VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY
INSTITUTE OF MATHEMATICS

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Coderivatives of Normal Cone Mappings and Applications

DOCTORAL DISSERTATION IN MATHEMATICS

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Supervisors:
1. Prof. Dr. Hab. Nguyen Dong Yen
2. Dr. Bui Trong Kien

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To my beloved parents and family members

Confirmation

This dissertation was written on the basis of my research works carried at Institute of Mathematics (VAST, Hanoi) under the supervision of Professor Nguyen Dong Yen and Dr. Bui Trong Kien. All the results presented have never been published by others.

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The author

Nguyen Thanh Qui

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Contents

Table of Notations

$\mathbb{N} := \{1, 2, \ldots\}$	set of positive natural numbers
$\emptyset$	empty set
$\mathbb{R}$	set of real numbers
$\mathbb{R}_{++}$	set of $x \in \mathbb{R}$ with $x > 0$
$\mathbb{R}_{+}$	set of $x \in \mathbb{R}$ with $x \geq 0$
$\mathbb{R}_{-}$	set of $x \in \mathbb{R}$ with $x \leq 0$
$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$	set of generalized real numbers
$\|x\|$	absolute value of $x \in \mathbb{R}$
$\mathbb{R}^n$	n-dimensional Euclidean vector space
$\\|x\\|$	norm of a vector x
$\mathbb{R}^{m\times n}$	set of $m \times n$-real matrices
$\det A$	determinant of a matrix A
$A^\top$	transposition of a matrix A
$\\|A\\|$	norm of a matrix A
X^*	topological dual of a norm space X
$\langle x^*, x\rangle$	canonical pairing
$\langle x, y\rangle$	canonical inner product
$\widehat{(u, v)}$	angle between two vectors u and v
$B(x, \delta)$	open ball with centered at x and radius δ
$\bar{B}(x, \delta)$	closed ball with centered at x and radius δ
B_X	open unit ball in a norm space X
$\bar{B}_X$	closed unit ball in a norm space X
$\text{pos}\Omega$	convex cone generated by Ω
$\text{span}\Omega$	linear subspace generated by Ω
$\text{dist}(x; \Omega)$	distance from x to Ω
$\{x_k\}$	sequence of vectors
$x_k \to x$	x_k converges to x in norm topology
$x_k^* \xrightarrow{w^*} x^*$	x_k^* converges to x^* in weak* topology

$\forall x$	for all x
$x := y$	x is defined by y
$\widehat{N}(x;\Omega)$	Fréchet normal cone to Ω at x
$N(x;\Omega)$	limiting normal cone to Ω at x
$f : X \to Y$	function from X to Y
$f'(x)$, $\nabla f(x)$	Fréchet derivative of f at x
$\varphi : X \to \overline{I\!R}$	extended-real-valued function
$\mathrm{dom}\varphi$	effective domain of φ
$\mathrm{epi}\varphi$	epigraph of φ
$\partial\varphi(x)$	limiting subdifferential of φ at x
$\partial^2\varphi(x,y)$	limiting second-order subdifferential of φ at x relative to y
$F : X \rightrightarrows Y$	multifunction from X to Y
$\mathrm{dom}F$	domain of F
$\mathrm{rge}F$	range of F
$\mathrm{gph}F$	graph of F
$\ker F$	kernel of F
$\widehat{D}^*F(x,y)$	Fréchet coderivative of F at (x,y)
$D^*F(x,y)$	Mordukhovich coderivative of F at (x,y)

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Introduction

Motivated by solving optimization problems, the concept of *derivative* was first introduced by Pierre de Fermat. It led to the *Fermat stationary principle*, which plays a crucial role in the development of *differential calculus* and serves as an effective tool in various applications. Nevertheless, many fundamental objects having no derivatives, no first-order approximations (defined by certain derivative mappings) occur naturally and frequently in mathematical models. The objects include nondifferentiable functions, sets with nonsmooth boundaries, and set-valued mappings. Since the classical differential calculus is inadequate for dealing with such functions, sets, and mappings, the appearance of generalized differentiation theories is an indispensable trend.

In the 1960s, differential properties of convex sets and convex functions have been studied. The fundamental contributions of J.-J. Moreau and R. T. Rockafellar have been widely recognized. Their results led to the beautiful theory of convex analysis [47]. The derivative-like structure for convex functions, called *subdifferential*, is one of the main concepts in this theory. In contrast to the singleton of derivatives, subdifferential is a collection of subgradients. Convex programming which is based on convex analysis plays a fundamental role in Mathematics and in applied sciences.

In 1973, F. H. Clarke defined basic concepts of a generalized differentiation theory, which works for locally Lipschitz functions, in his doctoral dissertation under the supervision of R. T. Rockafellar. In Clarke's theory, convexity is a key point; for instance, subdifferential in the sense of Clarke is always a closed convex set. In the later 1970s, the concepts of Clarke have been developed for lower semicontinuous extended-real-valued functions in the works of R. T. Rockafellar, J.-B. Hiriart-Urruty, J.-P. Aubin, and others. Although the theory of Clarke is beautiful due to the convexity used, as well as to the elegant proofs of many fundamental results, the Clarke subdifferential and the Clarke normal cone face with the challenge of being too big, so too rough, in complicated practical problems where nonconvexity is an inherent property. Despite to this, Clarke's theory has opened a new chapter in the development of nonlinear analysis and optimization theory (see, e.g., [8], [2]).

In the mid 1970s, to avoid the above-mentioned convexity limitations of the Clarke concepts, B. S. Mordukhovich introduced the notions of *limiting normal cone* and *limiting subdifferential* which are based entirely on dual-space constructions. His dual approach led to a modern theory of generalized differentiation [28] with a variety of applications [29]. Long before the publication of these books, Mordukhovich's contributions to Variational Analysis had been presented in the well-known monograph of R. T. Rockafellar and R. J.-B. Wets [48].

The limiting subdifferential is generally nonconvex and smaller than the Clarke subdifferential. Similarly, the limiting normal cone to a closed set in a Banach space is nonconvex in general and usually smaller than the Clarke normal cone. Therefore, necessary optimality conditions in nonlinear programming and optimal control in terms of the limiting subdifferential and limiting normal cone are much tighter than that given by the corresponding Clarke's concepts. Furthermore, the Mordukhovich criteria for the Lipchitz-like property (that is the pseudo-Lipschitz property in the original terminology of J.-P. Aubin [1], or the Aubin continuity as suggested by A. L. Dontchev and R. T. Rockafellar [11], [12]) and the metric regularity of multifunctions are remarkable tools to study stability of variational inequalities, generalized equations, and the Karush-Kuhn-Tucker point sets in parametric optimization problems. Note that if one uses Clarke's theory then only sufficient conditions for stability can be obtained. Meanwhile, Mordukhovich's theory provides one with both necessary and sufficient conditions for stability. Another advantage of the latter theory is that its system of calculus rules is much more developed than that of Clarke's theory. So, the wide range of applications and bright prospects of Mordukhovich's generalized differentiation theory are understandable.

In the late 1990s, V. Jeyakumar and D. T. Luc introduced the concepts of approximate Jacobian and corresponding generalized subdifferential. It can be seen [18] that using the approximate Jacobian one can establish conditions for stability, metric regularity, and local Lipschitz-like property of the solution maps of parametric inequality systems involving nonsmooth continuous functions and closed convex sets. Calculus rules and various applications of the approximate Jacobian can be found in the monograph [17]. It is worthy to study relationships between the concepts of coderivative and approximate Jacobian. In [33], the authors show that the Mordukhovich coderivative and the approximate Jacobian have a little in common. These concepts are very different, and they require different methods of study and lead to results in different forms.

As far as we understand, Variational Analysis is a new name of a mathematical discipline which unifies Nonsmooth Analysis, Set-Valued Analysis with applications to Optimization Theory and equilibrium problems. Many aspects of the theory can be seen in [2], [4], [8], [28], [29], [48].

Let X, W_1, W_2 are Banach spaces, $\varphi : X \times W_1 \to I\!R$ is a continuously Fréchet differentiable function, $\Theta : W_2 \rightrightarrows X$ is a multifunction (i.e., a set-valued map) with closed convex values. Consider the minimization problem

$$\min\{\varphi(x, w_1)|\ x \in \Theta(w_2)\} \tag{1}$$

depending on the parameters $w = (w_1, w_2)$, which is given by the data set $\{\varphi, \Theta\}$. According to the generalized Fermat rule (see, for instance, [20, pp. 85–86]), if $\bar{x}$ is a local solution of (1) then

$$0 \in f(\bar{x}, w_1) + N(\bar{x}; \Theta(w_2)),$$

where $f(\bar{x}, w_1) = \nabla_x \varphi(\bar{x}, w_1)$ denotes the partial derivative of φ with respect to $\bar{x}$ at $(\bar{x}, w_1)$ and

$$N(\bar{x}; \Theta(w_2)) = \{x^* \in X^*|\ \langle x^*, x - \bar{x}\rangle \leq 0,\ \forall x \in \Theta(w_2)\},$$

with X^* being the dual space of X, stands for the normal cone of $\Theta(w_2)$. This means that $\bar{x}$ is a solution of the following generalized equation

$$0 \in f(x, w_1) + \mathcal{F}(x, w_2), \tag{2}$$

where $\mathcal{F}(x, w_2) := N(x; \Theta(w_2))$ for every $x \in \Theta(w_2)$ and $\mathcal{F}(x, w_2) := \emptyset$ for every $x \notin \Theta(w_2)$, is the parametric normal cone mapping related to the multifunction $\Theta(\cdot)$. Equilibrium problems of the form (2) have been investigated intensively in the literature (see, e.g., [11], [12], [24], [27], [28, Chapter 4], [43]). Necessary and sufficient conditions for the Lipschitz-like property of the solution map $(w_1, w_2) \mapsto S(w_1, w_2)$ of (2) can be characterized by using the Mordukhovich criterion. According to the method proposed by A. L. Dontchev and R. T. Rockafellar [11], which has been developed by A. B. Levy and B. S. Mordukhovich [24] and by G. M. Lee and N. D. Yen

[22], one has to compute the Fréchet and the Mordukhovich coderivatives of $\mathcal{F} : X \times W_2 \rightrightarrows X^*$. Such a computation has been done in [11] for the case $\Theta(w_2)$ is a fixed polyhedral convex set in $I\!R^n$, and in [54] for the case where $\Theta(w_2)$ is a fixed smooth-boundary convex set. The problem is rather difficult if $\Theta(w_2)$ depends on w_2.

J.-C. Yao and N. D. Yen [52], [53] first studied the case $\Theta(w_2) = \Theta(b) := \{x \in I\!R^n |\ Ax \leq b\}$ where A is an $m \times n$ matrix, b is a parameter. Some arguments from these papers have been used by R. Henrion, B. S. Mordukhovich and N. M. Nam [13] to compute coderivatives of the normal cone mappings to a fixed polyhedral convex set in Banach space. N. M. Nam [32] showed that the results of [52], [53] on normal cone mappings to linearly perturbed polyhedra can be extended to an infinite dimensional setting. N. T. Q. Trang [50] proposed some developments and refinements of the results of [32].

G. M. Lee and N. D. Yen [23] computed the Fréchet coderivatives of the normal cone mappings to a perturbed Euclidean balls and derived from the results a stability criterion for the Karush-Kuhn-Tucker point set mapping of parametric trust-region subproblems.

As concerning normal cone mappings to nonlinearly perturbed polyhedra, we would like to mention a recent paper [9] where the authors have computed coderivatives of the normal cone to a rotating closed half-space.

The normal cone mapping considered in [23] is a special case of the normal cone mapping to the solution set $\Theta(w_2) = \Theta(p) := \{x \in X |\ \psi(x,p) \leq 0\}$ where $\psi : X \times P \to I\!R$ is a $\mathcal{C}^2$-smooth function defined on the product space of Banach spaces X and P.

More generally, for the solution map

$$\Theta(w_2) = \Theta(p) := \{x \in X |\ \Psi(x,p) \in K\}$$

of a parametric generalized equality system with $\Psi : X \times P \to Y$ being a $\mathcal{C}^2$-smooth vector function which maps the product space $X \times P$ into a Banach space Y, $K \subset Y$ a closed convex cone, the problems of computing the Fréchet coderivative (respectively, the Mordukhovich coderivative) of the Fréchet normal cone mapping $(x, w_2) \mapsto \widehat{N}(x; \Theta(w_2))$ (respectively, of the limiting normal cone mapping $(x, w_2) \mapsto N(x; \Theta(w_2))$), are interesting, but very difficult. All the above-mentioned normal cone mappings are special cases of the last two normal cone mappings. It will take some time before significant advances on these general problems can be done. Some aspects of this question have been investigated by [14].

It is worthy to stress that coderivatives of normal cone mappings are nothing else as the second-order subdifferentials of the indicator functions of the set in question. The concepts of Fréchet and/or limiting second-order subdifferentials of extended-real-valued functions are discussed in [28], [37], [30], [5], [6], [7], [31] from different points of views.

This dissertation studies some problems related to the generalized differentiation theory of Mordukhovich and its applications. Our main efforts concentrate on computing or estimating the Fréchet coderivative and the Mordukhovich coderivative of the normal cone mappings to

a) linearly perturbed polyhedra in finite dimensional spaces, as well as in infinite dimensional reflexive Banach spaces,

b) nonlinearly perturbed polyhedra in finite dimensional spaces,

c) perturbed Euclidean balls.

Applications of the obtained results are used to study the metric regularity property and/or the Lipschitz-like property of the solution maps of some classes of parametric variational inequalities as well as parametric generalized equations.

Our results develop certain aspects of the preceding works [11], [52], [53], [13], [32], and [23]. The four open questions raised in [52] and [23] have been solved in this dissertation. Some of our techniques are new.

The dissertation has four chapters and a list of references.

Chapter 1 collects several basic concepts and facts on generalized differentiation, together with the well-known dual characterizations of the two fundamental properties of multifunctions: the local Lipschitz-like property defined by J.-P. Aubin and the metric regularity which has origin in Ljusternik's theorem [16, p. 30].

Chapter 2 studies generalized differentiability properties of the normal cone mappings associated to perturbed polyhedral convex sets in reflexive Banach spaces. The obtained results lead to solution stability criteria for a class of variational inequalities in finite dimensional spaces under linear perturbations. This chapter also answers the two open questions in [52].

Chapter 3 computes the Fréchet and the Mordukhovich coderivatives of the normal cone mappings studied in the previous chapter with respect to total perturbations. As a consequence, solution stability of affine variational inequalities under nonlinear perturbations in finite dimensional spaces can be addressed by means of the Mordukhovich criterion and the coderivative formula for implicit multifunctions due to A. B. Levy and B. S. Mordukhovich [24, Theorem 2.1].

Based on a recent paper of G. M. Lee and N. D. Yen [23], Chapter 4 presents a comprehensive study of the solution stability of a class of linear generalized equations connected with the parametric trust-region subproblems which are well-known in nonlinear programming. We show that exact formulas for the coderivatives of the normal cone mappings associated to perturbed Euclidean balls can be obtained. Then, combining the formulas with the necessary and the sufficient conditions for the local Lipschitz-like property of implicit multifunctions from a paper by G. M. Lee and N. D. Yen [22], we get new results on stability of the Karush-Kuhn-Tucker point set maps of parametric trust-region subproblems. This chapter also solves the two open questions in [23].

The results of Chapter 2 and Chapter 3 were published on the journals *Nonlinear Analysis* [38], *Journal of Mathematics and Applications* [39], *Acta Mathematica Vietnamica* [40], *Journal of Optimization Theory and Applications* [41]. Chapter 4 is written on the basis of a joint paper by N. T. Qui and N. D. Yen, which has been accepted for publication on *SIAM Journal on Optimization* [42].

Chapter 1

Preliminary

In this chapter we review some background material of Variational Analysis; see, e.g., [1], [12], [20], [28], [29], [35], [48] for more details and references. The basic concepts of generalized differentiation of set-valued mappings and extended-real-valued functions are presented in this chapter are taken from Mordukhovich [28], [29].

1.1 Basic Definitions and Conventions

Let X be a norm space with the norm usually denoted by $\|\cdot\|$. For each $x_0 \in X$ and $\delta > 0$, we denote by $B(x_0, \delta)$ the open ball $\{x \in X|\ \|x - x_0\| < \delta\}$, and let $\bar{B}(x_0, \delta)$ stand for the corresponding closed ball. We will write B_X and $\bar{B}_X$ respectively for $B(0_X, 1)$ and $\bar{B}(0_X, 1)$. Unless otherwise stated, *every norm in question in a product norm space is a sum norm.* Let Ω be a subset of X. When $\Omega \neq \emptyset$, $\operatorname{dist}(x; \Omega)$ is the distance from $x \in X$ to the nonempty set Ω, that is

$$\operatorname{dist}(x; \Omega) = \inf_{u \in \Omega} \|x - u\|.$$

If $\Omega = \emptyset$, we put $\operatorname{dist}(x; \Omega) = +\infty$ by convention. The *negative dual cone* of $\Omega \subset X$ is defined by

$$\Omega^* := \{x^* \in X^*|\ \langle x^*, v\rangle \leq 0,\ \forall v \in \Omega\}$$

with X^* being the dual space of X, and $\langle \cdot, \cdot \rangle$ standing for the *canonical pairing* between X^* and X. For each $u^* \in X^*$, we define

$$\{u^*\}^{\perp} := \{v \in X|\ \langle u^*, v\rangle = 0\}.$$

When X is a finite dimensional Euclidean space, the notation $\langle\cdot,\cdot\rangle$ also stands for the *canonical inner product* in X. In working with X, we keep to the Euclidean norm given by $\|x\| = \sqrt{\langle x, x\rangle}$ for every $x \in X$. In the sequel, $x \xrightarrow{\Omega} \bar{x}$ means $x \to \bar{x}$ with $x \in \Omega$.

Let $F : X \rightrightarrows Y$ be a *set-valued mapping/multifunction* between nonempty sets X and Y. Denote respectively by

$$\begin{aligned} &\text{dom}F := \big\{x \in X |\ F(x) \neq \emptyset\big\}, \\ &\text{rge}F := \big\{y \in Y |\ y \in F(x) \text{ for some } x \in X\big\} \end{aligned}$$

the *domain* and the *range* of F. The multifunction $F : X \rightrightarrows Y$ is uniquely associated with its *graph*

$$\text{gph}F := \big\{(x, y) \in X \times Y |\ y \in F(x)\big\}$$

in the product set $X \times Y$. Note that if X and Y are Banach spaces, then $X \times Y$ is also a Banach space with respect to the sum norm $\|(x, y)\| = \|x\| + \|y\|$ imposed on $X \times Y$ unless otherwise stated. In this case, the *kernel* of F is defined by

$$\text{ker}F := \big\{x \in X |\ 0 \in F(x)\big\}.$$

The *image* of a set $\Omega \subset X$ and the *inverse image* of a set $\Theta \subset Y$ under F are defined in succession by setting

$$F(\Omega) := \big\{y \in Y |\ y \in F(x) \text{ for some } x \in X\big\}$$

and

$$F^{-1}(\Theta) := \big\{x \in X |\ F(x) \cap \Theta \neq \emptyset\big\}.$$

The *inverse mapping* to $F : X \rightrightarrows Y$ is the multifunction $F^{-1} : Y \rightrightarrows X$ with

$$F^{-1}(y) := \{x \in X |\ y \in F(x)\}.$$

Observe that $\text{dom}F^{-1} = \text{rge}F$, $\text{rge}F^{-1} = \text{dom}F$, and

$$\text{gph}F^{-1} = \big\{(y, x) \in Y \times X |\ (x, y) \in \text{gph}F\big\}.$$

A multifunction between Banach spaces $F : X \rightrightarrows Y$ is said to be *positively homogeneous* if $0 \in F(0)$ and $F(\alpha x) \supset \alpha F(x)$ for all $x \in X$ and $\alpha > 0$. The latter is equivalent to saying that the graph of F is a cone in $X \times Y$. The norm of a positively homogeneous multifunction F is defined by

$$\|F\| := \sup\Big\{\|y\| \mid y \in F(x) \text{ with } \|x\| \leq 1\Big\}.$$

1.2 Normal and Tangent Cones

In this section, we recall the concepts of normals and tangents to sets in Banach spaces and discuss their properties and relationships.

Let $F : X \rightrightarrows Y$ be a multifunction between topological spaces X and Y. Following [28] and [48], the *sequential Painlevé-Kuratowski upper/outer limit* of F as $x \to \bar{x}$ is defined by

$$\mathop{\rm Lim\,sup}_{x\to\bar{x}} F(x) = \Big\{ y \in Y \Big|\ \text{exist sequences } x_k \to \bar{x} \text{ and } y_k \to y \text{ with } y_k \in F(x_k) \text{ for all } k \in I\!\!N \Big\}. \tag{1.1}$$

Note that the limits in expression (1.1) are understood in the sequential sense which contrast to net/topological limits in general topological spaces. When $F : X \rightrightarrows X^*$ is a multifunction between a Banach space X and its dual X^*, we always understand the *sequential Painlevé-Kuratowski upper limit* of F as $x \to \bar{x}$ with respect to the norm topology of X and the weak* topology of X^*. The latter means that

$$\mathop{\rm Lim\,sup}_{x\to\bar{x}} F(x) = \Big\{ x^* \in X^* \Big|\ \text{exist sequences } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k) \text{ for all } k \in I\!\!N \Big\}. \tag{1.2}$$

In what follows, *all the reference spaces are real Banach spaces.*

Definition 1.1 (See [28, Definition 1.1]) Let Ω be a nonempty subset of a Banach space X.

(i) Given $\bar{x} \in \Omega$ and $\varepsilon \ge 0$, we define the set of *ε-normals* to Ω at $\bar{x}$ by

$$\widehat{N}_\varepsilon(\bar{x};\Omega) := \Big\{ x^* \in X^* \Big|\ \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \le \varepsilon \Big\}. \tag{1.3}$$

When $\varepsilon = 0$, elements of (1.3) are called *Fréchet normals* and their collection, denoted by $\widehat{N}(\bar{x};\Omega)$, is the *Fréchet normal cone* to Ω at $\bar{x}$. If $\bar{x} \notin \Omega$, we put $\widehat{N}_\varepsilon(\bar{x};\Omega) = \emptyset$ for all $\varepsilon \ge 0$.

(ii) For $\bar{x} \in \Omega$, a vector $x^* \in X^*$ is called *limiting normal* to Ω at $\bar{x}$ if there are sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* \xrightarrow{w^*} x^*$ such that $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k;\Omega)$ for all $k \in I\!\!N$. The collection of such normals

$$N(\bar{x};\Omega) := \mathop{\rm Lim\,sup}_{\substack{x\to\bar{x}\\ \varepsilon\downarrow 0}} \widehat{N}_\varepsilon(x;\Omega) \tag{1.4}$$

is the *limiting normal cone* to Ω at $\bar{x}$. We put $N(\bar{x};\Omega) = \emptyset$ when $\bar{x} \notin \Omega$.

We see that for each $\varepsilon \geq 0$ the ε-normal set $\widehat{N}_\varepsilon(\bar{x};\Omega)$ is convex and closed in the norm topology of X^*. In contrast to the ε-normal sets, the limiting normal cone may be nonconvex. For instance, given a subset $\Omega := \{(x_1,x_2) \in I\!R^2|\ x_2 \geq -|x_1|\} \subset I\!R^2$, we have $\widehat{N}\big((0,0);\Omega\big) = \{0\}$ while

$$N\big((0,0);\Omega\big) = \{(v,v)|\ v \leq 0\} \cup \{(v,-v)|\ v \geq 0\}$$

is a nonconvex set. Since duality implies convexity, the example shows that the limiting normal cone to a set Ω at a given point $\bar{x}$ cannot be dual to any tangential approximation of Ω at $\bar{x}$ in the primal space. Example 1.7 in [28] shows that, in general, the limiting normal cone may not be norm closed in the dual space X^* (hence it is not weakly* closed).

A set $\Omega \subset X$ is said to be *normally regular* at $\bar{x} \in \Omega$ if $N(\bar{x};\Omega) = \widehat{N}(\bar{x};\Omega)$. From (1.3) and (1.4) it follows that $\widehat{N}(\bar{x};\Omega) \subset N(\bar{x};\Omega)$ for any $\Omega \subset X$ and $\bar{x} \in \Omega$. If Ω is convex, then by Propositions 1.3 and 1.5 in [28] it holds

$$\widehat{N}(\bar{x};\Omega) = N(\bar{x};\Omega) = \Big\{x^* \in X^*|\ \langle x^*, x-\bar{x}\rangle \leq 0,\ \forall x \in \Omega\Big\}.$$

In this case, both the Fréchet and the limiting normal cones coincide with the normal cone of Convex Analysis; thus Ω is normally regular at $\bar{x}$.

One says that a set $\Omega \subset X$ is *locally closed* around $\bar{x} \in \Omega$ if there exists $\delta > 0$ for which $\Omega \cap \bar{B}(\bar{x},\delta)$ is closed. The next theorem establishes a special representation of the limiting normal cone to closed subsets of finite dimensional spaces.

Theorem 1.1 (See [28, Theorem 1.6]) *Let $\Omega \subset I\!R^n$ be locally closed around $\bar{x} \in \Omega$. Then it holds that*

$$N(\bar{x};\Omega) = \operatorname*{Limsup}_{x \to \bar{x}} \widehat{N}(x;\Omega). \tag{1.5}$$

Asplund spaces, which are specific Banach spaces, have important role [28, Chapter 3] in Variational Analysis. If X is an Asplund space, the expression on the right-hand side of the formula (1.4) can be also simplified similarly as (1.5).

Definition 1.2 (See [28, Definition 2.17] and [35, Definition 1.22]) A Banach space X is *Asplund*, or it has the *Asplund property*, if every convex continuous function $\varphi : U \to I\!R$ defined on an open convex subset U of X is Fréchet differentiable on a dense subset of U.

An interesting characterization of Asplund spaces is that *X is Asplund if and only if every separable closed subspace of X has a separable dual.* The next theorem provides us with a formula for computing of the limiting normal cones to closed subsets of Asplund spaces.

Theorem 1.2 (See [28, Theorem 2.35]) *Let X be a Banach space. The following properties are equivalent:*

(i) *X is Asplund.*

(ii) *For every closed set $\Omega \subset X$ and every $\bar{x} \in \Omega$ one has the representation*

$$N(\bar{x};\Omega) = \operatorname*{Limsup}_{x \to \bar{x}} \widehat{N}(x;\Omega). \tag{1.6}$$

The Fréchet normal cone has a tight connection with the concepts of contingent tangent cone and of weak contingent cone.

Definition 1.3 (See [28, Definition 1.8]) Let Ω be a subset of a Banach space X and $\bar{x} \in \Omega$.

(i) The set $T(\bar{x};\Omega) \subset X$ defined by

$$T(\bar{x};\Omega) := \operatorname*{Limsup}_{t \downarrow 0} \frac{\Omega - \bar{x}}{t}, \tag{1.7}$$

where the "Limsup" is taken with respect to the norm topology of X, is called the *contingent cone* to Ω at $\bar{x}$.

(ii) If the "Limsup" in (i) is taken with respect to the weak topology of X, then the resulting construction, denoted by $T_W(\bar{x};\Omega)$, is called the *weak contingent cone* to Ω at $\bar{x}$.

The contingent cone $T(\bar{x};\Omega)$ in Definition 1.7 was introduced by Bouligand, and it was also introduced independently by Severi. Hence, another, better name for this cone would be the *Bouligand-Severi tangent cone.* Note that when Ω is convex, the contingent cone $T(\bar{x};\Omega)$ coincides with the notion of *tangent cone* in the sense of Convex Analysis. This means that $T(\bar{x};\Omega)$ is the topological closure of the cone $\{\lambda(x - \bar{x})|\ x \in \Omega, \lambda \geq 0\}$.

In contrast to the limiting normal cone, the Fréchet normal cone can be dual of a tangent cone to a set in the primal space. Relationships between the Fréchet normal cone and the contingent cones are described as follows.

Proposition 1.1 (See [28, Corollary 1.11]) *Let X be a reflexive space and $\Omega \subset X$ with $\bar{x} \in \Omega$. Then the Fréchet normal cone to Ω at $\bar{x}$ is computed by*

$$\widehat{N}(\bar{x};\Omega) = \big(T_W(\bar{x};\Omega)\big)^* = \Big\{x^* \in X^* \big|\ \langle x^*, v\rangle \leq 0,\ \forall v \in T_W(\bar{x};\Omega)\Big\}.$$

Thus, when X is finite dimensional, one has

$$\widehat{N}(\bar{x};\Omega) = \big(T(\bar{x};\Omega)\big)^*.$$

1.3 Coderivatives and Subdifferential

The Fréchet and the Mordukhovich coderivatives of multifunctions [28] are two basic concepts of the generalized differentiation theory constructed by the dual-space approach. They are defined via the concepts of Fréchet normal cone and limiting normal cone.

Definition 1.4 (See [28, Definition 1.32]) Let $F : X \rightrightarrows Y$ be a multifunction between Banach spaces X and Y.

(i) For any $(\bar{x},\bar{y}) \in X \times Y$ and $\varepsilon \geq 0$, *ε-coderivative* of F at $(\bar{x},\bar{y})$ is the multifunction $\widehat{D}^*_\varepsilon F(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$ defined by

$$\widehat{D}^*_\varepsilon F(\bar{x},\bar{y})(y^*) = \Big\{x^* \in X^* \big|\ (x^*,-y^*) \in \widehat{N}_\varepsilon\big((\bar{x},\bar{y}); \mathrm{gph}F\big)\Big\},\ \forall y^* \in Y^*. \tag{1.8}$$

The mapping $\widehat{D}^*_\varepsilon F(\bar{x},\bar{y})$ with $\varepsilon = 0$ is said to be the *Fréchet coderivative* of F at $(\bar{x},\bar{y})$ and is denoted by $\widehat{D}^* F(\bar{x},\bar{y})$.

(ii) The *Mordukhovich coderivative* (or the *normal coderivative*) of F at $(\bar{x},\bar{y}) \in \mathrm{gph}F$ is the multifunction $D^*F(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$ given by

$$D^*F(\bar{x},\bar{y})(\bar{y}^*) = \mathop{\mathrm{Limsup}}_{\substack{(x,y)\to(\bar{x},\bar{y})\\ y^* \xrightarrow{w^*} \bar{y}^*\\ \varepsilon\downarrow 0}} \widehat{D}^*_\varepsilon F(x,y)(y^*). \tag{1.9}$$

If $(\bar{x},\bar{y}) \notin \mathrm{gph}F$, we put $D^*F(\bar{x},\bar{y})(y^*) = \emptyset$ for all $y^* \in Y^*$.

It follows from (1.8) that $\widehat{D}^*_\varepsilon F(\bar{x},\bar{y})(y^*) = \emptyset$ for all $\varepsilon \geq 0$ and $y^* \in Y^*$ when $(\bar{x},\bar{y}) \notin \mathrm{gph}F$. From (1.9) we see that $D^*F(\bar{x},\bar{y})(\bar{y}^*)$ is the collection of such $\bar{x}^* \in X^*$ for which there are sequences $\varepsilon_k \downarrow 0$, $(x_k,y_k) \to (\bar{x},\bar{y})$, and $(x^*_k,y^*_k) \xrightarrow{w^*} (\bar{x}^*,\bar{y}^*)$ with $(x_k,y_k) \in \mathrm{gph}F$ and $x^*_k \in \widehat{D}^*_{\varepsilon_k} F(x_k,y_k)(y^*_k)$ for all $k \in I\!N$. Note that the multifunction $D^*F(\bar{x},\bar{y})$ in (1.9) is uniquely determined by the limiting normal cone to the graph of F at the point $(\bar{x}, \bar{y})$. Namely, the Mordukhovich coderivative of F at the point $(\bar{x}, \bar{y})$ is the multifunction $D^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$, where

$$D^*F(\bar{x}, \bar{y})(y^*) := \Big\{x^* \in X^* \big|\ (x^*, -y^*) \in N\big((\bar{x}, \bar{y}); \mathrm{gph}F\big)\Big\},\ \forall y^* \in Y^*. \tag{1.10}$$

From (1.6) and (1.10) it is clear that the computation of the Fréchet normal cone to the graph of a multifunction between Asplund spaces is a crucial step towards a complete differentiation of that multifunction.

One says that F is *graphically regular* at a given point $(\bar{x}, \bar{y}) \in \mathrm{gph}F$ if

$$D^*F(\bar{x}, \bar{y})(y^*) = \widehat{D}^*F(\bar{x}, \bar{y})(y^*),\ \ \forall y^* \in Y^*.$$

When $F \equiv f$ is a single-valued mapping and $\bar{y} = f(\bar{x})$, one writes respectively $\widehat{D}^*f(\bar{x})$ and $D^*f(\bar{x})$ for $\widehat{D}^*F(\bar{x}, \bar{y})$ and $D^*F(\bar{x}, \bar{y})$.

By definition, $f : X \to Y$ is *Fréchet differentiable* at $\bar{x}$ if there is a continuous linear operator $\nabla f(\bar{x}) : X \to Y$, called the *Fréchet derivative* of f at $\bar{x}$, such that

$$\lim_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x})}{\|x - \bar{x}\|} = 0.$$

Function $f : X \to Y$ is said to be *strictly differentiable* [28, Definition 1.13] at $\bar{x}$ with the *strict derivative* denoted by $\nabla f(\bar{x})$ if

$$\lim_{\substack{x \to \bar{x} \\ u \to \bar{x}}} \frac{f(x) - f(u) - \nabla f(\bar{x})(x - u)}{\|x - u\|} = 0.$$

According to [28, Theorem 1.38], if $f : X \to Y$ is Fréchet differentiable at $\bar{x}$, then $\widehat{D}^*f(\bar{x})(y^*) = \{\nabla f(\bar{x})^* y^*\}$ for all $y^* \in Y^*$ with $\nabla f(\bar{x})^*$ being the *adjoint operator* of $\nabla f(\bar{x})$. Similarly, if f is strictly differentiable at $\bar{x}$ (in particular, if f is continuously Fréchet differentiable in a neighborhood of $\bar{x}$) with the strict derivative $\nabla f(\bar{x})$, then

$$D^*f(\bar{x})(y^*) = \widehat{D}^*f(\bar{x})(y^*) = \{\nabla f(\bar{x})^* y^*\},\ \ \forall y^* \in Y^*.$$

Thus the Fréchet coderivative (resp., the Mordukhovich coderivative) of multifunctions is a natural extension of the adjoint of the Fréchet derivative (resp., of the strict derivative) of a single-valued mapping.

Let $\varphi : X \to \overline{I\!R}$ be an extended-real-valued function defined on a Banach space X. If $\varphi(x) > -\infty$ for all $x \in X$ and its *effective domain*

$$\mathrm{dom}\varphi := \{x \in X |\ \varphi(x) < \infty\}$$

is nonempty, then φ is said to be a *proper* function. To φ we associate the *epigraph*

$$\text{epi}\varphi := \{(x, \alpha) \in X \times I\!R |\ \alpha \geq \varphi(x)\}.$$

Definition 1.5 (See [28, Definition 1.77 and 1.118]) Let $\varphi : X \to \overline{I\!R}$ be finite at $\bar{x} \in X$.

(i) The *limiting subdifferential* of φ at $\bar{x}$ is the set

$$\partial\varphi(\bar{x}) := \{x^* \in X^* |\ (x^*, -1) \in N\big((\bar{x}, \varphi(\bar{x})); \text{epi}\varphi\big)\},$$

and its elements are called *limiting subgradients* of φ at this point. When $\varphi(\bar{x}) = \infty$, one puts $\partial\varphi(\bar{x}) = \emptyset$.

(ii) For any $\bar{y} \in \partial\varphi(\bar{x})$, the mapping $\partial^2\varphi(\bar{x}, \bar{y}) : X^{**} \rightrightarrows X^*$ with the values

$$\partial^2\varphi(\bar{x}, \bar{y})(u) = (D^*\partial\varphi)(\bar{x}, \bar{y})(u),\ \forall u \in X^{**},$$

is said to be the *limiting second-order subdifferential* of φ at $\bar{x}$ relative to $\bar{y}$.

In a finite dimensional setting, as well as in an infinite dimensional setting, the limiting subdifferential theory has been developed successfully; see e.g. [4], [28], [48]. Meanwhile, the limiting second-order subdifferential theory still requires further investigations, although many interesting theoretical results and applications can be found in [5], [6], [30], [31], and the references therein.

For each subset $\Omega \subset X$, an extended-real-valued function $\delta(\cdot\,; \Omega) : X \to \overline{I\!R}$,

$$\delta(x; \Omega) = \begin{cases} 0 & \text{if } x \in \Omega \\ \infty & \text{if } x \notin \Omega, \end{cases}$$

is called the *indicator function* of Ω.

Proposition 1.2 (See [28, Proposition 1.79]) *Consider a nonempty subset $\Omega \subset X$. Then for any $\bar{x} \in \Omega$ one has $\partial\delta(\bar{x}; \Omega) = N(\bar{x}; \Omega)$.*

The multifunction $F : X \rightrightarrows X^*$ with $F(x) = N(x; \Omega)$ for all $x \in X$ is called the *normal cone mapping* to Ω. From Proposition 1.2 it follows that if F is the normal cone mapping to Ω and $(\bar{x}, \bar{x}^*) \in \text{gph}F$, then we have

$$D^*F(\bar{x}, \bar{x}^*)(u) = \big(D^*\partial\delta(\cdot\,; \Omega)\big)(\bar{x}, \bar{x}^*)(u) = \partial^2\delta(\cdot\,; \Omega)(\bar{x}, \bar{x}^*)(u),\ \forall u \in X^{**}.$$

The latter implies that the problem of computing the limiting second-order subdifferential of the indicator function of a set reduces to that of computing coderivatives of the normal cone mapping.

1.4 Lipschitzian Properties and Metric Regularity

Lipschitzian properties of multifunctions play a principal role in many aspects of variational analysis and its applications.

Consider a multifunction $F : X \rightrightarrows Y$ between Banach spaces X and Y. One says that F is *Lipschitz continuous* on X if there exists a constant $\ell > 0$ such that

$$F(x) \subset F(u) + \ell\|x - u\|\bar{B}_Y, \ \ \forall x, u \in X. \tag{1.11}$$

If (1.11) holds for all x, u from a neighborhood U of $\bar{x} \in X$, then F is called *locally Lipschitz* at $\bar{x}$. The multifunction F is said to be *locally upper Lipschitz* at $\bar{x} \in X$ with the modulus ℓ if there exists a neighborhood U of $\bar{x} \in X$ with

$$F(x) \subset F(\bar{x}) + \ell\|x - \bar{x}\|\bar{B}_Y, \ \ \forall x \in U.$$

The *local Lipschitz-like property* (called also the *pseudo-Lipschitz property* [1], or the *Aubin property* property [12]) of multifunctions plays a fundamental role in Variational Analysis.

Definition 1.6 (See [28, Definition 1.40]) Let $F : X \rightrightarrows Y$ with $\mathrm{dom}F \neq \emptyset$. Given $(\bar{x}, \bar{y}) \in \mathrm{gph}F$, we say that F is *locally Lipschitz-like* around $(\bar{x}, \bar{y})$ with modulus $\ell \geq 0$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x) \cap V \subset F(u) + \ell\|x - u\|\bar{B}_Y, \ \ \forall x, u \in U. \tag{1.12}$$

The infimum of all such moduli $\{\ell\}$ is called the *exact Lipschitzian bound* of F around $(\bar{x}, \bar{y})$ and is denoted by $\mathrm{lip}F(\bar{x}, \bar{y})$.

The following theorem provides us a criterion for the local Lipschitz-like property of multifunctions between finite dimensional spaces.

Theorem 1.3 (See [28, Theorem 4.10]) *Let $F : X \rightrightarrows Y$ be a multifunction between finite dimensional spaces with its graph being locally closed around $(\bar{x}, \bar{y}) \in \mathrm{gph}F$. Then the following are equivalent:*

(i) *F is locally Lipschitz-like around $(\bar{x}, \bar{y})$.*

(ii) $D^*F(\bar{x}, \bar{y})(0) = \{0\}$.

Moreover, in this case it holds $\mathrm{lip}F(\bar{x}, \bar{y}) = \|D^*F(\bar{x}, \bar{y})\| < \infty$.

Metric regularity is another important property of multifunctions; see [28], [29] for more details. This property is closely related to Lipschitzian properties of inverse mappings.

Definition 1.7 (See [28, Definition 1.47]) Let $F : X \rightrightarrows Y$ with $\text{dom}F \neq \emptyset$. We say that F is *locally metrically regular* around a given point $(\bar{x}, \bar{y}) \in \text{gph}F$ with modulus $\mu > 0$ if there exist some neighborhoods U of $\bar{x}$ and V of $\bar{y}$, and $\gamma > 0$ such that

$$\text{dist}(x; F^{-1}(y)) \leq \mu \, \text{dist}(y; F(x)) \tag{1.13}$$

for all $x \in U$ and $y \in V$ satisfying $\text{dist}(y; F(x)) \leq \gamma$. The infimum of all such moduli $\{\mu\}$, denoted by $\text{reg}F(\bar{x}, \bar{y})$, is called the *exact regularity bound* of F around $(\bar{x}, \bar{y})$.

A criterion for the metric regularity property of multifunctions between finite dimensional spaces is provided in the next theorem.

Theorem 1.4 (See [28, Theorem 4.18]) *Let $F : X \rightrightarrows Y$ be a multifunction between finite dimensional spaces with its graph being locally closed around $(\bar{x}, \bar{y}) \in \text{gph}F$. Then the following are equivalent:*

(i) *F is locally metrically regular around $(\bar{x}, \bar{y})$.*

(ii) $D^*F^{-1}(\bar{y}, \bar{x})(0) = \{0\}$.

Relationships between the local Lipschitz-like property of a multifunction F and the metric regularity property of its inverse F^{-1} are described as follows.

Theorem 1.5 (See [28, Theorem 1.49]) *Let $F : X \rightrightarrows Y$ with $\text{dom}F \neq \emptyset$. Then F is locally Lipschitz-like around $(\bar{x}, \bar{y}) \in \text{gph}F$ if and only if its inverse mapping $F^{-1} : Y \rightrightarrows X$ is locally metrically regular around $(\bar{y}, \bar{x}) \in \text{gph}F^{-1}$ with the same modulus. Moreover, the latter equivalent to the existence of neighborhoods U of $\bar{x}$, V of $\bar{y}$ and a number $\ell > 0$ such that*

$$F(x) \cap V \subset F(u) + \ell \|x - u\| \bar{B}_Y, \ \ \forall u \in U, \ \forall x \in X. \tag{1.14}$$

In this case one has the equality $\text{lip}F(\bar{x}, \bar{y}) = \text{reg}F^{-1}(\bar{y}, \bar{x})$.

