VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY
INSTITUTE OF MATHEMATICS

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SOME CONTRIBUTIONS TO THE THEORY OF GENERALIZED POLYHEDRAL OPTIMIZATION PROBLEMS

Speciality: Applied Mathematics

Speciality code: 9 46 01 12

SUMMARY

DOCTORAL DISSERTATION IN MATHEMATICS

HANOI - 2019

The dissertation was written on the basis of the author's research works carried at Institute of Mathematics, Vietnam Academy of Science and Technology.

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Second referee: ....
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Third referee: ....
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To be defended at the Jury of Institute of Mathematics, Vietnam Academy of Science and Technology:

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on ...., at ....o'clock ....

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Introduction

Vector optimization has a rich history and diverse applications. *Vector optimization* (sometimes called *multiobjective optimization*) is a natural development of *scalar optimization.* F.Y. Edgeworth (1881) and V. Pareto (1906) defined a notion, which later was called *Pareto solution.* This solution concept remains the most important in vector optimization. Other basic solution concepts of this theory are *weak Pareto solution* and *proper solution.* The latter has been defined in different ways by A.M. Geoffrion, J.M. Borwein, H.P. Benson, M.I. Henig, and other authors.

One calls a vector optimization problem (VOP) *linear* if the objective functions are linear (affine) functions and the constraint set is polyhedral convex (i.e., it is a intersection of a finite number of closed half-spaces). If at least one of the objective functions is *nonlinear* (non-affine, to be more precise) or the constraint set is not a polyhedral convex set (for example, it is merely a closed convex set or, more general, a solution set of a system of nonlinear inequalities), then the VOP is said to be *nonlinear.*

Linear VOPs have been considered in many books and in numerous papers. The classical Arrow-Barankin-Blackwell Theorem asserts that, for a linear vector optimization problem, the Pareto solution set and the weak Pareto solution set are connected by line segments and are the unions of finitely many faces of the constraint set. This is an example of qualitative properties of vector optimization problems.

This dissertation focuses on linear VOPs and several related nonlinear scalar optimization problems, as well as nonlinear vector optimization problems. Namely, apart from linear VOPs in locally convex Hausdorff topological vector spaces, which are the main subjects of our research, we will study polyhedral convex optimization problems and piecewise linear vector optimization problems. The fundamental concepts used in this dissertation are polyhedral convex set and polyhedral convex function on locally convex Hausdorff topological vector spaces. About one half of the dissertation is devoted to these concepts. Another half of the dissertation shows how our new results on polyhedral convex sets and polyhedral convex functions can be applied to scalar optimization problems and VOPs.

According to Bonnans and Shapiro (2000), a subset of a locally convex Hausdorff topological vector space is said to be a generalized polyhedral convex set, if it is the intersection of finitely many closed half-spaces and a closed affine subspace of that topological vector space. When the affine subspace can be chosen as the whole space, the generalized polyhedral convex set is called a polyhedral convex set

Many applications of polyhedral convex sets and piecewise linear functions in normed spaces to vector optimization can be found in the papers of Yang and Yen (2010), Zheng (2009), Zheng and Ng (2014), Zheng and Yang (2008).

Numerous applications of generalized polyhedral convex sets and generalized polyhedral multifunctions in Banach spaces to variational analysis, optimization problems, and variational inequalities can be found in the works by Henrion, Mordukhovich, and Nam (2010), Ban, Mordukhovich, and Song (2011), Gfrerer (2013, 2014), Ban and Song (2016).

The introduction of these concepts poses an interesting problem. Namely, since the entire Section 19 of the book *"Convex Analysis"* of Rockafellar (1970) is devoted to establishing a variety of basic properties of polyhedral convex sets and polyhedral convex functions which have numerous applications afterwards, one may ask whether a similar study can be done for generalized polyhedral convex sets and generalized polyhedral convex functions, or not.

The systematic study of generalized polyhedral convex sets and generalized polyhedral convex function in this dissertation can serve as a basis for further investigations on minimization of a generalized polyhedral convex function on a generalized polyhedral convex set – a *generalized polyhedral convex optimization problem*, which is a special infinite-dimensional convex programming problem.

Piecewise linear vector optimization problem (PLVOP) is a natural development of *polyhedral convex optimization*. The study of the structures and characteristic properties of these solution sets of PLVOPs is can be found in the papers of Zheng and Yang (2008), Yang and Yen (2010), Fang, Meng, and Yang (2012), Fang, Huang and Yang (2012), Fang, Meng and Yang (2015) Zheng and Ng (2014).

The dissertation has five chapters, a list of the related papers of the author, a section of general conclusions, and a list of references.

Chapter 1 gives a series of fundamental properties of generalized polyhedral convex sets.

In Chapter 2, we discuss some basic properties of generalized polyhedral convex functions.

Chapter 3 is devoted to several dual constructions including the concepts of conjugate function and subdifferential of a generalized polyhedral convex function.

Generalized polyhedral convex optimization problems in locally convex Hausdorff topological vector spaces are studied systematically in Chapter 4. We establish solution existence theorems, necessary and sufficient optimality conditions, weak and strong duality theorems. In particular, we show that the dual problem has the same structure as the primal problem, and the strong duality relation holds under three different sets of conditions.

Chapter 5 discusses structure of efficient solutions sets of linear vector optimization problems and piecewise linear vector optimization problems.

Chapter 1

Generalized Polyhedral Convex Sets

In this chapter, we first establish a representation formula for generalized convex polyhedra. A series of fundamental properties of generalized polyhedral convex sets are obtained in Sections 2-5. In Section 6, by using the representation formulas for generalized polyhedral convex sets we prove solution existence theorems in generalized linear programming.

The main theorems of Section 1 below (see Theorems 1.2 and 1.5), which can be considered as *geometrical descriptions* of generalized convex polyhedra and convex polyhedra, are not formal extensions of Theorem 19.1 from a book of Rockafellar (1970) and Corollary 2.1 of a paper of Zheng (2009). Recently, Yen and Yang (2018) have used Theorem 1.2 to study infinite-dimensional affine variational inequalities (AVIs) on normed spaces. It is shown that infinite-dimensional quadratic programming problems and infinite-dimensional linear fractional vector optimization problems can be studied by using AVIs. They have obtained two basic facts about infinite-dimensional AVIs: the Lagrange multiplier rule and the solution set decomposition.

1.1 Preliminaries

From now on, if not otherwise stated, X is a *locally convex Hausdorff topological vector space* over the reals. We denote by X^* the dual space of X and by $\langle x^*, x\rangle$ the value of $x^* \in X^*$ at $x \in X$.

For a subset $\Omega \subset X$ of a locally convex Hausdorff topological vector space, we denote its *interior* by $\operatorname{int}\Omega$, and its topological closure by $\overline{\Omega}$. The convex hull of a subset Ω is denoted by $\operatorname{conv}\Omega$.

One says that a nonempty subset $K \subset X$ is a cone if $tK \subset K$ for every $t > 0$. A cone $K \subset X$ is said to be a pointed cone if $\ell(K) = \{0\}$, where $\ell(K) := K \cap (-K)$. For a subset $\Omega \subset X$, by $\operatorname{cone}\Omega$ we denote the smallest convex cone containing Ω.

1.2 Representation Formulas for Generalized Convex Polyhedra

The following definition of generalized polyhedral convex set is due to Bonnans and Shapiro (2000).

Definition 1.1 (see Bonnans and Shapiro (2000)) A subset $D \subset X$ is said to be a *generalized polyhedral convex set*, or a *generalized convex polyhedron*, if there exist some $x_i^* \in X^*$, $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, p$, and a closed affine subspace $L \subset X$, such that

$$D = \Big\{x \in X \mid x \in L,\ \langle x_i^*, x\rangle \le \alpha_i,\ i = 1, \ldots, p\Big\}. \tag{1.1}$$

If D can be represented in the form (1.1) with $L = X$, then we say that it is a *polyhedral convex set*, or a *convex polyhedron*.

Let D be given as in (1.1). Then there exists a continuous surjective linear mapping A from X to a locally convex Hausdorff topological vector space Y and a vector $y \in Y$ such that $L = \Big\{x \in X \mid A(x) = y\Big\}$; then

$$D = \Big\{x \in X \mid A(x) = y,\ \langle x_i^*, x\rangle \le \alpha_i,\ i = 1, \ldots, p\Big\}. \tag{1.2}$$

Our further investigations are motivated by the following fundamental result about polyhedral convex sets in finite-dimensional topological vector spaces, which has origin in the works of Minkowski (1910) and Weyl (1934) (see also Klee (1959) and Rockafellar (1970)).

