$ will be used for identifying the coefficient q in (1.1)–(1.2) and the coefficient a in (1.3)–(1.4), respectively. In Chapters 2, 3, and 4, we apply the Tikhonov regularization to these convex functionals and correspondingly obtain convergence rates for three forms of regularization

$$\mathcal{R}(\cdot) = \|\cdot\|^2_{L^2(\Omega)},$$
$$\mathcal{R}(\cdot) = \int_\Omega |\nabla(\cdot)|,$$

and

$$\mathcal{R}(\cdot) = \frac{1}{2}\|\cdot\|^2_{L^2(\Omega)} + \int_\Omega |\nabla(\cdot)|$$

under weak source conditions without the smallness requirements on the source functions.

This chapter was written on the basis of the papers

[59] Dinh Nho Hào and Tran Nhan Tam Quyen (2010), Convergence rates for Tikhonov regularization of coefficient identification problems in Laplace-type equations, *Inverse Problems* **26**, 125014 (23pp).

[62] Dinh Nho Hào and Tran Nhan Tam Quyen (2012), Convergence rates for Tikhonov regularization of a two-coefficient identification problem in an elliptic boundary value problem, *Numerische Mathematik* **120**, pp. 45–77.

Chapter 2

L^2-regularization

In this chapter the convex functionals $J_{z^\delta}(\cdot)$ and $G_{z^\delta}(\cdot)$ defined by (1.12) and (1.22) are used for identifying the coefficient q and a in (1.1)–(1.2) and (1.3)–(1.4), respectively. We apply L^2-regularization to these functionals and obtain convergence rates $\mathcal{O}(\sqrt{\delta})$ of regularized solutions in the $L^2(\Omega)$-norm.

2.1 Convergence rates for L^2-regularization of the diffusion coefficient identification problem

2.1.1. L^2-regularization

In this section we use the functional $J_{z^\delta}(\cdot)$ with L^2-regularization to solve the problem of identifying the coefficient q in (1.1)–(1.2) in a stable way. Namely, we solve the minimization problem

$$\min_{q\in Q} J_{z^\delta}(q) + \rho\|q - q^*\|^2_{L^2(\Omega)}, \qquad (\mathbf{P}^q_{\rho,\delta})$$

where $\rho > 0$ is the regularization parameter and q^* is an a-priori estimate of the true coefficient which is identified. We note that q^* plays the role of a selection criterion, it need not belong to the admissible set Q and may be an element of $L^2(\Omega)$. Further, since $q \in Q$, the term $\|q-q^*\|^2_{L^2(\Omega)}$ has a meaning. We will see that the cost functional of problem $(\mathbf{P}^q_{\rho,\delta})$ is weakly lower semicontinuous (weakly l.s.c) in the $L^2(\Omega)$-norm and strictly convex (see Lemma 1.1.5), therefore it attains a *unique solution* q^δ_ρ on the nonempty, convex, bounded and closed in the $L^2(\Omega)$-norm and hence weakly compact set Q (see Theorem 2.1.2 below) which we consider as the regularized solution of our identification problem.

Before going to prove the main results of this subsection, we introduce the notion of *q^*-minimum norm solution.*

Lemma 2.1.1. *The set*

$$\Pi_Q(\overline{u}) := \{q \in Q \mid U(q) = \overline{u}\}$$

is nonempty, convex, bounded and closed in the $L^2(\Omega)$-norm. Hence there is a unique solution $q^\dagger$ of problem

$$\min_{q\in\Pi_Q(\overline{u})} \|q - q^*\|^2_{L^2(\Omega)} \qquad (\boldsymbol{K}^q)$$

which is called by the q^*-minimum norm solution of the identification problem.

Proof. It is clear that $\Pi_Q(\overline{u})$ is a nonempty, convex and bounded set. Suppose that the sequence $(q_n) \subset \Pi_Q(\overline{u})$ converges to q in the $L^2(\Omega)$-norm. We prove that $q \in \Pi_Q(\overline{u})$. First, since Q is closed in the $L^2(\Omega)$-norm, we have $q \in Q$. Further, since $\mathrm{mes}(\Omega) < \infty$, (q_n) also converges to q in the $L^1(\Omega)$-norm. Due to Lemma 1.1.3, we get $\overline{u} = U(q)$. Thus, $q \in \Pi_Q(\overline{u})$. The lemma is proved. □

Our goal is to investigate the convergence rate of regularized solutions q_ρ^δ to the q^*-minimum norm solution $q^\dagger$ of the equation $U(q) = \overline{u}$. In doing so, first we prove that problem $(\mathbf{P}^q_{\rho,\delta})$ has a unique solution which is stable with respect to the regularization parameter ρ and to the data z^δ.

Now, we are in a position to prove main results of this subsection.

Theorem 2.1.2. *There exists a unique solution q_ρ^δ of problem $(\boldsymbol{P}^q_{\rho,\delta})$.*

Proof. Due to Lemmas 1.1.4 and 1.1.5, the functional $J_{z^\delta}(\cdot)$ is convex and continuous on Q with respect to the $L^2(\Omega)$-norm, so it is weakly l.s.c. Therefore, the cost functional of $(\mathbf{P}^q_{\rho,\delta})$ is strictly convex and weakly l.s.c on Q, too. On the other hand, since Q is nonempty, convex, bounded and closed in the $L^2(\Omega)$-norm, it is weakly compact. Hence there exists a unique solution of $(\mathbf{P}^q_{\rho,\delta})$. The theorem is proved. □

Theorem 2.1.3. *For a fixed regularization parameter $\rho > 0$, let (z^{δ_n}) be a sequence which converges to z^δ in the $H^1(\Omega)$ and $(q_\rho^{\delta_n})$ be unique minimizers of problems*

$$\min_{q\in Q} J_{z^{\delta_n}}(q) + \rho\|q - q^*\|^2_{L^2(\Omega)}.$$

Then, $(q_\rho^{\delta_n})$ converges to the unique solution q_ρ^δ of $(\boldsymbol{P}^q_{\rho,\delta})$ in the $L^2(\Omega)$-norm.

Proof. By the definition of $(q_\rho^{\delta_n})$, we have

$$\begin{aligned} J_{z^{\delta_n}}(q_\rho^{\delta_n}) + \rho\|q_\rho^{\delta_n} - q^*\|^2_{L^2(\Omega)} &\le J_{z^{\delta_n}}(q) + \rho\|q - q^*\|^2_{L^2(\Omega)} \\ &\le \overline{q}\left(\|U(q)\|^2_{H^1(\Omega)} + C\right) + \rho\|q - q^*\|^2_{L^2(\Omega)} \end{aligned} \tag{2.1}$$

for any $q \in Q$ and $n \in \mathbb{N}$, where the positive constant C is independent of n such that $\|z^{\delta_n}\|^2_{H^1(\Omega)} \le C$ for all $n \in \mathbb{N}$. It follows that $(q_\rho^{\delta_n})$ is bounded in the $L^2(\Omega)$-norm. Therefore, there exist a subsequence $(q_\rho^{\delta_{1n}})$ of $(q_\rho^{\delta_n})$ and an element $q_\rho^\delta \in L^2(\Omega)$ such that $(q_\rho^{\delta_{1n}})$ weakly converges to q_ρ^δ in $L^2(\Omega)$. By the convexity and closedness of Q in the $L^2(\Omega)$-norm, we conclude that $q_\rho^\delta \in Q$. On the other hand, since $J_{z^\delta}(\cdot)$ and the norm $\|\cdot\|_{L^2(\Omega)}$ are weakly l.s.c., we have

$$J_{z^\delta}(q_\rho^\delta) \le \liminf_n J_{z^\delta}(q_\rho^{\delta_{1n}}) \tag{2.2}$$

and

$$\|q_\rho^\delta - q^*\|^2_{L^2(\Omega)} \le \liminf_n \|q_\rho^{\delta_{1n}} - q^*\|^2_{L^2(\Omega)}. \tag{2.3}$$

Besides, applying Lemma 1.1.6, we get

$$\liminf_n J_{z^\delta}(q_\rho^{\delta_{1n}}) = \liminf_n J_{z^{\delta_{1n}}}(q_\rho^{\delta_{1n}}). \tag{2.4}$$

From (2.2), (2.3) and (2.4) we get

$$\begin{aligned}
J_{z^\delta}(q_\rho^\delta) + \rho\|q_\rho^\delta - q^*\|^2_{L^2(\Omega)} &\le \liminf_n J_{z^{\delta_{1n}}}(q_\rho^{\delta_{1n}}) + \rho \liminf_n \|q_\rho^{\delta_{1n}} - q^*\|^2_{L^2(\Omega)} \\
&\le \liminf_n \Big(J_{z^{\delta_{1n}}}(q_\rho^{\delta_{1n}}) + \rho\|q_\rho^{\delta_{1n}} - q^*\|^2_{L^2(\Omega)} \Big) \\
&\le \limsup_n \Big(J_{z^{\delta_{1n}}}(q_\rho^{\delta_{1n}}) + \rho\|q_\rho^{\delta_{1n}} - q^*\|^2_{L^2(\Omega)} \Big) \\
&\qquad \text{(in virtue of (2.1))} \\
&\le \limsup_n \Big(J_{z^{\delta_{1n}}}(q) + \rho\|q - q^*\|^2_{L^2(\Omega)} \Big) \\
&= J_{z^\delta}(q) + \rho\|q - q^*\|^2_{L^2(\Omega)} \qquad (2.5)
\end{aligned}$$

for any $q \in Q$. This means that q_ρ^δ is a (unique) solution of $(\mathbf{P}^q_{\rho,\delta})$. By contradiction we show that $(q_\rho^{\delta_{1n}})$ converges to q_ρ^δ in the $L^2(\Omega)$-norm. In fact, assume that $q_\rho^{\delta_{1n}} \nrightarrow q_\rho^\delta$. This and (2.3) follow that

$$\epsilon := \limsup_n \|q_\rho^{\delta_{1n}} - q^*\|^2_{L^2(\Omega)} > \|q_\rho^\delta - q^*\|^2_{L^2(\Omega)}. \qquad (2.6)$$

Therefore, there exists a subsequence $(q_\rho^{\delta_{2n}})$ of $(q_\rho^{\delta_{1n}})$ such that

$$q_\rho^{\delta_{2n}} \rightharpoonup q_\rho^\delta \text{ weakly in } L^2(\Omega) \text{ and } \|q_\rho^{\delta_{2n}} - q^*\|^2_{L^2(\Omega)} \to \epsilon. \qquad (2.7)$$

Choosing $q = q_\rho^\delta$ in (2.5) and by (2.7), we get

$$\begin{aligned}
J_{z^\delta}(q_\rho^\delta) + \rho\|q_\rho^\delta - q^*\|^2_{L^2(\Omega)} &= \lim_n \Big(J_{z^{\delta_{2n}}}(q_\rho^{\delta_{2n}}) + \rho\|q_\rho^{\delta_{2n}} - q^*\|^2_{L^2(\Omega)} \Big) \\
&\ge \liminf_n J_{z^{\delta_{2n}}}(q_\rho^{\delta_{2n}}) + \rho\epsilon \\
&= \liminf_n J_{z^\delta}(q_\rho^{\delta_{2n}}) + \rho\epsilon,
\end{aligned}$$

by Lemma 1.1.6. Combining the last inequality with (2.6), we arrive at

$$\begin{aligned}
\liminf_n J_{z^\delta}(q_\rho^{\delta_{2n}}) &\le J_{z^\delta}(q_\rho^\delta) + \rho\Big(\|q_\rho^\delta - q^*\|^2_{L^2(\Omega)} - \epsilon\Big) \\
&< J_{z^\delta}(q_\rho^\delta).
\end{aligned}$$

Since $\liminf_n J_{z^\delta}(q_\rho^{\delta_{1n}}) \le \liminf_n J_{z^\delta}(q_\rho^{\delta_{2n}})$, the last inequality gives a contradiction to (2.2). Thus, $(q_\rho^{\delta_{1n}})$ converges to q_ρ^δ in the $L^2(\Omega)$-norm. Since the element q_ρ^δ is unique, we conclude that the whole sequence $(q_\rho^{\delta_n})$ also converges to q_ρ^δ in the $L^2(\Omega)$-norm. The theorem therefore is proved. □

Theorem 2.1.4. *For any positive sequence $(\delta_n) \to 0$, let $\rho_n := \rho(\delta_n)$ be such that*

$$\rho_n \to 0 \text{ and } \frac{\delta_n^2}{\rho_n} \to 0 \text{ as } n \to \infty.$$

Moreover, let (z^{δ_n}) be a sequence satisfying $\|\overline{u} - z^{\delta_n}\|_{H^1(\Omega)} \le \delta_n$ and $\big(q_{\rho_n}^{\delta_n}\big)$ be the unique minimizers of the problems

$$\min_{q\in Q} J_{z^{\delta_n}}(q) + \rho_n\|q - q^*\|^2_{L^2(\Omega)}.$$

Then, $\big(q_{\rho_n}^{\delta_n}\big)$ converges to the unique solution $q^\dagger$ of problem $(\boldsymbol{K}^q)$ in the $L^2(\Omega)$-norm.

Proof. For all $n \in \mathbb{N}$, by the definition of $q_{\rho_n}^{\delta_n}$, we have

$$\begin{aligned}
J_{z^{\delta_n}}(q_{\rho_n}^{\delta_n}) + \rho_n \|q_{\rho_n}^{\delta_n} - q^*\|^2_{L^2(\Omega)} &\le J_{z^{\delta_n}}(q^\dagger) + \rho_n \|q^\dagger - q^*\|^2_{L^2(\Omega)} \\
&\le \frac{\overline{q}}{2} \int_\Omega |\nabla (U(q^\dagger) - z^{\delta_n})|^2 + \rho_n \|q^\dagger - q^*\|^2_{L^2(\Omega)} \\
&\le \frac{\overline{q}}{2} \|\overline{u} - z^{\delta_n}\|^2_{H^1(\Omega)} + \rho_n \|q^\dagger - q^*\|^2_{L^2(\Omega)} \\
&\le \frac{\overline{q}}{2} \delta_n^2 + \rho_n \|q^\dagger - q^*\|^2_{L^2(\Omega)}. \qquad (2.8)
\end{aligned}$$

By the assumption $\delta_n^2/\rho_n \to 0$, the last inequality yields

$$\limsup_n \|q_{\rho_n}^{\delta_n} - q^*\|^2_{L^2(\Omega)} \le \|q^\dagger - q^*\|^2_{L^2(\Omega)}. \qquad (2.9)$$

It follows from the last estimate that there exist a subsequence $(q_{\rho_{1_n}}^{\delta_{1_n}})$ of $(q_{\rho_n}^{\delta_n})$ and $\widehat{q} \in Q$ such that $(q_{\rho_{1_n}}^{\delta_{1_n}})$ weakly converges to $\widehat{q}$ in $L^2(\Omega)$. On the other hand, since $J_{\bar{u}}(\cdot)$ is weakly l.s.c, it follows that

$$J_{\bar{u}}(\widehat{q}) \le \liminf_n J_{\bar{u}}(q_{\rho_{1_n}}^{\delta_{1_n}}).$$

By Lemma 1.1.7, the last estimate follows that

$$\frac{\alpha}{2} \|U(\widehat{q}) - \overline{u}\|^2_{H^1(\Omega)} \le J_{\bar{u}}(\widehat{q}) \le \liminf_n J_{\bar{u}}(q_{\rho_{1_n}}^{\delta_{1_n}}). \qquad (2.10)$$

On the other hand, applying Lemma 1.1.6, we get

$$\liminf_n J_{\bar{u}}(q_{\rho_{1_n}}^{\delta_{1_n}}) = \liminf_n J_{z^{\delta_{1_n}}}(q_{\rho_{1_n}}^{\delta_{1_n}}).$$

Thus, by (2.8),

$$\begin{aligned}
\liminf_n J_{\bar{u}}(q_{\rho_{1_n}}^{\delta_{1_n}}) &\le \liminf_n \left(\frac{\overline{q}}{2} \delta_{1_n}^2 + \rho_{1_n} \|q^\dagger - q^*\|^2_{L^2(\Omega)} \right) \\
&= 0. \qquad (2.11)
\end{aligned}$$

It follows from inequalities (2.10) and (2.11) that $U(\widehat{q}) = \overline{u}$. Therefore, replacing $q^\dagger$ in (2.8) by $\widehat{q}$, we also get

$$\limsup_n \left\|q_{\rho_n}^{\delta_n} - q^*\right\|^2_{L^2(\Omega)} \le \|\widehat{q} - q^*\|^2_{L^2(\Omega)}.$$

Since $(q_{\rho_{1_n}}^{\delta_{1_n}})$ weakly converges to $\widehat{q}$ in $L^2(\Omega)$, we get

$$\begin{aligned}
\|\widehat{q} - q^*\|^2_{L^2(\Omega)} &\le \liminf_n \|q_{\rho_{1_n}}^{\delta_{1_n}} - q^*\|^2_{L^2(\Omega)} \\
&\le \limsup_n \|q_{\rho_{1_n}}^{\delta_{1_n}} - q^*\|^2_{L^2(\Omega)} \\
&\le \limsup_n \|q_{\rho_n}^{\delta_n} - q^*\|^2_{L^2(\Omega)}.
\end{aligned}$$

Therefore,

$$\lim_n \|q_{\rho_{1_n}}^{\delta_{1_n}} - q^*\|^2_{L^2(\Omega)} = \|\widehat{q} - q^*\|^2_{L^2(\Omega)}.$$

Now, by the definition of $q^\dagger$ and (2.9), we obtain that

$$\begin{aligned}
\|q^\dagger - q^*\|^2_{L^2(\Omega)} &\le \|\widehat{q} - q^*\|^2_{L^2(\Omega)} \\
&= \lim_n \|q_{\rho_{1_n}}^{\delta_{1_n}} - q^*\|^2_{L^2(\Omega)} \\
&\le \|q^\dagger - q^*\|^2_{L^2(\Omega)}.
\end{aligned}$$

Hence $\|q^\dagger - q^*\|^2_{L^2(\Omega)} = \|\widehat{q} - q^*\|^2_{L^2(\Omega)}$ or $q^\dagger = \widehat{q}$, by the uniqueness of $q^\dagger$. The theorem is proved. □

2.1.2. Convergence rates

Now we state our main result on convergence rates for L^2-regularization of the problem of estimating the coefficient q in the Neumann problem (1.1)–(1.2).

We remark that since $L^\infty(\Omega) = L^1(\Omega)^* \subset L^\infty(\Omega)^*$, any $q \in L^\infty(\Omega)$ can be considered as an element in $L^\infty(\Omega)^*$, the dual space of $L^\infty(\Omega)$, by

$$\langle q, h\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} = \int_\Omega qh \tag{2.12}$$

for all $h \in L^\infty(\Omega)$ and

$$\|q\|_{(L^\infty(\Omega))^*} \le \mathrm{mes}(\Omega)\|q\|_{L^\infty(\Omega)}.$$

Besides, for $q \in Q$, the mapping

$$U'(q) : L^\infty(\Omega) \to H^1_\diamond(\Omega)$$

is a continuous linear operator (see Lemma 1.1.1). Denote by

$$U'(q)^* : H^1_\diamond(\Omega)^* \to L^\infty(\Omega)^*$$

the dual operator of $U'(q)$. Then,

$$\left\langle U'(q)^* w^*, h\right\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} = \langle w^*, U'(q)h\rangle_{\left(H^1_\diamond(\Omega)^*, H^1_\diamond(\Omega)\right)} \tag{2.13}$$

for all $w^* \in H^1_\diamond(\Omega)^*$ and $h \in L^\infty(\Omega)$.

Now we state one of the main results of this chapter.

Theorem 2.1.5. *Assume that there exists a function $w^* \in H^1_\diamond(\Omega)^*$ such that*

$$q^\dagger - q^* = U'(q^\dagger)^* w^*. \tag{2.14}$$

Then,

$$\|q^\delta_\rho - q^\dagger\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta}) \text{ and } \|U(q^\delta_\rho) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

Remark 2.1.1. Our source condition is almost identical in structure to (A_3) in § 0.3.9. However, in our condition the source function is in $H^1_\diamond(\Omega)^*$, but not in the Hilbert space $\mathcal{U}$ as in (A_3). Moreover, we do not require the "small enough condition" on the source function (0.31) which is hard to check (see, e.g. [41, 42]).

Denote by $C^\infty(\overline{\Omega})$ the set of infinitely differentiable functions on $\overline{\Omega}$. To prove Theorem 2.1.5 we need the following auxiliary results.

Lemma 2.1.6. *The set of functions $C^\infty(\overline{\Omega}) \cap H^1_\diamond(\Omega)$ is everywhere dense in the space $H^1_\diamond(\Omega)$.*

Proof. Let Φ be any function belonging to $H^1_\diamond(\Omega)$. For an arbitrary positive constant ϵ but fixed, we choose $f_\epsilon \in C^\infty(\overline{\Omega})$ such that

$$\|\Phi - f_\epsilon\|_{H^1(\Omega)} \le \frac{\epsilon}{2}.$$

Set

$$g_\epsilon = f_\epsilon - \frac{1}{\text{mes}(\Omega)} \int_\Omega f_\epsilon.$$

Then, $g_\epsilon \in C^\infty(\overline{\Omega}) \cap H^1_\diamond(\Omega)$ and

$$\begin{aligned}
\|\Phi - g_\epsilon\|^2_{H^1(\Omega)} &= \int_\Omega (\Phi - g_\epsilon)^2 + \int_\Omega |\nabla(\Phi - g_\epsilon)|^2 \\
&= \int_\Omega \left(\Phi - f_\epsilon + \frac{1}{\text{mes}(\Omega)} \int_\Omega f_\epsilon\right)^2 + \int_\Omega |\nabla(\Phi - f_\epsilon)|^2 \\
&\le 2\int_\Omega (\Phi - f_\epsilon)^2 + \frac{2}{\text{mes}(\Omega)} \left(\int_\Omega f_\epsilon\right)^2 + \int_\Omega |\nabla(\Phi - f_\epsilon)|^2 \\
&= 2\int_\Omega (\Phi - f_\epsilon)^2 + \frac{2}{\text{mes}(\Omega)} \left(\int_\Omega (\Phi - f_\epsilon)\right)^2 + \int_\Omega |\nabla(\Phi - f_\epsilon)|^2 \\
&\le 4\int_\Omega (\Phi - f_\epsilon)^2 + \int_\Omega |\nabla(\Phi - f_\epsilon)|^2 \\
&\le 4\|\Phi - f_\epsilon\|^2_{H^1(\Omega)} \\
&\le \epsilon^2.
\end{aligned}$$

The lemma is proved. □

Lemma 2.1.7. *Let $(\psi_\rho)_{\rho\in(0,1)}$ be a family of functions belonging to $C^\infty(\overline{\Omega})\cap H^1_\diamond(\Omega)$ bounded in the $H^1(\Omega)$-norm. Let $(\phi_\rho)_{\rho\in(0,1)}$ be the family of unique solutions in $H^1_\diamond(\Omega)$ of the following Neumann problems*

$$\begin{aligned}
-div(q\nabla\phi_\rho) &= \Delta\psi_\rho - \psi_\rho \text{ in } \Omega, \\
q\frac{\partial\phi_\rho}{\partial n} &= -\frac{\partial\psi_\rho}{\partial n} \text{ on } \partial\Omega
\end{aligned}$$

with a fixed element $q \in Q$. Then, there exists a positive constant C such that

$$\int_\Omega |\nabla\phi_\rho|^2 \le C^2$$

for all $\rho \in (0,1)$.

Proof. For ρ fixed in $(0,1)$, the weak form of this Neumann problem is

$$\begin{aligned}
\int_\Omega q\nabla\phi_\rho\nabla v &= \int_\Omega \Delta\psi_\rho v - \int_\Omega \psi_\rho v - \int_{\partial\Omega} \frac{\partial\psi_\rho}{\partial n} v \\
&= -\int_\Omega \nabla\psi_\rho\nabla v - \int_\Omega \psi_\rho v
\end{aligned}$$

for all $v \in H^1_\diamond(\Omega)$. Taking $v = \phi_\rho$, by the definition of the set Q, we have

$$\begin{aligned}
\underline{q}\int_\Omega |\nabla\phi_\rho|^2 &\le \int_\Omega q|\nabla\phi_\rho|^2 \\
&= -\int_\Omega \nabla\psi_\rho\nabla\phi_\rho - \int_\Omega \psi_\rho\phi_\rho.
\end{aligned}$$

Using the Cauchy-Schwarz inequality and the Poincaré-Friedrichs inequality (1.10), we get

$$\begin{aligned}\underline{q}\int_\Omega|\nabla\phi_\rho|^2 &\le \left(\int_\Omega|\nabla\psi_\rho|^2\right)^{1/2}\left(\int_\Omega|\nabla\phi_\rho|^2\right)^{1/2}+\left(\int_\Omega\psi_\rho^2\right)^{1/2}\left(\int_\Omega\phi_\rho^2\right)^{1/2}\\ &\le \frac{\underline{q}}{4}\int_\Omega|\nabla\phi_\rho|^2+\frac{1}{\underline{q}}\int_\Omega|\nabla\psi_\rho|^2+\frac{\underline{q}C_\Omega}{4}\int_\Omega\phi_\rho^2+\frac{1}{\underline{q}C_\Omega}\int_\Omega\psi_\rho^2\\ &\le \frac{\underline{q}}{2}\int_\Omega|\nabla\phi_\rho|^2+\frac{1}{\underline{q}}\int_\Omega|\nabla\psi_\rho|^2+\frac{1}{\underline{q}C_\Omega}\int_\Omega\psi_\rho^2.\end{aligned}$$

Therefore,

$$\int_\Omega|\nabla\phi_\rho|^2\le 2\frac{1+C_\Omega}{\underline{q}^2C_\Omega}\|\psi_\rho\|^2_{H^1(\Omega)}.$$

By the assumption, there is a positive constant C such that $\|\psi_\rho\|_{H^1(\Omega)}\le C$ for all $\rho\in(0,1)$. This and the last estimate yield the desired inequality. The lemma is proved. □

Proof of Theorem 2.1.5. By the definition of the regularized solution q_ρ^δ, we have

$$J_{z^\delta}(q_\rho^\delta)+\rho\|q_\rho^\delta-q^*\|^2_{L^2(\Omega)}\le J_{z^\delta}(q^\dagger)+\rho\|q^\dagger-q^*\|^2_{L^2(\Omega)}.$$

Therefore,

$$\begin{aligned}&J_{z^\delta}(q_\rho^\delta)+\rho\|q_\rho^\delta-q^\dagger\|^2_{L^2(\Omega)}\\ &\quad\le J_{z^\delta}(q^\dagger)+\rho\left(\|q^\dagger-q^*\|^2_{L^2(\Omega)}-\|q_\rho^\delta-q^*\|^2_{L^2(\Omega)}+\|q_\rho^\delta-q^\dagger\|^2_{L^2(\Omega)}\right)\\ &\quad= J_{z^\delta}(q^\dagger)+2\rho\langle q^\dagger-q^*,q^\dagger-q_\rho^\delta\rangle_{L^2(\Omega)}\\ &\quad\le \frac{\overline{q}}{2}\delta^2+2\rho\langle q^\dagger-q^*,q^\dagger-q_\rho^\delta\rangle_{L^2(\Omega)}.\end{aligned}\tag{2.15}$$

We now treat the second term in the right hand side of inequality (2.15). By (2.12) and (2.14), we have

$$\begin{aligned}2\rho\langle q^\dagger-q^*,q^\dagger-q_\rho^\delta\rangle_{L^2(\Omega)}&=2\rho\int_\Omega\left(q^\dagger-q^*\right)\left(q^\dagger-q_\rho^\delta\right)\\ &=2\rho\left\langle q^\dagger-q^*,q^\dagger-q_\rho^\delta\right\rangle_{(L^\infty(\Omega)^*,L^\infty(\Omega))}\\ &=2\rho\left\langle U'(q^\dagger)^*w^*,q^\dagger-q_\rho^\delta\right\rangle_{(L^\infty(\Omega)^*,L^\infty(\Omega))}.\end{aligned}$$

From the last equality and (2.13) we get

$$2\rho\left\langle q^\dagger-q^*,q^\dagger-q_\rho^\delta\right\rangle_{L^2(\Omega)}=2\rho\left\langle w^*,U'(q^\dagger)(q^\dagger-q_\rho^\delta)\right\rangle_{\left(H^1_\diamond(\Omega)^*,H^1_\diamond(\Omega)\right)}.\tag{2.16}$$

By the Riesz representation theorem, there exists an element $w\in H^1_\diamond(\Omega)$ such that

$$\left\langle w^*,U'(q^\dagger)(q^\dagger-q_\rho^\delta)\right\rangle_{\left(H^1_\diamond(\Omega)^*,H^1_\diamond(\Omega)\right)}=\left\langle w,U'(q^\dagger)(q^\dagger-q_\rho^\delta)\right\rangle_{H^1(\Omega)}.\tag{2.17}$$

Now, using Lemma 2.1.6, for ρ fixed in $(0,1)$, we choose $\psi_\rho\in C^\infty(\overline{\Omega})\cap H^1_\diamond(\Omega)$ such that

$$\|w-\psi_\rho\|_{H^1(\Omega)}\le\rho.\tag{2.18}$$

Consider the Neumann problem for the following elliptic equation

$$\begin{aligned}-\mathrm{div}(q^\dagger\nabla\phi_\rho)&=\Delta\psi_\rho-\psi_\rho\ \text{in}\ \Omega,\\ q^\dagger\frac{\partial\phi_\rho}{\partial n}&=-\frac{\partial\psi_\rho}{\partial n}\ \text{on}\ \partial\Omega.\end{aligned}$$

Since $q^\dagger \in Q$ and $\psi_\rho \in C^\infty(\overline{\Omega})$, we conclude that this Neumann problem has a unique solution $\phi_\rho \in H^1_\diamond(\Omega)$ in the sense that

$$\int_\Omega q^\dagger \nabla\phi_\rho \nabla v = \int_\Omega \Delta\psi_\rho v - \int_\Omega \psi_\rho v - \int_{\partial\Omega} \frac{\partial\psi_\rho}{\partial n} v$$

for all $v \in H^1_\diamond(\Omega)$. By integration by parts, the last equation leads to

$$\begin{aligned}\int_\Omega q^\dagger \nabla\phi_\rho \nabla v &= \int_{\partial\Omega} \frac{\partial\psi_\rho}{\partial n} v - \int_\Omega \nabla\psi_\rho \nabla v - \int_\Omega \psi_\rho v - \int_{\partial\Omega} \frac{\partial\psi_\rho}{\partial n} v \\ &= - \langle \psi_\rho, v\rangle_{H^1(\Omega)}\end{aligned}$$

for all $v \in H^1_\diamond(\Omega)$. Taking $v = U'(q^\dagger)(q^\dagger - q^\delta_\rho) \in H^1_\diamond(\Omega)$ in the last equality, we obtain that

$$\int_\Omega q^\dagger \nabla\phi_\rho \nabla U'(q^\dagger)(q^\dagger - q^\delta_\rho) = - \left\langle \psi_\rho, U'(q^\dagger)(q^\dagger - q^\delta_\rho)\right\rangle_{H^1(\Omega)}.$$

It follows from the last equality and (2.16), (2.17) that

$$\begin{aligned}2\rho \left\langle q^\dagger - q^*, q^\dagger - q^\delta_\rho\right\rangle_{L^2(\Omega)} &= 2\rho \left\langle w, U'(q^\dagger)(q^\dagger - q^\delta_\rho)\right\rangle_{H^1(\Omega)} \\ &= 2\rho \left\langle \psi_\rho, U'(q^\dagger)(q^\dagger - q^\delta_\rho)\right\rangle_{H^1(\Omega)} \\ &\quad + 2\rho \left\langle w - \psi_\rho, U'(q^\dagger)(q^\dagger - q^\delta_\rho)\right\rangle_{H^1(\Omega)} \\ &= -2\rho \int_\Omega q^\dagger \nabla\phi_\rho \nabla U'(q^\dagger)(q^\dagger - q^\delta_\rho) \\ &\quad + 2\rho \left\langle w - \psi_\rho, U'(q^\dagger)(q^\dagger - q^\delta_\rho)\right\rangle_{H^1(\Omega)}.\end{aligned}$$

Applying inequality (2.18) to the last one, we get

$$\begin{aligned}2\rho\langle q^\dagger - q^*, q^\dagger - q^\delta_\rho\rangle_{L^2(\Omega)} &\le -2\rho \int_\Omega q^\dagger \nabla\phi_\rho \nabla U'(q^\dagger)(q^\dagger - q^\delta_\rho) \\ &\quad + 2\rho^2 \|U'(q^\dagger)(q^\dagger - q^\delta_\rho)\|_{H^1(\Omega)} \\ &:= \Sigma + \Lambda. \qquad (2.19)\end{aligned}$$

From (1.14) and the definition of the set Q we have

$$\begin{aligned}\Lambda &= 2\rho^2 \|U'(q^\dagger)(q^\dagger - q^\delta_\rho)\|_{H^1(\Omega)} \\ &\le \rho^2 \frac{2\Lambda_\alpha}{\alpha} \Big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \Big) \|q^\dagger - q^\delta_\rho\|_{L^\infty(\Omega)} \\ &\le \rho^2 \frac{2\Lambda_\alpha}{\alpha} \Big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \Big) \Big(\|q^\dagger\|_{L^\infty(\Omega)} + \|q^\delta_\rho\|_{L^\infty(\Omega)} \Big) \\ &\le \frac{4\overline{q}\Lambda_\alpha}{\alpha} \Big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \Big) \rho^2. \qquad (2.20)\end{aligned}$$

From (1.13) and (1.6) we have

$$\begin{aligned}\Sigma &= -2\rho \int_\Omega q^\dagger \nabla\phi_\rho \nabla U'(q^\dagger)(q^\dagger - q^\delta_\rho) \\ &= 2\rho \int_\Omega (q^\dagger - q^\delta_\rho) \nabla U(q^\dagger) \nabla\phi_\rho \\ &= 2\rho \int_\Omega q^\dagger \nabla U(q^\dagger) \nabla\phi_\rho - 2\rho \int_\Omega q^\delta_\rho \nabla U(q^\dagger) \nabla\phi_\rho \\ &= 2\rho \int_\Omega q^\delta_\rho \nabla U(q^\delta_\rho) \nabla\phi_\rho - 2\rho \int_\Omega q^\delta_\rho \nabla U(q^\dagger) \nabla\phi_\rho \\ &= 2\rho \int_\Omega q^\delta_\rho \nabla \left(U(q^\delta_\rho) - U(q^\dagger) \right) \nabla\phi_\rho.\end{aligned}$$

Using the Cauchy-Schwarz inequality and the definition $U(q^\dagger) = \overline{u}$, we get

$$\begin{aligned}
\Sigma &= 2\rho \int_\Omega q_\rho^\delta \nabla \left(U(q_\rho^\delta) - z^\delta \right) \nabla \phi_\rho + 2\rho \int_\Omega q_\rho^\delta \nabla \left(z^\delta - U(q^\dagger) \right) \nabla \phi_\rho \\
&\le 2\rho \left(\int_\Omega |\nabla (U(q^\dagger) - z^\delta)|^2 \right)^{1/2} \left(\int_\Omega (q_\rho^\delta)^2 |\nabla \phi_\rho|^2 \right)^{1/2} \\
&+ 2\rho \left(\int_\Omega q_\rho^\delta |\nabla (U(q_\rho^\delta) - z^\delta)|^2 \right)^{1/2} \left(\int_\Omega q_\rho^\delta |\nabla \phi_\rho|^2 \right)^{1/2} \\
&\le 2\rho \|\overline{u} - z^\delta\|_{H^1(\Omega)} \left(\overline{q}^2 \int_\Omega |\nabla \phi_\rho|^2 \right)^{1/2} \\
&+ 2\rho \left(\int_\Omega q_\rho^\delta |\nabla (U(q_\rho^\delta) - z^\delta)|^2 \right)^{1/2} \left(\overline{q} \int_\Omega |\nabla \phi_\rho|^2 \right)^{1/2}.
\end{aligned}$$

It follows from inequality (1.5) that

$$\Sigma \le 2\overline{q}\delta\rho \left(\int_\Omega |\nabla \phi_\rho|^2 \right)^{1/2} + 2\rho\sqrt{\overline{q}} \left(2J_{z^\delta}(q_\rho^\delta) \right)^{1/2} \left(\int_\Omega |\nabla \phi_\rho|^2 \right)^{1/2}.$$

A further application of the Cauchy-Schwarz inequality yields

$$\Sigma \le 2\overline{q}\delta\rho \left(\int_\Omega |\nabla \phi_\rho|^2 \right)^{1/2} + 4\rho^2 \overline{q} \int_\Omega |\nabla \phi_\rho|^2 + \frac{1}{2} J_{z^\delta}(q_\rho^\delta).$$

Combining (2.19) and (2.20) with the last estimate, we get

$$\begin{aligned}
2\rho \langle q^\dagger - q^*, q^\dagger - q_\rho^\delta \rangle_{L^2(\Omega)} &\le \frac{4\overline{q}\Lambda_\alpha}{\alpha} \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right) \rho^2 + 2\overline{q}\delta\rho \left(\int_\Omega |\nabla \phi_\rho|^2 \right)^{1/2} \\
&+ 4\rho^2 \overline{q} \int_\Omega |\nabla \phi_\rho|^2 + \frac{1}{2} J_{z^\delta}(q_\rho^\delta).
\end{aligned} \tag{2.21}$$

It follows from inequalities (2.15), (2.21), and an application of Lemma 1.1.7 that

$$\begin{aligned}
&\frac{\alpha}{4} \|U(q_\rho^\delta) - z^\delta\|_{H^1(\Omega)}^2 + \rho \|q_\rho^\delta - q^\dagger\|_{L^2(\Omega)}^2 \\
&\qquad \le \frac{\overline{q}}{2} \delta^2 + \frac{4\overline{q}\Lambda_\alpha}{\alpha} \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right) \rho^2 \\
&\qquad + 2\overline{q} \left(\int_\Omega |\nabla \phi_\rho|^2 \right)^{1/2} \delta\rho + 4\overline{q} \left(\int_\Omega |\nabla \phi_\rho|^2 \right) \rho^2.
\end{aligned}$$

By (2.18), the set $(\psi_\rho)_{\rho \in (0,1)}$ is bounded in the $H^1(\Omega)$-norm. Hence, by Lemma 2.1.7, there exists a constant $C > 0$ depending only on Ω such that

$$\int_\Omega |\nabla \phi_\rho|^2 \le C^2$$

for all $\rho \in (0, 1)$. From this and the last estimate we obtain that

$$\frac{\alpha}{4} \|U(q_\rho^\delta) - z^\delta\|_{H^1(\Omega)}^2 + \rho \|q_\rho^\delta - q^\dagger\|_{L^2(\Omega)}^2 = \mathcal{O}(\delta^2)$$

as $\delta \to 0$ and $\rho \sim \delta$. The proof is completed. □

2.1.3. Discussion of the source condition

Now we discuss the source condition (2.14). The crucial assumption in our result on establishing the convergence rate of regularized solutions q_ρ^δ to the q^*-minimum norm solution $q^\dagger$ is the existence of a source element $w^* \in H^1_\diamond(\Omega)^*$ such that

$$q^\dagger - q^* = U'(q^\dagger)^* w^*. \tag{2.22}$$

This is a weak source condition and it does not require any smoothness of $q^\dagger$. Moreover, the smallness requirement on source functions of the general convergence theory for nonlinear ill-posed problems in [41, 42], which is hard to check, is liberated in our source condition.