In contrast to (1.12), there is no restriction on x in (1.14) because of the localization in both domain and range of F.

1.5 Conclusions

This chapter collects several basic concepts and facts on generalized differentiation, together with the well-known dual characterizations of the two fundamental properties of multifunctions: the local Lipschitz-like property and the metric regularity.

Chapter 2

Linear Perturbations of Polyhedral Normal Cone Mappings

Generalized differentiability properties of the normal cone mappings allow us to get useful information about solution sensitivity/stability of variational inequalities with polyhedral convex constraint sets; see e.g. [11], [13], [15], [25], [32], [38], [39], [40], [45], [46], [50], [52], [53]. In this chapter, we differentiate the normal cone mappings to linearly perturbed polyhedral convex sets and apply the results to solution stability of affine variational inequalities. We will answer two open questions stated by Yao and Yen in [52].

This chapter is written on the basis of the results in [38], [39], and [40].

2.1 The Normal Cone Mapping $\mathcal{F}(x, b)$

Let X be a real Banach space with the dual denoted by X^*. Consider an index set $T = \{1, 2, \ldots, m\}$, a vector system $\{\mathfrak{a}_i^* \in X^* |\ i \in T\}$, and a polyhedral convex set

$$\Theta(b) = \Big\{x \in X |\ \langle \mathfrak{a}_i^*, x\rangle \le b_i,\ \forall i \in T\Big\} \tag{2.1}$$

depending on the parameter $b = (b_1, \ldots, b_m) \in I\!R^m$. The numbers $b_1, \ldots, b_m$ are interpreted as the *right-hand side perturbations* of the linear inequality system

$$\langle \mathfrak{a}_i^*, x\rangle \le b_i,\ \ i \in T. \tag{2.2}$$

For a pair $(x, b) \in X \times I\!R^m$, we call

$$I(x, b) = \{i \in T |\ \langle \mathfrak{a}_i^*, x\rangle = b_i\} \tag{2.3}$$

the *active index set* of $\Theta(b)$ at x. For any $I \subset T$, put $\bar{I} = T \backslash I$. By b_I we denote the vector with the components b_i where $i \in I$. We will write $b_I \leq 0$ (resp., $b_I \geq 0$, $b_I = 0$) if $b_i \leq 0$ (resp., $b_i \geq 0$, $b_i = 0$) for all $i \in I$.

The multifunction $\mathcal{F} : X \times I\!\!R^m \rightrightarrows X^*$ defined by setting

$$\mathcal{F}(x, b) = N(x; \Theta(b)), \ \forall (x, b) \in X \times I\!\!R^m, \tag{2.4}$$

is said to be the *linearly perturbed polyhedral normal cone mapping* to the perturbed polyhedron $\Theta(b)$ (or, the *normal cone mapping* $\mathcal{F}(\cdot)$, for short). Here, the set

$$N(x; \Theta(b)) = \begin{cases} \{x^* \in X^* | \ \langle x^*, u - x \rangle \leq 0, \ \forall u \in \Theta(b)\}, & \text{if } x \in \Theta(b) \\ \emptyset, & \text{if } x \notin \Theta(b) \end{cases}$$

denotes the *normal cone* to $\Theta(b)$ at x in the sense of convex analysis.

It is well-known that the problem of computing the Fréchet coderivative and the Mordukhovich coderivative [28] of the normal cone mapping of a system of linear inequalities was solved by Dontchev and Rockafellar [11]. The obtained coderivative formulas were used to establish a complete characterization of the Aubin property (i.e., the local Lipschitz-like property [28]) of the solution map of parametric affine variational inequalities. Recently, extending the results of [11], Yao and Yen [52] have given some upper estimates for the Fréchet normal cone and the limiting normal cone to the graph of the normal cone mapping of a system of linear inequalities under linear perturbations. The results in [52] are applied to stability analysis of parametric variational inequalities, whose constraint sets are linearly perturbed polyhedra [53].

Further developments of the studies [11], [52], and [53] can be seen in [13], [32], [40], [38], [39]. Note that Henrion, Mordukhovich and Nam [13] have obtained exact formulas for the Fréchet and the Mordukhovich coderivatives of the normal cone mapping of a system of linear inequalities in reflexive Banach spaces. These coderivative formulas help one to study robust stability of variational inequalities in infinite dimensional spaces [13].

Nam [32] extended the results to the case where the linear inequality system in question undergoes linear perturbations. To investigate the Lipschitz stability of parametric variational inequalities in reflexive Banach spaces, Nam [32] has also established exact formulas for the Fréchet and the Mordukhovich coderivatives of $\mathcal{F}(x, b)$ under a *linear independence assumption* which is imposed on the normal vectors of the active constraints.

Relaxing and then removing that assumption of Nam [32], we will prove an exact formula for the Fréchet coderivative and several estimates for the Mordukhovich coderivative of $\mathcal{F}(x,b)$ in the next sections. In the rest of this section we provide some facts related to the normal cone mapping $\mathcal{F}(x,b)$.

Let V be a vector space over the real and let $J=\{1,2,\ldots,r\}$. For a vector system $\{v_j|\ j\in J\}\subset V$, the convex cone generated by $\{v_j|\ j\in J\}$ is denoted by $\mathrm{pos}\{v_j|\ j\in J\}$. This means that

$$\mathrm{pos}\{v_j|\ j\in J\}=\left\{\sum_{j\in J}\lambda_j v_j\,\middle|\ \lambda_j\geq 0,\ \forall j\in J\right\}.$$

In the sequel, to cover also the case $J=\emptyset$, we use the convention $\mathrm{pos}\,\emptyset=\{0\}$. The following proposition was proved in [32] by using a generalized version of the Farkas lemma [3].

Proposition 2.1 (See [32, Lemma 3.1]) *Let $\bar b\in I\!\!R^m$, $\Theta(\bar b)$ be given by* (2.1), *and let $\bar x\in\Theta(\bar b)$, $I(\bar x,\bar b)$ be defined by* (2.3). *Then*

$$\begin{aligned}N(\bar x;\Theta(\bar b))&=\mathrm{pos}\big\{\mathfrak{a}_i^*|\ i\in I(\bar x,\bar b)\big\}\\&=\left\{\sum_{i\in I(\bar x,\bar b)}\lambda_i\mathfrak{a}_i^*\,\middle|\ \lambda_i\geq 0,\ \forall i\in I(\bar x,\bar b)\right\}\end{aligned}\tag{2.5}$$

and

$$T(\bar x;\Theta(\bar b))=\Big\{v\in X|\ \langle\mathfrak{a}_i^*,v\rangle\leq 0,\ \forall i\in I(\bar x,\bar b)\Big\}.\tag{2.6}$$

From Proposition 2.1 it follows that

$$\mathcal{F}(x,b)=\mathrm{pos}\big\{\mathfrak{a}_i^*|\ i\in I(x,b)\big\},\ \forall (x,b)\in X\times I\!\!R^m.\tag{2.7}$$

We now show that Lemma 2.1 of [36] can be stated for vector systems in an arbitrary vector space. The proof is similar to that of [36].

Lemma 2.1 *Consider a vector system $\{v_j|\ j\in J\}\subset V$. For each nonzero $u\in\mathrm{pos}\{v_j|\ j\in J\}$, there exists a subset $\widetilde J\subset J$ such that $\{v_j|\ j\in\widetilde J\}$ are linearly independent and $u\in\mathrm{pos}\{v_j|\ j\in\widetilde J\}$.*

Proof. There is nothing to prove if $\{v_j|\ j\in J\}$ are linearly independent. Now, we consider the case where $\{v_j|\ j\in J\}$ are linearly dependent. Let

$$u=\sum_{j\in J}\lambda_j v_j,\quad \lambda_j\geq 0\ \text{ for }\ j\in J,$$

and $u \neq 0$. Without loss of generality we may assume that $\lambda_j > 0$ for all $j \in J$. Since $\{v_j |\ j \in J\}$ are linearly dependent, there exist $\alpha_j \in I\!R$, $j \in J$, not all zero, such that

$$\sum_{j \in J} \alpha_j v_j = 0.$$

Choose an index $j_0 \in J$ satisfying

$$\frac{|\alpha_{j_0}|}{\lambda_{j_0}} = \max\left\{ \frac{|\alpha_j|}{\lambda_j} \,\Big|\, j \in J \right\} > 0.$$

For each $j \in J$, put

$$\mu_j = \lambda_j - \frac{\lambda_{j_0}}{\alpha_{j_0}} \alpha_j.$$

We see that $\mu_{j_0} = 0$ and

$$\mu_j = \lambda_j \left(1 - \frac{\lambda_{j_0}}{\alpha_{j_0}} \cdot \frac{\alpha_j}{\lambda_j} \right) \geq \lambda_j \left(1 - \frac{\lambda_{j_0}}{|\alpha_{j_0}|} \cdot \frac{|\alpha_j|}{\lambda_j} \right) \geq 0 \quad \text{for } j \in J \backslash \{j_0\}.$$

Hence

$$\sum_{j \in J \backslash \{j_0\}} \mu_j v_j = \sum_{j \in J} \mu_j v_j = \sum_{j \in J} \lambda_j v_j - \frac{\lambda_{j_0}}{\alpha_{j_0}} \sum_{j \in J} \alpha_j v_j = \sum_{j \in J} \lambda_j v_j = u.$$

This shows that $u \in \text{pos}\big\{v_j |\ j \in J\backslash\{j_0\}\big\}$. If the vectors $\big\{v_j |\ j \in J\backslash\{j_0\}\big\}$ are linearly dependent, using again the above arguments we can find a proper subset $J' \subset J\backslash\{j_0\}$ such that $u \in \text{pos}\{v_j |\ j \in J'\}$. Since J is finite and $u \neq 0$, there must exist an index subset $\widetilde{J} \subset J$ such that $\{v_j |\ j \in \widetilde{J}\}$ are linearly independent and $u \in \text{pos}\{v_j |\ j \in \widetilde{J}\}$. □

The following lemma shows that the graph of the normal cone mapping $\mathcal{F}(\cdot)$ is closed in the product space $X \times I\!R^m \times X^*$. This property allows us to calculate the limiting normal cone $N\big((\bar{x}, \bar{b}, \bar{x}^*); \text{gph}\mathcal{F}\big)$ by formula (1.6).

Lemma 2.2 *Let $\mathcal{F}(\cdot)$ be given by* (2.4)*. Then, the graph of $\mathcal{F}(\cdot)$ is closed in the sum norm topology of the product space $X \times I\!R^m \times X^*$.*

Proof. Suppose that $\{(x_k, b_k, x_k^*)\}_{k \in I\!N} \subset \text{gph}\mathcal{F}$ and $(x_k, b_k, x_k^*) \to (\bar{x}, \bar{b}, \bar{x}^*)$. We have to show that $(\bar{x}, \bar{b}, \bar{x}^*) \in \text{gph}\mathcal{F}$. If $\bar{x}^* = 0$, then $(\bar{x}, \bar{b}, \bar{x}^*) \in \text{gph}\mathcal{F}$ because $0 \in N(\bar{x}, \Theta(\bar{b})) = \mathcal{F}(\bar{x}, \bar{b})$. We now consider the case that $\bar{x}^* \neq 0$. We have $\bar{x}^* \neq 0$ and $x_k^* \to \bar{x}^*$, so $x_k^* \neq 0$ for all k large enough. Since $(x_k, b_k) \to (\bar{x}, \bar{b})$, we have $I(x_k, b_k) \subset I(\bar{x}, \bar{b})$ for sufficiently large indexes $k \in I\!N$. Thus, without loss of generality we may assume that $x_k^* \neq 0$ and

$I(x_k, b_k) = \widetilde{I} \subset I(\bar{x}, \bar{b})$ for all $k \in I\!N$. For each $k \in I\!N$, since $x_k^* \in \mathcal{F}(x_k, b_k)$, by (2.7) it holds

$$x_k^* \in \text{pos}\{\mathfrak{a}_i^* |\ i \in I(x_k, b_k)\} = \text{pos}\{\mathfrak{a}_i^* |\ i \in \widetilde{I}\}.$$

As finitely generated convex cones are closed and $x_k^* \to \bar{x}^*$, it follows that

$$\bar{x}^* \in \text{pos}\{\mathfrak{a}_i^* |\ i \in \widetilde{I}\} \subset \text{pos}\{\mathfrak{a}_i^* |\ i \in I(\bar{x}, \bar{b})\} = \mathcal{F}(\bar{x}, \bar{b}).$$

Hence, $(\bar{x}, \bar{b}, \bar{x}^*) \in \text{gph}\mathcal{F}$. We have shown that $\text{gph}\mathcal{F}$ is closed. □

2.2 The Fréchet Coderivative of $\mathcal{F}(x, b)$

Let $(\bar{x}, \bar{b}, \bar{x}^*) \in \text{gph}\mathcal{F}$ with $\mathcal{F}$ being defined by (2.4). For simplicity, we will write I for $I(\bar{x}, \bar{b})$. Since $\bar{x}^* \in \mathcal{F}(\bar{x}, \bar{b})$, from (2.7) it follows that

$$\bar{x}^* = \sum_{i \in I(\bar{x}, \bar{b})} \lambda_i \mathfrak{a}_i^*, \quad \text{for some } \lambda_i \geq 0,\ i \in I(\bar{x}, \bar{b}).$$

We define respectively the multiplier set and the index set corresponding to the point $(\bar{x}, \bar{b}, \bar{x}^*)$ as follows

$$\Xi(\bar{x}, \bar{b}, \bar{x}^*) = \Big\{ (\lambda_i)_{i \in I} \in I\!R^{|I|} \Big|\ \bar{x}^* = \sum_{i \in I} \lambda_i \mathfrak{a}_i^*,\ \ \lambda_i \geq 0\ \ \forall i \in I \Big\}, \tag{2.8}$$

$$I_1(\bar{x}, \bar{b}, \bar{x}^*) = \Big\{ i \in I \Big|\ \lambda_i = 0 \text{ for some } (\lambda_j)_{j \in I} \in \Xi(\bar{x}, \bar{b}, \bar{x}^*) \Big\}, \tag{2.9}$$

and construct the sets

$$\begin{aligned} H(\bar{x}, \bar{b}, \bar{x}^*) = \Big\{ (x^*, b^*, v) \Big|\ & x^* \in \big(T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp \big)^*, \\ & v\ \in\ T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp, \\ & x^* = -\textstyle\sum_{i \in I} b_i^* \mathfrak{a}_i^*,\ b_{\bar{I}}^* = 0,\ b_{I_1}^* \leq 0 \Big\}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} E_0(\bar{x}, \bar{b}, \bar{x}^*) = \Big\{ (x^*, b^*, v) \Big|\ & x^* \in \big(T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp \big)^*, \\ & v\ \in\ T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp, \\ & x^* = -\textstyle\sum_{i \in I} b_i^* a_i^*,\ b_{\bar{I}}^* = 0,\ b_I^* \leq 0 \Big\}, \end{aligned} \tag{2.11}$$

where $b^* = (b_1^*, \ldots, b_m^*) \in I\!R^m$ and $\bar{I} = T \setminus I$. For each $\lambda = (\lambda_i)_{i \in I} \in \Xi(\bar{x}, \bar{b}, \bar{x}^*)$, we consider

$$I_0(\lambda) = \big\{ i \in I |\ \lambda_i = 0 \big\}, \tag{2.12}$$

and

$$E_\lambda(\bar{x}, \bar{b}, \bar{x}^*) = \left\{ (x^*, b^*, v) \middle|\ \begin{array}{l} x^* \in \big(T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp\big)^*, \\ v \ \in \ T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp, \\ x^* = -\sum_{i \in I} b_i^* \mathfrak{a}_i^*, \ b_{\bar{I}}^* = 0, \ b_{I_0(\lambda)}^* \le 0 \end{array} \right\}. \tag{2.13}$$

Lemma 2.3 *For any $(\bar{x}, \bar{b}, \bar{x}^*) \in \mathrm{gph}\mathcal{F}$, we have*

$$H(\bar{x}, \bar{b}, \bar{x}^*) = \bigcap_{\lambda \in \Xi(\bar{x}, \bar{b}, \bar{x}^*)} E_\lambda(\bar{x}, \bar{b}, \bar{x}^*). \tag{2.14}$$

Proof. Pick any $(x^*, b^*, v) \in H(\bar{x}, \bar{b}, \bar{x}^*)$. For each $\lambda \in \Xi(\bar{x}, \bar{b}, \bar{x}^*)$, we have $(x^*, b^*, v) \in E_\lambda(\bar{x}, \bar{b}, \bar{x}^*)$ whenever $b_{I_0(\lambda)}^* \le 0$. Since $(x^*, b^*, v) \in H(\bar{x}, \bar{b}, \bar{x}^*)$, $b_{I_1}^* \le 0$. We have $I_0(\lambda) \subset I_1$, so $b_{I_0(\lambda)}^* \le 0$. Thus, $H(\bar{x}, \bar{b}, \bar{x}^*) \subset E_\lambda(\bar{x}, \bar{b}, \bar{x}^*)$ for all $\lambda \in \Xi(\bar{x}, \bar{b}, \bar{x}^*)$ which justifies the inclusion " $\subset$ " in (2.14). Now, fix any (x^*, b^*, v) in the set on the right-hand side of (2.14). Clearly, one has $(x^*, b^*, v) \in H(\bar{x}, \bar{b}, \bar{x}^*)$ if $b_{I_1}^* \le 0$. For each $i \in I_1$, we have $\lambda_i = 0$ for some $\lambda = (\lambda_j)_{j \in I} \in \Xi(\bar{x}, \bar{b}, \bar{x}^*)$. Hence, $i \in I_0(\lambda)$. Since $(x^*, b^*, v) \in E_\lambda(\bar{x}, \bar{b}, \bar{x}^*)$, it holds $b_{I_0(\lambda)}^* \le 0$. Consequently, $b_i^* \le 0$ which implies $b_{I_1}^* \le 0$. □

The inclusion (2.15) below is proved in Proposition 3.2 of [32] without using any regularity assumption. Under an auxiliary assumption, the inclusion will become an equality.

Theorem 2.1 (See [32, Proposition 3.2]) *Let $(\bar{x}, \bar{b}, \bar{x}^*) \in \mathrm{gph}\mathcal{F}$. Then, for every $\lambda \in \Xi(\bar{x}, \bar{b}, \bar{x}^*)$, the following inclusion holds*

$$\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big) \subset E_\lambda(\bar{x}, \bar{b}, \bar{x}^*), \tag{2.15}$$

where $E_\lambda(\bar{x}, \bar{b}, \bar{x}^*)$ is given by (2.13). Besides, if the vectors $\{\mathfrak{a}_i^* |\ i \in I\}$ are linearly independent, then $\Xi(\bar{x}, \bar{b}, \bar{x}^*)$ has only one element $\bar{\lambda}$ and

$$\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big) = E_{\bar{\lambda}}(\bar{x}, \bar{b}, \bar{x}^*). \tag{2.16}$$

For the case $X = I\!R^n$, the upper estimate (2.15) was obtained in [52]. Concerning the above mentioned upper estimate for the Fréchet normal cone given in [52], an open question was stated in the same paper. In our notation, Question 1 of [52] can be restated as follows.

Question 2.1 *For any* $\lambda \in \Xi(\bar{x}, \bar{b}, \bar{x}^*)$, *does the inclusion* (2.15) *always hold as an equality?*

Definition 2.1 Let $\{v_j\}_{j\in J}$ be a family of finitely many vectors of a vector space V over the reals. We say that $\{v_j\}_{j\in J}$ is *positively linearly independent* if from the conditions $\sum_{j\in J} \lambda_j v_j = 0$ and $\lambda_j \geq 0$ for all $j \in J$ it follows that $\lambda_j = 0$ for all $j \in J$.

The second assertion of Theorem 2.1 solves Question 2.1 in the affirmative under the condition that the vectors $\{\mathfrak{a}_i^* |\ i \in I(\bar{x}, \bar{b})\}$ are linearly independent. We are going to show that, in general, the inclusion (2.15) does not hold as an equality. Moreover, even under positive linear independence assumption of $\{\mathfrak{a}_i^* |\ i \in I(\bar{x}, \bar{b})\}$, the inclusion (2.15) is also not an equality.

Proposition 2.2 *The inclusion* (2.15) *may be strict in some cases.*

Proof. Let $X = I\!R^2$, $\mathfrak{a}_1^* = (0,1)$, $\mathfrak{a}_2^* = (0,2) \in X^* = I\!R^2$, $\bar{b} = (0,0) \in I\!R^2$, and $\bar{x} = (0,0) \in X$. We have

$$\begin{aligned}
\Theta(\bar{b}) &= \Big\{x \in I\!R^2 |\ \langle \mathfrak{a}_i^*, x\rangle \leq 0,\ i = 1,2\Big\} = I\!R \times (-\infty, 0],\\
I(\bar{x}, \bar{b}) &= \{i\,|\, \langle \mathfrak{a}_i^*, \bar{x}\rangle = 0\} = \{1,2\},\\
\mathcal{F}(\bar{x}, \bar{b}) &= N(\bar{x}; \Theta(\bar{b})) = \mathrm{pos}\{\mathfrak{a}_1^*, \mathfrak{a}_2^*\} = \Big\{\lambda_1\mathfrak{a}_1^* + \lambda_2\mathfrak{a}_2^* |\ \lambda_i \geq 0,\ i = 1,2\Big\}\\
&= \{0\} \times [0, +\infty),\\
T(\bar{x}; \Theta(\bar{b})) &= \big(N(\bar{x}; \Theta(\bar{b}))\big)^* = \Big\{v \in I\!R^2 |\ \langle \mathfrak{a}_i^*, v\rangle \leq 0,\ i = 1,2\Big\}\\
&= I\!R \times (-\infty, 0].
\end{aligned}$$

For any $\alpha > 0$, put $\bar{x}^* = (0, \alpha)$. Since $\bar{x}^* \in \{0\} \times [0, +\infty) = \mathcal{F}(\bar{x}, \bar{b})$, it holds $(\bar{x}, \bar{b}, \bar{x}^*) \in \mathrm{gph}\mathcal{F}$. We observe that

$$\begin{aligned}
&\{\bar{x}^*\}^\perp = \{(0,\alpha)\}^\perp = I\!R \times \{0\},\\
&T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp = I\!R \times \{0\},\\
&\big(T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp\big)^* = \{0\} \times I\!R.
\end{aligned}$$

If we use the representation

$$\bar{x}^* = (0, \alpha) = \alpha\mathfrak{a}_1^* + 0\mathfrak{a}_2^*,$$

then $\lambda = (\alpha, 0)$ and $I_0(\lambda) = \{2\}$. Choose $b^* = (b_1^*, b_2^*) \in I\!R^2$, where $b_2^* \le 0$ and the value b_1^* will be determined later. For any $\gamma \in I\!R$, we have

$$v = (\gamma, 0) \in T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp.$$

Put

$$x^* = -(b_1^*\mathfrak{a}_1^* + b_2^*\mathfrak{a}_2^*) = (0, -b_1^* - 2b_2^*) \in \big(T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp\big)^*.$$

Hence, $(x^*, b^*, v) \in E_\lambda(\bar{x}, \bar{b}, \bar{x}^*)$.

We now show that (x^*, b^*, v) does not belong to the cone $\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big)$. Consider the sequences $\{b^k\}$ and $\{x^k\}$, where $b^k = (\frac{1}{k}, \frac{1}{k})$ and $x^k = (\frac{1}{2k}, \frac{1}{2k})$, $k \in I\!N$. We see that $b^k \to \bar{b}$ and $x^k \to \bar{x}$ as $k \to \infty$. For every $k \in I\!N$, it holds that

$$\Theta(b^k) = \left\{(x_1, x_2) \in I\!R^2 \,\middle|\, x_2 \le \frac{1}{k},\ 2x_2 \le \frac{1}{k}\right\} = I\!R \times \left(-\infty, \frac{1}{2k}\right].$$

Then, $x^k \in \Theta(b^k)$, $I(x^k, b^k) = \{2\}$, and

$$\mathcal{F}(x^k, b^k) = N(x^k; \Theta(b^k)) = \mathrm{pos}\{\mathfrak{a}_2^*\} = \{0\} \times [0, +\infty),$$

for all $k \in I\!N$. Now, consider the sequence $\{u_k^*\}_{k \in I\!N}$ where $u_k^* = (0, \alpha + \frac{1}{2k})$ for all $k \in I\!N$. For every $k \in I\!N$, we have $u_k^* \in \mathcal{F}(x^k, b^k)$, and $u_k^* \to (0, \alpha) = \bar{x}^*$ as $k \to \infty$. Note that

$$\begin{aligned}
&\limsup_{(x,b,u^*) \xrightarrow{\mathrm{gph}\mathcal{F}} (\bar{x},\bar{b},\bar{x}^*)} \frac{\langle x^*, x - \bar{x}\rangle + \langle b^*, b - \bar{b}\rangle + \langle v, u^* - \bar{x}^*\rangle}{\|x - \bar{x}\| + \|b - \bar{b}\| + \|u^* - \bar{x}^*\|} \\
&\ge \limsup_{k \to \infty} \frac{\langle x^*, x^k - \bar{x}\rangle + \langle b^*, b^k - \bar{b}\rangle + \langle v, u_k^* - \bar{x}^*\rangle}{\|x^k - \bar{x}\| + \|b^k - \bar{b}\| + \|u_k^* - \bar{x}^*\|} \\
&= \limsup_{k \to \infty} \frac{\big\langle (0, -b_1^* - 2b_2^*), (\frac{1}{2k}, \frac{1}{2k})\big\rangle + \big\langle (b_1^*, b_2^*), (\frac{1}{k}, \frac{1}{k})\big\rangle + \big\langle (\gamma, 0), (0, \frac{1}{2k})\big\rangle}{\|(\frac{1}{2k}, \frac{1}{2k})\| + \|(\frac{1}{k}, \frac{1}{k})\| + \|(0, \frac{1}{2k})\|} \\
&= \lim_{k \to \infty} \frac{\frac{-b_1^* - 2b_2^*}{2k} + \frac{b_1^* + b_2^*}{k}}{\frac{\sqrt{2}}{2k} + \frac{\sqrt{2}}{k} + \frac{1}{2k}} = \frac{b_1^*}{\sqrt{2} + 2\sqrt{2} + 1} =: \mu.
\end{aligned}$$

Choosing $b_1^* = 1$, we have

$$\mu = \frac{1}{\sqrt{2} + 2\sqrt{2} + 1} > 0.$$

This implies that $(x^*, b^*, v) \notin \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big)$, and hence

$$E_\lambda(\bar{x}, \bar{b}, \bar{x}^*) \not\subset \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big).$$

The proof is complete. □

Remark 2.1 We have seen that without the linear independence condition of $\{\mathfrak{a}_i^*|\ i \in I(\bar{x}, \bar{b})\}$ the equality (2.16) may fail to hold. The vectors $\{\mathfrak{a}_1^*, \mathfrak{a}_2^*\}$ in the above proof are not linearly independent, but they are positively linearly independent. We have thus shown that *the inclusion* (2.15) *may be strict even in the case the vectors* $\{\mathfrak{a}_i^*|\ i \in I(\bar{x}, \bar{b})\}$ *are positively linearly independent.*

Remark 2.2 As usual, we say that the inequality system (2.2) satisfies the *Slater condition* if there exists $x^0 \in X$ with $\langle \mathfrak{a}_i^*, x^0\rangle < b_i$ for all $i \in T$. This condition is a significant sign of the stability of the given inequality system. One may hope that the equality (2.16) holds when the Slater condition is satisfied. Nevertheless, the proof of Proposition 2.2 overturns the hope. Indeed, taking $x^0 = (0, -1)$ we have $\langle \mathfrak{a}_i^*, x^0\rangle < \bar{b}_i$ for $i = 1, 2$ but, as shown in the proof, $E_\lambda(\bar{x}, \bar{b}, \bar{x}^*) \not\subset \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big)$.

Using the set $E_0(\bar{x}, \bar{b}, \bar{x}^*)$ defined by (2.11), we now provide a lower estimate for the Fréchet normal cone $\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big)$. Our result is an extension of [52, Lemma 4.2] where it was assumed that $X = I\!R^n$.

Theorem 2.2 *If* $(\bar{x}, \bar{b}, \bar{x}^*) \in \mathrm{gph}\mathcal{F}$, *then the following inclusion holds*

$$E_0(\bar{x}, \bar{b}, \bar{x}^*) \subset \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big). \tag{2.17}$$

Proof. Let $(x^*, b^*, v) \in E_0(\bar{x}, \bar{b}, \bar{x}^*)$. In order to show that

$$(x^*, b^*, v) \in \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big),$$

we need to verify the inequality

$$\limsup_{(x,b,u^*)\xrightarrow{\mathrm{gph}\mathcal{F}}(\bar{x},\bar{b},\bar{x}^*)} \frac{\langle x^*, x - \bar{x}\rangle + \langle b^*, b - \bar{b}\rangle + \langle v, u^* - \bar{x}^*\rangle}{\|x - \bar{x}\| + \|b - \bar{b}\| + \|u^* - \bar{x}^*\|} \le 0. \tag{2.18}$$

Let there be given any sequence $(x_k, b_k, u_k^*) \xrightarrow{\mathrm{gph}\mathcal{F}} (\bar{x}, \bar{b}, \bar{x}^*)$. Then, because $(x_k, b_k) \to (\bar{x}, \bar{b})$, we must have $I(x_k, b_k) \subset I(\bar{x}, \bar{b})$ for all k sufficiently large. We put $I = I(\bar{x}, \bar{b})$ for short. Since

$$u_k^* \in \mathrm{pos}\big\{\mathfrak{a}_i^*|\ i \in I(x_k, b_k)\big\} \subset \mathrm{pos}\big\{\mathfrak{a}_i^*|\ i \in I\big\} = N(\bar{x}; \Theta(\bar{b})) = \mathcal{F}(\bar{x}, \bar{b}),$$

the condition $v \in T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp$ implies that

$$\langle v, u_k^* - \bar{x}^*\rangle = \langle v, u_k^*\rangle \le 0. \tag{2.19}$$

From the equalities $x^* = -\sum_{i\in I} b_i^* \mathfrak{a}_i^*$ and $b_{\bar{I}}^* = 0$ we deduce that

$$\begin{aligned}\langle x^*, x_k - \bar{x}\rangle + \langle b^*, b_k - \bar{b}\rangle &= \Big\langle -\sum_{i\in I} b_i^* \mathfrak{a}_i^*, x_k - \bar{x}\Big\rangle + \langle b_I^*, b_k - \bar{b}\rangle \\ &= \sum_{i\in I} b_i^*\Big(\langle \mathfrak{a}_i^*, \bar{x}\rangle - \langle \mathfrak{a}_i^*, x_k\rangle\Big) + \sum_{i\in I} b_i^*\Big((b_k)_i - \bar{b}_i\Big) \\ &= \sum_{i\in I} b_i^*\Big(\langle \mathfrak{a}_i^*, \bar{x}\rangle - \bar{b}_i + (b_k)_i - \langle \mathfrak{a}_i^*, x_k\rangle\Big) \\ &= \sum_{i\in I} b_i^*\Big((b_k)_i - \langle \mathfrak{a}_i^*, x_k\rangle\Big).\end{aligned}$$

Since $b_I^* \le 0$ and $\langle \mathfrak{a}_i^*, x_k\rangle \le (b_k)_i$ for all $i \in I$, this implies that

$$\langle x^*, x_k - \bar{x}\rangle + \langle b^*, b_k - \bar{b}\rangle \le 0. \tag{2.20}$$

From (2.19) and (2.20) we obtain

$$\limsup_{k\to\infty} \frac{\langle x^*, x_k - \bar{x}\rangle + \langle b^*, b_k - \bar{b}\rangle + \langle v, u_k^* - \bar{x}^*\rangle}{\|x_k - \bar{x}\| + \|b_k - \bar{b}\| + \|u_k^* - \bar{x}^*\|} \le 0,$$

which yields (2.18) because the sequence $(x_k, b_k, u_k^*) \xrightarrow{\text{gph}\mathcal{F}} (\bar{x}, \bar{b}, \bar{x}^*)$ was given arbitrarily. The proof is complete. $\square$

Our next goal is to solve the following question which is the second open question of Yao and Yen in [52], where X is a finite dimensional Euclidean space.

Question 2.2 *Does the inclusion* $(x^*, b^*, v) \in \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \text{gph}\mathcal{F}\big)$ *imply* $b_I^* \le 0$ *where* $I = I(\bar{x}, \bar{b})$*? In other words, does* (2.17) *always hold as an equality?*

Proposition 2.3 *The inclusion* (2.17) *may be strict in some cases.*

Proof. Let $X = I\!R^2$, $\mathfrak{a}_1^* = (1, 0), \mathfrak{a}_2^* = (0, 1) \in X^* = I\!R^2$, $\bar{b} = (0, 0) \in I\!R^2$, and $\bar{x} = (0, 0) \in X$. We observe that

$$\begin{aligned}\Theta(\bar{b}) &= \Big\{x \in I\!R^2 \mid \langle \mathfrak{a}_i^*, x\rangle \le 0,\ i = 1, 2\Big\} = (-\infty, 0] \times (-\infty, 0], \\ I(\bar{x}, \bar{b}) &= \Big\{i \mid \langle \mathfrak{a}_i^*, \bar{x}\rangle = 0\Big\} = \{1, 2\}, \\ \mathcal{F}(\bar{x}, \bar{b}) &= N(\bar{x}; \Theta(\bar{b})) = \text{pos}\{\mathfrak{a}_1^*, \mathfrak{a}_2^*\} = [0, +\infty) \times [0, +\infty).\end{aligned}$$

Given an arbitrary $\alpha > 0$, we have $\bar{x}^* = (0, \alpha) \in \mathcal{F}(\bar{x}, \bar{b})$. This means that $(\bar{x}, \bar{b}, \bar{x}^*) \in \text{gph}\mathcal{F}$. We want to find a triplet $(x^*, b^*, v) \in \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \text{gph}\mathcal{F}\big)$

with $b_I^* \not\leq 0$, i.e., there exists $i \in I = I(\bar{x}, \bar{b})$ such that $b_i^* > 0$. Note that

$$\begin{aligned}
&\{\bar{x}^*\}^\perp = \{(0,\alpha)\}^\perp = I\!R \times \{0\},\\
&T(\bar{x};\Theta(\bar{b})) = (-\infty,0] \times (-\infty,0],\\
&T(\bar{x};\Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp = (-\infty,0] \times \{0\},\\
&\big(T(\bar{x};\Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp\big)^* = [0,+\infty) \times I\!R.
\end{aligned}$$

We have $\bar{x}^* = (0,\alpha) = 0\mathfrak{a}_1^* + \alpha\mathfrak{a}_2^*$. Hence, $I_0(\bar{\lambda}) = \{1\}$, where $\bar{\lambda} = (0,\alpha)$. Observe that $\{\mathfrak{a}_1^*, \mathfrak{a}_2^*\}$ are linearly independent, thus $\Xi(\bar{x}, \bar{b}, \bar{x}^*) = \{\bar{\lambda}\}$. Choose

$$x^* = (1,-1) \in \big(T(\bar{x};\Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp\big)^*,$$

$b^* = (b_1^*, b_2^*)$ with $b_1^* = -1 \leq 0$, $b_2^* = 1$, and

$$v = (\gamma, 0) \in T(\bar{x};\Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp,$$

where $\gamma \leq 0$. By the choice of (x^*, b^*, v) above, we have the representation $x^* = -(b_1^*\mathfrak{a}_1^* + b_2^*\mathfrak{a}_2^*)$ and $b^*_{I_0(\bar{\lambda})} \leq 0$. According to Theorem 2.1, we can infer that $(x^*, b^*, v) \in \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big)$. Since $b_2^* = 1 > 0$ and $2 \in I \backslash I_0(\bar{\lambda})$, we have shown that the inclusion $(x^*, b^*, v) \in \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big)$ does not imply $b_I^* \leq 0$. The proof is complete. □

The following fact plays an important role in our subsequent investigations.

Theorem 2.3 (See [38, Theorem 3.2]) *Let $(\bar{x}, \bar{b}, \bar{x}^*) \in \mathrm{gph}\mathcal{F}$ and $I = I(\bar{x}, \bar{b})$. If the vectors $\{\mathfrak{a}_i^* |\ i \in I\}$ are positively linearly independent, then*

$$\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big) = H(\bar{x}, \bar{b}, \bar{x}^*), \tag{2.21}$$

where $H(\bar{x}, \bar{b}, \bar{x}^*)$ is defined by (2.10).

Let us prove that (2.21) holds without any additional assumption. Thus, the positive linear independence assumption on the vectors $\{\mathfrak{a}_i^* |\ i \in I\}$ can be removed from the formulation of Theorem 2.3

Theorem 2.4 *For any $(\bar{x}, \bar{b}, \bar{x}^*) \in \mathrm{gph}\mathcal{F}$, the equality* (2.21) *is valid.*

Proof. First, we prove the inclusion " $\subset$ " in (2.21). For every $\lambda \in \Xi(\bar{x}, \bar{b}, \bar{x}^*)$, by Theorem 2.1 we deduce that $\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big) \subset E_\lambda(\bar{x}, \bar{b}, \bar{x}^*)$. Hence,

$$\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big) \subset \bigcap_{\lambda \in \Xi(\bar{x}, \bar{b}, \bar{x}^*)} E_\lambda(\bar{x}, \bar{b}, \bar{x}^*).$$

From this and Lemma 2.3 we get $\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big) \subset H(\bar{x}, \bar{b}, \bar{x}^*)$.

To justify the opposite inclusion in (2.21), fix any $(x^*, b^*, v) \in H(\bar{x}, \bar{b}, \bar{x}^*)$. We have to show that $(x^*, b^*, v) \in \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big)$. We suppose to the contrary that $(x^*, b^*, v) \notin \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big)$. Then, by the definition of the Fréchet normal cone $\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big)$, there exist $\delta > 0$ and a sequence $(x_k, b_k, x_k^*) \xrightarrow{\mathrm{gph}\mathcal{F}} (\bar{x}, \bar{b}, \bar{x}^*)$ such that

$$\frac{\langle x^*, x_k - \bar{x}\rangle + \langle b^*, b_k - \bar{b}\rangle + \langle v, x_k^* - \bar{x}^*\rangle}{\|x_k - \bar{x}\| + \|b_k - \bar{b}\| + \|x_k^* - \bar{x}^*\|} \geq \delta > 0, \quad \forall k \in I\!N. \tag{2.22}$$

Since $I(x_k, b_k) \subset T$ for all $k \in I\!N$ and $(x_k, b_k) \to (\bar{x}, \bar{b})$, we may assume that $I(x_k, b_k) = \widetilde{I} \subset I$ for all $k \in I\!N$. This implies that

$$x_k^* \in N(x_k, \Theta(b_k)) \subset N(\bar{x}, \Theta(\bar{b})), \quad \forall k \in I\!N.$$

Hence, by the inclusion $(x^*, b^*, v) \in H(\bar{x}, \bar{b}, \bar{x}^*)$ and by (2.10),

$$\langle v, x_k^* - \bar{x}^*\rangle = \langle v, x_k^*\rangle \leq 0. \tag{2.23}$$

Besides, the relations $x^* = -\sum_{i\in I} b_i^* \mathfrak{a}_i^*$ and $b_{\bar{I}}^* = 0$ imply that

$$\begin{aligned}
&\langle x^*, x_k - \bar{x}\rangle + \langle b^*, b_k - \bar{b}\rangle \\
&\quad= \sum_{i\in I} b_i^*\Big(\langle \mathfrak{a}_i^*, \bar{x}\rangle - \langle \mathfrak{a}_i^*, x_k\rangle\Big) + \sum_{i\in I} b_i^*\Big((b_k)_i - \bar{b}_i\Big) \\
&\quad= \sum_{i\in I} b_i^*\Big((b_k)_i - \langle \mathfrak{a}_i^*, x_k\rangle\Big) \\
&\quad= \sum_{i\in I_1} b_i^*\Big((b_k)_i - \langle \mathfrak{a}_i^*, x_k\rangle\Big) + \sum_{i\in I\setminus I_1} b_i^*\Big((b_k)_i - \langle \mathfrak{a}_i^*, x_k\rangle\Big).
\end{aligned} \tag{2.24}$$

Since $b_i^* \leq 0$ and $\langle \mathfrak{a}_i^*, x_k\rangle \leq (b_k)_i$ for all $i \in I_1$, we have

$$\sum_{i\in I_1} b_i^*\Big((b_k)_i - \langle \mathfrak{a}_i^*, x_k\rangle\Big) \leq 0. \tag{2.25}$$

Let us show that $\bar{x}^* \in \mathrm{pos}\{\mathfrak{a}_i^* |\ i \in \widetilde{I}\}$. If $\bar{x}^* = 0$ then it is obvious that $\bar{x}^* \in \mathrm{pos}\{\mathfrak{a}_i^* |\ i \in \widetilde{I}\}$. Suppose now that $\bar{x}^* \neq 0$. Since $x_k^* \to \bar{x}^*$, there exists $k_0 > 0$ such that $x_k^* \neq 0$ for all $k \geq k_0$. For every $k \in I\!N$, the equality $I(x_k, b_k) = \widetilde{I}$ together with inclusion $x_k^* \in \mathcal{F}(x_k, b_k) = N(x_k; \Theta(b_k))$ and formula (2.5) yield $x_k^* \in \mathrm{pos}\{\mathfrak{a}_i^* |\ i \in \widetilde{I}\}$. By Lemma 2.1, one can find $J_k \subset \widetilde{I}$ such that $\{\mathfrak{a}_i^* |\ i \in J_k\}$ are linearly independent and $x_k^* \in \mathrm{pos}\{\mathfrak{a}_i^* |\ i \in J_k\}$. Since $\widetilde{I}$ is finite, the set below is finite

$$\Gamma := \Big\{J \subset \widetilde{I} \,\big|\, \mathfrak{a}_i^*,\ i \in J, \text{ are linearly independent}\Big\}.$$

Consequently, there must exist $\widetilde{J} \in \Gamma$ and a subsequence $\{k_\ell\}$ of $\{k\}$ such that $x^*_{k_\ell} \in \mathrm{pos}\{\mathfrak{a}^*_i |\ i \in \widetilde{J}\}$ for all $\ell \in I\!N$. This means that

$$x^*_{k_\ell} = \sum_{i \in \widetilde{J}} \lambda_i^{k_\ell} \mathfrak{a}^*_i \quad \text{for some } \lambda_i^{k_\ell} \geq 0,\ i \in \widetilde{J}. \tag{2.26}$$

Combining (2.26) with the linear independence of $\{\mathfrak{a}^*_i |\ i \in \widetilde{J}\}$ we infer that

$$\bar{x}^* = \lim_{\ell \to \infty} x^*_{k_\ell} = \lim_{\ell \to \infty} \sum_{i \in \widetilde{J}} \lambda_i^{k_\ell} \mathfrak{a}^*_i = \sum_{i \in \widetilde{J}} \Big(\lim_{\ell \to \infty} \lambda_i^{k_\ell} \Big) \mathfrak{a}^*_i = \sum_{i \in \widetilde{J}} \lambda_i \mathfrak{a}^*_i,$$

where $\lambda_i := \lim_{\ell \to \infty} \lambda_i^{k_\ell} \geq 0$ for all $i \in \widetilde{J}$. Thus,

$$\bar{x}^* \in \mathrm{pos}\{\mathfrak{a}^*_i |\ i \in \widetilde{J}\} \subset \mathrm{pos}\{\mathfrak{a}^*_i |\ i \in \widetilde{I}\}.$$

We have $I \backslash I_1 = \widetilde{I} \backslash I_1$. Indeed, since $\widetilde{I} \subset I$, it holds $\widetilde{I} \backslash I_1 \subset I \backslash I_1$. Conversely, by the inclusion $\widetilde{J} \subset \widetilde{I}$ we have

$$\bar{x}^* = \sum_{i \in \widetilde{J}} \lambda_i \mathfrak{a}^*_i = \sum_{i \in \widetilde{I}} \lambda_i \mathfrak{a}^*_i \tag{2.27}$$

provided that we put $\lambda_i = 0$ for all $i \in \widetilde{I} \backslash \widetilde{J}$. By (2.27) and definition of $I_1 = I_1(\bar{x}, \bar{b}, \bar{x}^*)$ in (2.9), we see that $I \backslash \widetilde{I} \subset I_1$. Since $\widetilde{I} \subset I$, this implies that $I \backslash I_1 \subset I \backslash (I \backslash \widetilde{I}) = \widetilde{I}$. Hence $I \backslash I_1 \subset \widetilde{I} \backslash I_1$. The equality $I \backslash I_1 = \widetilde{I} \backslash I_1$ has been proved. Furthermore, we have $\langle \mathfrak{a}^*_i, x_k \rangle = (b_k)_i$ for any $k \in I\!N$ and $i \in I(x_k, b_k) = \widetilde{I}$. Thus,

$$\sum_{i \in I \backslash I_1} b^*_i \Big((b_k)_i - \langle \mathfrak{a}^*_i, x_k \rangle \Big) = \sum_{i \in \widetilde{I} \backslash I_1} b^*_i \Big((b_k)_i - \langle \mathfrak{a}^*_i, x_k \rangle \Big) = 0. \tag{2.28}$$

From (2.24), (2.25) and (2.28), it follows that

$$\langle x^*, x_k - \bar{x} \rangle + \langle b^*, b_k - \bar{b} \rangle \leq 0.$$

Combining the latter with (2.23), we obtain

$$\frac{\langle x^*, x_k - \bar{x} \rangle + \langle b^*, b_k - \bar{b} \rangle + \langle v, x^*_k - \bar{x}^* \rangle}{\|x_k - \bar{x}\| + \|b_k - \bar{b}\| + \|x^*_k - \bar{x}^*\|} \leq 0, \quad \forall k \in I\!N.$$

This contradicts (2.22). Therefore, we have $(x^*, b^*, v) \in \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big)$. The equality (2.21) has been established. □

Let us consider an example to see how Theorem 2.4 can be used for practical computation of the Fréchet normal cone $\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big)$.