Theorem 1.1 (see Rockafellar (1970)) *For any nonempty convex set C in $\mathbb{R}^n$, the following properties are equivalent:*

(a) *C is a convex polyhedron;*

(b) *C is finitely generated, i.e., C can be represented as*

$$C = \Big\{ \sum_{i=1}^{k} \lambda_i u_i + \sum_{j=1}^{\ell} \mu_j v_j \mid \lambda_i \geq 0,\ \forall i = 1, \ldots, k, \\ \sum_{i=1}^{k} \lambda_i = 1,\ \ \mu_j \geq 0,\ \forall j = 1, \ldots, \ell \Big\}, \tag{1.3}$$

for some $u_i \in \mathbb{R}^n$, $i = 1, \ldots, k$, and $v_j \in \mathbb{R}^n$, $j = 1, \ldots, \ell$;

(c) *C is closed and it has only a finite number of faces.*

A natural question arises: *Is there any analogue of the representation* (1.3) *for convex polyhedra in locally convex Hausdorff topological vector spaces, or not?*

The following proposition extends a result of Zheng (2009), which was given in a normed spaces setting, to the case of convex polyhedra in locally convex Hausdorff topological vector spaces.

Proposition 1.1 *A nonempty subset $D \subset X$ is a convex polyhedron if only if there exist closed linear subspaces X_0, X_1 of X and a convex polyhedron $D_1 \subset X_1$ such that $X = X_0 + X_1$, $X_0 \cap X_1 = \{0\}$, $\dim X_1 < +\infty$, and $D = D_1 + X_0$.*

The main result of this section is formulated as follows.

Theorem 1.2 *A nonempty subset $D \subset X$ is a generalized convex polyhedron if and only if there exist $u_1, \ldots, u_k \in X$, $v_1, \ldots, v_\ell \in X$, and a closed linear subspace $X_0 \subset X$ such that*

$$D = \Big\{ \sum_{i=1}^{k} \lambda_i u_i + \sum_{j=1}^{\ell} \mu_j v_j \mid \lambda_i \geq 0,\ \forall i = 1, \ldots, k, \\ \sum_{i=1}^{k} \lambda_i = 1,\ \ \mu_j \geq 0,\ \forall j = 1, \ldots, \ell \Big\} + X_0. \tag{1.4}$$

Combining Theorem 1.2 with Proposition 1.1, we get a representation formula for convex polyhedra.

Theorem 1.3 *A nonempty subset $D \subset X$ is a convex polyhedron if and only if there exist $u_1, \ldots, u_k \in X$, $v_1, \ldots, v_\ell \in X$, and a closed linear subspace $X_0 \subset X$ of finite codimension such that* (1.4) *is valid.*

Some illustrative examples for Theorem 1.3 are given in the dissertation.

From Theorem 1.2 we can obtain a representation formula for generalized polyhedral convex cones.

Theorem 1.4 *A nonempty set $K \subset X$ is a generalized polyhedral convex cone if and only if there exist $v_j \in K, j = 1, \ldots, \ell$, and a closed linear subspace X_0 such that*

$$K = \left\{ \sum_{j=1}^{\ell} \mu_j v_j \mid \mu_j \geq 0,\ \forall j = 1, \ldots, \ell \right\} + X_0. \tag{1.5}$$

Combining Theorem 1.3 with Theorem 1.4, we obtain a representation formula for polyhedral convex cones.

Theorem 1.5 *A nonempty set $K \subset X$ is a polyhedral convex cone if and only if there exist $v_j \in K, j = 1, \ldots, \ell$, and a closed linear subspace $X_0 \subset X$ of finite codimension such that* (1.5) *is valid.*

1.3 Characterizations via the Finiteness of the Faces

Definition 1.2 (see Bonnans and Shapiro (2000)) The *relative interior* $\operatorname{ri} C$ of a convex subset $C \subset X$ is the interior of C in the induced topology of the closed affine hull $\overline{\operatorname{aff}}\, C$ of C.

If $C \subset X$ is a nonempty generalized polyhedral convex set, then $\operatorname{ri} C \neq \emptyset$ (see Bonnans and Shapiro (2000)). The latter fact shows that generalized polyhedral convex sets have a nice topological structure.

Definition 1.3 (see Rockafellar (1970)) A convex subset F of a convex set $C \subset X$ is said to be a *face* of C if for every x^1, x^2 in C satisfying $(1-\lambda)x^1 + \lambda x^2 \in F$ with $\lambda \in (0,1)$ one has $x^1 \in F$ and $x^2 \in F$.

Definition 1.4 (see Rockafellar (1970)) A convex subset F of a convex set $C \subset X$ is said to be an *exposed face* of C if there exists $x^* \in X^*$ such that $F = \big\{u \in C \mid \langle x^*, u\rangle = \inf_{x \in C} \langle x^*, x\rangle \big\}$.

In the spirit of Theorem 1.1, for a nonempty convex subset $D \subset X$, we are interested in establishment of relations between the following properties:

(a) *D is a generalized polyhedral convex set*;

(b) *D is closed and has only a finite number of faces.*

The next theorems shows that a generalized polyhedral convex set can be characterized via the finiteness of the number of its faces.

Theorem 1.6 *Every generalized polyhedral convex set has a finite number of faces and all the nonempty faces are exposed.*

Theorem 1.7 *Let $D \subset X$ be a closed convex set with nonempty relative interior. If D has finitely many faces, then D is a generalized polyhedral convex set.*

1.4 Images via Linear Mappings and Sums of Generalized Polyhedral Convex Sets

Let us consider the following question: *Given locally convex Hausdorff topological vector spaces X and Y, whether the image of a generalized polyhedral convex set via a linear mapping from X to Y is a generalized polyhedral convex set, or not?* The answers in the affirmative for the case where X and Y are finite-dimensional (see Rockafellar (1970)), for the case where X is a Banach space and Y is finite-dimensional (see Zheng and Yang (2008)).

The following proposition extends a lemma from the paper of Zheng and Yang (2008), which was given in a normed space setting, to the case of convex polyhedra in locally convex Hausdorff topological vector spaces.

Proposition 1.2 *If $T : X \to Y$ is a linear mapping between locally convex Hausdorff topological vector spaces with Y being a space of finite dimension and if $D \subset X$ is a generalized polyhedral convex set, then $T(D)$ is a convex polyhedron of Y.*

One may wonder: *Whether the assumption on the finite dimensionality of Y can be removed from Proposition 1.2, or not?* In the dissertation, some examples have been given to show that if Y is a infinite-dimensional space then $T(D)$ may not be a generalized polyhedral convex set.

Proposition 1.3 *Suppose that $T : X \to Y$ is a linear mapping between locally convex Hausdorff topological vector spaces and $D \subset X$, $Q \subset Y$ are nonempty generalized polyhedral convex sets. Then, $\overline{T(D)}$ is a generalized polyhedral convex set. If T is continuous, then $T^{-1}(Q)$ is a generalized polyhedral convex set.*

Proposition 1.4 *If $D_1, \ldots, D_m$ are nonempty generalized polyhedral convex sets in X, so is $\overline{D_1 + \cdots + D_m}$.*

One may ask: *Whether the statement of Corollary 1.4 is valid also for the sum of the sets D_i, $i = 1, \ldots, m$, without the closure operation.* When X is a finite-dimensional space, the sum of finitely many polyhedral convex sets in X is a polyhedral convex set (see Klee (1959)). However, when X is an infinite-dimensional space, the sum of a finite number of generalized polyhedral convex sets may be not a generalized polyhedral convex set. Concerning this question, in the two following propositions we shall describe some situations where the closure sign can be dropped.

Proposition 1.5 *If D_1, D_2 are generalized polyhedral convex sets of X and $\operatorname{aff} D_1$ is finite-dimensional, then $D_1 + D_2$ is a generalized polyhedral convex set.*

Proposition 1.6 *If $D_1 \subset X$ is a polyhedral convex set and $D_2 \subset X$ is a generalized polyhedral convex set, then $D_1 + D_2$ is a polyhedral convex set.*

The next result is an extension of a result from Rockafellar (1970) to an infinite-dimensional setting.

