VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY
INSTITUTE OF MATHEMATICS

NGUYEN NGOC LUAN

SOME CONTRIBUTIONS TO THE THEORY OF GENERALIZED POLYHEDRAL OPTIMIZATION PROBLEMS

DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

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Confirmation

This dissertation was written on the basis of my research works carried out at Institute of Mathematics, Vietnam Academy of Science and Technology under the supervision of Prof. Dr.Sc. Nguyen Dong Yen. All the presented results have never been published by others.

August 06, 2019
The author

Nguyen Ngoc Luan

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Contents

Table of Notations

$\mathbb{R}$	the set of real numbers
$\bar{\mathbb{R}}$	the extended real line
$\emptyset$	the empty set
$A \subset B$	A is a subset of B (the case $A = B$ is not excluded)
$\|\|x\|\|$	the norm of a vector x
$\operatorname{int} A$	the topological interior of A
$\bar{A}$	the closure of a set A
$A^{\perp}$	the annihilator of a set A
$\operatorname{cone} A$	the convex cone generated by A
$\operatorname{conv} A$	the convex hull of A
$C[a, b]$	the linear space of continuous real-valued functions on the interval $[a, b]$
$\operatorname{dom} f$	the effective domain of a function f
$\operatorname{epi} f$	the epigraph of f
$\sup_{x \in D} f(x)$	the supremum of the set $\{f(x) \mid x \in D\}$
$\inf_{x \in D} f(x)$	the infimum of the set $\{f(x) \mid x \in D\}$
$T_C(x)$	the tangent cone of C at x
$\partial f(x)$	the subdifferential of f at x
$N_C(x)$	the normal cone of C at x
$f'(x; h)$	the directional derivative of f at x with respect to a direction h
$M : X \to Y$	an operator from X to Y
$M^* : Y^* \to X^*$	the adjoint operator of M
$\ker M$	the kernel of M
$\operatorname{span} \{x_j \mid j = 1, \ldots, m\}$	the linear subspace generated by vectors x_j, $j = 1, \ldots, m$
resp.	respectively

w.r.t.	with respect to
l.s.c.	lower semicontinuous
lcHtvs	locally convex Hausdorff topological vector space
pcs	polyhedral convex set
gpcs	generalized polyhedral convex set
pcf	polyhedral convex function
gpcf	generalized polyhedral convex function
VOP	vector optimization problem
PLVOP	piecewise linear vector optimization problem

Introduction

Vector optimization has a rich history and diverse applications. *Vector optimization* (sometimes called *multiobjective optimization*) is a natural development of *scalar optimization*. F.Y. Edgeworth (1881) and V. Pareto (1906) defined a notion, which later was called *Pareto solution*. This solution concept remains the most important in vector optimization. Other basic solution concepts of this theory are *weak Pareto solution* and *proper solution*. The latter has been defined in different ways by A.M. Geoffrion, J.M. Borwein, H.P. Benson, M.I. Henig, and other authors.

Vector optimization has numerous applications in economics, management science, and engineering; see, e.g., [2, 22, 41, 67].

One calls a vector optimization problem (VOP) *linear* if the objective functions are linear (affine) functions and the constraint set is polyhedral convex (i.e., it is a intersection of a finite number of closed half-spaces). If at least one of the objective functions is *nonlinear* (non-affine, to be more precise) or the constraint set is not a polyhedral convex set (for example, it is merely a closed convex set or, more general, a solution set of a system of nonlinear inequalities), then the VOP is said to be *nonlinear*.

Linear VOPs have been considered in many books (see, e.g., [50, 51]) and in numerous papers (see, e.g., [3, 34, 38, 39]). The classical Arrow-Barankin-Blackwell Theorem (the ABB Theorem; see, e.g., [3, 50]) asserts that, for a linear vector optimization problem, the Pareto solution set and the weak Pareto solution set are connected by line segments and are the unions of finitely many faces of the constraint set. This is an example of qualitative properties of vector optimization problems. Quantitative aspects (i.e., solution methods) are also very important in vector optimization. Observe that, the second part of the recent book [51] of D.T. Luc on linear vector optimization discusses qualitative properties, while the entire third part is devoted to quantitative aspects of the problems in question.

Nonlinear VOPs have been considered in many books (see, e.g., [2, 41, 50]) and research papers (see, e.g., [37, 75, 76, 79, 82]).

This dissertation focuses on linear VOPs and several related nonlinear scalar optimization problems, as well as nonlinear vector optimization problems. Namely, apart from linear VOPs in locally convex Hausdorff topological vector spaces, which are the main subjects of our research, we will study polyhedral convex optimization problems and piecewise linear vector optimization problems.

The dissertation is put on the framework of functional analysis, convex analysis, and convex optimization. The book by Rudin [65] is main source of the facts from functional analysis used herein. Observe that comprehensive results on convex analysis and convex optimization in locally convex Hausdorff topological vector spaces can be found in the books by Ioffe and Tihomirov [40], Zălinescu [78].

The fundamental concepts used in this dissertation are polyhedral convex set and polyhedral convex function on locally convex Hausdorff topological vector spaces. About one half of the dissertation is devoted to these concepts. Another half of the dissertation shows how our new results on polyhedral convex sets and polyhedral convex functions can be applied to scalar optimization problems and VOPs.