Example 2.1 Let $X = I\!R^2$ and let $\{\mathfrak{a}_i^* |\ i \in T\} \subset X^*$ where $T = \{1, 2, 3\}$, and

$$\mathfrak{a}_1^* = (1, 0),\ \ \mathfrak{a}_2^* = (0, 1),\ \ \mathfrak{a}_3^* = (1, 2).$$

For $\bar{b} = (0, 0, 0) \in I\!R^3,\ \bar{x} = (0, 0) \in X$, we have

$$\begin{aligned}
\Theta(\bar{b}) \ &= \big\{x \in I\!R^2 |\ \langle \mathfrak{a}_i^*, x\rangle \le 0,\ \ i \in T\big\} = (-\infty, 0] \times (-\infty, 0],\\
I(\bar{x}, \bar{b}) \ &= \big\{i\,|\ \langle \mathfrak{a}_i^*, \bar{x}\rangle = \bar{b}_i\big\} = \{1, 2, 3\},\\
\mathcal{F}(\bar{x}; \bar{b}) \ &= N(\bar{x}; \Theta(\bar{b})) = \mathrm{pos}\{\mathfrak{a}_1^*, \mathfrak{a}_2^*, \mathfrak{a}_3^*\} = [0, +\infty) \times [0, +\infty).
\end{aligned}$$

For $\alpha > 0$, one has $\bar{x}^* = (0, \alpha) \in \mathcal{F}(\bar{x}; \bar{b})$, thus $(\bar{x}, \bar{b}, \bar{x}^*) \in \mathrm{gph}\mathcal{F}$. We observe that

$$\begin{aligned}
\{\bar{x}^*\}^\perp \ &= \{(0, \alpha)\}^\perp = I\!R \times \{0\},\\
T(\bar{x}; \Theta(\bar{b})) \ &= \big(N(\bar{x}; \Theta(\bar{b}))\big)^* = (-\infty, 0] \times (-\infty, 0],\\
T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp \ &= (-\infty, 0] \times \{0\},\\
\big(T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp\big)^* \ &= [0, +\infty) \times I\!R.
\end{aligned}$$

Since $I := I(\bar{x}, \bar{b}) = \{1, 2, 3\}$, it holds $\bar{I} = T \backslash I = \emptyset$. There is only one way to represent $\bar{x}^* = \sum_{i \in I} \lambda_i \mathfrak{a}_i^*$ where $\lambda_i \ge 0$ for $i \in I$ as follows

$$\bar{x}^* = (0, \alpha) = 0(1, 0) + \alpha(0, 1) + 0(1, 2) = 0\mathfrak{a}_1^* + \alpha\mathfrak{a}_2^* + 0\mathfrak{a}_3^*.$$

Hence, $I_1 := I_1(\bar{x}, \bar{b}, \bar{x}^*) = \{1, 3\}$. By (2.21) and (2.10), we obtain

$$\begin{aligned}
&\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big)\\
&\quad = \Big\{(x^*, b^*, v)\Big|\ x^* \in [0, +\infty) \times I\!R,\ \ v \in (-\infty, 0] \times \{0\},\\
&\qquad\qquad\qquad x^* = (-b_1^* - b_3^*, -b_2^* - 2b_3^*),\ \ b_1^* \le 0,\ \ b_3^* \le 0\Big\}\\
&\quad = \Big\{\Big((-\beta_1 - \beta_3, -\beta_2 - 2\beta_3), (\beta_1, \beta_2, \beta_3), (\gamma, 0)\Big)\Big|\ \beta_1, \beta_3, \gamma \in I\!R_-\Big\}.
\end{aligned}$$

The next statement is immediate from Theorem 2.4 and the definition of the Fréchet coderivative.

Theorem 2.5 *For any* $(\bar{x}, \bar{b}, \bar{x}^*) \in \mathrm{gph}\mathcal{F}$, *the Fréchet coderivative of* $\mathcal{F}(\cdot)$ *at* *the point* $(\bar{x}, \bar{b}, \bar{x}^*)$ *is computed by the formula*

$$\begin{aligned}
&\widehat{D}^*\mathcal{F}(\bar{x}, \bar{b}, \bar{x}^*) : X^{**} \rightrightarrows X^* \times I\!R^m,\\
&\widehat{D}^*\mathcal{F}(\bar{x}, \bar{b}, \bar{x}^*)(v)\\
&\quad = \Big\{(x^*, b^*) \in X^* \times I\!R^m \Big|\ (x^*, b^*, -v) \in \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big)\Big\}\\
&\quad = \Big\{(x^*, b^*) \in X^* \times I\!R^m \Big|\ x^* = -\sum_{i\in I} b_i^* \mathfrak{a}_i^*,\ b_{\bar{I}}^* = 0,\ b_{I_1}^* \le 0,\\
&\qquad\quad (x^*, -v) \in \big(T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^{\perp}\big)^* \times \big(T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^{\perp}\big)\Big\}
\end{aligned} \tag{2.29}$$

for every $v \in X^{**}$.

Example 2.2 Let X, T, $\{\mathfrak{a}_i^* |\ i \in T\}$, $(\bar{x}, \bar{b}, \bar{x}^*)$ be the same as in Example 2.1. It follows from (2.29) and the results obtained in Example 2.1 that

$$\begin{aligned}
&\widehat{D}^*\mathcal{F}(\bar{x}, \bar{b}, \bar{x}^*)(v)\\
&\quad = \Big\{(x^*, b^*) \in I\!R^2 \times I\!R^3 \Big|\ (x^*, b^*, -v) \in \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big)\Big\}\\
&\quad = \begin{cases} \Omega, & \text{if } v = (v_1, v_2) \in I\!R_+ \times \{0\}\\ \emptyset, & \text{if } v \notin I\!R_+ \times \{0\}, \end{cases}
\end{aligned}$$

where

$$\Omega = \Big\{\Big((-\beta_1 - \beta_3, -\beta_2 - 2\beta_3), (\beta_1, \beta_2, \beta_3)\Big)\Big|\ \beta_1, \beta_3 \in I\!R_-\Big\}.$$

2.3 The Mordukhovich Coderivative of $\mathcal{F}(x, b)$

Following Henrion, Mordukhovich, and Nam [13], for any sets P, Q with $P \subset Q \subset T$, we put

$$\mathcal{A}_{Q,P} = \mathrm{span}\{\mathfrak{a}_i^* |\ i \in P\} + \mathrm{pos}\{\mathfrak{a}_i^* |\ i \in Q\backslash P\}, \tag{2.30}$$

where the notation spanΩ stands for the linear subspace generated by Ω (when $\Omega = \emptyset$, we use the convention span$\emptyset = \{0\}$), and

$$\mathcal{B}_{Q,P} = \Big\{x \in X \Big|\ \langle \mathfrak{a}_i^*, x\rangle = 0\ \ \forall i \in P,\ \langle \mathfrak{a}_i^*, x\rangle \le 0\ \ \forall i \in Q\backslash P\Big\}. \tag{2.31}$$

Note that $\mathcal{A}_{Q,P} \subset X^*$ and $\mathcal{B}_{Q,P} \subset X$.

Lemma 2.4 (See [13, Lemma 3.3]) *If $P \subset Q \subset T$, then*

$$\left(\mathcal{B}_{Q,P}\right)^* = \mathcal{A}_{Q,P} \tag{2.32}$$

with $\left(\mathcal{B}_{Q,P}\right)^* := \left\{u^* \in X^* | \ \langle u^*, x\rangle \leq 0, \ \forall x \in \mathcal{B}_{Q,P}\right\}$.

Lemma 2.5 *If X is reflexive, then $\mathcal{B}_{Q,P}$ which is embedded in X^{**} via the canonical embedding $X \hookrightarrow X^{**}$ is a weakly* closed set in X^{**}, and $\mathcal{A}_{Q,P}$ is a weakly* closed set in X^*.*

Proof. It is well-known that any closed convex set in an arbitrary locally convex topological vector space is weakly closed. Since $X^{**} = X$, the weak* topology $\sigma(X^{**}, X^*)$ on X^{**} coincides with the weak topology $\sigma(X, X^*)$ on X. Note that $\mathcal{B}_{Q,P}$ is closed and convex. Hence, $\mathcal{B}_{Q,P}$ is weakly closed in X, and thus $\mathcal{B}_{Q,P}$ is weakly* closed in X^{**}. By Lemma 2.4,

$$\mathcal{A}_{Q,P} = \left(\mathcal{B}_{Q,P}\right)^* = \left\{x^* \in X^* | \ \langle x^*, x\rangle \leq 0, \ \forall x \in \mathcal{B}_{Q,P}\right\}.$$

It follows that $\mathcal{A}_{Q,P}$ is weakly* closed. □

The following lemma develops a result in [13, Theorem 3.4], where the parameter b is fixed.

Lemma 2.6 *Let $(\bar{x}, \bar{b}, \bar{x}^*) \in \mathrm{gph}\mathcal{F}$, $\lambda = (\lambda_i)_{i\in I} \in \Xi(\bar{x}, \bar{b}, \bar{x}^*)$ with $I = I(\bar{x}, \bar{b})$. For $K = \{i \in I | \ \lambda_i > 0\}$, it holds*

$$\left(T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp\right)^* \times \left(T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp\right) = \mathcal{A}_{I,K} \times \mathcal{B}_{I,K}. \tag{2.33}$$

Proof. Let us prove that

$$T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp = \mathcal{B}_{I,K}. \tag{2.34}$$

Given any $x \in \mathcal{B}_{I,K}$, by the construction of $\mathcal{B}_{I,K}$ we have $\langle \mathfrak{a}_i^*, x\rangle \leq 0$ for all $i \in I$. Hence, $x \in T(\bar{x}; \Theta(\bar{b}))$. Since $\langle \mathfrak{a}_i^*, x\rangle = 0$ for all $i \in K$ and $\bar{x}^* = \sum_{i\in K} \lambda_i \mathfrak{a}_i^*$, it holds

$$\langle \bar{x}^*, x\rangle = \sum_{i\in K} \lambda_i \langle \mathfrak{a}_i^*, x\rangle = 0.$$

Therefore, $x \in T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp$. So, $\mathcal{B}_{I,K} \subset T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp$. Now, fixing any $x \in T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp$, we have $\langle \mathfrak{a}_i^*, x\rangle \leq 0$ for all $i \in I$, and $\langle \bar{x}^*, x\rangle = 0$. Since $(\bar{x}, \bar{b}, \bar{x}^*) \in \mathrm{gph}\mathcal{F}$ and $\lambda = (\lambda_i)_{i\in I} \in \Xi(\bar{x}, \bar{b}, \bar{x}^*)$ where $I = I(\bar{x}, \bar{b})$, by (2.8) we get $\bar{x}^* = \sum_{i\in I} \lambda_i \mathfrak{a}_i^*$. As $K = \{i \in I | \ \lambda_i > 0\}$, it follows from the last equality and the condition $\langle \bar{x}^*, x\rangle = 0$ that

$$0 = \langle \bar{x}^*, x\rangle = \sum_{i\in K} \lambda_i \langle \mathfrak{a}_i^*, x\rangle.$$

Since $\langle \mathfrak{a}_i^*, x\rangle \le 0$ for all $i \in I$, we see that $\langle \mathfrak{a}_i^*, x\rangle = 0$ for all $i \in K$. Thus $x \in \mathcal{B}_{I,K}$. We have shown that $\mathcal{B}_{I,K} \supset T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp$. Hence the equality (2.34) is valid.

By Lemma 2.4 and by (2.34), we obtain

$$\big(T(\bar{x}; \Theta(\bar{b})) \cap \{\bar{x}^*\}^\perp\big)^* = \big(\mathcal{B}_{I,K}\big)^* = \mathcal{A}_{I,K},$$

hence establishing (2.33). □

Combining the formula for computing $\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big)$ in Theorem 2.4 with Lemma 2.6 we obtain the following statement which unlike Corollary 4.1 in [38], does not require the assumption that $\{\mathfrak{a}_i^* |\ i \in I(\bar{x}, \bar{b})\}$ are linearly independent.

Lemma 2.7 *Let $(\bar{x}, \bar{b}, \bar{x}^*)$, $I = I(\bar{x}, \bar{b})$, $\lambda = (\lambda_i)_{i\in I}$, and K be the same as in Lemma 2.6. Then*

$$\begin{aligned}\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big) = \Big\{(x^*, b^*, v)\Big|\ &(x^*, v) \in \mathcal{A}_{I,K} \times \mathcal{B}_{I,K},\\ &x^* = -\sum_{i\in I} b_i^* \mathfrak{a}_i^*,\ b_{\bar{I}}^* = 0,\ b_{I_1}^* \le 0\Big\},\end{aligned} \tag{2.35}$$

where $\bar{I} = T\backslash I$ and $I_1 = I_1(\bar{x}, \bar{b}, \bar{x}^*)$.

For each $(x, b, x^*) \in \mathrm{gph}\mathcal{F}$, we put

$$\mathcal{I}(x, b, x^*) = \Big\{P \subset I(x, b)\big|\ P \ne \emptyset,\ x^* \in \mathrm{pos}\{\mathfrak{a}_i^* |\ i \in P\}\Big\},$$

$$\mathcal{J}(x, b, x^*) = \Big\{P \in \mathcal{I}(x, b, x^*)\big|\ \mathfrak{a}_i^*,\ i \in P,\ \text{are linearly independent}\Big\},$$

and

$$\widehat{\mathcal{I}}(x, b, x^*) = \begin{cases} \mathcal{J}(x, b, x^*) & \text{if } x^* \ne 0\\ \mathcal{J}(x, b, x^*) \cup \{\emptyset\} & \text{if } x^* = 0.\end{cases}$$

For every $Q \subset T$, we define a *pseudo-face* of $\Theta(b)$ by putting

$$\mathfrak{F}_Q(b) = \Big\{x \in X\big|\ \langle \mathfrak{a}_i^*, x\rangle = b_i\ \ \forall i \in Q,\ \ \langle \mathfrak{a}_i^*, x\rangle < b_i\ \ \forall i \in T\backslash Q\Big\}.$$

Now, let $(\bar{x}, \bar{b}, \bar{x}^*) \in \mathrm{gph}\mathcal{F}$, $I = I(\bar{x}, \bar{b})$, $J = I\backslash I_1(\bar{x}, \bar{b}, \bar{x}^*)$, $\mathcal{I} = \mathcal{I}(\bar{x}, \bar{b}, \bar{x}^*)$, and $\widehat{\mathcal{I}} = \widehat{\mathcal{I}}(\bar{x}, \bar{b}, \bar{x}^*)$. Define

$$\begin{aligned}\Sigma(\bar{x}, \bar{b}, \bar{x}^*) = \bigcup_{P\subset Q\subset I,\, P\in\widehat{\mathcal{I}}} \Big\{(x^*, b^*, v)\Big|\ &(x^*, v) \in \mathcal{A}_{Q,P} \times \mathcal{B}_{Q,P},\\ &x^* = -\sum_{i\in Q} b_i^* \mathfrak{a}_i^*,\ b_{\overline{Q}}^* = 0,\ b_{Q\backslash P}^* \le 0\Big\},\end{aligned} \tag{2.36}$$

and

$$\Sigma_0(\bar{x},\bar{b},\bar{x}^*) = \bigcup_{\substack{P\subset Q\subset I,\, P\in\mathcal{I}\\ \mathfrak{F}_Q(\bar{b})\neq\emptyset}} \Big\{(x^*,b^*,v)\Big|\ (x^*,v)\in\mathcal{A}_{Q,P}\times\mathcal{B}_{Q,P},$$
$$x^* = -\sum_{i\in Q} b_i^*\mathfrak{a}_i^*,\ b_{\overline{Q}}^* = 0,\ b_{Q\setminus J}^* \le 0\Big\}. \tag{2.37}$$

We will show that the sets given by (2.36) and (2.37) are respectively an upper estimate and a lower estimate for the limiting normal cone $N\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)$. Since the Fréchet normal cone $\widehat{N}\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)$ is a subset of the cone $N\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)$, it serves readily as a lower estimate of the latter. In the sequel, we will prove that $\Sigma_0(\bar{x},\bar{b},\bar{x}^*)$ is not only a lower estimate for $N\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)$ but it is also an upper estimate for $\widehat{N}\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)$ in the case $\bar{x}^*\neq 0$. Moreover, by constructing a suitable example, we will show that the inclusion $\widehat{N}\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)\subset\Sigma_0(\bar{x},\bar{b},\bar{x}^*)$ may be strict if $\bar{x}^*\neq 0$.

In comparison with Theorem 4.1 in [38], the estimates provided by the following theorem are tighter. Moreover, we do not assume that the vectors $\{\mathfrak{a}_i^*|\ i\in I(\bar{x},\bar{b})\}$ are positively linearly independent.

Theorem 2.6 *For any $(\bar{x},\bar{b},\bar{x}^*)\in\mathrm{gph}\mathcal{F}$, the estimates*

$$\Sigma_0(\bar{x},\bar{b},\bar{x}^*)\subset N\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)\subset\Sigma(\bar{x},\bar{b},\bar{x}^*), \tag{2.38}$$

where $\Sigma(\bar{x},\bar{b},\bar{x}^*)$ and $\Sigma_0(\bar{x},\bar{b},\bar{x}^*)$ are given respectively by (2.36) *and* (2.37)*, hold. Besides, if $\bar{x}^*\neq 0$, then*

$$\widehat{N}\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)\subset\Sigma_0(\bar{x},\bar{b},\bar{x}^*)\subset N\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big). \tag{2.39}$$

Proof. Note that $\mathrm{gph}\mathcal{F}$ is closed by Lemma 2.2 and X is reflexive. Hence, according to [28, Theorem 2.35],

$$N\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big) = \operatorname*{Limsup}_{(x_k,b_k,x_k^*)\xrightarrow{\mathrm{gph}\mathcal{F}}(\bar{x},\bar{b},\bar{x}^*)} \widehat{N}\big((x_k,b_k,x_k^*);\mathrm{gph}\mathcal{F}\big).$$

To obtain the second inclusion in (2.38), we fix an arbitrary element (x^*,b^*,v) of $N\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)$. We have to show that $(x^*,b^*,v)\in\Sigma(\bar{x},\bar{b},\bar{x}^*)$. By the definition of the limiting normal cone, one can find $(x_k,b_k,x_k^*)\xrightarrow{\mathrm{gph}\mathcal{F}}(\bar{x},\bar{b},\bar{x}^*)$ and $(u_k^*,\eta_k^*,v_k)\xrightarrow{w^*}(x^*,b^*,v)$ such that

$$(u_k^*,\eta_k^*,v_k)\in\widehat{N}\big((x_k,b_k,x_k^*);\mathrm{gph}\mathcal{F}\big)$$

for all $k\in I\!N$. Since $I(x_k,b_k)\subset T$ and $(x_k,b_k)\to(\bar{x},\bar{b})$ for all k, we may suppose that $I(x_k,b_k)=Q$ for all k, where $Q\subset I(\bar{x},\bar{b})$ is a fixed index set.

Then, by Proposition 2.1,

$$x_k^* \in N(x_k; \Theta(b_k)) = \text{pos}\{\mathfrak{a}_i^* |\ i \in Q\} \quad \text{for all } k \in I\!N.$$

Due to Lemma 2.1 and the Dirichlet principle, by considering a subsequence of $\{x_k^*\}$ if necessary, we may assume that there exists a subset $\widetilde{P} \subset Q$ such that the vectors $\{\mathfrak{a}_i^* |\ i \in \widetilde{P}\}$ are linearly independent and

$$x_k^* \in \text{pos}\{\mathfrak{a}_i^* |\ i \in \widetilde{P}\} \quad \text{for all } k \in I\!N.$$

This means that

$$x_k^* = \sum_{i \in \widetilde{P}} \lambda_i^k \mathfrak{a}_i^* \quad \text{for some } \lambda_i^k \geq 0,\ i \in \widetilde{P}.$$

Using again the Dirichlet principle, we can find a subsequence $\{k_\ell\}$ of $\{k\}$ and a subset $P \subset \widetilde{P}$ such that

$$\{i \in \widetilde{P} |\ \lambda_i^{k_\ell} > 0\} = P \quad \text{for all } \ell \in I\!N. \tag{2.40}$$

For each $\ell \in I\!N$, since $(u_{k_\ell}^*, \eta_{k_\ell}^*, v_{k_\ell}) \in \widehat{N}\big((x_{k_\ell}, b_{k_\ell}, x_{k_\ell}^*); \text{gph}\mathcal{F}\big)$, by Lemma 2.7 we get

$$(u_{k_\ell}^*, v_{k_\ell}) \in \mathcal{A}_{Q,P} \times \mathcal{B}_{Q,P},\ u_{k_\ell}^* = -\sum_{i \in Q} (\eta_{k_\ell}^*)_i \mathfrak{a}_i^*,\ (\eta_{k_\ell}^*)_{\overline{Q}} = 0,\ (\eta_{k_\ell}^*)_{I_1^{k_\ell}} \leq 0, \tag{2.41}$$

where $I_1^{k_\ell} := I_1(x_{k_\ell}, b_{k_\ell}, x_{k_\ell}^*)$. If $P = \emptyset$, then $x_{k_\ell}^* = 0$ for every $\ell \in I\!N$. (In this case, we have $\bar{x}^* = 0$ because $\bar{x}^* = \lim\limits_{\ell \to \infty} x_{k_\ell}^*$.) By the definition of $I_1(x_{k_\ell}, b_{k_\ell}, x_{k_\ell}^*)$ we get $I_1(x_{k_\ell}, b_{k_\ell}, x_{k_\ell}^*) = Q$, and thus $Q \backslash P = I_1(x_{k_\ell}, b_{k_\ell}, x_{k_\ell}^*)$. If $P \neq \emptyset$, then

$$x_{k_\ell}^* = \sum_{i \in P} \lambda_i^{k_\ell} \mathfrak{a}_i^* \quad \text{for all } \ell \in I\!N. \tag{2.42}$$

Hence, from (2.40) and (2.42) we deduce that

$$Q \backslash P = I(x_{k_\ell}, b_{k_\ell}) \backslash P \subset I_1(x_{k_\ell}, b_{k_\ell}, x_{k_\ell}^*) = I_1^{k_\ell}.$$

Since $(\eta_{k_\ell}^*)_{I_1^{k_\ell}} \leq 0$ for every $\ell \in I\!N$, we also have

$$(\eta_{k_\ell}^*)_{Q \backslash P} \leq 0 \ \text{ for every } \ell \in I\!N.$$

Note that the sets $\mathcal{A}_{Q,P}$ and $\mathcal{B}_{Q,P}$ are weakly* closed by Lemma 2.5. By letting $\ell \to \infty$ and invoking the first inclusion in (2.41), from the relation $(u_{k_\ell}^*, v_{k_\ell}) \xrightarrow{w^*} (x^*, v)$ we get

$$(x^*, v) \in \mathcal{A}_{Q,P} \times \mathcal{B}_{Q,P}. \tag{2.43}$$

Taking into account that $(u^*_{k_\ell}, \eta^*_{k_\ell}, v_{k_\ell}) \xrightarrow{w^*} (x^*, b^*, v)$, we have $(\eta^*_{k_\ell})_i \to b^*_i$ for each $i \in T$. Therefore, by letting $\ell \to \infty$ and by using (2.41) we obtain

$$x^* = -\sum_{i\in Q} b^*_i \mathfrak{a}^*_i, \;\; b^*_{\overline{Q}} = 0, \;\; b^*_{Q\backslash P} \leq 0. \tag{2.44}$$

By the choice of P and Q, it holds $P \subset Q \subset I$ and $P \in \widehat{\mathcal{I}}$. Hence, by (2.43), (2.44) and (2.36) we can assert that $(x^*, b^*, v) \in \Sigma(\bar{x}, \bar{b}, \bar{x}^*)$. This implies that $N\big((\bar{x}, \bar{b}, \bar{x}^*); \mathrm{gph}\mathcal{F}\big) \subset \Sigma(\bar{x}, \bar{b}, \bar{x}^*)$.

Now, to verify the first inclusion in (2.38), we take any (x^*, b^*, v) from $\Sigma_0(\bar{x}, \bar{b}, \bar{x}^*)$. Choose P, Q with $P \subset Q \subset I$, $P \in \mathcal{I}$, and $\mathfrak{F}_Q(\bar{b}) \neq \emptyset$ such that

$$(x^*, v) \in \mathcal{A}_{Q,P} \times \mathcal{B}_{Q,P}, \;\; x^* = -\sum_{i\in Q} b^*_i \mathfrak{a}^*_i, \;\; b^*_{\overline{Q}} = 0, \;\; b^*_{Q\backslash J} \leq 0. \tag{2.45}$$

Fix any $\widetilde{x} \in \mathfrak{F}_Q(\bar{b})$ and consider the sequence

$$x_k = k^{-1}\widetilde{x} + (1 - k^{-1})\bar{x}.$$

It is clear that $x_k \in \mathfrak{F}_Q(\bar{b})$ and $x_k \to \bar{x}$ as $k \to \infty$. For every k, by putting $b_k = \bar{b}$ we have $I(x_k, b_k) = Q$. Since $P \in \mathcal{I}$, it follows that $P \neq \emptyset$ and

$$\bar{x}^* = \sum_{i\in P} \lambda_i \mathfrak{a}^*_i \quad \text{for some } \lambda_i \geq 0, \; i \in P.$$

For each k, put

$$x^*_k = \sum_{i\in P} (\lambda_i + k^{-1}) \mathfrak{a}^*_i \in N(x_k; \Theta(b_k)) = \mathcal{F}(x_k, b_k). \tag{2.46}$$

Since $x^*_k \to \bar{x}^*$ as $k \to \infty$, we have $(x_k, b_k, x^*_k) \xrightarrow{\mathrm{gph}\mathcal{F}} (\bar{x}, \bar{b}, \bar{x}^*)$. From (2.46) we infer that $Q\backslash P \subset I_1(x_k, b_k, x^*_k) \subset I(x_k, b_k) = Q$ for all $k \in I\!N$. By considering a subsequence of $\{k\}$, if necessary, we can assume that $I_1(x_k, b_k, x^*_k) = \widetilde{I}_1$ for all $k \in I\!N$. The inclusion $\widetilde{I}_1 \subset Q\backslash J$ holds. To show this, fix any $i \in \widetilde{I}_1$. For every $k \in I\!N$, there exist some $\mu^k_j \geq 0$ for $j \in Q\backslash\{i\}$ satisfying

$$x^*_k = \sum_{j\in Q} \mu^k_j \mathfrak{a}^*_j \in \mathrm{pos}\big\{\mathfrak{a}^*_j | \; j \in Q\backslash\{i\}\big\},$$

where $\mu^k_i := 0$. By Lemma 2.1, one can find a subset $K \subset Q\backslash\{i\}$ and a subsequence $\{k_\ell\}$ of $\{k\}$ such that $\{\mathfrak{a}^*_j | \; j \in K\}$ are linearly independent and $x^*_{k_\ell} \in \mathrm{pos}\{\mathfrak{a}^*_j | \; j \in K\}$ for all $\ell \in I\!N$. Since $\{\mathfrak{a}^*_j | \; j \in K\}$ are linearly independent and $x^*_{k_\ell} \to \bar{x}^*$, we obtain

$$\bar{x}^* \in \mathrm{pos}\{\mathfrak{a}^*_j | \; j \in K\}. \tag{2.47}$$

Since $i \notin K$, by the definition of $I_1(\bar{x}, \bar{b}, \bar{x}^*)$ and (2.47) we have $i \in I_1(\bar{x}, \bar{b}, \bar{x}^*)$. As $J = I \backslash I_1(\bar{x}, \bar{b}, \bar{x}^*)$, it follows that $i \in Q \backslash J$. Therefore, the inclusion $\widetilde{I}_1 \subset Q \backslash J$ is valid. Since we have $I_1(x_k, b_k, x_k^*) = \widetilde{I}_1 \subset Q \backslash J$ for each $k \in I\!\!N$, from (2.45) it follows that

$$(x^*, v) \in \mathcal{A}_{Q,P} \times \mathcal{B}_{Q,P},\ x^* = -\sum_{i \in Q} b_i^* \mathfrak{a}_i^*,\ b_{\overline{Q}}^* = 0,\ b_{\widetilde{I}_1}^* \leq 0.$$

Hence, by Lemma 2.7 we deduce that $(x^*, b^*, v) \in \widehat{N}\big((x_k, b_k, x_k^*); \text{gph}\mathcal{F}\big)$ for all $k \in I\!\!N$. Combining this with the fact that $(x_k, b_k, x_k^*) \xrightarrow{\text{gph}\mathcal{F}} (\bar{x}, \bar{b}, \bar{x}^*)$ yields $(x^*, b^*, v) \in N\big((\bar{x}, \bar{b}, \bar{x}^*); \text{gph}\mathcal{F}\big)$. Since (x^*, b^*, v) can be chosen arbitrary in $\Sigma_0(\bar{x}, \bar{b}, \bar{x}^*)$, we obtain $\Sigma_0(\bar{x}, \bar{b}, \bar{x}^*) \subset N\big((\bar{x}, \bar{b}, \bar{x}^*); \text{gph}\mathcal{F}\big)$.

Finally, let us prove the first inclusion in (2.39) under the assumption that $\bar{x}^* \neq 0$. Fix any $\lambda = (\lambda_i)_{i \in I} \in \Xi(\bar{x}, \bar{b}, \bar{x}^*)$, where $\Xi(\bar{x}, \bar{b}, \bar{x}^*)$ is given by (2.8). Choosing $Q = I$ and $P = \{i \in I |\ \lambda_i > 0\}$, we get $I_1 = I \backslash J = Q \backslash J$. According to Lemma 2.7,

$$\begin{aligned} \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \text{gph}\mathcal{F}\big) &= \Big\{(x^*, b^*, v) \Big|\ (x^*, v) \in \mathcal{A}_{Q,P} \times \mathcal{B}_{Q,P}, \\ &\qquad x^* = -\textstyle\sum_{i \in Q} b_i^* \mathfrak{a}_i^*,\ b_{\overline{Q}}^* = 0,\ b_{I_1}^* \leq 0\Big\} \\ &= \Big\{(x^*, b^*, v) \Big|\ (x^*, v) \in \mathcal{A}_{Q,P} \times \mathcal{B}_{Q,P}, \\ &\qquad x^* = -\textstyle\sum_{i \in Q} b_i^* \mathfrak{a}_i^*,\ b_{\overline{Q}}^* = 0,\ b_{Q \backslash J}^* \leq 0\Big\}. \end{aligned} \tag{2.48}$$

Since $\bar{x}^* \neq 0$, we infer that $P \neq \emptyset$. By our choice of P and Q, it holds $P \subset Q \subset I$, $P \in \mathcal{I}$. In addition, we have $\mathfrak{F}_Q(\bar{b}) \neq \emptyset$ because $\bar{x} \in \mathfrak{F}_Q(\bar{b})$. Hence, $\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \text{gph}\mathcal{F}\big) \subset \Sigma_0(\bar{x}, \bar{b}, \bar{x}^*)$. The proof is complete. □

Remark 2.3 One necessary condition for obtaining precise formulas for the limiting normal cone $N\big((\bar{x}, \bar{b}, \bar{x}^*); \text{gph}\mathcal{F}\big)$ is that $\mathfrak{F}_Q(\bar{b}) \neq \emptyset$ for $Q \subset I = I(\bar{x}, \bar{b})$ (see e.g. [13, Theorem 4.2] and [32, Theorem 4.1]). The linear independence assumption of $\{\mathfrak{a}_i^* |\ i \in I\}$ ensures that $\mathfrak{F}_Q(\bar{b}) \neq \emptyset$ for every $Q \subset I$. Indeed, since $\{\mathfrak{a}_i^* |\ i \in I\}$ are linearly independent, by [13, Theorem 4.2] there exists $\widehat{x} \in X$ such that $\langle \mathfrak{a}_i^*, \widehat{x} \rangle = 0$ for $i \in Q$ and $\langle \mathfrak{a}_i^*, \widehat{x} \rangle < 0$ for $i \in T \setminus Q$. This implies that $\widehat{x} + \bar{x} \in \mathfrak{F}_Q(\bar{b})$. We have seen that there is no regularity assumption for $\{\mathfrak{a}_i^* |\ i \in I\}$ in Theorem 4.1. Hence in that theorem only upper and lower estimates can be obtained for $N\big((\bar{x}, \bar{b}, \bar{x}^*); \text{gph}\mathcal{F}\big)$.

The next example is designed to show how Theorem 2.6 can be used for getting an upper estimate and a lower estimate of the limiting normal cone $N\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)$. In this example, we will see that the first inclusion of (2.39) is strict in general. This means that the lower estimate $\Sigma_0(\bar{x},\bar{b},\bar{x}^*)\subset N\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)$ provided by Theorem 2.6 for $N\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)$ is better than the natural estimate $\widehat{N}\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)\subset N\big((\bar{x},\bar{b},\bar{x}^*);\mathrm{gph}\mathcal{F}\big)$.

Example 2.3 Let X, T, and $\{\mathfrak{a}_i^*|\ i\in T\}$ be the same as in Example 2.1. The calculations already done in Example 2.1 assure that $(\bar{x},\bar{b},\bar{x}^*)\in\mathrm{gph}\mathcal{F}$, where

$$\bar{b}=(0,0,0)\in I\!R^3,\ \ \bar{x}=(0,0)\in X,\ \text{ and } \bar{x}^*=(0,\alpha) \text{ for } \alpha>0.$$

Besides, $I=I(\bar{x},\bar{b})=\{1,2,3\}$, $\bar{I}=T\backslash I=\emptyset$, $I_1=I_1(\bar{x},\bar{b},\bar{x}^*)=\{1,3\}$, and $J=I\backslash I_1=\{2\}$.

Note that the unique way to represent $\bar{x}^*=\sum_{i\in I}\lambda_i\mathfrak{a}_i^*$, $\lambda_i\geq 0$ for $i\in I$, is that

$$\bar{x}^*=(0,\alpha)=0(1,0)+\alpha(0,1)+0(1,2)=0\mathfrak{a}_1^*+\alpha\mathfrak{a}_2^*+0\mathfrak{a}_3^*.$$

So, we have

$$\mathcal{I}=\mathcal{I}(\bar{x},\bar{b},\bar{x}^*)=\Big\{P\subset I\big|\ P\neq\emptyset,\ \bar{x}^*\in\mathrm{pos}\{\mathfrak{a}_i^*|\ i\in P\}\Big\}=\Big\{P\subset I\big|\ 2\in P\Big\}.$$

Hence, P and Q satisfy the conditions $P\subset Q\subset I$ and $P\in\mathcal{I}$ if and only if one of the following cases occurs:

(a) $Q=\{1,2,3\}$ and $P=\{1,2,3\}$, $\quad(0,0)\in\mathfrak{F}_Q(\bar{b})\neq\emptyset$;

(b) $Q=\{1,2,3\}$ and $P=\{1,2\}$, $\quad(0,0)\in\mathfrak{F}_Q(\bar{b})\neq\emptyset$;

(c) $Q=\{1,2,3\}$ and $P=\{2,3\}$, $\quad(0,0)\in\mathfrak{F}_Q(\bar{b})\neq\emptyset$;

(d) $Q=\{1,2,3\}$ and $P=\{2\}$, $\quad(0,0)\in\mathfrak{F}_Q(\bar{b})\neq\emptyset$;

(e) $Q=\{1,2\}$ and $P=\{1,2\}$, $\quad\mathfrak{F}_Q(\bar{b})=\emptyset$;

(f) $Q=\{1,2\}$ and $P=\{2\}$, $\quad\mathfrak{F}_Q(\bar{b})=\emptyset$;

(g) $Q=\{2,3\}$ and $P=\{2,3\}$, $\quad\mathfrak{F}_Q(\bar{b})=\emptyset$;

(h) $Q=\{2,3\}$ and $P=\{2\}$, $\quad\mathfrak{F}_Q(\bar{b})=\emptyset$;

(i) $Q=\{2\}$ and $P=\{2\}$, $\quad(-1,0)\in\mathfrak{F}_Q(\bar{b})\neq\emptyset$.

For each $t \in \Gamma := \{a, b, c, d, e, f, g, h, i\}$, define P, Q as in the case (t) above and put

$$\Lambda_{(t)} = \Big\{(x^*, b^*, v)\big|\ (x^*, v) \in \mathcal{A}_{Q,P} \times \mathcal{B}_{Q,P},\ x^* = -\sum_{i\in Q} b_i^* \mathfrak{a}_i^*,\ b_{\overline{Q}}^* = 0,\ b_{Q\setminus P}^* \le 0\Big\},$$

$$\widehat{\Lambda}_{(t)} = \Big\{(x^*, b^*, v)\big|\ (x^*, v) \in \mathcal{A}_{Q,P} \times \mathcal{B}_{Q,P},\ x^* = -\sum_{i\in Q} b_i^* \mathfrak{a}_i^*,\ b_{\overline{Q}}^* = 0,\ b_{Q\setminus J}^* \le 0\Big\},$$

$$\Lambda_{(t)}^0 = \begin{cases} \widehat{\Lambda}_{(t)} & \text{if} \quad \mathfrak{F}_Q(\bar{b}) \neq \emptyset \\ \emptyset & \text{if} \quad \mathfrak{F}_Q(\bar{b}) = \emptyset. \end{cases}$$

Since $\bar{x}^* = (0, \alpha) \neq 0_{I\!R^2}$, we have $\emptyset \notin \widehat{\mathcal{I}}(\bar{x}, \bar{b}, \bar{x}^*)$. We see that $\{\mathfrak{a}_i^*|\ i \in P\}$ are linearly independent if and only if P is given as in the case (t) with $t \in \Gamma\setminus\{a\}$. Hence

$$\Sigma(\bar{x}, \bar{b}, \bar{x}^*) = \bigcup_{t\in\Gamma\setminus\{a\}} \Lambda_{(t)} \quad \text{and} \quad \Sigma_0(\bar{x}, \bar{b}, \bar{x}^*) = \bigcup_{t\in\Gamma} \Lambda_{(t)}^0. \tag{2.49}$$

The sets $\Lambda_{(t)}$ and $\Lambda_{(t)}^0$ can be computed through various realizations of the inclusion $t \in \Gamma$ as follows.