Corollary 1.1 *Suppose that $D_1 \subset X$ is a polyhedral convex set and $D_2 \subset X$ is a generalized polyhedral convex set. If $D_1 \cap D_2 = \emptyset$, then there exists $x^* \in X^*$ such that*

$$\sup\{\langle x^*, u\rangle \mid u \in D_1\} < \inf\{\langle x^*, v\rangle \mid v \in D_2\}.$$

1.5 Convex Hulls and Conic Hulls

As in Rockafellar (1970), the *recession cone* 0^+C of a convex set $C \subset X$ is given by

$$0^+C = \Big\{v \in X \mid x + tv \in C,\ \forall x \in C,\ \forall t \geq 0\Big\}.$$

Theorem 1.8 *Suppose that $D_1, \ldots, D_m$ are generalized polyhedral convex sets in X. Let D be the smallest closed convex subset of X that contains D_i for all $i = 1, \ldots, m$. Then D is a generalized polyhedral convex set. If at least one of the sets $D_1, \ldots, D_m$ is polyhedral convex, then D is a polyhedral convex set.*

From Theorem 1.8 we obtain the following corollary.

Corollary 1.2 *If a convex subset $D \subset X$ is the union of a finite number of generalized polyhedral convex sets (resp., of polyhedral convex sets) in X, then D is a generalized polyhedral convex (resp., polyhedral convex) set.*

It turns out that the closure of the cone generated by a generalized polyhedral convex set is a generalized polyhedral convex cone. Hence, next proposition extends a theorem from Rockafellar (1970) to a locally convex Hausdorff topological vector spaces setting.

Proposition 1.7 *If a nonempty subset $D \subset X$ is generalized polyhedral convex, then $\overline{\text{cone}\, D}$ is a generalized polyhedral convex cone. In addition, if $0 \in D$ then $\text{cone}\, D$ is a generalized polyhedral convex cone; hence $\text{cone}\, D$ is closed.*

An analogue of Proposition 1.7 for polyhedral convex sets can be formulated as follows.

Proposition 1.8 *If a nonempty subset $D \subset X$ is polyhedral convex, then $\overline{\text{cone}\, D}$ is a polyhedral convex cone. In addition, if $0 \in D$ then $\text{cone}\, D$ is a polyhedral convex cone; hence $\text{cone}\, D$ is closed.*

In convex analysis, to every convex set and a point belonging to it, one associates a tangent cone. The forthcoming proposition shows that the tangent cone to a generalized polyhedral convex set at a given point is a generalized polyhedral convex cone. By definition, the (Bouligand-Severi) *tangent cone* $T_D(x)$ to a closed subset $D \subset X$ at $x \in D$ is the set of all $v \in X$ such that there exist sequences $t_k \to 0^+$ and $v_k \to v$ such that $x + t_k v_k \in D$ for every k. If D is convex, then $T_D(x) = \overline{\text{cone}\,(D - x)}$.

Proposition 1.9 *If $D \subset X$ is a generalized polyhedral convex set (resp., a polyhedral convex set) and if $x \in D$, then $T_D(x)$ is a generalized polyhedral convex cone (resp., a polyhedral convex cone) and one has $T_D(x) = \text{cone}\,(D - x)$.*

1.6 Relative Interiors of Polyhedral Convex Cones

In this section, we obtain a formula for the relative interiors of a generalized polyhedral convex cone and the dual cone of a polyhedral convex cone.

Theorem 1.9 *Suppose that $C \subset X$ is a generalized polyhedral convex cone in a locally convex Hausdorff topological vector space. If* $C = \left\{ \sum_{i=1}^{p} \lambda_i u_i \mid \lambda_i \geq 0,\ i = 1, \ldots, p \right\}$, *where $u_i \in X$ for $i = 1, \ldots, p$, then* $\operatorname{ri} C = \left\{ \sum_{i=1}^{p} \lambda_i u_i \mid \lambda_i > 0,\ i = 1, \ldots, p \right\}$.

Let Y be a locally convex Hausdorff topological vector space. Suppose that $K \subset Y$ is a polyhedral convex cone defined by

$$K = \left\{ y \in Y \mid \langle y_j^*, y \rangle \leq 0,\ j = 1, \ldots, q \right\},$$

where $y_j^* \in Y^* \setminus \{0\}$ for all $j = 1, \ldots, q$. The positive dual cone of a cone $K \subset Y$ is given by $K^* := \left\{ y^* \in Y^* \mid \langle y^*, y \rangle \geq 0 \ \ \forall y \in K \right\}$. By using the set $K \setminus \ell(K)$, we can be describe the relative interior of the dual cone K^* as follows.

Theorem 1.10 *If K is not a linear subspace of Y, then a vector $y^* \in Y^*$ belongs to* $\operatorname{ri} K^*$ *if and only if $\langle y^*, y \rangle > 0$ for all $y \in K \setminus \ell(K)$.*

1.7 Solution Existence in Linear Optimization

Our aim in this section is to apply the representation formula for generalized polyhedral convex sets to proving solution existence theorems for generalized linear programming problems.

Consider a *generalized linear programming problem*

$$\text{(LP)} \qquad \min \{ \langle x^*, x \rangle \mid x \in D \}$$

where, as before, X is a locally convex Hausdorff topological vector space, $D \subset X$ is a generalized polyhedral convex set, $x^* \in X^*$.

The following two existence theorems for (LP) are known results. Actually, in combination, they express the contents of a theorem of Bonnans and Shapiro (2000). Our simple proofs show how Theorem 1.2 can be used to study the solution existence of generalized linear programs.

Theorem 1.11 (The Eaves-type existence theorem; see Bonnans and Shapiro (2000)) *If D is nonempty, then* (LP) *has a solution if and only if $\langle x^*, v \rangle \geq 0$ for every $v \in 0^+D$.*

Theorem 1.12 (The Frank–Wolfe-type existence theorem; see Bonnans and Shapiro (2000)) *If D is nonempty, then* (LP) *has a solution if and only if there is a real number γ such that $\langle x^*, x \rangle \geq \gamma$ for every $x \in D$.*

We are interested in studying the region G of all x^* for which (LP) has a nonempty solution set, assuming that the constraint set D is nonempty and fixed.

Proposition 1.10 *If D has the form* (1.4)*, then G is a generalized polyhedral convex cone of X^* which has the representation* $G = X_0^{\perp} \cap \{ x^* \in X^* \mid \langle x^*, v_j \rangle \geq 0,\ j = 1, \ldots, \ell \}$.

Next, for each $x^* \in G$, we want to describe the solution set of (LP), which is denoted by $S(x^*)$. For doing so, let us suppose that D is given by (1.4) and consider the index sets

$$I(x^*) := \{ i_0 \in \{1, \ldots, k\} \mid \langle x^*, u_{i_0} \rangle \leq \langle x^*, u_i \rangle \ \ \forall i = 1, \ldots, k \},$$

and $J(x^*) := \{ j_0 \in \{1, \ldots, \ell\} \mid \langle x^*, v_{j_0} \rangle = 0 \}$.

Proposition 1.11 *If $x^* \in G$ and D is given by (1.4), then*

$$S(x^*) = \Big\{ \sum_{i\in I(x^*)} \lambda_i u_i + \sum_{j\in J(x^*)} \mu_j v_j \mid \ \lambda_i \geq 0 \ \ \forall i \in I(x^*), \sum_{i\in I(x^*)} \lambda_i = 1, \ \mu_j \geq 0 \ \ \forall j \in J(x^*) \Big\} + X_0.$$

In particular, $S(x^*)$ is a generalized polyhedral convex set.

1.8 Conclusions

We have studied basic properties of generalized polyhedral convex sets in locally convex Hausdorff topological vector spaces. Adopting an approach very different from that of Zheng, we have obtained a representation formula for generalized polyhedral convex sets in locally convex Hausdorff topological vector spaces, which is a comprehensive infinite-dimensional analogue of the celebrated theorem of Minkowski and Weyl. In this chapter, the formula has been used for proving solution existence theorems in generalized linear programming. We have shown that a generalized polyhedral convex set can be characterized via the finiteness of the number of its faces. Our results can be considered as adequate extensions of the corresponding classical results on polyhedral convex sets in Rockafellar (1970).

Chapter 2

Generalized Polyhedral Convex Functions

As the title indicates, the present chapter deals with the concept of generalized polyhedral convex functions. The latter is based on the notion of generalized polyhedral convex set, which has been considered in details in Chapter 1.