The notions of *polyhedral convex set* – also called a *convex polyhedron*, and *generalized polyhedral convex set* – also called a *generalized convex polyhedron*, stand in the crossroad of several mathematical theories.

First, let us briefly review some basic facts about polyhedral convex set in a finite-dimensional setting. By definition, *a polyhedral convex set in a finite-dimensional Euclidean space is the intersection of a finite family of closed half-spaces.* (By convention, the intersection of an empty family of closed half-spaces is the whole space. Therefore, emptyset and the whole space are two special polyhedra.) So, a polyhedral convex set is the solution set of a system of finitely many inhomogenous linear inequalities. This is the analytical definition of a polyhedral convex set.

According to Klee [46, Theorem 2.12] and Rockafellar [63, Theorem 19.1], *for every given convex polyhedron one can find a finite number of points and* *a finite number of directions such that the polyhedron can be represented as the sum of the convex hull of those points and the convex cone generated by those directions. The converse is also true.* This celebrated theorem, which is a very deep geometrical characterization of polyhedral convex set, is attributed [63, p. 427] primarily to Minkowski [55] and Weyl [73, 74]. By using the result, it is easy to derive fundamental solution existence theorems in linear programming. It is worthy to stress that the above cited representation formula for finite-dimensional polyhedral convex set has many other applications in mathematics. As an example, we refer to the elegant proofs of the necessary and sufficient second-oder conditions for a local solution and for a locally unique solution in quadratic programming, which were given by Contesse [18] in 1980; see [49, pp. 50–63] for details.

For polyhedral convex sets, there is another important characterization: *A closed convex set is a polyhedral convex set if and only if it has finitely many faces*; see [46, Theorem 2.12] and [63, Theorem 19.1] for details.

A bounded polyhedral convex set is called a *polytope*. Leonhard Euler's Theorem stating *a relation between the numbers of faces of different dimensions of a polytope* is a profound classical result. The reader is referred to [33, pp. 130–142b] for a comprehensive exposition of that theorem and some related results.

Functions can be identified with their epigraphs, while sets can be identified with their indicator functions. As explained by Rockafellar [63, p. xi], *"These identifications make it easy to pass back and forth between a geometric approach and an analytic approach".* In that spirit, it seems reasonable to call a function *generalized polyhedral convex* when its epigraph is a generalized polyhedral convex set.

Now, let us discuss the existing facts about polyhedral convex sets and generalized polyhedral convex sets in an infinite-dimensional setting. According to Bonnans and Shapiro [14, Definition 2.195], a subset of a locally convex Hausdorff topological vector space (lcHtvs) is said to be a generalized polyhedral convex set (gpcs), or a generalized convex polyhedron, if it is the intersection of finitely many closed half-spaces and a closed affine subspace of that topological vector space. When the affine subspace can be chosen as the whole space, the generalized polyhedral convex set is called a polyhedral convex set (pcs), or a convex polyhedron. The theories of generalized linear programming in locally convex Hausdorff topological vector spaces and quadratic programming in Banach spaces (see [14, Sections 2.5.7 and 3.4.3]) are based on the concept of generalized convex polyhedron. It is worthy to stress that this concept allows one to obtain such beautiful and important results as Hoffman's lemma for systems of equalities and inequalities in Banach spaces [14, Theorem 2.200], the generalized Farkas lemma [14, Proposition 2.201], an analogue of the Walkup-Wets theorem in a Banach space setting (see [72] and [14, Theorem 2.207]), Robinson's theorem on the local upper Lipschitzian property for polyhedral multifunctions in a Banach space setting (see [62] and [14, Theorem 2.207]), an extension of Frank-Wolfe's and Eaves' solution existence theorems for quadratic programming in a Hilbert space setting (see [14, Theorem 3.128] and [49]). Theorem 3.128 of [14] requires that the quadratic form must be a Legendre form. Recently, by constructing an elegant example, Dong and Tam [19, Example 3.3] have shown that the requirement cannot be dropped.

Many applications of polyhedral convex sets and piecewise linear functions in normed spaces to vector optimization can be found in the papers by Yang and Yen [75], Zheng [80], Zheng and Ng [81], Zheng and Yang [82].

Numerous applications of generalized polyhedral convex sets and generalized polyhedral multifunctions in Banach spaces to variational analysis, optimization problems, and variational inequalities can be found in the works by Henrion, Mordukhovich, and Nam [36], Ban, Mordukhovich, and Song [7], Gfrerer [29, 30], Ban and Song [8].

In 2009, using a result related to the Banach open mapping theorem (see, e.g., [65, Theorem 5.20]), Zheng [80, Corollary 2.1] has clarified the relationships between convex polyhedra in Banach spaces and the finite-dimensional convex polyhedra.

It is well known that any infinite-dimensional normed space equipped with the *weak topology* is not metrizable, but it is a locally convex Hausdorff topological vector space. Similarly, the dual space of any infinite-dimensional normed space equipped with the *weak* topology* is not metrizable, but it is a locally convex Hausdorff topological vector space. Actually, the just mentioned two models provide us with the most typical examples of locally convex Hausdorff topological vector spaces, whose topologies cannot be given by norms. It is clear that Zheng's results in [80] cannot be used neither for a infinite-dimensional normed space equipped with the weak topology, nor for the dual space of any infinite-dimensional normed space equipped with the weak* topology.