1. If $t = a$, then $Q = \{1, 2, 3\}$ and $P = \{1, 2, 3\}$. We have

   $$\mathcal{A}_{Q,P} = \operatorname{span}\{\mathfrak{a}_i^*|\ i \in P\} = I\!R^2, \quad \mathcal{B}_{Q,P} = (\mathcal{A}_{Q,P})^* = \{0_{I\!R^2}\}.$$

   Note that $\mathfrak{F}_Q(\bar{b}) \neq \emptyset$. Let $x^* \in \mathcal{A}_{Q,P}$ be such that $x^* = -\sum_{i\in Q} b_i^* \mathfrak{a}_i^*$ and $b_{Q\setminus J}^* \le 0$. Then,

   $$x^* = -b_1^*\mathfrak{a}_1^* - b_2^*\mathfrak{a}_2^* - b_3^*\mathfrak{a}_3^* = (-b_1^* - b_3^*, -b_2^* - 2b_3^*) \quad \text{and} \quad b_1^* \le 0,\ b_3^* \le 0.$$

   Therefore,

   $$\Lambda_{(a)}^0 = \Big\{\Big((-\beta_1 - \beta_3, -\beta_2 - 2\beta_3), (\beta_1, \beta_2, \beta_3), (0, 0)\Big)\Big|\ \beta_1, \beta_3 \in I\!R_-\Big\}.$$

2. If $t = b$, then $Q = \{1, 2, 3\}$ and $P = \{1, 2\}$. We have

   $$\mathcal{A}_{Q,P} = \operatorname{span}\{\mathfrak{a}_1^*, \mathfrak{a}_2^*\} + \operatorname{pos}\{\mathfrak{a}_3^*\} = I\!R^2, \quad \mathcal{B}_{Q,P} = (\mathcal{A}_{Q,P})^* = \{0_{I\!R^2}\}.$$

   Let $x^* \in \mathcal{A}_{Q,P}$ be such that $x^* = -\sum_{i\in Q} b_i^* \mathfrak{a}_i^*$ and $b_{Q\setminus P}^* \le 0$. Then,

   $$x^* = -b_1^*\mathfrak{a}_1^* - b_2^*\mathfrak{a}_2^* - b_3^*\mathfrak{a}_3^* = (-b_1^* - b_3^*, -b_2^* - 2b_3^*) \quad \text{and} \quad b_3^* \le 0.$$

   Therefore,

   $$\Lambda_{(b)} = \Big\{\Big((-\beta_1 - \beta_3, -\beta_2 - 2\beta_3), (\beta_1, \beta_2, \beta_3), (0, 0)\Big)\Big|\ \beta_3 \in I\!R_-\Big\}.$$

Note that $\mathfrak{F}_Q(\bar{b}) \neq \emptyset$. Let $x^* \in \mathcal{A}_{Q,P}$ be such that $x^* = -\sum_{i\in Q} b_i^* \mathfrak{a}_i^*$ and $b^*_{Q\setminus J} \leq 0$. Then,

$$x^* = (-b_1^* - b_3^*, -b_2^* - 2b_3^*) \quad \text{and} \quad b_1^* \leq 0,\ b_3^* \leq 0.$$

Therefore,

$$\Lambda^0_{(b)} = \left\{ \Big((-\beta_1 - \beta_3, -\beta_2 - 2\beta_3), (\beta_1, \beta_2, \beta_3), (0,0) \Big) \Big| \ \beta_1, \beta_3 \in I\!R_- \right\}.$$

Treating the cases $t = c, \ldots, t = h$ in a similar manner, we obtain the following results.

3. For $t = c$,

$$\Lambda_{(c)} = \left\{ \Big((-\beta_1 - \beta_3, -\beta_2 - 2\beta_3), (\beta_1, \beta_2, \beta_3), (0,0) \Big) \Big| \ \beta_1 \in I\!R_- \right\},$$
$$\Lambda^0_{(c)} = \left\{ \Big((-\beta_1 - \beta_3, -\beta_2 - 2\beta_3), (\beta_1, \beta_2, \beta_3), (0,0) \Big) \Big| \ \beta_1, \beta_3 \in I\!R_- \right\}.$$

4. For $t = d$,

$$\Lambda_{(d)} = \left\{ \Big((-\beta_1 - \beta_3, -\beta_2 - 2\beta_3), (\beta_1, \beta_2, \beta_3), (\gamma,0) \Big) \Big| \ \beta_1, \beta_3, \gamma \in I\!R_- \right\},$$
$$\Lambda^0_{(d)} = \Lambda_{(d)}.$$

5. For $t = e$,

$$\Lambda_{(e)} = \left\{ \Big((-\beta_1, -\beta_2), (\beta_1, \beta_2, 0), (0,0) \Big) \Big| \ \beta_1, \beta_2 \in I\!R \right\},$$
$$\Lambda^0_{(e)} = \emptyset.$$

6. For $t = f$,

$$\Lambda_{(f)} = \left\{ \Big((-\beta_1, -\beta_2), (\beta_1, \beta_2, 0), (\gamma,0) \Big) \Big| \ \beta_1, \gamma \in I\!R_- \right\},$$
$$\Lambda^0_{(f)} = \emptyset.$$

7. For $t = g$,

$$\Lambda_{(g)} = \left\{ \Big((-\beta_3, -\beta_2 - 2\beta_3), (0, \beta_2, \beta_3), (0,0) \Big) \Big| \ \beta_2, \beta_3 \in I\!R \right\},$$
$$\Lambda^0_{(g)} = \emptyset.$$

8. For $t = h$,

$$\Lambda_{(h)} = \Big\{ \Big((-\beta_3, -\beta_2 - 2\beta_3), (0, \beta_2, \beta_3), (\gamma, 0) \Big) \Big| \ \beta_3, \gamma \in I\!R_- \Big\},$$

$$\Lambda^0_{(h)} = \emptyset.$$

The case $t = i$ deserves a special treatment. Detailed arguments are given below.

9. If $t = i$, then $Q = \{2\}$ and $P = \{2\}$. We have

$$\mathcal{A}_{Q,P} = \text{span}\{\mathfrak{a}^*_2\} = \{0\} \times I\!R, \quad \mathcal{B}_{Q,P} = (\mathcal{A}_{Q,P})^* = I\!R \times \{0\}.$$

Let $x^* \in \mathcal{A}_{Q,P}$ be such that $x^* = -\sum_{i \in Q} b^*_i \mathfrak{a}^*_i$, $b^*_{\overline{Q}} = 0$, and $b^*_{Q \setminus P} \leq 0$. Then,

$$x^* = -b^*_2 \mathfrak{a}^*_2 = (0, -b^*_2), \quad b^*_1 = b^*_3 = 0.$$

Hence, $b^* \in \{0\} \times I\!R \times \{0\}$ and $x^* \in \{0\} \times I\!R$. Therefore,

$$\Lambda_{(i)} = \Big\{ \Big((0, -\beta_2), (0, \beta_2, 0), (\gamma, 0) \Big) \Big| \ \beta_2, \gamma \in I\!R \Big\}.$$

We have $\mathfrak{F}_Q(\bar{b}) \neq \emptyset$ and $Q \setminus P = Q \setminus J$, thus

$$\Lambda^0_{(i)} = \Lambda_{(i)} = \Big\{ \Big((0, -\beta_2), (0, \beta_2, 0), (\gamma, 0) \Big) \Big| \ \beta_2, \gamma \in I\!R \Big\}.$$

On the basis of the above listed results and (2.49), we obtain the exact formulas for $\Sigma(\bar{x}, \bar{b}, \bar{x}^*)$ and $\Sigma_0(\bar{x}, \bar{b}, \bar{x}^*)$. Since $\bar{x}^* \neq 0$, by (2.39) we have $\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \text{gph}\mathcal{F}\big) \subset \Sigma_0(\bar{x}, \bar{b}, \bar{x}^*)$. According to (2.49),

$$\Lambda^0_{(i)} = \Big\{ \Big((0, -\beta_2), (0, \beta_2, 0), (\gamma, 0) \Big) \Big| \ \beta_2, \gamma \in I\!R \Big\} \subset \Sigma_0(\bar{x}, \bar{b}, \bar{x}^*).$$

As shown in Example 2.1, we have

$$\begin{aligned} &\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \text{gph}\mathcal{F}\big) \\ &\quad = \Big\{ \Big((-\beta_1 - \beta_3, -\beta_2 - 2\beta_3), (\beta_1, \beta_2, \beta_3), (\gamma, 0) \Big) \Big| \ \beta_1, \beta_3, \gamma \in I\!R_- \Big\}. \end{aligned}$$

Since $\Lambda^0_{(i)} \not\subset \widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \text{gph}\mathcal{F}\big)$, we deduce that

$$\widehat{N}\big((\bar{x}, \bar{b}, \bar{x}^*); \text{gph}\mathcal{F}\big) \subsetneqq \Sigma_0(\bar{x}, \bar{b}, \bar{x}^*).$$

From Theorem 2.6 we obtain easily some upper and lower estimates for values of the Mordukhovich coderivative $D^*\mathcal{F}(x,b,x^*) : X^{**} \rightrightarrows X^* \times I\!\!R^m$ of $\mathcal{F}$ at a point $(x,b,x^*) \in \mathrm{gph}\mathcal{F}$. Namely, putting

$$\mathbf{\Omega}(x,b,x^*)(v) = \Big\{(u^*,\eta^*) \in X^* \times I\!\!R^m \big|\ (u^*,\eta^*,-v) \in \Sigma(x,b,x^*)\Big\}$$

and

$$\mathbf{\Omega}_0(x,b,x^*)(v) = \Big\{(u^*,\eta^*) \in X^* \times I\!\!R^m \big|\ (u^*,\eta^*,-v) \in \Sigma_0(x,b,x^*)\Big\},$$

we have

$$\mathbf{\Omega}_0(x,b,x^*)(v) \subset D^*\mathcal{F}(x,b,x^*)(v) \subset \mathbf{\Omega}(x,b,x^*)(v) \quad \text{for all } v \in X^{**}.$$

The obtained generalized differentiation properties of the mapping $\mathcal{F}(x,b)$ will be applied to establish the conditions for solution stability of parametric affine variational inequalities (AVIs, for short) under linear perturbations in the next section.

2.4 AVIs under Linear Perturbations

In this section we investigate solution stability of variational inequalities with polyhedral convex constraint sets under linear perturbations. We establish necessary and sufficient conditions for the local Lipschitz-like and the metric regularity properties of the solution maps of such variational inequalities in finite dimensional spaces.

Consider the implicit multifunction $S : I\!\!R^m \times I\!\!R^n \rightrightarrows I\!\!R^n$ defined by

$$S(b,q) = \{x \in I\!\!R^n |\ q \in Mx + \mathcal{F}(x,b)\} \tag{2.50}$$

with $M \in I\!\!R^{n\times n}$, $(b,q) \in I\!\!R^m \times I\!\!R^n$, and $\mathcal{F}(x,b) = N(x;\Theta(b))$ being given by (2.4), where $X = I\!\!R^n$. For any pair $(b',q') \in I\!\!R^m \times I\!\!R^n$, it holds

$$\begin{aligned} &D^*S^{-1}(\bar{x},\bar{b},\bar{q})(b',q') \\ &\quad = \{x' \in I\!\!R^n |\ (x',-b',-q') \in N((\bar{x},\bar{b},\bar{q}); \mathrm{gph}S^{-1})\} \\ &\quad = \{x' \in I\!\!R^n |\ (-b',-q',x') \in N((\bar{b},\bar{q},\bar{x}); \mathrm{gph}S)\} \\ &\quad = \{x' \in I\!\!R^n |\ (-b',-q') \in D^*S(\bar{b},\bar{q},\bar{x})(-x')\}. \end{aligned}$$

Hence, we have

$$D^*S^{-1}(\bar{x},\bar{b},\bar{q})(0_{I\!\!R^{m+n}}) = \{x' \in I\!\!R^n |\ 0_{I\!\!R^{m+n}} \in D^*S(\bar{b},\bar{q},\bar{x})(-x')\}. \tag{2.51}$$

Remark 2.4 Setting $w = -q$ and $f(x,w) = Mx - q$, we see that $S(b,q)$ coincides with the solution map of the parametric variational inequality

$$\text{Find } x \in \Theta(b) \quad \text{subject to } \langle f(x,w), u - x\rangle \geq 0, \ \forall u \in \Theta(b).$$

Note that the inclusion $q \in Mx + \mathcal{F}(x,b)$ in (2.50) is a special form of the following generalized equation

$$0 \in f(x,w) + N(x; \Theta(b)).$$

Here, $f(x,w) = Mx - q$ is an affine operator.

Remark 2.5 We can represent $S(b,q)$ from (2.50) as the solution set of a standard *parametric affine variational inequality* as follows

$$S(b,q) = \{x \in \Theta(b) | \ \langle Mx - q, y - x\rangle \geq 0, \ \forall y \in \Theta(b)\}.$$

Lemma 2.8 *Let $\widetilde{\mathcal{F}}(x,b,q) = Mx - q + \mathcal{F}(x,b)$. Then, the graph of $\widetilde{\mathcal{F}}(\cdot)$ is closed in the product space $I\!R^n \times I\!R^m \times I\!R^n \times I\!R^n$.*

Proof. Suppose that $\{(x_k, b_k, q_k, x_k^*)\}_{k\in I\!N}$ is an arbitrary sequence in $\mathrm{gph}\widetilde{\mathcal{F}}$ and $(x_k, b_k, q_k, x_k^*) \to (\bar{x}, \bar{b}, \bar{q}, \bar{x}^*)$. We have to show that $(\bar{x}, \bar{b}, \bar{q}, \bar{x}^*) \in \mathrm{gph}\widetilde{\mathcal{F}}$. For every $k \in I\!N$, since $(x_k, b_k, q_k, x_k^*) \in \mathrm{gph}\widetilde{\mathcal{F}}$, it holds

$$z_k^* := x_k^* - Mx_k + q_k \in \mathcal{F}(x_k, b_k). \tag{2.52}$$

We have $z_k^* \to \bar{z}^* := \bar{x}^* - M\bar{x} + \bar{q}$. If $\bar{z}^* = 0$, then $\bar{z}^* \in N(\bar{x}, \Theta(\bar{b})) = \mathcal{F}(\bar{x}, \bar{b})$. It follows that $\bar{x}^* \in M\bar{x} - \bar{q} + \mathcal{F}(\bar{x}, \bar{b}) = \widetilde{\mathcal{F}}(\bar{x}, \bar{b}, \bar{q})$. Hence, $(\bar{x}, \bar{b}, \bar{q}, \bar{x}^*) \in \mathrm{gph}\widetilde{\mathcal{F}}$. We now consider the case that $\bar{z}^* \neq 0$. We have $\bar{z}^* \neq 0$ and $z_k^* \to \bar{z}^*$, so $z_k^* \neq 0$ for all k large enough. Since $(x_k, b_k) \to (\bar{x}, \bar{b})$, one has $I(x_k, b_k) \subset I(\bar{x}, \bar{b})$ for sufficiently large indexes $k \in I\!N$. Thus, without loss of generality we may assume that $z_k^* \neq 0$ and $I(x_k, b_k) = \widetilde{I} \subset I(\bar{x}, \bar{b})$ for every $k \in I\!N$. For each $k \in I\!N$, from condition (2.52) it follows that

$$z_k^* \in \mathrm{pos}\{\mathfrak{a}_i^* | \ i \in I(x_k, b_k)\} = \mathrm{pos}\{\mathfrak{a}_i^* | \ i \in \widetilde{I}\}.$$

As $\mathrm{pos}\{\mathfrak{a}_i^* | \ i \in \widetilde{I}\}$ is closed and $z_k^* \to \bar{z}^*$, it holds

$$\bar{z}^* \in \mathrm{pos}\{\mathfrak{a}_i^* | \ i \in \widetilde{I}\} \subset \mathrm{pos}\{\mathfrak{a}_i^* | \ i \in I(\bar{x}, \bar{b})\} = \mathcal{F}(\bar{x}, \bar{b}).$$

Hence, $\bar{x}^* \in M\bar{x} - \bar{q} + \mathcal{F}(\bar{x}, \bar{b}) = \widetilde{\mathcal{F}}(\bar{x}, \bar{b}, \bar{q})$. Consequently, $(\bar{x}, \bar{b}, \bar{q}, \bar{x}^*) \in \mathrm{gph}\widetilde{\mathcal{F}}$ which completes the proof. □

Let any $(\bar{b}, \bar{q}, \bar{x}) \in \text{gph}S$. Clearly, $(\bar{x}, \bar{b}, \bar{x}^*) \in \text{gph}\mathcal{F}$ with $\bar{x}^* := \bar{q} - M\bar{x}$. For every $x' \in I\!R^n$, we define the following sets

$$\widehat{\mathbf{K}}_{M,\bar{q}}(x') = \bigcup_{v' \in I\!R^n} \Big\{ (b', q') \in I\!R^{m+n} \Big| \, (-x', b', q') \in M^\top v' \times \{0_{I\!R^{m+n}}\} - \{0_{I\!R^{n+m}}\} \times \{v'\} + \widehat{D}^* \mathcal{F}(\bar{x}, \bar{b}, \bar{x}^*)(v') \times \{0_{I\!R^n}\} \Big\},$$

$$\mathbf{K}_{M,\bar{q}}(x') = \bigcup_{v' \in I\!R^n} \Big\{ (b', q') \in I\!R^{m+n} \Big| \, (-x', b', q') \in M^\top v' \times \{0_{I\!R^{m+n}}\} - \{0_{I\!R^{n+m}}\} \times \{v'\} + D^* \mathcal{F}(\bar{x}, \bar{b}, \bar{x}^*)(v') \times \{0_{I\!R^n}\} \Big\},$$

and

$$\mathbf{L}_{M,\bar{q}}(x') = \bigcup_{v' \in I\!R^n} \Big\{ (b', q') \in I\!R^{m+n} \Big| \, (-x', b', q') \in M^\top v' \times \{0_{I\!R^{m+n}}\} - \{0_{I\!R^{n+m}}\} \times \{v'\} + \mathbf{\Omega}(\bar{x}, \bar{b}, \bar{x}^*)(v') \times \{0_{I\!R^n}\} \Big\}.$$

Remark 2.6 We have $D^*\mathcal{F}(\bar{x}, \bar{b}, \bar{x}^*)(v) \subset \mathbf{\Omega}(\bar{x}, \bar{b}, \bar{x}^*)(v)$ for all $v \in I\!R^n$, thus the inclusion $\mathbf{K}_{M,\bar{q}}(x') \subset \mathbf{L}_{M,\bar{q}}(x')$ holds for every $x' \in I\!R^n$.

Theorem 2.7 *The following estimates*

$$\widehat{\mathbf{K}}_{M,\bar{q}}(x') \subset \widehat{D}^* S(\bar{b}, \bar{q}, \bar{x})(x') \subset D^* S(\bar{b}, \bar{q}, \bar{x})(x') \subset \mathbf{K}_{M,\bar{q}}(x') \subset \mathbf{L}_{M,\bar{q}}(x')$$

hold for all $x' \in I\!R^n$. Moreover, if $\mathcal{F}(\cdot)$ is graphically regular at $(\bar{x}, \bar{b}, \bar{x}^*)$, then

$$\widehat{\mathbf{K}}_{M,\bar{q}}(x') = \widehat{D}^* S(\bar{b}, \bar{q}, \bar{x})(x') = D^* S(\bar{b}, \bar{q}, \bar{x})(x') = \mathbf{K}_{M,\bar{q}}(x') \subset \mathbf{L}_{M,\bar{q}}(x')$$

for every $x' \in I\!R^n$.

Proof. Let $\widetilde{\mathcal{F}}(x, b, q) = Mx - q + \mathcal{F}(x, b)$ and let $\bar{z} = (\bar{x}, \bar{b}, \bar{q}, 0_{I\!R^n})$. Since $(\bar{b}, \bar{q}, \bar{x}) \in \text{gph}S$, it holds $\bar{q} \in M\bar{x} + \mathcal{F}(\bar{x}, \bar{b})$. Therefore, $\bar{z} \in \text{gph}\widetilde{\mathcal{F}}$. Using the coderivative sum rules [28, Theorem 1.62], we obtain

$$\widehat{D}^* \widetilde{\mathcal{F}}(\bar{z})(v') = M^\top v' \times \{0_{I\!R^{m+n}}\} - \{0_{I\!R^{n+m}}\} \times \{v'\} + \widehat{D}^* \mathcal{F}(\bar{x}, \bar{b}, \bar{x}^*)(v') \times \{0_{I\!R^n}\},$$

$$D^* \widetilde{\mathcal{F}}(\bar{z})(v') = M^\top v' \times \{0_{I\!R^{m+n}}\} - \{0_{I\!R^{n+m}}\} \times \{v'\} + D^* \mathcal{F}(\bar{x}, \bar{b}, \bar{x}^*)(v') \times \{0_{I\!R^n}\},$$

for every $v' \in I\!R^n$. Note that the graph of $\widetilde{\mathcal{F}}(\cdot)$ is closed by Lemma 2.8. In addition, it is obvious that $\ker D^* \widetilde{\mathcal{F}}(\bar{z}) = \{0\}$. Applying Theorem 3.1 in [22] to the implicit multifunction

$$S(b, q) = \big\{ x \in I\!R^n \,|\, 0 \in \widetilde{\mathcal{F}}(x, b, q) \big\}$$

and using Remark 2.6, we get all the assertions of the theorem. □

Following Definition 1.7, the map $S(\cdot)$ is locally metrically regular around $(\bar{b}, \bar{q}, \bar{x}) \in \text{gph}S$ if there are $\mu > 0$, $\gamma > 0$, and neighborhoods U of $\bar{x}$, V of $(\bar{b}, \bar{q})$ such that

$$\text{dist}\big((b, q); S^{-1}(x)\big) \leq \mu\, \text{dist}\big(x; S(b, q)\big)$$

for all $x \in U$ and $(b, q) \in V$ satisfying $\text{dist}\big(x; S(b, q)\big) \leq \gamma$. Suppose that $S(\cdot)$ is locally closed around $(\bar{b}, \bar{q}, \bar{x})$. Then, by Theorem 1.4, $S(\cdot)$ is locally metrically regular around $(\bar{b}, \bar{q}, \bar{x})$ if and only if $D^*S^{-1}(\bar{x}, \bar{b}, \bar{q})(0) = \{0\}$. By virtue of (2.51), the last equality is equivalent to

$$\ker D^*S(\bar{b}, \bar{q}, \bar{x}) = \{0\}. \tag{2.53}$$

Theorem 2.8 *The following assertions hold*

(i) *If $S(\cdot)$ is locally metrically regular around $(\bar{b}, \bar{q}, \bar{x})$, then $\ker\widehat{\mathbf{K}}_{M,\bar{q}} = \{0\}$. The last equality means that*

$$\Big[(-x', 0, 0) \in M^\top v' \times \{0_{I\!R^{m+n}}\} - \{0_{I\!R^{n+m}}\} \times \{v'\} + \widehat{D}^*\mathcal{F}(\bar{x}, \bar{b}, \bar{x}^*)(v') \times \{0_{I\!R^n}\}\Big] \Longrightarrow \Big[x' = 0\Big].$$

(ii) *If $\ker\mathbf{L}_{M,\bar{q}} = \{0\}$, or equivalently as*

$$\Big[(-x', 0, 0) \in M^\top v' \times \{0_{I\!R^{m+n}}\} - \{0_{I\!R^{n+m}}\} \times \{v'\} + D^*\mathcal{F}(\bar{x}, \bar{b}, \bar{x}^*)(v') \times \{0_{I\!R^n}\}\Big] \Longrightarrow \Big[x' = 0\Big],$$

then $S(\cdot)$ is locally metrically regular around $(\bar{b}, \bar{q}, \bar{x})$.

(iii) *If $\mathcal{F}(\cdot)$ is graphically regular at $(\bar{x}, \bar{b}, \bar{x}^*)$, then $S(\cdot)$ is locally metrically regular around $(\bar{b}, \bar{q}, \bar{x})$ if and only if $\ker\widehat{\mathbf{K}}_{M,\bar{q}} = \{0\}$.*

Proof. (i) According to Lemma 2.8, the graph of $\widetilde{\mathcal{F}}(\cdot)$ is closed. Hence, there exists $\eta > 0$ such that

$$\text{gph}\widetilde{\mathcal{F}} \cap \big\{(x, b, q, v)\big|\ \|x - \bar{x}\| + \|b - \bar{b}\| + \|q - \bar{q}\| + \|v\| \leq \eta\big\}$$

is closed. Thus, $\text{gph}\widetilde{\mathcal{F}} \cap \big\{(x, b, q, 0)\big|\ \|x - \bar{x}\| + \|b - \bar{b}\| + \|q - \bar{q}\| \leq \eta\big\}$ is also closed. This implies that

$$\text{gph}S \cap \big\{(b, q, x)\big|\ \ \|b - \bar{b}\| + \|q - \bar{q}\| + \|x - \bar{x}\| \leq \eta\big\}$$

is closed. This meas that $S(\cdot)$ is locally closed around $(\bar{b}, \bar{q}, \bar{x})$. Since $S(\cdot)$ is locally metrically regular around $(\bar{b}, \bar{q}, \bar{x})$, the equality (2.53) holds. By Theorem 2.7, we deduce that $\ker\widehat{\mathbf{K}}_{M,\bar{q}} = \{0\}$.

(ii) From the assumption $\ker\mathbf{L}_{M,\bar{q}} = \{0\}$ and the estimates in Theorem 2.7 we obtain $\ker D^*S(\bar{b}, \bar{q}, \bar{x}) = \{0\}$. This shows that $S(\cdot)$ is locally metrically regular around $(\bar{b}, \bar{q}, \bar{x})$.

(iii) Since $\mathcal{F}(\cdot)$ is graphically regular at $(\bar{x}, \bar{b}, \bar{x}^*)$, by Theorem 2.7 we get the following equalities

$$\widehat{\mathbf{K}}_{M,\bar{q}}(x') = \widehat{D}^*S(\bar{b}, \bar{q}, \bar{x})(x') = D^*S(\bar{b}, \bar{q}, \bar{x})(x') = \mathbf{K}_{M,\bar{q}}(x')$$

for every $x' \in I\!R^n$. These equalities implies that $\ker\widehat{\mathbf{K}}_{M,\bar{q}} = \ker D^*S(\bar{b}, \bar{q}, \bar{x})$. Consequently, $S(\cdot)$ is locally metrically regular around $(\bar{b}, \bar{q}, \bar{x})$ if and only if $\ker\widehat{\mathbf{K}}_{M,\bar{q}} = \ker D^*S(\bar{b}, \bar{q}, \bar{x}) = \{0\}$. □

According to Definition 1.6, the map $S(\cdot)$ is locally Lipschitz-like around $(\bar{b}, \bar{q}, \bar{x}) \in \text{gph}S$ if there exist $\ell > 0$, and some neighborhoods U of $\bar{x}$ and V of $(\bar{b}, \bar{q})$ such that

$$S(b, q) \cap U \subset S(b', q') + \ell\|(b, q) - (b', q')\|\bar{B}_{I\!R^n}$$

holds for all $(b, q), (b', q') \in V$, where $\bar{B}_{I\!R^n}$ denotes the unit closed ball in $I\!R^n$. If $S(\cdot)$ is locally closed around $(\bar{b}, \bar{q}, \bar{x})$, then Theorem 1.3 asserts that $S(\cdot)$ is locally Lipschitz-like around $(\bar{b}, \bar{q}, \bar{x})$ if and only if

$$D^*S(\bar{b}, \bar{q}, \bar{x})(0) = \{0\}. \tag{2.54}$$

Theorem 2.9 *The following assertions are valid*

(i) *If $S(\cdot)$ is locally Lipschitz-like around $(\bar{b}, \bar{q}, \bar{x})$, then $\widehat{\mathbf{K}}_{M,\bar{q}}(0) = \{0\}$.*

(ii) *If $\mathbf{L}_{M,\bar{q}}(0) = \{0\}$, then $S(\cdot)$ is locally Lipschitz-like around $(\bar{b}, \bar{q}, \bar{x})$.*

(iii) *If $\mathcal{F}(\cdot)$ is graphically regular at $(\bar{x}, \bar{b}, \bar{x}^*)$, then $S(\cdot)$ is locally Lipschitz-like around $(\bar{b}, \bar{q}, \bar{x})$ if and only if $\widehat{\mathbf{K}}_{M,\bar{q}}(0) = \{0\}$.*

Proof. (i) Applying the argument of the proof of Theorem 2.8, we infer that $S(\cdot)$ is locally closed around $(\bar{b}, \bar{q}, \bar{x})$. Since $S(\cdot)$ is locally Lipschitz-like around $(\bar{b}, \bar{q}, \bar{x})$, we have $D^*S(\bar{b}, \bar{q}, \bar{x})(0) = \{0\}$. By Theorem 2.7, it follows that $\widehat{\mathbf{K}}_{M,\bar{q}}(0) = \{0\}$.

(ii) Using the estimates in Theorem 2.7, from the equality $\mathbf{L}_{M,\bar{q}}(0) = \{0\}$ we deduce that $D^*S(\bar{b}, \bar{q}, \bar{x})(0) = \{0\}$. The last equality means that $S(\cdot)$ is locally Lipschitz-like around $(\bar{b}, \bar{q}, \bar{x})$.

(iii) If $\mathcal{F}(\cdot)$ is graphically regular at $(\bar{x}, \bar{b}, \bar{x}^*)$, Theorem 2.7 asserts that

$$\widehat{\mathbf{K}}_{M,\bar{q}}(x') = \widehat{D}^* S(\bar{b}, \bar{q}, \bar{x})(x') = D^* S(\bar{b}, \bar{q}, \bar{x})(x') = \mathbf{K}_{M,\bar{q}}(x')$$

for every $x' \in I\!R^n$. Therefore, $S(\cdot)$ is locally Lipschitz-like around $(\bar{b}, \bar{q}, \bar{x})$ if and only if $\widehat{\mathbf{K}}_{M,\bar{q}}(0) = D^* S(\bar{b}, \bar{q}, \bar{x})(0) = \{0\}$. □

2.5 Conclusions

Theorem 2.4 in this chapter gives an exact formula for the Fréchet normal cone to the graph of the normal cone mappings $\mathcal{F}(x, b)$ in reflexive Banach spaces, and Theorem 2.6 establishes upper and lower estimates for the limiting normal cone to the graph of $\mathcal{F}(x, b)$. Based on the results, an exact formula for the Fréchet coderivative and upper and lower estimates for the values of the Mordukhovich coderivative of $\mathcal{F}(x, b)$ are derived. These formulas yield the necessary conditions and the sufficient conditions for the local metric regularity in Theorem 2.8, as well as the necessary conditions and the sufficient conditions in Theorem 2.9 for the locally Lipschitz-like property of the solution maps of affine variational inequalities under linear perturbations. Proposition 2.2 and Proposition 2.3 respectively answer the first and the second open questions raised by Yao and Yen in [52].

Chapter 3

Nonlinear Perturbations of Polyhedral Normal Cone Mappings

As a continuation of the study of generalized differentiation of the normal cone mappings presented in the previous chapter, this chapter is devoted to the estimation of the Fréchet and the limiting normal cones to the graphs of the normal cone mappings to nonlinearly perturbed polyhedral convex sets in finite dimensional spaces. The obtained estimates are applied to solution stability of affine variational inequalities under nonlinear perturbations.

The presentation given below comes from the results in [41].

3.1 The Normal Cone Mapping $\mathcal{F}(x, A, b)$

Let $T = \{1, 2, \ldots, m\}$ be a given index set. For each pair $(A, b) \in I\!R^{m\times n} \times I\!R^m$, we consider the perturbed polyhedral convex set

$$\Theta(A, b) = \{x \in I\!R^n |\ Ax \le b\}, \tag{3.1}$$

where $A = (a_{ij})_{m\times n} \in I\!R^{m\times n}$ and $b = (b_1, \ldots, b_m) \in I\!R^m$ are parameters. We interpret $A_1^\top, \ldots, A_m^\top$, where $A_i = (a_{i1} \ldots a_{in})$ is the i-th row of the matrix A and superscript $^\top$ denotes transposition, as *nonlinear perturbations* and $b_1, \ldots, b_m$ as the *right-hand side perturbations* of the system

$$\langle A_i^\top, x\rangle \le b_i, \quad i \in T. \tag{3.2}$$

For every $(x, A, b) \in I\!R^n \times I\!R^{m\times n} \times I\!R^m$ with $A = (a_{ij})_{m\times n}$, $b = (b_1, \ldots, b_m)$, and $x \in \Theta(A, b)$, let

$$I(x, A, b) = \{i \in T |\ A_i x = b_i\} \tag{3.3}$$

denote the *active index set* corresponding to the triplet (x, A, b). For any subset $\Gamma \subset T$, we put $\overline{\Gamma} := T \backslash \Gamma$. The equality $A_\Gamma = 0$ means that $A_i = 0$ for all $i \in \Gamma$. If $\Gamma = \{i_1, \ldots, i_r\}$, then by $a_{\Gamma,j}$ we denote the column vector

$$a_{\Gamma,j} = \begin{pmatrix} a_{i_1 j} \\ \vdots \\ a_{i_r j} \end{pmatrix} \quad \text{for any } j \in \{1, \ldots, n\}.$$

The notation $a_{\Gamma,j} \leq 0$ (resp., $a_{\Gamma,j} \geq 0$, $a_{\Gamma,j} = 0$) means that $a_{ij} \leq 0$ (resp., $a_{ij} \geq 0$, $a_{ij} = 0$) for all $i \in \Gamma$. The notation $b_\Gamma \leq 0$ (resp., $b_\Gamma \geq 0$, $b_\Gamma = 0$) is used whenever $b_i \leq 0$ (resp., $b_i \geq 0$, $b_i = 0$) for all $i \in \Gamma$. As usual, the norm of a matrix $A \in I\!R^{m \times n}$ is defined by

$$\|A\| = \max\big\{\|Ax\| \mid x \in I\!R^n, \|x\| = 1\big\},$$

where $\|x\|$ and $\|Ax\|$ denote, respectively, the Euclidean norms of $x \in I\!R^n$ and $Ax \in I\!R^m$. The scalar product of two matrices $A = (a_{ij})$, $\widetilde{A} = (\widetilde{a}_{ij})$ from $I\!R^{m \times n}$ is given by

$$\langle A, \widetilde{A} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \widetilde{a}_{ij}.$$

The multifunction $\mathcal{F} : I\!R^n \times I\!R^{m \times n} \times I\!R^m \rightrightarrows I\!R^n$ given by

$$\mathcal{F}(x, A, b) = N(x; \Theta(A, b)), \ \ \forall (x, A, b) \in I\!R^n \times I\!R^{m \times n} \times I\!R^m, \tag{3.4}$$

is said to be the *nonlinearly perturbed polyhedral normal cone mapping* to the perturbed polyhedron $\Theta(A, b)$ (or, the *normal cone mapping* $\mathcal{F}(\cdot)$, for short), where the formula

$$N(x; \Theta(A, b)) = \begin{cases} \big\{\xi^* \in I\!R^n \big| \ \langle \xi^*, u - x \rangle \leq 0, \ \forall u \in \Theta(A, b)\big\}, & \text{if } x \in \Theta(A, b) \\ \emptyset, & \text{if } x \notin \Theta(A, b) \end{cases}$$

defines the normal cone to the set $\Theta(A, b)$ at x in the sense of convex analysis.

Specializing Lemma 3.1 from [32], which has been obtained by using a generalized version of the Farkas lemma [3], to the linear inequalities (3.2) we have the following statement.

Proposition 3.1 *Let* $\Theta(A, b)$ *be defined by* (3.1). *For any point* $x \in \Theta(A, b)$, *let* $I = I(x, A, b)$ *with* $I(x, A, b)$ *being given by* (3.3). *Then*

$$\begin{aligned} N(x; \Theta(A, b)) &= \text{pos}\big\{A_i^\top \,|\, i \in I\big\} \\ &= \left\{ \sum_{i \in I} \lambda_i A_i^\top \,\middle|\, \lambda_i \geq 0, \ \forall i \in I \right\}, \end{aligned}$$

and

$$T(x;\Theta(A,b))=\Big\{v\in I\!R^n \big|\ \langle A_i^\top,v\rangle\le 0,\ \forall i\in I\Big\}.$$

The forthcoming lemma shows that the positive linear independence property of a finite system of vectors is stable under small nonlinear perturbations of the vectors.

Lemma 3.1 *Let $\bar A\in I\!R^{m\times n}$ and $\Gamma\subset T$. If the vectors $\{\bar A_i^\top|\ i\in\Gamma\}$ are positively linearly independent, then there exists $\delta>0$ such that for any $A\in\bar B(\bar A,\delta)$ the vectors $\{A_i^\top|\ i\in\Gamma\}$ are also positively linearly independent.*

Proof. To obtain a contradiction, suppose that for any $\delta>0$ there exists some A in $\bar B(\bar A,\delta)$ with $\{A_i^\top|\ i\in\Gamma\}$ being not positively linearly independent. This implies that for every $k\in I\!N$ there exist $A_k\in\bar B(\bar A,k^{-1})$ and $\lambda_i^k\ge 0$ $(i\in\Gamma)$, which are not all zero, such that $\sum_{i\in\Gamma}\lambda_i^k(A_k)_i^\top=0$. For each $k\in I\!N$, we put

$$\varrho^k:=\sum_{i\in\Gamma}\lambda_i^k>0,\quad \text{and}\quad \mu_i^k:=\frac{\lambda_i^k}{\varrho^k}\in[0,1]\quad\text{for all } i\in\Gamma.$$

Without any loss of generality, we can assume that there exists a subsequence $\{k_\ell\}$ of $\{k\}$ such that $\lim_{\ell\to\infty}\mu_i^{k_\ell}=\mu_i\ge 0$ for all $i\in\Gamma$. Note that

$$\sum_{i\in\Gamma}\mu_i^k(A_k)_i^\top=\sum_{i\in\Gamma}\frac{\lambda_i^k}{\varrho^k}(A_k)_i^\top=\frac{1}{\varrho^k}\sum_{i\in\Gamma}\lambda_i^k(A_k)_i^\top=0\quad\text{for all } k\in I\!N.$$

Hence,

$$\sum_{i\in\Gamma}\mu_i\bar A_i^\top=\lim_{\ell\to\infty}\sum_{i\in\Gamma}\mu_i^{k_\ell}(A_{k_\ell})_i^\top=0.$$

This and the assumed positive linear independence of $\{\bar A_i^\top|\ i\in\Gamma\}$ force $\mu_i=0$ for all $i\in\Gamma$. As

$$\sum_{i\in\Gamma}\mu_i=\lim_{\ell\to\infty}\sum_{i\in\Gamma}\mu_i^{k_\ell}=1,$$

we arrive at a contradiction, which completes the proof. □

The next proposition shows that the graph of the normal cone mapping $\mathcal{F}(x,A,b)$ is locally closed in the product space $I\!R^n\times I\!R^{m\times n}\times I\!R^m\times I\!R^n$ under the positive linear independence condition on the normal vectors of the active constraints. This property allows us to calculate the limiting normal cone $N\big((\bar x,\bar A,\bar b,\bar\xi^*);\mathrm{gph}\mathcal{F}\big)$ via formula (1.5).

Proposition 3.2 *For any* $(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) \in \text{gph}\mathcal{F}$*, if* $\{\bar{A}_i^\top |\ i \in I(\bar{x}, \bar{A}, \bar{b})\}$ *are positively linearly independent, then* $\text{gph}\mathcal{F}$ *is locally closed around* $(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)$.

Proof. First, note that there exists $\delta > 0$ such that $I(x, A, b) \subset I(\bar{x}, \bar{A}, \bar{b})$ for all $(x, A, b) \in \bar{B}(\bar{x}, \delta) \times \bar{B}(\bar{A}, \delta) \times \bar{B}(\bar{b}, \delta)$. Indeed, suppose on the contrary that for any $\delta > 0$ there exists $(x, A, b) \in \bar{B}(\bar{x}, \delta) \times \bar{B}(\bar{A}, \delta) \times \bar{B}(\bar{b}, \delta)$ such that $I(x, A, b) \backslash I(\bar{x}, \bar{A}, \bar{b}) \neq \emptyset$. Then, for each $k \in I\!N$, we can find (x_k, A_k, b_k) in $\bar{B}(\bar{x}, k^{-1}) \times \bar{B}(\bar{A}, k^{-1}) \times \bar{B}(\bar{b}, k^{-1})$ such that $I(x_k, A_k, b_k) \backslash I(\bar{x}, \bar{A}, \bar{b}) \neq \emptyset$. It follows that, for every $k \in I\!N$, there is an index $i \in T \backslash I(\bar{x}, \bar{A}, \bar{b})$ satisfying $\langle (A_k)_i^\top, x_k \rangle = (b_k)_i$. By the Dirichlet principle there exist a subsequence $\{k_\ell\}$ of $\{k\}$ and an index $i_0 \in T \backslash I(\bar{x}, \bar{A}, \bar{b})$ with $\langle (A_{k_\ell})_{i_0}^\top, x_{k_\ell} \rangle = (b_{k_\ell})_{i_0}$ for all $\ell \in I\!N$. This implies that

$$\langle \bar{A}_{i_0}^\top, \bar{x} \rangle = \langle \lim_{\ell\to\infty} (A_{k_\ell})_{i_0}^\top, \lim_{\ell\to\infty} x_{k_\ell} \rangle = \lim_{\ell\to\infty} \langle (A_{k_\ell})_{i_0}^\top, x_{k_\ell} \rangle = \lim_{\ell\to\infty} (b_{k_\ell})_{i_0} = \bar{b}_{i_0}.$$

Therefore, we obtain $i_0 \in I(\bar{x}, \bar{A}, \bar{b})$, a contradiction.

Due to Lemma 3.1, we can assume without loss of generality that vectors $\{A_i^\top |\ i \in I(\bar{x}, \bar{A}, \bar{b})\}$ are positively linearly independent for all $A \in \bar{B}(\bar{A}, \delta)$.