2.1 Generalized Polyhedral Convex Function as a Maximum of Finitely Many Affine Functions

Let X be a locally convex Hausdorff topological vector space and f a function from X to $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. The *effective domain* and the *epigraph* of f are defined, respectively, by setting $\operatorname{dom} f = \{x \in X \mid f(x) < +\infty\}$ and $\operatorname{epi} f = \big\{(x, \alpha) \in X \times \mathbb{R} \mid x \in \operatorname{dom} f, \ f(x) \leq \alpha\big\}$. If $\operatorname{dom} f$ is nonempty and $f(x) > -\infty$ for all $x \in X$, then f is said to be *proper*. We say that f is *convex* if $\operatorname{epi} f$ is a convex set in $X \times \mathbb{R}$.

According to Rockafellar (1970), a real-valued function defined on $\mathbb{R}^n$ is called polyhedral convex if its epigraph is a polyhedral convex set in $\mathbb{R}^{n+1}$. The following notion of generalized polyhedral convex function appears naturally in that spirit.

Definition 2.1 Let X be a locally convex Hausdorff topological vector space. A function $f : X \to \bar{\mathbb{R}}$ is called *generalized polyhedral convex* (resp., *polyhedral convex*) if its epigraph is a generalized polyhedral convex set (resp., a polyhedral convex set) in $X \times \mathbb{R}$. If $-f$ is a generalized polyhedral convex function (resp., a polyhedral convex function), then f is said to be a *generalized polyhedral concave function* (resp., a *polyhedral concave function*).

Complete characterizations of a generalized polyhedral convex function (resp., a polyhedral convex function) in the form of the maximum of a finite family of continuous affine functions over a certain generalized polyhedral convex set (resp., a polyhedral convex set) are given in next theorem.

Theorem 2.1 *Suppose that $f : X \to \bar{\mathbb{R}}$ is a proper function. Then f is generalized polyhedral convex (resp., polyhedral convex) if and only if $\operatorname{dom} f$ is a generalized polyhedral convex set (resp., a polyhedral convex set) in X and there exist $v_k^* \in X^*$, $\beta_k \in \mathbb{R}$, for $k = 1, \ldots, m$, such that*

$$f(x) = \begin{cases} \max\left\{\langle v_k^*, x\rangle + \beta_k \mid k = 1, \ldots, m\right\} & \text{if } x \in \operatorname{dom} f, \\ +\infty & \text{if } x \notin \operatorname{dom} f. \end{cases} \tag{2.1}$$

2.2 Piecewise Linearity of Generalized Polyhedral Convex Functions and an Application

We will need the following infinite-dimensional generalization of the concept of piecewise linear function on $\mathbb{R}^n$ of Rockafellar and Wets (1998).

Definition 2.2 A proper function $f : X \to \bar{\mathbb{R}}$, which is defined on a locally convex Hausdorff topological vector space, is said to be *generalized piecewise linear* (resp., *piecewise linear*) if there exist generalized polyhedral convex sets (resp., polyhedral convex sets) $D_1, \ldots, D_m$ in X, $v_1^*, \ldots, v_m^* \in X^*$, and $\beta_1, \ldots, \beta_m \in \mathbb{R}$ such that $\operatorname{dom} f = \bigcup_{k=1}^{m} D_k$ and $f(x) = \langle v_k^*, x\rangle + \beta_k$ for all $x \in D_k$, $k = 1, \ldots, m$.

Theorem 2.1 provides us with a general formula for any generalized polyhedral convex function on a locally convex Hausdorff topological vector space. For polyhedral convex functions on $\mathbb{R}^n$, there is another important characterization: *A proper convex function f is polyhedral convex if and only if f is piecewise linear* (see Rockafellar and Wets (1998)). It is of interest to obtain analogous results for generalized polyhedral convex functions and polyhedral convex functions on a locally convex Hausdorff topological vector space.

The forthcoming theorem clarifies the relationships between generalized polyhedral convex functions and generalized piecewise linear functions.

Theorem 2.2 *Suppose that $f : X \to \bar{\mathbb{R}}$ is a proper convex function. Then the function f is generalized polyhedral convex (resp., polyhedral convex) if and only if f is generalized piecewise linear (resp., piecewise linear).*

Based on Theorem 2.2, we can prove that the class of generalized polyhedral convex functions (resp., the class of polyhedral convex functions) is invariant with respect to the addition of functions.

Theorem 2.3 *Let f_1, f_2 be two proper functions on X. If f_1, f_2 are generalized polyhedral convex (resp., polyhedral convex) and* $(\mathrm{dom}\, f_1) \cap (\mathrm{dom}\, f_2)$ *is nonempty, then $f_1 + f_2$ is a proper generalized polyhedral convex function (resp., a polyhedral convex function).*

2.3 Directional Derivatives

In convex analysis, it is well known that the concept of directional derivative has an important role. We are going to discuss a property of the directional derivative mapping of a generalized polyhedral convex function (resp., a polyhedral convex function) at a given point.

If $f : X \to \bar{\mathbb{R}}$ is a proper convex function and $x \in X$ is such that $f(x)$ is finite, the *directional derivative* $f'(x; h) := \lim_{t \to 0^+} \dfrac{f(x + th) - f(x)}{t}$ of f at x with respect to a direction $h \in X$, always exists (it can take values $-\infty$ or $+\infty$). Moreover, the closure of the epigraph of $f'(x; \cdot)$ coincides with the tangent cone to $\mathrm{epi}\, f$ at $(x, f(x))$, i.e.,

$$\overline{\mathrm{epi}\, f'(x; \cdot)} = T_{\mathrm{epi}\, f}(x, f(x)). \tag{2.2}$$

We know that if $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is proper polyhedral convex, then the closure sign in (2.2) can be omitted and $f'(x; \cdot)$ is a proper polyhedral convex function. The last two facts can be extended to polyhedral convex functions on locally convex Hausdorff topological vector spaces and generalized polyhedral convex functions as follows.

Theorem 2.4 *Let f be a proper generalized polyhedral convex function (resp., a proper polyhedral convex function) on a locally convex Hausdorff topological vector space X. For any $x \in \mathrm{dom}\, f$, $f'(x; \cdot)$ is a proper generalized polyhedral convex function (resp., a proper polyhedral convex function). In particular,* $\mathrm{epi}\, f'(x; \cdot)$ *is closed and, by* (2.2) *one has*

$$\mathrm{epi}\, f'(x; \cdot) = T_{\mathrm{epi}\, f}(x, f(x)).$$

2.4 Infimal Convolutions

In this section, we are interested in the concept of infimal convolution function, which was introduced by Fenchel (1953). According to Rockafellar (1970), the infimal convolution operation is analogous to the classical formula for integral convolution and, in a sense, is dual to the operation of addition of convex functions.

Although the infimal convolution of a finite family of functions can be defined (see Ioffe and Tihomirov (1979)), for simplicity, we will only consider the infimal convolution of two functions. By induction, one can easily extends the result obtained in Proposition 2.1 below to infimal convolutions of finite families of generalized polyhedral convex functions, provided that one of them is polyhedral convex.

Definition 2.3 (see Ioffe and Tihomirov (1979)) Let f_1, f_2 be two proper functions on a locally convex Hausdorff topological vector space X. The *infimal convolution* of f_1, f_2 is the function defined by

$$(f_1 \Box f_2)(x) := \inf \{f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x\}.$$

Proposition 2.1 *Let f_1, f_2 be two proper functions. If f_1 is polyhedral convex and f_2 is generalized polyhedral convex, then $f_1 \Box f_2$ is a polyhedral convex function.*

2.5 Conclusions

We have studied basic properties of generalized polyhedral convex functions on locally convex Hausdorff topological vector spaces, and the related constructions such as sum of functions, directional derivative, infimal convolution. It is also proved that the infimal convolution of a generalized polyhedral convex function and a polyhedral convex function is a polyhedral convex function. Our results can be considered as adequate extensions of the corresponding classical results on polyhedral convex functions in Rockafellar (1970).

Chapter 3

Dual Constructions

Various properties of normal cones to and polars of generalized polyhedral convex sets, conjugates of generalized polyhedral convex functions, and subdifferentials of generalized polyhedral convex functions are studied in this chapter

3.1 Normal Cones

As before, X is a locally convex Hausdorff topological vector space and X^* is the dual space of X. According to Rudin (1991), the weak*–topology turns X^* into a locally convex Hausdorff topological vector space whose dual space is X.