We note that the source condition (2.22) is fulfilled if and only if there exists a function $w \in H^1_\diamond(\Omega)$ such that

$$\langle q^\dagger - q^*, h\rangle_{L^2(\Omega)} = \langle w, U'(q^\dagger)h\rangle_{H^1(\Omega)} \tag{2.23}$$

for all h belonging to $L^\infty(\Omega)$.

In the following, as q^* is only an a-priori estimate of $q^\dagger$, for simplicity, we assume that $q^* \in H^1(\Omega)$. The following result gives a sufficient condition for (2.23) with a quite weak hypothesis about the regularity of the sought coefficient.

Theorem 2.1.8. *Assume that the boundary $\partial\Omega$ is of class C^1 and $q^\dagger$ belongs to $H^1(\Omega)$. Moreover, suppose that the exact $\overline{u} \in W^{2,\infty}(\Omega)$ and $|\nabla \overline{u}| \geq \gamma$ a.e. on Ω, where γ is a positive constant. Then, the condition* (2.23) *is fulfilled and hence a convergence rate $\mathcal{O}(\sqrt{\delta})$ of L^2-regularization is obtained.*

We remark that the hypothesis $|\nabla \overline{u}| \geq \gamma$ on Ω is quite natural, as if $|\nabla \overline{u}|$ vanishes in a subregion of Ω, then it is impossible to determine q on it. This is one of the reasons why our coefficient identification problem is ill-posed. Further, this assumption guarantees that the set $\Pi_Q(\overline{u})$ is a singleton (see, for example, [83, 110]).

To prove this theorem we need the following results, which are generalizations of that in [88] and [16].

Lemma 2.1.9. *Assume that $\zeta = (\zeta^1, ..., \zeta^d) \in W^{1,\infty}(\Omega)^d$ and the boundary $\partial\Omega$ is of class C^1. Then, for λ sufficiently large, more precisely $\lambda > \frac{1}{2}\|div\, \zeta\|_{L^\infty(\Omega)} + d\|\zeta\|_{W^{1,\infty}(\Omega)^d}$, where $\|\zeta\|_{W^{1,\infty}(\Omega)^d} := \sum_{k=1}^d \|\zeta^k\|_{W^{1,\infty}(\Omega)}$, the equation*

$$\lambda\vartheta + \zeta\nabla\vartheta = p \tag{2.24}$$

has a solution $\vartheta \in H^1(\Omega)$ for each $p \in H^1(\Omega)$. Further, there exists a positive constant C independent of p such that

$$\|\vartheta\|_{H^1(\Omega)} \leq C\|p\|_{H^1(\Omega)}. \tag{2.25}$$

Proof. We associate with Ω an arbitrary fixed open set $\widetilde{\Omega}$ such that $\overline{\Omega} \subset \widetilde{\Omega}$. By the result on the extension of functions (see [117], p. 53), there exist a closed set Π with $\Omega \subset \Pi \subset \widetilde{\Omega}$ and $\widetilde{\zeta} = (\widetilde{\zeta}^1, ..., \widetilde{\zeta}^d) \in W^{1,\infty}(\widetilde{\Omega})^d$ and $\widetilde{p} \in H^1(\widetilde{\Omega})$ such that supp $\widetilde{\zeta}^k \subset \Pi$, supp $\widetilde{p} \subset \Pi$ and $\widetilde{\zeta}^k{}_{|\Omega} = \zeta^k$, $\widetilde{p}_{|\Omega} = p$ for all $k = 1, ..., d$. For each $\epsilon > 0$, we consider the Dirichlet problem

$$-\epsilon\Delta\widetilde{\vartheta}^\epsilon + \widetilde{\zeta}\nabla\widetilde{\vartheta}^\epsilon + \lambda\widetilde{\vartheta}^\epsilon = \widetilde{p} \text{ in } \widetilde{\Omega}, \tag{2.26}$$

$$\widetilde{\vartheta}^\epsilon = 0 \text{ on } \partial\widetilde{\Omega}. \tag{2.27}$$

By the assumption, we conclude that the weak solution $\widetilde{\vartheta}^\epsilon$ to (2.26)–(2.27) is an element of $H^2(\widetilde{\Omega}) \cap H_0^1(\widetilde{\Omega})$ (see, e.g., [93], p. 71). Multiplying the both sides of (2.26) by $-\Delta\widetilde{\vartheta}^\epsilon$ and then integrating by parts, we get

$$\epsilon \int_{\widetilde{\Omega}} (\Delta\widetilde{\vartheta}^\epsilon)^2 - \int_{\widetilde{\Omega}} (\widetilde{\zeta}\nabla\widetilde{\vartheta}^\epsilon)\Delta\widetilde{\vartheta}^\epsilon + \lambda \int_{\widetilde{\Omega}} |\nabla\widetilde{\vartheta}^\epsilon|^2 = \int_{\widetilde{\Omega}} \nabla\widetilde{p}\nabla\widetilde{\vartheta}^\epsilon, \tag{2.28}$$

where we used the fact that $\widetilde{\vartheta}^\epsilon = 0$ and $\widetilde{p} = 0$ on $\partial\widetilde{\Omega}$. Since $\widetilde{\zeta} = 0$ on $\partial\widetilde{\Omega}$, we get

$$\begin{aligned}
-\int_{\widetilde{\Omega}} (\widetilde{\zeta}\nabla\widetilde{\vartheta}^\epsilon)\Delta\widetilde{\vartheta}^\epsilon &= -\int_{\partial\widetilde{\Omega}} \widetilde{\zeta}\nabla\widetilde{\vartheta}^\epsilon(\nabla\widetilde{\vartheta}^\epsilon n) + \int_{\widetilde{\Omega}} \nabla(\widetilde{\zeta}\nabla\widetilde{\vartheta}^\epsilon)\nabla\widetilde{\vartheta}^\epsilon \\
&= \int_{\widetilde{\Omega}} \nabla(\widetilde{\zeta}\nabla\widetilde{\vartheta}^\epsilon)\nabla\widetilde{\vartheta}^\epsilon \\
&= \sum_{i=1}^d \sum_{k=1}^d \int_{\widetilde{\Omega}} (\widetilde{\zeta}^k {\widetilde{\vartheta}^\epsilon}_{x_k x_i} {\widetilde{\vartheta}^\epsilon}_{x_i} + \widetilde{\zeta}^k_{x_i} {\widetilde{\vartheta}^\epsilon}_{x_k} {\widetilde{\vartheta}^\epsilon}_{x_i}) \\
&= \sum_{i=1}^d \sum_{k=1}^d \int_{\widetilde{\Omega}} \widetilde{\zeta}^k {\widetilde{\vartheta}^\epsilon}_{x_k x_i} {\widetilde{\vartheta}^\epsilon}_{x_i} + \sum_{i=1}^d \sum_{k=1}^d \int_{\widetilde{\Omega}} \widetilde{\zeta}^k_{x_i} {\widetilde{\vartheta}^\epsilon}_{x_k} {\widetilde{\vartheta}^\epsilon}_{x_i} \\
&:= \Sigma + \Lambda.
\end{aligned} \tag{2.29}$$

Integrating by parts, we have

$$\begin{aligned}
\Sigma &:= \sum_{i=1}^d \sum_{k=1}^d \int_{\widetilde{\Omega}} \widetilde{\zeta}^k {\widetilde{\vartheta}^\epsilon}_{x_k x_i} {\widetilde{\vartheta}^\epsilon}_{x_i} \\
&= \frac{1}{2} \sum_{i=1}^d \sum_{k=1}^d \int_{\widetilde{\Omega}} \widetilde{\zeta}^k ({\widetilde{\vartheta}^\epsilon}^2_{x_i})_{x_k} \\
&= -\frac{1}{2} \sum_{i=1}^d \sum_{k=1}^d \int_{\widetilde{\Omega}} \widetilde{\zeta}^k_{x_k} {\widetilde{\vartheta}^\epsilon}^2_{x_i} \\
&= -\frac{1}{2} \int_{\widetilde{\Omega}} \mathrm{div}\widetilde{\zeta} |\nabla\widetilde{\vartheta}^\epsilon|^2 \\
&\geq -\frac{1}{2} \|\mathrm{div}\widetilde{\zeta}\|_{L^\infty(\widetilde{\Omega})} \int_{\widetilde{\Omega}} |\nabla\widetilde{\vartheta}^\epsilon|^2.
\end{aligned} \tag{2.30}$$

Further,

$$\begin{aligned}
\left| \int_{\widetilde{\Omega}} \widetilde{\zeta}^k_{x_i} {\widetilde{\vartheta}^\epsilon}_{x_k} {\widetilde{\vartheta}^\epsilon}_{x_i} \right| &\leq \|\widetilde{\zeta}^k\|_{W^{1,\infty}(\widetilde{\Omega})} \left(\int_{\widetilde{\Omega}} {\widetilde{\vartheta}^\epsilon}^2_{x_k} \right)^{1/2} \left(\int_{\widetilde{\Omega}} {\widetilde{\vartheta}^\epsilon}^2_{x_i} \right)^{1/2} \\
&\leq \|\widetilde{\zeta}^k\|_{W^{1,\infty}(\widetilde{\Omega})} \int_{\widetilde{\Omega}} |\nabla\widetilde{\vartheta}^\epsilon|^2.
\end{aligned}$$

Therefore, the second term in the right hand side of (2.29) can be estimated as follows

$$\begin{aligned}
\Lambda &:= \sum_{i=1}^d \sum_{k=1}^d \int_{\widetilde{\Omega}} \widetilde{\zeta}^k_{x_i} {\widetilde{\vartheta}^\epsilon}_{x_k} {\widetilde{\vartheta}^\epsilon}_{x_i} \\
&\geq -\sum_{i=1}^d \sum_{k=1}^d \|\widetilde{\zeta}^k\|_{W^{1,\infty}(\widetilde{\Omega})} \int_{\widetilde{\Omega}} |\nabla\widetilde{\vartheta}^\epsilon|^2 \\
&= -d\|\widetilde{\zeta}\|_{W^{1,\infty}(\widetilde{\Omega})^d} \int_{\widetilde{\Omega}} |\nabla\widetilde{\vartheta}^\epsilon|^2.
\end{aligned} \tag{2.31}$$

From (2.29) and inequalities (2.30), (2.31) we get

$$-\int_{\widetilde{\Omega}} (\widetilde{\zeta}\nabla\widetilde{\vartheta}^{\epsilon})\Delta\widetilde{\vartheta}^{\epsilon} \geq -\left(\frac{1}{2}\|\mathrm{div}\widetilde{\zeta}\|_{L^{\infty}(\widetilde{\Omega})} + d\|\widetilde{\zeta}\|_{W^{1,\infty}(\widetilde{\Omega})^d}\right)\int_{\widetilde{\Omega}} |\nabla\widetilde{\vartheta}^{\epsilon}|^2. \tag{2.32}$$

It follows from (2.28) and (2.32) that

$$\begin{aligned}\epsilon\int_{\widetilde{\Omega}} (\Delta\widetilde{\vartheta}^{\epsilon})^2 + \left(\lambda - \frac{1}{2}\|\mathrm{div}\widetilde{\zeta}\|_{L^{\infty}(\widetilde{\Omega})} - d\|\widetilde{\zeta}\|_{W^{1,\infty}(\widetilde{\Omega})^d}\right)\int_{\widetilde{\Omega}} |\nabla\widetilde{\vartheta}^{\epsilon}|^2 \\ \leq \left(\int_{\widetilde{\Omega}} |\nabla\widetilde{p}|^2\right)^{1/2}\left(\int_{\widetilde{\Omega}} |\nabla\widetilde{\vartheta}^{\epsilon}|^2\right)^{1/2}.\end{aligned} \tag{2.33}$$

Since $\lambda > \frac{1}{2}\|\mathrm{div}\ \zeta\|_{L^{\infty}(\Omega)} + d\|\zeta\|_{W^{1,\infty}(\Omega)^d}$, the last inequality yields

$$\int_{\widetilde{\Omega}} |\nabla\widetilde{\vartheta}^{\epsilon}|^2 \leq \frac{1}{\left(\lambda - \frac{1}{2}\|\mathrm{div}\widetilde{\zeta}\|_{L^{\infty}(\widetilde{\Omega})} - d\|\widetilde{\zeta}\|_{W^{1,\infty}(\widetilde{\Omega})^d}\right)^2}\int_{\widetilde{\Omega}} |\nabla\widetilde{p}|^2. \tag{2.34}$$

On the other hand, it follows from (2.34) and the Poincaré-Friedrichs inequality that the set $(\widetilde{\vartheta}^{\epsilon})_{\epsilon>0}$ is bounded with respect to $H_0^1(\widetilde{\Omega})$-norm. Hence there exist a subsequence of it denoted by the same symbol and $\widetilde{\vartheta} \in H_0^1(\widetilde{\Omega})$ such that $\widetilde{\vartheta}^{\epsilon} \to \widetilde{\vartheta}$ in $H_0^1(\widetilde{\Omega})$ weakly as ϵ tends to zero. Multiplying the both sides of (2.26) by $w \in C_c^1(\widetilde{\Omega})$, the set of differentiable functions with compact support lying in $\widetilde{\Omega}$, we obtain

$$-\epsilon\int_{\widetilde{\Omega}} w\Delta\widetilde{\vartheta}^{\epsilon} + \left(\int_{\widetilde{\Omega}} w\widetilde{\zeta}\nabla\widetilde{\vartheta}^{\epsilon} + \lambda\int_{\widetilde{\Omega}} w\widetilde{\vartheta}^{\epsilon}\right) = \int_{\widetilde{\Omega}} w\widetilde{p}. \tag{2.35}$$

Integrating by parts, we get

$$\begin{aligned}\left|\int_{\widetilde{\Omega}} w\Delta\widetilde{\vartheta}^{\epsilon}\right| &= \left|\int_{\widetilde{\Omega}} \nabla w\nabla\widetilde{\vartheta}^{\epsilon}\right| \\ &\leq \left(\int_{\widetilde{\Omega}} |\nabla w|^2\right)^{1/2}\left(\int_{\widetilde{\Omega}} |\nabla\widetilde{\vartheta}_{\epsilon}|^2\right)^{1/2}.\end{aligned}$$

It follows from the last inequality and (2.34) that the first term in the left hand side of (2.35) tends to zero as ϵ goes to zero. Besides, since $\widetilde{\vartheta}^{\epsilon} \to \widetilde{\vartheta}$ in $H_0^1(\widetilde{\Omega})$ weakly, we have

$$\int_{\widetilde{\Omega}} w\widetilde{\zeta}\nabla\widetilde{\vartheta}^{\epsilon} + \lambda\int_{\widetilde{\Omega}} w\widetilde{\vartheta}^{\epsilon} \to \int_{\widetilde{\Omega}} w\widetilde{\zeta}\nabla\widetilde{\vartheta} + \lambda\int_{\widetilde{\Omega}} w\widetilde{\vartheta}$$

when ϵ goes to zero.

Thus,

$$\int_{\widetilde{\Omega}} w\widetilde{\zeta}\nabla\widetilde{\vartheta} + \int_{\widetilde{\Omega}} \lambda w\widetilde{\vartheta} = \int_{\widetilde{\Omega}} w\widetilde{p}$$

for all $w \in C_c^1(\widetilde{\Omega})$. This means that

$$\lambda\widetilde{\vartheta} + \widetilde{\zeta}\nabla\widetilde{\vartheta} = \widetilde{p} \tag{2.36}$$

a.e. on $\widetilde{\Omega}$. Then, $\widetilde{\vartheta}_{|\Omega} := \vartheta$ is a solution of equation (2.24). Since $\widetilde{\vartheta}^{\epsilon} \to \widetilde{\vartheta}$ in $H_0^1(\widetilde{\Omega})$ weakly, it follows from (2.34) that

$$\left(\lambda - \frac{1}{2}\|\mathrm{div}\widetilde{\zeta}\|_{L^{\infty}(\widetilde{\Omega})} - d\|\widetilde{\zeta}\|_{W^{1,\infty}(\widetilde{\Omega})^d}\right)\left(\int_{\widetilde{\Omega}} |\nabla\widetilde{\vartheta}|^2\right)^{1/2} \leq \left(\int_{\widetilde{\Omega}} |\nabla\widetilde{p}|^2\right)^{1/2}$$

for all the open set $\widetilde{\Omega}$ satisfying $\overline{\Omega} \subset \widetilde{\Omega}$. Hence

$$\left(\lambda - \frac{1}{2}\|\mathrm{div}\widetilde{\zeta}\|_{L^\infty(\Omega)} - d\|\widetilde{\zeta}\|_{W^{1,\infty}(\Omega)^d}\right)\left(\int_\Omega |\nabla\vartheta|^2\right)^{1/2} \le \left(\int_\Omega |\nabla p|^2\right)^{1/2}. \tag{2.37}$$

Now integrating by parts, we get

$$\begin{aligned}
\int_{\widetilde{\Omega}} (\widetilde{\zeta}\nabla\widetilde{\vartheta})\widetilde{\vartheta} &= \sum_{k=1}^d \int_{\widetilde{\Omega}} (\widetilde{\zeta}^k\widetilde{\vartheta})\widetilde{\vartheta}_{x_k} \\
&= -\sum_{k=1}^d \int_{\widetilde{\Omega}} (\widetilde{\zeta}^k\widetilde{\vartheta})_{x_k}\widetilde{\vartheta} \\
&= -\sum_{k=1}^d \int_{\widetilde{\Omega}} \widetilde{\zeta}^k_{x_k}\widetilde{\vartheta}^2 - \sum_{k=1}^d \int_{\widetilde{\Omega}} (\widetilde{\zeta}^k\widetilde{\vartheta})\widetilde{\vartheta}_{x_k} \\
&= -\int_{\widetilde{\Omega}} \mathrm{div}\widetilde{\zeta}\widetilde{\vartheta}^2 - \int_{\widetilde{\Omega}} (\widetilde{\zeta}\nabla\widetilde{\vartheta})\widetilde{\vartheta}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\widetilde{\Omega}} (\widetilde{\zeta}\nabla\widetilde{\vartheta})\widetilde{\vartheta} &= -\frac{1}{2}\int_{\widetilde{\Omega}} \mathrm{div}\widetilde{\zeta}\widetilde{\vartheta}^2 \\
&\ge -\frac{1}{2}\|\mathrm{div}\widetilde{\zeta}\|_{L^\infty(\widetilde{\Omega})}\int_{\widetilde{\Omega}} \widetilde{\vartheta}^2.
\end{aligned} \tag{2.38}$$

Multiplying the both sides of (2.36) by $\widetilde{\vartheta}$ and then integrating over $\widetilde{\Omega}$, by the aid of (2.38), we obtain

$$\begin{aligned}
\left(\lambda - \frac{1}{2}\|\mathrm{div}\widetilde{\zeta}\|_{L^\infty(\widetilde{\Omega})}\right)\int_{\widetilde{\Omega}} \widetilde{\vartheta}^2 &\le \lambda\int_{\widetilde{\Omega}} \widetilde{\vartheta}^2 + \int_{\widetilde{\Omega}} (\widetilde{\zeta}\nabla\widetilde{\vartheta})\widetilde{\vartheta} \\
&= \int_{\widetilde{\Omega}} \widetilde{\vartheta}\widetilde{p} \\
&\le \left(\int_{\widetilde{\Omega}} \widetilde{\vartheta}^2\right)^{1/2}\left(\int_{\widetilde{\Omega}} \widetilde{p}^2\right)^{1/2}
\end{aligned}$$

for all the open set $\widetilde{\Omega}$ satisfying $\overline{\Omega} \subset \widetilde{\Omega}$. Then,

$$\left(\lambda - \frac{1}{2}\|\mathrm{div}\widetilde{\zeta}\|_{L^\infty(\Omega)}\right)\left(\int_\Omega \vartheta^2\right)^{1/2} \le \left(\int_\Omega p^2\right)^{1/2}. \tag{2.39}$$

Combining (2.37) and (2.39), we arrive at (2.25). The proof is completed. □

Lemma 2.1.10. *Assume that the boundary $\partial\Omega$ is of class C^1 and $u \in W^{2,\infty}(\Omega)$ and $|\nabla u| \ge \gamma$ a.e. on Ω, where γ is a positive constant. Then, for any element $\widetilde{q} \in H^1(\Omega)$, there exists $v \in H^1(\Omega)$ satisfying*

$$\nabla u \cdot \nabla v = \widetilde{q}. \tag{2.40}$$

Further, there exists a positive constant C independent of $\widetilde{q}$ such that

$$\|v\|_{H^1(\Omega)} \le C\|\widetilde{q}\|_{H^1(\Omega)}. \tag{2.41}$$

Proof. We denote by $\zeta := \frac{\nabla u}{|\nabla u|^2}$. By the assumption on the function u, we have $\zeta \in W^{1,\infty}(\Omega)^d$ and $\left(\frac{1}{e^{\lambda u}|\nabla u|^2}\right)\widetilde{q} \in H^1(\Omega)$. Then, by Lemma 2.1.9, for λ sufficiently large, there exists $\vartheta \in H^1(\Omega)$ such that

$$\lambda\vartheta + \frac{\nabla u}{|\nabla u|^2}\nabla\vartheta = \left(\frac{1}{e^{\lambda u}|\nabla u|^2}\right)\widetilde{q}$$

a.e. on Ω and $\|\vartheta\|_{H^1(\Omega)} \le C\|\widetilde{q}\|_{H^1(\Omega)}$, where the positive constant C is independent of $\widetilde{q}$. Put $v := e^{\lambda u}\vartheta$, we conclude that v belongs to $H^1(\Omega)$ and satisfies (2.40) and (2.41). The lemma is proved. □

Proof of Theorem 2.1.8. It follows from Lemma 2.1.10 that there exists a function $v \in H^1(\Omega)$ such that $\nabla U(q^\dagger)\cdot\nabla v = q^\dagger - q^*$. Set

$$\widehat{v} := \frac{\int_\Omega v}{\text{mes}(\Omega)} - v.$$

Then,

$$-\nabla U(q^\dagger)\cdot\nabla\widehat{v} = q^\dagger - q^* \text{ and } \widehat{v} \in H^1_\diamond(\Omega).$$

Hence for all h in $L^\infty(\Omega)$, we have

$$\begin{aligned}\langle q^\dagger - q^*, h\rangle_{L^2(\Omega)} &= -\int_\Omega h\nabla U(q^\dagger)\nabla\widehat{v} \\ &\quad \text{(in virtue of (1.13))} \\ &= \int_\Omega q^\dagger\nabla U'(q^\dagger)h\nabla\widehat{v} \\ &= \left\langle w, U'(q^\dagger)h\right\rangle_{H^1(\Omega)}\end{aligned}$$

for some $w \in H^1_\diamond(\Omega)$ independent of $h \in L^\infty(\Omega)$. The theorem is proved. □

2.2 Convergence rates for L^2-regularization of the reaction coefficient identification problem

2.2.1. L^2-regularization

Now we use the functional $G_{z^\delta}(\cdot)$ with L^2-regularization to solve the problem of identifying the coefficient a in (1.3)–(1.4). Namely, we solve the strictly convex minimization problem

$$\min_{a\in A} G_{z^\delta}(a) + \rho\|a - a^*\|^2_{L^2(\Omega)} \qquad (\mathbf{P}^a_{\rho,\delta})$$

with $\rho > 0$ being the regularization parameter, a^* an a-priori estimate of the true coefficient. We will see that the cost functional of problem $(\mathbf{P}^a_{\rho,\delta})$ is weakly l.s.c in the $L^2(\Omega)$-norm and strictly convex, therefore it attains a *unique solution* a^δ_ρ on the weakly compact set A. Furthermore, we will prove that the unique solution a^δ_ρ of $(\mathbf{P}^a_{\rho,\delta})$ is stable in the $L^2(\Omega)$-norm with respect to the regularization parameter ρ and to the data z^δ.

Now we introduce the notion of a^*-*minimum norm solution.*

Lemma 2.2.1. *The set*

$$\Pi_A(\overline{u}) := \{a \in A \mid U(a) = \overline{u}\}$$

is nonempty, convex, bounded and closed in the $L^2(\Omega)$-norm. Hence there is a unique solution $a^\dagger$ of problem

$$\min_{a\in\Pi_A(\overline{u})} \|a - a^*\|^2_{L^2(\Omega)} \tag{$\boldsymbol{K}^a$}$$

which is called by the a^*-minimum norm solution of the identification problem.

Proof. It is clear that $\Pi_A(\overline{u})$ is a nonempty, convex and bounded set. Suppose that the sequence $(a_n) \subset \Pi_A(\overline{u})$ converges to a in the $L^2(\Omega)$-norm. Since A is closed in the $L^2(\Omega)$-norm, we have $a \in A$. Further, applying Lemma 1.2.2, we conclude that $\overline{u} = U(a)$. Therefore, $a \in \Pi_A(\overline{u})$. The lemma is proved. □

Now, we are in a position to prove main results of this subsection.

Theorem 2.2.2. *There exists a unique solution a^δ_ρ of problem $(\boldsymbol{P}^a_{\rho,\delta})$.*

Proof. By Lemmas 1.2.3 and 1.2.4, we see that the cost function of $(\mathbf{P}^a_{\rho,\delta})$ is strictly convex and weakly l.s.c on the weakly compact set A in the $L^2(\Omega)$-norm. Then, the assertion of the theorem directly follows. □

Theorem 2.2.3. *For a fixed regularization parameter $\rho > 0$, let (z^{δ_n}) be a sequence converging to z^δ in $H^1(\Omega)$ and $(a^{\delta_n}_\rho)$ be unique minimizers of problems*

$$\min_{a\in A} G_{z^{\delta_n}}(a) + \rho\|a - a^*\|^2_{L^2}.$$

Then, $(a^{\delta_n}_\rho)$ converges to the unique solution a^δ_ρ of $(\boldsymbol{P}^a_{\rho,\delta})$ in the $L^2(\Omega)$-norm.

Proof. By the definition of $(a^{\delta_n}_\rho)$ we have

$$\begin{aligned} G_{z^{\delta_n}}(a^{\delta_n}_\rho) + \rho\|a^{\delta_n}_\rho - a^*\|^2_{L^2(\Omega)} &\le G_{z^{\delta_n}}(a) + \rho\|a - a^*\|^2_{L^2(\Omega)} \\ &\le \max\{1, \overline{a}\}\Big(\|U(a)\|^2_{H^1(\Omega)} + C\Big) + \rho\|a - a^*\|^2_{L^2(\Omega)} \end{aligned} \tag{2.42}$$

for any $a \in A$ and $n \in \mathbb{N}$, where the positive constant C is independent of n such that $\|z^{\delta_n}\|^2_{H^1(\Omega)} \le C$ for all $n \in \mathbb{N}$. It follows that $(a^{\delta_n}_\rho)$ is bounded in the $L^2(\Omega)$-norm. Therefore, there exist a subsequence $(a^{\delta_{1n}}_\rho)$ of $(a^{\delta_n}_\rho)$ and an element $a^\delta_\rho \in L^2(\Omega)$ such that $(a^{\delta_{1n}}_\rho)$ weakly converges to a^δ_ρ in $L^2(\Omega)$. By the convexity and closedness of A in the $L^2(\Omega)$-norm, we conclude that $a^\delta_\rho \in A$. On the other hand, since $G_{z^\delta}(\cdot)$ and the norm $\|\cdot\|_{L^2(\Omega)}$ are weakly l.s.c., we have

$$G_{z^\delta}(a^\delta_\rho) \le \liminf_n G_{z^\delta}(a^{\delta_{1n}}_\rho) \tag{2.43}$$

and

$$\|a^\delta_\rho - a^*\|^2_{L^2(\Omega)} \le \liminf_n \|a^{\delta_{1n}}_\rho - a^*\|^2_{L^2(\Omega)}. \tag{2.44}$$

Besides, applying Lemma 1.2.5, we get

$$\liminf_n G_{z^\delta}(a^{\delta_{1n}}_\rho) = \liminf_n G_{z^{\delta_{1n}}}(a^{\delta_{1n}}_\rho). \tag{2.45}$$

From (2.43), (2.44) and (2.45) we get

$$\begin{aligned}
G_{z^\delta}(a_\rho^\delta) + \rho\|a_\rho^\delta - a^*\|^2_{L^2(\Omega)} &\le \liminf_n G_{z^{\delta_{1n}}}(a_\rho^{\delta_{1n}}) + \rho \liminf_n \|a_\rho^{\delta_{1n}} - a^*\|^2_{L^2(\Omega)} \\
&\le \liminf_n \Big(G_{z^{\delta_{1n}}}(a_\rho^{\delta_{1n}}) + \rho\|a_\rho^{\delta_{1n}} - a^*\|^2_{L^2(\Omega)} \Big) \\
&\le \limsup_n \Big(G_{z^{\delta_{1n}}}(a_\rho^{\delta_{1n}}) + \rho\|a_\rho^{\delta_{1n}} - a^*\|^2_{L^2(\Omega)} \Big) \\
&\quad \text{(in virtue of (2.42))} \\
&\le \limsup_n \Big(G_{z^{\delta_{1n}}}(a) + \rho\|a - a^*\|^2_{L^2(\Omega)} \Big) \\
&= G_{z^\delta}(a) + \rho\|a - a^*\|^2_{L^2(\Omega)} \qquad (2.46)
\end{aligned}$$

for any $a \in A$. This means that a_ρ^δ is a (unique) solution of $(\mathbf{P}^a_{\rho,\delta})$. By contradiction we show that $(a_\rho^{\delta_{1n}})$ converges to a_ρ^δ in the $L^2(\Omega)$-norm. In fact, assume that $a_\rho^{\delta_{1n}} \nrightarrow a_\rho^\delta$. This and (2.44) follow that

$$\epsilon := \limsup_n \|a_\rho^{\delta_{1n}} - a^*\|^2_{L^2(\Omega)} > \|a_\rho^\delta - a^*\|^2_{L^2(\Omega)}. \qquad (2.47)$$

Therefore, there exists a subsequence $(a_\rho^{\delta_{2n}})$ of $(a_\rho^{\delta_{1n}})$ such that

$$a_\rho^{\delta_{2n}} \rightharpoonup a_\rho^\delta \text{ weakly in } L^2(\Omega) \text{ and } \|a_\rho^{\delta_{2n}} - a^*\|^2_{L^2(\Omega)} \to \epsilon. \qquad (2.48)$$

Choosing $a = a_\rho^\delta$ in (2.46) and by (2.48), we get

$$\begin{aligned}
G_{z^\delta}(a_\rho^\delta) + \rho\|a_\rho^\delta - a^*\|^2_{L^2(\Omega)} &= \lim_n \Big(G_{z^{\delta_{2n}}}(a_\rho^{\delta_{2n}}) + \rho\|a_\rho^{\delta_{2n}} - a^*\|^2_{L^2(\Omega)} \Big) \\
&\ge \liminf_n G_{z^{\delta_{2n}}}(a_\rho^{\delta_{2n}}) + \rho\epsilon \\
&= \liminf_n G_{z^\delta}(a_\rho^{\delta_{2n}}) + \rho\epsilon,
\end{aligned}$$

by Lemma 1.2.5. Combining the last inequality with (2.45) and by (2.47), we arrive at

$$\begin{aligned}
\liminf_n G_{z^\delta}(a_\rho^{\delta_{2n}}) &\le G_{z^\delta}(a_\rho^\delta) + \rho\Big(\|a_\rho^\delta - a^*\|^2_{L^2(\Omega)} - \epsilon\Big) \\
&< G_{z^\delta}(a_\rho^\delta).
\end{aligned}$$

Since $\liminf_n G_{z^\delta}(a_\rho^{\delta_{1n}}) \le \liminf_n G_{z^\delta}(a_\rho^{\delta_{2n}})$, the last inequality gives a contradiction to (2.43). Thus, $(a_\rho^{\delta_{1n}})$ converges to a_ρ^δ in the $L^2(\Omega)$-norm. Since the element a_ρ^δ is unique, we conclude that the whole sequence $(a_\rho^{\delta_n})$ also converges to a_ρ^δ in the $L^2(\Omega)$-norm. The theorem is proved. □

Theorem 2.2.4. *For any positive sequence $(\delta_n) \to 0$, let $\rho_n := \rho(\delta_n)$ be such that*

$$\rho_n \to 0 \text{ and } \frac{\delta_n^2}{\rho_n} \to 0 \text{ as } n \to \infty.$$

Moreover, let (z^{δ_n}) be a sequence satisfying $\|\overline{u} - z^{\delta_n}\|_{H^1(\Omega)} \le \delta_n$ and $\big(a_{\rho_n}^{\delta_n}\big)$ be the unique minimizers of the problems

$$\min_{a \in A} G_{z^{\delta_n}}(a) + \rho_n\|a - a^*\|^2_{L^2(\Omega)}.$$

Then, $\big(a_{\rho_n}^{\delta_n}\big)$ converges to the unique solution $a^\dagger$ of problem $(\boldsymbol{K}^a)$ in the $L^2(\Omega)$-norm.