We now prove that $\mathcal{G} := \text{gph}\mathcal{F} \cap \big(\bar{B}(\bar{x}, \delta) \times \bar{B}(\bar{A}, \delta) \times \bar{B}(\bar{b}, \delta) \times I\!R^n\big)$ is a closed set. Suppose that $\{(x_k, A_k, b_k, \xi_k^*)\}$ is a sequence in $\mathcal{G}$ and (x_k, A_k, b_k, ξ_k^*) converges to $(\widehat{x}, \widehat{A}, \widehat{b}, \widehat{\xi}^*)$. We have to show that $(\widehat{x}, \widehat{A}, \widehat{b}, \widehat{\xi}^*) \in \mathcal{G}$. Note that $(\widehat{x}, \widehat{A}, \widehat{b}) \in \bar{B}(\bar{x}, \delta) \times \bar{B}(\bar{A}, \delta) \times \bar{B}(\bar{b}, \delta)$ because $\bar{B}(\bar{x}, \delta) \times \bar{B}(\bar{A}, \delta) \times \bar{B}(\bar{b}, \delta)$ is closed in $I\!R^n \times I\!R^{m\times n} \times I\!R^m$ and (x_k, A_k, b_k) converges to $(\widehat{x}, \widehat{A}, \widehat{b})$. According to the observation stated at the beginning of this proof, we have $I(x_k, A_k, b_k) \subset I(\bar{x}, \bar{A}, \bar{b})$ for sufficiently large indexes $k \in I\!N$. Without loss of generality, we can assume that $I(x_k, A_k, b_k) = \widetilde{I} \subset I(\bar{x}, \bar{A}, \bar{b})$ for all $k \in I\!N$. For each $k \in I\!N$, it holds

$$\xi_k^* = \sum_{i\in\widetilde{I}} \lambda_i^k (A_k)_i^\top \quad \text{for some } \lambda_i^k \geq 0,\ i \in \widetilde{I}. \tag{3.5}$$

We see that the sequences $\{\lambda_i^k\}_{k\in I\!N}$, $i \in \widetilde{I}$, are bounded. Indeed, suppose on the contrary that there exists $i_0 \in \widetilde{I}$ such that $\{\lambda_{i_0}^k\}_{k\in I\!N}$ is unbounded. Then, for every $\ell \in I\!N$ there is $k_\ell \geq \ell$ satisfying $\lambda_{i_0}^{k_\ell} > \ell$. There is no loss of generality in assuming that $k_{\ell+1} > k_\ell$ for all $\ell \in I\!N$. Since $\sum_{j\in\widetilde{I}} \lambda_j^{k_\ell} \geq \lambda_{i_0}^{k_\ell} \to +\infty$ as $\ell \to \infty$, $\varrho^{k_\ell} := \sum_{j\in\widetilde{I}} \lambda_j^{k_\ell}$ converges $+\infty$ as $\ell \to \infty$. Note that

$$\mu_i^{k_\ell} := \frac{\lambda_i^{k_\ell}}{\varrho^{k_\ell}} = \frac{\lambda_i^{k_\ell}}{\sum_{j\in\widetilde{I}} \lambda_j^{k_\ell}} \in [0, 1] \quad \text{for all } i \in \widetilde{I}.$$

Thus, for each index $i \in \widetilde{I}$, any subsequence of $\{\mu_i^{k_\ell}\}$ possesses a convergent subsequence. We can assume that $\lim_{\ell\to\infty} \mu^{k_\ell} = \mu_i \geq 0$ for every $i \in \widetilde{I}$. As $\xi_k^* \to \widehat{\xi}^*$, from (3.5) we deduce that

$$0 = \lim_{\ell\to\infty} \frac{1}{\varrho^{k_\ell}} \xi_{k_\ell}^* = \lim_{\ell\to\infty} \sum_{i\in\widetilde{I}} \mu_i^{k_\ell} (A_{k_\ell})_i^\top = \sum_{i\in\widetilde{I}} \mu_i \widehat{A}_i^\top.$$

On one hand, since $\widehat{A} \in \bar{B}(\bar{A}, \delta)$ and $\widetilde{I} \subset I(\bar{x}, \bar{A}, \bar{b})$, the vectors $\{\widehat{A}_i^\top \mid i \in \widetilde{I}\}$ are positively linearly independent. So, the equality $\sum_{i\in\widetilde{I}} \mu_i \widehat{A}_i^\top = 0$ implies $\mu_i = 0$ for all $i \in \widetilde{I}$. On the other hand, the property $\sum_{i\in\widetilde{I}} \mu_i^{k_\ell} = 1$, which holds for all $\ell \in I\!N$, yields

$$\sum_{i\in\widetilde{I}} \mu_i = \sum_{i\in\widetilde{I}} \lim_{\ell\to\infty} \mu_i^{k_\ell} = \lim_{\ell\to\infty} \sum_{i\in\widetilde{I}} \mu_i^{k_\ell} = 1.$$

We have arrived at a contradiction.

By the boundedness of all the sequences $\{\lambda_i^k\}_{k\in I\!N}$ $(i \in \widetilde{I})$, we can assume that there exists a subsequence $\{k_\ell\}$ of $\{k\}$ with $\lim_{\ell\to\infty} \lambda_i^{k_\ell} = \lambda_i \geq 0$ for all $i \in \widetilde{I}$. Combining this with (3.5) yields

$$\widehat{\xi}^* = \lim_{\ell\to\infty} \xi_{k_\ell}^* = \lim_{\ell\to\infty} \sum_{i\in\widetilde{I}} \lambda_i^{k_\ell} (A_{k_\ell})_i^\top = \sum_{i\in\widetilde{I}} \lambda_i \widehat{A}_i^\top \in N(\widehat{x}; \Theta(\widehat{A}, \widehat{b})) = \mathcal{F}(\widehat{x}, \widehat{A}, \widehat{b}).$$

We have shown that $(\widehat{x}, \widehat{A}, \widehat{b}, \widehat{\xi}^*) \in \mathrm{gph}\mathcal{F}$. Moreover, we obtain

$$(\widehat{x}, \widehat{A}, \widehat{b}, \widehat{\xi}^*) \in \mathrm{gph}\mathcal{F} \cap \Big(\bar{B}(\bar{x}, \delta) \times \bar{B}(\bar{A}, \delta) \times \bar{B}(\bar{b}, \delta) \times I\!R^n\Big) = \mathcal{G},$$

which establishes the closedness of $\mathcal{G}$. □

On the basis of generalized differentiability properties of the normal cone mapping $\mathcal{F}(x, A, b)$, we discuss solution stability of the following parametric affine variational inequality problem

$$\text{Find } x \in \Theta(A, b) \quad \text{subject to } \langle Mx - q, u - x\rangle \geq 0, \ \forall u \in \Theta(A, b), \tag{3.6}$$

where $M \in I\!R^{n\times n}$ is a fixed matrix, and $A \in I\!R^{m\times n}$, $b \in I\!R^m$, $q \in I\!R^n$ are subject to change. Let $S(A, b, q)$ be the solution set of the problem (3.6) with respect to a parametric triple (A, b, q). As in [43] and [44], the problem (3.6) can be regarded as a linear generalized equation of the form

$$0 \in Mx - q + \mathcal{F}(x, A, b), \tag{3.7}$$

where the multifunction $\mathcal{F}(x, A, b)$ is given by (3.4). Hence, $S(A, b, q)$ is also the solution set of the linear generalized equation (3.7). We first provide an evaluation of the Fréchet normal cone to the graph of the normal cone mapping $\mathcal{F}(x, A, b)$. Then an upper estimate for the limiting normal cone to the graph of that normal cone mapping is obtained under a *positive linear independence assumption* on the normal vectors of the active constraints of (3.2). These results allow us to evaluate the values of the Mordukhovich coderivative of the normal cone mapping under consideration. Finally, using a standard coderivative sum rule, we are able to establish some criteria for solution stability of the problem (3.6) under nonlinear perturbations.

3.2 Estimation of the Fréchet Normal Cone to $\mathrm{gph}\mathcal{F}$

We now provide an upper estimate for the Fréchet normal cone to the graph of the normal cone mapping $\mathcal{F}(\cdot)$ given by (3.4). This is a major step to establish an upper estimate for the limiting normal cone to the graph of $\mathcal{F}(\cdot)$ which will be discussed in next section.

For any $(x, A, b, \xi^*) \in \mathrm{gph}\mathcal{F}$, we put

$$\Xi(x, A, b, \xi^*) := \Big\{(\lambda_i)_{i\in I}\Big|\ \xi^* = \sum_{i\in I} \lambda_i A_i^\top,\ \lambda_i \geq 0\ \forall i \in I\Big\}, \tag{3.8}$$

and

$$I_1(x, A, b, \xi^*) := \Big\{i \in I\big|\ \lambda_i = 0 \text{ for some } (\lambda_j)_{j\in I} \in \Xi(x, A, b, \xi^*)\Big\}, \tag{3.9}$$

where $I := I(x, A, b)$ is defined by (3.3). The dual of a set $\Omega \subset I\!R^n$ is given by

$$\Omega^* := \big\{u^* \in I\!R^n|\ \langle u^*, v\rangle \leq 0\ \ \forall v \in \Omega\big\}.$$

For each $u \in I\!R^n$, set

$$\{u\}^\perp := \big\{v \in I\!R^n|\ \langle u, v\rangle = 0\big\}.$$

For every $\lambda = (\lambda_i)_{i\in I} \in \Xi(x, A, b, \xi^*)$, we define

$$I_0(\lambda) := \{i \in I|\ \lambda_i = 0\}, \tag{3.10}$$

and

$$
\begin{aligned}
&E_\lambda(x,A,b,\xi^*)\\
&\quad=\Big\{(x^*,A^*,b^*,\xi)\Big|\ x^*\in\big(T(x;\Theta(A,b))\cap\{\xi^*\}^\perp\big)^*,\\
&\qquad\qquad x^*=-\textstyle\sum_{i\in I}b_i^*A_i^\top,\ \xi\in T(x;\Theta(A,b))\cap\{\xi^*\}^\perp,\\
&\qquad\qquad a^*_{I_0(\lambda),j}\le 0\ \text{ if }\ x_j<0,\ a^*_{I_0(\lambda),j}\ge 0\ \text{ if }\ x_j>0,\\
&\qquad\qquad a^*_{I_0(\lambda),j}=0\ \text{ if }\ x_j=0,\ A^*_{\bar I}=0,\ b^*_{\bar I}=0,\ b^*_{I_0(\lambda)}\le 0\Big\},
\end{aligned}
\tag{3.11}
$$

where $A^*=(a^*_{ij})_{m\times n}\in I\!R^{m\times n}$, and $b^*=(b_1^*\ \dots\ b_m^*)^\top\in I\!R^m$. Using the abbreviation $I_1:=I_1(x,A,b,\xi^*)$ with $I_1(x,A,b,\xi^*)$ being given by (3.9), we construct the set

$$
\begin{aligned}
&\mathcal{H}(x,A,b,\xi^*)\\
&\quad=\Big\{(x^*,A^*,b^*,\xi)\Big|\ x^*\in\big(T(x;\Theta(A,b))\cap\{\xi^*\}^\perp\big)^*,\\
&\qquad\qquad x^*=-\textstyle\sum_{i\in I}b_i^*A_i^\top,\ \xi\in T(x;\Theta(A,b))\cap\{\xi^*\}^\perp,\\
&\qquad\qquad a^*_{I_1,j}\le 0\ \text{ if }\ x_j<0,\ a^*_{I_1,j}\ge 0\ \text{ if }\ x_j>0,\\
&\qquad\qquad a^*_{I_1,j}=0\ \text{ if }\ x_j=0,\ A^*_{\bar I}=0,\ b^*_{\bar I}=0,\ b^*_{I_1}\le 0\Big\}.
\end{aligned}
\tag{3.12}
$$

Although the construction of (3.12) is a little bit complicated, it is clear that the set $\mathcal{H}(x,A,b,\xi^*)$ can be computed explicitly.

We first clarify a relationship between the family $\big\{E_\lambda(x,A,b,\xi^*)\big\}_{\lambda\in\Xi(x,A,b,\xi^*)}$ and $\mathcal{H}(x,A,b,\xi^*)$. Later, in Theorem 3.1 it will be proved that $\mathcal{H}(x,A,b,\xi^*)$ is an upper estimate for the Fréchet normal cone $\widehat{N}\big((x,A,b,\xi^*);\mathrm{gph}\mathcal{F}\big)$.

Lemma 3.2 *For any $(\bar x,\bar A,\bar b,\bar\xi^*)\in\mathrm{gph}\mathcal{F}$, it holds*

$$
\mathcal{H}(\bar x,\bar A,\bar b,\bar\xi^*)=\bigcap_{\lambda\in\Xi(\bar x,\bar A,\bar b,\bar\xi^*)}E_\lambda(\bar x,\bar A,\bar b,\bar\xi^*). \tag{3.13}
$$

Proof. From (3.9) and (3.10) we infer that

$$
I_1(\bar x,\bar A,\bar b,\bar\xi^*)=\bigcup_{\lambda\in\Xi(\bar x,\bar A,\bar b,\bar\xi^*)}I_0(\lambda).
$$

By (3.11) and (3.12), the equality (3.13) follows from this fact. □

For any $\bar A\in I\!R^{m\times n}$, we consider the multifunction $F_{\bar A}:I\!R^n\times I\!R^m\rightrightarrows I\!R^n$ defined by setting

$$
F_{\bar A}(x,b):=N(x;\Theta(\bar A,b)). \tag{3.14}
$$

The following lemma gives us an upper estimate for the Fréchet normal cone to the graph of $F_{\bar{A}}$ at a given point $(x, b, \xi^*) \in \mathrm{gph} F_{\bar{A}}$.

Lemma 3.3 (See [52, Lemma 4.1]) *If $(x^*, b^*, \xi) \in \widehat{N}\big((x, b, \xi^*); \mathrm{gph} F_{\bar{A}}\big)$, then*

$$(x^*, \xi) \in \big(T(x; \Theta(\bar{A}, b)) \cap \{\xi^*\}^{\perp}\big)^* \times \big(T(x; \Theta(\bar{A}, b)) \cap \{\xi^*\}^{\perp}\big),$$

$$x^* = -\sum_{i \in I} b_i^* \bar{A}_i^\top, \quad \textit{and} \quad b_{\bar{I}}^* = 0,$$

where $I = \{i \in T |\ \bar{A}_i x = b_i\}$ and $\bar{I} = T \setminus I$. Moreover, if $\xi^* = \sum_{i \in I} \lambda_i \bar{A}_i^\top$ with $\lambda_i \geq 0$ for all $i \in I$, and $I_0 = \{i \in I |\ \lambda_i = 0\}$, then $b_{I_0}^* \leq 0$.

The main result can be formulated as follows.

Theorem 3.1 *For any $(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) \in \mathrm{gph}\mathcal{F}$, it holds*

$$\widehat{N}\big((\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*); \mathrm{gph}\mathcal{F}\big) \subset \mathcal{H}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*). \tag{3.15}$$

Proof. According to Lemma 3.2, (3.15) is equivalent to the inclusion

$$\widehat{N}\big((\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*); \mathrm{gph}\mathcal{F}\big) \subset \bigcap_{\lambda \in \Xi(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)} E_\lambda(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*). \tag{3.16}$$

To obtain (3.16), let us pick any $(x^*, A^*, b^*, \xi) \in \widehat{N}\big((\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*); \mathrm{gph}\mathcal{F}\big)$ and $\lambda = (\lambda_i)_{i \in I} \in \Xi(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)$ with $I = I(\bar{x}, \bar{A}, \bar{b})$ being given by (3.3). The proof will be completed if we can show that

$$(x^*, A^*, b^*, \xi) \in E_\lambda(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*). \tag{3.17}$$

Since $(x^*, A^*, b^*, \xi) \in \widehat{N}\big((\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*); \mathrm{gph}\mathcal{F}\big)$, we have

$$\limsup_{(x, A, b, \xi^*) \xrightarrow{\mathrm{gph}\mathcal{F}} (\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)} \frac{\langle x^*, x - \bar{x}\rangle + \langle A^*, A - \bar{A}\rangle + \langle b^*, b - \bar{b}\rangle + \langle \xi, \xi^* - \bar{\xi}^*\rangle}{\|x - \bar{x}\| + \|A - \bar{A}\| + \|b - \bar{b}\| + \|\xi^* - \bar{\xi}^*\|} \leq 0. \tag{3.18}$$

Note that $(x, \bar{A}, b, \xi^*) \in \mathrm{gph}\mathcal{F}$ if and only if $\xi^* \in N(x; \Theta(\bar{A}, b)) = F_{\bar{A}}(x, b)$, where $F_{\bar{A}}$ is given by (3.14). Hence, $(x, \bar{A}, b, \xi^*) \in \mathrm{gph}\mathcal{F}$ if and only if $(x, b, \xi^*) \in \mathrm{gph} F_{\bar{A}}$. Taking $A = \bar{A}$, by (3.18) we deduce that

$$\limsup_{(x, b, \xi^*) \xrightarrow{\mathrm{gph} F_{\bar{A}}} (\bar{x}, \bar{b}, \bar{\xi}^*)} \frac{\langle x^*, x - \bar{x}\rangle + \langle b^*, b - \bar{b}\rangle + \langle \xi, \xi^* - \bar{\xi}^*\rangle}{\|x - \bar{x}\| + \|b - \bar{b}\| + \|\xi^* - \bar{\xi}^*\|} \leq 0.$$

This means that $(x^*, b^*, \xi) \in \widehat{N}\big((\bar{x}, \bar{b}, \bar{\xi}^*); \mathrm{gph} F_{\bar{A}}\big)$. Applying Lemma 3.3, we obtain

$$(x^*, \xi) \in \big(T(\bar{x}; \Theta(\bar{A}, \bar{b})) \cap \{\bar{\xi}^*\}^{\perp}\big)^* \times \big(T(\bar{x}; \Theta(\bar{A}, \bar{b})) \cap \{\bar{\xi}^*\}^{\perp}\big),$$

and

$$x^* = -\sum_{i\in I} b_i^* \bar{A}_i^\top,\ b_{\bar{I}}^* = 0,\ b_{I_0(\lambda)}^* \le 0,$$

where $\bar{I} = T\backslash I$ and $I_0(\lambda) = \{i \in I|\ \lambda_i = 0\}$. By virtue of these relations, based on (3.11) we see that the proof of (3.17) now reduces to establishing the following properties

$$\begin{aligned} &a_{I_0(\lambda),j}^* \le 0 \ \text{ if } \ x_j < 0,\\ &a_{I_0(\lambda),j}^* \ge 0 \ \text{ if } \ x_j > 0,\\ &a_{I_0(\lambda),j}^* = 0 \ \text{ if } \ x_j = 0,\\ &A_{\bar{I}}^* = 0. \end{aligned} \tag{3.19}$$

First, we prove that $A_{\bar{I}}^* = 0$. Fix any pair $(r,s) \in \bar{I} \times \{1,\ldots,n\}$. Let $b = \bar{b}$ and let $a_{ij} = \bar{a}_{ij}$ for all $i \in T$ and $j \in \{1,\ldots,n\}$ with $(i,j) \ne (r,s)$. Choose $a_{rs} \in (\bar{a}_{rs} - \varepsilon, \bar{a}_{rs} + \varepsilon)$ and let $A = (a_{ij})$, where $\varepsilon > 0$ is as small as

$$\bar{A}_r\bar{x} - \varepsilon\bar{x}_s < b_r \quad \text{and} \quad \bar{A}_r\bar{x} + \varepsilon\bar{x}_s < b_r. \tag{3.20}$$

Note that there is $t \in (0,1)$ such that $a_{rs} = t(\bar{a}_{rs} - \varepsilon) + (1-t)(\bar{a}_{rs} + \varepsilon)$. Combining this with (3.20), we get

$$A_r\bar{x} = t(\bar{A}_r\bar{x} - \varepsilon\bar{x}_s) + (1-t)(\bar{A}_r\bar{x} + \varepsilon\bar{x}_s) < b_r.$$

By the construction of A and b, we obtain

$$A_i\bar{x} = b_i \ \ \forall i \in I \quad \text{and} \quad A_i\bar{x} < b_i \ \ \forall i \in \bar{I}.$$

Hence $x := \bar{x}$ belongs to $\Theta(A,b)$, and $\xi^* := \bar{\xi}^*$ satisfies the relation

$$\xi^* \in \operatorname{pos}\{\bar{A}_i^\top|\ i \in I\} = \operatorname{pos}\{A_i^\top|\ i \in I\} = N(x;\Theta(A,b)).$$

From (3.18) it follows that

$$\limsup_{a_{rs}\to\bar{a}_{rs}} \frac{a_{rs}^*(a_{rs} - \bar{a}_{rs})}{|a_{rs} - \bar{a}_{rs}|} \le 0.$$

Since $a_{rs} \in (\bar{a}_{rs} - \varepsilon, \bar{a}_{rs} + \varepsilon)$ can be chosen arbitrarily, this implies that $a_{rs}^* = 0$. Thus, we deduce that $A_{\bar{I}}^* = 0$.

Now, fix any pair $(r,s) \in I_0(\lambda) \times \{1,\ldots,n\}$. Choose $x = \bar{x},\ b = \bar{b},\ \xi^* = \bar{\xi}^*$, $a_{ij} = \bar{a}_{ij}$ for all $i \in T$ and $j \in \{1,\ldots,n\}$ satisfying $(i,j) \ne (r,s)$. Let $A = (a_{ij})$, where a_{rs} is chosen as follows:

α) If $\bar{x}_s = 0$, then $a_{rs} \in (-\infty,+\infty)$ is taken arbitrary;

β) If $\bar{x}_s > 0$, then we choose $a_{rs} \in (-\infty, \bar{a}_{rs})$;

γ) If $\bar{x}_s < 0$, then we choose $a_{rs} \in (\bar{a}_{rs}, +\infty)$.

Denote the set of all $a_{rs} \in I\!R$ satisfying these conditions by Ω. By the choice of (x, A, b, ξ^*) and the condition $r \in I_0(\lambda)$, we infer that $(x, A, b, \xi^*) \in \text{gph}\mathcal{F}$. By (3.18), it holds

$$\limsup_{a_{rs} \xrightarrow{\Omega} \bar{a}_{rs}} \frac{a^*_{rs}(a_{rs} - \bar{a}_{rs})}{|a_{rs} - \bar{a}_{rs}|} \le 0. \tag{3.21}$$

From (3.21) it follows that

α_1) If $\bar{x}_s = 0$, then $a^*_{rs} = 0$;

β_1) If $\bar{x}_s > 0$, then $a^*_{rs} \ge 0$;

γ_1) If $\bar{x}_s < 0$, then $a^*_{rs} \le 0$.

Thus, the properties in (3.19) are valid. The proof is complete. □

To have an idea about how the estimation formula (3.15) can be used in practical calculations, consider the following numerical example, which has the origin in [49, Example 2].

Example 3.1 Let $T = \{1, 2, 3\}$ and $\mathcal{F} : I\!R^2 \times I\!R^{3\times 2} \times I\!R^3 \rightrightarrows I\!R^2$ be given by $\mathcal{F}(x, A, b) = N(x; \Theta(A, b))$ for any $(x, A, b) \in I\!R^2 \times I\!R^{3\times 2} \times I\!R^3$. Let

$$\bar{A} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \bar{x} = \begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \end{pmatrix} \in \Theta(\bar{A}, \bar{b}),$$

where $\Theta(A, b) = \{x \in I\!R^2 |\ Ax \le b\}$.

We have $I = I(\bar{x}, \bar{A}, \bar{b}) = \{i \in T |\ \bar{A}_i \bar{x} = b_i\} = \{1\}$, $\bar{I} = T \setminus I = \{2, 3\}$. Choosing $\bar{\xi}^* = (\frac{2}{3}, \frac{2}{3})$, we observe that

$$\bar{\xi}^* = \frac{2}{3}\bar{A}_1^\top \in \text{pos}\{\bar{A}_1^\top\} = N(\bar{x}; \Theta(\bar{A}, \bar{b})).$$

Hence, $(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)$ belongs to gph$\mathcal{F}$. Note that

$$\begin{aligned} \Xi(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) &= \left\{ (\lambda_i)_{i\in I} \middle|\ \xi^* = \textstyle\sum_{i\in I} \lambda_i A_i^\top,\ \lambda_i \ge 0\ \forall i \in I \right\} \\ &= \left\{ \lambda_1 \middle|\ \bar{\xi}^* = \lambda_1 \bar{A}_1^\top,\ \lambda_1 \ge 0 \right\} = \left\{ \frac{2}{3} \right\}. \end{aligned}$$

Thus,

$$I_1 = I_1(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) = \Big\{ i \in I \big|\ \lambda_i = 0 \text{ for some } (\lambda_j)_{j\in I} \in \Xi(x, A, b, \xi^*) \Big\} = \emptyset.$$

Besides, it holds

$$\begin{aligned}
\{\bar{\xi}^*\}^\perp &= \big\{(\tfrac{2}{3}, \tfrac{2}{3})\big\}^\perp = \big\{(v_1, v_2)|\ v_1 + v_2 = 0\big\},\\
T(\bar{x}; \Theta(\bar{A}, \bar{b})) &= \big\{v \in I\!R^2|\ \langle \bar{A}_1^\top, v\rangle \le 0\big\} = \big\{(v_1, v_2)|\ v_1 + v_2 \le 0\big\},\\
T(\bar{x}; \Theta(\bar{A}, \bar{b})) \cap \{\bar{\xi}^*\}^\perp &= \big\{(v_1, v_2)|\ v_1 + v_2 = 0\big\} = \big\{(v_1, -v_1)|\ v_1 \in I\!R\big\},\\
\big(T(\bar{x}; \Theta(\bar{A}, \bar{b})) \cap \{\bar{\xi}^*\}^\perp\big)^* &= \big\{(u_1^*, u_1^*)|\ u_1^* \in I\!R\big\}.
\end{aligned}$$

Fix an element $(x^*, A^*, b^*, \xi) \in \mathcal{H}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)$, where $\mathcal{H}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)$ is defined via (3.12). Since $I_1 = \emptyset$, one cannot find any index $i \in I_1$ such that $b_i^* \le 0$. Besides, as $\bar{I} = \{2, 3\}$, $b_{\bar{I}}^* = (b_2^*, b_3^*) = (0, 0)$. Hence $b^* = (b_1^*, 0, 0)$, where $b_1^* \in I\!R$. In the accordance with (3.12), taking account of the fact that $I_1 = \emptyset$ we have

$$x^* = -b_1^* \bar{A}_1^\top = \begin{pmatrix} -b_1^* \\ -b_1^* \end{pmatrix} \text{ and } A^* = \begin{pmatrix} a_{11}^* & a_{12}^* \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with } b_1^*, a_{11}^*, a_{12}^* \in I\!R.$$

Consequently, from (3.12) we derive the formula for $\mathcal{H}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)$ as follows

$$\mathcal{H}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) = \left\{ \left(\begin{pmatrix} -b_1^* \\ -b_1^* \end{pmatrix}, \begin{pmatrix} a_{11}^* & a_{12}^* \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b_1^* \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \xi_1 \\ -\xi_1 \end{pmatrix} \right) \middle|\ b_1^*, a_{11}^*, a_{12}^*, \xi_1 \in I\!R \right\}.$$

Combining this with (3.15) we get the following upper estimate for the Fréchet normal cone $\widehat{N}\big((\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*); \mathrm{gph}\mathcal{F}\big)$:

$$\widehat{N}\big((\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*); \mathrm{gph}\mathcal{F}\big) \subset \left\{ \left(\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} \beta & \gamma \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -\alpha \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu \\ -\mu \end{pmatrix} \right) \middle|\ \alpha, \beta, \gamma, \mu \in I\!R \right\}.$$

The problem of finding an exact formula for the computation of the Fréchet normal cone $\widehat{N}\big((\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*); \mathrm{gph}\mathcal{F}\big)$ remains open.

3.3 Estimation of the Limiting Normal Cone to $\mathrm{gph}\mathcal{F}$

The main goal of this section is to establish an upper estimate for the limiting normal cone to the graph of the normal cone mapping $\mathcal{F}(\cdot)$ given by (3.4). As a direct consequence, we get an upper estimate for values of the Mordukhovich coderivative of such mapping at a given point on $\mathrm{gph}\mathcal{F}$.

Given any matrix $A \in I\!R^{m\times n}$ and subsets P, Q of T satisfying $P \subset Q$, following [13] we put

$$\mathcal{A}_{Q,P}(A) = \mathrm{span}\{A_i^\top \,|\, i \in P\} + \mathrm{pos}\{A_i^\top \,|\, i \in Q \setminus P\},$$

and

$$\mathcal{B}_{Q,P}(A) = \Big\{v \in I\!R^n \Big|\ \langle A_i^\top, v\rangle = 0\ \forall i \in P,\ \langle A_i^\top, v\rangle \le 0\ \forall i \in Q \setminus P\Big\}.$$

Lemma 3.4 (See [13, Lemma 3.3]) *For any $A \in I\!R^{m\times n}$ and $P \subset Q \subset T$, we have*

$$\big(\mathcal{B}_{Q,P}(A)\big)^* = \mathcal{A}_{Q,P}(A),$$

where $\big(\mathcal{B}_{Q,P}(A)\big)^* = \{u \in I\!R^n \,|\, \langle u, v\rangle \le 0\ \ \forall v \in \mathcal{B}_{Q,P}(A)\}$.

Lemma 3.5 *Let $(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) \in \mathrm{gph}\mathcal{F}$, $I = I(\bar{x}, \bar{A}, \bar{b})$, $(\lambda_i)_{i\in I} \in \Xi(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)$, and $K = \{i \in I \,|\, \lambda_i > 0\}$. Then, it holds*

$$\begin{cases} \mathcal{A}_{I,K}(\bar{A}) = \big(T(\bar{x}; \Theta(\bar{A}, \bar{b})) \cap \{\bar{\xi}^*\}^\perp\big)^* \\ \mathcal{B}_{I,K}(\bar{A}) = T(\bar{x}; \Theta(\bar{A}, \bar{b})) \cap \{\bar{\xi}^*\}^\perp. \end{cases} \tag{3.22}$$

Proof. By setting $\mathfrak{a}_i^* = \bar{A}_i^\top$ for all $i \in T$ and arguing the same as in the proof of Lemma 2.6, we get the formula (3.22). □

Combining (3.12) with (3.22) we obtain the next lemma.

Lemma 3.6 *Let $(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)$, I, $(\lambda_i)_{i\in I}$, and K be the same as in Lemma 3.5. Then, we have*

$$\begin{aligned} &\mathcal{H}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) \\ &\quad = \Big\{(x^*, A^*, b^*, \xi)\Big|\ (x^*, \xi) \in \mathcal{A}_{I,K}(\bar{A}) \times \mathcal{B}_{I,K}(\bar{A}), \\ &\qquad\qquad x^* = -\textstyle\sum_{i\in I} b_i^* \bar{A}_i^\top,\ \ b_{\bar{I}}^* = 0,\ \ b_{I_1}^* \le 0, \\ &\qquad\qquad A_{\bar{I}}^* = 0,\ \ a_{I_1,j}^* = 0\ \text{ if }\ \bar{x}_j = 0, \\ &\qquad\qquad a_{I_1,j}^* \le 0\ \text{ if }\ \bar{x}_j < 0,\ \ a_{I_1,j}^* \ge 0\ \text{ if }\ \bar{x}_j > 0\Big\}. \end{aligned}$$

For each $(x,A,b,\xi^*)\in \mathrm{gph}\mathcal{F}$, we put

$$\mathcal{I}(x,A,b,\xi^*):=\Big\{P\subset I(x,A,b)\big|\ P\neq\emptyset,\ \xi^*\in\mathrm{pos}\{A_i^\top|\ i\in P\}\Big\},\tag{3.23}$$

$$\mathcal{J}(x,A,b,\xi^*):=\Big\{P\in\mathcal{I}\big|\ A_i^\top,\ i\in P,\ \text{are linearly independent}\Big\}\tag{3.24}$$

with $\mathcal{I}=\mathcal{I}(x,A,b,\xi^*)$, and

$$\widehat{\mathcal{I}}(x,A,b,\xi^*):=\begin{cases}\mathcal{J}(x,A,b,\xi^*), & \text{if } \xi^*\neq 0,\\ \mathcal{J}(x,A,b,\xi^*)\cup\{\emptyset\}, & \text{if } \xi^*=0.\end{cases}\tag{3.25}$$

Using the abbreviations $I:=I(x,A,b)$ and $\widehat{\mathcal{I}}:=\widehat{\mathcal{I}}(x,A,b,\xi^*)$, we define

$$\begin{aligned}\mathbf{\Sigma}(x,A,b,\xi^*):=\bigcup_{P\subset Q\subset I,\,P\in\widehat{\mathcal{I}}}\Big\{(x^*,A^*,b^*,\xi)\big|\ &(x^*,\xi)\in\mathcal{A}_{Q,P}(A)\times\mathcal{B}_{Q,P}(A),\\ &x^*=-\textstyle\sum_{i\in Q}b_i^*A_i^\top,\\ &b^*_{\overline{Q}}=0,\ \ b^*_{Q\setminus P}\le 0,\ \ A^*_{\overline{Q}}=0,\\ &a^*_{Q\setminus P,j}\le 0 \text{ if } x_j<0,\ \ a^*_{Q\setminus P,j}\ge 0 \text{ if } x_j>0\Big\}.\end{aligned}\tag{3.26}$$

Note that the set in (3.26) can be computed explicitly. The forthcoming statement, the result of this section, describes an upper estimate for the limiting normal cone to $\mathrm{gph}\mathcal{F}$ at a given point.

Theorem 3.2 *Let any $(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)\in\mathrm{gph}\mathcal{F}$ and $I=I(\bar{x},\bar{A},\bar{b})$. If the vectors $\{\bar{A}_i^\top|\ i\in I\}$ are positively linearly independent, then*

$$N\big((\bar{x},\bar{A},\bar{b},\bar{\xi}^*);\mathrm{gph}\mathcal{F}\big)\subset\mathbf{\Sigma}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*).\tag{3.27}$$

Proof. Since the vectors $\{\bar{A}_i^\top|\ i\in I\}$ are positively linearly independent, $\mathrm{gph}\mathcal{F}$ is locally closed around $(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)$ by Proposition 3.2. According to [28, Theorem 2.35], it holds

$$N\big((\bar{x},\bar{A},\bar{b},\bar{\xi}^*);\mathrm{gph}\mathcal{F}\big)=\operatorname*{Limsup}_{(x,A,b,\xi^*)\xrightarrow{\mathrm{gph}\mathcal{F}}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)}\widehat{N}\big((x,A,b,\xi^*);\mathrm{gph}\mathcal{F}\big).\tag{3.28}$$

Fix any $(x^*,A^*,b^*,\xi)\in N\big((\bar{x},\bar{A},\bar{b},\bar{\xi}^*);\mathrm{gph}\mathcal{F}\big)$. To justify (3.27), we have to prove that $(x^*,A^*,b^*,\xi)\in\mathbf{\Sigma}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)$. By (3.28), there exist sequences $(x_k,A_k,b_k,\xi_k^*)\xrightarrow{\mathrm{gph}\mathcal{F}}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)$ and $(u_k^*,A_k^*,\eta_k^*,\xi_k)\in\widehat{N}\big((x_k,A_k,b_k,\xi_k^*);\mathrm{gph}\mathcal{F}\big)$ for all $k\in I\!N$ such that $(u_k^*,A_k^*,\eta_k^*,\xi_k)$ converges to (x^*,A^*,b^*,ξ). Since $(x_k,A_k,b_k)\to(\bar{x},\bar{A},\bar{b})$ and $I(x_k,A_k,b_k)\subset T$ for every $k\in I\!N$, we can assume that $I(x_k, A_k, b_k) = Q$ for all $k \in I\!N$, where Q is a fixed index set and $Q \subset I(\bar{x}, \bar{A}, \bar{b})$. By Proposition 3.1 we have

$$\xi_k^* \in N(x_k; \Theta(A_k, b_k)) = \mathrm{pos}\{(A_k)_i^\top |\ i \in Q\} \quad \text{for all } k \in I\!N.$$

Due to Lemma 2.1 and the Dirichlet principle, by considering a subsequence of $\{k\}$ if necessary, we can assume that there is a subset $\widetilde{P} \subset Q$ such that $\{(A_k)_i^\top |\ i \in \widetilde{P}\}$ are linearly independent and

$$\xi_k^* \in \mathrm{pos}\{(A_k)_i^\top |\ i \in \widetilde{P}\} \quad \text{for all } k \in I\!N.$$

Thus, for every $k \in I\!N$, it holds

$$\xi_k^* = \sum_{i \in \widetilde{P}} \lambda_i^k (A_k)_i^\top \quad \text{for some } \lambda_i^k \geq 0,\ i \in \widetilde{P}.$$

Using again the Dirichlet principle, we find a subsequence $\{k_\ell\}$ of $\{k\}$ and a subset $P \subset \widetilde{P}$ such that

$$\{i \in \widetilde{P} |\ \lambda_i^{k_\ell} > 0\} = P \quad \text{for every } \ell \in I\!N.$$

For each $\ell \in I\!N$, as $(u_{k_\ell}^*, A_{k_\ell}^*, \eta_{k_\ell}^*, \xi_{k_\ell}) \in \widehat{N}\big((x_{k_\ell}, A_{k_\ell}, b_{k_\ell}, \xi_{k_\ell}^*); \mathrm{gph}\mathcal{F}\big)$, it follows from Theorem 3.1 that

$$(u_{k_\ell}^*, A_{k_\ell}^*, \eta_{k_\ell}^*, \xi_{k_\ell}) \in \mathcal{H}(x_{k_\ell}, A_{k_\ell}, b_{k_\ell}, \xi_{k_\ell}^*).$$

Therefore, by Lemma 3.6 we have

$$(u_{k_\ell}^*, \xi_{k_\ell}) \in \mathcal{A}_{Q,P}(A_{k_\ell}) \times \mathcal{B}_{Q,P}(A_{k_\ell}), \quad u_{k_\ell}^* = -\sum_{i \in Q} (\eta_{k_\ell}^*)_i (A_{k_\ell})_i^\top,$$

$$(\eta_{k_\ell}^*)_{\overline{Q}} = 0, \quad (\eta_{k_\ell}^*)_{I_1^{k_\ell}} \leq 0,$$

$$(A_{k_\ell}^*)_{\overline{Q}} = 0, \quad (a_{k_\ell}^*)_{I_1^{k_\ell}, j} = 0 \text{ if } (x_{k_\ell})_j = 0,$$

$$(a_{k_\ell}^*)_{I_1^{k_\ell}, j} \leq 0 \text{ if } (x_{k_\ell})_j < 0, \quad (a_{k_\ell}^*)_{I_1^{k_\ell}, j} \geq 0 \text{ if } (x_{k_\ell})_j > 0,$$

where $A_{k_\ell}^* = ((a_{k_\ell}^*)_{ij})$ and $I_1^{k_\ell} := I_1(x_{k_\ell}, A_{k_\ell}, b_{k_\ell}, \xi_{k_\ell}^*)$.

If $P = \emptyset$, then $\xi_{k_\ell}^* = 0$ for all $\ell \in I\!N$. (In this case, $\bar{\xi}^* = \lim_{\ell \to \infty} \xi_{k_\ell}^* = 0$.) By the definition of $I_1^{k_\ell}$, we get $I_1^{k_\ell} = Q$, and thus $Q \setminus P = I_1^{k_\ell}$. If $P \neq \emptyset$, then

$$\xi_{k_\ell}^* = \sum_{i \in P} \lambda_i^{k_\ell} (A_{k_\ell})_i^\top \quad \text{for every } \ell \in I\!N. \tag{3.29}$$

From (3.29) and the definition of $I_1^{k_\ell}$ it holds $Q \setminus P = I(x_{k_\ell}, A_{k_\ell}, b_{k_\ell}) \setminus P \subset I_1^{k_\ell}$ for every $\ell \in I\!N$. Consequently, since $(\eta_{k_\ell}^*)_{I_1^{k_\ell}} \leq 0$ for every $\ell \in I\!N$, we get

$$(\eta_{k_\ell}^*)_{Q \setminus P} \leq 0 \quad \text{for every } \ell \in I\!N.$$

As $(u_k^*, A_k^*, \eta_k^*, \xi_k) \to (x^*, A^*, b^*, \xi)$, we have $(\eta_k^*)_i \to b_i^*$ as $k \to \infty$, for each $i \in T$. Because $(\eta_{k_\ell}^*)_{\overline{Q}} = 0$ and $(\eta_{k_\ell}^*)_{Q\setminus P} \le 0$ for all $\ell \in I\!N$, by letting $\ell \to \infty$ we obtain

$$b_{\overline{Q}}^* = 0 \quad \text{and} \quad b_{Q\setminus P}^* \le 0. \tag{3.30}$$

Since $u_{k_\ell}^* = -\sum_{i\in Q}(\eta_{k_\ell}^*)_i (A_{k_\ell})_i^\top \to -\sum_{i\in Q} b_i^* \bar{A}_i^\top$ and $u_{k_\ell}^* \to x^*$, it holds

$$x^* = \sum_{i\in Q}(-b_i^*)\bar{A}_i^\top \in \text{span}\{\bar{A}_i^\top \mid i \in P\} + \text{pos}\{\bar{A}_i^\top \mid i \in Q \setminus P\} = \mathcal{A}_{Q,P}(\bar{A}).$$

As $\xi_{k_\ell} \in \mathcal{B}_{Q,P}(A_{k_\ell})$, by the definition of the latter we have $\langle (A_{k_\ell})_i^\top, \xi_{k_\ell}\rangle = 0$ for $i \in P$ and $\langle (A_{k_\ell})_i^\top, \xi_{k_\ell}\rangle \le 0$ for $i \in Q\setminus P$. Since $(A_{k_\ell})_i^\top \to \bar{A}_i^\top$ for all $i \in T$ and $\xi_{k_\ell} \to \xi$, we infer that $\langle \bar{A}_i^\top, \xi\rangle = 0$ for $i \in P$ and $\langle \bar{A}_i^\top, \xi\rangle \le 0$ for $i \in Q \setminus P$. This means that $\xi \in \mathcal{B}_{Q,P}(\bar{A})$. We have thus shown that

$$(x^*, \xi) \in \mathcal{A}_{Q,P}(\bar{A}) \times \mathcal{B}_{Q,P}(\bar{A}), \quad x^* = -\sum_{i\in Q} b_i^* \bar{A}_i^\top. \tag{3.31}$$

Since $(u_k^*, A_k^*, \eta_k^*, \xi_k) \to (x^*, A^*, b^*, \xi)$, using the representations $A_k^* = ((a_k^*)_{ij})$ and $A^* = (a_{ij}^*)$ we deduce that $(a_k^*)_{ij} \to a_{ij}^*$ as $k \to \infty$, for all $i \in T$ and $j \in \{1, \ldots, n\}$. Combining this with $(A_{k_\ell}^*)_{\overline{Q}} = 0$ for all $\ell \in I\!N$ yields $A_{\overline{Q}}^* = 0$. We now consider an arbitrary index $j \in \{1, \ldots, n\}$. If $\bar{x}_j < 0$ (resp., $\bar{x}_j > 0$), then we have $(x_{k_\ell})_j < 0$ (resp., $(x_{k_\ell})_j > 0$) for all ℓ large enough because $x_{k_\ell} \to \bar{x}$ as $\ell \to \infty$. Remembering $Q \setminus P \subseteq I_1^{k_\ell}$, from these properties we get $(a_{k_\ell}^*)_{Q\setminus P, j} \le 0$ (resp., $(a_{k_\ell}^*)_{Q\setminus P,j} \ge 0$) for all ℓ large enough. By letting $\ell \to \infty$, we have $a_{Q\setminus P,j}^* \le 0$ if $\bar{x}_j < 0$ and $a_{Q\setminus P,j}^* \ge 0$ if $\bar{x}_j > 0$. Therefore,

$$A_{\overline{Q}}^* = 0, \quad a_{Q\setminus P,j}^* \le 0 \ \text{ if } \bar{x}_j < 0, \quad a_{Q\setminus P,j}^* \ge 0 \ \text{ if } \bar{x}_j > 0. \tag{3.32}$$

From (3.30), (3.31), and (3.32) we conclude that $(x^*, A^*, b^*, \xi) \in \mathbf{\Sigma}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)$. The proof is complete. □

By Theorem 3.2, setting

$$\mathbf{\Lambda}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi) = \Big\{(x^*, A^*, b^*) \big| \ (x^*, A^*, b^*, -\xi) \in \mathbf{\Sigma}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)\Big\} \tag{3.33}$$

for every $\xi \in I\!R^n$ and recalling that

$$D^*\mathcal{F}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi) = \Big\{(x^*, A^*, b^*) \big| \ (x^*, A^*, b^*, -\xi) \in N\big((\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*); \text{gph}\mathcal{F}\big)\Big\},$$

we have

$$D^*\mathcal{F}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi) \subset \mathbf{\Lambda}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi) \quad \text{for all } \xi \in I\!R^n. \tag{3.34}$$

The inclusion (3.34) provides us with an upper estimate for the values of the Mordukhovich coderivative of $\mathcal{F}(\cdot)$ at the given point $(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) \in \text{gph}\mathcal{F}$.