The introduction of these concepts poses an interesting problem. Namely, since the entire Section 19 of [63] is devoted to establishing a variety of basic properties of polyhedral convex sets and polyhedral convex functions which have numerous applications afterwards, one may ask whether a similar study can be done for generalized polyhedral convex sets and generalized polyhedral convex functions, or not.

The systematic study of generalized polyhedral convex sets and generalized polyhedral convex function in this dissertation can serve as a basis for further investigations on minimization of a generalized polyhedral convex function on a generalized polyhedral convex set – a *generalized polyhedral convex optimization problem*, which is a special infinite-dimensional convex programming problem. If the objective function is linear, then the just mentioned problem collapse to the *generalized linear programming problem* introduced and treated in detail by Bonnans and Shapiro [14, Chapter 2 and p. 571]. The concepts of polyhedral convex optimization problem have attracted much attention from researchers (see Rockafellar and Wets [64], Bertsekas, Nedíc, and Ozdaglar [12], Boyd and Vandenberghe [15], Bertsekas [10, 11], and the references therein). As observed by Bonnans and Shapiro [14, p. 133], such problems can be viewed as particular cases of *conic linear problems* when the ordering cones in the primal and image spaces are generalized polyhedral convex. It is worthy to stress that semi-infinite linear programs, the mass-transfer problem, maximal flow in a dynamic network, continuous linear programs, and other infinite linear programs can be viewed as conic linear problems (see Anderson and Nash [1]).

Piecewise linear vector optimization problem (PLVOP) is a natural development of *polyhedral convex optimization*. The study of the structures and characteristic properties of these solution sets of PLVOPs is useful in the design of efficient algorithms for solving these PLVOPs. Zheng and Yang [82] have proved that for a PLVOP, where the spaces are normed and the constrain set is a polyhedral convex set, the weak efficient solutions set is the union of finitely many polyhedral convex sets. Moreover, if the objective function is convex w.r.p cone, then the weak efficient solutions set is connected by line segments. In order to describe the structure of the efficient solutions set of PLVOP and obtain sufficient conditions for its connectedness, Yang and Yen [75] have applied the *image space approach* [31, 32] to optimization problems and variational systems and proposed the notion of *semi-closed polyhedral convex set*. On account of [75, Theorem 2.1], if the spaces are normed, the image space is of finite dimension, the ordering cone is a pointed cone, and the constrain set is a polyhedral convex set, then the efficient solution set is the union of finitely many semi-closed polyhedra. In this setting, if the objective function is convex with respect to a cone, then the efficient solutions set is the union of finitely many polyhedra and it is connected by line segments; see [75, Theorem 2.2]. Observe that the main tool for proving the latter results is the representation formula for convex polyhedra in $\mathbb{R}^n$ via a finite number of points and a finite number of directions. Theorem 2.3 of [75] is an infinite-dimensional version of the classical Arrow-Barankin-Blackwell Theorem.

Fang, Meng, and Yang [24] have studied multiobjective optimization problems with either continuous or discontinuous piecewise linear objective functions and polyhedral convex constraint sets. They obtained an algebraic representation of a semi-closed polyhedron and apply it to show that the image of a semi-closed polyhedron under a continuous linear function is always a semi-closed polyhedron. They proposed an algorithm for finding the Pareto point set of a continuous piecewise linear bi-criteria program and generalized it to the discontinuous case. The authors applied that algorithm to solve discontinuous bi-criteria portfolio selection problems with an ℓ_∞ risk measure and transaction costs. Some examples with the historical data of the Hong Kong Stock Exchange are discussed. Other results in this direction were given in [23] and [25]. Later, Zheng and Ng [81] have investigated the metric subregularity of piecewise polyhedral multifunctions and applied this property to piecewise linear multiobjective optimization.

The dissertation has five chapters, a list of the related papers of the author, a section of general conclusions, and a list of references.

Chapter 1 gives a series of fundamental properties of generalized polyhedral convex sets.

In Chapter 2, we discuss some basic properties of generalized polyhedral convex functions.

Chapter 3 is devoted to several dual constructions including the concepts of conjugate function and subdifferential of a generalized polyhedral convex function.

Generalized polyhedral convex optimization problems in locally convex Hausdorff topological vector spaces are studied systematically in Chapter 4. We establish solution existence theorems, necessary and sufficient optimality conditions, weak and strong duality theorems. In particular, we show that the dual problem has the same structure as the primal problem, and the strong duality relation holds under three different sets of conditions.

Chapter 5 discusses structure of efficient solutions sets of linear vector optimization problems and piecewise linear vector optimization problems.

The dissertation is written on the basis of 5 papers in the List of Author's Related Papers on page 113: Paper [A1] published in *Optimization*, paper [A2] published in *Applicable Analysis*, paper [A3] published in *Numerical Functional Analysis and Optimization*, paper [A4] published in *Journal of Global Optimization*, and paper [A5] published in *Acta Mathematica Vietnamica*.

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- The 16th Workshop on "Optimization and Scientific Computing" (April 19–21, 2018, Ba Vi, Hanoi);

- The 9th Vietnam Mathematical Congress (August 14–18, 2018, Nha Trang, Khanh Hoa).