Now, suppose that $C \subset X$ is a nonempty convex set. The *normal cone* to C at $x \in C$ is the set $N_C(x) := \big\{x^* \in X^* \mid \langle x^*, u - x\rangle \le 0, \forall u \in C\big\}$. The formula

$$C^{\perp} := \{x^* \in X^* \mid \langle x^*, u\rangle = 0, \ \forall u \in C\}$$

defines the *annihilator* of C.

In this chapter, if not otherwise stated, $D \subset X$ is a nonempty generalized polyhedral convex set given by (1.2). Let $I = \{1, \ldots, p\}$. For every $x \in D$, we define the active index set $I(x) := \big\{i \in I \mid \langle x_i^*, x\rangle = \alpha_i\big\}$. If D is a polyhedral convex set, then one can choose $Y = \{0\}$, $A \equiv 0$, and $y = 0$.

Normal cones to a generalized polyhedral convex set also share the polyhedral structure.

Theorem 3.1 *If $D \subset X$ is a generalized polyhedral convex set and if $x \in D$, then $N_D(x)$ is a generalized polyhedral convex cone.*

During the course of the proof of Theorem 3.1, we have obtained the following result.

Proposition 3.1 *Suppose that $D \subset X$ is a generalized polyhedral convex set given by* (1.2). *Then, for every $x \in D$, one has $N_D(x) = \mathrm{cone}\,\{x_i^* \mid i \in I(x)\} + (\ker A)^{\perp}$.*

In connection with Theorem 3.1, one may ask: *Given a polyhedral convex set $D \subset X$ and $x \in D$, whether $N_D(x)$ is a polyhedral convex set, or not?* An answer for that question is given in next statement.

Proposition 3.2 *Suppose that $D \subset X$ is a polyhedral convex set and $x \in D$. Then, $N_D(x)$ is a polyhedral convex cone in X^* if and only if X is finite-dimensional.*

One has the following analogue of Proposition 3.1 for polyhedral convex sets.

Proposition 3.3 *Let $D \subset X$ be a polyhedral convex set of the form* (1.2), *where $Y = \{0\}$, $A \equiv 0$, and $y = 0$. Then, for every $x \in D$, one has $N_D(x) = \mathrm{cone}\,\{x_i^* \mid i \in I(x)\}$.*

Theorem 3.2 *Let D_1 and D_2 be two generalized polyhedral convex sets of X. For every $x \in D_1 \cap D_2$, one has $N_{D_1 \cap D_2}(x) = \overline{N_{D_1}(x) + N_{D_2}(x)}$.*

Theorem 3.3 *Suppose that $D_1 \subset X$ is a polyhedral convex set and $D_2 \subset X$ is a generalized polyhedral convex set. Then, For every $x \in D_1 \cap D_2$, one has $N_{D_1 \cap D_2}(x) = N_{D_1}(x) + N_{D_2}(x)$.*

3.2 Polars

Following Robertson (1964), we define the *polar* of a nonempty set C by

$$C^o := \Big\{x^* \in X^* \mid \langle x^*, x\rangle \le 1,\ \forall x \in C\Big\}.$$

The forthcoming proposition extends a result from Rockafellar (1970) to a locally convex Hausdorff topological vector spaces setting.

Proposition 3.4 *The polar of a nonempty generalized polyhedral convex set is a generalized polyhedral convex set.*

3.3 Conjugate Functions

According to Ioffe and Tihomirov (1979), the *conjugate function* (or the *Young-Fenchel transform function*) of a function $f : X \to \bar{\mathbb{R}}$ is the function $f^* : X^* \to \bar{\mathbb{R}}$ given by

$$f^*(x^*) = \sup \Big\{\langle x^*, x\rangle - f(x) \mid x \in X\Big\}.$$

Theorem 3.4 *The conjugate function of a proper generalized polyhedral convex function is a proper generalized polyhedral convex function.*

3.4 Subdifferentials

In this section, we will study subdifferentials of generalized polyhedral convex functions. It is well known that the subdifferential of a convex function is the basis for optimality conditions and other issues in convex programming. A linear functional $x^* \in X^*$ is said to be a *subgradient* of a proper convex function f at $x \in \mathrm{dom}\, f$ if

$$\langle x^*, u - x\rangle \le f(u) - f(x) \quad (u \in X).$$

The *subdifferential* of f at x, denoted by $\partial f(x)$, is the set of all the subgradients of f at x. If C is a nonempty convex subset of X then, for any $x \in C$, one has $\partial\delta(x, C) = N_C(x)$, where $\delta(\cdot, C)$ is the indicator function of C.

The next theorem provides us with a formula for the subdifferential of a generalized polyhedral convex function.

Theorem 3.5 *Suppose that f is a proper generalized polyhedral convex function with*

$$\mathrm{dom}\, f = \{x \in X \mid A(x) = y,\ \langle x_i^*, x\rangle \le \alpha_i,\ i = 1, \ldots, p\}$$

and $f(x) = \max\big\{\langle v_j^*, x\rangle + \beta_j \mid j = 1, \ldots, m\big\}$ *for all* $x \in \mathrm{dom}\, f$), *where A is a continuous linear mapping from the space X to a locally convex Hausdorff topological vector space Y, $y \in Y$, $x_i^* \in X^*, \alpha_i \in \mathbb{R}$, $i = 1, \ldots, p$, $v_j^* \in X^*, \beta_j \in \mathbb{R}$, $j = 1, \ldots, m$. Then, for every $x \in \mathrm{dom}\, f$,*

$$\partial f(x) = \mathrm{conv}\,\{v_j^* \mid j \in J(x)\} + \mathrm{cone}\,\{x_i^* \mid i \in I(x)\} + (\ker A)^{\perp},$$

where $I(x) = \big\{i \in \{1, \ldots, p\} \mid \langle x_i^*, x\rangle = \alpha_i\big\}$ *and*

$$J(x) = \big\{j \in \{1, \ldots, m\} \mid \langle v_j^*, x\rangle + \beta_j = f(x)\big\}.$$

In particular, if $Y = \{0\}$, $A \equiv 0$ and $y = 0$ (the case where $\mathrm{dom}\, f$ is a polyhedral convex set) then, for any $x \in \mathrm{dom}\, f$,

$$\partial f(x) = \mathrm{conv}\,\{v_j^* \mid j \in J(x)\} + \mathrm{cone}\,\{x_i^* \mid i \in I(x)\}.$$

By the definition of subdifferential, if $f_1, \ldots, f_m$ are proper convex functions on X then, for every $x \in \bigcap_{i=1}^{m} \mathrm{dom}\, f_i$,

$$\partial f_1(x) + \cdots + \partial f_m(x) \subset \partial(f_1 + \cdots + f_m)(x). \tag{3.1}$$

The Moreau–Rockafellar theorem tells us that (3.1) holds with equality if there exists $x_0 \in \bigcap_{i=1}^{m} \mathrm{dom}\, f_i$ such that all the functions $f_1, \ldots, f_m$ except, possibly, one are continuous at x_0. The specific structure of generalized polyhedral convex functions allows one to have a subdifferential sum rule without the continuity assumption.

Theorem 3.6 *Let $f_1, \ldots, f_m$ be proper generalized polyhedral convex functions. Then, for any $x \in \bigcap_{i=1}^{m} \mathrm{dom}\, f_i$,*

$$\partial(f_1 + f_2 + \cdots + f_m)(x) = \overline{\partial f_1(x) + \partial f_2(x) + \cdots + \partial f_m(x)}.$$

Theorem 3.7 *Suppose that f_1 is a proper polyhedral convex function and f_2 is a proper generalized polyhedral convex function. Then, for any $x \in (\mathrm{dom}\, f_1) \cap (\mathrm{dom}\, f_2)$,*

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x).$$

3.5 Conclusions

We have studied several dual constructions including the concepts normal cone, conjugate function, and subdifferential. Among other things, we obtain a formula to compute the normal cone of generalized polyhedral convex sets, the subdifferential of a generalized polyhedral convex function at a point. Moreover, we have shown that the specific structure of generalized polyhedral convex functions allows one to have a subdifferential sum rule without any assumption on continuity.

Chapter 4

Generalized Polyhedral Convex Optimization Problems

This chapter is devoted to studying generalized polyhedral convex optimization problems.