Remark 3.1 In connection with (3.27), it is worthy to note that $\mathcal{H}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)$ is a subset of $\mathbf{\Sigma}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)$ for any $(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)\in \text{gph}\mathcal{F}$. Indeed, if $\bar{\xi}^*=0$, we choose $P=\emptyset$ and $Q=I$ with $I=I(\bar{x},\bar{A},\bar{b})$. We now consider the case that $\bar{\xi}^*\neq 0$. Since $\bar{\xi}^*\neq 0$ and $\bar{\xi}^*\in \text{pos}\{\bar{A}_i^\top|\ i\in I\}$, by Lemma 2.1 there exists $\widetilde{I}\subset I$ such that $\{\bar{A}_i^\top|\ i\in\widetilde{I}\}$ are linear independent and $\bar{\xi}^*\in\text{pos}\{\bar{A}_i^\top|\ i\in\widetilde{I}\}$. This implies that there exist multipliers $\lambda_i\geq 0$ for $i\in\widetilde{I}$ such that $\bar{\xi}^*=\sum_{i\in\widetilde{I}}\lambda_i\bar{A}_i^\top$. In this case, we choose $P=\{i\in\widetilde{I}|\ \lambda_i>0\}$ and $Q=I$. In both two cases of $\bar{\xi}^*$, by the definition of $I_1:=I_1(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)$ we have $Q\setminus P\subseteq I_1$. By the choice of P and Q, applying Lemma 3.6 we get

$$
\begin{aligned}
\mathcal{H}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)=\Big\{(x^*,A^*,b^*,\xi)\Big|\ &(x^*,\xi)\in\mathcal{A}_{Q,P}(\bar{A})\times\mathcal{B}_{Q,P}(\bar{A}),\\
&x^*=-\textstyle\sum_{i\in Q}b_i^*\bar{A}_i^\top,\ \ b^*_{\overline{Q}}=0,\ \ b^*_{I_1}\leq 0,\\
&A^*_{\overline{Q}}=0,\ \ a^*_{I_1,j}=0\ \text{ if }\ \bar{x}_j=0,\\
&a^*_{I_1,j}\leq 0\ \text{ if }\ \bar{x}_j<0,\ \ a^*_{I_1,j}\geq 0\ \text{ if }\ \bar{x}_j>0\Big\}\\
\subset\Big\{(x^*,A^*,b^*,\xi)\Big|\ &(x^*,\xi)\in\mathcal{A}_{Q,P}(\bar{A})\times\mathcal{B}_{Q,P}(\bar{A}),\\
&x^*=-\textstyle\sum_{i\in Q}b_i^*\bar{A}_i^\top,\\
&b^*_{\overline{Q}}=0,\ \ b^*_{Q\setminus P}\leq 0,\ \ A^*_{\overline{Q}}=0,\\
&a^*_{Q\setminus P,j}\leq 0\ \text{ if }\ \bar{x}_j<0,\ \ a^*_{Q\setminus P,j}\geq 0\ \text{ if }\ \bar{x}_j>0\Big\}.
\end{aligned}
\tag{3.35}
$$

Note that $P\subset Q\subset I$ and $P\in\widehat{\mathcal{I}}$ with $\widehat{\mathcal{I}}=\widehat{\mathcal{I}}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)$ given by (3.25). Hence, by the formula (3.26) the last set in (3.35) is a subset of $\mathbf{\Sigma}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)$. Therefore, from (3.35) we conclude that $\mathcal{H}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)\subset\mathbf{\Sigma}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)$.

The next example is designed to show how the estimate (3.27) can be used in practical calculations.

Example 3.2 Let T, $\mathcal{F}$, $\bar{A}$, $\bar{b}$, $\bar{x}$, and $\bar{\xi}^*$ be given as in Example 3.1. Recall that $I=I(\bar{x},\bar{A},\bar{b})=\{1\}$, $\bar{I}=T\setminus I=\{2,3\}$, and $I_1=I_1(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)=\emptyset$, where $\bar{\xi}^*=(\frac{2}{3},\frac{2}{3})=\frac{2}{3}\bar{A}_1^\top\in\text{pos}\{\bar{A}_1^\top\}=N(\bar{x};\Theta(\bar{A},\bar{b}))$.

We now use the formula (3.27) to established an upper estimate for the limiting normal cone $N\big((\bar{x},\bar{A},\bar{b},\bar{\xi}^*);\text{gph}\mathcal{F}\big)$. Let $\mathcal{I}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)$, $\mathcal{J}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)$, and $\widehat{\mathcal{I}}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)$ be defined via (3.23), (3.24), and (3.25) respectively. It is easy to verify that

$$\widehat{\mathcal{I}}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)=\mathcal{J}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)=\mathcal{I}(\bar{x},\bar{A},\bar{b},\bar{\xi}^*)=\{I\},$$

where $I = \{1\}$. Thus, the conditions $P \in \widehat{\mathcal{I}}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)$ and $P \subset Q \subset I$ are satisfied if and only if $P = I$ and $Q = I$. Therefore, by (3.26) we obtain

$$\mathbf{\Sigma}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) =$$

$$\left\{ \left(\begin{pmatrix} -b_1^* \\ -b_1^* \end{pmatrix}, \begin{pmatrix} a_{11}^* & a_{12}^* \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b_1^* \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \xi_1 \\ -\xi_1 \end{pmatrix} \right) \middle| \; b_1^*, a_{11}^*, a_{12}^*, \xi_1 \in I\!R \right\}.$$

Comparing this with the result of Example 3.1, we receive the equality $\mathbf{\Sigma}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) = \mathcal{H}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)$. Since $\{\bar{A}_i^\top |\; i \in I\} = \{\bar{A}_1^\top\}$ is positively linearly independent, applying (3.27) we get

$$N\big((\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*); \mathrm{gph}\mathcal{F}\big) \subset \mathbf{\Sigma}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) = \mathcal{H}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)$$

$$= \left\{ \left(\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} \beta & \gamma \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -\alpha \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu \\ -\mu \end{pmatrix} \right) \middle| \; \alpha, \beta, \gamma, \mu \in I\!R \right\}.$$

In what follows, we apply the results of this and of the preceding section to study solution stability of parametric affine variational inequalities (AVIs, for short), where both the basic operator and the constraint set undergo nonlinear perturbations.

3.4 AVIs under Nonlinear Perturbations

Let us first recall an upper estimate for the values of the Mordukhovich coderivative of the implicit multifunction

$$S(p) := \{x \in I\!R^n |\; 0 \in F(x, p)\},$$

where $F : I\!R^n \times I\!R^d \rightrightarrows I\!R^m$ is an arbitrary multifunction. Given any point $(\bar{p}, \bar{x}) \in \mathrm{gph}S$, we set $\bar{z} := (\bar{x}, \bar{p}, 0) \in I\!R^{n+d+m}$. It is clear that $\bar{z} \in \mathrm{gph}F$. For each $x' \in I\!R^n$, following [22] we put

$$\Omega(x') := \bigcup_{v' \in I\!R^m} \{p' \in I\!R^d |\; (-x', p') \in D^*F(\bar{z})(v')\}.$$

Theorem 3.3 (See [24, Theorem 2.1(a)] and [22, Theorem 3.1]) *Let F be locally closed around $\bar{z} \in \mathrm{gph}F$, i.e., there is $\rho > 0$ such that the intersection of $\mathrm{gph}F$ with the closed ball centered at $\bar{z}$ with radius ρ is a closed set. If $\ker D^*F(\bar{z}) = \{0\}$, then*

$$D^*S(\bar{p}, \bar{x})(x') \subset \Omega(x')$$

is valid for every $x' \in I\!R^n$*.*

In [22], Lee and Yen have shown that the inclusion $D^*S(\bar{p},\bar{x})(x') \subset \Omega(x')$ may not hold if the regularly condition $\ker D^*F(\bar{z}) = \{0\}$ is violated.

We now consider the implicit multifunction $S : I\!R^{m\times n} \times I\!R^m \times I\!R^n \rightrightarrows I\!R^n$ defined by setting

$$S(A,b,q) = \big\{x \in I\!R^n |\ 0 \in Mx - q + \mathcal{F}(x,A,b)\big\} \tag{3.36}$$

with $M \in I\!R^{n\times n}$ being a fixed matrix, $(A,b,q) \in I\!R^{m\times n}\times I\!R^m\times I\!R^n$ a parameter, and $\mathcal{F}(x,A,b) = N(x;\Theta(A,b))$ given by (3.4). Note that $S(A,b,q)$ can be represented as the solution set of a *parametric affine variational inequality* as follows

$$S(A,b,q) = \big\{x \in \Theta(A,b) |\ \langle Mx - q, u - x\rangle \geq 0,\ \forall u \in \Theta(A,b)\big\}.$$

Fix any $\bar{x} \in S(\bar{A},\bar{b},\bar{q})$ and note that $\bar{\vartheta} := (\bar{x},\bar{A},\bar{b},\bar{q})$ belongs to $\mathrm{gph}S^{-1}$. For every $(A^*,b^*,q^*) \in I\!R^{m\times n} \times I\!R^m \times I\!R^n$, by the definition of the Mordukhovich coderivative we have

$$\begin{aligned} &D^*S^{-1}(\bar{\vartheta})(A^*,b^*,q^*)\\ &\quad = \big\{x^* \in I\!R^n |\ (x^*,-A^*,-b^*,-q^*) \in N((\bar{x},\bar{A},\bar{b},\bar{q});\mathrm{gph}S^{-1})\big\}\\ &\quad = \big\{x^* \in I\!R^n |\ (-A^*,-b^*,-q^*,x^*) \in N((\bar{A},\bar{b},\bar{q},\bar{x});\mathrm{gph}S)\big\}\\ &\quad = \big\{x^* \in I\!R^n |\ (-A^*,-b^*,-q^*) \in D^*S(\bar{A},\bar{b},\bar{q},\bar{x})(-x^*)\big\}. \end{aligned}$$

Hence,

$$D^*S^{-1}(\bar{\vartheta})(\mathbf{0}) = \big\{x^* \in I\!R^n |\ \mathbf{0} \in D^*S(\bar{A},\bar{b},\bar{q},\bar{x})(-x^*)\big\}, \tag{3.37}$$

where $\mathbf{0} := (0_{I\!R^{m\times n}}, 0_{I\!R^m}, 0_{I\!R^n})$.

The following statement is a generalization of Proposition 3.2 where the case $M = 0_{I\!R^{n\times n}}$ and $q = 0_{I\!R^n}$ was treated.

Lemma 3.7 *Let* $\widetilde{\mathcal{F}}(x,A,b,q) = Mx - q + \mathcal{F}(x,A,b)$ *and* $(\bar{x},\bar{A},\bar{b},\bar{q},\bar{\xi}^*) \in \mathrm{gph}\widetilde{\mathcal{F}}$*. If* $\{\bar{A}_i^\top |\ i \in I(\bar{x},\bar{A},\bar{b})\}$ *are positively linearly independent, then there exists* $\delta > 0$ *such that* $\widetilde{\mathcal{G}} := \mathrm{gph}\widetilde{\mathcal{F}} \cap \Big(\bar{B}(\bar{x},\delta) \times \bar{B}(\bar{A},\delta) \times \bar{B}(\bar{b},\delta) \times I\!R^n \times I\!R^n\Big)$ *is a closed set.*

Proof. Arguing as in the first part of the proof of Proposition 3.2, we can find $\delta > 0$ such that $\{A_i^\top |\ i \in I(\bar{x},\bar{A},\bar{b})\}$ are positively linearly independent for all $A \in \bar{B}(\bar{A},\delta)$, and $I(x,A,b) \subset I(\bar{x},\bar{A},\bar{b})$ whenever (x,A,b) belongs to $\bar{B}(\bar{x},\delta)\times\bar{B}(\bar{A},\delta)\times\bar{B}(\bar{b},\delta)$. We now prove that $\widetilde{\mathcal{G}}$ is closed. Given any sequence

$\{(x_k, A_k, b_k, q_k, \xi_k^*)\}$ in $\widetilde{\mathcal{G}}$ with $(x_k, A_k, b_k, q_k, \xi_k^*) \to (\widehat{x}, \widehat{A}, \widehat{b}, \widehat{q}, \widehat{\xi}^*)$, we are going to show that $(\widehat{x}, \widehat{A}, \widehat{b}, \widehat{q}, \widehat{\xi}^*) \in \widetilde{\mathcal{G}}$. Since (x_k, A_k, b_k) converges to $(\widehat{x}, \widehat{A}, \widehat{b})$ and $\bar{B}(\bar{x}, \delta) \times \bar{B}(\bar{A}, \delta) \times \bar{B}(\bar{b}, \delta)$ is closed, $(\widehat{x}, \widehat{A}, \widehat{b}) \in \bar{B}(\bar{x}, \delta) \times \bar{B}(\bar{A}, \delta) \times \bar{B}(\bar{b}, \delta)$. Hence, $I(x_k, A_k, b_k) \subset I(\widehat{x}, \widehat{A}, \widehat{b}) \subset I(\bar{x}, \bar{A}, \bar{b})$ for every k sufficiently large. We can assume that $I(x_k, A_k, b_k) = \widetilde{I} \subset I(\widehat{x}, \widehat{A}, \widehat{b}) \subset I(\bar{x}, \bar{A}, \bar{b})$ for all $k \in I\!N$, where $\widetilde{I}$ is a fixed index set. As $(x_k, A_k, b_k, q_k, \xi_k^*) \in \text{gph}\widetilde{\mathcal{F}}$, it holds

$$\xi_k^* + q_k - Mx_k \in \mathcal{F}(x_k, A_k, b_k) = N(x_k; \Theta(A_k, b_k)) = \text{pos}\{(A_k)_i^\top |\ i \in \widetilde{I}\}$$

for every $k \in I\!N$. Thus, for each $k \in I\!N$, there exist $\lambda_i^k \geq 0$ for $i \in \widetilde{I}$ such that

$$\sum_{i \in \widetilde{I}} \lambda_i^k (A_k)_i^\top = \xi_k^* + q_k - Mx_k \to \widehat{\xi}^* + \widehat{q} - M\widehat{x}. \tag{3.38}$$

Since $\{\bar{A}_i^\top |\ i \in I(\bar{x}, \bar{A}, \bar{b})\}$ are positively linearly independent, arguing as in the second part of the proof of Proposition 3.2 we can find a subsequence k_ℓ of $\{k\}$ such that

$$\lim_{\ell \to \infty} \sum_{i \in \widetilde{I}} \lambda_i^{k_\ell} (A_{k_\ell})_i^\top = \sum_{i \in \widetilde{I}} \lambda_i \widehat{A}_i^\top \in N(\widehat{x}; \Theta(\widehat{A}, \widehat{b})) = \mathcal{F}(\widehat{x}, \widehat{A}, \widehat{b}). \tag{3.39}$$

From (3.38) and (3.39) we deduce that $\widehat{\xi}^* + \widehat{q} - M\widehat{x} \in \mathcal{F}(\widehat{x}, \widehat{A}, \widehat{b})$. This means that $\widehat{\xi}^* \in \widetilde{\mathcal{F}}(\widehat{x}, \widehat{A}, \widehat{b}, \widehat{q})$. Therefore, $(\widehat{x}, \widehat{A}, \widehat{b}, \widehat{q}, \widehat{\xi}^*) \in \widetilde{\mathcal{G}}$. □

Let $\bar{w} = (\bar{A}, \bar{b}, \bar{q}, \bar{x}) \in \text{gph}S$, where the multifunction $S(\cdot)$ is defined by (3.36). Setting $\bar{\xi}^* = \bar{q} - M\bar{x}$, we have $(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) \in \text{gph}\mathcal{F}$. For each $x^* \in I\!R^n$, we put

$$\begin{aligned}\mathbf{K}(\bar{w})(x^*) = \bigcup_{\xi \in I\!R^n} \Big\{ (A^*, b^*, q^*) \Big|\ & (-x^*, A^*, b^*, q^*) \in M^\top \xi \times \{(0_{I\!R^{m\times n}}, 0_{I\!R^m}, 0_{I\!R^n})\} \\ & + \{(0_{I\!R^n}, 0_{I\!R^{m\times n}}, 0_{I\!R^m})\} \times \{-\xi\} \\ & + D^*\mathcal{F}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi) \times \{0_{I\!R^n}\} \Big\}\end{aligned}$$

and

$$\begin{aligned}\mathbf{L}(\bar{w})(x^*) = \bigcup_{\xi \in I\!R^n} \Big\{ (A^*, b^*, q^*) \Big|\ & (-x^*, A^*, b^*, q^*) \in M^\top \xi \times \{(0_{I\!R^{m\times n}}, 0_{I\!R^m}, 0_{I\!R^n})\} \\ & + \{(0_{I\!R^n}, 0_{I\!R^{m\times n}}, 0_{I\!R^m})\} \times \{-\xi\} \\ & + \mathbf{\Lambda}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi) \times \{0_{I\!R^n}\} \Big\},\end{aligned}$$

where $\mathbf{\Lambda}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi)$ is given by (3.33).

Remark 3.2 According to (3.34), the inclusion $\mathbf{K}(\bar{w})(x^*) \subset \mathbf{L}(\bar{w})(x^*)$ holds if the vectors $\{\bar{A}_i^\top |\, i \in I(\bar{x}, \bar{A}, \bar{b})\}$ are positively linearly independent.

Theorem 3.4 *If $\{\bar{A}_i^\top |\, i \in I(\bar{x}, \bar{A}, \bar{b})\}$ are positively linearly independent, we obtain the estimates*

$$D^*S(\bar{w})(x^*) \subset \mathbf{K}(\bar{w})(x^*) \subset \mathbf{L}(\bar{w})(x^*) \tag{3.40}$$

for every $x^* \in \mathbb{R}^n$.

Proof. As in Lemma 3.7, we define $\widetilde{\mathcal{F}}(x, A, b, q) = Mx - q + \mathcal{F}(x, A, b)$. Let $\bar{z} = (\bar{x}, \bar{A}, \bar{b}, \bar{q}, 0_{\mathbb{R}^n}) \in \text{gph}\widetilde{\mathcal{F}}$. Then, according to the coderivative sum rules [28, Theorem 1.62], we obtain

$$\begin{aligned} D^*\widetilde{\mathcal{F}}(\bar{z})(\xi) = M^\top\xi &\times \{(0_{\mathbb{R}^{m\times n}}, 0_{\mathbb{R}^m}, 0_{\mathbb{R}^n})\} \\ &+ \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}^{m\times n}}, 0_{\mathbb{R}^m})\} \times \{-\xi\} \\ &+ D^*\mathcal{F}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi) \times \{0_{\mathbb{R}^n}\} \end{aligned} \tag{3.41}$$

for every $\xi \in \mathbb{R}^n$, where $\bar{\xi}^* = \bar{q} - M\bar{x}$. By (3.41) it follows that $0 \in D^*\widetilde{\mathcal{F}}(\bar{z})(\xi)$ if and only if $\xi = 0$. This means that $\ker D^*\widetilde{\mathcal{F}}(\bar{z}) = \{0\}$. Since the vectors $\{\bar{A}_i^\top |\, i \in I(\bar{x}, \bar{A}, \bar{b})\}$ are positively linearly independent, $\text{gph}\widetilde{\mathcal{F}}$ is locally closed around $\bar{z}$ by Lemma 3.7. Applying Theorem 3.3 to the implicit multifunction

$$S(A, b, q) = \big\{x \in \mathbb{R}^n |\, 0 \in \widetilde{\mathcal{F}}(x, A, b, q)\big\},$$

which is exactly the multifunction $S(\cdot)$ given by (3.36), and recalling (3.41), we get the inclusion $D^*S(\bar{w})(x^*) \subset \mathbf{K}(\bar{w})(x^*)$. From Remark 3.2 it holds $\mathbf{K}(\bar{w})(x^*) \subset \mathbf{L}(\bar{w})(x^*)$. Thus, the inclusions in (3.40) are valid. □

As defined in Chapter 1, $S(\cdot)$ is locally metrically regular around the point $(\bar{A}, \bar{b}, \bar{q}, \bar{x}) \in \text{gph}S$ if there are $\mu > 0$, $\gamma > 0$, and neighborhoods U of $\bar{x}$ and V of $(\bar{A}, \bar{b}, \bar{q})$ such that

$$\text{dist}\big((A, b, q); S^{-1}(x)\big) \le \mu\, \text{dist}\big(x; S(A, b, q)\big)$$

for all $x \in U$ and $(A, b, q) \in V$ satisfying $\text{dist}\big(x; S(A, b, q)\big) \le \gamma$. If $S(\cdot)$ is locally closed around $(\bar{A}, \bar{b}, \bar{q}, \bar{x})$, then Theorem 1.4 tells us that $S(\cdot)$ is locally metrically regular around $(\bar{A}, \bar{b}, \bar{q}, \bar{x})$ if and only if

$$D^*S^{-1}(\bar{x}, \bar{A}, \bar{b}, \bar{q})(\mathbf{0}) = \{0\}$$

with $\mathbf{0} := (0_{\mathbb{R}^{m\times n}}, 0_{\mathbb{R}^m}, 0_{\mathbb{R}^n})$. By (3.37), the last equality can be rewritten equivalently as

$$\ker D^*S(\bar{A}, \bar{b}, \bar{q}, \bar{x}) = \{0\}. \tag{3.42}$$

Recall that $S(\cdot)$ is locally Lipschitz-like around $(\bar{A}, \bar{b}, \bar{q}, \bar{x}) \in \mathrm{gph}S$ if there exist a constant $\ell > 0$ and neighborhoods U of $\bar{x}$, V of $(\bar{A}, \bar{b}, \bar{q})$ such that

$$S(A, b, q) \cap U \subset S(A', b', q') + \ell\|(A, b, q) - (A', b', q')\|\bar{B}_{I\!R^n}$$

for all $(A, b, q), (A', b', q') \in V$, where $\bar{B}_{I\!R^n}$ denotes the closed unit ball in $I\!R^n$. If $S(\cdot)$ is locally closed around $(\bar{A}, \bar{b}, \bar{q}, \bar{x})$, then Theorem 1.3 asserts that $S(\cdot)$ is locally Lipschitz-like around $(\bar{A}, \bar{b}, \bar{q}, \bar{x})$ if and only if

$$D^*S(\bar{A}, \bar{b}, \bar{q}, \bar{x})(0) = \{0\}. \tag{3.43}$$

Theorem 3.5 *Let* $\bar{w} = (\bar{A}, \bar{b}, \bar{q}, \bar{x}) \in \mathrm{gph}S$ *and the vectors* $\{\bar{A}_i^\top |\ i \in I(\bar{x}, \bar{A}, \bar{b})\}$ *be positively linearly independent. The following assertions are valid:*

(i) *If* $\ker\mathbf{L}(\bar{w}) = \{0\}$*, then* $S(\cdot)$ *is locally metrically regular around* $\bar{w}$*.*

(ii) *If* $\mathbf{L}(\bar{w})(0) = \{0\}$*, then* $S(\cdot)$ *is locally Lipschitz-like around* $\bar{w}$*.*

Proof. The vectors $\{\bar{A}_i^\top |\ i \in I(\bar{x}, \bar{A}, \bar{b})\}$ are positively linearly independent, thus the multifunction $\widetilde{\mathcal{F}}(x, A, b, q) = Mx - q + \mathcal{F}(x, A, b)$ is locally closed around $(\bar{x}, \bar{A}, \bar{b}, \bar{q}, 0_{I\!R^n}) \in \mathrm{gph}\widetilde{\mathcal{F}}$ by Lemma 3.7. Fix $\rho > 0$ such that the set

$$\mathrm{gph}\widetilde{\mathcal{F}} \cap \big\{(x, A, b, q, \xi^*)\big|\ \|x - \bar{x}\| + \|A - \bar{A}\| + \|b - \bar{b}\| + \|q - \bar{q}\| + \|\xi^*\| \le \rho\big\}$$

is closed. Then, the set

$$\mathrm{gph}\widetilde{\mathcal{F}} \cap \big\{(x, A, b, q, 0_{I\!R^n})\big|\ \|x - \bar{x}\| + \|A - \bar{A}\| + \|b - \bar{b}\| + \|q - \bar{q}\| \le \rho\big\}$$

is also closed. Since $(A, b, q, x) \in \mathrm{gph}S$ if and only if $(x, A, b, q, 0_{I\!R^n}) \in \mathrm{gph}\widetilde{\mathcal{F}}$, the latter implies the closedness of

$$\mathrm{gph}S \cap \big\{(A, b, q, x)\big|\ \|A - \bar{A}\| + \|b - \bar{b}\| + \|q - \bar{q}\| + \|x - \bar{x}\| \le \rho\big\}.$$

This means that $S(\cdot)$ is locally closed around $\bar{w}$. The assumption that $\{\bar{A}_i^\top |\ i \in I(\bar{x}, \bar{A}, \bar{b})\}$ are positively linearly independent guaranties (3.40).

(i) Combining the condition $\ker\mathbf{L}(\bar{w}) = \{0\}$ with (3.40) yields (3.42) which establishes the local metric regularity of $S(\cdot)$ around $\bar{w}$.

(ii) Taking account of (3.40) and the assumption $\mathbf{L}(\bar{w})(0) = \{0\}$ we obtain (3.43) which ensures that $S(\cdot)$ is locally Lipschitz-like around $\bar{w}$. □

Theorem 3.6 *Let* $\bar{w} = (\bar{A}, \bar{b}, \bar{q}, \bar{x}) \in \mathrm{gph}S$*. If the vectors* $\{\bar{A}_i^\top |\ i \in I(\bar{x}, \bar{A}, \bar{b})\}$ *are positively linearly independent, then there exists* $\delta > 0$ *such that* $S(\cdot)$ *is locally metrically regular around any point* $w \in \bar{B}(\bar{w}, \delta) \cap \mathrm{gph}S$*.*

Proof. By Lemma 3.1, the positive linear independence of the vectors $\{\bar{A}_i^\top |\ i \in I(\bar{x}, \bar{A}, \bar{b})\}$ implies that there is $\delta > 0$ such that $\{A_i^\top |\ i \in I(\bar{x}, \bar{A}, \bar{b})\}$ are positively linearly independent for any $A \in \bar{B}(\bar{A}, \delta)$. Choose δ as small as $I(x, A, b) \subset I(\bar{x}, \bar{A}, \bar{b})$ for every $(A, b, q, x) \in \bar{B}(\bar{w}, \delta) \cap \text{gph}S$. For any $\widehat{w} = (\widehat{A}, \widehat{b}, \widehat{q}, \widehat{x}) \in \bar{B}(\bar{w}, \delta) \cap \text{gph}S$, since

$$\|\widehat{A} - \bar{A}\| \le \|\widehat{A} - \bar{A}\| + \|\widehat{b} - \bar{b}\| + \|\widehat{q} - \bar{q}\| + \|\widehat{x} - \bar{x}\| = \|\widehat{w} - \bar{w}\| \le \delta,$$

the vectors $\{\widehat{A}_i^\top |\ i \in I(\bar{x}, \bar{A}, \bar{b})\}$ are positively linearly independent. Because $I(\widehat{x}, \widehat{A}, \widehat{b}) \subset I(\bar{x}, \bar{A}, \bar{b})$, the vectors $\{\widehat{A}_i^\top |\ i \in I(\widehat{x}, \widehat{A}, \widehat{b})\}$ are also positively linearly independent.

We now consider an arbitrary $x^* \in \ker\mathbf{L}(\widehat{w})$. The inclusion $x^* \in \ker\mathbf{L}(\widehat{w})$ means that

$$\begin{aligned} \mathbf{0} &\in \mathbf{L}(\widehat{w})(x^*) \\ &= \bigcup_{\xi \in I\!R^n} \Big\{ (A^*, b^*, q^*) \Big|\ (-x^*, A^*, b^*, q^*) \in M^\top \xi \times \{(0_{I\!R^{m\times n}}, 0_{I\!R^m}, 0_{I\!R^n})\} \\ &\qquad + \{(0_{I\!R^n}, 0_{I\!R^{m\times n}}, 0_{I\!R^m})\} \times \{-\xi\} + \mathbf{\Lambda}(\widehat{x}, \widehat{A}, \widehat{b}, \widehat{\xi}^*)(\xi) \times \{0_{I\!R^n}\} \Big\}, \end{aligned}$$

where $\mathbf{0} = (0_{I\!R^{m\times n}}, 0_{I\!R^m}, 0_{I\!R^n})$ and $\widehat{\xi}^* = \widehat{q} - M\widehat{x}$. The last inclusion implies that $0_{I\!R^n} \in \{-\xi\}$ and

$$(-x^*, 0_{I\!R^{m\times n}}, 0_{I\!R^m}) \in M^\top \xi \times \{0_{I\!R^{m\times n}}\} \times \{0_{I\!R^m}\} + \mathbf{\Lambda}(\widehat{x}, \widehat{A}, \widehat{b}, \widehat{\xi}^*)(\xi).$$

Thus, it holds $(-x^*, 0_{I\!R^{m\times n}}, 0_{I\!R^m}) \in \mathbf{\Lambda}(\widehat{x}, \widehat{A}, \widehat{b}, \widehat{\xi}^*)(0_{I\!R^n})$. By (3.33), the last inclusion yields $(-x^*, 0_{I\!R^{m\times n}}, 0_{I\!R^m}, 0_{I\!R^n}) \in \mathbf{\Sigma}(\widehat{x}, \widehat{A}, \widehat{b}, \widehat{\xi}^*)$. From this and (3.26) we deduce that there exists $Q \subset I(\widehat{x}, \widehat{A}, \widehat{b})$ such that $x^* = \sum_{i \in Q} b_i^* \widehat{A}_i^\top$, where $b^* = (b_1^*, \ldots, b_m^*) = 0_{I\!R^m}$. Hence, $x^* = 0$. We have shown that $\ker\mathbf{L}(\widehat{w}) = \{0\}$.

The positive linear independence of the vectors $\{\widehat{A}_i^\top |\ i \in I(\widehat{x}, \widehat{A}, \widehat{b})\}$ and the property $\ker\mathbf{L}(\widehat{w}) = \{0\}$ allow us to apply Theorem 3.5(i) to assert that $S(\cdot)$ is locally metrically regular around $\widehat{w}$. Since $\widehat{w}$ can be chosen arbitrary in $\bar{B}(\bar{w}, \delta) \cap \text{gph}S$, the proof is complete. □

Example 3.3 As in [49, Example 2], we consider the parametric nonconvex quadratic programming problem (P_ε) of minimizing the function

$$f_\varepsilon(x) = -\frac{1}{2}x_1^2 - x_2^2 + x_1 - \varepsilon x_2$$

on $\Delta = \{x \in I\!R^2 |\ x \ge 0, x_1 + x_2 \le 2\}$, where ε is a parameter. Denote by $\widehat{S}(\varepsilon)$ the Karush-Kuhn-Tucker (KKT) point set of (P_ε). The KKT point set $\widehat{S}(\varepsilon)$

is the solution set of the following parametric affine variational inequality problem

$$\text{Find} \quad x \in \Delta \quad \text{subject to} \quad \langle \nabla f_\varepsilon(x), u - x \rangle \geq 0, \ \forall u \in \Delta, \tag{3.44}$$

where $\nabla f_\varepsilon(x) = Mx - q_\varepsilon$ with

$$M = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{and} \quad q_\varepsilon = \begin{pmatrix} -1 \\ \varepsilon \end{pmatrix}.$$

We see that $\bar{x}$ is a solution of (3.44) if and only if $\bar{x}$ is a solution of the following generalized equation

$$0 \in M\bar{x} - q_\varepsilon + N(\bar{x}; \Delta).$$

By Example 2 in [49], we have

$$\widehat{S}(0) = \left\{ (0,0), (1,0), (2,0), \left(\frac{5}{3}, \frac{1}{3}\right), (0,2) \right\},$$

$$\widehat{S}(\varepsilon) = \left\{ (2,0), \left(\frac{5+\varepsilon}{3}, \frac{1-\varepsilon}{3}\right), (0,2) \right\}$$

for $\varepsilon > 0$ small enough, and the multifunction $\varepsilon \mapsto \widehat{S}(\varepsilon)$ is not lower semicontinuous at $\varepsilon = 0$. We consider the implicit multifunction defined via (3.36) as follows

$$S : I\!R^{3\times 2} \times I\!R^3 \times I\!R^2 \rightrightarrows I\!R^2,$$
$$S(A, b, q) = \{x \in I\!R^2 |\ 0 \in Mx - q + \mathcal{F}(x, A, b)\},$$

where $\Theta(A, b) = \{x \in I\!R^2 |\ Ax \leq b\}$ and $\mathcal{F}(x, A, b) = N(x; \Theta(A, b))$. Let

$$\bar{A} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \bar{q} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Then, $\Theta(\bar{A}, \bar{b}) = \Delta$ and $S(\bar{A}, \bar{b}, \bar{q}) \equiv \widehat{S}(0)$. Hence, $\bar{x} = (\frac{5}{3}, \frac{1}{3}) \in S(\bar{A}, \bar{b}, \bar{q})$ and $\bar{w} := (\bar{A}, \bar{b}, \bar{q}, \bar{x}) \in \text{gph}S$. We use the criterion Theorem 3.5(ii) to verify the local Lipschitz-like property of S around $\bar{w}$. For any $x^* \in I\!R^2$, we have

$$\begin{aligned}\mathbf{L}(\bar{w})(x^*) = \bigcup_{\xi \in I\!R^2} \Big\{ (A^*, b^*, q^*) |\ & (-x^*, A^*, b^*, q^*) \in M^\top \xi \times \{(0_{I\!R^{3\times 2}}, 0_{I\!R^3}, 0_{I\!R^2})\} \\ & + \{(0_{I\!R^2}, 0_{I\!R^{3\times 2}}, 0_{I\!R^3})\} \times \{-\xi\} \\ & + \mathbf{\Lambda}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi) \times \{0_{I\!R^2}\} \Big\},\end{aligned}$$

where $\bar{\xi}^* = \bar{q} - M\bar{x}$. Hence, the inclusion $(A^*, b^*, q^*) \in \mathbf{L}(\bar{w})(0_{I\!R^2})$ means that there exists $\xi \in I\!R^2$ such that

$$\begin{cases} (-M^\top \xi, A^*, b^*) \in \mathbf{\Lambda}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*)(\xi) \\ q^* = -\xi. \end{cases} \tag{3.45}$$

By (3.33), we can rewrite (3.45) as follows

$$\begin{cases} (-M^\top \xi, A^*, b^*, -\xi) \in \mathbf{\Sigma}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) \\ q^* = -\xi. \end{cases} \tag{3.46}$$

According to Example 3.2, we obtain

$$\mathbf{\Sigma}(\bar{x}, \bar{A}, \bar{b}, \bar{\xi}^*) = \left\{ \left(\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} \beta & \gamma \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -\alpha \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu \\ -\mu \end{pmatrix} \right) \middle| \alpha, \beta, \gamma, \mu \in I\!R \right\}.$$

Therefore, the condition (3.46) is satisfied if and only if

$$-M^\top \xi = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}, \quad A^* = \begin{pmatrix} \beta & \gamma \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad b^* = \begin{pmatrix} -\alpha \\ 0 \\ 0 \end{pmatrix}, \quad q^* = -\xi = \begin{pmatrix} \mu \\ -\mu \end{pmatrix}, \tag{3.47}$$

where $\alpha, \beta, \gamma, \mu \in I\!R$. It is easy to verify that the condition (3.47) forces $\xi = 0$, $\alpha = 0$, $\mu = 0$. However, β and γ can be arbitrary chosen. Consequently, $b^* = 0_{I\!R^3}$, $q^* = 0_{I\!R^2}$, and A^* may be different from $0_{I\!R^{3\times 2}}$. This implies that $\mathbf{L}(\bar{w})(0_{I\!R^2}) \neq \{(0_{I\!R^{3\times 2}}, 0_{I\!R^3}, 0_{I\!R^2})\}$. The latter shows that by using the criterion Theorem 3.5(ii) we cannot conclude about the locally Lipschitz-like property of $S(\cdot)$ around $\bar{w}$.

3.5 Conclusions

In this chapter, nonlinearly perturbed polyhedral normal cone mappings in finite dimensional spaces have been studied. Upper estimates for the Fréchet and the limiting normal cones to the graphs of such normal cone mappings were given respectively in Theorems 3.1 and 3.2. New results on solution stability of parametric affine variational inequalities under nonlinear perturbations, shown in Theorems 3.5 and 3.6, are derived from these estimates. The problem of finding exact formulas for computing the Fréchet normal cone and the limiting normal cone remains open.

Chapter 4

A Class of Linear Generalized Equations

Solution stability of a class of linear generalized equations in finite dimensional Euclidean spaces is investigated in this chapter by means of generalized differentiation. We establish exact formulas for the Fréchet and the Mordukhovich coderivatives of the normal cone mappings to perturbed Euclidean balls. Necessary and sufficient conditions for the local Lipschitz-like property of the solution maps of such the linear generalized equations are derived from the coderivative formulas for the normal cone mappings. These conditions lead to new results on stability of the parametric trust-region subproblems. The two open problems stated by Lee and Yen in [23] are solved.

The results presented below are taken from [42].

4.1 Linear Generalized Equations

The concept of *generalized equation* introduced by Robinson [43] has been recognized as an efficient tool for dealing with various questions in optimization theory. It is also a unified framework for studying equilibrium problems. When the basic single-valued operator of the generalized equation is affine and the accompanying set-valued map is the normal cone operator of a fixed closed convex set called the constraint set, one has a linear generalized equation (linear GE for brevity). Robinson [43, Theorem 2] proved that if a linear GE is monotone and the solution set is nonempty and bounded, then the solution map is locally upper Lipschitzian with respect to the parameters describing the affine operator. This important result has found many applications (see, e.g., [55]).

Linear GEs with perturbed constraint sets have been studied in [20] and [25] (see also the references therein).

In connection with the solution methods [34], [51] and the qualitative study [21] for the trust-region subproblems, we are interested in the linear GEs of the form

$$0 \in Ax + b + N(x; E(\alpha)), \tag{4.1}$$

where symmetric $n \times n$ matrix $A \in \mathbb{R}^{n\times n}$, vector $b \in \mathbb{R}^n$, and real number $\alpha > 0$ are parameters, $E(\alpha) := \{x \in \mathbb{R}^n |\ \|x\| \le \alpha\}$, and

$$N(x; E(\alpha)) := \begin{cases} \{v \in \mathbb{R}^n |\ \langle v, y - x\rangle \le 0,\ \forall y \in E(\alpha)\}, & \text{if } x \in E(\alpha) \\ \emptyset, & \text{if } x \notin E(\alpha) \end{cases} \tag{4.2}$$

is the normal cone to $E(\alpha)$ at x. The solution set of (4.1) is denoted by $S(A, b, \alpha)$. Note that (4.1) is a linear GE where the perturbation of the constraint set $E(\alpha)$ is described by parameter $\alpha \in (0, +\infty)$. Here $E(\alpha)$ is a ball centered at 0 with radius α.

If x is a local solution of the optimization problem

$$\min \left\{ f(x) = \frac{1}{2} x^\top A x + b^\top x \,\middle|\, x \in E(\alpha) \right\}, \tag{4.3}$$

which is called the *trust-region subproblem*, then (4.1) holds due to the generalized Fermat rule (see, e.g., [20, p. 85]). The trust-region subproblems given by (4.3) are at a frequent use in the development of the trust-region methods [10]. Here and in the sequel, the apex $^\top$ denotes matrix transposition. It is well-known [26] that if $x \in E(\alpha)$ is a local minimum of (4.3), then there exists a Lagrange multiplier $\lambda \ge 0$ such that

$$(A + \lambda I)x = -b, \quad \lambda(\|x\| - \alpha) = 0, \tag{4.4}$$

where I denotes the $n \times n$ unit matrix. If $x \in E(\alpha)$ and there exists $\lambda \ge 0$ satisfying (4.4), x is said to be a *Karush-Kuhn-Tucker point* (or a *KKT point*) of (4.3) and (x, λ) is called a KKT pair. For each KKT point x, the Lagrange multiplier λ is defined uniquely (see, e.g., [21]). Recall [19] that x is a KKT point of (4.3) if and only if

$$\langle Ax + b, y - x\rangle \ge 0,\ \ \forall y \in E(\alpha).$$

Thus, the solution set of (4.1) coincides with the *Karush-Kuhn-Tucker point set* of (4.3).