Proof. For all $n \in \mathbb{N}$, by the definition of $a^{\delta_n}_{\rho_n}$, we have

$$\begin{aligned} G_{z^{\delta_n}}(a^{\delta_n}_{\rho_n}) + \rho_n \|a^{\delta_n}_{\rho_n} - a^*\|^2_{L^2(\Omega)} &\leq G_{z^{\delta_n}}(a^\dagger) + \rho_n \|a^\dagger - a^*\|^2_{L^2(\Omega)} \\ &\leq \frac{\max\{1, \overline{a}\}}{2} \delta_n^2 + \rho_n \|a^\dagger - a^*\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.49}$$

By the assumption $\delta_n^2/\rho_n \to 0$, the last inequality yields

$$\limsup_n \|a^{\delta_n}_{\rho_n} - a^*\|^2_{L^2(\Omega)} \leq \|a^\dagger - a^*\|^2_{L^2(\Omega)}. \tag{2.50}$$

It follows from estimate (2.50) that there exist a subsequence $(a^{\delta_{1_n}}_{\rho_{1_n}})$ of $(a^{\delta_n}_{\rho_n})$ and $\widehat{a} \in A$ such that $(a^{\delta_{1_n}}_{\rho_{1_n}})$ weakly converges to $\widehat{a}$ in $L^2(\Omega)$. Thus,

$$G_{\bar{u}}(\widehat{a}) \leq \liminf_n G_{\bar{u}}(a^{\delta_{1_n}}_{\rho_{1_n}}).$$

By Lemma 1.2.6, the last estimate follows that

$$\frac{\beta}{2} \|U(\widehat{a}) - \overline{u}\|^2_{H^1(\Omega)} \leq \liminf_n G_{\bar{u}}(a^{\delta_{1_n}}_{\rho_{1_n}}). \tag{2.51}$$

Now, applying Lemma 1.2.5 and by (2.49), we have

$$\begin{aligned} \liminf_n G_{\bar{u}}(a^{\delta_{1_n}}_{\rho_{1_n}}) &= \liminf_n G_{z^{\delta_{1_n}}}(a^{\delta_{1_n}}_{\rho_{1_n}}) \\ &\leq \liminf_n \left(\frac{\max\{1, \overline{a}\}}{2} \delta_{1_n}^2 + \rho_{1_n} \|a^\dagger - a^*\|^2_{L^2(\Omega)} \right) \\ &= 0. \end{aligned} \tag{2.52}$$

It follows from inequalities (2.51) and (2.52) that $U(\widehat{a}) = \overline{u}$. Therefore, replacing $a^\dagger$ in (2.49) by $\widehat{a}$, we also get

$$\limsup_n \|a^{\delta_n}_{\rho_n} - a^*\|^2_{L^2(\Omega)} \leq \|\widehat{a} - a^*\|^2_{L^2(\Omega)}. \tag{2.53}$$

We have

$$\begin{aligned} \limsup_n \|a^{\delta_{1_n}}_{\rho_{1_n}} - \widehat{a}\|^2_{L^2(\Omega)} &= \limsup_n \|a^{\delta_{1_n}}_{\rho_{1_n}} - a^* + a^* - \widehat{a}\|^2_{L^2(\Omega)} \\ &= \limsup_n \left(\|a^{\delta_{1_n}}_{\rho_{1_n}} - a^*\|^2_{L^2(\Omega)} + \|\widehat{a} - a^*\|^2_{L^2(\Omega)} \right) \\ &\quad + 2 \lim_n \left\langle a^{\delta_{1_n}}_{\rho_{1_n}} - a^*, a^* - \widehat{a} \right\rangle_{L^2(\Omega)}. \end{aligned}$$

It follows from the last inequality, (2.53), and the weak convergence of $(a^{\delta_{1_n}}_{\rho_{1_n}})$ to $\widehat{a}$ in $L^2(\Omega)$ that

$$\begin{aligned} \limsup_n \|a^{\delta_{1_n}}_{\rho_{1_n}} - \widehat{a}\|^2_{L^2(\Omega)} &\leq \|\widehat{a} - a^*\|^2_{L^2(\Omega)} + \|\widehat{a} - a^*\|^2_{L^2(\Omega)} + 2\langle \widehat{a} - a^*, a^* - \widehat{a} \rangle_{L^2(\Omega)} \\ &= 0. \end{aligned}$$

Now, by the definition of $a^\dagger$ and (2.50), we obtain that

$$\begin{aligned} \|a^\dagger - a^*\|^2_{L^2(\Omega)} &\leq \|\widehat{a} - a^*\|^2_{L^2(\Omega)} \\ &= \lim_n \|a^{\delta_{1_n}}_{\rho_{1_n}} - a^*\|^2_{L^2(\Omega)} \\ &\leq \|a^\dagger - a^*\|^2_{L^2(\Omega)}. \end{aligned}$$

Hence $\|a^\dagger - a^*\|^2_{L^2(\Omega)} = \|\widehat{a} - a^*\|^2_{L^2(\Omega)}$ or $a^\dagger = \widehat{a}$, by the uniqueness of $a^\dagger$. The theorem is proved. □

2.2.2. Convergence rates

We now investigate the convergence rate of regularized solutions a_ρ^δ to the a^*-minimum norm solution $a^\dagger$ of the equation $U(a) = \overline{u}$.

Theorem 2.2.5. *Assume that there exists a function $w^* \in H^1(\Omega)^*$ such that*

$$a^\dagger - a^* = U'(a^\dagger)^* w^*. \tag{2.54}$$

Then,

$$\|a_\rho^\delta - a^\dagger\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta}) \text{ and } \|U(a_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

To prove Theorem 2.2.5, we need the following auxiliary result.

Lemma 2.2.6. *Let $(\psi_\rho)_{\rho\in(0,1)}$ be a family of functions belonging to $C^\infty(\overline{\Omega})$ bounded in the $H^1(\Omega)$-norm. Let $(\phi_\rho)_{\rho\in(0,1)}$ be the family of (unique) solutions in $H^1(\Omega)$ of the following Neumann problems*

$$-\Delta\phi_\rho + a\phi_\rho = \psi_\rho - \Delta\psi_\rho \text{ in } \Omega, \tag{2.55}$$

$$\frac{\partial\phi_\rho}{\partial n} = \frac{\partial\psi_\rho}{\partial n} \text{ on } \partial\Omega \tag{2.56}$$

with a fixed element $a \in A$. Then, there exists a positive constant C such that

$$\max\left\{\int_\Omega \phi_\rho^2, \int_\Omega |\nabla\phi_\rho|^2\right\} \le C^2$$

for all $\rho \in (0,1)$.

Proof. The weak form of (2.55)–(2.56) is

$$\int_\Omega \nabla\phi_\rho\nabla v + \int_\Omega a\phi_\rho v = \int_\Omega \psi_\rho v + \int_\Omega \nabla\psi_\rho\nabla v$$

for all $v \in H^1(\Omega)$. Taking $v = \phi_\rho$, by the definition of the set A, we have

$$\begin{aligned}\int_\Omega |\nabla\phi_\rho|^2 + \underline{a}\int_\Omega \phi_\rho^2 &\le \int_\Omega |\nabla\phi_\rho|^2 + \int_\Omega a\phi_\rho^2 \\ &= \int_\Omega \psi_\rho\phi_\rho + \int_\Omega \nabla\psi_\rho\nabla\phi_\rho.\end{aligned}$$

Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\int_\Omega |\nabla\phi_\rho|^2 + \underline{a}\int_\Omega \phi_\rho^2 &\le \left(\int_\Omega \psi_\rho^2\right)^{1/2}\left(\int_\Omega \phi_\rho^2\right)^{1/2} + \left(\int_\Omega |\nabla\psi_\rho|^2\right)^{1/2}\left(\int_\Omega |\nabla\phi_\rho|^2\right)^{1/2} \\ &\le \frac{1}{2\underline{a}}\int_\Omega \psi_\rho^2 + \frac{\underline{a}}{2}\int_\Omega \phi_\rho^2 + \frac{1}{2}\int_\Omega |\nabla\psi_\rho|^2 + \frac{1}{2}\int_\Omega |\nabla\phi_\rho|^2.\end{aligned}$$

Hence

$$\frac{1}{2}\int_\Omega |\nabla\phi_\rho|^2 + \frac{\underline{a}}{2}\int_\Omega \phi_\rho^2 \le \frac{1}{2\underline{a}}\int_\Omega \psi_\rho^2 + \frac{1}{2}\int_\Omega |\nabla\psi_\rho|^2. \tag{2.57}$$

By the assumption, there exists a positive constant C such that

$$\|\psi_\rho\|_{H^1(\Omega)}^2 \le C$$

for all $\rho \in (0,1)$. This and inequality (2.57) yield

$$\max\left\{\int_\Omega \phi_\rho^2, \int_\Omega |\nabla\phi_\rho|^2\right\} \le \frac{1+\underline{a}}{\underline{a}\min\{1,\underline{a}\}}C$$

for all $\rho \in (0,1)$. The lemma is proved. □

Proof of Theorem 2.2.5. By the definition of regularized solutions a_ρ^δ and (1.5), we have

$$\begin{aligned} G(a_\rho^\delta) + \rho\|a_\rho^\delta - a^*\|^2_{L^2(\Omega)} &\le G(a^\dagger) + \rho\|a^\dagger - a^*\|^2_{L^2(\Omega)} \\ &\le \frac{\max\{1,\overline{a}\}}{2}\delta^2 + \rho\|a^\dagger - a^*\|^2_{L^2(\Omega)}. \end{aligned}$$

Therefore,

$$G(a_\rho^\delta) + \rho\|a^\dagger - a_\rho^\delta\|^2_{L^2(\Omega)} \le \frac{\max\{1,\overline{a}\}}{2}\delta^2 + 2\rho\langle a^\dagger - a^*, a^\dagger - a_\rho^\delta\rangle_{L^2(\Omega)}. \tag{2.58}$$

By (2.12), (2.13), (2.54) and the Riesz representation theorem, there exists an element $w \in H^1(\Omega)$ such that

$$2\rho\langle a^\dagger - a^*, a^\dagger - a_\rho^\delta\rangle_{L^2(\Omega)} = 2\rho\langle w, U'(a^\dagger)(a^\dagger - a_\rho^\delta)\rangle_{H^1(\Omega)}.$$

For each $\rho \in (0,1)$, take $\psi_\rho \in C^\infty(\overline{\Omega})$ such that

$$\|w - \psi_\rho\|_{H^1(\Omega)} \le \rho. \tag{2.59}$$

Then,

$$\begin{aligned} 2\rho\langle a^\dagger - a^*, a^\dagger - a_\rho^\delta\rangle_{L^2(\Omega)} &= 2\rho\langle \psi_\rho, U'(a^\dagger)(a^\dagger - a_\rho^\delta)\rangle_{H^1(\Omega)} \\ &\quad + 2\rho\langle w - \psi_\rho, U'(a^\dagger)(a^\dagger - a_\rho^\delta)\rangle_{H^1(\Omega)} \\ &\le 2\rho\langle \psi_\rho, U'(a^\dagger)(a^\dagger - a_\rho^\delta)\rangle_{H^1(\Omega)} \\ &\quad + 2\rho^2\|U'(a^\dagger)(a^\dagger - a_\rho^\delta)\|_{H^1(\Omega)} \\ &:= \Sigma + \Lambda. \end{aligned} \tag{2.60}$$

As $a^\dagger$ and a^δ belong to A, we have

$$\Lambda = 2\rho^2\|U'(a^\dagger)(a^\dagger - a_\rho^\delta)\|_{H^1(\Omega)} \le \frac{4\overline{a}\Lambda_\beta}{\beta}\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right)\rho^2. \tag{2.61}$$

For each $\rho \in (0,1)$, consider the following Neumann problem

$$\begin{aligned} -\Delta\phi_\rho + a^\dagger\phi_\rho &= \psi_\rho - \Delta\psi_\rho \text{ in } \Omega, \\ \frac{\partial\phi_\rho}{\partial n} &= \frac{\partial\psi_\rho}{\partial n} \text{ on } \partial\Omega. \end{aligned}$$

The unique solution in $H^1(\Omega)$ of the last problem satisfies the identity

$$\int_\Omega \nabla\phi_\rho\nabla v + \int_\Omega a^\dagger\phi_\rho v = \int_\Omega(\psi_\rho - \Delta\psi_\rho)v + \int_{\partial\Omega}\frac{\partial\psi_\rho}{\partial n}v$$

for all $v \in H^1(\Omega)$. Taking $v = U'(a^\dagger)(a^\dagger - a_\rho^\delta)$ and integrating by parts, we obtain

$$\begin{aligned} \Sigma &= 2\rho\langle \psi_\rho, U'(a^\dagger)(a^\dagger - a_\rho^\delta)\rangle_{H^1(\Omega)} \\ &= 2\rho\int_\Omega \nabla\phi_\rho\nabla U'(a^\dagger)(a^\dagger - a_\rho^\delta) + 2\rho\int_\Omega a^\dagger\phi_\rho U'(a^\dagger)(a^\dagger - a_\rho^\delta). \end{aligned}$$

By equality (1.23), we have

$$\Sigma = -2\rho \int_\Omega (a^\dagger - a_\rho^\delta) U(a^\dagger)\phi_\rho.$$

Taking $v = \phi_\rho$ in (1.17), we get

$$\begin{aligned}\int_\Omega \nabla U(a^\dagger)\nabla\phi_\rho + \int_\Omega a^\dagger U(a^\dagger)\phi_\rho &= \int_\Omega f\phi_\rho + \int_{\partial\Omega} g\phi_\rho \\ &= \int_\Omega \nabla U(a_\rho^\delta)\nabla\phi_\rho + \int_\Omega a_\rho^\delta U(a_\rho^\delta)\phi_\rho.\end{aligned}$$

Therefore,

$$\begin{aligned}\Sigma &= 2\rho\int_\Omega a_\rho^\delta U(a^\dagger)\phi_\rho - 2\rho\int_\Omega a^\dagger U(a^\dagger)\phi_\rho \\ &= 2\rho\int_\Omega a_\rho^\delta U(a^\dagger)\phi_\rho - 2\rho\int_\Omega a_\rho^\delta U(a_\rho^\delta)\phi_\rho \\ &+ 2\rho\int_\Omega \nabla U(a^\dagger)\nabla\phi_\rho - 2\rho\int_\Omega \nabla U(a_\rho^\delta)\cdot\nabla\phi_\rho \\ &= 2\rho\int_\Omega a_\rho^\delta (U(a^\dagger) - z^\delta)\phi_\rho + 2\rho\int_\Omega a_\rho^\delta(z^\delta - U(a_\rho^\delta))\phi_\rho \\ &+ 2\rho\int_\Omega \nabla(U(a^\dagger) - z^\delta)\nabla\phi_\rho + 2\rho\int_\Omega \nabla(z^\delta - U(a_\rho^\delta))\nabla\phi_\rho.\end{aligned}$$

Using the Cauchy-Schwartz inequality, we get

$$\begin{aligned}\Sigma &\le 2\rho\left(\int_\Omega (U(a^\dagger) - z^\delta)^2\right)^{1/2}\left(\int_\Omega (a_\rho^\delta)^2\phi_\rho^2\right)^{1/2} \\ &+ 2\rho\left(\int_\Omega a_\rho^\delta(z^\delta - U(a_\rho^\delta))^2\right)^{1/2}\left(\int_\Omega a_\rho^\delta\phi_\rho^2\right)^{1/2} \\ &+ 2\rho\left(\int_\Omega |\nabla(U(a^\dagger) - z^\delta)|^2\right)^{1/2}\left(\int_\Omega |\nabla\phi_\rho|^2\right)^{1/2} \\ &+ 2\rho\left(\int_\Omega \left|\nabla\left(z^\delta - U(a_\rho^\delta)\right)\right|^2\right)^{1/2}\left(\int_\Omega |\nabla\phi_\rho|^2\right)^{1/2}.\end{aligned}$$

Then, by (1.5),

$$\begin{aligned}\Sigma &\le 2\rho\delta\overline{a}\left(\int_\Omega \phi_\rho^2\right)^{1/2} + 2\rho\sqrt{\overline{a}}\left(\int_\Omega a_\rho^\delta(z^\delta - U(a_\rho^\delta))^2\right)^{1/2}\left(\int_\Omega \phi_\rho^2\right)^{1/2} \\ &+ 2\rho\delta\left(\int_\Omega |\nabla\phi_\rho|^2\right)^{1/2} + 2\rho\left(\int_\Omega |\nabla(z^\delta - U(a_\rho^\delta))|^2\right)^{1/2}\left(\int_\Omega |\nabla\phi_\rho|^2\right)^{1/2}.\end{aligned}$$

An application of the Cauchy-Schwarz inequality yields

$$\begin{aligned}\Sigma &\le 2\rho\delta\overline{a}\left(\int_\Omega \phi_\rho^2\right)^{1/2} + 4\rho^2\overline{a}\int_\Omega \phi_\rho^2 + \frac{1}{4}\int_\Omega a_\rho^\delta\left(z^\delta - U(a_\rho^\delta)\right)^2 \\ &+ 2\rho\delta\left(\int_\Omega |\nabla\phi_\rho|^2\right)^{1/2} + 4\rho^2\int_\Omega |\nabla\phi_\rho|^2 + \frac{1}{4}\int_\Omega \left|\nabla\left(z^\delta - U(a_\rho^\delta)\right)\right|^2. \qquad (2.62)\end{aligned}$$

By inequality (2.59), it implies that the family $(\psi_\rho)_{\rho\in(0,1)}$ is bounded in the $H^1(\Omega)$-norm. From Lemma 2.2.6, there exists a positive constant C such that

$$\max\left\{\int_\Omega \phi_\rho^2, \int_\Omega |\nabla\phi_\rho|^2\right\} \le C^2$$

for all $\rho \in (0,1)$. From this and inequality (2.62), we have

$$\Sigma \le 2\rho\delta\overline{a}C + 4\rho^2\overline{a}C^2 + 2\rho\delta C + 4\rho^2 C^2 + \frac{1}{2}G_{z^\delta}(a_\rho^\delta). \tag{2.63}$$

By the estimate in Lemma 1.2.6, we conclude from (2.58), (2.60), (2.61) and (2.63) that

$$\begin{aligned}\frac{\beta}{2}\|U(a_\rho^\delta) - z^\delta\|^2_{H^1(\Omega)} + \rho\|a^\dagger - a_\rho^\delta\|^2_{L^2(\Omega)}&\\ &\le \frac{\max\{1,\overline{a}\}}{2}\delta^2 + 2\rho\delta\overline{a}C + 4\rho^2\overline{a}C^2\\ &+ 2\rho\delta C + 4\rho^2C^2 + \frac{4\overline{a}\Lambda_\beta}{\beta}\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right)\rho^2\\ &= \mathcal{O}(\delta^2)\end{aligned}$$

as $\delta \to 0$ and $\rho \sim \delta$. The theorem is proved. □

2.2.3. Discussion of the source condition

Now we discuss the source condition (2.54) which is equivalent to the following one: there exists a function $w \in H^1(\Omega)$ such that

$$\langle a^\dagger - a^*, h\rangle_{L^2(\Omega)} = \langle w, U'(a^\dagger)h\rangle_{H^1(\Omega)} \tag{2.64}$$

for all $h \in L^\infty(\Omega)$. We see that this condition is satisfied under a weak hypothesis about the regularity of the sought coefficient. Further, the smallness requirement on the source functions of the general convergence theory for nonlinear ill-posed problems in [41, 42] is liberated in our source condition.

Theorem 2.2.7. *Assume that $\frac{a^\dagger - a^*}{U(a^\dagger)}$ is an element of $H^1(\Omega)$. Then, the condition (2.64) is fulfilled and hence a convergence rate $\mathcal{O}(\sqrt{\delta})$ of L^2-regularization is obtained.*

Proof. By the hypothesis that $\frac{a^\dagger - a^*}{U(a^\dagger)}$ belongs to $H^1(\Omega)$ we can choose $v \in H^1(\Omega)$ such that $-U(a^\dagger)v = a^\dagger - a^*$. Thus, for all h in $L^\infty(\Omega)$, we get

$$\begin{aligned}\langle a^\dagger - a^*, h\rangle_{L^2(\Omega)} &= -\int_\Omega hU(a^\dagger)v\\ &\quad\text{(in virtue of (1.23))}\\ &= \int_\Omega \nabla U'(a^\dagger)h\nabla v + a^\dagger U'(a^\dagger)hv\\ &= \langle w, U'(a^\dagger)h\rangle_{H^1(\Omega)}\end{aligned}$$

for some $w \in H^1(\Omega)$ independent of $h \in L^\infty(\Omega)$. The theorem is proved. □

We close this section by the following note.

Remark 2.2.1. The hypothesis that $\frac{a^\dagger - a^*}{U(a^\dagger)}$ belongs to $H^1(\Omega)$ is satisfied if there exists a positive constant γ such that $|U(a^\dagger)| \ge \gamma$ a.e. on Ω and $a^\dagger - a^*$ is an element of $H^1(\Omega)$.

Conclusions

In this chapter, we investigate convergence rates for the Tikhonov regularization of ill-posed nonlinear inverse problems of identifying the diffusion coefficient and the reaction coefficient in the Neumann problems for the elliptic equation (1.1)–(1.2) and (1.3)–(1.4), respectively when the exact solution $\overline{u}$ is imprecisely given by observed data z^δ satisfying (1.5). We apply L^2-regularization to convex functionals $J_{z^\delta}(\cdot)$ and $G_{z^\delta}(\cdot)$ defined by (1.12) and (1.22) and obtain convergence rates $\mathcal{O}(\sqrt{\delta})$ of regularized solutions in the $L^2(\Omega)$-norm. Our source conditions are simple and very weak, since we remove the so-called "small enough condition" upon the source functions which is popularized in the theory of regularization of nonlinear ill-posed problems but very hard to check. Furthermore, our source conditions can be easily interpreted as a requirment of being a member of a certain Sobolev space, and our results are valid for multi-dimensional identification problems. The previous result by Engl, Kunisch and Neubauer [42] requires the small enough condition of the source functions, and high regularities on the sought coefficients. It is applicable only to one-dimensional problems.

This chapter was written on the basis of the papers

[59] Dinh Nho Hào and Tran Nhan Tam Quyen (2010), Convergence rates for Tikhonov regularization of coefficient identification problems in Laplace-type equations, *Inverse Problems* **26**, 125014 (23pp).

[62] Dinh Nho Hào and Tran Nhan Tam Quyen (2012), Convergence rates for Tikhonov regularization of a two-coefficient identification problem in an elliptic boundary value problem, *Numerische Mathematik* **120**, pp. 45–77.

Chapter 3

Total variation regularization

In this chapter we apply total variation regularization to the convex functionals $J_{z^\delta}(\cdot)$ and $G_{z^\delta}(\cdot)$ respectively defined by (1.12) and (1.22) and obtain convergence rates $\mathcal{O}(\delta)$ of regularized solutions to solutions of our identification problems in the sense of the Bregman distance.

3.1 Convergence rates for total variation regularization of the diffusion coefficient identification problem

3.1.1. Regularization by the total variation

To estimate coefficients that may discontinuous or highly oscillating, we apply and arrive at the total variation regularization method, the convex minimization problem

$$\min_{q \in Q_{ad}} J_{z^\delta}(q) + \rho \int_\Omega |\nabla q| \tag{$\mathcal{P}^q_{\rho,\delta}$}$$

for identifying the coefficient q in (1.1)–(1.2), where

$$Q_{ad} := Q \cap BV(\Omega) \tag{3.1}$$

is the admissible set of coefficients, $BV(\Omega)$ is the space of functions with bounded total variation (see § 0.3.9.) and $\rho > 0$ is the regularization parameter.

In the following we will see that problem $(\mathcal{P}^q_{\rho,\delta})$ has a solution q^δ_ρ (see Theorem 3.1.4). Further, the problem

$$\min_{q \in \Pi_{Q_{ad}}(\overline{u})} \int_\Omega |\nabla q| \tag{$\mathcal{K}^q$}$$

also has a solution which is called the *total variation-minimizing solution* of the equation $U(q) = \overline{u}$, where

$$\Pi_{Q_{ad}}(\overline{u}) := \{q \in Q_{ad} \mid U(q) = \overline{u}\}. \tag{3.2}$$

Our aim in this section is to investigate the convergence rates of regularized solutions q^δ_ρ to the total variation-minimizing solution $q^\dagger$ of equation $U(q) = \overline{u}$.

The following results are useful.

Lemma 3.1.1. *([50, pp. 7–17])*

(i) Let (q_n) be a bounded sequence in the $BV(\Omega)$-norm. Then, there exist a subsequence (q_{1_n}) of it and an element $q \in BV(\Omega)$ such that (q_{1_n}) converges to q in the $L^1(\Omega)$-norm.

(ii) Let (q_n) be a sequence in $BV(\Omega)$ which converges to q in the $L^1(\Omega)$-norm. Then, $q \in BV(\Omega)$ and

$$\int_\Omega |\nabla q| \leq \liminf_n \int_\Omega |\nabla q_n|.$$

Lemma 3.1.2. *The total variation is continuous on $BV(\Omega)$, i.e., if $(q_n) \subset BV(\Omega)$ converges to $q \in BV(\Omega)$, then*

$$\lim_n \int_\Omega |\nabla q_n| = \int_\Omega |\nabla q|.$$

Proof. The affirmation of the lemma follows from the definition of the $BV(\Omega)$-norm and the following inequality

$$\left| \int_\Omega |\nabla p| - \int_\Omega |\nabla q| \right| \leq \int_\Omega |\nabla (p - q)|$$

for all $p, q \in BV(\Omega)$. □

Lemma 3.1.3. *Let (z^{δ_n}) be a sequence which converges to z^δ in the $H^1(\Omega)$-norm and (q_n) be a sequence in the set Q defined by (1.7) convergent to q in the $L^1(\Omega)$-norm. Then,*

$$\lim_n J_{z^{\delta_n}}(q_n) = \lim_n J_{z^\delta}(q).$$

Proof. Applying Lemma 1.1.3, we get that the sequence $(U(q_n))$ weakly converges to $U(q)$ in $H^1(\Omega)$. On the other hand, using (1.6), we rewrite (1.12) in the form

$$\begin{aligned} 2J_{z^{\delta_n}}(q_n) &= \int_\Omega q_n \nabla U(q_n) \nabla (U(q_n) - z^{\delta_n}) - \int_\Omega q_n \nabla (U(q_n) - z^{\delta_n}) \nabla z^{\delta_n} \\ &= \int_\Omega f(U(q_n) - z^{\delta_n}) + \int_{\partial\Omega} g(U(q_n) - z^{\delta_n}) \\ &- \int_\Omega q_n \nabla (U(q_n) - z^{\delta_n}) \nabla z^{\delta_n}. \end{aligned}$$

Since $(U(q_n))$ weakly converges to $U(q)$ in $H^1(\Omega)$, we get

$$\begin{aligned} &\int_\Omega f(U(q_n) - z^{\delta_n}) + \int_{\partial\Omega} g(U(q_n) - z^{\delta_n}) \\ &\qquad \to \int_\Omega f(U(q) - z^\delta) + \int_{\partial\Omega} g(U(q) - z^\delta). \end{aligned}$$

Now we have

$$\begin{aligned} \int_\Omega q_n \nabla (U(q_n) - z^{\delta_n}) \nabla z^{\delta_n} &= \int_\Omega q_n \nabla (U(q_n) - z^{\delta_n}) \nabla (z^{\delta_n} - z^\delta) \\ &+ \int_\Omega q \nabla (U(q_n) - z^{\delta_n}) \nabla z^\delta \\ &+ \int_\Omega (q_n - q) \nabla (U(q_n) - z^{\delta_n}) \nabla z^\delta. \end{aligned}$$

Since (z^{δ_n}) converges to z^δ in the $H^1(\Omega)$-norm, we have

$$\int_\Omega q_n \nabla(U(q_n) - z^{\delta_n})\nabla(z^{\delta_n} - z^\delta) \to 0$$

and

$$\int_\Omega q\nabla(U(q_n) - z^{\delta_n})\nabla z^\delta \to \int_\Omega q\nabla(U(q) - z^\delta)\nabla z^\delta.$$

On the other hand, using Lemma 1.1.2, we obtain

$$\int_\Omega (q_n - q)\nabla(U(q_n) - z^{\delta_n})\nabla z^\delta \to 0.$$

Thus,

$$\begin{aligned} 2\lim_n J_{z^{\delta_n}}(q_n) &= \int_\Omega f(U(q) - z^\delta) + \int_{\partial\Omega} g(U(q) - z^\delta) + \int_\Omega q\nabla(U(q) - z^\delta)\nabla z^\delta \\ &= 2J_{z^\delta}(q). \end{aligned}$$

The lemma is proved. □

Now, we are in a position to prove main results of this subsection.

Theorem 3.1.4. *(i) There exists a solution q_ρ^δ of problem $(\mathcal{P}^q_{\rho,\delta})$.*

(ii) There exists a solution $q^\dagger$ of problem $(\mathcal{K}^q)$.

Proof. The proof of part (i) can be found, for example, in [31]. It remains to prove (ii). Let (q_n) be a sequence in $\Pi_{Q_{ad}}(\overline{u})$ such that

$$\lim_n \int_\Omega |\nabla q_n| = \inf_{q\in\Pi_{Q_{ad}}(\overline{u})} \int_\Omega |\nabla q|\,. \tag{3.3}$$

Since $(q_n) \subset Q$, it is bounded in the $L^1(\Omega)$-norm. Hence it follows from the last equality that (q_n) is bounded in the $BV(\Omega)$-norm. By Lemma 3.1.1, we see that there exist a subsequence (q_{1_n}) of (q_n) and an element $q^\dagger \in BV(\Omega)$ such that (q_{1_n}) converges to $q^\dagger$ in the $L^1(\Omega)$-norm and

$$\int_\Omega \left|\nabla q^\dagger\right| \le \liminf_n \int_\Omega |\nabla q_{1_n}|\,. \tag{3.4}$$

Since Q is closed in the $L^1(\Omega)$-norm, we obtain that $q^\dagger \in Q$ and so $q^\dagger \in Q \cap BV(\Omega) = Q_{ad}$. On the other hand, using Lemma 1.1.3, we have $U(q^\dagger) = \overline{u}$ or $q^\dagger \in \Pi_{Q_{ad}}(\overline{u})$. Further, inequalities (3.3) and (3.4) lead to

$$\int_\Omega \left|\nabla q^\dagger\right| \le \inf_{q\in\Pi_{Q_{ad}}(\overline{u})} \int_\Omega |\nabla q|\,.$$

The theorem is proved. □

In the following we denote by

$$\mathfrak{X} := L^\infty(\Omega) \cap BV(\Omega).$$

Then, $\mathfrak{X}$ is a Banach space with the norm

$$\|q\|_{\mathfrak{X}} := \|q\|_{L^\infty(\Omega)} + \|q\|_{BV(\Omega)}.$$

Further,

$$L^\infty(\Omega)^* \subset \mathfrak{X}^* \quad \text{and} \quad BV(\Omega)^* \subset \mathfrak{X}^*.$$

In addition, we will write $\mathfrak{X}_{BV(\Omega)} := \big(\mathfrak{X}, \|\cdot\|_{BV(\Omega)}\big)$ $\big(\mathfrak{X}_{L^\infty(\Omega)} := \big(\mathfrak{X}, \|\cdot\|_{L^\infty(\Omega)}\big)\big)$ to denote the space $\mathfrak{X}$ with respect to the $BV(\Omega)$-norm (the $L^\infty(\Omega)$-norm).