4.1 Motivations

Let X be a locally convex Hausdorff topological vector space, $\varphi : X \to \bar{\mathbb{R}}$ a proper convex function, and $C \subset X$ a nonempty convex set. For each index $i \in \{1, \ldots, m\}$, where m is a positive integer, select some $x_i \in \operatorname{dom} \varphi$ and suppose that there exists some $x_i^* \in \partial\varphi(x_i)$. Then we have

$$\varphi(x) \geq \varphi(x_i) + \langle x_i^*, x - x_i \rangle, \quad \forall x \in X.$$

So,

$$\varphi(x) \geq \psi(x) := \max\{\varphi(x_i) + \langle x_i^*, x - x_i \rangle \mid i = 1, \ldots, m\}, \quad \forall x \in X. \tag{4.1}$$

Therefore, the polyhedral convex function ψ defined in (4.1) is a *lower piecewise convex approximation* of φ. Recall that the *relative interior* $\operatorname{ri} C$ of C is the interior of C in the induced topology of the closed affine hull $\overline{\operatorname{aff}}\, C$ of C. Select some points $u_1, \ldots, u_k$ in the boundary of C in the induced topology of $\overline{\operatorname{aff}}\, C$. Suppose that $\operatorname{ri} C$ is nonempty. Then, one can find $u_j^* \in N_C(u_j) \setminus \{0\}$ for $j = 1, \ldots, k$. Since

$$\langle u_j^*, x - u_j \rangle \leq 0, \quad \forall j = 1, \ldots, k, \ \forall x \in C,$$

one has

$$C \subset \widetilde{C} := \{x \in X \mid x \in \overline{\operatorname{aff}}\, C, \ \langle u_j^*, x \rangle \leq \langle u_j^*, u_j \rangle, \ \ j = 1, \ldots, k\}. \tag{4.2}$$

In other words, *the generalized polyhedral convex set* $\widetilde{C}$ *defined in* (4.2) *is an outer approximation of* C. This outer approximation of C and the above construction of a lower convex approximation of φ allow us to consider the *generalized polyhedral convex optimization problem*

$$\min \{\psi(x) \mid x \in \widetilde{C}\} \tag{4.3}$$

an approximation of the convex optimization problem

$$\text{(P)} \qquad \min\{\varphi(x) \mid x \in C\}.$$

Since the optimal value of (4.3) is smaller than that of (P), it can serve as a lower bound for the latter. By increasing m and k, one attains tighter approximations of (P) in the form (4.3). Therefore, in this sense, *one can approximate any convex optimization problem by a generalized polyhedral convex optimization problem.* Linearization techniques in optimization theory are the subjects of many books and research papers (see Pshenichnyj 1994, Bertsekas and Yu).

4.2 Solution Existence Theorems

Let $D \subset X$ be a nonempty generalized polyhedral convex set of the form (1.2). Set $I = \{1, \ldots, p\}$ and $I(x) = \left\{i \in I \mid \langle x_i^*, x\rangle = \alpha_i\right\}$ for $x \in D$. If D is a polyhedral convex set, then one can choose $Y = \{0\}$, $A \equiv 0$, and $y = 0$.

Consider a *generalized polyhedral convex optimization problem*

$$(\mathcal{P}) \qquad \min\{f(x) \mid x \in D\}$$

where, as before, X is a locally convex Hausdorff topological vector space, $D \subset X$ a nonempty generalized polyhedral convex set, and $f : X \to \bar{\mathbb{R}}$ a proper generalized polyhedral convex function. We say that $u \in D$ is a solution of $(\mathcal{P})$ if $f(u)$ is finite and $f(u) \le f(x)$ for all $x \in D$. The solution set of $(\mathcal{P})$ is denoted by $\mathrm{Sol}(\mathcal{P})$.

From now on, if not otherwise stated, the constraint set D is given by (1.2)*, and the objective function f is defined by* (2.1)*.*

Since $\mathrm{dom}\, f$ is a generalized polyhedral convex set, it admits the representation

$$\mathrm{dom}\, f = \left\{x \in X \mid B(x) = z,\ \langle u_j^*, x\rangle \le \gamma_j,\ j = 1, \ldots, q\right\}, \tag{4.4}$$

where B is a continuous linear mapping from X to a locally convex Hausdorff topological vector space Z, $z \in Z$, $u_j^* \in X^*$, $\gamma_j \in \mathbb{R}$, $j = 1, \ldots, q$. Set $J = \{1, \ldots, q\}$. For each $x \in \mathrm{dom}\, f$, let $J(x) = \left\{j \in J \mid \langle u_j^*, x\rangle = \gamma_j\right\}$ and $\Theta(x) = \left\{k \in \{1, \ldots, m\} \mid \langle v_k^*, x\rangle + \beta_k = f(x)\right\}$. If f is a polyhedral convex function, then $\mathrm{dom}\, f$ is polyhedral convex by Theorem 2.1; hence, we can choose $Z = \{0\}$, $B \equiv 0$, and $z = 0$.

Following Rockafellar (1970), we define the *recession function* $f0^+$ of a proper convex function $f : X \to \bar{\mathbb{R}}$ by the formula

$$f0^+(v) = \inf\left\{\mu \in \mathbb{R} \mid (v, \mu) \in 0^+(\mathrm{epi} f)\right\} \quad (v \in X).$$

Several solution existence theorems for generalized polyhedral convex optimization problems will be obtained in this section.

Theorem 4.1 (A Frank–Wolfe-type existence theorem) *If $D \cap \mathrm{dom}\, f$ is nonempty then, $(\mathcal{P})$ has a solution if and only if there is a real value γ such that $f(x) \ge \gamma$ for every $x \in D$.*

Theorem 4.2 (An Eaves-type existence theorem) *Suppose that $D \cap \mathrm{dom}\, f$ is nonempty. Then $(\mathcal{P})$ has a solution if and only if $f0^+(v) \ge 0$ for every $v \in 0^+D$.*

We now give an explicit criterion for $(\mathcal{P})$ to have a solution.

Theorem 4.3 *Let D be given by* (1.2)*, the function f be defined by* (2.1) *with* $\mathrm{dom}\, f$ *be given by* (4.4)*. Suppose that $D \cap \mathrm{dom}\, f$ is nonempty. Then* $(\mathcal{P})$ *has a solution if and only if*

$$0 \in \mathrm{conv}\,\{v_k^* \mid k = 1, \ldots, m\} + \mathrm{cone}\,\{u_j^* \mid j = 1, \ldots, q\} \\ +\mathrm{cone}\,\{x_i^* \mid i = 1, \ldots, p\} + (\ker A \cap \ker B)^{\perp}.$$

Corollary 4.1 *In the notations of Theorem 4.3, suppose that* $\mathrm{dom}\, f \subset D$*. Then,* $(\mathcal{P})$ *has a solution if and only if* $0 \in \mathrm{conv}\,\{v_k^* \mid k = 1, \ldots, m\} + \mathrm{cone}\,\{u_j^* \mid j = 1, \ldots, q\} + (\ker B)^{\perp}$.

Corollary 4.2 *Suppose that $D = X$ and f is given by* (2.1) *with* $\mathrm{dom}\, f = X$*. Then* $(\mathcal{P})$ *has a solution if and only if* $0 \in \mathrm{conv}\,\{v_k^* \mid k = 1, \ldots, m\}$.

Next, we will describe the solution set of $(\mathcal{P})$.

Proposition 4.1 $\mathrm{Sol}(\mathcal{P})$ *is a generalized polyhedral convex set. If D and* $\mathrm{dom}\, f$ *are polyhedral convex, so is* $\mathrm{Sol}(\mathcal{P})$.

One illustrative example for the results in this section has been given in the dissertation.

4.3 Optimality Conditions

We now obtain some optimality conditions for $(\mathcal{P})$.

Theorem 4.4 (Optimality condition I) *A vector $x \in D \cap \mathrm{dom}\, f$ is a solution of* $(\mathcal{P})$ *if and only if*

$$0 \in \overline{\partial f(x) + N_D(x)}. \tag{4.5}$$

One may ask: *The closure sign in* (4.5) *can be omitted, or not?* Example 4.2 in the dissertation shows that the closure sign in (4.5) is essential.

Theorem 4.5 (Optimality condition II) *Assume that either f is a proper polyhedral convex function or D is polyhedral convex set. Then, $x \in D \cap \mathrm{dom}\, f$ is a solution of* $(\mathcal{P})$ *if and only if* $0 \in \partial f(x) + N_D(x)$.

Under the assumptions of Theorem 4.5, by Proposition 1.6 we know that $D - \mathrm{dom}\, f$ is a polyhedral convex set in X. We want to have an analogue of Theorem 4.5 in a Banach space setting for the case $D - \mathrm{dom}\, f$ is a generalized polyhedral convex set.