The purpose of this chapter is to investigate the stability of (4.1) with respect to the perturbations of all the three components of its data set $\{A, b, \alpha\}$. Our main tools are the Mordukhovich criterion (see [28, Theorem 4.10] and [48, Theorem 9.40], or Theorem 1.3) for the local Lipschitz-like property of multifunctions between finite dimensional normed spaces and some lower and upper estimates for coderivatives of implicit multifunctions from [22]. Our results develop furthermore the preceding work of Lee and Yen [23] on the stability of (4.1). To be more precise, we provide a complete solution for the open problems raised in [23, Remarks 3.6 and 3.13] by giving exact formulas for the Fréchet an the Mordukhovich coderivatives of the normal cone mapping $(x, \alpha) \mapsto N(x; E(\alpha))$. Moreover, we complement the sufficient conditions for stability of the solution set of (4.1) given in [23, Theorem 5.1] by a more comprehensive necessary and sufficient conditions for stability.

This chapter shows how the generalized differentiation theory [28], [48] can be applied with a success for analyzing a typical polynomial optimization problem of the form (4.3).

Our approach to the analysis of the parametric problem (4.3) is quite different from that one adopted by Lee, Tam and Yen [21]. It is worthy to stress that the focus point of [21] is the lower semicontinuity of the solution map of (4.1), while our aim is to characterize the local Lipschitz-like property of that map. The latter is stronger than the inner semicontinuity of the solution map, which is the basis for defining the above-mentioned lower semicontinuity. It is still unclear to us whether the inner semicontinuity property [28, p. 42] of a multifunction can be characterized by using coderivatives, or not.

4.2 Formulas for Coderivatives

The normal cone $N(x; E(\alpha))$ can be computed explicitly. Namely, we have the formula

$$N(x; E(\alpha)) = \begin{cases} \{0\}, & \text{if } \|x\| < \alpha \\ \{\mu x \mid \mu \geq 0\}, & \text{if } \|x\| = \alpha \\ \emptyset, & \text{if } \|x\| > \alpha. \end{cases} \tag{4.5}$$

For every $(x, \alpha) \in I\!R^n \times I\!R$, we put

$$\mathcal{N}(x, \alpha) = \begin{cases} N(x; E(\alpha)), & \text{if } \alpha > 0 \\ \emptyset, & \text{if } \alpha \le 0, \end{cases} \tag{4.6}$$

where $N(x; E(\alpha))$ is given by (4.2). Thus, $\mathcal{N} : I\!R^n \times I\!R \rightrightarrows I\!R^n$ is a multifunction with closed convex values. It is called the *normal cone mapping* of the closed ball $E(\alpha)$. Setting $y = -b$, $w = (A, \alpha)$, $G(x, w) = Ax$, and $M(x, w) = \mathcal{N}(x, \alpha)$, we can rewrite (4.1) equivalently as

$$y \in G(x, w) + M(x, w). \tag{4.7}$$

It is clear that

$$S(A, b, \alpha) = \widetilde{S}(w, y) := \{x \in I\!R^n \big|\ y \in G(x, w) + M(x, w)\}. \tag{4.8}$$

Hence, the solution map

$$S : H(n) \times I\!R^n \times I\!R \rightrightarrows I\!R^n, \quad (A, b, \alpha) \mapsto S(A, b, \alpha),$$

of (4.1) can be interpreted as the implicit multifunction

$$\widetilde{S} : W \times I\!R^n \rightrightarrows I\!R^n, \quad (w, y) \mapsto \widetilde{S}(w, y), \tag{4.9}$$

where $W := H(n) \times I\!R$ with $H(n) \subset I\!R^{n\times n}$ being the linear subspace of symmetric $n \times n$ matrices of $I\!R^{n\times n}$.

By $\widehat{(u, v)}$ we denote the angle between nonzero vectors u and v in $I\!R^n$, i.e., $\widehat{(u, v)} \in [0, \pi]$ and $\langle u, v\rangle = \|u\| \cdot \|v\| \cos \widehat{(u, v)}$. For each pair $u, v \in I\!R^n$ with $u = (u_1, \dots, u_n)^\top$ and $v = (v_1, \dots, v_n)^\top$, we define the vector $\overrightarrow{uv}$ in $I\!R^n$ by setting $\overrightarrow{uv} = (v_1 - u_1, \dots, v_n - u_n)^\top$. For any $x, y, z \in I\!R^n$, we call $\widehat{xyz}$ the angle between $\overrightarrow{yx}$ and $\overrightarrow{yz}$, provided the latter vectors are nonzero.

We are going to obtain exact formulas for the Fréchet and the Mordukhovich coderivatives of the normal cone mapping $\mathcal{N}(x, \alpha)$ given by (4.6).

Fix any point $(x, \alpha, v) \in \mathrm{gph}\mathcal{N}$.

4.2.1 The Fréchet Coderivative of $\mathcal{N}(x, \alpha)$

The following results are due to Lee and Yen [23].

Lemma 4.1 (See [23, Lemma 3.1]) *If $\|x\| < \alpha$, then $v = 0$ and*

$$\widehat{D}^*\mathcal{N}(x, \alpha, v)(v') = \{(0_{I\!R^n}, 0_{I\!R})\},$$

for every $v' \in I\!R^n$.

Lemma 4.2 (See [23, Lemma 3.2]) *If* $\|x\| = \alpha$ *and* $v \neq 0$*, then* $v = \mu x$ *for some* $\mu > 0$*. If* $(x', \alpha') \in \widehat{D}^*\mathcal{N}(x, \alpha, v)(v')$*, then*

$$x' = -\frac{\alpha'}{\alpha}x + \mu v', \quad \langle v', x\rangle = 0.$$

Lemma 4.1 describes the Fréchet coderivative $\widehat{D}^*\mathcal{N}(x, \alpha, v)$ in the case $\|x\| < \alpha$. Lemma 4.2 gives an upper estimate for the Fréchet coderivative value $\widehat{D}^*(x, \alpha, v)(v')$ in the case $\|x\| = \alpha$, $v \neq 0$. The first part of the open problem raised in [23, Remark 3.6] can be reformulated as follows: *Is the upper estimate provided by Lemma 4.2 an exact one?* The next statement, which answers this question in the affirmative, establishes an exact formula for computing the coderivative $\widehat{D}^*\mathcal{N}(x, \alpha, v)$ in the situation $\|x\| = \alpha$ and $v \neq 0$.

Theorem 4.1 *If* $\|x\| = \alpha$ *and* $v \neq 0$*, then* $v = \mu x$ *with* $\mu = \|v\| \cdot \|x\|^{-1}$ *and, for every* $v' \in I\!R^n$*,*

$$\widehat{D}^*\mathcal{N}(x, \alpha, v)(v') = \begin{cases} \{(x', \alpha') \in I\!R^n \times I\!R \,|\, x' = -\frac{\alpha'}{\alpha}x + \mu v'\}, & \text{if } \langle v', x\rangle = 0 \\ \emptyset, & \text{if } \langle v', x\rangle \neq 0. \end{cases} \tag{4.10}$$

Proof. The property $v = \mu x$ with $\mu = \|v\| \cdot \|x\|^{-1}$ and the inclusion " $\subset$ " of (4.10) are immediate from Lemma 4.2.

To prove the opposite inclusion of (4.10), suppose to the contrary that there exists (x', α') belonging to the set described by the right-hand side of (4.10) with $(x', \alpha') \notin \widehat{D}^*\mathcal{N}(x, \alpha, v)(v')$. Then $(x', \alpha', -v') \notin \widehat{N}((x, \alpha, v); \text{gph}\mathcal{N})$. So there exist a sequence $(x_k, \alpha_k, v_k) \xrightarrow{\text{gph}\mathcal{N}} (x, \alpha, v)$ and a constant $\delta > 0$ such that

$$P_k := \frac{\langle x', x_k - x\rangle + \alpha'(\alpha_k - \alpha) - \langle v', v_k - v\rangle}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|} \geq \delta, \tag{4.11}$$

for all $k \in I\!N$. By the choice of (x', α'), we have

$$\begin{aligned} P_k &= \frac{\langle -\frac{\alpha'}{\alpha}x + \mu v', x_k - x\rangle + \alpha'(\alpha_k - \alpha) - \langle v', v_k - v\rangle}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|} \\ &= \frac{\langle -\frac{\alpha'}{\alpha}x, x_k - x\rangle + \alpha'(\alpha_k - \alpha)}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|} + \frac{\langle \mu v', x_k - x\rangle - \langle v', v_k - v\rangle}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|} \\ &= \frac{\langle -\frac{\alpha'}{\alpha}x, x_k - x\rangle + \alpha'(\alpha_k - \alpha)}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|} + \frac{\langle \mu v', x_k\rangle - \langle v', v_k\rangle}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|} \\ &= Q_k + R_k, \end{aligned}$$

where

$$Q_k := \frac{\langle -\frac{\alpha'}{\alpha}x, x_k - x\rangle + \alpha'(\alpha_k - \alpha)}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|},$$

and

$$R_k := \frac{\langle \mu v', x_k\rangle - \langle v', v_k\rangle}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|}.$$

Since $v = \mu x \neq 0$ and $v_k \to v$, we may assume that $v_k \neq 0$ for all k. Since $v_k \in \mathcal{N}(x_k, \alpha_k)$, by (4.6) and (4.5) we have $v_k = \mu_k x_k$ with $\mu_k > 0$. As $\mu_k = \|v_k\| \cdot \|x_k\|^{-1}$ and $x_k \to x$, we must have $\mu_k \to \mu$ as $k \to \infty$.

If $x_k = x$ then $\alpha_k = \alpha$ and $v_k = \mu_k x_k = \mu_k x$. Combining this with the properties $v = \mu x$ and $\langle v', x\rangle = 0$, we get $P_k = 0$, contradicting (4.11). We have thus shown that $x_k \neq x$ for all $k \in I\!N$.

It holds that $\limsup_{k\to\infty} R_k \leq 0$. Indeed, otherwise there exist a subsequence $\{k_\ell\}$ of $\{k\}$ and a constant $\rho > 0$ such that

$$R_{k_\ell} = \frac{\langle \mu v', x_{k_\ell}\rangle - \langle v', v_{k_\ell}\rangle}{\|x_{k_\ell} - x\| + |\alpha_{k_\ell} - \alpha| + \|v_{k_\ell} - v\|} \geq \rho, \quad \forall \ell \in I\!N. \tag{4.12}$$

Then we have

$$\begin{aligned} R_{k_\ell} &\leq \frac{\langle \mu v', x_{k_\ell}\rangle - \langle v', v_{k_\ell}\rangle}{\|x_{k_\ell} - x\| + \|v_{k_\ell} - v\|} \\ &= \frac{\langle \mu v', x_{k_\ell}\rangle - \langle v', \mu_{k_\ell} x_{k_\ell}\rangle}{\|x_{k_\ell} - x\| + \|v_{k_\ell} - v\|} = \frac{(1 - \mu_{k_\ell}\mu^{-1})\langle \mu v', x_{k_\ell}\rangle}{\|x_{k_\ell} - x\| + \|v_{k_\ell} - v\|}. \end{aligned}$$

Since $\langle v', x\rangle = 0$, it holds that

$$\begin{aligned} R_{k_\ell} &\leq \frac{(1 - \mu_{k_\ell}\mu^{-1})\langle \mu v', x_{k_\ell} - x\rangle}{\|x_{k_\ell} - x\| + \|v_{k_\ell} - v\|} \\ &\leq \frac{(1 - \mu_{k_\ell}\mu^{-1})\langle \mu v', x_{k_\ell} - x\rangle}{\|x_{k_\ell} - x\|} \\ &= (1 - \mu_{k_\ell}\mu^{-1})\big\langle \mu v', \|x_{k_\ell} - x\|^{-1}(x_{k_\ell} - x)\big\rangle. \end{aligned}$$

There is no loss of generality in assuming that $\|x_{k_\ell} - x\|^{-1}(x_{k_\ell} - x) \to \xi$ with $\|\xi\| = 1$. Since $\mu_{k_\ell} \to \mu$, we get

$$R_{k_\ell} \leq (1 - \mu_{k_\ell}\mu^{-1}) \left\langle \mu v', \frac{x_{k_\ell} - x}{\|x_{k_\ell} - x\|} \right\rangle \to 0$$

as $\ell \to \infty$. This contradicts (4.12), hence there must exist $N_0 > 0$ such that $R_k \leq \delta/2$ for all $k \geq N_0$. Since (4.11) is satisfied and $P_k = Q_k + R_k$ for all

$k \in I\!N$, this implies that $Q_k \geq \delta/2$ for all $k \geq N_0$. For each $k \geq N_0$, we have

$$\frac{\delta}{2} \leq Q_k = \frac{\langle -\frac{\alpha'}{\alpha}x, x_k - x\rangle + \alpha'(\alpha_k - \alpha)}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|} = \frac{\frac{\alpha'}{\alpha}\langle x, x\rangle - \frac{\alpha'}{\alpha}\langle x, x_k\rangle + \alpha'(\alpha_k - \alpha)}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|}$$
$$= \frac{\frac{\alpha'}{\alpha}\alpha^2 - \frac{\alpha'}{\alpha}\langle x, x_k\rangle + \alpha'\alpha_k - \alpha'\alpha}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|} = \frac{-\frac{\alpha'}{\alpha}\langle x, x_k\rangle + \alpha'\alpha_k}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|}$$
$$= \frac{\alpha'\left(\alpha_k - \frac{\langle x, x_k\rangle}{\alpha}\right)}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|}. \tag{4.13}$$

Hence, if $\alpha' = 0$ then $Q_k = 0$, which is impossible. Thus $\alpha' \neq 0$. If $\alpha' < 0$, then it follows from (4.13) that $\alpha_k - \frac{\langle x, x_k\rangle}{\alpha} < 0$. Consequently, we have

$$\alpha_k < \frac{\langle x, x_k\rangle}{\alpha} \leq \frac{\|x\| \cdot \|x_k\|}{\alpha} = \frac{\alpha\alpha_k}{\alpha} = \alpha_k,$$

a contradiction. It remains to consider the case where $\alpha' > 0$. The subsequent analytical arguments are based on a geometrical construction. Define the intersection of the ray Ox_k with the sphere $\partial E(\alpha) := \{x \in I\!R^n |\ \|x\| = \alpha\}$ by z_k. Let u_k be the orthogonal projection of x on the ray Ox_k. (Since $x_k \to x \neq 0$, u_k is well defined for $k \geq N_0$ large enough.) Since $x_k \neq 0$ for all k, we have

$$\frac{\delta}{2} \leq Q_k = \frac{\alpha'\left(\alpha_k - \frac{\langle x, x_k\rangle}{\alpha}\right)}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|}$$
$$\leq \frac{\alpha'\left(\alpha_k - \frac{\langle x, x_k\rangle}{\alpha}\right)}{\|x_k - x\|} = \frac{\alpha'\alpha_k\left(1 - \frac{\langle x, x_k\rangle}{\alpha\alpha_k}\right)}{\|x_k - x\|}$$
$$= \frac{\alpha'\alpha_k\left(1 - \frac{\langle x, x_k\rangle}{\|x\| \cdot \|x_k\|}\right)}{\|x_k - x\|} = \frac{\alpha'\alpha_k\left(1 - \cos\left(\widehat{x, x_k}\right)\right)}{\|x_k - x\|}$$
$$= \frac{2\alpha'\alpha_k \sin^2\left(\frac{1}{2}\left(\widehat{x, x_k}\right)\right)}{\|x_k - x\|} = \frac{2\alpha'\alpha_k\left(\frac{\|z_k - x\|}{2}\alpha^{-1}\right)^2}{\|x_k - x\|} = \frac{\alpha'\alpha_k}{2\alpha^2} \cdot \frac{\|z_k - x\|^2}{\|x_k - x\|}.$$

The equality $z_k = x$ yields $\delta/2 \leq Q_k \leq 0$, an absurd. Thus $\|z_k - x\| \neq 0$ for all $k \geq N_0$ sufficient large. From the above it follows that

$$\frac{\delta}{2} \leq Q_k \leq \frac{\alpha'\alpha_k}{2\alpha^2} \cdot \frac{\|z_k - x\|^2}{\|u_k - x\|} = \frac{\alpha'\alpha_k}{2\alpha^2} \cdot \frac{\|z_k - x\|}{\|u_k - x\| \cdot \|z_k - x\|^{-1}}$$
$$= \frac{\alpha'\alpha_k}{2\alpha^2} \cdot \frac{\|z_k - x\|}{\sin \widehat{Oz_kx}} < \frac{\alpha'}{\alpha} \cdot \frac{\|z_k - x\|}{\sin \widehat{Oz_kx}}.$$

Thus, for all k large enough,

$$0 < \frac{\delta\alpha}{2\alpha'} < \frac{\|z_k - x\|}{\sin \widehat{Oz_kx}}. \tag{4.14}$$

Note that since the triangle Oz_kx is isosceles and $z_k \to x$, the angle $\widehat{Oz_kx}$ tends to $\pi/2$ as $k \to \infty$. Hence, from (4.14) we deduce that

$$0 < \frac{\delta\alpha}{2\alpha'} \le 0,$$

an absurd. Thus, the inclusion " $\supset$ " of (4.10) is valid. The formula (4.10) has been established. □

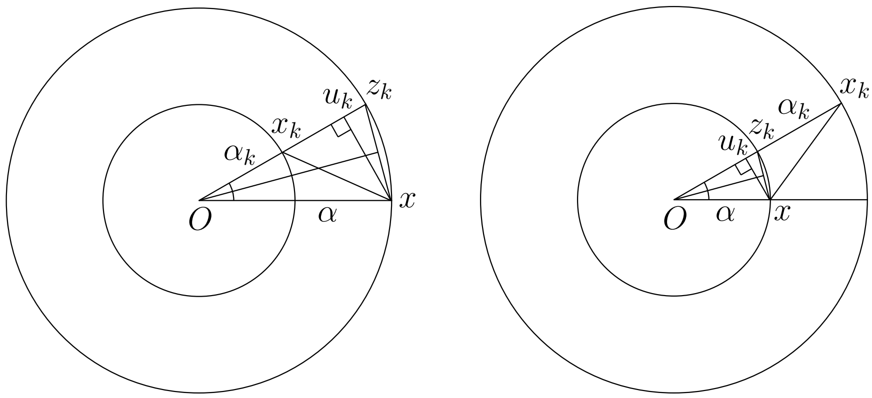

Figure 4.1: The sequences $\{(x_k, \alpha_k)\}_{k\in I\!N}$, $\{z_k\}_{k\in I\!N}$, and $\{u_k\}_{k\in I\!N}$

Remark 4.1 For the second part of the proof of Theorem 4.1, let us present another argument dealing with the case where $\alpha' > 0$. In this case we have

$$\frac{\delta}{2} \le Q_k = \frac{\alpha'\left(\alpha_k - \frac{\langle x, x_k\rangle}{\alpha}\right)}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|} \le \frac{\alpha'\alpha^{-1}(\alpha\alpha_k - \langle x, x_k\rangle)}{\|x_k - x\|}.$$

It follows that

$$\begin{aligned}
\frac{\delta\alpha}{\alpha'} &\le \frac{2(\alpha\alpha_k - \langle x, x_k\rangle)}{\|x_k - x\|} = \frac{2(\|x\| \cdot \|x_k\| - \langle x, x_k\rangle)}{\|x_k - x\|} \\
&= \frac{\langle x_k - x, x_k - x\rangle + 2\|x\| \cdot \|x_k\| - \|x\|^2 - \|x_k\|^2}{\|x_k - x\|} \\
&= \|x_k - x\| - \Big|\|x_k\| - \|x\|\Big| \cdot \frac{\Big|\|x_k\| - \|x\|\Big|}{\|x_k - x\|}.
\end{aligned} \tag{4.15}$$

Since $\Big|\|x_k\| - \|x\|\Big| \le \|x_k - x\|$, we have

$$0 \le \Big|\|x_k\| - \|x\|\Big| \cdot \frac{\Big|\|x_k\| - \|x\|\Big|}{\|x_k - x\|} \le \|x_k - x\|.$$

Therefore, passing k to infinity, from (4.15) we deduce that $0 < \frac{\delta\alpha}{\alpha'} \le 0$. We have arrived at a contradiction.

Remark 4.2 Formula (4.10) shows that if $\langle v', x\rangle = 0$, then $\widehat{D}^*\mathcal{N}(x,\alpha,v)(v')$ is a straight line in $I\!R^n \times I\!R$ passing through the point $(\mu v', 0)$. To see this, it suffices to put first $\alpha' = 0$ to get $x' = \mu v'$, then let α' take an arbitrary real value and compute $x' = -\frac{\alpha'}{\alpha}x + \mu v'$ for each $\alpha' \in I\!R$.

In the case where $\|x\| = \alpha$ and $v = 0$, the following result has been obtained in [23].

Lemma 4.3 (See [23, Lemma 3.3]) *If $\|x\| = \alpha$, $v = 0$, and $(x', \alpha') \in \widehat{D}^*\mathcal{N}(x,\alpha,v)(v')$, then $\langle v', x\rangle \ge 0$, and there exists $\gamma \in I\!R$ such that $x' = \gamma x$.*

The upper estimate for the Fréchet coderivative value $\widehat{D}^*\mathcal{N}(x,\alpha,v)(v')$ provided by Lemma 4.3 can be rewritten formally as

$$\widehat{D}^*\mathcal{N}(x,\alpha,v)(v') \subset \Big\{(x',\alpha') \in I\!R^n \times I\!R \big|\ x' = \gamma x \ \text{ for some } \gamma \in I\!R\Big\} \tag{4.16}$$

when $\langle v', x\rangle \ge 0$, and $\widehat{D}^*\mathcal{N}(x,\alpha,v)(v') = \emptyset$ if $\langle v', x\rangle < 0$. The estimate (4.16) may be strict.

Example 4.1 Let $n = 2$. In this case, $\mathcal{N}$ is a multifunction between $I\!R^2 \times I\!R$ and $I\!R^2$. For $\alpha = 1$, $x = (1,0)^\top$, and $v = (0,0)^\top$, we have $(x,\alpha,v) \in \mathrm{gph}\mathcal{N}$ because $v \in N(x; E(\alpha))$. Choosing $\alpha_k = \alpha = 1$, $x_k = (1 - k^{-1}, 0)^\top$, and $v_k = v = (0,0)^\top$, we see at once that $(x_k, \alpha_k, v_k) \xrightarrow{\mathrm{gph}\mathcal{N}} (x,\alpha,v)$. Select $v' = (1,0)^\top$, $x' = (-1,0)^\top$, $\alpha' \in I\!R$, and observe that

$$\langle v', x\rangle > 0 \quad \text{and} \quad x' = \gamma x,$$

where $\gamma = -1$. However, $(x',\alpha') \notin \widehat{D}^*\mathcal{N}(x,\alpha,v)(v')$. To see this, it suffices to note that

$$\begin{aligned}
&\limsup_{(\widetilde{x},\widetilde{\alpha},\widetilde{v}) \xrightarrow{\mathrm{gph}\mathcal{N}} (x,\alpha,v)} \frac{\langle x', \widetilde{x} - x\rangle + \alpha'(\widetilde{\alpha} - \alpha) - \langle v', \widetilde{v} - v\rangle}{\|\widetilde{x} - x\| + |\widetilde{\alpha} - \alpha| + \|\widetilde{v} - v\|} \\
&\ge \limsup_{k\to\infty} \frac{\langle x', x_k - x\rangle + \alpha'(\alpha_k - \alpha) - \langle v', v_k - v\rangle}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k - v\|} = 1 > 0;
\end{aligned}$$

hence $(x', \alpha', -v') \notin \widehat{N}((x,\alpha,v); \mathrm{gph}\mathcal{N})$.

Tightening the estimate (4.16) we can get an exact formula for the coderivative $\widehat{D}^*\mathcal{N}(x,\alpha,v)$ in the case $\|x\| = \alpha$ and $v = 0$ as follows.

Theorem 4.2 *If $\|x\| = \alpha$ and $v = 0$, then*

$$\widehat{D}^*\mathcal{N}(x,\alpha,v)(v') = \begin{cases} \{(x',\alpha') \in I\!R^n \times I\!R \mid x' = -\frac{\alpha'}{\alpha}x,\ \alpha' \le 0\}, & \text{if } \langle v', x\rangle \ge 0 \\ \emptyset, & \text{if } \langle v', x\rangle < 0, \end{cases} \tag{4.17}$$

for every $v' \in I\!R^n$.

Proof. Fix any $v' \in I\!R^n$. If $\langle v', x\rangle < 0$, then $\widehat{D}^*\mathcal{N}(x,\alpha,v)(v') = \emptyset$ by Lemma 4.3. If $\langle v', x\rangle \ge 0$ and $(x',\alpha') \in \widehat{D}^*\mathcal{N}(x,\alpha,v)(v')$, then by Lemma 4.3 we can select a $\gamma \in I\!R$ such that

$$x' = \gamma x. \tag{4.18}$$

We are going to show that $\gamma = -\frac{\alpha'}{\alpha}$ and $\alpha' \le 0$. As $(x',\alpha') \in \widehat{D}^*\mathcal{N}(x,\alpha,v)(v')$, we have

$$\limsup_{(\widetilde{x},\widetilde{\alpha},\widetilde{v}) \overset{\mathrm{gph}\mathcal{N}}{\longrightarrow} (x,\alpha,v)} \frac{\langle x', \widetilde{x}-x\rangle + \alpha'(\widetilde{\alpha}-\alpha) - \langle v', \widetilde{v}-v\rangle}{\|\widetilde{x}-x\| + |\widetilde{\alpha}-\alpha| + \|\widetilde{v}-v\|} \le 0. \tag{4.19}$$

Choosing $\alpha_k = \alpha$, $x_k = (1-k^{-1})x$, and $v_k = 0$ for every $k \in I\!N$, we can infer that $(x_k,\alpha_k,v_k) \overset{\mathrm{gph}\mathcal{N}}{\longrightarrow} (x,\alpha,v)$. Hence, in accordance with (4.19) and (4.18),

$$\begin{aligned} 0 &\ge \lim_{k\to\infty} \frac{\langle x', x_k - x\rangle + \alpha'(\alpha_k-\alpha) - \langle v', v_k - v\rangle}{\|x_k - x\| + |\alpha_k-\alpha| + \|v_k - v\|} \\ &= \lim_{k\to\infty} \frac{\langle x', x_k - x\rangle}{\|x_k - x\|} = \lim_{k\to\infty} \frac{\langle \gamma x, (1-k^{-1})x - x\rangle}{\|(1-k^{-1})x - x\|} = \frac{-\gamma\langle x, x\rangle}{\|x\|} = -\gamma\alpha. \end{aligned}$$

Combining this with the condition $\alpha > 0$, we get $\gamma \ge 0$. Now, for every $k \in I\!N$, let $x_k = \alpha_k\alpha^{-1}x$ and $v_k = v = 0$, where α_k will be chosen so that $\alpha_k \to \alpha$. As $(x_k,\alpha_k,v_k) \overset{\mathrm{gph}\mathcal{N}}{\longrightarrow} (x,\alpha,v)$, by (4.19) we have

$$\limsup_{k\to\infty} \frac{\langle x', x_k - x\rangle + \alpha'(\alpha_k-\alpha)}{\|x_k - x\| + |\alpha_k-\alpha|} \le 0.$$

Thus, for any $\varepsilon > 0$, there exists $k_\varepsilon \in I\!N$ satisfying

$$\langle x', x_k - x\rangle + \alpha'(\alpha_k-\alpha) \le \varepsilon(\|x_k - x\| + |\alpha_k-\alpha|), \quad \forall k \ge k_\varepsilon.$$

Since $x' = \gamma x$, the latter implies that

$$\gamma\langle x, x_k - x\rangle + \alpha'(\alpha_k-\alpha) \le \varepsilon(\|x_k - x\| + |\alpha_k-\alpha|), \quad \forall k \ge k_\varepsilon.$$

Hence,

$$\begin{aligned} \alpha'(\alpha_k-\alpha) &\le \gamma(\langle x, x\rangle - \langle x, x_k\rangle) + \varepsilon(\|x_k - x\| + |\alpha_k-\alpha|) \\ &= \gamma(\alpha^2 - \langle x, \alpha_k\alpha^{-1}x\rangle) + \varepsilon(\|\alpha_k\alpha^{-1}x - x\| + |\alpha_k-\alpha|) \\ &= \gamma(\alpha^2 - \alpha_k\alpha) + 2\varepsilon|\alpha_k-\alpha|. \end{aligned}$$

Therefore,

$$-\frac{\alpha'}{\alpha}(\alpha-\alpha_k)\le\gamma(\alpha-\alpha_k)+\frac{2\varepsilon}{\alpha}|\alpha_k-\alpha|. \tag{4.20}$$

Letting $\alpha_k\uparrow\alpha$ as $k\to\infty$, from (4.20) we get $-\frac{\alpha'}{\alpha}\le\gamma+\frac{2\varepsilon}{\alpha}$. Letting $\alpha_k\downarrow\alpha$ as $k\to\infty$, from (4.20) we obtain $-\frac{\alpha'}{\alpha}\ge\gamma-\frac{2\varepsilon}{\alpha}$. Since $\varepsilon>0$ can be chosen arbitrary, it follows that $\gamma=-\alpha'\alpha^{-1}$. As $\gamma\ge 0$ and $\alpha>0$, we must have $\alpha'\le 0$. Since $x'=\gamma x=-\alpha'\alpha^{-1}x$ by virtue of (4.18), we have proved that

$$\widehat{D}^*\mathcal{N}(x,\alpha,v)(v')\subset\left\{(x',\alpha')\in I\!R^{n+1}\,\middle|\, x'=-\frac{\alpha'}{\alpha}x,\ \alpha'\le 0\right\}. \tag{4.21}$$

Let us check the opposite inclusion of (4.21) in the case $\langle v',x\rangle\ge 0$. If one could find an element (x',α') from the set on the right-hand side of (4.21) with $(x',\alpha')\notin\widehat{D}^*\mathcal{N}(x,\alpha,v)(v')$, then there would exist a sequence (x_k,α_k,v_k) and a constant $\delta>0$ such that $(x_k,\alpha_k,v_k)\xrightarrow{\text{gph}\mathcal{N}}(x,\alpha,v)$ as $k\to\infty$ and

$$P_k:=\frac{\langle x',x_k-x\rangle+\alpha'(\alpha_k-\alpha)-\langle v',v_k-v\rangle}{\|x_k-x\|+|\alpha_k-\alpha|+\|v_k-v\|}\ge\delta,\quad\forall k\in I\!N. \tag{4.22}$$

Since $x'=-\alpha'\alpha^{-1}x$ with $\alpha'\le 0$ and since $v=0$, we have

$$P_k=Q_k-R_k \tag{4.23}$$

where

$$Q_k:=\frac{\langle -\frac{\alpha'}{\alpha}x,x_k-x\rangle+\alpha'(\alpha_k-\alpha)}{\|x_k-x\|+|\alpha_k-\alpha|+\|v_k\|},\quad R_k:=\frac{\langle v',v_k\rangle}{\|x_k-x\|+|\alpha_k-\alpha|+\|v_k\|}.$$

We distinguish two cases: (i) $\langle v',x\rangle=0$, (ii) $\langle v',x\rangle>0$.

Case (i): $\langle v',x\rangle=0$. In this case $R_k\to 0$ as $k\to\infty$. Indeed, if $v_k=0$ for all large k, then $R_k=0$ for all k large enough; hence $\lim_{k\to\infty}R_k=0$. Otherwise, we may assume that $v_k\ne 0$ for all k. For every k, since $v_k\in\mathcal{N}(x_k,\alpha_k)\setminus\{0\}$, by (4.6) and (4.5) there exists $\mu_k>0$ such that $v_k=\mu_k x_k$. Then, we have $\|v_k\|=\mu_k\|x_k\|\ne 0$. Consequently,

$$\begin{aligned}|R_k|&=\frac{|\langle v',v_k\rangle|}{\|x_k-x\|+|\alpha_k-\alpha|+\|v_k\|}\\&\le\frac{|\langle v',v_k\rangle|}{\|v_k\|}=\frac{|\langle v',x_k\rangle|}{\|x_k\|}\to\frac{|\langle v',x\rangle|}{\|x\|}=0\quad(\text{as }k\to\infty).\end{aligned}$$

We have seen that $R_k\to 0$ as $k\to\infty$.

Case (ii): $\langle v',x\rangle>0$. Since $x_k\to x$, this strict inequality yields $\langle v',x_k\rangle>0$ for all k large enough. If there is $\bar{k}\in I\!N$ such that $v_k=0$ for all $k\ge\bar{k}$, then

$R_k = 0$ for all $k \geq \bar{k}$. If there exists a subsequence $\{v_{k_\ell}\}$ of $\{v_k\}$ with $v_{k_\ell} \neq 0$ for all $\ell \in I\!N$, then $v_{k_\ell} = \mu_{k_\ell} x_{k_\ell}$, where $\mu_{k_\ell} > 0$ for all ℓ. Since $\langle v', x_{k_\ell}\rangle > 0$ for all ℓ sufficiently large, we have

$$\langle v', v_{k_\ell}\rangle = \langle v', \mu_{k_\ell} x_{k_\ell}\rangle = \mu_{k_\ell}\langle v', x_{k_\ell}\rangle > 0 \quad (\ell \text{ is large enough}).$$

This implies that

$$\begin{aligned} 0 \leq R_{k_\ell} &= \frac{\langle v', v_{k_\ell}\rangle}{\|x_{k_\ell} - x\| + |\alpha_{k_\ell} - \alpha| + \|v_{k_\ell}\|} \\ &\leq \frac{\langle v', v_{k_\ell}\rangle}{\|v_{k_\ell}\|} = \frac{\langle v', x_{k_\ell}\rangle}{\|x_{k_\ell}\|} \to \frac{\langle v', x\rangle}{\|x\|} \quad (\text{as } \ell \to \infty). \end{aligned}$$

Since the last property of $\{R_k\}$ is valid for any subsequence $\{v_{k_\ell}\}$ of $\{v_k\}$ with $v_{k_\ell} \neq 0$ for all ℓ, we can assert that $0 \leq R_k \leq \|x\|^{-1}\langle v', x\rangle + 1$ for all k large enough.

From the above analysis we see that, in both the cases (i) and (ii), there exists an index k_0 such that $R_k \geq -\delta/2$ for all $k \geq k_0$. Then, by (4.22) and (4.23),

$$Q_k = P_k + R_k \geq \delta + R_k \geq \frac{\delta}{2}, \quad \forall k \geq k_0.$$

Since $\alpha' \leq 0$ and $\|x_k\| \leq \alpha_k$, for each $k \geq k_0$ we have

$$\begin{aligned} \frac{\delta}{2} \leq Q_k &= \frac{\langle -\frac{\alpha'}{\alpha}x, x_k - x\rangle + \alpha'(\alpha_k - \alpha)}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k\|} = \frac{-\frac{\alpha'}{\alpha}\langle x, x_k\rangle + \frac{\alpha'}{\alpha}\alpha^2 + \alpha'\alpha_k - \alpha'\alpha}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k\|} \\ &= \frac{\alpha'\alpha_k - \frac{\alpha'}{\alpha}\langle x, x_k\rangle}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k\|} \leq \frac{\alpha'\alpha_k - \frac{\alpha'}{\alpha}\|x\| \cdot \|x_k\|}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k\|} \\ &\leq \frac{\alpha'\alpha_k - \alpha'\alpha_k}{\|x_k - x\| + |\alpha_k - \alpha| + \|v_k\|} = 0. \end{aligned}$$

This contradiction completes the proof of the opposite inclusion in (4.21), hence establishes (4.17). □

Remark 4.3 Theorem 4.2 gives a complete solution for the second part of the open problem raised in [23, Remark 3.6].

4.2.2 The Mordukhovich Coderivative of $\mathcal{N}(x, \alpha)$

Based on the obtained formulas for $\widehat{D}^*\mathcal{N}(x, \alpha, v)(\cdot)$, we provide exact formulas for the Mordukhovich coderivative $D^*\mathcal{N}(x, \alpha, v)(\cdot)$ of the normal cone mapping $\mathcal{N}(\cdot)$ in various cases. In the next two lemmas, we recall some existing results.

Lemma 4.4 (See [23, Lemma 4.4]) *The set* gph$\mathcal{N}$ *is locally closed in the product space* $I\!R^n \times I\!R \times I\!R^n$.

Lemma 4.5 (See [23, Lemma 3.7]) *If* $\|x\| < \alpha$, *then* $v = 0$ *and*

$$D^*\mathcal{N}(x, \alpha, v)(v') = \widehat{D}^*\mathcal{N}(x, \alpha, v)(v') = \{(0_{I\!R^n}, 0_{I\!R})\},$$

for every $v' \in I\!R^n$.

By Lemma 4.5, the normal cone mapping $\mathcal{N}(\cdot)$ is graphically regular at any point $(x, \alpha, v) \in \text{gph}\mathcal{N}$ with $\|x\| < \alpha$. The forthcoming theorem shows that $\mathcal{N}(\cdot)$ is also graphically regular at any point $(x, \alpha, v) \in \text{gph}\mathcal{N}$ with $\|x\| = \alpha$ and $v \neq 0$.

Theorem 4.3 *If* $\|x\| = \alpha$ *and if* $v \neq 0$, *then we have*

$$\begin{aligned} D^*\mathcal{N}(x, \alpha, v)(v') &= \widehat{D}^*\mathcal{N}(x, \alpha, v)(v') \\ &= \begin{cases} \{(x', \alpha') \in I\!R^n \times I\!R \,|\, x' = -\frac{\alpha'}{\alpha}x + \mu v'\}, & \text{if } \langle v', x\rangle = 0 \\ \emptyset, & \text{if } \langle v', x\rangle \neq 0 \end{cases} \end{aligned} \tag{4.24}$$

for every $v' \in I\!R^n$, *where* $\mu := \|v\| \cdot \|x\|^{-1}$.

Proof. Fix any $v' \in I\!R^n$ and let $(x', \alpha') \in D^*\mathcal{N}(x, \alpha, v)(v')$ be given arbitrary. By the definition of the Mordukhovich coderivative, there exist sequences $(x_k, \alpha_k, v_k) \xrightarrow{\text{gph}\mathcal{N}} (x, \alpha, v)$ and $(x'_k, \alpha'_k, v'_k) \to (x', \alpha', v')$ such that

$$(x'_k, \alpha'_k) \in \widehat{D}^*\mathcal{N}(x_k, \alpha_k, v_k)(v'_k), \quad \forall k \in I\!N. \tag{4.25}$$

Since $v \neq 0$, we have $v_k \neq 0$ for all k large enough. For those k, according to Theorem 4.1, (4.25) holds if and only if $\|x_k\| = \alpha_k$,

$$\langle v'_k, x_k\rangle = 0 \quad \text{and} \quad x'_k = -\frac{\alpha'_k}{\alpha_k}x_k + \frac{\|v_k\|}{\|x_k\|}v'_k. \tag{4.26}$$

Passing (4.26) to limit as $k \to \infty$ and remembering that $x_k \to x$, $\alpha_k \to \alpha$, $v_k \to v$, $x'_k \to x'$, $\alpha'_k \to \alpha'$, and $v'_k \to v'$, we obtain

$$\langle v', x\rangle = 0 \quad \text{and} \quad x' = -\frac{\alpha'}{\alpha}x + \frac{\|v\|}{\|x\|}v'.$$

Thus $(x', \alpha') \in \widehat{D}^*\mathcal{N}(x, \alpha, v)(v')$ by Theorem 4.1. We have shown that $D^*\mathcal{N}(x, \alpha, v)(v') \subset \widehat{D}^*\mathcal{N}(x, \alpha, v)(v')$. Since the reverse inclusion is obvious, combining this with (4.10) we obtain (4.24) for every $v' \in I\!R^n$. □

The case $(x, \alpha, v) \in \text{gph}\mathcal{N}$ with $\|x\| = \alpha$, $v = 0$, is treated now. Combining the following theorem with Theorem 4.2, we see that $D^*\mathcal{N}(x, \alpha, v)(v') \neq \widehat{D}^*\mathcal{N}(x, \alpha, v)(v')$ for all v' from the closed half-space $\{v' \in I\!R^n |\ \langle v', x\rangle \leq 0\}$. So the multifunction $\mathcal{N}(\cdot)$ is graphically irregular at any point $(x, \alpha, v) \in \text{gph}\mathcal{N}$ where $\|x\| = \alpha$ and $v = 0$.