Chapter 1

Generalized Polyhedral Convex Sets

In this chapter, we first establish a representation formula for generalized convex polyhedra. A series of fundamental properties of generalized polyhedral convex sets will be obtained in Sections 2-5. In Section 6, by using the representation formulas for generalized polyhedral convex sets we will prove solution existence theorems in generalized linear programming.

The main theorems of Section 1 below (see Theorems 1.2 and 1.5), which can be considered as *geometrical descriptions* of generalized convex polyhedra and convex polyhedra, are not formal extensions of Theorem 19.1 from [63] and Corollary 2.1 of [80]. Recently, Yen and Yang [77] have used Theorem 1.2 to study infinite-dimensional affine variational inequalities (AVIs) on normed spaces. It is shown that infinite-dimensional quadratic programming problems and infinite-dimensional linear fractional vector optimization problems can be studied by using AVIs. They have obtained two basic facts about infinite-dimensional AVIs: the Lagrange multiplier rule and the solution set decomposition.

The present chapter is written on the basis of the papers [A1], [A2], and [A3] in the List of Author's Related Papers on page 113.

1.1 Preliminaries

From now on, if not otherwise stated, X is a *locally convex Hausdorff topological vector space* over the reals. This means (see, e.g., [65, Definitions 1.6, 1.8, and Theorem 1.12]) that

(a) *X is a vector space over the field $\mathbb{R}$ of reals number;*

(b) *X is equipped with a topology τ;*

(c) *The vector space operations are continuous with respect to τ;*

(d) *For any distinct points u, v in X, there exist a neighborhood U of u and a neighborhood V of v such that $U \cap V = \emptyset$;*

(e) *There is a base $\mathcal{B}$ of neighborhoods of 0 such that every neighborhood $U \in \mathcal{B}$ is a convex set.*

We denote by X^* the dual space of X and by $\langle x^*, x\rangle$ the value of $x^* \in X^*$ at $x \in X$. If X is a Hilbert space with the scalar product (x, y), then by the Riesz theorem one can identify X^* with X. Namely, for each $x^* \in X^*$ there exists a unique vector $y \in X$ such that, for all $x \in X$, $(y, x) = \langle x^*, x\rangle$. Taking account of the last identity, one would prefer to replace (y, x) by $\langle y, x\rangle$. This way of writing the scalar product in a Hilbert space or in an Euclidean space is used in the whole dissertation.

For a subset $\Omega \subset X$ of a locally convex Hausdorff topological vector space, we denote its *interior* by $\operatorname{int}\Omega$, and its topological closure by $\overline{\Omega}$. The convex hull of a subset Ω is denoted by $\operatorname{conv}\Omega$.

One says that a nonempty subset $K \subset X$ is a cone if $tK \subset K$ for every $t > 0$. A cone $K \subset X$ is said to be a pointed cone if $\ell(K) = \{0\}$, where $\ell(K) := K \cap (-K)$. For a subset $\Omega \subset X$, by $\operatorname{cone}\Omega$ we denote the smallest convex cone containing Ω, that is, $\operatorname{cone}\Omega = \{tx \mid t > 0,\ x \in \operatorname{conv}\Omega\}$.

Any normed space is a locally convex Hausdorff topological vector space. Its is also well known (see, e.g., [65, Sections 3.12, 3.14]) that if X is a normed space, then X (resp., X^*) equipped with the week topology (resp. the weak* topology) is a locally convex Hausdorff topological vector space.

1.2 Representation Formulas for Generalized Convex Polyhedra

We begin this section with the definition of generalized polyhedral convex set due to Bonnans and Shapiro [14].

Definition 1.1 (See [14, p. 133]) A subset $D \subset X$ is said to be a *generalized polyhedral convex set*, or a *generalized convex polyhedron*, if there exist some $x_i^* \in X^*$, $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, p$, and a closed affine subspace $L \subset X$, such that

$$D = \{x \in X \mid x \in L,\ \langle x_i^*, x\rangle \le \alpha_i,\ i = 1, \ldots, p\}. \tag{1.1}$$

If D can be represented in the form (1.1) with $L = X$, then we say that it is a *polyhedral convex set*, or a *convex polyhedron*. (Hence, the notion of polyhedral convex set is more specific than that of generalized polyhedral convex set.)

Let D be given as in (1.1). According to [14, Remark 2.196], there exists a continuous surjective linear mapping A from X to a locally convex Hausdorff topological vector space Y and a vector $y \in Y$ such that

$$L = \{x \in X \mid A(x) = y\};$$

then

$$D = \{x \in X \mid A(x) = y,\ \langle x_i^*, x\rangle \le \alpha_i,\ i = 1, \ldots, p\}. \tag{1.2}$$

Set $I = \{1, \ldots, p\}$ and $I(x) = \{i \in I \mid \langle x_i^*, x\rangle = \alpha_i\}$ for $x \in D$.

From Definition 1.1 it follows that every generalized polyhedral convex set is a closed set. If X is finite-dimensional, a subset $D \subset X$ is a generalized polyhedral convex set if and only if it is a polyhedral convex set. In that case, we can represent a given affine subspace $L \subset X$ as the solution set of a system of finitely many linear inequalities.