Theorem 4.6 (Optimality condition III) *Suppose that X is a Banach space and the set* $D - \mathrm{dom}\, f$ *is generalized polyhedral convex. Then, $x \in D \cap \mathrm{dom}\, f$ is a solution of* $(\mathcal{P})$ *if and only if* $0 \in \partial f(x) + N_D(x)$.

Turning back to the optimality condition given by Theorem 4.4, we observe that sometimes it is difficult to find the topological closure of $\partial f(x) + N_D(x)$. The forthcoming theorem gives a new optimality condition for $(\mathcal{P})$ in the general case, where no topological closure sign is needed.

Theorem 4.7 (Optimality condition IV) *A vector $x \in D \cap \mathrm{dom}\, f$ is a solution of* $(\mathcal{P})$ *if and only if*

$$0 \in \mathrm{conv}\,\{v_k^* \mid k \in \Theta(x)\} + \mathrm{cone}\,\{x_i^* \mid i \in I(x)\} \\ + \mathrm{cone}\,\{u_j^* \mid j \in J(x)\} + (\ker A \cap \ker B)^{\perp}.$$

4.4 Duality

In this section, we will use the general conjugate duality scheme presented in the book of Bonnans and Shapiro (2000) to construct a dual problem for $(\mathcal{P})$ and obtain several duality theorems.

We obtain the following *dual problem* of $(\widetilde{\mathcal{P}})$:

$$(\mathcal{D}) \qquad \max\{g(x^*) \mid x^* \in X^*\}$$

with $g(x^*) := -f^*(-x^*) - \delta^*(\cdot, D)(x^*)$. The objective function of $(\mathcal{D})$ is generalized polyhedral concave.

A *weak duality* relationship between $(\mathcal{P})$ and $(\mathcal{D})$ can be described as follows.

Theorem 4.8 (Weak duality theorem) *For every $u \in D$ and $u^* \in X^*$, the inequality $g(u^*) \leq f(u)$ holds. Hence, if $f(u) = g(u^*)$, then $u \in \mathrm{Sol}(\mathcal{P})$ and $u^* \in \mathrm{Sol}(\mathcal{D})$.*

The next statement can be interpreted as a sufficient optimality condition for $(\mathcal{P})$ and $(\mathcal{D})$.

Proposition 4.2 *If $u \in X$ and $u^* \in N_D(u) \cap (-\partial f(u))$, then $u \in \mathrm{Sol}(\mathcal{P})$ and $u^* \in \mathrm{Sol}(\mathcal{D})$. Moreover, the optimal values of $(\mathcal{P})$ and $(\mathcal{D})$ are equal.*

If the optimal value of $(\mathcal{D})$ equals to the optimal value of $(\mathcal{P})$, then one says that the *strong duality* relationship among the dual pair holds. We are going to show that if either f is polyhedral convex or D is polyhedral convex, then this property is available under a mild condition.

Theorem 4.9 (Strong duality theorem I) *Assume that either f is a proper polyhedral convex function and D is a nonempty generalized polyhedral convex set, or f is a proper generalized polyhedral convex function and D is a nonempty polyhedral convex set. If one of the two problems has a solution, then both of them have solutions and the optimal values are equal.*

The conclusion of Theorem 4.9 may not true in the general case, where one just assumes that f is a proper generalized polyhedral convex function and D is a nonempty generalized polyhedral convex set.

The assumption of Theorem 4.9 implies that $D - \mathrm{dom}\, f$ is a polyhedral convex set in X. In particular, $D - \mathrm{dom}\, f$ is closed. Interestingly, in a Banach space setting, the polyhedral convexity of $D - \mathrm{dom}\, f$ can be replaced by its closedness – a weaker property.

Theorem 4.10 (Strong duality theorem II) *Suppose that X is a Banach space and the set $D - \mathrm{dom}\, f$ is closed. If one of the two problems $(\mathcal{P})$ and $(\mathcal{D})$ has a solution, then both of them have solutions and the optimal values are equal.*

In optimization theory, a strong duality theorem can be formulated as a combined statement about the solution existence of the primal and dual problems when they have feasible points where the objective functions are finite, and the equality of the optimal values. In that spirit, for generalized polyhedral convex optimization problems we have next result.

Theorem 4.11 (Strong duality theorem III) *Suppose that the problems $(\mathcal{P})$ and $(\mathcal{D})$ have feasible points, at which the values of the object functions are finite. Then both problems have solutions. In addition, if either f or D is polyhedral convex, then there is no duality gap between the problems.*

Concerning Theorem 4.11, the following question seems to be interesting: *Whether the conclusion "there is no duality gap between two problems" is still true, if one drops the assumption "either f or D is polyhedral convex"?* Our attempts in constructing a counterexample have not achieved the goal, so far.

4.5 Conclusions

Generalized polyhedral convex optimization problems in locally convex Hausdorff topological vector spaces have been studied systematically in this chapter. We have established solution existence theorems, necessary and sufficient optimality conditions, weak and strong duality theorems. In particular, we have shown that the dual problem has the same structure as the primal problem, and the strong duality relation holds under three different sets of conditions.

Chapter 5

Linear and Piecewise Linear Vector Optimization Problems

In this chapter, we study structure of efficient solutions sets of linear vector optimization problems and piecewise linear vector optimization problems.

5.1 Preliminaries

Given two locally convex Hausdorff topological vector spaces X and Y, a vector-valued function $f : X \to Y$, a generalized polyhedral convex set $D \subset X$, and a polyhedral convex cone $K \subset Y$ with $K \neq Y$, we consider a *vector optimization problem*

$$\text{(VP)} \qquad \text{Min}_K \big\{ f(x) \mid x \in D \big\}.$$

A vector $u \in D$ is called an *efficient solution* (resp., a *weakly efficient solution*) of (VP) if no $x \in D$ such that $f(u) - f(x) \in K \setminus \ell(K)$ (resp., $f(u) - f(x) \in \text{int}\, K$). The efficient solution set and the weakly efficient solution set of (VP) are denoted, respectively, by Sol(VP) and Sol^w(VP).

In the terminology of Giannessi (1984), one says that f is a *K-function* on D if

$$(1 - \lambda)f(x_1) + \lambda f(x_2) - f((1 - \lambda)x_1 + \lambda x_2) \in K$$

for any x_1, x_2 in D and $\lambda \in [0, 1]$. When f is a K-function on D, we say that (VP) is a *convex problem.*

Similarly as in Zheng and Yang (2008), we say that a mapping $f : X \to Y$ is a *piecewise linear function* (or a piecewise affine function) if there exist polyhedral convex sets $P_1, \dots, P_m$

in X, continuous linear mappings $T_1, \ldots, T_m$ from X to Y, and vectors $b_1, \ldots, b_m$ in Y such that $X = \bigcup_{k=1}^{m} P_k$ and $f(x) = T_k(x) + b_k$ for all $x \in P_k$, $k = 1, \ldots, m$.

In the sequel, if not otherwise stated, f is a piecewise linear function.

5.2 The Weakly Efficient Solution Set in Linear Vector Optimization

Let us consider a special case of (VP). Namely, we suppose that $f(x) = M(x)$ with $M : X \to Y$ being a continuous linear mapping. Consider a *generalized linear vector optimization problem*

$$\text{(LVOP)} \qquad \mathrm{Min}_K \left\{ M(x) \mid x \in D \right\}.$$

The efficient solution set and the weakly efficient solution set of (LVOP) are denoted, respectively, by Sol(LVOP) and Sol^w(LVOP).

By a standard scalarization scheme in vector optimization, given any element $y^* \in Y^*$, we define the scalar problem

$$\text{(LP)}_{y^*} \qquad \min \left\{ \langle y^*, M(x) \rangle \mid x \in D \right\}.$$

Theorem 5.1 *Problem* (LVOP) *has a weakly efficient solution if and only if*

$$M^* \left(K^* \setminus \{0\} \right) \cap \left(0^+ D \right)^* \neq \emptyset.$$

In particular, if $M^* \left(K^* \right) \cap \left(0^+ D \right)^* \neq \{0\}$, *then* (LVOP) *has a weakly efficient solution.*

The following statement about the structure of Sol^w(LVOP) is applicable to the case where $K \subset Y$ is an arbitrary convex cone.