Note that the functional $J_{z^\delta}(\cdot)$ defined by (1.12) is Fréchet differentiable on Q in the $L^\infty(\Omega)$-norm. For each $q \in Q$, its Fréchet differential is defined by

$$J'_{z^\delta}(q)h = -\frac{1}{2}\int_\Omega h\nabla(U(q) - z^\delta)\nabla(U(q) + z^\delta), \ \forall h \in L^\infty(\Omega).$$

Since the functional $J'_{z^\delta}(q)(\cdot)$ is linear and continuous on $L^\infty(\Omega)$, so is it on $\mathfrak{X}$. Thus,

$$\begin{aligned} J'_{z^\delta}(q)h &= \langle J'_{z^\delta}(q), h\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} \\ &= \langle J'_{z^\delta}(q), h\rangle_{(\mathfrak{X}^*, \mathfrak{X})}, \ \forall h \in \mathfrak{X}. \end{aligned} \tag{3.5}$$

Further, for any $\ell \in \mathfrak{X}^*_{BV(\Omega)}$,

$$\langle \ell, h\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} = \langle \ell, h\rangle_{(\mathfrak{X}^*, \mathfrak{X})}, \ \forall h \in \mathfrak{X}. \tag{3.6}$$

Recall that for each $q \in \mathfrak{X}_{BV(\Omega)}$,

$$\partial\left(\int_\Omega |\nabla(\cdot)|\right)(q) = \Big\{q^* \in \mathfrak{X}^*_{BV(\Omega)} \mid \int_\Omega |\nabla p| \ge \int_\Omega |\nabla q| + \langle q^*, p - q\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \text{ for all } p \in \mathfrak{X}_{BV(\Omega)}\Big\}.$$

Moreover, it follows from Lemma 3.1.2 that $\partial\left(\int_\Omega |\nabla(\cdot)|\right)(q) \ne \emptyset$ for each q in $BV(\Omega)$.

The following result delivers a necessary and sufficient optimality condition for problems $(\mathcal{P}^q_{\rho,\delta})$ and $(\mathcal{K}^q)$.

Lemma 3.1.5. *(i) Let $\widetilde{q} \in Q_{ad}$. Then, $\widetilde{q}$ is a solution of $(\mathcal{P}^q_{\rho,\delta})$ if and only if for any $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(\widetilde{q})$, and the inequality*

$$J'_{z^\delta}(\widetilde{q})(q - \widetilde{q}) + \rho\langle \ell, q - \widetilde{q}\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \ge 0 \tag{3.7}$$

holds for all q in Q_{ad}.

(ii) Let $q^\dagger \in \Pi_{Q_{ad}}(\overline{u})$. Then, $q^\dagger$ is a solution of $(\mathcal{K}^q)$ if and only if for any $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger)$, and the inequality

$$\langle \ell, q - q^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \ge 0$$

holds for all q in $\Pi_{Q_{ad}}(\overline{u})$.

Proof. (i) Since $J_{z^\delta}(\cdot)$ and the total variation are convex, an element $\widetilde{q} \in Q_{ad}$ is a solution of $(\mathcal{P}^q_{\rho,\delta})$ if and only if

$$J'_{z^\delta}(\widetilde{q})(q - \widetilde{q}) + \rho\left(\int_\Omega |\nabla q| - \int_\Omega |\nabla \widetilde{q}|\right) \ge 0 \tag{3.8}$$

for all q in Q_{ad}. In fact, let $\widetilde{q} \in Q_{ad}$ satisfy (3.8). For all $q \in Q_{ad}$, we have

$$\begin{aligned} J_{z^\delta}(q) - J_{z^\delta}(\widetilde{q}) &\ge J'_{z^\delta}(\widetilde{q})(q - \widetilde{q}) \\ &\ge \rho\int_\Omega |\nabla \widetilde{q}| - \rho\int_\Omega |\nabla q|. \end{aligned}$$

This means that $\widetilde{q}$ is a solution of $(\mathcal{P}^q_{\rho,\delta})$. Now, for an arbitrary element $q \in Q_{ad}$ and $s \in (0,1]$ we get

$$\begin{aligned} J_{z^\delta}(\widetilde{q}) + \rho \int_\Omega |\nabla \widetilde{q}| &\le J_{z^\delta}(\widetilde{q} + s(q-\widetilde{q})) + \rho \int_\Omega |\nabla(\widetilde{q} + s(q-\widetilde{q}))| \\ &\le J_{z^\delta}(\widetilde{q} + s(q-\widetilde{q})) + \rho \int_\Omega |\nabla \widetilde{q}| + \rho s \left(\int_\Omega |\nabla q| - \int_\Omega |\nabla \widetilde{q}| \right). \end{aligned}$$

Hence we have

$$\rho \left(\int_\Omega |\nabla \widetilde{q}| - \int_\Omega |\nabla q| \right) \le \frac{J_{z^\delta}(\widetilde{q} + s(q-\widetilde{q})) - J_{z^\delta}(\widetilde{q})}{s}.$$

Taking the limit $s \to 0^+$, it yields

$$\rho \left(\int_\Omega |\nabla \widetilde{q}| - \int_\Omega |\nabla q| \right) \le J'_{z^\delta}(\widetilde{q})(q-\widetilde{q}).$$

By (3.5), inequality (3.8) means that

$$\langle J'_{z^\delta}(\widetilde{q}), q-\widetilde{q}\rangle_{(\mathfrak{X}^*,\mathfrak{X})} + \rho \left(\int_\Omega |\nabla q| - \int_\Omega |\nabla \widetilde{q}| \right) \ge 0, \ \ \forall q \in Q_{ad}$$

or

$$0 \in J'_{z^\delta}(\widetilde{q}) + \partial \left(\rho \int_\Omega |\nabla(\cdot)| + I_{Q_{ad}} \right)(\widetilde{q}) \subset \mathfrak{X}^*, \tag{3.9}$$

where $I_{Q_{ad}}$ is the indicator function of the set Q_{ad} defined by

$$I_{Q_{ad}}(q) := \begin{cases} 0 & \text{if } q \in Q_{ad}, \\ +\infty & \text{if } q \in \mathfrak{X} \setminus Q_{ad} \end{cases}$$

with

$$\partial I_{Q_{ad}}(\widetilde{q}) = \{\xi^* \in \mathfrak{X}^* \mid \langle \xi^*, q-\widetilde{q}\rangle_{(\mathfrak{X}^*,\mathfrak{X})} \le 0 \text{ for all } q \in Q_{ad}\}.$$

From Lemma 3.1.2, equality (3.9) is equivalent to

$$0 \in J'_{z^\delta}(\widetilde{q}) + \rho\partial \left(\int_\Omega |\nabla(\cdot)| \right)(\widetilde{q}) + \partial I_{Q_{ad}}(\widetilde{q}).$$

Hence for all $\ell \in \partial \left(\int_\Omega |\nabla(\cdot)| \right)(\widetilde{q})$, we get

$$\langle -J'_{z^\delta}(\widetilde{q}) - \rho\ell, q-\widetilde{q}\rangle_{(\mathfrak{X}^*,\mathfrak{X})} \le 0 \text{ for all } q \in Q_{ad}.$$

By equalities (3.5) and (3.6), the last inequality means (3.7).

(ii) Since the total variation is convex, an element $q^\dagger$ is a solution of $(\mathcal{K}^q)$ if and only if

$$\begin{aligned} 0 &\in \partial \left(\int_\Omega |\nabla(\cdot)| + I_{Q_{ad}} \right)(q^\dagger) \\ &= \partial \left(\int_\Omega |\nabla(\cdot)| \right)(q^\dagger) + \partial I_{Q_{ad}}(q^\dagger) \end{aligned}$$

which is equivalent to the inequality

$$\langle \ell, q-q^\dagger \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \ge 0$$

for any $\ell \in \partial \left(\int_\Omega |\nabla(\cdot)| \right)(q^\dagger)$ and $q \in \Pi_{Q_{ad}}(\overline{u})$. The lemma is proved. □

Theorem 3.1.6. *For a fixed regularization parameter $\rho > 0$, let (z^{δ_n}) be a sequence which converges to z^δ in the $H^1(\Omega)$-norm and $(q_\rho^{\delta_n})$ be minimizers of the problems*

$$\min_{q\in Q_{ad}} J_{z^{\delta_n}}(q) + \rho\int_\Omega |\nabla q|.$$

Then, there exist a subsequence $(q_\rho^{\delta_{1_n}})$ of $(q_\rho^{\delta_n})$ and $\widetilde{q} \in Q_{ad}$ such that $(q_\rho^{\delta_{1_n}})$ converges to $\widetilde{q}$ in the $L^1(\Omega)$-norm and

$$\lim_n \int_\Omega |\nabla q_\rho^{\delta_{1_n}}| = \int_\Omega |\nabla \widetilde{q}|. \tag{3.10}$$

Further, $\widetilde{q}$ is a solution to $(\mathcal{P}^q_{\rho,\delta})$.

Proof. For all $n \in \mathbb{N}$ and $q \in Q_{ad}$, we have

$$J_{z^{\delta_n}}(q_\rho^{\delta_n}) + \rho\int_\Omega |\nabla q_\rho^{\delta_n}| \le J_{z^{\delta_n}}(q) + \rho\int_\Omega |\nabla q|. \tag{3.11}$$

Since (z^{δ_n}) is a bounded sequence in the $H^1(\Omega)$-norm, inequality (3.11) follows that the sequence $(q_\rho^{\delta_n})$ is bounded in the $BV(\Omega)$-norm. By Lemma 3.1.1, there exists a subsequence $(q_\rho^{\delta_{1_n}})$ of $(q_\rho^{\delta_n})$ and $\widetilde{q} \in BV(\Omega)$ such that $(q_\rho^{\delta_{1_n}})$ converges to $\widetilde{q}$ in the $L^1(\Omega)$-norm and

$$\int_\Omega |\nabla \widetilde{q}| \le \liminf_n \int_\Omega |\nabla q_\rho^{\delta_{1_n}}|. \tag{3.12}$$

Since $q_\rho^{\delta_{1_n}} \in Q$ for all $n \in \mathbb{N}$, it follows that $\widetilde{q} \in Q$. Hence we have $\widetilde{q} \in BV(\Omega) \cap Q = Q_{ad}$. Now, applying Lemma 3.1.3, we get

$$\lim_n J_{z^{\delta_{1_n}}}(q_\rho^{\delta_{1_n}}) = J_{z^\delta}(\widetilde{q}). \tag{3.13}$$

By (3.12), (3.13) and (3.11), we obtain that

$$\begin{aligned}
J_{z^\delta}(\widetilde{q}) + \rho\int_\Omega |\nabla \widetilde{q}| &\le \liminf_n \left(J_{z^{\delta_{1_n}}}(q_\rho^{\delta_{1_n}}) + \rho \int_\Omega |\nabla q_\rho^{\delta_{1_n}}| \right)\\
&\le \liminf_n \left(J_{z^{\delta_{1_n}}}(q) + \rho\int_\Omega |\nabla q| \right)\\
&= J_{z^\delta}(q) + \rho\int_\Omega |\nabla q|
\end{aligned}$$

for all $q \in Q_{ad}$. This means that $\widetilde{q}$ is a solution to $(\mathcal{P}^q_{\rho,\delta})$.

Choosing $q = \widetilde{q}$ in (3.11) and again using (3.13), we have

$$\begin{aligned}
J_{z^\delta}(\widetilde{q}) + \rho\limsup_n \int_\Omega |\nabla q_\rho^{\delta_{1_n}}| &= \limsup_n \left(J_{z^{\delta_{1_n}}}(q_\rho^{\delta_{1_n}}) + \rho\int_\Omega |\nabla q_\rho^{\delta_{1_n}}| \right)\\
&\le \limsup_n \left(J_{z^{\delta_{1_n}}}(\widetilde{q}) + \rho\int_\Omega |\nabla \widetilde{q}| \right)\\
&= J_{z^\delta}(\widetilde{q}) + \rho\int_\Omega |\nabla \widetilde{q}|.
\end{aligned}$$

Hence

$$\limsup_n \int_\Omega |\nabla q_\rho^{\delta_{1_n}}| \le \int_\Omega |\nabla \widetilde{q}|.$$

From the last inequality and (3.12) we arrive at (3.10). The theorem is proved. □

Theorem 3.1.7. *For any positive sequence $(\delta_n) \to 0$, let $\rho_n := \rho(\delta_n)$ be such that*

$$\rho_n \to 0 \text{ and } \frac{\delta_n^2}{\rho_n} \to 0 \text{ as } \quad n \to \infty.$$

Moreover, let (z^{δ_n}) be a sequence satisfying $\|\overline{u} - z^{\delta_n}\|_{H^1(\Omega)} \le \delta_n$ and $(q_{\rho_n}^{\delta_n})$ be minimizers of problems

$$\min_{q \in Q_{ad}} J_{z^{\delta_n}}(q) + \rho_n \int_\Omega |\nabla q|.$$

Then, there exist a subsequence $(q_{\rho_{1n}}^{\delta_{1n}})$ of $(q_{\rho_n}^{\delta_n})$ and an element $\widehat{q} \in \Pi_{Q_{ad}}(\overline{u})$ such that

$$\lim_n \|q_{\rho_{1n}}^{\delta_{1n}} - \widehat{q}\|_{L^1(\Omega)} = 0 \text{ and } \lim_n \int_\Omega |\nabla q_{\rho_{1n}}^{\delta_{1n}}| = \int_\Omega |\nabla \widehat{q}|.$$

Further, $\widehat{q}$ is a solution to problem $(\mathcal{K}^q)$ and for all $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(\widehat{q})$

$$\lim_n D_{TV}^\ell(q_{\rho_{1n}}^{\delta_{1n}}, \widehat{q}) = 0.$$

We remark that if the solution $\widehat{q}$ to problem $(\mathcal{K}^q)$ is unique, then

$$\lim_n \|q_{\rho_n}^{\delta_n} - \widehat{q}\|_{L^1(\Omega)} = 0, \ \lim_n \int_\Omega |\nabla q_{\rho_n}^{\delta_n}| = \int_\Omega |\nabla \widehat{q}| \text{ and } \lim_n D_{TV}^\ell(q_{\rho_n}^{\delta_n}, \widehat{q}) = 0$$

for the whole sequence $(q_{\rho_n}^{\delta_n})$. This may be valid if, for example, we replace the $BV(\Omega)$ space by $W^{1,1}(\Omega)$ in our identification problem.

Proof of Theorem 3.1.7. For all $n \in \mathbb{N}$ and $q \in Q_{ad}$, by the definition of $q_{\rho_n}^{\delta_n}$, we have

$$J_{z^{\delta_n}}(q_{\rho_n}^{\delta_n}) + \rho_n \int_\Omega |\nabla q_{\rho_n}^{\delta_n}| \le J_{z^{\delta_n}}(q) + \rho_n \int_\Omega |\nabla q|.$$

In particular, for all $q \in \Pi_{Q_{ad}}(\overline{u})$,

$$\begin{aligned} J_{z^{\delta_n}}(q_{\rho_n}^{\delta_n}) + \rho_n \int_\Omega |\nabla q_{\rho_n}^{\delta_n}| &\le \frac{1}{2} \int_\Omega q |\nabla(\overline{u} - z^{\delta_n})|^2 + \rho_n \int_\Omega |\nabla q| \\ &\le \frac{\overline{q}}{2} \delta_n^2 + \rho_n \int_\Omega |\nabla q|. \end{aligned} \tag{3.14}$$

Thus,

$$\int_\Omega |\nabla q_{\rho_n}^{\delta_n}| \le \frac{\overline{q}\delta_n^2}{2\rho_n} + \int_\Omega |\nabla q|, \ \forall n \in \mathbb{N}, \forall q \in \Pi_{Q_{ad}}(\overline{u}). \tag{3.15}$$

Since $\delta_n^2/\rho_n \to 0$ as $n \to \infty$, the last inequality follows that the sequence $(\int_\Omega |\nabla q_{\rho_n}^{\delta_n}|)$ is bounded. By Lemmas 3.1.1, we conclude that there exist a subsequence $(q_{\rho_{1n}}^{\delta_{1n}})$ of $(q_{\rho_n}^{\delta_n})$ and $\widehat{q} \in Q_{ad}$ such that

$$q_{\rho_{1n}}^{\delta_{1n}} \to \widehat{q} \text{ in the } L^1(\Omega)\text{-norm}, \tag{3.16}$$

$$\int_\Omega |\nabla \widehat{q}| \le \liminf_n \int_\Omega |\nabla q_{\rho_{1n}}^{\delta_{1n}}|. \tag{3.17}$$

Applying Lemma 1.1.3, it follows from (3.16) that

$$U(q_{\rho_{1n}}^{\delta_{1n}}) \text{ weakly converges to } U(\widehat{q}) \text{ in } H^1(\Omega). \tag{3.18}$$

Besides, using the estimate in Lemma 1.1.7, we have

$$\alpha\|U(q_{\rho_{1_n}}^{\delta_{1_n}}) - z^{\delta_{1_n}}\|^2_{H^1(\Omega)} \leq J_{z^{\delta_{1_n}}}(q_{\rho_{1_n}}^{\delta_{1_n}}), \ \ \forall n \in \mathbb{N}.$$

with the positive constant α defined by (1.9). Hence it follows from the last inequality and (3.14) that

$$\frac{\alpha}{2}\|U(q_{\rho_{1_n}}^{\delta_{1_n}}) - z^{\delta_{1_n}}\|^2_{H^1(\Omega)} \leq \frac{\overline{q}}{2}\delta_{1_n}^2 + \rho_{1_n}\int_\Omega |\nabla q|, \ \ \forall q \in \Pi_{Q_{ad}}(\overline{u}).$$

Since $\delta_{1_n} \to 0$ and $\rho_{1_n} \to 0$ as $n \to \infty$, the last inequality follows that

$$\|U(q_{\rho_{1_n}}^{\delta_{1_n}}) - z^{\delta_{1_n}}\|_{H^1(\Omega)} \to 0 \text{ as } n \to \infty.$$

On the other hand, an application of the triangle inequality gives

$$\begin{aligned}\|U(q_{\rho_{1_n}}^{\delta_{1_n}}) - \overline{u}\|_{H^1(\Omega)} &\leq \|U(q_{\rho_{1_n}}^{\delta_{1_n}}) - z^{\delta_{1_n}}\|_{H^1(\Omega)} + \|\overline{u} - z^{\delta_{1_n}}\|_{H^1(\Omega)} \\ &\leq \|U(q_{\rho_{1_n}}^{\delta_{1_n}}) - z^{\delta_{1_n}}\|_{H^1(\Omega)} + \delta_{1_n} \to 0 \qquad (3.19)\end{aligned}$$

as $n \to \infty$. It follows from (3.18) and (3.19) that $U(\widehat{q}) = \overline{u}$ or $\widehat{q} \in \Pi_{Q_{ad}}(\overline{u})$. Further, by (3.17) and (3.15), for all $q \in \Pi_{Q_{ad}}(\overline{u})$, we have

$$\int_\Omega |\nabla \widehat{q}| \leq \liminf_n \left(\frac{\overline{q}\delta_{1_n}^2}{2\rho_{1_n}} + \int_\Omega |\nabla q|\right) = \int_\Omega |\nabla q|.$$

This means that $\widehat{q}$ is the solution to problem $(\mathcal{K}^q)$.

Again using (3.15), we get

$$\limsup_n \int_\Omega |\nabla q_{\rho_{1_n}}^{\delta_{1_n}}| \leq \int_\Omega |\nabla \widehat{q}|.$$

By the last inequality and (3.17), we obtain that

$$\int_\Omega |\nabla \widehat{q}| = \lim_n \int_\Omega |\nabla q_{\rho_{1_n}}^{\delta_{1_n}}|. \qquad (3.20)$$

Now, it follows from (3.16) and (3.20) that the sequence $(q_{\rho_{1_n}}^{\delta_{1_n}})$ weakly converges to $\widehat{q}$ in $BV(\Omega)$ (see [9], Proposition 10.1.2, p. 374). Thus, for all $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(\widehat{q})$, we have

$$\lim_n \langle \ell, q_{\rho_{1_n}}^{\delta_{1_n}} - \widehat{q}\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} = 0.$$

Therefore, by (3.20), we conclude that

$$\lim_n D_{TV}^\ell(q_{\rho_{1_n}}^{\delta_{1_n}}, \widehat{q}) = \lim_n \left(\int_\Omega |\nabla q_{\rho_{1_n}}^{\delta_{1_n}}| - \int_\Omega |\nabla \widehat{q}| - \langle \ell, q_{\rho_{1_n}}^{\delta_{1_n}} - \widehat{q}\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})}\right) = 0.$$

The theorem is proved. □

3.1.2. Convergence rates

We note that for all $w^* \in H^1_\diamond(\Omega)^*$, the element $U'(q)^* w^* \subset L^\infty(\Omega)^* \subset \mathfrak{X}^*$ and

$$\begin{aligned}\langle w^*, U'(q)h\rangle_{\left(H^1_\diamond(\Omega)^*, H^1_\diamond(\Omega)\right)} &= \left\langle U'(q)^* w^*, h\right\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} \\ &= \left\langle U'(q)^* w^*, h\right\rangle_{(\mathfrak{X}^*, \mathfrak{X})}, \ \ \forall h \in \mathfrak{X}. \qquad (3.21)\end{aligned}$$

Theorem 3.1.8. *Let $q^\dagger$ be a solution of $(\mathcal{K}^q)$. Assume that there exists a functional $w^* \in H^1_\diamond(\Omega)^*$ such that*

$$U'(q^\dagger)^* w^* \in \partial \left(\int_\Omega |\nabla(\cdot)| \right)(q^\dagger). \tag{3.22}$$

Then,

$$D_{TV}^{U'(q^\dagger)^* w^*}(q_\rho^\delta, q^\dagger) = \mathcal{O}(\delta) \quad \text{and} \quad \left| \int_\Omega |\nabla q_\rho^\delta| - \int_\Omega |\nabla q^\dagger| \right| = \mathcal{O}(\delta),$$

and

$$\|U(q_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

By the convexity of the function $J_{z^\delta}(\cdot)$, we get the following useful auxiliary result.

Lemma 3.1.9. *The estimate*

$$-\rho \left\langle \ell, q^\dagger - q_\rho^\delta \right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \le \frac{\overline{q}}{2}\delta^2 \tag{3.23}$$

holds for all q_ρ^δ being solutions of problem $(\mathcal{P}^q_{\rho,\delta})$ and $\ell \in \partial \left(\int_\Omega |\nabla(\cdot)| \right)(q_\rho^\delta)$.

Proof. By (3.7), we get

$$-\rho \left\langle \ell, q^\dagger - q_\rho^\delta \right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \le J'_{z^\delta}(q_\rho^\delta)(q^\dagger - q_\rho^\delta). \tag{3.24}$$

Since the function $J_{z^\delta}(\cdot)$ is convex and non-negative, we have

$$\begin{aligned} J'_{z^\delta}(q_\rho^\delta)(q^\dagger - q_\rho^\delta) &\le J_{z^\delta}(q^\dagger) - J_{z^\delta}(q_\rho^\delta) \\ &\le J_{z^\delta}(q^\dagger) \\ &\le \frac{\overline{q}}{2}\delta^2. \end{aligned} \tag{3.25}$$

From inequalities (3.24) and (3.25) we arive at (3.23). The lemma is proved. □

Proof of Theorem 3.1.8. By the definition of q_ρ^δ, we have

$$J_{z^\delta}(q_\rho^\delta) + \rho \int_\Omega |\nabla q_\rho^\delta| \le J_{z^\delta}(q^\dagger) + \rho \int_\Omega |\nabla q^\dagger|. \tag{3.26}$$

Then, by the definition of $q^\dagger$ and (1.5),

$$\begin{aligned} J_{z^\delta}(q_\rho^\delta) + \rho D_{TV}^{U'(q^\dagger)^* w^*}(q_\rho^\delta, q^\dagger) &= J_{z^\delta}(q_\rho^\delta) + \rho \int_\Omega |\nabla q_\rho^\delta| - \rho \int_\Omega |\nabla q^\dagger| \\ &\quad - \rho \langle U'(q^\dagger)^* w^*, q_\rho^\delta - q^\dagger \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \\ &\le J_{z^\delta}(q^\dagger) - \rho \langle U'(q^\dagger)^* w^*, q_\rho^\delta - q^\dagger \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \\ &\le \frac{1}{2}\overline{q}\delta^2 + \rho \langle U'(q^\dagger)^* w^*, q^\dagger - q_\rho^\delta \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)}. \end{aligned} \tag{3.27}$$

It follows from equalities (3.6) and (3.21), and the source condition (3.22) that

$$\begin{aligned}\langle U'(q^\dagger)^* w^*, q^\dagger - q^\delta_\rho\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} &= \langle U'(q^\dagger)^* w^*, q^\dagger - q^\delta_\rho\rangle_{(\mathfrak{X}^*, \mathfrak{X})} \\ &= \left\langle U'(q^\dagger)^* w^*, q^\dagger - q^\delta_\rho\right\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} \\ &= \left\langle w^*, U'(q^\dagger)(q^\dagger - q^\delta_\rho)\right\rangle_{\left(H^1_\diamond(\Omega)^*, H^1_\diamond(\Omega)\right)} \\ &= \left\langle w, U'(q^\dagger)(q^\dagger - q^\delta_\rho)\right\rangle_{H^1(\Omega)} \end{aligned} \tag{3.28}$$

for some $w \in H^1_\diamond(\Omega)$, by the Riesz representation theorem.

Since $q^\dagger \geq \underline{q} > 0$, the scalar product

$$[u, v]_{H^1(\Omega)} := \int_\Omega q^\dagger \nabla u \nabla v \quad \text{for all } u, v \in H^1_\diamond(\Omega)$$

is equivalent to $\langle u, v\rangle_{H^1(\Omega)}$ on $H^1_\diamond(\Omega)$. Hence there exists an element $\widehat{w} \in H^1_\diamond(\Omega)$ independent of q^δ_ρ such that

$$\left\langle w, U'(q^\dagger)(q^\dagger - q^\delta_\rho)\right\rangle_{H^1(\Omega)} = \int_\Omega q^\dagger \nabla \widehat{w} \nabla U'(q^\dagger)(q^\dagger - q^\delta_\rho). \tag{3.29}$$

From (3.28), (3.29) and (1.13) we have

$$\begin{aligned}\langle U'(q^\dagger)^* w^*, q^\dagger - q^\delta_\rho\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} &= \int_\Omega q^\dagger \nabla \widehat{w} \nabla U'(q^\dagger)(q^\dagger - q^\delta_\rho) \\ &= -\int_\Omega (q^\dagger - q^\delta_\rho) \nabla U(q^\dagger) \nabla \widehat{w} \\ &= \int_\Omega q^\delta_\rho \nabla U(q^\dagger) \nabla \widehat{w} - \int_\Omega q^\dagger \nabla U(q^\dagger) \nabla \widehat{w} \\ &= \int_\Omega q^\delta_\rho \nabla U(q^\dagger) \nabla \widehat{w} - \int_\Omega q^\delta_\rho \nabla U(q^\delta_\rho) \nabla \widehat{w} \\ &= \int_\Omega q^\delta_\rho \nabla \left(U(q^\dagger) - U(q^\delta_\rho)\right) \nabla \widehat{w}.\end{aligned}$$

Using the Cauchy-Schwarz inequality and the definition $U(q^\dagger) = \overline{u}$, from the last equality we have

$$\begin{aligned}\langle U'(q^\dagger)^* w^*, q^\dagger - q^\delta_\rho\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} &= \int_\Omega q^\delta_\rho \nabla \left(U(q^\dagger) - z^\delta\right) \nabla \widehat{w} \\ &\quad + \int_\Omega q^\delta_\rho \nabla \left(z^\delta - U(q^\delta_\rho)\right) \nabla \widehat{w} \\ &\leq \left(\int_\Omega |\nabla \left(U(q^\dagger) - z^\delta\right)|^2\right)^{1/2} \left(\int_\Omega \left(q^\delta_\rho\right)^2 |\nabla \widehat{w}|^2\right)^{1/2} \\ &\quad + \left(\int_\Omega q^\delta_\rho |\nabla \left(U(q^\delta_\rho) - z^\delta\right)|^2\right)^{1/2} \left(\int_\Omega q^\delta_\rho |\nabla \widehat{w}|^2\right)^{1/2} \\ &\leq \|\overline{u} - z^\delta\|_{H^1(\Omega)} \left(\overline{q}^2 \int_\Omega |\nabla \widehat{w}|^2\right)^{1/2} \\ &\quad + \left(\int_\Omega q^\delta_\rho |\nabla \left(U(q^\delta_\rho) - z^\delta\right)|^2\right)^{1/2} \left(\overline{q} \int_\Omega |\nabla \widehat{w}|^2\right)^{1/2}.\end{aligned}$$

It follows from inequality (1.5) that

$$\langle U'(q^\dagger)^* w^*, q^\dagger - q_\rho^\delta \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \leq \overline{q}\delta \left(\int_\Omega |\nabla \widehat{w}|^2 \right)^{1/2} + \sqrt{\overline{q}} \left(2 J_{z^\delta}(q_\rho^\delta) \right)^{1/2} \left(\int_\Omega |\nabla \widehat{w}|^2 \right)^{1/2}.$$

An application of the Cauchy-Schwarz inequality yields

$$\langle U'(q^\dagger)^* w^*, q^\dagger - q_\rho^\delta \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \leq \overline{q}\delta \left(\int_\Omega |\nabla \widehat{w}|^2 \right)^{1/2} + \overline{q}\rho \int_\Omega |\nabla \widehat{w}|^2 + \frac{1}{2\rho} J_{z^\delta}(q_\rho^\delta). \tag{3.30}$$

It follows from (3.27) and (3.30) that

$$\frac{1}{2} J_{z^\delta}(q_\rho^\delta) + \rho D_{TV}^{U'(q^\dagger)^* w^*}(q_\rho^\delta, q^\dagger) \leq \overline{q}\delta\rho \left(\int_\Omega |\nabla \widehat{w}|^2 \right)^{1/2} + \overline{q}\rho^2 \int_\Omega |\nabla \widehat{w}|^2 + \frac{1}{2}\overline{q}\delta^2. \tag{3.31}$$

By Lemma 1.1.7 and inequality (3.31), we conclude that

$$\begin{aligned} C_\Omega \| U(q_\rho^\delta) - z^\delta \|^2_{H^1(\Omega)} &+ \frac{4(1+C_\Omega)}{\underline{q}} \rho D_{TV}^{U'(q^\dagger)^* w^*}(q_\rho^\delta, q^\dagger) \\ &\leq \frac{4\overline{q}(1+C_\Omega)}{\underline{q}} \delta\rho \left(\int_\Omega |\nabla \widehat{w}|^2 \right)^{1/2} \\ &+ \frac{4\overline{q}(1+C_\Omega)}{\underline{q}} \rho^2 \int_\Omega |\nabla \widehat{w}|^2 + \frac{2\overline{q}(1+C_\Omega)}{\underline{q}} \delta^2 \\ &= \mathcal{O}(\delta^2) \quad \text{as } \delta \to 0 \text{ and } \rho \sim \delta. \end{aligned}$$

This means that

$$D_{TV}^{U'(q^\dagger)^* w^*}(q_\rho^\delta, q^\dagger) = \mathcal{O}(\delta) \quad \text{and} \quad \| U(q_\rho^\delta) - z^\delta \|_{H^1(\Omega)} = \mathcal{O}(\delta) \text{ as } \delta \to 0 \text{ and } \rho \sim \delta.$$

Now we establish the convergence rate

$$\left| \int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q_\rho^\delta| \right| = \mathcal{O}(\delta) \tag{3.32}$$

as $\delta \to 0$ and $\rho \sim \delta$. Take $\ell \in \partial \left(\int_\Omega |\nabla(\cdot)| \right)(q_\rho^\delta)$, it follows from Lemma 3.1.9 that

$$\begin{aligned} \int_\Omega |\nabla q_\rho^\delta| - \int_\Omega |\nabla q^\dagger| &\leq - \langle \ell, q^\dagger - q_\rho^\delta \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \\ &\leq \frac{\overline{q}\delta^2}{2\rho}. \end{aligned} \tag{3.33}$$

On the other hand, since $U'(q^\dagger)^* w^* \in \partial \left(\int_\Omega |\nabla(\cdot)| \right)(q^\dagger)$, we have

$$\begin{aligned} \int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q_\rho^\delta| &\leq -\langle U'(q^\dagger)^* w^*, q_\rho^\delta - q^\dagger \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \\ &= \langle U'(q^\dagger)^* w^*, q^\dagger - q_\rho^\delta \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)}. \end{aligned} \tag{3.34}$$

Now, using (3.30) and (3.31), we get

$$\langle U'(q^\dagger)^* w^*, q^\dagger - q^\delta_\rho \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \le 2\overline{q}\delta \left(\int_\Omega |\nabla \widehat{w}|^2 \right)^{1/2} + 2\overline{q}\rho \int_\Omega |\nabla \widehat{w}|^2 + \frac{\overline{q}\delta^2}{2\rho}. \tag{3.35}$$

From (3.33)–(3.35) we arrive at (3.32). The theorem is proved. □

3.1.3. Discussion of the source condition

Now we discuss the source condition (3.22). This is a weak source condition and it does not require any the smoothness of $q^\dagger$. Moreover, the smallness requirement on the source functions of the general convergence theory for nonlinear ill-posed problems in [41, 42, 108] is removed in our source condition.

We note that the source condition (3.22) is fulfilled if and only if there exists a functional $w^* \in H^1_\diamond(\Omega)^*$ such that

$$\int_\Omega |\nabla q| - \int_\Omega |\nabla q^\dagger| - \langle U'(q^\dagger)^* w^*, q - q^\dagger \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \ge 0 \tag{3.36}$$

for all $q \in \mathfrak{X}$. To further analyze our source condition, we assume that the sought coefficient belongs to $H^1(\Omega)$. Therefore, the admissible set of coefficients is restricted to

$$\widehat{Q_{ad}} = Q \cap H^1(\Omega) \subset Q \cap BV(\Omega).$$

We remark that, since $H^1(\Omega) \subset BV(\Omega)$, any ℓ in the dual space of $BV(\Omega)$ can be considered as an element of $H^1(\Omega)$ in the sense that there is a unique element in $H^1(\Omega)$ denoted by the same symbol such that

$$\langle \ell, q \rangle_{(BV(\Omega)^*, BV(\Omega))} = \langle \ell, q \rangle_{H^1(\Omega)}, \ \ \forall q \in H^1(\Omega).$$

In fact, since $\ell \in BV(\Omega)^*$, there exists a positive constant C such that for all $q \in H^1(\Omega)$,

$$\begin{aligned} \left| \langle \ell, q \rangle_{(BV(\Omega)^*, BV(\Omega))} \right| &\le C \|q\|_{BV(\Omega)} \\ &\le C\sqrt{2\mathrm{mes}(\Omega)} \|q\|_{H^1(\Omega)}. \end{aligned}$$

This means that ℓ belongs to $H^1(\Omega)^*$. Hence, by the Riesz representation theorem, there is a unique element $\widetilde{\ell} \in H^1(\Omega)$ such that $\langle \ell, q \rangle_{(BV(\Omega)^*, BV(\Omega))} = \langle \widetilde{\ell}, q \rangle_{H^1(\Omega)}$ for all $q \in H^1(\Omega)$.