Our further investigations are motivated by the following fundamental result [63, Theorem 19.1] about polyhedral convex sets in finite-dimensional topological vector spaces, which has origin in the works of Minkowski [55] and Weyl [73, 74] (see also Klee [46, Theorem 2.12]).

Theorem 1.1 (See [63, Theorem 19.1]) *For any nonempty convex set C in $\mathbb{R}^n$, the following properties are equivalent:*

(a) *C is a convex polyhedron;*

(b) *C is finitely generated, i.e., C can be represented as*

$$C = \left\{ \sum_{i=1}^{k} \lambda_i u_i + \sum_{j=1}^{\ell} \mu_j v_j \mid \lambda_i \geq 0,\ \forall i = 1, \ldots, k, \right.$$
$$\left. \sum_{i=1}^{k} \lambda_i = 1,\ \ \mu_j \geq 0,\ \forall j = 1, \ldots, \ell \right\}, \tag{1.3}$$

for some $u_i \in \mathbb{R}^n$, $i = 1, \ldots, k$, *and* $v_j \in \mathbb{R}^n$, $j = 1, \ldots, \ell$;

(c) *C is closed and it has only a finite number of faces.*

From (1.3) it follows that $u_i \in C$ for $i = 1, \ldots, k$.

A natural question arises: *Is there any analogue of the representation* (1.3) *for convex polyhedra in locally convex Hausdorff topological vector spaces, or not?* In order to give an answer in the affirmative to this question, we will need several results from functional analysis.

Lemma 1.1 (Closedness of the sum two linear subspaces; see [65, Theorem 1.42]) *Suppose* X_0 *and* X_1 *are linear subspaces of* X, X_0 *is closed, and* X_1 *has finite dimension. Then* $X_0 + X_1$ *is closed.*

Lemma 1.2 (The Hahn-Banach extension theorem; see [65, Theorem 3.6]) *If* x^* *is a continuous linear functional on a linear subspace* M *of* X, *then there exists* $\widetilde{x}^* \in X^*$ *such that* $\langle \widetilde{x}^*, x \rangle = \langle x^*, x \rangle$ *for all* $x \in M$.

The forthcoming lemma follows from a theorem in [65]. A proof is provided here for the sake of clarity of our presentation.

Lemma 1.3 *If* Y *and* Z *are Hausdorff finite-dimensional topological vector spaces of dimension* n *and if* $g : Y \to Z$ *is a linear bijective mapping, then* g *is a homeomorphism.*

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of the Euclidean space $\mathbb{R}^n$, which is equipped with the natural topology. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of Y. Setting $w_i = g(v_i)$ for $i = 1, \ldots, n$, we see that $\{w_1, w_2, \ldots, w_n\}$ is a basis of Z. Clearly, there is an unique linear bijection $\Phi : \mathbb{R}^n \to Y$ satisfying the conditions $\Phi(e_i) = v_i$ for all i. Similarly, there is an unique linear bijection $\Psi : \mathbb{R}^n \to Z$ with $\Psi(e_i) = w_i$ for all i. By [65, Theorem 1.21(a)], Φ and Ψ are homeomorphisms. (Note that the quoted result was obtained for $\mathbb{C}^n$ and topological vector spaces over the complex field $\mathbb{C}$. Nevertheless, the method of proof is valid for the case of $\mathbb{R}^n$ and topological vector spaces over $\mathbb{R}$.) Since $g = \Psi \circ \Phi^{-1}$ and $g^{-1} = \Phi \circ \Psi^{-1}$ by our construction, it follows that both g and g^{-1} are continuous mappings. □

We are now in a position to extend Corollary 2.1 from [80], which was given in a normed spaces setting, to the case of convex polyhedra in locally convex Hausdorff topological vector spaces.

Proposition 1.1 *A nonempty subset $D \subset X$ is a convex polyhedron if only if there exist closed linear subspaces X_0, X_1 of X and a convex polyhedron $D_1 \subset X_1$ such that*

$$X = X_0 + X_1, \quad X_0 \cap X_1 = \{0\}, \quad \dim X_1 < +\infty, \tag{1.4}$$

and

$$D = D_1 + X_0. \tag{1.5}$$

Proof. *Necessity:* If D is a convex polyhedron, then there exist $x_i^* \in X^*$, $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, p$, such that

$$D = \{x \in X \mid \langle x_i^*, x\rangle \le \alpha_i,\ i = 1, \ldots, p\}.$$

Let

$$X_0 := \{x \in X \mid \langle x_i^*, x\rangle = 0,\ i = 1, \ldots, p\}.$$

Because X_0 is a closed linear subspace of finite codimension, one can find a finite-dimensional linear subspace X_1 of X, such that $X = X_0 + X_1$ and $X_0 \cap X_1 = \{0\}$. By [65, Theorem 1.21(b)], X_1 is closed. Clearly,

$$D_1 := \{x \in X_1 \mid \langle x_i^*, x\rangle \le \alpha_i,\ i = 1, \ldots, p\}$$

is a convex polyhedron in X_1. It is easy to verify that $D_1 + X_0 \subset D$. The reverse inclusion is also true. Indeed, for each $x \in D$ there exist $x_0 \in X_0$ and $x_1 \in X_1$ satisfying $x = x_0 + x_1$. Since

$$\langle x_i^*, x_1\rangle = \langle x_i^*, x\rangle - \langle x_i^*, x_0\rangle = \langle x_i^*, x\rangle \le \alpha_i$$

for all $i = 1, \ldots, p$, it follows that $x_1 \in D_1$; hence $x = x_1 + x_0 \in D_1 + X_0$. We have thus proved that $D = D_1 + X_0$.