Theorem 5.2 *The weakly efficient solution set of* (LVOP) *is the union of finitely many generalized polyhedral convex sets.*

5.3 The Efficient Solution Set in Linear Vector Optimization

In the preceding section, in a locally convex Hausdorff topological vector space setting, we have obtained a scalarization formula for the weakly efficient solution set of a generalized linear vector optimization problem, and proved that the latter is the union of finitely many generalized polyhedral convex sets. It is reasonable to look for similar results for the corresponding efficient solution set.

Theorem 5.3 *If K is not a linear subspace of Y, then $u \in Y$ is an efficient solution of* (LVOP) *if and only if there exists $y^* \in \mathrm{ri}\, K^*$ satisfying $u \in \mathrm{argmin}\left((\mathrm{LP})_{y^*} \right)$. In other words,*

$$\mathrm{Sol(LVOP)} = \bigcup_{y^* \in \mathrm{ri}\, K^*} \mathrm{argmin}\left((\mathrm{LP})_{y^*} \right). \tag{5.1}$$

The scalarization formula (5.1) allows us to obtain the following result on the structure of the efficient solution set of (LVOP).

Theorem 5.4 *The efficient solution set of* (LVOP) *is the union of finitely many generalized polyhedral convex sets.*

If the spaces in question are finite dimensional, then the result in Theorem 5.4 expresses one assertion of the Arrow-Barankin-Blackwell Theorem. Another assertion of the latter says that Sol(LVOP) is connected by line segments. Recall that a subset $A \subset X$ is said to be *connected by line segments* if for any points u, v in A, there are some points $u_1, \ldots, u_r$ in A with $u_1 = u$ and $u_r = v$ such that $[u_i, u_{i+1}] \subset A$ for $i = 1, 2, \ldots, r-1$. A natural question arises: *Whether the efficient solution set of* (LVOP) *is connected by line segments, or not?* Next theorem answers this question.

Theorem 5.5 *The efficient solution set* Sol(LVOP) *of* (LVOP) *is connected by line segments.*

A similar result for the weakly efficient solution set of (LVOP) can be obtained.

Theorem 5.6 *If* $\operatorname{int} K \neq \emptyset$, *then the weakly efficient solution set* Sol^w(LVOP) *of* (LVOP) *is connected by line segments.*

5.4 Structure of the Solution Sets in the Convex Case

In this section, we study piecewise linear vector optimization problems whose object functions are convex.

The next result is an extension of a theorem of Yang and Yen (2010) and a theorem of Zheng and Yang (2008) to the locally convex Hausdorff topological vector space setting.

Theorem 5.7 *If f is a K-function on D, the efficient solution set and the weakly efficient solution set of* (VP) *are the unions of finitely many generalized polyhedral convex sets and they are connected by line segments.*

In the dissertation, an illustrative example for Theorem 5.7 has been given.

5.5 Structure of the Solution Sets in the Nonconvex Case

In Theorem 5.7, the assumption f is a K-function on D cannot be dropped (see Yang and Yen (2010) for an example about the efficient solution set, Zheng and Yang (2008) for an example about the weakly efficient solution set). For the case where X, Y are normed spaces, Y is of finite dimension, $K \subset Y$ is a pointed cone, and $D \subset X$ is a polyhedral convex set, the efficient solution set of (VP) is shown to be the union of finitely many semi-closed polyhedral convex sets (see Yang and Yen (2010)).

According to Yang and Yen (2010), a subset of a normed space is called a semi-closed polyhedron if it is the intersection of a finite family of (closed or open) half-spaces. The following definition appears naturally in that spirit.

Definition 5.1 A subset $D \subset X$ is said to be a *semi-closed generalized polyhedral convex set*, or a *semi-closed generalized convex polyhedron*, if there exist $x_i^* \in X^*$, $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, q$, with a positive integer $p \leq q$, and a closed affine subspace $L \subset X$, such that

$$D = \big\{x \in L \mid \langle x_i^*, x\rangle \leq \alpha_i,\ i = 1, \ldots, p;\ \langle x_i^*, x\rangle < \alpha_i,\ i = p+1, \ldots, q\big\}. \tag{5.2}$$

If D can be represented in the form (5.2) with $L = X$, then we say that it is a *semi-closed polyhedral convex set*, or a *semi-closed convex polyhedron.*

Theorem 2.1 in Yang and Yen (2010) can be extended to the locally convex Hausdorff topological vector space setting, which we are considering, as follows.

Theorem 5.8 *The efficient solution set of* (VP) *is the union of finitely many semi-closed generalized polyhedral convex sets.*

The next result is a generalization of Theorem 3.1 of Zheng and Yang (2008).

Theorem 5.9 *If* $\operatorname{int} K$ *is nonempty, then the weakly efficient solution set of* (VP) *is the union of finitely many generalized polyhedral convex sets.*

As an illustration for Theorems 5.8 and 5.9, one example has been designed and shown in the dissertation.

5.6 Conclusions

Linear and piecewise linear vector optimization problems in a locally convex Hausdorff topological vector spaces setting have been considered in this chapter. The efficient solution set of these problems are shown to be the unions of finitely many semi-closed generalized polyhedral convex sets. If, in addition, the problem is convex, then the efficient solution set and the weakly efficient solution set are the unions of finitely many generalized polyhedral convex sets and they are connected by line segments. Our results develop the preceding ones of Zheng and Yang (2008), and Yang and Yen (2010), which were established in a normed spaces setting.

General Conclusions

This dissertation has applied different tools from functional analysis, convex analysis, variational analysis, and optimization theory, to study generalized polyhedral convex structure on locally convex Hausdorff topological vector spaces setting.

The main results of the dissertation include:

1) A representation formula for generalized polyhedral convex sets and polyhedral convex sets in locally convex Hausdorff topological vector spaces.

2) A number of basic properties of generalized polyhedral convex sets in locally convex Hausdorff topological vector spaces.

3) Fundamental properties of generalized polyhedral convex functions on locally convex Hausdorff topological vector spaces.

4) Various properties of normal cones to and polars of generalized polyhedral convex sets, conjugates of generalized polyhedral convex functions, and subdifferentials of generalized polyhedral convex functions.

5) Solution existence theorems, necessary and sufficient optimality conditions, weak and strong duality theorems for generalized polyhedral convex optimization problems in locally convex Hausdorff topological vector spaces.

6) Several theorems describing the structures of the efficient and weakly efficient solutions sets of linear and piecewise linear vector optimization problems.

Developing a concept studied by Zheng (2009), we say that a multifunction between two locally convex Hausdorff topological vector spaces is *generalized polyhedral* if its graph is a union of finitely many generalized polyhedral convex sets. In the light of the theory of set-valued optimization as presented by Khan, Tammer, and Zălinescu (2015), we think that generalized polyhedral multifunctions and optimization problems with such multifunctions as the objective functions deserve a careful study.

List of Author's Related Papers

1. N.N. Luan and N.D. Yen, *A representation of generalized convex polyhedra and applications*, Optimization, DOI: 10.1080/02331934.2019.1614179. (SCIE)

2. N.N. Luan, *Efficient solutions in generalized linear vector optimization*, Applicable Analysis **98** (2019), 1694–1704. (SCIE)

3. N.N. Luan, J.-C. Yao, and N.D. Yen, *On some generalized polyhedral convex constructions*, Numerical Functional Analysis and Optimization **39** (2018), 537–570. (SCIE)

4. N.N. Luan and J.-C. Yao, *Generalized polyhedral convex optimization problems*, Journal of Global Optimization, DOI: 10.1007/s10898-019-00763-4. (SCI)

5. N.N. Luan, *Piecewise linear vector optimization problems on locally convex Hausdorff topological vector spaces*, Acta Mathematica Vietnamica **43** (2018), 289–308. (SCOPUS)

The results of this dissertation have been presented at

- The weekly seminar of the Department of Numerical Analysis and Scientific Computing, Institute of Mathematics, Vietnam Academy of Science and Technology;

- The 14$^{\text{th}}$ Workshop on "Optimization and Scientific Computing" (April 21–23, 2016, Ba Vi, Hanoi);

- The 15$^{\text{th}}$ Workshop on "Optimization and Scientific Computing" (April 20–22, 2017, Ba Vi, Hanoi);

- The 16$^{\text{th}}$ Workshop on "Optimization and Scientific Computing" (April 19–21, 2018, Ba Vi, Hanoi);

- The 9$^{\text{th}}$ Vietnam Mathematical Congress (August 14–18, 2018, Nha Trang, Khanh Hoa).