Lemma 3.1.10. *Denote by*

$$\mathcal{B} = \left\{ \ell \in BV(\Omega)^* \mid \exists \widehat{\ell} \in H^1(\Omega) : \ \ \langle \ell, q \rangle_{(BV(\Omega)^*, BV(\Omega))} = \langle \widehat{\ell}, q \rangle_{L^2(\Omega)}, \ \ \forall q \in H^1(\Omega) \right\}.$$

If the dimension $d \le 4$ and the boundary $\partial\Omega$ is of class C^1, then

$$\overline{\mathcal{B}} = H^1(\Omega),$$

where the bar denotes the closure in $H^1(\Omega)$.

Proof. For an arbitrary but fixed function λ in $H^1(\Omega)$, the linear mapping $\Lambda : H^1(\Omega) \to \mathbb{R}$ defined by

$$\Lambda(\varphi) = \langle \lambda, \varphi \rangle_{L^2(\Omega)} \tag{3.37}$$

is continuous on $H^1(\Omega)$ since

$$|\Lambda(\varphi)| = |\langle\lambda,\varphi\rangle_{L^2(\Omega)}| \leq \|\lambda\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} \leq \|\lambda\|_{L^2(\Omega)}\|\varphi\|_{H^1(\Omega)}. \tag{3.38}$$

Then, there exists a unique element $\tilde{\lambda} \in H^1(\Omega)$ such that

$$\Lambda(\varphi) = \langle\tilde{\lambda},\varphi\rangle_{H^1(\Omega)} \tag{3.39}$$

for all $\varphi \in H^1(\Omega)$ and

$$\|\tilde{\lambda}\|_{H^1(\Omega)} = \|\Lambda\|_{\mathcal{L}(H^1(\Omega),\mathbb{R})} \leq \|\lambda\|_{L^2(\Omega)}, \tag{3.40}$$

by inequality (3.38). Then, we can define the linear mapping

$$\tilde{\Lambda} : H^1(\Omega) \to H^1(\Omega)$$

by $\tilde{\Lambda}(\lambda) = \tilde{\lambda}$. Thus,

$$\langle\lambda,\varphi\rangle_{L^2(\Omega)} = \langle\tilde{\Lambda}(\lambda),\varphi\rangle_{H^1(\Omega)}$$

for all λ and φ in $H^1(\Omega)$.

Since for all $\lambda,\varphi \in H^1(\Omega)$,

$$\langle\tilde{\Lambda}(\lambda),\varphi\rangle_{H^1(\Omega)} = \langle\lambda,\varphi\rangle_{L^2(\Omega)} = \langle\varphi,\lambda\rangle_{L^2(\Omega)} = \langle\tilde{\Lambda}(\varphi),\lambda\rangle_{H^1(\Omega)} = \langle\lambda,\tilde{\Lambda}(\varphi)\rangle_{H^1(\Omega)},$$

it follows that $\tilde{\Lambda}$ is selfadjoint. Moreover, it directly follows from inequality (3.40) that $\tilde{\Lambda}$ is continuous.

Let $\tilde{\Lambda}(\lambda) = 0$ for some λ in $H^1(\Omega)$. From this and equalities (3.39), (3.37) we have $\langle\lambda,\varphi\rangle_{L^2(\Omega)} = 0$ for all $\varphi \in H^1(\Omega)$. Therefore, $\lambda = 0$. Hence the invertible operator $\Theta^{-1} : R(\tilde{\Lambda}) \to H^1(\Omega)$ exists. We note that $\overline{R(\tilde{\Lambda})} = H^1(\Omega)$, where the bar denotes the closure in $H^1(\Omega)$. In fact, assume that $\overline{R(\tilde{\Lambda})} \neq H^1(\Omega)$. Then, there exists $\psi_0 \in \overline{R(\tilde{\Lambda})}^{\perp}$ and $\psi_0 \neq 0$. For all $\varphi \in H^1(\Omega)$, we have $0 = \langle\tilde{\Lambda}(\varphi),\psi_0\rangle_{H^1(\Omega)} = \langle\varphi,\tilde{\Lambda}(\psi_0)\rangle_{H^1(\Omega)}$, since $\tilde{\Lambda}$ is selfadjoint. It follows that $\psi_0 = 0$. This is a contradiction. It remains to show that

$$R(\tilde{\Lambda}) \subset \mathcal{B}.$$

In fact, let $\tilde{\lambda} \in R(\tilde{\Lambda})$ be an arbitrary but fixed function. Then, there is a unique element $\widehat{\lambda} \in H^1(\Omega)$ such that

$$\langle\widehat{\lambda},\varphi\rangle_{L^2(\Omega)} = \langle\tilde{\lambda},\varphi\rangle_{H^1(\Omega)} \tag{3.41}$$

for all φ in $H^1(\Omega)$. Consider the Neumann problem for the following elliptic equation

$$-\Delta\psi + \psi = \widehat{\lambda} \text{ in } \Omega,$$
$$\frac{\partial\psi}{\partial n} = 0 \text{ on } \partial\Omega.$$

It is known that this problem has a unique solution $\psi \in H^1(\Omega)$ in the sense

$$\langle\psi,\varphi\rangle_{H^1(\Omega)} = \langle\widehat{\lambda},\varphi\rangle_{L^2(\Omega)} \tag{3.42}$$

for all φ in $H^1(\Omega)$. Further, since $\widehat{\lambda} \in H^1(\Omega)$, we conclude that $\psi \in H^3(\Omega)$. Thus, it follows from (3.41) and (3.42) that $\widetilde{\lambda} = \psi \in H^3(\Omega)$.

Since $d \leq 4$, due to the Sobolev embedding theorem (see, e.g. [9, Theorem 5.7.6, p. 205]), we obtain that $\widetilde{\lambda} \in C^1(\overline{\Omega})$. Therefore, for all φ in $H^1(\Omega)$,

$$\begin{aligned}\left|\langle\tilde{\lambda}, \varphi\rangle_{H^1(\Omega)}\right| &= \left|\int_\Omega \tilde{\lambda}\varphi + \sum_{i=1}^d \int_\Omega \frac{\partial\tilde{\lambda}}{\partial x_i}\frac{\partial\varphi}{\partial x_i}\right| \\ &\leq \|\tilde{\lambda}\|_{L^\infty(\Omega)} \int_\Omega |\varphi| + \|\nabla\tilde{\lambda}\|_{L^\infty(\Omega)} \int_\Omega |\nabla\varphi| \\ &\leq C\|\varphi\|_{BV(\Omega)},\end{aligned}$$

where the positive constant C is independent of φ. This means that $\widetilde{\lambda}$ is a continuous linear functional on $H^1(\Omega)$ equipped with the $BV(\Omega)$-norm. Thus,

$$\langle\tilde{\lambda}, \varphi\rangle_{H^1(\Omega)} = \langle\tilde{\lambda}, \varphi\rangle_{(BV(\Omega)^*, BV(\Omega))}, \ \forall\varphi \in H^1(\Omega).$$

It follows from the last inequality and (3.41) that $\tilde{\lambda} \in \mathcal{B}$. The lemma is proved. □

Theorem 3.1.11. *Let the boundary $\partial\Omega$ be of class C^1 and the dimension $d \leq 4$. Suppose that a solution $q^\dagger$ to $(\mathcal{K}^q)$ has the property that there is an element $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger)$ such that $\langle \ell, q\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} = \langle\widehat{\ell}, q\rangle_{L^2(\Omega)}$ for all $q \in L^\infty(\Omega) \cap H^1(\Omega)$, where $\widehat{\ell}$ is some element of $H^1(\Omega)$. Further, assume that the exact $\overline{u} \in W^{2,\infty}(\Omega)$, $|\nabla\overline{u}| \geq \gamma$ a.e. on Ω with γ being a positive constant. Then, the condition (3.36) is fulfilled and hence convergence rates*

$$D_{TV}^{U'(q^\dagger)^* w^*}(q_\rho^\delta, q^\dagger) = \mathcal{O}(\delta) \ \text{and} \ \left|\int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q_\rho^\delta|\right| = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$, of pure total variation regularization are obtained.

We remark that, from Lemma 3.1.10, the requirement on $q^\dagger$ of the theorem is fulfilled at least on a set which is everywhere dense on $H^1(\Omega)$ as the boundary $\partial\Omega$ is of class C^1 and the dimension $d \leq 4$.

Proof of Theorem 3.1.11. Since $\widehat{\ell} \in H^1(\Omega)$, it follows from Lemma 2.1.10 that there exists $v \in H^1(\Omega)$ satisfying

$$\nabla U(q^\dagger) \cdot \nabla v = \widehat{\ell}.$$

Set

$$\widehat{v} := \frac{\int_\Omega v}{\text{mes}(\Omega)} - v.$$

Then,

$$-\nabla U(q^\dagger) \cdot \nabla\widehat{v} = \widehat{\ell} \ \text{ and } \ \widehat{v} \in H^1_\diamond(\Omega).$$

Therefore,

$$\begin{aligned}\langle \ell, q\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} &= \langle\widehat{\ell}, q\rangle_{L^2(\Omega)} \\ &= -\int_\Omega q\nabla U(q^\dagger)\nabla\widehat{v} \end{aligned} \tag{3.43}$$

for all $q \in L^\infty(\Omega) \cap H^1(\Omega)$. By (1.13), we have

$$\begin{aligned}-\int_\Omega q\nabla U(q^\dagger)\nabla\widehat{v} &= \int_\Omega q^\dagger\nabla U'(q^\dagger)q\nabla\widehat{v} \\ &= \left\langle\widehat{w}, U'(q^\dagger)q\right\rangle_{H^1(\Omega)}.\end{aligned} \tag{3.44}$$

for some $\widehat{w} \in H^1_\diamond(\Omega)$ independent of $q \in L^\infty(\Omega) \cap H^1(\Omega)$. Equalities (3.43) and (3.44) lead to

$$\langle \ell, q \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} = \left\langle \widehat{w}, U'(q^\dagger) q \right\rangle_{H^1(\Omega)}.$$

It follows that there exists a functional $w^* \in H^1_\diamond(\Omega)^*$ such that

$$\begin{aligned} \langle \ell, q \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} &= \langle w^*, U'(q^\dagger) q \rangle_{\left(H^1_\diamond(\Omega)^*, H^1_\diamond(\Omega)\right)} \\ &= \left\langle U'(q^\dagger)^* w^*, q \right\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} \end{aligned} \tag{3.45}$$

for all $q \in L^\infty(\Omega) \cap H^1(\Omega)$. It remains to show that $U'(q^\dagger)^* w^*$ is linear and continuous on $L^\infty(\Omega) \cap H^1(\Omega)$ equipped with the $BV(\Omega)$-norm. In fact, since

$$\left\langle U'(q^\dagger)^* w^*, q \right\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} = \langle \widehat{\ell}, q \rangle_{L^2(\Omega)}$$

for all $q \in L^\infty(\Omega) \cap H^1(\Omega)$ and by the similar reasoning as in the proof of Lemma 3.1.10, we obtain that there exists a functional $\psi \in C^1(\overline{\Omega})$ such that

$$\begin{aligned} \left| \left\langle U'(q^\dagger)^* w^*, q \right\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} \right| &= \left| \langle \psi, q \rangle_{H^1(\Omega)} \right| \\ &\le \max \left\{ \|\psi\|_{L^\infty(\Omega)}, \|\nabla \psi\|_{L^\infty(\Omega)} \right\} \|q\|_{\mathfrak{X}_{BV(\Omega)}} \end{aligned}$$

for all $q \in L^\infty(\Omega) \cap H^1(\Omega)$. Thus, $U'(q^\dagger)^* w^* \in \mathfrak{X}^*_{BV(\Omega)}$ and

$$\left\langle U'(q^\dagger)^* w^*, q \right\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} = \left\langle U'(q^\dagger)^* w^*, q \right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)}$$

for all $q \in L^\infty(\Omega) \cap H^1(\Omega)$. Now, since $\ell \in \partial \left(\int_\Omega |\nabla(\cdot)| \right)(q^\dagger)$ and (3.45), we conclude that there exists a functional $w^* \in H^1_\diamond(\Omega)^*$ such that

$$\begin{aligned} &\int_\Omega |\nabla q| - \int_\Omega |\nabla q^\dagger| - \langle U'(q^\dagger)^* w^*, q - q^\dagger \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \\ &\qquad = \int_\Omega |\nabla q| - \int_\Omega |\nabla q^\dagger| - \langle \ell, q - q^\dagger \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \ge 0 \end{aligned}$$

for all $q \in L^\infty(\Omega) \cap H^1(\Omega)$. The proof of theorem is now completed. □

3.2 Convergence rates for total variation regularization of the reaction coefficient identification problem

3.2.1. Regularization by the total variation

For solving the problem of identifying the coefficient a in (1.3)–(1.4), in this section we solve the convex minimization problem

$$\min_{a \in A_{ad}} G_{z^\delta}(a) + \rho \int_\Omega |\nabla a|, \tag{$\mathcal{P}^a_{\rho,\delta}$}$$

where

$$A_{ad} := A \cap BV(\Omega) \tag{3.46}$$

is the admissible set of coefficients and $\rho > 0$ is the regularization parameter. We shall see that problem $(\mathcal{P}^a_{\rho,\delta})$ has a solution a^δ_ρ which is considered as the regularized solution to the inverse problem. On the other hand, the problem

$$\min_{a\in\Pi_{A_{ad}}(\overline{u})} \int_\Omega |\nabla a| \tag{$\mathcal{K}^a$}$$

also has a solution, which is called the *total variation-minimizing solution* of equation $U(a) = \overline{u}$, where

$$\Pi_{A_{ad}}(\overline{u}) := \{a \in A_{ad} \mid U(a) = \overline{u}\}. \tag{3.47}$$

In this section we investigate the convergence rates of a^δ_ρ to the total variation-minimizing solution $a^\dagger$ of equation $U(a) = \overline{u}$.

We note that the functional $G_{z^\delta}(\cdot)$ is convex and Fréchet differentiable on A in the $L^\infty(\Omega)$-norm (see Lemmas 1.2.1 and 1.2.4). Its Fréchet differential is defined by

$$G'_{z^\delta}(a)h = -\frac{1}{2}\int_\Omega h(U(a) - z^\delta)(U(a) + z^\delta), \ \ \forall h \in L^\infty(\Omega)$$

for each $a \in A$. Thus, any $a \in A$,

$$\begin{aligned} G'_{z^\delta}(a)h &= \langle G'_{z^\delta}(a), h\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} \\ &= \langle G'_{z^\delta}(a), h\rangle_{(\mathfrak{X}^*,\mathfrak{X})}, \ \ \forall h \in \mathfrak{X}. \end{aligned} \tag{3.48}$$

Lemma 3.2.1. *Let (z^{δ_n}) be a sequence which converges to z^δ in the $H^1(\Omega)$-norm and (a_n) be a sequence in the set A defined by (1.18) convergent to a in the $L^1(\Omega)$-norm. Then,*

$$\lim_n G_{z^{\delta_n}}(a_n) = \lim_n G_{z^\delta}(a).$$

Proof. By Lemma 1.2.2, we have that the sequence $(U(a_n))$ weakly converges to $U(a)$ in $H^1(\Omega)$. We rewrite

$$\begin{aligned} 2G_{z^{\delta_n}}(a_n) &= \int_\Omega \nabla U(a_n)\nabla(U(a_n) - z^{\delta_n}) + \int_\Omega a_n U(a_n)(U(a_n) - z^{\delta_n}) \\ &\quad - \int_\Omega \nabla(U(a_n) - z^{\delta_n})\nabla z^{\delta_n} - \int_\Omega a_n(U(a_n) - z^{\delta_n})z^{\delta_n}. \end{aligned}$$

Since $(U(a_n))$ weakly converges to $U(q)$ in $H^1(\Omega)$, we get

$$\begin{aligned} &\int_\Omega \nabla U(a_n)\nabla(U(a_n) - z^{\delta_n}) + \int_\Omega a_n U(a_n)(U(a_n) - z^{\delta_n}) \\ &\qquad = \int_\Omega f(U(a_n) - z^{\delta_n}) + \int_{\partial\Omega} g(U(a_n) - z^{\delta_n}) \\ &\qquad \to \int_\Omega f(U(q) - z^\delta) + \int_{\partial\Omega} g(U(q) - z^\delta). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\int_\Omega \nabla(U(a_n) - z^{\delta_n})\nabla z^{\delta_n} + \int_\Omega a_n(U(a_n) - z^{\delta_n})z^{\delta_n} \\ &\qquad = \int_\Omega \nabla(U(a_n) - z^{\delta_n})\nabla(z^{\delta_n} - z^\delta) + \int_\Omega \nabla(U(a_n) - z^{\delta_n})\nabla z^\delta \\ &\qquad + \int_\Omega a_n(U(a_n) - z^{\delta_n})(z^{\delta_n} - z^\delta) + \int_\Omega a(U(a_n) - z^{\delta_n})z^\delta \\ &\qquad + \int_\Omega (a_n - a)(U(a_n) - z^{\delta_n})z^\delta. \end{aligned}$$

Since (z^{δ_n}) converges to z^δ in the $H^1(\Omega)$-norm, we have

$$\int_\Omega \nabla(U(a_n) - z^{\delta_n})\nabla(z^{\delta_n} - z^\delta) + \int_\Omega a_n(U(a_n) - z^{\delta_n})(z^{\delta_n} - z^\delta) \to 0.$$

A further application of the fact that $(U(a_n))$ weakly converges to $U(q)$ in $H^1(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega \nabla(U(a_n) - z^{\delta_n})\nabla z^\delta &+ \int_\Omega a(U(a_n) - z^{\delta_n})z^\delta \\ &\to \int_\Omega \nabla(U(a) - z^\delta)\nabla z^\delta + \int_\Omega a(U(a) - z^\delta)z^\delta.\end{aligned}$$

Besides, using Lemma 1.1.2, we have

$$\int_\Omega (a_n - a)(U(a_n) - z^{\delta_n})z^\delta \to 0. \tag{3.49}$$

Therefore,

$$\begin{aligned}2\lim_n G_{z^{\delta_n}}(a_n) &= \int_\Omega f(U(q) - z^\delta) + \int_{\partial\Omega} g(U(q) - z^\delta) \\ &+ \int_\Omega \nabla(U(a) - z^\delta)\nabla z^\delta + \int_\Omega a(U(a) - z^\delta)z^\delta \\ &= 2G_{z^\delta}(a).\end{aligned}$$

The lemma is proved. ☐

Now we state and prove main results of this subsection.

Theorem 3.2.2. *(i) There exists a solution of problem* $(\mathcal{P}^a_{\rho,\delta})$*. Further, an element* $\widetilde{a}$ *in* A_{ad} *is the solution to* $(\mathcal{P}^a_{\rho,\delta})$ *if and only if for all* $\lambda \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(\widetilde{a})$*, and the inequality*

$$G'_{z^\delta}(\widetilde{a})(a - \widetilde{a}) + \rho\langle\lambda, a - \widetilde{a}\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \geq 0 \tag{3.50}$$

holds for all a *in* A_{ad}*.*

(ii) There exists a solution of problem $(\mathcal{K}^a)$*. Further, an element* $a^\dagger$ *in* $\Pi_{A_{ad}}(\overline{u})$ *is the solution to* $(\mathcal{K}^a)$ *if and only if for all* $\lambda \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(a^\dagger)$*, and the inequality*

$$\langle\lambda, a - a^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \geq 0$$

holds for all a *in* $\Pi_{A_{ad}}(\overline{u})$*.*

Proof. (i) First, we show that problem $(\mathcal{P}^a_{\rho,\delta})$ has a solution. In fact, let (a_n) be a sequence in A_{ad} such that

$$G_{z^\delta}(a_n) + \rho\int_\Omega |\nabla a_n| \to \inf_{a\in A_{ad}} G_{z^\delta}(a) + \rho\int_\Omega |\nabla a|.$$

Since $(a_n) \subset A$, it is bounded in the $L^1(\Omega)$-norm. Hence it follows from the last inequality that (a_n) is bounded in the $BV(\Omega)$-norm. By Lemma 3.1.1, we see that there exist a subsequence (a_{1_n}) of (a_n) and an element $\widetilde{a} \in BV(\Omega)$ such that (a_{1_n}) converges to $\widetilde{a}$ in the $L^1(\Omega)$-norm and

$$\int_\Omega |\nabla\widetilde{a}| \leq \liminf_n \int_\Omega |\nabla a_{1_n}|.$$

Since A is closed in the $L^1(\Omega)$-norm, we obtain that $\widetilde{a} \in A$ and so $\widetilde{a} \in A \cap BV(\Omega) = A_{ad}$. Applying Lemma 3.2.1, we have

$$\begin{aligned} G_{z^\delta}(\widetilde{a}) + \rho \int_\Omega |\nabla \widetilde{a}| &\le \lim_n G_{z^\delta}(a_{1_n}) + \rho \liminf_n \int_\Omega |\nabla a_{1_n}| \\ &= \liminf_n \left(G_{z^\delta}(a_{1_n}) + \rho \int_\Omega |\nabla a_{1_n}| \right) \\ &= \inf_{a \in A_{ad}} G_{z^\delta}(a) + \rho \int_\Omega |\nabla a|. \end{aligned}$$

Hence $\widetilde{a}$ is a solution of $(\mathcal{P}^a_{\rho,\delta})$.

Now, we note that an element $\widetilde{a} \in A_{ad}$ is a solution of $(\mathcal{P}^a_{\rho,\delta})$ if and only if

$$G'_{z^\delta}(\widetilde{a})(a - \widetilde{a}) + \rho \left(\int_\Omega |\nabla a| - \int_\Omega |\nabla \widetilde{a}| \right) \ge 0, \;\; \forall a \in A_{ad}$$

or

$$0 \in G'_{z^\delta}(\widetilde{a}) + \partial \left(\rho \int_\Omega |\nabla (\cdot)| + I_{A_{ad}} \right) (\widetilde{a})$$

which is equivalent to inequality (3.50).

(ii) Let (a_n) be a sequence in $\Pi_{A_{ad}}(\overline{u})$ such that

$$\lim_n \int_\Omega |\nabla a_n| = \inf_{a \in \Pi_{A_{ad}}(\overline{u})} \int_\Omega |\nabla a|.$$

It follows from the last equality that there exist a subsequence (a_{1_n}) of (a_n) and an element $a^\dagger \in A_{ad}$ such that (a_{1_n}) converges to $a^\dagger$ in the $L^1(\Omega)$-norm and

$$\begin{aligned} \int_\Omega |\nabla a^\dagger| &\le \liminf_n \int_\Omega |\nabla a_{1_n}| \\ &= \inf_{a \in \Pi_{A_{ad}}(\overline{u})} \int_\Omega |\nabla a|. \end{aligned}$$

Applying Lemma 1.2.2, we have $U(a^\dagger) = \overline{u}$ or $a^\dagger \in \Pi_{A_{ad}}(\overline{u})$. This means that $a^\dagger$ is a solution of problem $(\mathcal{K}^a)$.

Finally, we have that an element $a^\dagger \in \Pi_{A_{ad}}(\overline{u})$ is a solution of $(\mathcal{K}^a)$ if and only if

$$0 \in \partial \left(\int_\Omega |\nabla (\cdot)| + I_{A_{ad}} \right) (a^\dagger)$$

which is equivalent to the desired inequality. The theorem is proved. □

Theorem 3.2.3. *For a fixed regularization parameter $\rho > 0$, let (z^{δ_n}) be a sequence which converges to z^δ in the $H^1(\Omega)$-norm and $(a^{\delta_n}_\rho)$ be minimizers of the problems*

$$\min_{a \in A_{ad}} G_{z^{\delta_n}}(a) + \rho \int_\Omega |\nabla a|.$$

Then, there exist a subsequence $(a^{\delta_{1n}}_\rho)$ of $(a^{\delta_n}_\rho)$ and $\widetilde{a} \in A_{ad}$ such that $(a^{\delta_{1n}}_\rho)$ converges to $\widetilde{a}$ in the $L^1(\Omega)$-norm and

$$\lim_n \int_\Omega |\nabla a^{\delta_{1n}}_\rho| = \int_\Omega |\nabla \widetilde{a}|.$$

Further, $\widetilde{a}$ is a solution to $(\mathcal{P}^a_{\rho,\delta})$.

Proof. For all $n \in \mathbb{N}$ and $a \in A_{ad}$, we have

$$G_{z^{\delta_n}}(a_\rho^{\delta_n}) + \rho \int_\Omega |\nabla a_\rho^{\delta_n}| \leq G_{z^{\delta_n}}(a) + \rho \int_\Omega |\nabla a|. \tag{3.51}$$

By Lemma 3.1.1, the last inequality yields that there exist a subsequence $(a_\rho^{\delta_{1n}})$ of $(a_\rho^{\delta_n})$ and $\widetilde{a} \in BV(\Omega)$ such that $(a_\rho^{\delta_{1n}})$ converges to $\widetilde{a}$ in the $L^1(\Omega)$-norm and

$$\int_\Omega |\nabla \widetilde{a}| \leq \liminf_n \int_\Omega |\nabla a_\rho^{\delta_{1n}}|. \tag{3.52}$$

Since A is closed in the $L^1(\Omega)$-norm, it follows that $\widetilde{a} \in A$. Hence we have $\widetilde{a} \in A_{ad}$. Now, applying Lemma 3.2.1, we have

$$\lim_n G_{z^{\delta_{1n}}}(a_\rho^{\delta_{1n}}) = G_{z^\delta}(\widetilde{a}). \tag{3.53}$$

From the last equality and (3.51), (3.52), we have

$$\begin{aligned} G_{z^\delta}(\widetilde{a}) + \rho \int_\Omega |\nabla \widetilde{a}| &\leq \liminf_n \left(G_{z^{\delta_{1n}}}(a_\rho^{\delta_{1n}}) + \rho \int_\Omega |\nabla a_\rho^{\delta_{1n}}| \right) \\ &\leq \liminf_n \left(G_{z^{\delta_{1n}}}(a) + \rho \int_\Omega |\nabla a| \right) \\ &= G_{z^\delta}(a) + \rho \int_\Omega |\nabla a| \end{aligned}$$

for all $a \in A_{ad}$. Thus, $\widetilde{a}$ is a solution to $(\mathcal{P}^a_{\rho,\delta})$. On the other hand, choosing $a = \widetilde{a}$ in (3.51) and again using (3.53), we have

$$\begin{aligned} G_{z^\delta}(\widetilde{a}) + \rho \limsup_n \int_\Omega |\nabla a_\rho^{\delta_{1n}}| &= \limsup_n \left(G_{z^{\delta_{1n}}}(a_\rho^{\delta_{1n}}) + \rho \int_\Omega |\nabla a_\rho^{\delta_{1n}}| \right) \\ &\leq \limsup_n \left(G_{z^{\delta_{1n}}}(\widetilde{a}) + \rho \int_\Omega |\nabla \widetilde{a}| \right) \\ &= G_{z^\delta}(\widetilde{a}) + \rho \int_\Omega |\nabla \widetilde{a}|. \end{aligned}$$

Therefore,

$$\limsup_n \int_\Omega |\nabla a_\rho^{\delta_{1n}}| \leq \int_\Omega |\nabla \widetilde{a}|.$$

This and (3.52) yield $\lim_n \int_\Omega |\nabla a_\rho^{\delta_{1n}}| = \int_\Omega |\nabla \widetilde{a}|$. The theorem is proved. □

Theorem 3.2.4. *For any positive sequence $(\delta_n) \to 0$, let $\rho_n := \rho(\delta_n)$ be such that*

$$\rho_n \to 0 \text{ and } \frac{\delta_n^2}{\rho_n} \to 0 \text{ as } n \to \infty.$$

Moreover, let (z^{δ_n}) be a sequence satisfying $\|\overline{u} - z^{\delta_n}\|_{H^1(\Omega)} \leq \delta_n$ and $(a_{\rho_n}^{\delta_n})$ be minimizers of problems

$$\min_{a \in A_{ad}} G_{z^{\delta_n}}(a) + \rho_n \int_\Omega |\nabla a|.$$

Then, there exists a subsequence $(a_{\rho_{1n}}^{\delta_{1n}})$ of $(a_{\rho_n}^{\delta_n})$ and an element $\widehat{a} \in \Pi_{A_{ad}}(\overline{u})$ such that

$$\lim_n \|a_{\rho_{1n}}^{\delta_{1n}} - \widehat{a}\|_{L^1(\Omega)} = 0 \text{ and } \lim_n \int_\Omega |\nabla a_{\rho_{1n}}^{\delta_{1n}}| = \int_\Omega |\nabla \widehat{a}|.$$

Further, $\widehat{a}$ is a solution to problem ($\mathcal{K}^a$) and for all $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(\widehat{a})$

$$\lim_n D^\ell_{TV}(a^{\delta_{1_n}}_{\rho_{1_n}}, \widehat{a}) = 0.$$

Proof. For all $n \in \mathbb{N}$ and $a \in A_{ad}$, we have

$$G_{z^{\delta_n}}(a^{\delta_n}_{\rho_n}) + \rho_n \int_\Omega |\nabla a^{\delta_n}_{\rho_n}| \le G_{z^{\delta_n}}(a) + \rho_n \int_\Omega |\nabla a|.$$

Thus, for all $a \in \Pi_{A_{ad}}(\overline{u})$,

$$\begin{aligned} G_{z^{\delta_n}}(a^{\delta_n}_{\rho_n}) + \rho_n \int_\Omega |\nabla a^{\delta_n}_{\rho_n}| &\le \frac{1}{2}\int_\Omega |\nabla(\overline{u} - z^{\delta_n})|^2 + \frac{1}{2}\int_\Omega a(\overline{u} - z^{\delta_n})^2 + \rho_n \int_\Omega |\nabla a| \\ &\le \frac{\max\{1, \overline{a}\}}{2}\delta_n^2 + \rho_n \int_\Omega |\nabla a| \qquad (3.54) \end{aligned}$$

and

$$\int_\Omega |\nabla a^{\delta_n}_{\rho_n}| \le \frac{\max\{1, \overline{a}\}}{2}\frac{\delta_n^2}{\rho_n} + \int_\Omega |\nabla a|, \ \ \forall n \in \mathbb{N}, \forall a \in \Pi_{A_{ad}}(\overline{u}). \qquad (3.55)$$

Since $\delta_n^2/\rho_n \to 0$ as $n \to \infty$, the last inequality follows that the sequence $(a^{\delta_n}_{\rho_n})$ is bounded in the $BV(\Omega)$-norm. Hence there exist a subsequence $(a^{\delta_{1_n}}_{\rho_{1_n}})$ of $(a^{\delta_n}_{\rho_n})$ and $\widehat{a} \in A_{ad}$ such that

$$a^{\delta_{1_n}}_{\rho_{1_n}} \to \widehat{a} \text{ in the } L^1(\Omega)\text{-norm}, \qquad (3.56)$$

$$\int_\Omega |\nabla \widehat{a}| \le \liminf_n \int_\Omega |\nabla a^{\delta_{1_n}}_{\rho_{1_n}}|. \qquad (3.57)$$

Due to Lemma 1.2.2, the convergence (3.56) yields that

$$U(a^{\delta_{1_n}}_{\rho_{1_n}}) \text{ weakly converges to } U(\widehat{a}) \text{ in } H^1(\Omega). \qquad (3.58)$$

By Lemma 1.2.6, we have

$$\beta \|U(a^{\delta_{1_n}}_{\rho_{1_n}}) - z^{\delta_{1_n}}\|^2_{H^1(\Omega)} \le G_{z^{\delta_{1_n}}}(a^{\delta_{1_n}}_{\rho_{1_n}}), \ \ \forall n \in \mathbb{N}.$$

Thus, by (3.54), for all $n \in \mathbb{N}$, we obtain

$$\frac{\beta}{2}\|U(a^{\delta_{1_n}}_{\rho_{1_n}}) - z^{\delta_{1_n}}\|^2_{H^1(\Omega)} \le \frac{\max\{1, \overline{a}\}}{2}\delta_{1_n}^2 + \rho_{1_n}\int_\Omega |\nabla a|, \ \ \forall a \in \Pi_{A_{ad}}(\overline{u}).$$

Since $\delta_{1_n} \to 0$ and $\rho_{1_n} \to 0$ as $n \to \infty$, the last inequality follows that

$$\|U(a^{\delta_{1_n}}_{\rho_{1_n}}) - z^{\delta_{1_n}}\|_{H^1(\Omega)} \to 0 \text{ as } n \to \infty.$$

Besides, an application of the triangle inequality gives

$$\begin{aligned} \|U(a^{\delta_{1_n}}_{\rho_{1_n}}) - \overline{u}\|_{H^1(\Omega)} &\le \|U(a^{\delta_{1_n}}_{\rho_{1_n}}) - z^{\delta_{1_n}}\|_{H^1(\Omega)} + \|\overline{u} - z^{\delta_{1_n}}\|_{H^1(\Omega)} \\ &\le \|U(a^{\delta_{1_n}}_{\rho_{1_n}}) - z^{\delta_{1_n}}\|_{H^1(\Omega)} + \delta_{1_n} \to 0 \qquad (3.59) \end{aligned}$$

as $n \to \infty$. By inequalities (3.58) and (3.59), we have $U(\widehat{a}) = \overline{u}$ or $\widehat{a} \in \Pi_{A_{ad}}(\overline{u})$. Further, for all $a \in \Pi_{A_{ad}}(\overline{u})$, it follows from (3.57) and (3.55) that

$$\int_\Omega |\nabla \widehat{a}| \le \liminf_n \left(\frac{\max\{1, \overline{a}\}}{2}\frac{\delta_{1_n}^2}{\rho_{1_n}} + \int_\Omega |\nabla a|\right) = \int_\Omega |\nabla a|.$$

Thus, $\widehat{a}$ is the solution to problem $(\mathcal{K}^q)$. Now, again using (3.55), we get

$$\limsup_n \int_\Omega |\nabla a^{\delta_{1n}}_{\rho_{1n}}| \le \int_\Omega |\nabla \widehat{a}|$$

and hence

$$\int_\Omega |\nabla \widehat{a}| = \lim_n \int_\Omega |\nabla a^{\delta_{1n}}_{\rho_{1n}}|. \tag{3.60}$$

By equalities (3.56) and (3.60), we get that the sequence $(a^{\delta_{1n}}_{\rho_{1n}})$ weakly converges to $\widehat{a}$ in $BV(\Omega)$ (see [9], Proposition 10.1.2, p. 374). Thus, for all $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(\widehat{a})$, we conclude that

$$\lim_n D^\ell_{TV}(a^{\delta_{1n}}_{\rho_{1n}}, \widehat{a}) = \lim_n \left(\int_\Omega |\nabla a^{\delta_{1n}}_{\rho_{1n}}| - \int_\Omega |\nabla \widehat{a}| - \langle \ell, a^{\delta_{1n}}_{\rho_{1n}} - \widehat{a} \rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \right) = 0.$$

The theorem is proved. □

3.2.2. Convergence rates

Now we investigate convergence rates of a^δ_ρ to the total variation-minimizing solution $a^\dagger$ of equation $U(a) = \overline{u}$.