Sufficiency: Let X_0, X_1 be closed subspaces of X satisfying the conditions in (1.4). Let $D_1 \subset X_1$ be a convex polyhedron in X_1 and let D be defined by (1.5). Select $u_j^* \in X_1^*$ and $\beta_j \in \mathbb{R}$, $j = 1, \ldots, m$, such that

$$D_1 = \{u \in X_1 \mid \langle u_j^*, u\rangle \le \beta_j,\ j = 1, \ldots, m\}.$$

Let $\pi_0 : X \to X/X_0$, $x \mapsto x + X_0$ for all $x \in X$, be the canonical projection from X on the quotient space X/X_0. It is clear that the operator

$$\Phi_0 : X/X_0 \to X_1,\ x_1 + X_0 \mapsto x_1$$

for all $x_1 \in X_1$, is a linear bijective mapping. On one hand, by [65, Theorem 1.41(a)], π_0 is a linear continuous mapping. On the other hand, Φ_0 is a homeomorphism by Lemma 1.3. So, the operator $\pi := \Phi_0 \circ \pi_0 : X \to X_1$ is linear and continuous. Put $x_j^* = u_j^* \circ \pi$, $j = 1, \ldots, m$. Take any $x = x_1 + x_0$ with $x_1 \in D_1$ and $x_0 \in X_0$. It is clear that

$$\langle x_j^*, x\rangle = \langle u_j^* \circ \pi, x\rangle = \langle u_j^*, \pi(x)\rangle = \langle u_j^*, x_1\rangle \le \beta_j$$

for all $j = 1, \ldots, m$. Conversely, take any $x \in X$ satisfying $\langle x_j^*, x\rangle \le \beta_j$ for every $j = 1, \ldots, m$. Let $x_0 \in X_0$ and $x_1 \in X_1$ be such that $x = x_0 + x_1$. Since

$$\beta_j \ge \langle x_j^*, x_0 + x_1\rangle = \langle u_j^* \circ \Phi_0 \circ \pi_0, x_0 + x_1\rangle = \langle u_j^*, x_1\rangle$$

for all $j = 1, \ldots, m$, we see that $x_1 \in D_1$. Hence $x \in D_1 + X_0$. It follows that

$$D_1 + X_0 = \left\{x \in X \mid \langle x_j^*, x\rangle \le \beta_j,\ j = 1, \ldots, m\right\}.$$

Therefore $D = D_1 + X_0$ is a convex polyhedron in X. □

The main result of this section is formulated as follows.

Theorem 1.2 *A nonempty subset $D \subset X$ is a generalized convex polyhedron if and only if there exist $u_1, \ldots, u_k \in X$, $v_1, \ldots, v_\ell \in X$, and a closed linear subspace $X_0 \subset X$ such that*

$$\begin{aligned} D = \Bigg\{ \sum_{i=1}^{k} \lambda_i u_i + \sum_{j=1}^{\ell} \mu_j v_j \mid \lambda_i \ge 0,\ \forall i = 1, \ldots, k, \\ \sum_{i=1}^{k} \lambda_i = 1,\ \ \mu_j \ge 0,\ \forall j = 1, \ldots, \ell \Bigg\} + X_0. \end{aligned} \tag{1.6}$$

Proof. *Necessity:* Suppose that D is a generalized convex polyhedron. Then we have

$$D = \{x \in X \mid x \in L,\ \langle x_i^*, x\rangle \le \alpha_i,\ i = 1, 2, \ldots, p\},$$

where $L \subset X$ is a closed affine subspace, $x_i^* \in X^*$ and $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, p$. Select a locally convex Hausdorff topological vector space Y, a continuous linear mapping $A : X \to Y$, and a point $y \in Y$ such that

$$L = \{x \in X \mid A(x) = y\}.$$

Fix an element $x_0 \in D$ and set $D_0 = D - x_0$. It is easy to verify that

$$D_0 = \{u \in X \mid A(u) = 0,\ \langle x_i^*, u\rangle \le \alpha_i - \langle x_i^*, x_0\rangle,\ i = 1, \dots, p\}.$$

As D_0 is a convex polyhedron in $\ker A := \{u \in X \mid A(u) = 0\}$, by Proposition 1.1 we can find closed linear subspaces $X_{0,A}$ and $X_{1,A}$ of $\ker A$ and a convex polyhedron $D_{1,A} \subset X_{1,A}$ such that

$$\ker A = X_{0,A} + X_{1,A}, \quad X_{1,A} \cap X_{0,A} = \{0\}, \quad \dim X_{1,A} < +\infty,$$

and

$$D_0 = D_{1,A} + X_{0,A}.$$

Because $X_{1,A} \subset \ker A$ is closed and $\ker A$ is a closed linear subspace of X, $X_{1,A}$ is a closed linear subspace of X. Since $D_{1,A}$ is a convex polyhedron of the finite-dimensional space $X_{1,A}$, invoking Theorem 1.1 we can represent $D_{1,A}$ as