Theorem 4.4 *Suppose that $\|x\| = \alpha$ and $v = 0$. For every $v' \in I\!R^n$, the following hold*

(i) *If $\langle v', x\rangle \neq 0$, then*

$$D^*\mathcal{N}(x, \alpha, v)(v') = \begin{cases} \widehat{D}^*\mathcal{N}(x, \alpha, v)(v'), & \text{if } \langle v', x\rangle > 0 \\ \{(0_{I\!R^n}, 0_{I\!R})\}, & \text{if } \langle v', x\rangle < 0 \end{cases}$$

$$= \begin{cases} \{(x', \alpha') \in I\!R^{n+1} |\ x' = -\frac{\alpha'}{\alpha}x,\ \alpha' \leq 0\}, & \text{if } \langle v', x\rangle > 0 \\ \{(0_{I\!R^n}, 0_{I\!R})\}, & \text{if } \langle v', x\rangle < 0. \end{cases} \tag{4.27}$$

(ii) *If $\langle v', x\rangle = 0$, then*

$$D^*\mathcal{N}(x, \alpha, v)(v') = \left\{(x', \alpha') \in I\!R^n \times I\!R \,\middle|\, x' = -\frac{\alpha'}{\alpha}x,\ \alpha' \in I\!R\right\}. \tag{4.28}$$

Proof. Let $(x, \alpha, v) \in \text{gph}\mathcal{N}$, $\|x\| = \alpha$, $v = 0$, and $v' \in I\!R^n$. (i) If $\langle v', x\rangle < 0$, then $D^*\mathcal{N}(x, \alpha, v)(v') = \{(0_{I\!R^n}, 0_{I\!R})\}$. Indeed, for such v', given $(x', \alpha') \in D^*\mathcal{N}(x, \alpha, v)(v')$ one can find sequences $(x_k, \alpha_k, v_k) \xrightarrow{\text{gph}\mathcal{N}} (x, \alpha, v)$ and $(x'_k, \alpha'_k, v'_k) \to (x', \alpha', v')$ with

$$(x'_k, \alpha'_k) \in \widehat{D}^*\mathcal{N}(x_k, \alpha_k, v_k)(v'_k), \quad \forall k \in I\!N. \tag{4.29}$$

The condition $\langle v', x\rangle < 0$ implies that $\langle v'_k, x_k\rangle < 0$ for large k. Fix for a while such an index k. If $\|x_k\| = \alpha_k$ and if $v_k \neq 0$, then $\widehat{D}^*\mathcal{N}(x_k, \alpha_k, v_k)(v'_k) = \emptyset$ by Theorem 4.1 and by the equality $\langle v'_k, x_k\rangle < 0$. If $\|x_k\| = \alpha_k$ and if $v_k = 0$, then $\widehat{D}^*\mathcal{N}(x_k, \alpha_k, v_k)(v'_k) = \emptyset$ by Theorem 4.2 and by the equality $\langle v'_k, x_k\rangle < 0$. Therefore, the nonemptyness of $\widehat{D}^*\mathcal{N}(x_k, \alpha_k, v_k)(v'_k)$ shown in (4.29) yields $\|x_k\| < \alpha_k$. By Lemma 4.1, the latter implies that $\widehat{D}^*\mathcal{N}(x_k, \alpha_k, v_k)(v'_k) =$

$\{(0_{\mathbb{R}^n}, 0_{\mathbb{R}})\}$. Consequently, $(x'_k, \alpha'_k) = (0_{\mathbb{R}^n}, 0_{\mathbb{R}})$ for all k large enough; hence $(x', \alpha') = \lim_{k\to\infty}(x'_k, \alpha'_k) = (0_{\mathbb{R}^n}, 0_{\mathbb{R}})$. This justifies that $D^*\mathcal{N}(x,\alpha,v)(v') \subset \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}})\}$. To get the reverse inclusion, choose $x_k = (1-k^{-1})x$, $\alpha_k = \alpha$, $v_k = 0$, $x'_k = 0$, $\alpha'_k = 0$, and $v'_k = v'$ for every $k \in \mathbb{N}$. Then, $(x_k, \alpha_k, v_k) \in \mathrm{gph}\mathcal{N}$, $(x_k, \alpha_k, v_k) \to (x, \alpha, v)$, and $(x'_k, \alpha'_k, v'_k) \to (0_{\mathbb{R}^n}, 0_{\mathbb{R}}, v')$ as $k \to \infty$. The choice of x_k and α_k yields $\|x_k\| < \|x\| = \alpha = \alpha_k$. Hence, by Lemma 4.1 we have

$$(x'_k, \alpha'_k) \in \widehat{D}^*\mathcal{N}(x_k, \alpha_k, v_k)(v'_k) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}})\}, \quad \forall k \in \mathbb{N}.$$

This gives $(0_{\mathbb{R}^n}, 0_{\mathbb{R}}) \in D^*\mathcal{N}(x,\alpha,v)(v')$ and thus establishes the desired equality

$$D^*\mathcal{N}(x,\alpha,v)(v') = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}})\}.$$

Suppose now that $\langle v', x\rangle > 0$. Due to $\widehat{D}^*\mathcal{N}(x,\alpha,v)(v') \subset D^*\mathcal{N}(x,\alpha,v)(v')$ and Theorem 4.2, the proof of the equalities

$$D^*\mathcal{N}(x,\alpha,v)(v') = \widehat{D}^*\mathcal{N}(x,\alpha,v)(v') = \left\{(x', \alpha') \in \mathbb{R}^{n+1} \,\middle|\, x' = -\frac{\alpha'}{\alpha}x,\ \alpha' \le 0\right\},$$

which are stated in (4.27), reduces to checking the fulfilment of the inclusion

$$D^*\mathcal{N}(x,\alpha,v)(v') \subset \widehat{D}^*\mathcal{N}(x,\alpha,v)(v'). \tag{4.30}$$

For any $(x', \alpha') \in D^*\mathcal{N}(x,\alpha,v)(v')$, there exist $(x_k, \alpha_k, v_k) \xrightarrow{\mathrm{gph}\mathcal{N}} (x, \alpha, v)$ and $(x'_k, \alpha'_k, v'_k) \to (x', \alpha', v')$ such that (4.29) holds.

(a) Consider the situation $v_k = 0$ for all k sufficiently large. If $\|x_k\| < \alpha_k$ for all large k, then by Lemma 4.5 we get $(x'_k, \alpha'_k) = (0, 0)$ for large indexes k. Hence, $(x', \alpha') = (0, 0) \in \widehat{D}^*\mathcal{N}(x,\alpha,v)(v')$ (the last inclusion is ready by Theorem 4.2 and the assumptions $\|x\| = \alpha$, $v = 0$, and $\langle v', x\rangle > 0$). If there exists a subsequence $\{k_\ell\}$ of $\{k\}$ such that $\|x_{k_\ell}\| = \alpha_{k_\ell}$ for all $\ell \in \mathbb{N}$, then from (4.29) and Theorem 4.2 we can infer that

$$\langle v'_{k_\ell}, x_{k_\ell}\rangle \ge 0, \quad x'_{k_\ell} = -\frac{\alpha'_{k_\ell}}{\alpha_{k_\ell}}x_{k_\ell}, \quad \alpha'_{k_\ell} \le 0.$$

Taking the limits as $\ell \to \infty$, we obtain

$$\langle v', x\rangle \ge 0, \quad x' = -\frac{\alpha'}{\alpha}x, \quad \alpha' \le 0.$$

By virtue of (4.17), this yields $(x', \alpha') \in \widehat{D}^*\mathcal{N}(x,\alpha,v)(v')$.

(b) Suppose now that there is a subsequence $\{k_\ell\}$ of $\{k\}$ such that $v_{k_\ell} \ne 0$ for all $\ell \in \mathbb{N}$. Then, $\|x_{k_\ell}\| = \alpha_{k_\ell}$ for all ℓ. From (4.29) and Theorem 4.1 we obtain

$$\langle v'_{k_\ell}, x_{k_\ell}\rangle = 0 \quad \text{and} \quad x'_{k_\ell} = -\frac{\alpha'_{k_\ell}}{\alpha_{k_\ell}} x_{k_\ell} + \frac{\|v_{k_\ell}\|}{\|x_{k_\ell}\|} v'_{k_\ell}$$

for all ℓ. This obviously leads to $\langle v', x\rangle = 0$, a contradiction with the assumption that $\langle v', x\rangle > 0$. The inclusion (4.30) has been proved.

Thus, if $\langle v', x\rangle \neq 0$, then we get (4.27).

(ii) Suppose that $\langle v', x\rangle = 0$. For any $(x', \alpha') \in D^*\mathcal{N}(x, \alpha, v)(v')$, there exist sequences $(x_k, \alpha_k, v_k) \xrightarrow{\text{gph}\mathcal{N}} (x, \alpha, v)$ and $(x'_k, \alpha'_k, v'_k) \to (x', \alpha', v')$ such that (4.29) holds. If $v_k = 0$ for all k large enough then, arguing similarly as in the subcase (a) of the proof of assertion (i), we obtain

$$\langle v', x\rangle = 0, \quad x' = -\frac{\alpha'}{\alpha} x, \quad \alpha' \leq 0.$$

Hence (x', α') belongs to the set on the right-hand side of (4.28). If there is a subsequence $\{k_\ell\}$ of $\{k\}$ such that $v_{k_\ell} \neq 0$ for all $\ell \in I\!N$, then by repeating the arguments of subcase (b) of the proof of assertion (i) we have

$$\|x_{k_\ell}\| = \alpha_{k_\ell}, \quad \langle v'_{k_\ell}, x_{k_\ell}\rangle = 0, \quad \text{and} \quad x'_{k_\ell} = -\frac{\alpha'_{k_\ell}}{\alpha_{k_\ell}} x_{k_\ell} + \frac{\|v_{k_\ell}\|}{\|x_{k_\ell}\|} v'_{k_\ell},$$

for all ℓ. Since $v = 0$, this implies that

$$\langle v', x\rangle = 0, \quad x' = -\frac{\alpha'}{\alpha} x, \quad \text{and} \quad \alpha' \in I\!R.$$

Thus, the inclusion " $\subset$ " in (4.28) is valid.

To verify the inclusion " $\supset$ " in (4.28), fix any element (x', α') in the set on the right-hand side of (4.28). We have to show that $(x', \alpha') \in D^*\mathcal{N}(x, \alpha, v)(v')$. For every $k \in I\!N$, choose $x_k = x$, $\alpha_k = \alpha$, and $v_k = \mu_k x_k$, where $\mu_k := k^{-1}$. It is clear that $(x_k, \alpha_k, v_k) \in \text{gph}\mathcal{N}$ and $(x_k, \alpha_k, v_k) \to (x, \alpha, v)$ as $k \to \infty$. For every $k \in I\!N$, putting

$$v'_k = v', \quad \alpha'_k = \alpha', \quad \text{and} \quad x'_k = -\frac{\alpha'}{\alpha} x + k^{-1} v',$$

we have

$$\langle v'_k, x_k\rangle = \langle v', x\rangle = 0, \quad x'_k = -\frac{\alpha'_k}{\alpha_k} x_k + \mu_k v'_k.$$

Hence, in accordance with Theorem 4.1, $(x'_k, \alpha'_k) \in \widehat{D}^*\mathcal{N}(x_k, \alpha_k, v_k)(v'_k)$ for all k. Observing $(x'_k, \alpha'_k, v'_k) \to (x', \alpha', v')$ as $k \to \infty$, we obtain the inclusion $(x', \alpha') \in D^*\mathcal{N}(x, \alpha, v)(v')$ which completes the proof of (4.28). □

4.3 Necessary and Sufficient Conditions for Stability

Conditions for stability of the solution map $(A, b, \alpha) \mapsto S(A, b, \alpha)$ of the linear GE of the form (4.1) are obtained in this section.

4.3.1 Coderivatives of the KKT point set map

As in Section 4.1, we put $G(x, w) = Ax$ and $M(x, w) = \mathcal{N}(x, \alpha)$ for every $x \in I\!R^n$ and $w = (A, \alpha) \in W$ with $W = H(n) \times I\!R$. Fix a triplet $(\bar{A}, \bar{b}, \bar{\alpha}) \in H(n) \times I\!R^n \times I\!R$. Put $\bar{w} = (\bar{A}, \bar{\alpha})$, $\bar{y} = -\bar{b}$, and let $\bar{x} \in S(\bar{A}, \bar{b}, \bar{\alpha})$. Then we have $\bar{x} \in \widetilde{S}(\bar{w}, \bar{y})$ with $\widetilde{S}(\bar{w}, \bar{y})$ being given by (4.8). Let $\bar{v} = \bar{y} - G(\bar{x}, \bar{w}) = -\bar{b} - \bar{A}\bar{x}$.

We will need two more lemmas of [23].

Lemma 4.6 (See [23, Lemma 4.1]) *The Mordukhovich coderivative*

$$D^*M(\bar{x}, \bar{w}, \bar{v}) : I\!R^n \rightrightarrows I\!R^n \times H(n)^* \times I\!R$$

of the multifunction $M : I\!R^n \times W \rightrightarrows I\!R^n$, where $H(n)^*$ is the dual space of $H(n)$, is computed by the formula

$$D^*M(\bar{x}, \bar{w}, \bar{v})(v') = \left\{(x', 0_{H(n)^*}, \alpha') \,\middle|\, (x', \alpha') \in D^*\mathcal{N}(\bar{x}, \bar{\alpha}, \bar{v})(v')\right\}$$

for every $v' \in I\!R^n$.

Lemma 4.7 (See [23, Lemma 4.3]) *For every $v' \in I\!R^n$,*

$$\nabla G(\bar{x}, \bar{w})^*(v') = \{\bar{A}v'\} \times \{\tau(v', \bar{x})\} \times \{0_{I\!R}\},$$

where $\tau(v', \bar{x}) := (v'_i \bar{x}_j)$ is the $n \times n$ matrix whose ij-th element is $v'_i \bar{x}_j$.

Remark 4.4 Similarly as in Lemma 4.6, the Fréchet coderivative

$$\widehat{D}^*M(\bar{x}, \bar{w}, \bar{v}) : I\!R^n \rightrightarrows I\!R^n \times H(n)^* \times I\!R$$

of the multifunction $M : I\!R^n \times W \rightrightarrows I\!R^n$ is computed as follows

$$\widehat{D}^*M(\bar{x}, \bar{w}, \bar{v})(v') = \left\{(x', 0_{H(n)^*}, \alpha') \,\middle|\, (x', \alpha') \in \widehat{D}^*\mathcal{N}(\bar{x}, \bar{\alpha}, \bar{v})(v')\right\}.$$

For each $x' \in I\!R^n$, we put

$$\widehat{\Omega}_{G,\bar{y}}(x') = \bigcup_{v' \in I\!R^n} \Big\{(w', y') \in W \times I\!R^n \,\Big|\, (-x', w', y') \in \nabla G(\bar{x}, \bar{w})^*(v') \times \{0_{I\!R^n}\}$$
$$- \{0_{I\!R^n}\} \times \{0_W\} \times \{v'\} + \widehat{D}^*M(\bar{x}, \bar{w}, \bar{v})(v') \times \{0_{I\!R^n}\}\Big\},$$

and

$$\Omega_{G,\bar y}(x') = \bigcup_{v'\in I\!R^n} \Big\{(w',y') \in W \times I\!R^n \big|\ (-x',w',y') \in \nabla G(\bar x,\bar w)^*(v') \times \{0_{I\!R^n}\}$$
$$-\{0_{I\!R^n}\} \times \{0_W\} \times \{v'\} + D^*M(\bar x,\bar w,\bar v)(v') \times \{0_{I\!R^n}\}\Big\},$$

where $\bar v = \bar y - G(\bar x,\bar w) = -\bar b - \bar A\bar x$.

Since $M : I\!R^n \times W \rightrightarrows I\!R^n$ has a locally closed graph by Lemma 4.4, the next statement is an immediate corollary of [22, Theorem 4.3].

Theorem 4.5 *The inclusions*

$$\widehat\Omega_{G,\bar y}(x') \subset \widehat D^*\widetilde S(\bar w,\bar y,\bar x)(x') \subset D^*\widetilde S(\bar w,\bar y,\bar x)(x') \subset \Omega_{G,\bar y}(x') \tag{4.31}$$

hold for all $x' \in I\!R^n$. If, in addition, $M(\cdot)$ is graphically regular at $(\bar x,\bar w,\bar v) \in$ gph$M$, then

$$\widehat\Omega_{G,\bar y}(x') = \widehat D^*\widetilde S(\bar w,\bar y,\bar x)(x') = D^*\widetilde S(\bar w,\bar y,\bar x)(x') = \Omega_{G,\bar y}(x') \tag{4.32}$$

for every $x' \in I\!R^n$.

Combining (4.32) with Lemmas 4.6 and 4.7, Remark 4.4, Lemma 4.5, Theorem 4.3, and the first assertion of Theorem 4.4, we obtain exact formulas for computing the Fréchet and the Mordukhovich coderivatives of $\widetilde S(w,y) = S(A,b,\alpha)$ at the point $(\bar w,\bar y,\bar x) \in \text{gph}\widetilde S$ with the property that $M(x,w) = \mathcal N(x,\alpha)$ is graphically regular at $\bar\omega := (\bar x,\bar w,\bar v) \in \text{gph}M$. Similarly, invoking (4.31), Lemmas 4.6 and 4.7, Remark 4.4, Theorems 4.2 and 4.4, we get explicit estimates for the Fréchet and the Mordukhovich coderivatives of $\widetilde S(\cdot)$ at the point $(\bar w,\bar y,\bar x) \in \text{gph}\widetilde S$ where $\|\bar x\| = \bar\alpha$, $\bar v = -\bar b - \bar A\bar x = 0$.

4.3.2 The Lipschitz-like property

Since gph$\mathcal N$ is locally closed in the product space $I\!R^n \times I\!R \times I\!R^n$ by Lemma 4.4, gph$M$ is also locally closed in $I\!R^n \times W \times I\!R^n$. So, both gph$S$ and gph$\widetilde S$ are respectively locally closed in the product spaces $H(n) \times I\!R^n \times I\!R \times I\!R^n$ and $W \times I\!R^n \times I\!R^n$. Therefore, by the Mordukhovich criterion stated in Theorem 1.3, $\widetilde S(\cdot)$ is locally Lipschitz-like around $(\bar w,\bar y,\bar x)$ if and only if

$$D^*\widetilde S(\bar w,\bar y,\bar x)(0) = \{0\}. \tag{4.33}$$

By (4.8) we have

$$\Big[D^*S(\bar A,\bar b,\bar\alpha,\bar x)(0) = \{0\}\Big] \Longleftrightarrow \Big[D^*\widetilde S(\bar w,\bar y,\bar x)(0) = \{0\}\Big].$$

This implies that $S(\cdot)$ is locally Lipschitz-like around $(\bar{A},\bar{b},\bar{\alpha},\bar{x})$ if and only if $\widetilde{S}(\cdot)$ is locally Lipschitz-like around $(\bar{w},\bar{y},\bar{x})$. If $M(\cdot)$ is graphically regular at $\bar{\omega}$, then by (4.32) we see that (4.33) holds if and only if $\Omega_{G,\bar{y}}(0)=\widehat{\Omega}_{G,\bar{y}}(0)=\{0\}$. In the case where $M(\cdot)$ is graphically irregular at $\bar{\omega}$, by (4.31) we can infer that

$$\Big[\Omega_{G,\bar{y}}(0)=\{0\}\Big]\Longrightarrow(4.33)\Longrightarrow\Big[\widehat{\Omega}_{G,\bar{y}}(0)=\{0\}\Big]. \tag{4.34}$$

Theorem 4.6 *For any* $(\bar{A},\bar{b},\bar{\alpha},\bar{x})\in\mathrm{gph}S$*, the following assertions hold:*

(i) *If* $\|\bar{x}\|<\bar{\alpha}$*, then the map* $S(\cdot)$ *is locally Lipschitz-like around* $(\bar{A},\bar{b},\bar{\alpha},\bar{x})$ *if and only if* $\det\bar{A}\neq 0$.

(ii) *If* $\|\bar{x}\|=\bar{\alpha}$ *and* $\bar{A}\bar{x}+\bar{b}\neq 0$*, then* $S(\cdot)$ *is locally Lipschitz-like around* $(\bar{A},\bar{b},\bar{\alpha},\bar{x})$ *if and only if* $\det Q(\bar{A},\bar{b},\bar{\alpha},\bar{x})\neq 0$*, where*

$$Q(\bar{A},\bar{b},\bar{\alpha},\bar{x}):=\begin{pmatrix}\bar{A}+\mu I & -\frac{1}{\bar{\alpha}}\bar{x}\\ \bar{x}^\top & 0\end{pmatrix} \tag{4.35}$$

with μ *being the unique Lagrange multiplier associated to* $\bar{x}$.

Proof. (i) Suppose that $\|\bar{x}\|<\bar{\alpha}$. By Lemma 4.5, $\mathcal{N}(\cdot)$ is graphically regular at $(\bar{x},\bar{\alpha},\bar{v})$. Hence, $M(\cdot)$ is also graphically regular at $\bar{\omega}=(\bar{x},\bar{w},\bar{v})$. According to (4.32), $S(\cdot)$ is locally Lipschitz-like around $\bar{\omega}$ if and only if $\Omega_{G,\bar{y}}(0)=\{0\}$. We see that $(w',y')=(A',\alpha',y')$ belongs to $\Omega_{G,\bar{y}}(0)$ if and only if there exists $v'\in I\!R^n$ such that

$$\big(-\bar{A}v',A'-(v_i'\bar{x}_j),\alpha',y'+v'\big)\in D^*M(\bar{x},\bar{w},\bar{v})(v')\times\{0_{I\!R^n}\}.$$

According to Lemma 4.6, this is equivalent to

$$\big(-\bar{A}v',\alpha',A'-(v_i'\bar{x}_j),y'+v'\big)\in D^*\mathcal{N}(\bar{x},\bar{\alpha},\bar{v})(v')\times\{0_{H(n)^*}\}\times\{0_{I\!R^n}\}.$$

Since $D^*\mathcal{N}(\bar{x},\bar{\alpha},\bar{v})(v')=\{(0_{I\!R^n},0_{I\!R})\}$ by Lemma 4.5, the last inclusion means that

$$\big(-\bar{A}v',\alpha',A'-(v_i'\bar{x}_j),y'+v'\big)=(0_{I\!R^n},0_{I\!R},0_{H(n)^*},0_{I\!R^n}). \tag{4.36}$$

So, the equality $\Omega_{G,\bar{y}}(0)=\{0\}$ holds if and only if the fulfilment of (4.36) for some $v'\in I\!R^n$ yields $A'=0_{H(n)^*}$, $\alpha'=0_{I\!R}$, and $y'=0_{I\!R^n}$. The latter means that $\det\bar{A}\neq 0$. Indeed, if $\det\bar{A}=0$, then there is $v'\neq 0$ such that $-\bar{A}v'=0$. Setting $A'=\tau(v',\bar{x})=(v_i'\bar{x}_j)$, $\alpha'=0$, and $y'=-v'$, we get $(w',y')=(A',\alpha',y')\neq(0_{H(n)^*},0_{I\!R},0_{I\!R^n})$ satisfying (4.36). Thus, there exists

$v' \in I\!R^n$ such that the fulfilment of (4.36) does not yield $(w', y') = (0_W, 0_{I\!R^n})$. Conversely, if $\det \bar{A} \neq 0$, then (4.36) implies that $-\bar{A}v' = 0$; hence $v' = 0$. Substituting $v' = 0$ into (4.36) yields $A' = 0$, $\alpha' = 0$, and $y' = 0$.

(ii) Suppose that $\|\bar{x}\| = \bar{\alpha}$ and $\bar{A}\bar{x} + \bar{b} \neq 0$. As in the case (i), $S(\cdot)$ is locally Lipschitz-like around $\bar{\omega}$ if and only if $\Omega_{G,\bar{y}}(0) = \{0\}$. Moreover, $(w', y') \in \Omega_{G,\bar{y}}(0)$ if and only if there exists $v' \in I\!R^n$ such that

$$\big(-\bar{A}v', \alpha', A' - (v'_i\bar{x}_j), y' + v'\big) \in D^*\mathcal{N}(\bar{x}, \bar{\alpha}, \bar{v})(v') \times \{0_{H(n)^*}\} \times \{0_{I\!R^n}\}.$$

Since $\bar{v} = -\bar{b} - \bar{A}\bar{x} \neq 0$, Theorem 4.3 tells us that the last inclusion can be rewritten equivalently as

$$\begin{cases} -\bar{A}v' = -\frac{\alpha'}{\bar{\alpha}}\bar{x} + \mu v' \\ \langle v', \bar{x}\rangle = 0 \\ \alpha' \in I\!R, \ \ A' = (v'_i\bar{x}_j) \\ y' + v' = 0_{I\!R^n} \end{cases} \tag{4.37}$$

with $\mu := \|\bar{v}\| \cdot \|\bar{x}\|^{-1}$. If λ is the Lagrange multiplier corresponding to $\bar{x} \in S(\bar{A}, \bar{b}, \bar{\alpha})$, then $(\bar{A}\bar{x} + \lambda I)\bar{x} = -\bar{b}$. So, $\lambda\bar{x} = -\bar{b} - \bar{A}\bar{x} = \bar{v}$. It follows that $\lambda = \|\bar{v}\| \cdot \|\bar{x}\|^{-1}$. Thus, μ is the Lagrange multiplier corresponding to $\bar{x}$.

Clearly, $\Omega_{G,\bar{y}}(0) = \{0\}$ if and only if from (4.37), with $v' \in I\!R^n$ being chosen arbitrarily, it follows that $A' = 0_{H(n)^*}$, $\alpha' = 0_{I\!R}$, and $y' = 0_{I\!R^n}$. The latter is equivalent to saying that

$$\begin{cases} (\bar{A} + \mu I)v' - \frac{\alpha'}{\bar{\alpha}}\bar{x} = 0 \\ \bar{x}^\top v' = 0 \\ v' \in I\!R^n, \alpha' \in I\!R \end{cases} \Longrightarrow \begin{cases} v' = 0 \\ \alpha' = 0. \end{cases} \tag{4.38}$$

Since (4.38) can be rewritten equivalently as

$$\left[\begin{pmatrix} \bar{A} + \mu I & -\frac{1}{\bar{\alpha}}\bar{x} \\ \bar{x}^\top & 0 \end{pmatrix} \begin{pmatrix} v' \\ \alpha' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] \Longrightarrow \begin{cases} v' = 0 \\ \alpha' = 0, \end{cases}$$

the condition $\Omega_{G,\bar{y}}(0) = \{0\}$ means that $\det Q(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x})$ is nonzero, where $Q(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x})$ has been defined by (4.35).

The proof of the theorem is complete. □

Theorem 4.7 *Let* $(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) \in \mathrm{gph} S$ *be such that* $\|\bar{x}\| = \bar{\alpha}$ *and* $\bar{A}\bar{x} + \bar{b} = 0$. *Then, the following hold*

(i) *If* $S(\cdot)$ *is locally Lipschitz-like around* $(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x})$, *then the constraint qualification below is satisfied*

$$\begin{cases} \bar{A}v' - \frac{\alpha'}{\bar{\alpha}}\bar{x} = 0 \\ \langle v', \bar{x}\rangle \geq 0 \\ v' \in I\!R^n, \ \ \alpha' \leq 0 \end{cases} \Longrightarrow \begin{cases} v' = 0 \\ \alpha' = 0. \end{cases} \tag{4.39}$$

(ii) *If* $\det\bar{A} \neq 0$, $\det Q_1(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) \neq 0$, *where*

$$Q_1(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) := \begin{pmatrix} \bar{A} & -\frac{1}{\bar{\alpha}}\bar{x} \\ \bar{x}^\top & 0 \end{pmatrix}, \tag{4.40}$$

and (4.39) *is satisfied, then* $S(\cdot)$ *is locally Lipschitz-like around* $(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x})$.

Proof. (i) Suppose that $S(\cdot)$ is locally Lipschitz-like around $(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x})$. Then we have $D^*S(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x})(0) = \{0\}$. Thus, $D^*\widetilde{S}(\bar{w}, \bar{y}, \bar{x})(0) = \{0\}$. By (4.34), the latter implies that $\widehat{\Omega}_{G,\bar{y}}(0) = \{0\}$. Observe that $(w', y') \in \widehat{\Omega}_{G,\bar{y}}(0)$ if and only if there exists $v' \in I\!R^n$ such that

$$\big(- \bar{A}v', A' - (v_i'\bar{x}_j), \alpha', y' + v'\big) \in \widehat{D}^*M(\bar{x}, \bar{w}, \bar{v})(v') \times \{0_{I\!R^n}\}.$$

Due to Remark 4.4, the last inclusion means that

$$\big(- \bar{A}v', \alpha', A' - (v_i'\bar{x}_j), y' + v'\big) \in \widehat{D}^*\mathcal{N}(\bar{x}, \bar{\alpha}, \bar{v})(v') \times \{0_{H(n)^*}\} \times \{0_{I\!R^n}\}.$$

By virtue of Theorem 4.2, this means that

$$\begin{cases} -\bar{A}v' = -\frac{\alpha'}{\bar{\alpha}}\bar{x} \\ \langle v', \bar{x}\rangle \geq 0 \\ \alpha' \leq 0, \ \ A' = (v_i'\bar{x}_j) \\ y' = -v'. \end{cases} \tag{4.41}$$

Therefore, the condition $\widehat{\Omega}_{G,\bar{y}}(0) = \{0\}$ is equivalent to saying that (4.39) holds.

(ii) Suppose that $\det\bar{A} \neq 0$, $\det Q_1(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) \neq 0$ with $Q_1(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x})$ given by (4.40), and (4.39) is satisfied. As we have seen in the proof of Theorem 4.6(i), $(w', y') \in \Omega_{G,\bar{y}}(0)$ if and only if there exists $v' \in I\!R^n$ such that

$$\big(- \bar{A}v', A' - (v_i'\bar{x}_j), \alpha', y' + v'\big) \in D^*M(\bar{x}, \bar{w}, \bar{v})(v') \times \{0_{I\!R^n}\}$$

or, equivalently,

$$\big(- \bar{A}v', \alpha', A' - (v_i'\bar{x}_j), y' + v'\big) \in D^*\mathcal{N}(\bar{x}, \bar{\alpha}, \bar{v})(v') \times \{0_{H(n)^*}\} \times \{0_{I\!R^n}\}. \tag{4.42}$$

If $\langle v', \bar{x}\rangle < 0$, then $(-\bar{A}v', \alpha') \notin D^*\mathcal{N}(\bar{x}, \bar{\alpha}, \bar{v})(v')$. Indeed, the inequality $\langle v', \bar{x}\rangle < 0$ yields $v' \neq 0$. Hence $-\bar{A}v' \neq 0$ because $\det\bar{A} \neq 0$. By Theorem 4.4(i) and by the condition $\langle v', \bar{x}\rangle < 0$, we have $(-\bar{A}v', \alpha') \notin D^*\mathcal{N}(\bar{x}, \bar{\alpha}, \bar{v})(v')$. Therefore, (4.42) is equivalent to

$$\begin{cases} \langle v', \bar{x}\rangle \geq 0 \\ (-\bar{A}v', \alpha') \in D^*\mathcal{N}(\bar{x}, \bar{\alpha}, \bar{v})(v') \\ A' = (v'_i \bar{x}_j) \\ y' = -v'. \end{cases}$$

So, the equality $\Omega_{G,\bar{y}}(0) = \{0\}$ holds if and only if

$$\begin{cases} \langle v', \bar{x}\rangle > 0 \\ (-\bar{A}v', \alpha') \in D^*\mathcal{N}(\bar{x}, \bar{\alpha}, \bar{v})(v') \\ A' = (v'_i \bar{x}_j) \\ y' = -v' \end{cases} \Longrightarrow \begin{cases} A' = 0 \\ \alpha' = 0 \\ y' = 0 \end{cases} \tag{4.43}$$

and

$$\begin{cases} \langle v', \bar{x}\rangle = 0 \\ (-\bar{A}v', \alpha') \in D^*\mathcal{N}(\bar{x}, \bar{\alpha}, \bar{v})(v') \\ A' = (v'_i \bar{x}_j) \\ y' = -v' \end{cases} \Longrightarrow \begin{cases} A' = 0 \\ \alpha' = 0 \\ y' = 0. \end{cases} \tag{4.44}$$

By Theorem 4.4(i), (4.43) means that

$$\begin{cases} \bar{A}v' - \frac{\alpha'}{\bar{\alpha}}\bar{x} = 0 \\ \langle v', \bar{x}\rangle > 0 \\ v' \in I\!R^n, \ \alpha' \leq 0 \end{cases} \Longrightarrow \begin{cases} v' = 0 \\ \alpha' = 0. \end{cases} \tag{4.45}$$

Since (4.39) is satisfied by our assumptions, (4.45) is valid. By virtue of Theorem 4.4(ii), (4.44) is equivalent to

$$\begin{cases} \bar{A}v' - \frac{\alpha'}{\bar{\alpha}}\bar{x} = 0 \\ \bar{x}^\top v' = 0 \\ v' \in I\!R^n, \ \alpha' \in I\!R \end{cases} \Longrightarrow \begin{cases} v' = 0 \\ \alpha' = 0. \end{cases} \tag{4.46}$$

or, equivalently,

$$\left[\begin{pmatrix} \bar{A} & -\frac{1}{\bar{\alpha}}\bar{x} \\ \bar{x}^\top & 0 \end{pmatrix} \begin{pmatrix} v' \\ \alpha' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] \Longrightarrow \begin{cases} v' = 0 \\ \alpha' = 0. \end{cases}$$

The latter holds because $\det Q_1(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) \neq 0$ where $Q_1(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x})$ is given by (4.40). We have shown that under the assumptions made, the equality $\Omega_{G,\bar{y}}(0) = \{0\}$ holds. This implies that $S(\cdot)$ is locally Lipschitz-like around $(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x})$, and completes the proof. □

We now analyze Theorems 4.6 and 4.7 by four examples. The first two show how Theorems 4.6 can recognize stability/instability of $S(\cdot)$ in the situation where $\|\bar{x}\| = \bar{\alpha}$ and $\bar{A}\bar{x} + \bar{b} \neq 0$. The third example illustrates a good association of the necessary stability condition and the sufficient stability condition provided by Theorem 4.7 for the case where $\|\bar{x}\| = \bar{\alpha}$ and $\bar{A}\bar{x} + \bar{b} = 0$. The last example shows that the necessary stability condition given in Theorem 4.7(i) can recognize instability of many linear GEs.

Example 4.2 Following [51] and [23], we consider the problem

$$\min\Big\{f(x) = -4x_2^2 + x_1 \Big| \ x = (x_1, x_2)^\top \in I\!R^2, \ x_1^2 + x_2^2 \leq 1\Big\}. \tag{4.47}$$

Here we have

$$\bar{A} = \begin{pmatrix} 0 & 0 \\ 0 & -8 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{\alpha} = 1.$$

Using the necessary optimality condition (4.4) we find that

$$S(\bar{A}, \bar{b}, \bar{\alpha}) = \Big\{(-1, 0)^\top, (-1/8, \sqrt{63}/8)^\top, (-1/8, -\sqrt{63}/8)^\top\Big\}.$$

The Lagrange multiplier corresponding to the KKT point $\bar{x} := (-1/8, \sqrt{63}/8)^\top$ is $\lambda = 8$. Hence,

$$\det Q(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) = \det \begin{pmatrix} 8 & 0 & \frac{1}{8} \\ 0 & 0 & -\frac{\sqrt{63}}{8} \\ -\frac{1}{8} & \frac{\sqrt{63}}{8} & 0 \end{pmatrix} = \frac{63}{8},$$

and we see that the stability criterion (4.35) is satisfied. Observe that $\bar{A}\bar{x} + \bar{b} \neq 0$. Thanks to Theorem 4.6(ii), we can infer that $S(\cdot)$ is locally Lipschitz-like around $(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) \in \mathrm{gph}S$. By symmetry, we see at once that the assertions made for $\bar{x} = (-1/8, \sqrt{63}/8)^\top$ is also valid for the KKT point $(-1/8, -\sqrt{63}/8)^\top$. For the KKT point $\widehat{x} = (-1, 0)^\top$ and the associated Lagrange multiplier $\lambda = 1$, we find that $\bar{A}\widehat{x} + \bar{b} \neq 0$ and

$$\det Q(\bar{A}, \bar{b}, \bar{\alpha}, \widehat{x}) = \det \begin{pmatrix} 1 & 0 & 1 \\ 0 & -7 & 0 \\ -1 & 0 & 0 \end{pmatrix} = -7.$$

So, by Theorem 4.6(ii), $S(\cdot)$ is also locally Lipschitz-like around $(\bar{A}, \bar{b}, \bar{\alpha}, \widehat{x}) \in \text{gph}S$.

Example 4.3 As in [51] and [23], we consider the problem

$$\min\Big\{ f(x) = -4(x_2^2+x_3^2)+x_1 \,\Big|\, x = (x_1, x_2, x_3)^\top \in I\!R^3, x_1^2+x_2^2+x_3^2 \le 1 \Big\} \quad (4.48)$$

with the data tube $(\bar{A}, \bar{b}, \bar{\alpha})$ given by

$$\bar{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{\alpha} = 1.$$

Using (4.4) we obtain

$$S(\bar{A}, \bar{b}, \bar{\alpha}) = \Big\{(-1, 0, 0)^\top\Big\} \cup \Big\{(-1/8, x_2, x_3)^\top \,\Big|\, x_2^2 + x_3^2 = 63/64\Big\}$$

For the KKT point $\widehat{x} := (-1, 0, 0)^\top$ with the associated Lagrange multiplier $\lambda = 1$, a computation similar to that given in Example 4.2 shows that $S(\cdot)$ is locally Lipschitz-like around the point $(\bar{A}, \bar{b}, \bar{\alpha}, \widehat{x}) \in \text{gph}S$. To complete the stability analysis, fix any $t \in [0, 2\pi)$ and consider the KKT point

$$x_t := \big(-1/8, (\sqrt{63}/8)\sin t, (\sqrt{63}/8)\cos t\big)^\top$$

with the associated Lagrange multiplier $\lambda = 8$. Note that

$$\det Q(\bar{A}, \bar{b}, \bar{\alpha}, x_t) = \det \begin{pmatrix} 8 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & -\frac{\sqrt{63}}{8}\sin t \\ 0 & 0 & 0 & -\frac{\sqrt{63}}{8}\cos t \\ -\frac{1}{8} & \frac{\sqrt{63}}{8}\sin t & \frac{\sqrt{63}}{8}\cos t & 0 \end{pmatrix} = 0.$$

Since $\bar{A}\bar{x}_t + \bar{b} \ne 0$ for any $t \in [0, 2\pi)$, applying Theorem 4.6(ii) we deduce that the map $S(\cdot)$ is *not* locally Lipschitz-like around any point $(\bar{A}, \bar{b}, \bar{\alpha}, x_t)$ for $t \in [0, 2\pi)$.

Example 4.4 Consider (4.1) with $n = 2$, $\bar{A} = I$, $\bar{b} = -(\sqrt{2}, \sqrt{2})^\top$, $\bar{\alpha} = 2$, and $\bar{x} = (\sqrt{2}, \sqrt{2})^\top$. We have $\|\bar{x}\| = 2 = \bar{\alpha}$ and $\bar{v} := -\bar{b} - \bar{A}\bar{x} = 0$. The necessary condition for stability of $S(\cdot)$ provided by Theorem 4.7(i) is as follows:

$$\begin{cases} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} - \dfrac{\alpha'}{2}\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} = 0 \\ v_1' + v_2' \ge 0 \\ \alpha' \le 0, \ v' = (v_1', v_2')^\top \in I\!R^2 \end{cases} \Longrightarrow \begin{cases} v' = 0 \\ \alpha' = 0. \end{cases}$$

It is not difficult to see that this condition is satisfied. As $\det\bar{A} \neq 0$, the sufficient stability condition from Theorem 4.7(ii) reduces to $\det Q_1(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) \neq 0$, where the matrix $Q_1(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x})$ is given by (4.40). We have

$$\det Q_1(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) = \begin{pmatrix} 1 & 0 & -\sqrt{2}/2 \\ 0 & 1 & -\sqrt{2}/2 \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix} = 2 \neq 0.$$

Thus $S(\cdot)$ is locally Lipschitz-like around $(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) \in \text{gph}S$.

Example 4.5 (A class of unstable problems) Let $\bar{A} \in H(n)$ be not positive definite, i.e., among the eigenvalue set $\{\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n\}$ of $\bar{A}$ there is an element $\bar{\lambda}_{i_0} \leq 0$. Since $\det(\bar{A} - \bar{\lambda}_{i_0} I) = 0$, there exists $\bar{x} \in I\!R^n$ with $\|\bar{x}\| = 1$ such that $(\bar{A} - \bar{\lambda}_{i_0} I)\bar{x} = 0$. Let $\bar{b} = -\bar{A}\bar{x}$ and let $\bar{\alpha} = 1$. We claim that $S(\cdot)$ is not locally Lipschitz-like around $(\bar{A}, \bar{b}, \bar{\alpha}, \bar{x}) \in \text{gph}S$. To see this, it suffices to check that (4.39) is violated. For $v' := \bar{x}$, we have $v' \neq 0$ and $\langle v', \bar{x} \rangle > 0$. Choose $\alpha' = \bar{\lambda}_{i_0} \leq 0$. The condition $\bar{A}v' - \frac{\alpha'}{\bar{\alpha}}\bar{x} = 0$ in (4.39) is equivalent to $(\bar{A} - \bar{\lambda}_{i_0} I)\bar{x} = 0$. Since the latter is guaranteed by the choice of $\bar{x}$, we conclude that (4.39) fails to holds. Our claim has been proved.

4.4 Conclusions

Solution stability of a class of linear generalized equations in finite dimensional Euclidean spaces is studied by means of generalized differentiation. Exact formulas for the Fréchet and Mordukhovich coderivatives of the normal cone mappings of perturbed Euclidean balls are obtained in Theorems 4.1, 4.2, 4.3, and 4.4. These exact formulas solve the open problems raised by Lee and Yen in [23]. In Theorems 4.6 and 4.7, necessary and sufficient conditions for the local Lipschitz-like property of the solution maps of such linear generalized equations are derived from the obtained coderivative formulas. Since the trust-region subproblems in nonlinear programming can be regarded as linear generalized equations, these conditions lead to new results on stability of the parametric trust-region subproblems. A series of useful examples have been provided to illustrate the solution stability criteria for this type of linear generalized equations.

General Conclusions

The main results of this dissertation include:

1. An exact formula for the Fréchet coderivative and some upper and lower estimates for the Mordukhovich coderivative of the normal cone mappings to linearly perturbed polyhedral convex sets in reflexive Banach spaces.
2. Upper estimates for the Fréchet and the limiting normal cone to the graphs of the normal cone mappings to nonlinearly perturbed polyhedral convex sets in finite dimensional spaces.
3. Exact formulas for the Fréchet and the Mordukhovich coderivatives of the normal cone mappings to perturbed Euclidean balls.
4. Conditions for the local Lipschitz-like property and local metric regularity of the solution maps of parametric affine variational inequalities under linear/nonlinear perturbations, and conditions for the local Lipschitz-like property of the solution maps of a class of linear generalized equations in finite dimensional spaces.

The problem of finding coderivative estimates for nonlinearly perturbed polyhedral normal cone mappings requires further investigations, although it has been studied by several authors.

The general problem of computing the Fréchet coderivative (resp., the Mordukhovich coderivative) of the mapping $(x,p) \mapsto \widehat{N}(x;\Theta(p))$ (resp., of the mapping $(x,p) \mapsto N(x;\Theta(p))$), where

$$\Theta(p) := \{x \in X | \ \Psi(x,p) \in K\}$$

with $\Psi : X \times P \to Y$ being a $\mathcal{C}^2$-smooth vector function which maps the product $X \times P$ of two Banach spaces into another Banach space Y, and $K \subset Y$ is a closed convex cone, is our next target. In this direction, we have obtained some preliminary results on computing coderivatives of the normal cone mappings to parametric sets with smooth boundaries.

List of Author's Related Papers

1. N. T. Qui, *Linearly perturbed polyhedral normal cone mappings and applications*, Nonlinear Anal., **74** (2011), pp. 1676–1689.
2. N. T. Qui, *New results on linearly perturbed polyhedral normal cone mappings*, J. Math. Anal. Appl., **381** (2011), pp. 352–364.
3. N. T. Qui, *Upper and lower estimates for a Fréchet normal cone*, Acta Math. Vietnam., **36** (2011), pp. 601–610.
4. N. T. Qui, *Nonlinear perturbations of polyhedral normal cone mappings and affine variational inequalities*, J. Optim. Theory Appl., **153** (2012), pp. 98–122.
5. N. T. Qui and N. D. Yen, *A class of linear generalized equations*, SIAM J. Optim., (2014). [Accepted for publication]

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