Theorem 3.2.5. *Let $a^\dagger$ be a solution of $(\mathcal{K}^a)$. Assume that there exists a functional $w^* \in H^1(\Omega)^*$ such that*

$$U'(a^\dagger)^* w^* \in \partial \left(\int_\Omega |\nabla(\cdot)| \right)(a^\dagger). \tag{3.61}$$

Then,

$$D^{U'(a^\dagger)^* w^*}_{TV}(a^\delta_\rho, a^\dagger) = \mathcal{O}(\delta) \ \text{ and } \ \left| \int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a^\delta_\rho| \right| = \mathcal{O}(\delta)$$

and

$$\|U(a^\delta_\rho) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$.

To proving this result we need the following auxiliary lemma.

Lemma 3.2.6. *The estimate*

$$-\rho \left\langle \lambda, a^\dagger - a^\delta_\rho \right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \le \frac{\max\{1, \overline{a}\}}{2} \delta^2 \tag{3.62}$$

holds for all a^δ_ρ being solutions of problem $(\mathcal{P}^a_{\rho,\delta})$ and $\lambda \in \partial \left(\int_\Omega |\nabla(\cdot)| \right)(a^\delta_\rho)$.

Proof. By the convexity and non-negativity of the function $G_{z^\delta}(\cdot)$ and (3.50), we have

$$\begin{aligned} -\rho \left\langle \lambda, a^\dagger - a^\delta_\rho \right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} &\le G'_{z^\delta}(a^\delta_\rho)(a^\dagger - a^\delta_\rho) \\ &\le G_{z^\delta}(a^\dagger) - G_{z^\delta}(a^\delta_\rho) \\ &\le G_{z^\delta}(a^\dagger). \end{aligned}$$

The proof is now completed by noting that $G_{z^\delta}(a^\dagger) \le \frac{\max\{1,\overline{a}\}}{2}\delta^2$. □

Proof of Theorem 3.2.5. By the definition of a_ρ^δ, we have

$$G_{z^\delta}(a_\rho^\delta) + \rho \int_\Omega |\nabla a_\rho^\delta| \le G_{z^\delta}(a^\dagger) + \rho \int_\Omega |\nabla a^\dagger|.$$

Then,

$$\begin{aligned} G_{z^\delta}(a_\rho^\delta) + \rho D_{TV}^{U'(a^\dagger)^* w^*}(a_\rho^\delta, a^\dagger) \le \frac{\max\{\overline{a}, 1\}}{2}\delta^2 \\ - \rho \langle U'(a^\dagger)^* w^*, a_\rho^\delta - a^\dagger \rangle_{\left(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)}\right)}. \end{aligned} \tag{3.63}$$

It follows from equalities (3.6) and (3.21), and the source condition (3.61) that

$$\begin{aligned} -\langle U'(a^\dagger)^* w^*, a_\rho^\delta - a^\dagger \rangle_{\left(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)}\right)} &= \langle U'(a^\dagger)^* w^*, a^\dagger - a_\rho^\delta \rangle_{(\mathfrak{X}^*, \mathfrak{X})} \\ &= \left\langle U'(a^\dagger)^* w^*, a^\dagger - a_\rho^\delta \right\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} \\ &= \left\langle w^*, U'(a^\dagger)(a^\dagger - a_\rho^\delta) \right\rangle_{\left(H^1(\Omega)^*, H^1(\Omega)\right)} \\ &= \langle w, U'(a^\dagger)(a^\dagger - a_\rho^\delta) \rangle_{H^1(\Omega)} \end{aligned} \tag{3.64}$$

for some $w \in H^1(\Omega)$, by the Riesz representation theorem. Since $a^\dagger \ge \underline{a} > 0$, the scalar product

$$(u, v)_{H^1(\Omega)} := \int_\Omega \nabla u \nabla v + \int_\Omega a^\dagger uv$$

is equivalent to $\langle u, v \rangle_{H^1(\Omega)}$ on $H^1(\Omega)$. Hence there exist an element $\widehat{w} \in H^1(\Omega)$ independent of a_ρ^δ such that

$$\begin{aligned} \langle w, U'(a^\dagger)(a^\dagger - a_\rho^\delta) \rangle_{H^1(\Omega)} &= \int_\Omega \nabla \widehat{w} \nabla U'(a^\dagger)(a^\dagger - a_\rho^\delta) \\ &+ \int_\Omega a^\dagger \widehat{w} U'(a^\dagger)(a^\dagger - a_\rho^\delta). \end{aligned} \tag{3.65}$$

It follows from (3.64) and (3.65) that

$$\begin{aligned} -\langle U'(a^\dagger)^* w^*, a_\rho^\delta - a^\dagger \rangle_{\left(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)}\right)} &= \int_\Omega \nabla \widehat{w} \nabla U'(a^\dagger)(a^\dagger - a_\rho^\delta) \\ &+ \int_\Omega a^\dagger \widehat{w} U'(a^\dagger)(a^\dagger - a_\rho^\delta). \end{aligned}$$

By (1.23), the last equality leads to

$$-\langle U'(a^\dagger)^* w^*, a_\rho^\delta - a^\dagger \rangle_{\left(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)}\right)} = -\int_\Omega (a^\dagger - a_\rho^\delta) U(a^\dagger) \widehat{w}. \tag{3.66}$$

By equality (1.17), we have

$$\begin{aligned} \int_\Omega \nabla U(a^\dagger) \nabla \widehat{w} + \int_\Omega a^\dagger U(a^\dagger) \widehat{w} &= \int_\Omega f \widehat{w} \\ &= \int_\Omega \nabla U(a_\rho^\delta) \nabla \widehat{w} + \int_\Omega a_\rho^\delta U(a_\rho^\delta) \widehat{w}. \end{aligned}$$

Thus, by equality (3.66),

$$\begin{aligned}
-\langle U'(a^\dagger)^* w^*, a_\rho^\delta - a^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)} &= \int_\Omega a_\rho^\delta U(a^\dagger)\widehat{w} - \int_\Omega a^\dagger U(a^\dagger)\widehat{w} \\
&= \int_\Omega a_\rho^\delta U(a^\dagger)\widehat{w} - \int_\Omega a_\rho^\delta U(a_\rho^\delta)\widehat{w} \\
&+ \int_\Omega \nabla U(a^\dagger)\nabla\widehat{w} - \int_\Omega \nabla U(a_\rho^\delta)\nabla\widehat{w} \\
&= \int_\Omega a_\rho^\delta \left(U(a^\dagger) - z^\delta\right)\widehat{w} + \int_\Omega a_\rho^\delta \left(z^\delta - U(a_\rho^\delta)\right)\widehat{w} \\
&+ \int_\Omega \nabla\left(U(a^\dagger) - z^\delta\right)\nabla\widehat{w} \\
&+ \int_\Omega \nabla\left(z^\delta - U(a_\rho^\delta)\right)\nabla\widehat{w}.
\end{aligned}$$

Applying the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
-\langle U'(a^\dagger)^* w^*, a_\rho^\delta - a^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)} &\le \left(\int_\Omega (U(a^\dagger) - z^\delta)^2\right)^{1/2}\left(\int_\Omega (a_\rho^\delta)^2\widehat{w}^2\right)^{1/2} \\
&+ \left(\int_\Omega a_\rho^\delta (z^\delta - U(a_\rho^\delta))^2\right)^{1/2}\left(\int_\Omega a_\rho^\delta\widehat{w}^2\right)^{1/2} \\
&+ \left(\int_\Omega |\nabla(U(a^\dagger) - z^\delta)|^2\right)^{1/2}\left(\int_\Omega |\nabla\widehat{w}|^2\right)^{1/2} \\
&+ \left(\int_\Omega |\nabla\left(z^\delta - U(a_\rho^\delta)\right)|^2\right)^{1/2}\left(\int_\Omega |\nabla\widehat{w}|^2\right)^{1/2}.
\end{aligned}$$

Then, by (1.5),

$$\begin{aligned}
-\langle U'(a^\dagger)^* w^*, a_\rho^\delta - a^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)} &\le \delta\overline{a}\left(\int_\Omega \widehat{w}^2\right)^{1/2} + \delta\left(\int_\Omega |\nabla\widehat{w}|^2\right)^{1/2} \\
&+ \sqrt{\overline{a}}\left(\int_\Omega a_\rho^\delta (z^\delta - U(a_\rho^\delta))^2\right)^{1/2}\left(\int_\Omega \widehat{w}^2\right)^{1/2} \\
&+ \left(\int_\Omega |\nabla(z^\delta - U(a_\rho^\delta))|^2\right)^{1/2}\left(\int_\Omega |\nabla\widehat{w}|^2\right)^{1/2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
-\langle U'(a^\dagger)^* w^*, a_\rho^\delta - a^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)} &\le \delta\overline{a}\left(\int_\Omega \widehat{w}^2\right)^{1/2} + \delta\left(\int_\Omega |\nabla\widehat{w}|^2\right)^{1/2} \\
&+ \rho\overline{a}\int_\Omega \widehat{w}^2 + \frac{1}{4\rho}\int_\Omega a_\rho^\delta\left(z^\delta - U(a_\rho^\delta)\right)^2 \\
&+ \rho\int_\Omega |\nabla\widehat{w}|^2 + \frac{1}{4\rho}\int_\Omega |\nabla\left(z^\delta - U(a_\rho^\delta)\right)|^2. \qquad (3.67)
\end{aligned}$$

Hence inequalities (3.63), (3.64) and (3.67) yield

$$\begin{aligned}
\frac{1}{2}G_{z^\delta}(a_\rho^\delta) + \rho D_{TV}^{U'(a^\dagger)^* w^*}(a_\rho^\delta, a^\dagger) &\le \frac{\max\{\overline{a}, 1\}}{2}\delta^2 + \rho\delta\overline{a}\left(\int_\Omega \widehat{w}^2\right)^{1/2} + \rho^2\overline{a}\int_\Omega \widehat{w}^2 \\
&+ \rho\delta\left(\int_\Omega |\nabla\widehat{w}|^2\right)^{1/2} + \rho^2\int_\Omega |\nabla\widehat{w}|^2. \qquad (3.68)
\end{aligned}$$

By the estimate in Lemma 1.2.6, we obtain that

$$D_{TV}^{U'(a^\dagger)^* w^*}(a_\rho^\delta, a^\dagger) = \mathcal{O}(\delta) \text{ and } \|U(a_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta) \text{ as } \delta \to 0 \text{ and } \rho \sim \delta.$$

It remains to prove the convergence rate $\left|\int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a_\rho^\delta|\right| = \mathcal{O}(\delta)$ as $\delta \to 0$ and $\rho \sim \delta$. Since $\lambda \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(A_\rho^\delta)$, we get from Lemma 3.2.6 that

$$\begin{aligned}\int_\Omega |\nabla a_\rho^\delta| - \int_\Omega |\nabla a^\dagger| &\le -\langle \lambda, a^\dagger - a_\rho^\delta \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \\ &\le \frac{\max\{\overline{a}, 1\}}{2} \frac{\delta^2}{\rho}. \end{aligned} \tag{3.69}$$

On the other hand, since $U'(a^\dagger)^* w^* \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(a^\dagger)$, we have

$$\int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a_\rho^\delta| \le -\langle U'(a^\dagger)^* w^*, a_\rho^\delta - a^\dagger \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)}. \tag{3.70}$$

It follows from inequalities (3.67) and (3.68) that

$$\begin{aligned}-\rho\langle U'(a^\dagger)^* w^*, a_\rho^\delta - a^\dagger \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} &\le \frac{\max\{\overline{a}, 1\}}{2}\delta^2 + 2\rho\delta\overline{a}\left(\int_\Omega \widehat{w}^2\right)^{1/2} \\ &\quad + 2\rho^2\overline{a}\int_\Omega \widehat{w}^2 + 2\rho^2\int_\Omega |\nabla\widehat{w}|^2 \\ &\quad + 2\rho\delta\left(\int_\Omega |\nabla\widehat{w}|^2\right)^{1/2}. \end{aligned} \tag{3.71}$$

By inequalities (3.70) and (3.71), we conclude that

$$\begin{aligned}\int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a_\rho^\delta| &\le \frac{\max\{\overline{a}, 1\}}{2}\frac{\delta^2}{\rho} + 2\delta\overline{a}\left(\int_\Omega \widehat{w}^2\right)^{1/2} + 2\rho\overline{a}\int_\Omega \widehat{w}^2 \\ &\quad + 2\rho\int_\Omega |\nabla\widehat{w}|^2 + 2\delta\left(\int_\Omega |\nabla\widehat{w}|^2\right)^{1/2}. \end{aligned} \tag{3.72}$$

The proof of the theorem is now completed by estimates (3.69) and (3.72). □

3.2.3. Discussion of the source condition

Now we discuss the source condition (3.61). We have to show that there exists a function $w^* \in H^1(\Omega)^*$ such that

$$\int_\Omega |\nabla a| - \int_\Omega |\nabla a^\dagger| - \langle U'(a^\dagger)^* w^*, a - a^\dagger \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \ge 0 \tag{3.73}$$

for all $a \in \mathfrak{X}$. This is a weak source condition without the smallness requirement on the source functions. Furthermore, it does not require any the smoothness of $a^\dagger$. To further analyze this condition we assume that the admissible set of coefficients is restricted to

$$\widehat{A_{ad}} = A \cap H^1(\Omega) \subset A \cap BV(\Omega).$$

Theorem 3.2.7. *Let the boundary $\partial\Omega$ be of class C^1 and the dimension $d \le 4$. Suppose that a solution $a^\dagger$ to $(\mathcal{K}^a)$ has the property that there is an element $\lambda \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(a^\dagger)$ such that $\langle \lambda, a \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} = \langle \widehat{\lambda}, a \rangle_{L^2(\Omega)}$ for all $a \in L^\infty(\Omega) \cap H^1(\Omega)$, where $\widehat{\lambda}$ is some* *element of $H^1(\Omega)$. Furthermore, assume that there exists a positive constant γ such that $|\overline{u}| \ge \gamma$ a.e. on Ω. Then, the condition (3.73) is fulfilled and hence convergence rates*

$$D_{TV}^{U'(a^\dagger)^* w^*}(a_\rho^\delta, a^\dagger) = \mathcal{O}(\delta) \text{ and } \left| \int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a_\rho^\delta| \right| = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$, of pure total variation regularization are obtained.

Proof. Since $\widehat{\lambda} \in H^1(\Omega)$ and $|\overline{u}| = |U(a^\dagger)| \ge \gamma > 0$, we have $\frac{\widehat{\lambda}}{U(a^\dagger)} \in H^1(\Omega)$. Thus, we can choose $\vartheta \in H^1(\Omega)$ such that

$$-U(a^\dagger)\vartheta = \widehat{\lambda}.$$

Hence

$$-\int_\Omega aU(a^\dagger)\vartheta = \langle \widehat{\lambda}, a \rangle_{L^2(\Omega)} \tag{3.74}$$

for all $a \in L^\infty(\Omega) \cap H^1(\Omega)$. By (1.23), the last equality leads to

$$\begin{aligned} \langle \lambda, a \rangle_{\left(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)}\right)} &= \langle \widehat{\lambda}, a \rangle_{L^2(\Omega)} \\ &= \int_\Omega \nabla U'(a^\dagger)(a) \nabla \vartheta + \int_\Omega a^\dagger U'(a^\dagger)(a)\vartheta \\ &= \langle \widehat{w}, U'(a^\dagger)\,(a) \rangle_{H^1(\Omega)} \end{aligned}$$

for some $\widehat{w} \in H^1(\Omega)$ independent of $a \in L^\infty(\Omega) \cap H^1(\Omega)$. Therefore, there exist an element $w^* \in H^1(\Omega)^*$ such that

$$\begin{aligned} \langle \lambda, a \rangle_{\left(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)}\right)} &= \langle w^*, U'(a^\dagger)(a) \rangle_{(H^1(\Omega)^*, H^1(\Omega))} \\ &= \left\langle U'(a^\dagger)^* w^*, a \right\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))}. \end{aligned}$$

By the similar reasoning as in the proof of Theorem 3.1.11, we obtain that the functional $U'(a^\dagger)^* w^*$ is linear and continuous on $L^\infty(\Omega) \cap H^1(\Omega)$ equipped with the $BV(\Omega)$-norm and

$$\left\langle U'(a^\dagger)^* w^*, a \right\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} = \left\langle U'(a^\dagger)^* w^*, a \right\rangle_{\left(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)}\right)}$$

for all $a \in L^\infty(\Omega) \cap H^1(\Omega)$. Since $\lambda \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(a^\dagger)$, we conclude that there exists a functional $w^* \in H^1(\Omega)^*$ such that

$$\begin{aligned} &\int_\Omega |\nabla a| - \int_\Omega |\nabla a^\dagger| - \langle U'(a^\dagger)^* w^*, a - a^\dagger \rangle_{\left(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)}\right)} \\ &\qquad = \int_\Omega |\nabla a| - \int_\Omega |\nabla a^\dagger| - \langle \lambda, a - a^\dagger \rangle_{\left(\mathfrak{X}_{BV(\Omega)}^*, \mathfrak{X}_{BV(\Omega)}\right)} \ge 0 \end{aligned}$$

for all $a \in L^\infty(\Omega) \cap H^1(\Omega)$. The theorem is proved. □

Conclusions

In this chapter, we continuously investigate convergence rates for the Tikhonov regularization of ill-posed nonlinear inverse problems of identifying the diffusion coefficient and the reaction coefficient in the Neumann problems for the elliptic equation (1.1)–(1.2) and (1.3)–(1.4), respectively when the exact solution $\overline{u}$ is imprecisely given by observed data z^δ satisfying (1.5). We apply total variation regularization to convex functionals $J_{z^\delta}(\cdot)$ and $G_{z^\delta}(\cdot)$ defined by (1.12) and (1.22) and obtain convergence rates $\mathcal{O}(\delta)$ of regularized solutions in the sense of the Bregman distance. Our source conditions are simple and weak, since we remove the so-called "small enough condition" on the source functions that is standard in the theory of regularization of nonlinear ill-posed problems but very hard to check. Furthermore, our results are valid for multi-dimensional identification problems. They are the first results affirmatively answering the question whether total variation regularization can provide convergence rates for coefficient identification problems in partial differential equations.

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Chapter 4

Regularization of total variation combining with L^2-stabilization

In this chapter total variation regularization combining with L^2-stabilization is applied to the convex functionals $J_{z^\delta}(\cdot)$ and $G_{z^\delta}(\cdot)$ respectively defined by (1.12) and (1.22). We obtain convergence rates of regularized solutions to solutions of the identification problems in the sense of the Bregman distance and in the $L^2(\Omega)$-norm.

4.1 Convergence rates for total variation regularization combining with L^2-stabilization of the diffusion coefficient identification problem

4.1.1. Regularization by total variation combining with L^2-stabilization

For identifying the coefficient q in (1.1)–(1.2), in this section we solve the strictly convex minimization problem

$$\min_{q\in Q_{ad}} J_{z^\delta}(q) + \rho\left(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q|\right), \qquad (\mathbb{P}^q_{\rho,\delta})$$

where Q_{ad} defined by (3.1) is the admissible set of coefficients, $\rho > 0$ is the regularization parameter.

In the following we will see that problem $(\mathbb{P}^q_{\rho,\delta})$ has a *unique solution* q^δ_ρ on the nonempty, convex, bounded and closed in the $L^2(\Omega)$-norm set Q_{ad}, which is called *regularized solution* to our inverse problem (see Theorem 4.1.3). Due to the nonempty, convexity, closedness and boundedness in the $L^2(\Omega)$-norm of the set $\Pi_{Q_{ad}}(\overline{u})$ defined by (3.2) (see Lemma 4.1.1), we can conclude that there is a *unique solution* $q^\dagger$ of problem

$$\min_{q\in\Pi_{Q_{ad}}(\overline{u})} \left(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q|\right), \qquad (\mathbb{K}^q)$$

which we call *R-minimizing solution* to our inverse problem, where

$$R(\cdot) := \frac{1}{2}\|\cdot\|^2_{L^2(\Omega)} + \int_\Omega |\nabla(\cdot)|\,. \qquad (4.1)$$

Our aim in this section is to investigate convergence rates of q_ρ^δ to the R-minimizing solution $q^\dagger$ of the equation $U(q) = \overline{u}$.

The following results are useful.

Lemma 4.1.1. *The set $\Pi_{Q_{ad}}(\overline{u})$ is nonempty, convex, closed and bounded in the $L^2(\Omega)$-norm.*

Proof. It is clear that $\Pi_{Q_{ad}}(\overline{u})$ is a nonempty, convex and bounded set. Suppose that the sequence $(q_n) \subset \Pi_{Q_{ad}}(\overline{u})$ converges to q in the $L^2(\Omega)$-norm. We prove that $q \in \Pi_{Q_{ad}}(\overline{u})$. Since Q is closed in the $L^2(\Omega)$-norm, it follows that $q \in Q$. Since $\mathrm{mes}(\Omega) < \infty$, we obtain that (q_n) converges to q in the $L^1(\Omega)$-norm. Thus, by Lemma 3.1.1, $q \in BV(\Omega)$ and so $q \in Q \cap BV(\Omega) = Q_{ad}$. Further, due to Lemma 1.1.3, we get $\overline{u} = U(q)$. Hence $q \in \Pi_{Q_{ad}}(\overline{u})$. The lemma is proved. □

Lemma 4.1.2. *Let $\widehat{\mathcal{Q}} \subset BV(\Omega)$ be nonempty, convex, closed and bounded in the $L^2(\Omega)$-norm. Suppose that Ξ is a non-negative, strictly convex and continuous functional on $\widehat{\mathcal{Q}}$ in the $L^2(\Omega)$-norm. Then, the problem*

$$\min_{q \in \widehat{\mathcal{Q}}} \Xi(q) + \int_\Omega |\nabla q| \tag{4.2}$$

has a unique solution.

Proof. Let (q_n) be a sequence in $\widehat{\mathcal{Q}}$ such that

$$\lim_n \left(\Xi(q_n) + \int_\Omega |\nabla q_n| \right) = \inf_{q \in \widehat{\mathcal{Q}}} \left(\Xi(q) + \int_\Omega |\nabla q| \right).$$

It follows that the set $\left(\int_\Omega |\nabla q_n|\right)_{n \in \mathbb{N}}$ is bounded. Since (q_n) is bounded in the $L^2(\Omega)$-norm and $\mathrm{mes}(\Omega) < \infty$, it is bounded in the $L^1(\Omega)$-norm. Hence (q_n) is bounded in the $BV(\Omega)$-norm. By Lemma 3.1.1, we conclude that there exist a subsequence (q_{1_n}) of (q_n) and an element $\widehat{q} \in \widehat{\mathcal{Q}}$ such that (q_{1_n}) converges to $\widehat{q}$ in the $L^1(\Omega)$-norm, weakly in $L^2(\Omega)$ and $\int_\Omega |\nabla \widehat{q}| \le \liminf_n \int_\Omega |\nabla q_{1_n}|$. Since Ξ is convex and continuous on $\widehat{\mathcal{Q}}$ in the $L^2(\Omega)$-norm, it is weakly l.s.c in $L^2(\Omega)$. Therefore,

$$\begin{aligned} \Xi(\widehat{q}) + \int_\Omega |\nabla \widehat{q}| &\le \liminf_n \left(\Xi(q_{1_n}) + \int_\Omega |\nabla q_{1_n}| \right) \\ &= \inf_{q \in \widehat{\mathcal{Q}}} \left(\Xi(q) + \int_\Omega |\nabla q| \right) \end{aligned}$$

This means that $\widehat{q}$ is the (unique) solution of problem (4.2). The lemma is proved. □

Theorem 4.1.3. *(i) There exists a unique solution q_ρ^δ of problem $(\mathbb{P}^q_{\rho,\delta})$.*

(ii) There exists a unique solution $q^\dagger$ of problem $(\mathbb{K}^q)$.

Proof. The proposition of the theorem directly follows from Lemmas 1.1.4, 1.1.5, 4.1.1 and 4.1.2. □

We note that, for each $q \in Q$ and any $h \in \mathfrak{X}$ (see § 3.1.1. for the definition of the space $\mathfrak{X}$), since

$$\begin{aligned} \left| \langle q, h \rangle_{L^2(\Omega)} \right| &\le \|q\|_{L^\infty(\Omega)} \|h\|_{L^1(\Omega)} \\ &\le \|q\|_{L^\infty(\Omega)} \|h\|_{\mathfrak{X}_{BV(\Omega)}} \\ &\le \|q\|_{L^\infty(\Omega)} \|h\|_{\mathfrak{X}}, \end{aligned}$$

we get

$$\begin{aligned}\langle q,h\rangle_{L^2(\Omega)} &= \langle q,h\rangle_{\left(L^1(\Omega)^*,L^1(\Omega)\right)}\\ &= \langle q,h\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)}\\ &= \langle q,h\rangle_{(\mathfrak{X}^*,\mathfrak{X})}. \end{aligned} \tag{4.3}$$

Now we give results on a necessary and sufficient optimality condition for problems $(\mathbb{P}^q_{\rho,\delta})$ and $(\mathbb{K}^q)$.

Lemma 4.1.4. *(i) Let $q^\delta_\rho \in Q_{ad}$. Then, q^δ_ρ is a unique solution of $(\mathbb{P}^q_{\rho,\delta})$ if and only if for all $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\delta_\rho)$, the inequality*

$$J'_{z^\delta}(q^\delta_\rho)(q-q^\delta_\rho) + \rho\langle q^\delta_\rho, q-q^\delta_\rho\rangle_{L^2(\Omega)} + \rho\langle \ell, q-q^\delta_\rho\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)} \ge 0 \tag{4.4}$$

is satisfied for all q in Q_{ad}.

(ii) Let $q^\dagger \in \Pi_{Q_{ad}}(\overline{u})$. Then, $q^\dagger$ is a unique solution of $(\mathbb{K}^q)$ if and only if for all $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger)$, the inequality

$$\langle q^\dagger, q-q^\dagger\rangle_{L^2(\Omega)} + \langle \ell, q-q^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)} \ge 0$$

holds for all q in $\Pi_{Q_{ad}}(\overline{u})$.

Proof. (i) Since the cost functional of $(\mathbb{P}^q_{\rho,\delta})$ is convex, an element $q^\delta_\rho \in Q_{ad}$ is a solution of $(\mathbb{P}^q_{\rho,\delta})$ if and only if

$$J'_{z^\delta}(q^\delta_\rho)(q-q^\delta_\rho) + \rho\langle q^\delta_\rho, q-q^\delta_\rho\rangle_{L^2(\Omega)} + \rho\left(\int_\Omega |\nabla q| - \int_\Omega |\nabla q^\delta_\rho|\right) \ge 0$$

for all q in Q_{ad}. By (3.5) and (4.3), the last inequality means that

$$\langle J'_{z^\delta}(q^\delta_\rho) + \rho q^\delta_\rho, q-q^\delta_\rho\rangle_{(\mathfrak{X}^*,\mathfrak{X})} + \rho\left(\int_\Omega |\nabla q| - \int_\Omega |\nabla q^\delta_\rho|\right) \ge 0, \ \ \forall q \in Q_{ad}$$

or

$$0 \in J'_{z^\delta}(q^\delta_\rho) + \rho q^\delta_\rho + \partial\left(\rho\int_\Omega |\nabla(\cdot)| + I_{Q_{ad}}\right)(q^\delta_\rho) \subset \mathfrak{X}^*$$

with $I_{Q_{ad}}$ being the indicator function of the set Q_{ad}. Due to Lemma 3.1.2, the last equality is equivalent to the desired inequality.

(ii) An element $q^\dagger$ is a solution of $(\mathbb{K}^q)$ if and only if

$$\begin{aligned}0 &\in q^\dagger + \partial\left(\int_\Omega |\nabla(\cdot)| + I_{Q_{ad}}\right)(q^\dagger)\\ &= q^\dagger + \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger) + \partial I_{Q_{ad}}(q^\dagger)\end{aligned}$$

which is equivalent to the inequality

$$\langle q^\dagger, q-q^\dagger\rangle_{L^2(\Omega)} + \langle \ell, q-q^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)} \ge 0$$

for any $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger)$ and $q \in \Pi_{Q_{ad}}(\overline{u})$. The lemma is proved. □

Now, we state and prove stability results for our regularization method.

Theorem 4.1.5. *For a fixed regularization parameter $\rho > 0$, let (z^{δ_n}) be a sequence which converges to z^δ in the $H^1(\Omega)$-norm and $(q_\rho^{\delta_n})$ be the unique minimizers of problems*

$$\min_{q \in Q_{ad}} J_{z^{\delta_n}}(q) + \rho \left(\frac{1}{2} \|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q| \right).$$

Then, $(q_\rho^{\delta_n})$ converges to the unique solution q_ρ^δ of $(\mathbb{P}^q_{\rho,\delta})$ in the $L^2(\Omega)$-norm. Further,

$$\lim_n \int_\Omega |\nabla q_\rho^{\delta_n}| = \int_\Omega |\nabla q_\rho^\delta|. \tag{4.5}$$

Proof. For all $n \in \mathbb{N}$ and $q \in Q_{ad}$, by the definition of $q_\rho^{\delta_n}$, we have

$$J_{z^{\delta_n}}(q_\rho^{\delta_n}) + \frac{\rho}{2} \|q_\rho^{\delta_n}\|^2_{L^2(\Omega)} + \rho \int_\Omega |\nabla q_\rho^{\delta_n}| \le J_{z^{\delta_n}}(q) + \frac{\rho}{2} \|q\|^2_{L^2(\Omega)} + \rho \int_\Omega |\nabla q|. \tag{4.6}$$

It follows from the last inequality that $(q_\rho^{\delta_n})$ is bounded in the $L^2(\Omega)$-norm (and so in the $L^1(\Omega)$-norm, since $\mathrm{mes}(\Omega) < \infty$) and the sequence $\left(\int_\Omega |\nabla q_\rho^{\delta_n}| \right)$ is bounded, too. By Lemma 3.1.1, there exist a subsequence $(q_\rho^{\delta_{1n}})$ of $(q_\rho^{\delta_n})$ and $q_\rho^\delta \in Q_{ad}$ such that

$$(q_\rho^{\delta_{1n}}) \text{ converges to } q_\rho^\delta \text{ in } L^1(\Omega), \tag{4.7}$$

$$(q_\rho^{\delta_{1n}}) \ \text{ weakly converges to } q_\rho^\delta \text{ in } L^2(\Omega), \text{ and} \tag{4.8}$$

$$\int_\Omega |\nabla q_\rho^\delta| \le \liminf_n \int_\Omega |\nabla q_\rho^{\delta_{1n}}|. \tag{4.9}$$

Due to Lemma 3.1.3, it follows from (4.7) and the hypothesis that (z^{δ_n}) converges to z^δ in the $H^1(\Omega)$-norm that

$$\lim_n J_{z^{\delta_{1n}}}(q_\rho^{\delta_{1n}}) = J_{z^\delta}(q_\rho^\delta). \tag{4.10}$$

On the other hand, it follows from (4.8) that

$$\|q_\rho^\delta\|^2_{L^2(\Omega)} \le \liminf_n \|q_\rho^{\delta_{1n}}\|^2_{L^2(\Omega)}. \tag{4.11}$$

Therefore, by (4.9)–(4.11) and (4.6),

$$\begin{aligned}
J_{z^\delta}(q_\rho^\delta) &+ \frac{\rho}{2} \|q_\rho^\delta\|^2_{L^2(\Omega)} + \rho \int_\Omega |\nabla q_\rho^\delta| \\
&\le \liminf_n \left(J_{z^{\delta_{1n}}}(q_\rho^{\delta_{1n}}) + \frac{\rho}{2} \|q_\rho^{\delta_{1n}}\|^2_{L^2(\Omega)} + \rho \int_\Omega |\nabla q_\rho^{\delta_{1n}}| \right) \\
&\le \limsup_n \left(J_{z^{\delta_{1n}}}(q_\rho^{\delta_{1n}}) + \frac{\rho}{2} \|q_\rho^{\delta_{1n}}\|^2_{L^2(\Omega)} + \rho \int_\Omega |\nabla q_\rho^{\delta_{1n}}| \right) \\
&\le \limsup_n \left(J_{z^{\delta_{1n}}}(q) + \frac{\rho}{2} \|q\|^2_{L^2(\Omega)} + \rho \int_\Omega |\nabla q| \right) \\
&= J_{z^\delta}(q) + \frac{\rho}{2} \|q\|^2_{L^2(\Omega)} + \rho \int_\Omega |\nabla q|
\end{aligned} \tag{4.12}$$

for all $q \in Q_{ad}$. This means that q_ρ^δ is a (unique) solution to $(\mathbb{P}^q_{\rho,\delta})$.