$$D_{1,A} = \left\{ \sum_{i=1}^{k} \lambda_i u_i + \sum_{j=1}^{\ell} \mu_j v_j \mid \lambda_i \ge 0,\ \forall i = 1, \dots, k, \right.$$
$$\left. \sum_{i=1}^{k} \lambda_i = 1,\ \ \mu_j \ge 0,\ \forall j = 1, \dots, \ell \right\},$$

where $u_i \in D_{1,A}$ for $i = 1, \dots, k$ and $v_j \in X_{1,A}$ for $j = 1, \dots, \ell$. Therefore

$$D = \left\{ \sum_{i=1}^{k} \lambda_i (u_i + x_0) + \sum_{j=1}^{\ell} \mu_j v_j \mid \lambda_i \ge 0,\ \forall i = 1, \dots, k, \right.$$
$$\left. \sum_{i=1}^{k} \lambda_i = 1,\ \mu_j \ge 0,\ \forall j = 1, \dots, \ell \right\} + X_{0,A}.$$

We have thus found a representation of the form (1.6) for D.

Sufficiency: Suppose that D is of the form (1.6). Let

$$X_1 = \mathrm{span}\{u_1, \dots, u_k, v_1, \dots, v_\ell\}$$

be the linear subspace generated by the vectors $u_1, \dots, u_k, v_1, \dots, v_\ell$. Put

$$D_1 := \left\{ \sum_{i=1}^{k} \lambda_i u_i + \sum_{j=1}^{\ell} \mu_j v_j \mid \lambda_i \ge 0,\ \forall i = 1, \dots, k, \right.$$
$$\left. \sum_{i=1}^{k} \lambda_i = 1, \mu_j \ge 0,\ \forall j = 1, \dots, \ell \right\}.$$

By Lemma 1.1, $W := X_1 + X_0$ is a closed linear subspace of X. Because X_0 is a closed subspace of finite codimension of W, one can find a finite-dimensional linear subspace $W_1 \subset W$, such that $W = X_0 + W_1$ and $X_0 \cap W_1 = \{0\}$. Consider the linear mapping $\pi : W \to W_1$ be defined by $\pi(x) = w_1$, where $x = x_0 + w_1$ with $(w_1, x_0) \in W_1 \times X_0$. We have

$$\pi(D_1) = \left\{ \sum_{i=1}^{k} \lambda_i \pi(u_i) + \sum_{j=1}^{\ell} \mu_j \pi(v_j) \mid \lambda_i \geq 0,\ \forall i = 1, \ldots, k, \right.$$
$$\left. \sum_{i=1}^{k} \lambda_i = 1,\ \mu_j \geq 0,\ \forall j = 1, \ldots, \ell \right\}.$$

By Theorem 1.1, $\pi(D_1)$ is a polyhedral convex set of W_1. We have

$$D_1 + X_0 = \pi(D_1) + X_0.$$

Indeed, if $x = x_1 + x_0$ where $x_1 \in D_1$ and $x_0 \in X_0$, then

$$x_{1,0} := x_1 - \pi(x_1) \in X_0.$$

So $x = \pi(x_1) + x_{1,0} + x_0$ belongs to the set $\pi(D_1) + X_0$. Conversely, for any $x = \pi(z_1) + x_0$ with $z_1 \in D_1$ and $x_0 \in X_0$, we have

$$x = z_1 + \pi(z_1) - z_1 + x_0 = z_1 + (x_0 - (z_1 - \pi(z_1))) \in D_1 + X_0.$$

Since $D = D_1 + X_0 = \pi(D_1) + X_0$, D is a convex polyhedron in W by Proposition 1.1. Hence there exist $w_1^*, \ldots, w_m^* \in W^*$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ such that

$$D = \{x \in W \mid \langle w_i^*, x \rangle \leq \alpha_i,\ i = 1, \ldots, m\}.$$

According to Lemma 1.2, there exist $x_i^* \in X^*$, $i = 1, \ldots, m$, such that $\langle x_i^*, x \rangle = \langle w_i^*, x \rangle$ for all $x \in W$. Therefore

$$D = \{x \in W \mid \langle x_i^*, x \rangle \leq \alpha_i,\ i = 1, \ldots, m\}.$$

It follows that D is a generalized polyhedral convex set in X. □

Combining Theorem 1.2 with Proposition 1.1, we get a representation formula for convex polyhedra.

Theorem 1.3 *A nonempty subset $D \subset X$ is a convex polyhedron if and only if there exist $u_1, \ldots, u_k \in X$, $v_1, \ldots, v_\ell \in X$, and a closed linear subspace $X_0 \subset X$ of finite codimension such that* (1.6) *is valid.*

The next example is an illustration for Theorem 1.3.