By contradiction we show that $(q_\rho^{\delta_{1n}})$ converges to q_ρ^δ in the $L^2(\Omega)$-norm. In fact, assume that $(q_\rho^{\delta_{1n}}) \nrightarrow q_\rho^\delta$ in the $L^2(\Omega)$-norm. This and (4.11) follow that

$$\epsilon := \limsup_n \|q_\rho^{\delta_{1n}}\|^2_{L^2(\Omega)} > \|q_\rho^\delta\|^2_{L^2(\Omega)}. \tag{4.13}$$

Therefore, there exists a subsequence $(q_\rho^{\delta_{2n}})$ of $(q_\rho^{\delta_{1n}})$ such that

$$q_\rho^{\delta_{2n}} \rightharpoonup q_\rho^\delta \text{ weakly in } L^2(\Omega) \text{ and } \|q_\rho^{\delta_{2n}}\|^2_{L^2(\Omega)} \to \epsilon. \tag{4.14}$$

Choosing $q = q_\rho^\delta$ in (4.12), we get

$$J_{z^{\delta_{1n}}}(q_\rho^{\delta_{1n}}) + \frac{\rho}{2}\|q_\rho^{\delta_{1n}}\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla q_\rho^{\delta_{1n}}| \to J_{z^\delta}(q_\rho^\delta) + \frac{\rho}{2}\|q_\rho^\delta\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla q_\rho^\delta|. \tag{4.15}$$

It follows from (4.13), (4.14) and (4.10) that

$$\begin{aligned}
J_{z^\delta}(q_\rho^\delta) &+ \frac{\rho}{2}\|q_\rho^\delta\|^2_{L^2(\Omega)} + \rho\limsup_n \int_\Omega |\nabla q_\rho^{\delta_{2n}}| \\
&< J_{z^\delta}(q_\rho^\delta) + \frac{\rho}{2}\epsilon + \rho\limsup_n \int_\Omega |\nabla q_\rho^{\delta_{2n}}| \\
&= \lim_n J_{z^{\delta_{2n}}}(q_\rho^{\delta_{2n}}) + \frac{\rho}{2}\lim_n \|q_\rho^{\delta_{2n}}\|^2_{L^2(\Omega)} + \rho\limsup_n \int_\Omega |\nabla q_\rho^{\delta_{2n}}| \\
&= \limsup_n \left(J_{z^{\delta_{2n}}}(q_\rho^{\delta_{2n}}) + \frac{\rho}{2}\|q_\rho^{\delta_{2n}}\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla q_\rho^{\delta_{2n}}| \right).
\end{aligned}$$

By (4.15) and (4.9), the last inequality leads to

$$\begin{aligned}
\limsup_n \int_\Omega |\nabla q_\rho^{\delta_{2n}}| &< \int_\Omega |\nabla q_\rho^\delta| \\
&\le \liminf_n \int_\Omega |\nabla q_\rho^{\delta_{1n}}| \\
&\le \liminf_n \int_\Omega |\nabla q_\rho^{\delta_{2n}}|,
\end{aligned}$$

which is a contradiction. Thus, $(q_\rho^{\delta_{1n}})$ converges to q_ρ^δ in the $L^2(\Omega)$-norm. Since the solution q_ρ^δ of $(\mathbb{P}^q_{\rho,\delta})$ is unique, we conclude that the whole sequence $(q_\rho^{\delta_n})$ also converges to q_ρ^δ in the $L^2(\Omega)$-norm.

Now, this and (4.6) follow that

$$\begin{aligned}
J_{z^\delta}(q_\rho^\delta) &+ \frac{\rho}{2}\|q_\rho^\delta\|^2_{L^2(\Omega)} + \rho\limsup_n \int_\Omega |\nabla q_\rho^{\delta_{1n}}| \\
&= \limsup_n \left(J_{z^{\delta_{1n}}}(q_\rho^{\delta_{1n}}) + \frac{\rho}{2}\|q_\rho^{\delta_{1n}}\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla q_\rho^{\delta_{1n}}| \right) \\
&\le \limsup_n \left(J_{z^{\delta_{1n}}}(q_\rho^\delta) + \frac{\rho}{2}\|q_\rho^\delta\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla q_\rho^\delta| \right) \\
&= J_{z^\delta}(q_\rho^\delta) + \frac{\rho}{2}\|q_\rho^\delta\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla q_\rho^\delta|.
\end{aligned}$$

By (4.9), it follows from the last estimate that $\lim_n \int_\Omega |\nabla q_\rho^{\delta_{1n}}| = \int_\Omega |\nabla q_\rho^\delta|$ and so (4.5) holds. The proof of Theorem 4.1.5 is now completed. □

Theorem 4.1.6. *For any positive sequence* $(\delta_n) \to 0$, *let* $\rho_n := \rho(\delta_n)$ *be such that*

$$\rho_n \to 0 \text{ and } \frac{\delta_n^2}{\rho_n} \to 0 \text{ as } n \to \infty.$$

Moreover, let (z^{δ_n}) *be a sequence satisfying* $\|\overline{u} - z^{\delta_n}\|_{H^1(\Omega)} \le \delta_n$ *and* $(q_{\rho_n}^{\delta_n})$ *be the unique minimizers of problems*

$$\min_{q \in Q_{ad}} J_{z^{\delta_n}}(q) + \rho_n \left(\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q| \right).$$

Then, $\left(q_{\rho_n}^{\delta_n}\right)$ *converges to the unique solution* $q^\dagger$ *of problem* $(\mathbb{K}^q)$ *in the* $L^2(\Omega)$*-norm. Further,*

$$\lim_n \int_\Omega |\nabla q_{\rho_n}^{\delta_n}| = \int_\Omega |\nabla q^\dagger| \text{ and } \lim_n D^\ell_{TV}(q_{\rho_n}^{\delta_n}, q^\dagger) = 0$$

for all $\ell \in \partial \left(\int_\Omega |\nabla(\cdot)| \right)(q^\dagger)$.

Proof. For all $n \in \mathbb{N}$, by the definition of $q_{\rho_n}^{\delta_n}$, we have

$$\begin{aligned} J_{z^{\delta_n}}(q_{\rho_n}^{\delta_n}) &+ \frac{\rho_n}{2}\|q_{\rho_n}^{\delta_n}\|^2_{L^2(\Omega)} + \rho_n \int_\Omega |\nabla q_{\rho_n}^{\delta_n}| \\ &\le J_{z^{\delta_n}}(q^\dagger) + \frac{\rho_n}{2}\|q^\dagger\|^2_{L^2(\Omega)} + \rho_n \int_\Omega |\nabla q^\dagger| \\ &\le \frac{\overline{q}}{2}\delta_n^2 + \frac{\rho_n}{2}\|q^\dagger\|^2_{L^2(\Omega)} + \rho_n \int_\Omega |\nabla q^\dagger|. \end{aligned} \tag{4.16}$$

By the assumption $\delta_n^2/\rho_n \to 0$, the last inequality yields

$$\limsup_n \left(\frac{1}{2}\|q_{\rho_n}^{\delta_n}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q_{\rho_n}^{\delta_n}| \right) \le \frac{1}{2}\|q^\dagger\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q^\dagger|. \tag{4.17}$$

Thus, since $\text{mes}(\Omega) < +\infty$,

$$\sup_{n\in\mathbb{N}} \|q_{\rho_n}^{\delta_n}\|^2_{L^2(\Omega)} < +\infty \text{ and } \sup_{n\in\mathbb{N}} \left(\|q_{\rho_n}^{\delta_n}\|_{L^1(\Omega)} + \int_\Omega |\nabla q_{\rho_n}^{\delta_n}| \right) < +\infty.$$

It follows from the last estimates that there exist a subsequence $(q_{\rho_{1n}}^{\delta_{1n}})$ of $(q_{\rho_n}^{\delta_n})$ and $\widehat{q} \in Q_{ad}$ such that

$$(q_{\rho_{1n}}^{\delta_{1n}}) \text{ converges to } \widehat{q} \text{ in } L^1(\Omega), \tag{4.18}$$

$$(q_{\rho_{1n}}^{\delta_{1n}}) \text{ weakly converges to } \widehat{q} \text{ in } L^2(\Omega), \text{ and} \tag{4.19}$$

$$\int_\Omega |\nabla \widehat{q}| \le \liminf_n \int_\Omega |\nabla q_{\rho_{1n}}^{\delta_{1n}}|. \tag{4.20}$$

By Lemma 1.1.7, the convergence in (4.19) leads to the following estimates

$$\begin{aligned} \frac{\alpha}{2}\|U(\widehat{q}) - \overline{u}\|^2_{H^1(\Omega)} &\le J_{\bar{u}}(\widehat{q}) \\ &\le \liminf_n J_{\bar{u}}(q_{\rho_{1n}}^{\delta_{1n}}). \end{aligned}$$

An application of Lemma 1.1.6 and (4.16) yield

$$\begin{aligned} \liminf_n J_{\bar{u}}(q_{\rho_{1n}}^{\delta_{1n}}) &= \liminf_n J_{z^{\delta_{1n}}}(q_{\rho_{1n}}^{\delta_{1n}}) \\ &\le \liminf_n \left(\frac{\overline{q}}{2}\delta_{1n}^2 + \frac{\rho_{1n}}{2}\|q^\dagger\|^2_{L^2(\Omega)} + \rho_{1n} \int_\Omega |\nabla q^\dagger| \right) \\ &= 0. \end{aligned}$$

Therefore, $U(\widehat{q}) = \overline{u}$. Replacing $q^\dagger$ in (4.16) by $\widehat{q}$, we also get

$$\limsup_n \left(\frac{1}{2}\|q^{\delta_n}_{\rho_n}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q^{\delta_n}_{\rho_n}| \right) \le \frac{1}{2}\|\widehat{q}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla \widehat{q}|. \tag{4.21}$$

Now, we have

$$\begin{aligned}\limsup_n \int_\Omega |\nabla q^{\delta_{1_n}}_{\rho_{1_n}}| &\le \limsup_n \left(\frac{1}{2}\|q^{\delta_{1_n}}_{\rho_{1_n}} - \widehat{q}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q^{\delta_{1_n}}_{\rho_{1_n}}| \right) \\ &\le \limsup_n \left(\frac{1}{2}\|q^{\delta_{1_n}}_{\rho_{1_n}}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q^{\delta_{1_n}}_{\rho_{1_n}}| \right) \\ &\quad + \limsup_n \left(\frac{1}{2}\|\widehat{q}\|^2_{L^2(\Omega)} - \langle q^{\delta_{1_n}}_{\rho_{1_n}}, \widehat{q}\rangle_{L^2(\Omega)} \right).\end{aligned}$$

It follows from the last inequality, (4.21) and (4.19) that

$$\begin{aligned}\limsup_n \int_\Omega |\nabla q^{\delta_{1_n}}_{\rho_{1_n}}| &\le \frac{1}{2}\|\widehat{q}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla \widehat{q}| + \frac{1}{2}\|\widehat{q}\|^2_{L^2(\Omega)} - \langle \widehat{q}, \widehat{q}\rangle_{L^2(\Omega)} \\ &= \int_\Omega |\nabla \widehat{q}|.\end{aligned}$$

This and (4.20) yield

$$\lim_n \int_\Omega |\nabla q^{\delta_{1_n}}_{\rho_{1_n}}| = \int_\Omega |\nabla \widehat{q}|. \tag{4.22}$$

It follows from the last equality and (4.21) that

$$\begin{aligned}\limsup_n \frac{1}{2}\|q^{\delta_{1_n}}_{\rho_{1_n}} - \widehat{q}\|^2_{L^2(\Omega)} &= \limsup_n \frac{1}{2}\|q^{\delta_{1_n}}_{\rho_{1_n}} - \widehat{q}\|^2_{L^2(\Omega)} + \lim_n \left(\int_\Omega |\nabla q^{\delta_{1_n}}_{\rho_{1_n}}| - \int_\Omega |\nabla \widehat{q}| \right) \\ &= \limsup_n \left(\frac{1}{2}\|q^{\delta_{1_n}}_{\rho_{1_n}}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q^{\delta_{1_n}}_{\rho_{1_n}}| \right) \\ &\quad + \lim_n \left(\frac{1}{2}\|\widehat{q}\|^2_{L^2(\Omega)} - \int_\Omega |\nabla \widehat{q}| - \langle q^{\delta_{1_n}}_{\rho_{1_n}}, \widehat{q}\rangle_{L^2(\Omega)} \right) \\ &= 0.\end{aligned} \tag{4.23}$$

Now, by the definition of $q^\dagger$, (4.22) and (4.23), we obtain that

$$\begin{aligned}\frac{1}{2}\|q^\dagger\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q^\dagger| &\le \frac{1}{2}\|\widehat{q}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla \widehat{q}| \\ &= \lim_n \left(\frac{1}{2}\|q^{\delta_{1_n}}_{\rho_{1_n}}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q^{\delta_{1_n}}_{\rho_{1_n}}| \right) \\ &\le \frac{1}{2}\|q^\dagger\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q^\dagger|.\end{aligned}$$

Hence

$$\frac{1}{2}\|q^\dagger\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q^\dagger| = \frac{1}{2}\|\widehat{q}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla \widehat{q}|$$

or $q^\dagger = \widehat{q}$, by the uniqueness of $q^\dagger$.

Finally, again using (4.22) and (4.23), we see that the sequence $(q^{\delta_{1_n}}_{\rho_{1_n}})$ weakly converges to $q^\dagger$ in $BV(\Omega)$ (see [9], Proposition 10.1.2, p. 374). Thus, for all $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(q^\dagger)$, we conclude that

$$\lim_n D^\ell_{TV}(q^{\delta_{1_n}}_{\rho_{1_n}}, q^\dagger) = \lim_n \left(\int_\Omega |\nabla q^{\delta_{1_n}}_{\rho_{1_n}}| - \int_\Omega |\nabla q^\dagger| - \langle \ell, q^{\delta_{1_n}}_{\rho_{1_n}} - q^\dagger\rangle_{(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)})} \right) = 0.$$

The theorem is proved. □

4.1.2. Convergence rates

We note that

$$\partial R(q^\dagger) = q^\dagger + \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger) \subset \mathfrak{X}^*,$$

where the functional $R(\cdot)$ is defined by (4.1).

Theorem 4.1.7. *Assume that there exists a functional $w^* \in H^1_\diamond(\Omega)^*$ such that*

$$U'(q^\dagger)^* w^* = q^\dagger + \ell \in \partial R(q^\dagger). \tag{4.24}$$

for some element ℓ in $\partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger)$. Then,

$$\|q^\delta_\rho - q^\dagger\|^2_{L^2(\Omega)} + D^\ell_{TV}(q^\delta_\rho, q^\dagger) = \mathcal{O}(\delta) \ \ and \ \ \|U(q^\delta_\rho) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$. Moreover, if $\ell \in \mathfrak{X}^*$ can be identified with an element of $L^2(\Omega)$, then the following convergence rate is obtained

$$\left|\int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q^\delta_\rho|\right| = \mathcal{O}(\sqrt{\delta}) \ \ as\ \delta \to 0 \ and\ \rho \sim \delta. \tag{4.25}$$

To prove this result we need the following auxiliary result, which is based on the convexity of the functional $J_{z^\delta}(\cdot)$.

Lemma 4.1.8. *The estimate*

$$-\rho\left\langle \ell, q^\dagger - q^\delta_\rho\right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \le \frac{\overline{q}}{2}\delta^2 + \overline{q}\|q^\delta_\rho - q^\dagger\|_{L^2(\Omega)}\rho \tag{4.26}$$

holds for q^δ_ρ being the solution of problem $(\mathbb{P}^q_{\rho,\delta})$ and all $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\delta_\rho)$.

Proof. By (4.4), for all $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\delta_\rho)$, we get

$$\begin{aligned}
-\rho\left\langle \ell, q^\dagger - q^\delta_\rho\right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} &\le J'_{z^\delta}(q^\delta_\rho)(q^\dagger - q^\delta_\rho) + \rho\langle q^\delta_\rho, q^\dagger - q^\delta_\rho\rangle_{L^2(\Omega)} \\
&\le J'_{z^\delta}(q^\delta_\rho)(q^\dagger - q^\delta_\rho) + \overline{q}\|q^\delta_\rho - q^\dagger\|_{L^2(\Omega)}\rho.
\end{aligned} \tag{4.27}$$

Since the function $J_{z^\delta}(\cdot)$ is convex and non-negative, we obtain that

$$J'_{z^\delta}(q^\delta_\rho)(q^\dagger - q^\delta_\rho) \le J_{z^\delta}(q^\dagger) - J_{z^\delta}(q^\delta_\rho) \le J_{z^\delta}(q^\dagger) \le \frac{\overline{q}}{2}\delta^2. \tag{4.28}$$

From inequalities (4.27) and (4.28) we arive at (4.26). □

Proof of Theorem 4.1.7. By the definition of q^δ_ρ, we have

$$J_{z^\delta}(q^\delta_\rho) + \frac{\rho}{2}\|q^\delta_\rho\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla q^\delta_\rho| \le J_{z^\delta}(q^\dagger) + \frac{\rho}{2}\|q^\dagger\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla q^\dagger|. \tag{4.29}$$

Then,

$$\begin{aligned}
J_{z^\delta}(q^\delta_\rho) + \frac{\rho}{2}\|q^\delta_\rho - q^\dagger\|^2_{L^2(\Omega)} &\le J_{z^\delta}(q^\dagger) + \frac{\rho}{2}\left(\|q^\dagger\|^2_{L^2(\Omega)} - \|q^\delta_\rho\|^2_{L^2(\Omega)} + \|q^\delta_\rho - q^\dagger\|^2_{L^2(\Omega)}\right) \\
&\quad + \rho\left(\int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q^\delta_\rho|\right) \\
&= J_{z^\delta}(q^\dagger) + \rho\left(\langle q^\dagger, q^\dagger - q^\delta_\rho\rangle_{L^2(\Omega)} + \int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q^\delta_\rho|\right).
\end{aligned}$$

By (1.5), for any $\ell \in \partial \left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger)$, the last inequality leads to

$$\begin{aligned}
&J_{z^\delta}(q_\rho^\delta) + \frac{\rho}{2}\|q_\rho^\delta - q^\dagger\|^2_{L^2(\Omega)} + \rho D^\ell_{TV}(q_\rho^\delta, q^\dagger)\\
&\quad\le \frac{\overline{q}}{2}\delta^2 + \rho\left(\langle q^\dagger, q^\dagger - q_\rho^\delta\rangle_{L^2(\Omega)} + \int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q_\rho^\delta|\right)\\
&\quad+ \rho\left(\int_\Omega |\nabla q_\rho^\delta| - \int_\Omega |\nabla q^\dagger| - \left\langle \ell, q_\rho^\delta - q^\dagger\right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)}\right)\\
&\quad= \frac{\overline{q}}{2}\delta^2 + \rho\left(\langle q^\dagger, q^\dagger - q_\rho^\delta\rangle_{L^2(\Omega)} + \left\langle \ell, q^\dagger - q_\rho^\delta\right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)}\right). \qquad (4.30)
\end{aligned}$$

By (3.6) and (4.3), we get

$$\langle q^\dagger, q^\dagger - q_\rho^\delta\rangle_{L^2(\Omega)} = \langle q^\dagger, q^\dagger - q_\rho^\delta\rangle_{(\mathfrak{X}^*, \mathfrak{X})}$$

and

$$\left\langle \ell, q^\dagger - q_\rho^\delta\right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} = \left\langle \ell, q^\dagger - q_\rho^\delta\right\rangle_{(\mathfrak{X}^*, \mathfrak{X})}.$$

Hence, by the source condition (4.24), we have

$$\langle q^\dagger, q^\dagger - q_\rho^\delta\rangle_{L^2(\Omega)} + \left\langle \ell, q^\dagger - q_\rho^\delta\right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} = \langle U'(q^\dagger)^* w^*, q^\dagger - q_\rho^\delta\rangle_{(\mathfrak{X}^*, \mathfrak{X})}.$$

It follows from the last equality and (3.21) that

$$\begin{aligned}
&\langle q^\dagger, q^\dagger - q_\rho^\delta\rangle_{L^2(\Omega)} + \left\langle \ell, q^\dagger - q_\rho^\delta\right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)}\\
&\qquad= \left\langle U'(q^\dagger)^* w^*, q^\dagger - q_\rho^\delta\right\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))}\\
&\qquad= \left\langle w^*, U'(q^\dagger)(q^\dagger - q_\rho^\delta)\right\rangle_{\left(H^1_\diamond(\Omega)^*, H^1_\diamond(\Omega)\right)}. \qquad (4.31)
\end{aligned}$$

By the Riesz representation theorem, there exists an element $w \in H^1_\diamond(\Omega)$ such that

$$\left\langle w^*, U'(q^\dagger)(q^\dagger - q_\rho^\delta)\right\rangle_{\left(H^1_\diamond(\Omega)^*, H^1_\diamond(\Omega)\right)} = \left\langle w, U'(q^\dagger)(q^\dagger - q_\rho^\delta)\right\rangle_{H^1(\Omega)}. \qquad (4.32)$$

By the similar reasonings as in the proof of Theorem 3.1.8, we get the following estimate

$$\left\langle w, U'(q^\dagger)(q^\dagger - q_\rho^\delta)\right\rangle_{H^1(\Omega)} \le \overline{q}\delta\left(\int_\Omega |\nabla \widehat{w}|^2\right)^{1/2} + \overline{q}\rho\int_\Omega |\nabla \widehat{w}|^2 + \frac{1}{2\rho}J_{z^\delta}(q_\rho^\delta) \qquad (4.33)$$

for some $\widehat{w} \in H^1_\diamond(\Omega)$. It follows from (4.30) –(4.33) that

$$\begin{aligned}
&\frac{1}{2}J_{z^\delta}(q_\rho^\delta) + \frac{\rho}{2}\|q_\rho^\delta - q^\dagger\|^2_{L^2(\Omega)} + \rho D^\ell_{TV}(q_\rho^\delta, q^\dagger)\\
&\qquad\le \frac{\overline{q}}{2}\delta^2 + \overline{q}\delta\rho\left(\int_\Omega |\nabla \widehat{w}|^2\right)^{1/2} + \overline{q}\rho^2\int_\Omega |\nabla \widehat{w}|^2. \qquad (4.34)
\end{aligned}$$

By Lemma 1.1.7, the last inequality leads to the following convergence rates

$$\|q_\rho^\delta - q^\dagger\|^2_{L^2(\Omega)} + D^\ell_{TV}(q_\rho^\delta, q^\dagger) = \mathcal{O}(\delta) \text{ and } \|U(q_\rho^\delta) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta) \qquad (4.35)$$

as $\delta \to 0$ and $\rho \sim \delta$. It remains to establish the convergence rate (4.25). Take $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q_\rho^\delta)$, from Lemma 4.1.8, we get

$$\begin{aligned}
\int_\Omega |\nabla q_\rho^\delta| - \int_\Omega |\nabla q^\dagger| &\le -\left\langle \ell, q^\dagger - q_\rho^\delta\right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)}\\
&\le \frac{\overline{q}}{2}\frac{\delta^2}{\rho} + \overline{q}\|q_\rho^\delta - q^\dagger\|_{L^2(\Omega)}. \qquad (4.36)
\end{aligned}$$

On the other hand, for all $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger)$, we have

$$\begin{aligned}\int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q_\rho^\delta| &\le -\langle \ell, q_\rho^\delta - q^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \\ &= -\langle \ell, q_\rho^\delta - q^\dagger\rangle_{(\mathfrak{X}^*, \mathfrak{X})}. \end{aligned} \tag{4.37}$$

By the assumption that $\ell \in \mathfrak{X}^*$ can be identified with an element of $L^2(\Omega)$, there exists an element of $L^2(\Omega)$ denoted by the same symbol such that

$$\langle \ell, q_\rho^\delta - q^\dagger\rangle_{(\mathfrak{X}^*, \mathfrak{X})} = \langle \ell, q_\rho^\delta - q^\dagger\rangle_{L^2(\Omega)}.$$

The last equality and (4.37) yield

$$\begin{aligned}\int_\Omega |\nabla q^\dagger| - \int_\Omega |\nabla q_\rho^\delta| &\le -\langle \ell, q_\rho^\delta - q^\dagger\rangle_{L^2(\Omega)} \\ &\le \|\ell\|_{L^2(\Omega)} \|q_\rho^\delta - q^\dagger\|_{L^2(\Omega)}. \end{aligned} \tag{4.38}$$

Since (4.35), it follows that $\|q_\rho^\delta - q^\dagger\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta})$ as $\delta \to 0$ and $\rho \sim \delta$. Hence inequalities (4.36) and (4.38) yield (4.25). The theorem is proved. □

4.1.3. Discussion of the source condition

Now we discuss the source condition (4.24), which ensures the convergence rate

$$\|q_\rho^\delta - q^\dagger\|^2_{L^2(\Omega)} + D^\ell_{TV}(q_\rho^\delta, q^\dagger) = \mathcal{O}(\delta) \tag{4.39}$$

of the regularized solutions q_ρ^δ to the R-minimizing solution $q^\dagger$ of our inverse problem, where $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger)$. We remark that this source condition does not require any the regularity on $q^\dagger$ and the smallness of the source function. Further, condition (4.24) is fulfilled if and only if there exists a function $w^* \in H^1_\diamond(\Omega)^*$ such that

$$\begin{aligned}&\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q| - \frac{1}{2}\|q^\dagger\|^2_{L^2(\Omega)} \\ &\quad - \int_\Omega |\nabla q^\dagger| - \langle U'(q^\dagger)^* w^*, q - q^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \ge 0\end{aligned}$$

for all $q \in \mathfrak{X}$. To further analyze this condition we assume that the sought coefficient belongs to $H^1(\Omega)$. Therefore, the admissible set of sought coefficients is restricted to

$$\widehat{Q_{ad}} = Q \cap H^1(\Omega) \subset Q \cap BV(\Omega).$$

Moreover, if ℓ can be identified with an element of $L^2(\Omega)$, i.e., there exists an element $\widetilde{\ell}$ in $L^2(\Omega)$ such that

$$\langle \ell, q\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} = \langle \widetilde{\ell}, q\rangle_{L^2(\Omega)}$$

for all q in $\mathfrak{X}$, then the convergence rate

$$\left| \int_\Omega |\nabla q_\rho^\delta| - \int_\Omega |\nabla q^\dagger| \right| = \mathcal{O}(\sqrt{\delta}) \tag{4.40}$$

is also established.

Theorem 4.1.9. *Let the boundary $\partial\Omega$ be of class C^1 and the dimension $d \le 4$. Assume that $q^\dagger$ has the property that there is an element $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger)$ such that $\langle \ell, q\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)} = \langle \widehat{\ell}, q\rangle_{L^2(\Omega)}$ for all $q \in L^\infty(\Omega)\cap H^1(\Omega)$, where $\widehat{\ell}$ is some element of $H^1(\Omega)$. Further, suppose that the exact $\overline{u} \in W^{2,\infty}(\Omega)$, $|\nabla\overline{u}| \ge \gamma$ a.e. on Ω with γ being a positive constant. Then, the convergence rates (4.39) and (4.40) are obtained.*

Recall that as the boundary $\partial\Omega$ is of class C^1 and the dimension $d \le 4$, the requirement on $q^\dagger$ of the theorem is fulfilled at least on a set which is everywhere dense on $H^1(\Omega)$ (see Lemma 3.1.10).

Proof of Theorem 4.1.9. Due to Lemma 2.1.10, there exists $\psi \in H^1(\Omega)$ satisfying

$$\nabla\overline{u}\cdot\nabla\psi = \nabla U(q^\dagger)\cdot\nabla\psi = q^\dagger + \widehat{\ell}.$$

Set

$$\widehat{\psi} := \frac{\int_\Omega v}{\mathrm{mes}(\Omega)} - \psi.$$

Then,

$$-\nabla U(q^\dagger)\cdot\nabla\widehat{\psi} = q^\dagger + \widehat{\ell} \quad \text{and} \quad \widehat{\psi} \in H^1_\diamond(\Omega).$$

By (4.3), for all $q \in L^\infty(\Omega)\cap H^1(\Omega)$, we have

$$\begin{aligned}
\langle \ell + q^\dagger, q\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)} &= \langle \ell, q\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)} + \langle q^\dagger, q\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)}\\
&= \langle \widehat{\ell}, q\rangle_{L^2(\Omega)} + \langle q^\dagger, q\rangle_{L^2(\Omega)}\\
&= \langle q^\dagger + \widehat{\ell}, q\rangle_{L^2(\Omega)}\\
&= -\int_\Omega q\nabla U(q^\dagger)\nabla\widehat{\psi}.\\
&= \int_\Omega q^\dagger\nabla U'(q^\dagger)q\nabla\widehat{\psi}, \quad \text{by (1.13)},\\
&= \langle \widehat{w}, U'(q^\dagger)q\rangle_{H^1(\Omega)}
\end{aligned}$$

for some $\widehat{w} \in H^1_\diamond(\Omega)$ independent of $q \in L^\infty(\Omega)\cap H^1(\Omega)$. Thus, there exists a function $w^* \in H^1_\diamond(\Omega)^*$ such that

$$\begin{aligned}
\langle \ell + q^\dagger, q\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)} &= \langle w^*, U'(q^\dagger)q\rangle_{(H^1_\diamond(\Omega)^*,H^1_\diamond(\Omega))}\\
&= \left\langle U'(q^\dagger)^* w^*, q\right\rangle_{(L^\infty(\Omega)^*,L^\infty(\Omega))}.
\end{aligned} \tag{4.41}$$

Since

$$\left\langle U'(q^\dagger)^* w^*, q\right\rangle_{(L^\infty(\Omega)^*,L^\infty(\Omega))} = \langle q^\dagger + \widehat{\ell}, q\rangle_{L^2(\Omega)}$$

with $q^\dagger, \widehat{\ell} \in H^1(\Omega)$, the boundary $\partial\Omega$ being of class C^1 and the dimension $d \le 4$, it follows from the Sobolev embedding theorem that $U'(q^\dagger)^* w^*$ is linear and continuous on $L^\infty(\Omega)\cap H^1(\Omega)$ equipped with the $BV(\Omega)$-norm and

$$\left\langle U'(q^\dagger)^* w^*, q\right\rangle_{(L^\infty(\Omega)^*,L^\infty(\Omega))} = \left\langle U'(q^\dagger)^* w^*, q\right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)}$$

for all $q \in L^\infty(\Omega) \cap H^1(\Omega)$. Since $q^\dagger + \ell \in \partial R(q^\dagger)$, it follows from the last equality and (4.41) that

$$\begin{aligned}\frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q| - \frac{1}{2}\|q^\dagger\|^2_{L^2(\Omega)} - \int_\Omega |\nabla q^\dagger| - \langle U'(q^\dagger)^* w^*, q - q^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \\ = \frac{1}{2}\|q\|^2_{L^2(\Omega)} + \int_\Omega |\nabla q| - \frac{1}{2}\|q^\dagger\|^2_{L^2(\Omega)} - \int_\Omega |\nabla q^\dagger| \\ - \langle q^\dagger + \ell, q - q^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \\ \geq 0\end{aligned}$$

for all $q \in L^\infty(\Omega) \cap H^1(\Omega)$. The theorem is proved. □

4.2 Convergence rates for total variation regularization combining with L^2-stabilization of the reaction coefficient identification problem

4.2.1. Regularization by the total variation combining with L^2-stabilization

For identifying the coefficient a in (1.3)–(1.4), we solve the strictly convex minimization problem

$$\min_{a \in A_{ad}} G_{z^\delta}(a) + \rho\left(\frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a|\right), \tag{$\mathbb{P}^a_{\rho,\delta}$}$$

where A_{ad} defined by (3.46) and $\rho > 0$ is the regularization parameter.

We remark that problem ($\mathbb{P}^a_{\rho,\delta}$) has a *unique solution* a^δ_ρ on the nonempty, convex, bounded and closed in the $L^2(\Omega)$-norm set A_{ad}, which is considered as regularized solution to our inverse problem. On the other hand, due to the nonempty, convexity, closedness and boundedness in the $L^2(\Omega)$-norm of the set $\Pi_{A_{ad}}(\overline{u})$ defined by (3.47), we can conclude that there exists a unique solution $a^\dagger$ of problem

$$\min_{a \in \Pi_{A_{ad}}(\overline{u})} \frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a|, \tag{$\mathbb{K}^a$}$$

which we also call *R-minimizing solution* to our inverse problem, where the functional $R(\cdot)$ is defined by (4.1).

In this section we investigate the convergence rates of a^δ_ρ to the solution $a^\dagger$ of the equation $U(a) = \overline{u}$.

Theorem 4.2.1. *(i) There exists a unique solution of problem ($\mathbb{P}^a_{\rho,\delta}$). Further, an element a^δ_ρ in A_{ad} is a solution to ($\mathbb{P}^a_{\rho,\delta}$) if and only if for any $\lambda \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(a^\delta_\rho)$, the inequality*

$$G'_{z^\delta}(a^\delta_\rho)(a - a^\delta_\rho) + \rho\langle a^\delta_\rho, a - a^\delta_\rho\rangle_{L^2(\Omega)} + \rho\langle\lambda, a - a^\delta_\rho\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \geq 0 \tag{4.42}$$

holds for all a in A_{ad}.

(ii) There exists a unique solution of problem ($\mathbb{K}^a$). Further, an element $a^\dagger$ in $\Pi_{A_{ad}}(\overline{u})$ is a solution of problem ($\mathbb{K}^a$) if and only if for any $\lambda \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(a^\dagger)$, the inequality

$$\langle a^\dagger, a - a^\dagger\rangle_{L^2(\Omega)} + \langle\lambda, a - a^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \geq 0$$

holds for all a in $\Pi_{A_{ad}}(\overline{u})$.

Proof. (i) The existence of a unique solution to $(\mathbb{P}^a_{\rho,\delta})$ directly follows from Lemmas 1.2.3, 1.2.4 and 4.1.2. Further, an element $a^\delta_\rho \in A_{ad}$ is a solution of $(\mathbb{P}^a_{\rho,\delta})$ if and only if

$$G'_{z^\delta}(a^\delta_\rho)(a - a^\delta_\rho) + \rho\langle a^\delta_\rho, a - a^\delta_\rho\rangle_{L^2(\Omega)} + \rho\left(\int_\Omega |\nabla a| - \int_\Omega |\nabla a^\delta_\rho|\right) \geq 0$$

for all a in a_{ad}. Due to (3.48) and (4.3), the last inequality means that

$$\langle G'_{z^\delta}(a^\delta_\rho) + \rho a^\delta_\rho, a - a^\delta_\rho\rangle_{(\mathfrak{X}^*,\mathfrak{X})} + \rho\left(\int_\Omega |\nabla a| - \int_\Omega |\nabla a^\delta_\rho|\right) \geq 0, \ \ \forall a \in A_{ad}$$

or the inclusion

$$0 \in G'_{z^\delta}(a^\delta_\rho) + \rho a^\delta_\rho + \partial\left(\rho\int_\Omega |\nabla(\cdot)| + I_{A_{ad}}\right)(a^\delta_\rho) \subset \mathfrak{X}^*.$$

This is equivalent to the desired inequality.

(ii) The existence of a unique solution to $\Pi_{A_{ad}}(\overline{u})$ directly follows from Lemma 4.1.2. An element $a^\dagger$ is a solution of $(\mathbb{K}^a)$ if and only if

$$0 \in a^\dagger + \partial\left(\int_\Omega |\nabla(\cdot)| + I_{A_{ad}}\right)(a^\dagger)$$

which is equivalent to the desired inequality. The theorem is proved. □

Theorem 4.2.2. *For a fixed regularization parameter $\rho > 0$, let (z^{δ_n}) be a sequence which converges to z^δ in $H^1(\Omega)$ and $(a^{\delta_n}_\rho)$ be the unique minimizers of problems*

$$\min_{a\in A_{ad}} G_{z^{\delta_n}}(a) + \rho\left(\frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a|\right).$$

Then, $(a^{\delta_n}_\rho)$ converges to the unique solution a^δ_ρ of $(\mathbb{P}^a_{\rho,\delta})$ in the $L^2(\Omega)$-norm. Further,

$$\lim_n \int_\Omega |\nabla a^{\delta_n}_\rho| = \int_\Omega |\nabla a^\delta_\rho|.$$

Proof. By the definition of $a^{\delta_n}_\rho$, for all $n \in \mathbb{N}$ and $a \in A_{ad}$, we have

$$G_{z^{\delta_n}}(a^{\delta_n}_\rho) + \frac{\rho}{2}\|a^{\delta_n}_\rho\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla a^{\delta_n}_\rho| \leq G_{z^{\delta_n}}(a) + \frac{\rho}{2}\|a\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla a|. \tag{4.43}$$

Thus, the sequence $(a^{\delta_n}_\rho)$ is bounded in the $L^2(\Omega)$-norm and in the $BV(\Omega)$-norm, too. Hence there exist a subsequence $(a^{\delta_{1_n}}_\rho)$ of $(a^{\delta_n}_\rho)$ and $a^\delta_\rho \in A_{ad}$ such that $(a^{\delta_{1_n}}_\rho)$ converges to a^δ_ρ in the $L^1(\Omega)$-norm, weakly in $L^2(\Omega)$ and

$$\int_\Omega |\nabla a^\delta_\rho| \leq \liminf_n \int_\Omega |\nabla a^{\delta_{1_n}}_\rho|. \tag{4.44}$$

Applying Lemma 3.2.1, we have

$$\lim_n G_{z^{\delta_{1_n}}}(a^{\delta_{1_n}}_\rho) = G_{z^\delta}(a^\delta_\rho). \tag{4.45}$$

Further,

$$\|a^\delta_\rho\|^2_{L^2(\Omega)} \leq \liminf_n \|a^{\delta_{1_n}}_\rho\|^2_{L^2(\Omega)}. \tag{4.46}$$

Therefore, by (4.43)–(4.46), we have

$$
\begin{aligned}
G_{z^\delta}(a^\delta_\rho) + \frac{\rho}{2}\|a^\delta_\rho\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla a^\delta_\rho| \\
&\le \liminf_n \left(G_{z^{\delta_{1n}}}(a^{\delta_{1n}}_\rho) + \frac{\rho}{2}\|a^{\delta_{1n}}_\rho\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla a^{\delta_{1n}}_\rho| \right) \\
&\le \limsup_n \left(G_{z^{\delta_{1n}}}(a^{\delta_{1n}}_\rho) + \frac{\rho}{2}\|a^{\delta_{1n}}_\rho\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla a^{\delta_{1n}}_\rho| \right) \\
&\le \limsup_n \left(G_{z^{\delta_{1n}}}(a) + \frac{\rho}{2}\|a\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla a| \right) \\
&= G_{z^\delta}(a) + \frac{\rho}{2}\|a\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla a| \qquad (4.47)
\end{aligned}
$$

for all $a \in A_{ad}$. This means that a^δ_ρ is a (unique) solution to $(\mathbb{P}^a_{\rho,\delta})$. On the other hand, by contradiction we can show that $(a^{\delta_{1n}}_\rho)$ converges to a^δ_ρ in the $L^2(\Omega)$-norm. Since the solution a^δ_ρ of $(\mathbb{P}^a_{\rho,\delta})$ is unique, we conclude that the whole sequence $(a^{\delta_n}_\rho)$ also converges to a^δ_ρ in the $L^2(\Omega)$-norm. Now, it follows from (4.43) that

$$
\begin{aligned}
G_{z^\delta}(a^\delta_\rho) + \frac{\rho}{2}\|a^\delta_\rho\|^2_{L^2(\Omega)} + \rho\limsup_n \int_\Omega |\nabla a^{\delta_{1n}}_\rho| \\
&= \limsup_n \left(G_{z^{\delta_{1n}}}(a^{\delta_{1n}}_\rho) + \frac{\rho}{2}\|a^{\delta_{1n}}_\rho\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla a^{\delta_{1n}}_\rho| \right) \\
&\le \limsup_n \left(G_{z^{\delta_{1n}}}(a^\delta_\rho) + \frac{\rho}{2}\|a^\delta_\rho\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla a^\delta_\rho| \right) \\
&= G_{z^\delta}(a^\delta_\rho) + \frac{\rho}{2}\|a^\delta_\rho\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla a^\delta_\rho|.
\end{aligned}
$$

By (4.44), the last estimate and the uniqueness of a^δ_ρ we have $\lim_n \int_\Omega |\nabla a^{\delta_n}_\rho| = \int_\Omega |\nabla a^\delta_\rho|$. The theorem is proved. ☐

Theorem 4.2.3. *For any positive sequence $(\delta_n) \to 0$, let $\rho_n := \rho(\delta_n)$ be such that*

$$\rho_n \to 0 \text{ and } \frac{\delta_n^2}{\rho_n} \to 0 \text{ as } n \to \infty.$$

Moreover, let (z^{δ_n}) be a sequence satisfying $\|\overline{u} - z^{\delta_n}\|_{H^1(\Omega)} \le \delta_n$ and $(a^{\delta_n}_{\rho_n})$ be the unique minimizers of problems

$$\min_{a\in A_{ad}} G_{z^{\delta_n}}(a) + \rho_n \left(\frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a| \right).$$

Then, $(a^{\delta_n}_{\rho_n})$ converges to the unique solution $a^\dagger$ of problem $(\mathbb{K}^a)$ in the $L^2(\Omega)$-norm. Further,

$$\lim_n \int_\Omega |\nabla a^{\delta_n}_{\rho_n}| = \int_\Omega |\nabla a^\dagger| \text{ and } \lim_n D^\ell_{TV}(a^{\delta_n}_{\rho_n}, a^\dagger) = 0$$

for all $\ell \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(a^\dagger)$.

Proof. By the definition of $q_{\rho_n}^{\delta_n}$, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} G_{z^{\delta_n}}(a_{\rho_n}^{\delta_n}) + \frac{\rho_n}{2}\|a_{\rho_n}^{\delta_n}\|^2_{L^2(\Omega)} + \rho_n \int_\Omega |\nabla a_{\rho_n}^{\delta_n}| & \\ &\le G_{z^{\delta_n}}(a^\dagger) + \frac{\rho_n}{2}\|a^\dagger\|^2_{L^2(\Omega)} + \rho_n \int_\Omega |\nabla a^\dagger| \\ &\le \frac{\max\{1, \overline{a}\}}{2}\delta_n^2 + \frac{\rho_n}{2}\|a^\dagger\|^2_{L^2(\Omega)} + \rho_n \int_\Omega |\nabla a^\dagger|. \end{aligned} \tag{4.48}$$

By the assumption $\delta_n^2/\rho_n \to 0$, the last inequality follows that the sequence $(a_{\rho_n}^{\delta_n})$ is bounded in the $L^2(\Omega)$-norm and in the $BV(\Omega)$-norm, too. Consequently, there exist a subsequence $(a_{\rho_{1_n}}^{\delta_{1_n}})$ of $(a_{\rho_n}^{\delta_n})$ and $\widehat{a} \in A_{ad}$ such that $(a_{\rho_n}^{\delta_n})$ converges to $\widehat{a}$ in the $L^1(\Omega)$-norm, weakly in $L^2(\Omega)$ and

$$\int_\Omega |\nabla \widehat{a}| \le \liminf_n \int_\Omega |\nabla a_{\rho_{1_n}}^{\delta_{1_n}}|.$$

Moreover, since $G_{\bar{u}}(\cdot)$ is weakly l.s.c, it follows that

$$G_{\bar{u}}(\widehat{a}) \le \liminf_n G_{\bar{u}}(a_{\rho_{1_n}}^{\delta_{1_n}}).$$

An application of Lemma 1.2.5 and by (4.48), we get

$$\begin{aligned} \liminf_n G_{\bar{u}}(a_{\rho_{1_n}}^{\delta_{1_n}}) &= \liminf_n G_{z^{\delta_{1_n}}}(a_{\rho_{1_n}}^{\delta_{1_n}}) \\ &\le \liminf_n \left(\frac{\max\{1, \overline{a}\}}{2}\delta_{1_n}^2 + \frac{\rho_{1_n}}{2}\|a^\dagger\|^2_{L^2(\Omega)} + \rho_{1_n} \int_\Omega |\nabla a^\dagger| \right) \\ &= 0. \end{aligned}$$

On the other hand, using the estimate in Lemma 1.2.6, we get

$$\frac{\beta}{2}\|U(\widehat{a}) - \overline{u}\|^2_{H^1(\Omega)} \le G_{\bar{u}}(\widehat{a}) \le \liminf_n G_{\bar{u}}(a_{\rho_{1_n}}^{\delta_{1_n}}) = 0.$$

This follows that $U(\widehat{a}) = \overline{u}$. Thus, replacing $a^\dagger$ in (4.48) by $\widehat{a}$, we also get

$$\limsup_n \left(\frac{1}{2}\|a_{\rho_n}^{\delta_n}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a_{\rho_n}^{\delta_n}| \right) \le \frac{1}{2}\|\widehat{a}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla \widehat{a}|. \tag{4.49}$$

Now, we have

$$\begin{aligned} \limsup_n \int_\Omega |\nabla a_{\rho_{1_n}}^{\delta_{1_n}}| &\le \limsup_n \left(\frac{1}{2}\|a_{\rho_{1_n}}^{\delta_{1_n}} - \widehat{a}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a_{\rho_{1_n}}^{\delta_{1_n}}| \right) \\ &\le \limsup_n \left(\frac{1}{2}\|a_{\rho_{1_n}}^{\delta_{1_n}}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a_{\rho_{1_n}}^{\delta_{1_n}}| \right) \\ &\quad + \limsup_n \left(\frac{1}{2}\|\widehat{a}\|^2_{L^2(\Omega)} - \langle a_{\rho_{1_n}}^{\delta_{1_n}}, \widehat{a}\rangle_{L^2(\Omega)} \right). \end{aligned}$$

It follows from the last inequality and (4.49) that

$$\begin{aligned} \limsup_n \int_\Omega |\nabla a_{\rho_{1_n}}^{\delta_{1_n}}| &\le \frac{1}{2}\|\widehat{a}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla \widehat{a}| + \frac{1}{2}\|\widehat{a}\|^2_{L^2(\Omega)} - \langle \widehat{a}, \widehat{a}\rangle_{L^2(\Omega)} \\ &= \int_\Omega |\nabla \widehat{a}| \end{aligned}$$

and hence

$$\lim_n \int_\Omega |\nabla a^{\delta_{1n}}_{\rho_{1n}}| = \int_\Omega |\nabla \widehat{a}|. \tag{4.50}$$

Therefore, again using (4.49), we have

$$\begin{aligned}\limsup_n \frac{1}{2}\|a^{\delta_{1n}}_{\rho_{1n}} - \widehat{a}\|^2_{L^2(\Omega)} &= \limsup_n \frac{1}{2}\|a^{\delta_{1n}}_{\rho_{1n}} - \widehat{a}\|^2_{L^2(\Omega)} + \lim_n \left(\int_\Omega |\nabla a^{\delta_{1n}}_{\rho_{1n}}| - \int_\Omega |\nabla \widehat{a}| \right) \\ &= \limsup_n \left(\frac{1}{2}\|a^{\delta_{1n}}_{\rho_{1n}}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a^{\delta_{1n}}_{\rho_{1n}}| \right) \\ &\quad + \lim_n \left(\frac{1}{2}\|\widehat{a}\|^2_{L^2(\Omega)} - \int_\Omega |\nabla \widehat{a}| - \langle a^{\delta_{1n}}_{\rho_{1n}}, \widehat{a}\rangle_{L^2(\Omega)} \right) \\ &= 0. \end{aligned} \tag{4.51}$$

By the definition of $a^\dagger$ and (4.50) and (4.51), we obtain that

$$\begin{aligned}\frac{1}{2}\|a^\dagger\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a^\dagger| &\le \frac{1}{2}\|\widehat{a}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla \widehat{a}| \\ &= \lim_n \left(\frac{1}{2}\|a^{\delta_{1n}}_{\rho_{1n}}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a^{\delta_{1n}}_{\rho_{1n}}| \right) \\ &\le \frac{1}{2}\|a^\dagger\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a^\dagger|. \end{aligned}$$

This means that

$$\frac{1}{2}\|a^\dagger\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a^\dagger| = \frac{1}{2}\|\widehat{a}\|^2_{L^2(\Omega)} + \int_\Omega |\nabla \widehat{a}|$$

or $a^\dagger = \widehat{a}$, by the uniqueness of $a^\dagger$. Now, using (4.50) and (4.51), we see that the sequence $(a^{\delta_{1n}}_{\rho_{1n}})$ weakly converges to $a^\dagger$ in $BV(\Omega)$ (see [9], Prop. 10.1.2, p. 374). Thus, for all $\ell \in \partial(\int_\Omega |\nabla(\cdot)|)(a^\dagger)$, we conclude that $\lim_n D^\ell_{TV}(a^{\delta_{1n}}_{\rho_{1n}}, a^\dagger) = 0$. The theorem is proved. □

4.2.2. Convergence rates

Now we state and prove our result on convergence rates of regularized solutions a^δ_ρ to the R-minimizing solution $a^\dagger$ of the equation $U(a) = \overline{u}$.

Theorem 4.2.4. *Assume that there exists a function $w^* \in H^1(\Omega)^*$ such that*

$$U'(a^\dagger)^* w^* = a^\dagger + \lambda \in \partial R(a^\dagger) \tag{4.52}$$

for some element λ in $\partial\left(\int_\Omega |\nabla(\cdot)|\right)(a^\dagger)$. Then,

$$\|a^\delta_\rho - a^\dagger\|^2_{L^2(\Omega)} + D^\lambda_{TV}(a^\delta_\rho, a^\dagger) = \mathcal{O}(\delta) \ \text{ and } \ \|U(a^\delta_\rho) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$. Further, if $\lambda \in \mathfrak{X}^*$ can be identified with an element of $L^2(\Omega)$, then the convergence rate

$$\left| \int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a^\delta_\rho| \right| = \mathcal{O}(\sqrt{\delta}) \ \text{ as } \delta \to 0 \text{ and } \rho \sim \delta, \tag{4.53}$$

is also established.

We need the following lemma.

Lemma 4.2.5. *The estimate*

$$-\rho\left\langle \lambda, a^\dagger - a_\rho^\delta\right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)} \le \frac{\max\{\overline{a},1\}}{2}\delta^2 + \overline{a}\|a_\rho^\delta - a^\dagger\|_{L^2(\Omega)}\rho \tag{4.54}$$

holds for all a_ρ^δ being the solutions of problems $(\mathbb{P}^a_{\rho,\delta})$ and $\lambda \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(a_\rho^\delta)$.

Proof. By the convexity of the function $G_{z^\delta}(\cdot)$ and the inequality (4.42), we obtain that

$$\begin{aligned}
-\rho\left\langle \lambda, a^\dagger - a_\rho^\delta\right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)} &\le G'_{z^\delta}(a_\rho^\delta)(a^\dagger - a_\rho^\delta) + \rho\langle a_\rho^\delta, a^\dagger - a_\rho^\delta\rangle_{L^2(\Omega)}\\
&\le G_{z^\delta}(a^\dagger) - G_{z^\delta}(a_\rho^\delta) + \rho\langle a_\rho^\delta, a^\dagger - a_\rho^\delta\rangle_{L^2(\Omega)}\\
&\le G_{z^\delta}(a^\dagger) + \overline{a}\|a_\rho^\delta - a^\dagger\|_{L^2(\Omega)}\rho.
\end{aligned}$$

Since $G_{z^\delta}(a^\dagger) \le \frac{\max\{\overline{a},1\}}{2}\delta^2$, the last inequality yields (4.54). The lemma is proved. □

Proof of Theorem 4.2.4. By the definition of a_ρ^δ, we have

$$\begin{aligned}
G_{z^\delta}(a_\rho^\delta) + \frac{\rho}{2}\|a_\rho^\delta\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla a_\rho^\delta|&\\
\le G_{z^\delta}(a^\dagger) + \frac{\rho}{2}\|a^\dagger\|^2_{L^2(\Omega)} + \rho\int_\Omega |\nabla a^\dagger|&\\
\le \frac{\max\{\overline{a},1\}}{2}\delta^2 + \rho\left(\frac{1}{2}\|a^\dagger\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a^\dagger|\right).&
\end{aligned} \tag{4.55}$$

Take $\lambda \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(a^\dagger)$ such that $U'(a^\dagger)^* w^* = a^\dagger + \lambda$. Then,

$$\begin{aligned}
G_{z^\delta}(a_\rho^\delta) + \frac{\rho}{2}\|a_\rho^\delta - a^\dagger\|^2_{L^2(\Omega)} + \rho D^\lambda_{TV}(a_\rho^\delta, a^\dagger)&\\
\le \frac{\max\{\overline{a},1\}}{2}\delta^2 + \rho\langle a^\dagger, a^\dagger - a_\rho^\delta\rangle_{L^2(\Omega)}&\\
+ \rho\left\langle \lambda, a^\dagger - a_\rho^\delta\right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)}.&
\end{aligned} \tag{4.56}$$

Now we have

$$\begin{aligned}
\langle a^\dagger, a^\dagger - a_\rho^\delta\rangle_{L^2(\Omega)} + \left\langle \lambda, a^\dagger - a_\rho^\delta\right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)}&\\
= \langle a^\dagger, a^\dagger - a_\rho^\delta\rangle_{(\mathfrak{X}^*,\mathfrak{X})} + \left\langle \lambda, a^\dagger - a_\rho^\delta\right\rangle_{(\mathfrak{X}^*,\mathfrak{X})}&\\
= \left\langle a^\dagger + \lambda, a^\dagger - a_\rho^\delta\right\rangle_{(\mathfrak{X}^*,\mathfrak{X})}&\\
= \left\langle U'(a^\dagger)^* w^*, a^\dagger - a_\rho^\delta\right\rangle_{(\mathfrak{X}^*,\mathfrak{X})}&\\
= \left\langle U'(a^\dagger)^* w^*, a^\dagger - a_\rho^\delta\right\rangle_{(L^\infty(\Omega)^*,L^\infty(\Omega))}&\\
= \left\langle w^*, U'(a^\dagger)(a^\dagger - a_\rho^\delta)\right\rangle_{\left(H^1(\Omega)^*,H^1(\Omega)\right)}.&
\end{aligned}$$

By the Riesz representation theorem, the last equation follows that there exists an element $w \in H^1(\Omega)$ such that

$$\langle a^\dagger, a^\dagger - a_\rho^\delta\rangle_{L^2(\Omega)} + \left\langle \lambda, a^\dagger - a_\rho^\delta\right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)},\mathfrak{X}_{BV(\Omega)}\right)} = \langle w, U'(a^\dagger)(a^\dagger - a_\rho^\delta)\rangle_{H^1(\Omega)}. \tag{4.57}$$

By the similar reasonings as in the proof of Theorem 3.2.5, we get the following estimate

$$\begin{aligned}\langle w, U'(a^\dagger)(a^\dagger - a^\delta_\rho)\rangle_{H^1(\Omega)} &\le \delta\overline{a}\left(\int_\Omega \widehat{w}^2\right)^{1/2} + \rho\overline{a}\int_\Omega \widehat{w}^2 + \frac{1}{4\rho}\int_\Omega a^\delta_\rho (z^\delta - U(a^\delta_\rho))^2\\ &+ \delta\left(\int_\Omega |\nabla\widehat{w}|^2\right)^{1/2} + \rho\int_\Omega |\nabla\widehat{w}|^2 + \frac{1}{4\rho}\int_\Omega |\nabla(z^\delta - U(a^\delta_\rho)|^2.\end{aligned}$$

for some $\widehat{w} \in H^1(\Omega)$. Thus,

$$\begin{aligned}\rho\langle w, U'(a^\dagger)(a^\dagger - a^\delta_\rho)\rangle_{H^1(\Omega)} &\le \delta\rho\overline{a}\left(\int_\Omega \widehat{w}^2\right)^{1/2} + \rho^2\overline{a}\int_\Omega \widehat{w}^2\\ &+ \delta\rho\left(\int_\Omega |\nabla\widehat{w}|^2\right)^{1/2} + \rho^2\int_\Omega |\nabla\widehat{w}|^2 + \frac{1}{2}G_{z^\delta}(a^\delta_\rho).\end{aligned} \tag{4.58}$$

It follows from inequalities (4.56), (4.57) and (4.58) that

$$\begin{aligned}&\frac{1}{2}G_{z^\delta}(a^\delta_\rho) + \frac{\rho}{2}\|a^\delta_\rho - a^\dagger\|^2_{L^2(\Omega)} + \rho D^\lambda_{TV}(a^\delta_\rho, a^\dagger)\\ &\qquad\le \frac{\max\{\overline{a}, 1\}}{2}\delta^2 + \delta\rho\overline{a}\left(\int_\Omega \widehat{w}^2\right)^{1/2} + \rho^2\overline{a}\int_\Omega \widehat{w}^2\\ &\qquad+ \delta\rho\left(\int_\Omega |\nabla\widehat{w}|^2\right)^{1/2} + \rho^2\int_\Omega |\nabla\widehat{w}|^2.\end{aligned}$$

By Lemma 1.2.6, we obtain the following convergence rates

$$\|a^\delta_\rho - a^\dagger\|^2_{L^2(\Omega)} + D^\lambda_{TV}(a^\delta_\rho, a^\dagger) = \mathcal{O}(\delta) \quad\text{and}\quad \|U(a^\delta_\rho) - z^\delta\|_{H^1(\Omega)} = \mathcal{O}(\delta)$$

as $\delta \to 0$ and $\rho \sim \delta$. It remains to prove the convergence rate (4.53). Take $\lambda \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(a^\delta_\rho)$, we get from Lemma 4.2.5 that

$$\begin{aligned}\int_\Omega |\nabla a^\delta_\rho| - \int_\Omega |\nabla a^\dagger| &\le -\langle\lambda, a^\dagger - a^\delta_\rho\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)}\\ &\le \frac{\max\{\overline{a}, 1\}}{2}\frac{\delta^2}{\rho} + \overline{a}\|a^\delta_\rho - a^\dagger\|_{L^2(\Omega)}.\end{aligned} \tag{4.59}$$

On the other hand, since $\lambda \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(q^\dagger)$, by the assumption that $\lambda \in \mathfrak{X}^*$ can be identified with an element of $L^2(\Omega)$ which is denoted by the same symbol, we get

$$\begin{aligned}\int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a^\delta_\rho| &\le -\langle\lambda, a^\delta_\rho - a^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)}\\ &= -\langle\lambda, a^\delta_\rho - a^\dagger\rangle_{(\mathfrak{X}^*, \mathfrak{X})}\\ &= -\langle\lambda, a^\delta_\rho - a^\dagger\rangle_{L^2(\Omega)}.\end{aligned}$$

The last inequality yields

$$\int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a^\delta_\rho| \le \|\lambda\|_{L^2(\Omega)}\|a^\delta_\rho - a^\dagger\|_{L^2(\Omega)}. \tag{4.60}$$

From inequalities (4.59) and (4.60), and the fact that $\|a^\delta_\rho - a^\dagger\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta})$ we arrive at (4.53). The theorem is proved. □

4.2.3. Discussion of the source condition

Now we discuss the source condition (4.52) which is equivalent to the following one: there exists a function $w^* \in H^1(\Omega)^*$ such that

$$\frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a| - \frac{1}{2}\|a^\dagger\|^2_{L^2(\Omega)} - \int_\Omega |\nabla a^\dagger| - \langle U'(a^\dagger)^* w^*, a - a^\dagger\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \geq 0 \tag{4.61}$$

for all $a \in \mathfrak{X}$. To further analyze our condition we assume that the admissible set of coefficients is restricted to

$$\widehat{A_{ad}} = A \cap H^1(\Omega) \subset A \cap BV(\Omega).$$

Theorem 4.2.6. *Let the boundary $\partial\Omega$ be of class C^1 and the dimension $d \leq 4$. Suppose that $a^\dagger$ has the property that there is an element $\lambda \in \partial\left(\int_\Omega |\nabla(\cdot)|\right)(a^\dagger)$ such that $\langle \lambda, a\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} = \langle \widehat{\lambda}, a\rangle_{L^2(\Omega)}$ for all $a \in L^\infty(\Omega) \cap H^1(\Omega)$, where $\widehat{\lambda}$ is some element of $H^1(\Omega)$. Further, assume that there exists a positive constant γ such that $|\overline{u}| \geq \gamma$ a.e. on Ω. Then, the condition (4.61) is fulfilled and hence convergence rates*

$$\|a^\delta_\rho - a^\dagger\|^2_{L^2(\Omega)} + D^\lambda_{TV}(a^\delta_\rho, a^\dagger) = \mathcal{O}(\delta) \ \text{ and } \ \left|\int_\Omega |\nabla a^\dagger| - \int_\Omega |\nabla a^\delta_\rho|\right| = \mathcal{O}(\sqrt{\delta})$$

are obtained.

Proof. For any $a \in L^\infty(\Omega) \cap H^1(\Omega)$, we have

$$\begin{aligned}
\langle a^\dagger + \lambda, a\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} &= \langle a^\dagger, a\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} + \langle \lambda, a\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \\
&= \langle a^\dagger, a\rangle_{L^2(\Omega)} + \langle \lambda, a\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \\
&= \langle a^\dagger + \widehat{\lambda}, a\rangle_{L^2(\Omega)}.
\end{aligned} \tag{4.62}$$

Since $a^\dagger + \widehat{\lambda} \in H^1(\Omega)$ and $|\overline{u}| = |U(a^\dagger)| \geq \gamma > 0$, we have $\psi := -\frac{a^\dagger + \widehat{\lambda}}{U(a^\dagger)} \in H^1(\Omega)$. Hence

$$-\int_\Omega a U(a^\dagger)\psi = \langle a^\dagger + \widehat{\lambda}, a\rangle_{L^2(\Omega)} \tag{4.63}$$

for all $a \in L^\infty(\Omega) \cap H^1(\Omega)$. It follows from (4.62), (4.63) and (1.23) that

$$\begin{aligned}
\langle a^\dagger + \lambda, a\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} &= \int_\Omega \nabla U'(a^\dagger)a\nabla\psi + \int_\Omega a^\dagger U'(a^\dagger)a\psi \\
&= \langle \widehat{w}, U'(a^\dagger)a\rangle_{H^1(\Omega)}
\end{aligned}$$

for some $\widehat{w} \in H^1(\Omega)$ independent of $a \in L^\infty(\Omega) \cap H^1(\Omega)$. Therefore, there exists an element $w^* \in H^1(\Omega)^*$ such that

$$\begin{aligned}
\langle a^\dagger + \lambda, a\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} &= \langle w^*, U'(a^\dagger)a\rangle_{(H^1(\Omega)^*, H^1(\Omega))} \\
&= \left\langle U'(a^\dagger)^* w^*, a\right\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))}.
\end{aligned}$$

Since $\left\langle U'(a^\dagger)^* w^*, a\right\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} = \langle a^\dagger + \widehat{\lambda}, a\rangle_{L^2(\Omega)}$ with $a^\dagger + \widehat{\lambda} \in H^1(\Omega)$, the boundary $\partial\Omega$ being of class C^1, the dimension $d \leq 4$ and the Sobolev embedding theorem, we obtain that $U'(a^\dagger)^* w^*$ is linear and continuous on $L^\infty(\Omega) \cap H^1(\Omega)$ equipped with the $BV(\Omega)$-norm and

$$\left\langle U'(a^\dagger)^* w^*, a \right\rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} = \left\langle U'(a^\dagger)^* w^*, a \right\rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)}$$

for all $a \in L^\infty(\Omega) \cap H^1(\Omega)$. Therefore, since $a^\dagger + \lambda \in \partial R(a^\dagger)$, we conclude that there exists a functional $w^* \in H^1(\Omega)^*$ such that

$$\begin{aligned} \frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a| - \frac{1}{2}\|a^\dagger\|^2_{L^2(\Omega)} - \int_\Omega |\nabla a^\dagger| - \langle U'(a^\dagger)^* w^*, a - a^\dagger \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \\ = \frac{1}{2}\|a\|^2_{L^2(\Omega)} + \int_\Omega |\nabla a| - \frac{1}{2}\|a^\dagger\|^2_{L^2(\Omega)} - \int_\Omega |\nabla a^\dagger| \\ - \langle a^\dagger + \lambda, a - a^\dagger \rangle_{\left(\mathfrak{X}^*_{BV(\Omega)}, \mathfrak{X}_{BV(\Omega)}\right)} \\ \geq 0 \end{aligned}$$

for all $a \in L^\infty(\Omega) \cap H^1(\Omega)$. The theorem is proved. □

Conclusions

In the last chapter, we apply total variation regularization combining with L^2 stabilization to convex functionals $J_{z^\delta}(\cdot)$ and $G_{z^\delta}(\cdot)$ defined by (1.12) and (1.22) for identifying the diffusion coefficient and the reaction coefficient in the Neumann problems for the elliptic equation (1.1)–(1.2) and (1.3)–(1.4), respectively when the exact solution $\overline{u}$ is imprecisely given by observed data z^δ satisfying (1.5). We obtain convergence rates not only in the sense of the Bregman distance but also in the $L^2(\Omega)$-norm. Our source conditions are simple and weak, since we remove the so-called "small enough condition" on the source functions that is standard in the theory of regularization of nonlinear ill-posed problems but very hard to check. Furthermore, our results are valid for multi-dimensional identification problems. They are the first results affirmatively answering the question whether total variation regularization can provide convergence rates for coefficient identification problems in partial differential equations.

This chapter was written on the basis of the paper

[61] Dinh Nho Hào and Tran Nhan Tam Quyen (2012), Convergence rates for total variation regularization of coefficient identification problems in elliptic equations II, *Journal of Mathematical Analysis and Applications* **388**, pp. 593–616.

General Conclusions

Let Ω be an open bounded connected domain in $\mathbb{R}^d, d \geq 1$, with the Lipschitz boundary $\partial\Omega$, $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ be given. In this work we investigate ill-posed nonlinear inverse problems of identifying the diffusion coefficient q in the Neumann problem for the elliptic equation

$$\begin{aligned} -\mathrm{div}(q\nabla u) &= f \text{ in } \Omega, \\ q\frac{\partial u}{\partial n} &= g \text{ on } \partial\Omega \end{aligned}$$

and the reaction coefficient a in the Neumann problem for the elliptic equation

$$\begin{aligned} -\Delta u + au &= f \text{ in } \Omega, \\ \frac{\partial u}{\partial n} &= g \text{ on } \partial\Omega, \end{aligned}$$

when u is imprecisely given by z^δ with $\|u - z^\delta\|_{H^1(\Omega)} \leq \delta$ and $\delta > 0$. These problems frequently accounter in practice and attracted great attention from many researchers during the last 50 years or so. They are difficult due to their nonlinearity and ill-posedness, therefore regularization methods for them are required. However, up to now there have been very few results on convergence rates of suggested regularization methods. The famous one (and the only one) by Engl, Kunisch and Neubauer required the small enough condition of the source functions which is very difficult to check and applicable only to one-dimensional above identification problems. The drawback of this work and many related ones is that the authors follow the least-squares approach and thus they are faced with nonconvex minimization problems, the global minima of which are impossible to find. In this dissertation, we do not follow this approach, but regularize the above identification problems by correspondingly minimizing the (strictly) convex functionals

$$\frac{1}{2}\int_\Omega q|\nabla(U(q) - z^\delta)|^2 + \rho\mathcal{R}(q)$$

and

$$\frac{1}{2}\int_\Omega |\nabla(U(a) - z^\delta)|^2 + \frac{1}{2}\int_\Omega a(U(a) - z^\delta)^2 + \rho\mathcal{R}(a)$$

over the admissible sets of coefficients, where $U(q)$ ($U(a)$) is the solution of the first (second) Neumann boundary value problem, $\rho > 0$ is the regularization parameter and either

$$\mathcal{R}(\cdot) = \|\cdot\|^2_{L^2(\Omega)}$$

or

$$\mathcal{R}(\cdot) = \int_\Omega |\nabla(\cdot)|$$

or

$$\mathcal{R}(\cdot) = \frac{1}{2}\|\cdot\|^2_{L^2(\Omega)} + \int_\Omega |\nabla(\cdot)|.$$

One of the advantage of our approach is that the above minimization problems are convex and we can find their global minima. Furthermore, taking their solutions as the regularized solutions to the corresponding identification problems, we obtain the convergence rates of them to solutions of our inverse problems under weak source conditions

$$\text{there exists a function } w^* \in H^1_\diamond(\Omega)^* \text{ such that } U'(q^\dagger)^* w^* \in \partial\mathcal{R}(q^\dagger)$$

for the first problem and

$$\text{there exists a function } w^* \in H^1(\Omega)^* \text{ such that } U'(a^\dagger)^* w^* \in \partial\mathcal{R}(a^\dagger)$$

for the second problem with $q^\dagger$ and $a^\dagger$ respectively being the total $\mathcal{R}$-minimizing solutions of the coefficient identification problems. Our source conditions are simple and weak, since we remove the so-called "small enough condition" on the source functions that is standard in the theory of regularization of nonlinear ill-posed problems but very hard to check. Furthermore, our results are valid for multi-dimensional identification problems. They are the first results affirmatively answering the question whether total variation regularization can provide convergence rates for coefficient identification problems in partial differential equations.

List of the author's publications related to the dissertation

[1] Dinh Nho Hào and Tran Nhan Tam Quyen (2010), Convergence rates for Tikhonov regularization of coefficient identification problems in Laplace-type equations, *Inverse Problems* **26**, 125014 (23pp).

[2] Dinh Nho Hào and Tran Nhan Tam Quyen (2011), Convergence rates for total variation regularization of coefficient identification problems in elliptic equations I, *Inverse Problems* **27**, 075008 (28pp).

[3] Dinh Nho Hào and Tran Nhan Tam Quyen (2012), Convergence rates for total variation regularization of coefficient identification problems in elliptic equations II, *Journal of Mathematical Analysis and Applications* **388**, pp. 593–616.

[4] Dinh Nho Hào and Tran Nhan Tam Quyen (2012), Convergence rates for Tikhonov regularization of a two-coefficient identification problem in an elliptic boundary value problem, *Numerische Mathematik* **120**, pp. 45–77.

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1) Mini Special Semester on Inverse Problems, Johann Radom Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Jun 29th – Jul 2nd, 2010.

2) Scientific Conference in Celebration of the 35-Year History of Vietnam Academy of Science and Technology, Oct 25, 2010.

3) International Conference on Analysis and Applied Mathematics, Sai Gon University, Viet Nam, Mar 14, 2011.

and by Tran Nhan Tam Quyen

4) PhD. Students Conference, Hanoi Institute of Mathematics, Oct 30, 2009.

5) PhD. Students Conference, Hanoi Institute of Mathematics, Oct 29, 2010.

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