Example 1.1 Let $X = C[a,b]$ be the linear space of continuous real valued functions on the interval $[a,b]$ with the norm defined by

$$||x|| = \max_{t\in[a,b]} |x(t)|.$$

The Riesz representation theorem (see, e.g., [47, Theorem 6, p. 374] and [53, Theorem 1, p. 113]) asserts that the dual space of X is $X^* = NBV[a,b]$, the *normalized space of functions of bounded variation* on $[a,b]$, i.e., functions $y : [a,b] \to \mathbb{R}$ of bounded variation, $y(a) = 0$, and $y(\cdot)$ is continuous from the right at every point of (a,b). Let $x_1^*, x_2^* \in X^*$ be defined by

$$\langle x_1^*, x\rangle = \int_a^b \omega_1(t)x(t)dt, \quad \langle x_2^*, x\rangle = \int_a^b \omega_2(t)x(t)dt, \tag{1.7}$$

where ω_1, ω_2 in $X \setminus \{0\}$ are chosen such that the vectors ω_1, ω_2 are linearly independent. The integrals in (1.7) are Riemannian. They equal respectively to the Riemann-Stieltjes integrals (see [47, p. 367]) $\int_a^b x(t)dy_1(t)$ and $\int_a^b x(t)dy_2(t)$, which are given by the C^1-smooth functions $y_i(t) = \int_a^t \omega_i(\tau)d\tau$, $i = 1,2$. Set

$$\gamma_{i,j} := \int_a^b \omega_i(t)\omega_j(t)dt,$$

for $i,j \in \{1,2\}$. It is clear that $\gamma_{1,2} = \gamma_{2,1}, \gamma_{1,1} > 0, \gamma_{2,2} > 0$. The Cauchy-Schwarz inequality

$$\left(\int_a^b x^2(t)dt\right)^{1/2} \left(\int_a^b y^2(t)dt\right)^{1/2} \geq \left|\int_a^b x(t)y(t)dt\right|,$$

which is valid for any functions $x(\cdot), y(\cdot) \in C[a,b] \subset L_2[a,b]$, implies that

$$\delta := \gamma_{1,1}.\gamma_{2,2} - \gamma_{1,2}^2 \geq 0.$$

As the vectors ω_1, ω_2 are linearly independent, we must have $\delta > 0$. Given any real numbers α_1, α_2, we want to find a representation of form (1.6) for the convex polyhedron

$$D := \{x \in X \mid \langle x_1^*, x\rangle \leq \alpha_1,\ \langle x_2^*, x\rangle \leq \alpha_2\}. \tag{1.8}$$

It is clear that $X_0 := \{x \in X \mid \langle x_1^*, x\rangle = 0,\ \langle x_2^*, x\rangle = 0\}$ is a closed linear subspace of finite codimension of X. For $x = \eta_1\omega_1 + \eta_2\omega_2$ with $\eta_1, \eta_2 \in \mathbb{R}$, we

have
$$\langle x_1^*, x\rangle = \eta_1 \int\limits_a^b \omega_1^2(t)dt + \eta_2 \int\limits_a^b \omega_1(t)\omega_2(t)dt = \eta_1\gamma_{1,1} + \eta_2\gamma_{1,2},$$
and
$$\langle x_2^*, x\rangle = \eta_1 \int\limits_a^b \omega_1(t)\omega_2(t)dt + \eta_2 \int\limits_a^b \omega_2^2(t)dt = \eta_1\gamma_{1,2} + \eta_2\gamma_{2,2}.$$
Since $\delta > 0$, there exists a unique pair of real numbers (η_1, η_2) satisfying
$$\begin{cases} \eta_1\gamma_{1,1} + \eta_2\gamma_{1,2} = \alpha_1 \\ \eta_1\gamma_{1,2} + \eta_2\gamma_{2,2} = \alpha_2. \end{cases}$$
Let the point u and the directions $v_1, v_2 \in X$ be defined by
$$u = \eta_1\omega_1 + \eta_2\omega_2, \quad v_1 = \gamma_{1,2}\omega_1 - \gamma_{1,1}\omega_2, \quad v_2 = -\gamma_{2,2}\omega_1 + \gamma_{1,2}\omega_2.$$
It is easy to verify that $\langle x_i^*, u\rangle = \alpha_i$ for $i = 1, 2$, and
$$\langle x_1^*, v_1\rangle = 0, \quad \langle x_1^*, v_2\rangle = -\delta, \quad \langle x_2^*, v_1\rangle = -\delta, \quad \langle x_2^*, v_2\rangle = 0.$$
Let us show that
$$D = \{u + \mu_1 v_1 + \mu_2 v_2 \mid \mu_j \geq 0,\ j = 1, 2\} + X_0. \tag{1.9}$$
Take any $x = u + \mu_1 v_1 + \mu_2 v_2 + x_0$ with $\mu_1, \mu_2 \in \mathbb{R}_+$ and $x_0 \in X_0$. Because
$$\langle x_1^*, x\rangle = \langle x_1^*, u\rangle + \mu_1\langle x_1^*, v_1\rangle + \mu_2\langle x_1^*, v_2\rangle + \langle x_1^*, x_0\rangle = \alpha_1 - \mu_2\delta \leq \alpha_1$$
and
$$\langle x_2^*, x\rangle = \langle x_2^*, u\rangle + \mu_1\langle x_2^*, v_1\rangle + \mu_2\langle x_2^*, v_2\rangle + \langle x_2^*, x_0\rangle = \alpha_2 - \mu_1\delta \leq \alpha_2,$$
we have $x \in D$. Now, take any $x \in D$. Put
$$\mu_1 = \delta^{-1}\left(\alpha_2 - \langle x_2^*, x\rangle\right), \quad \mu_2 = \delta^{-1}\left(\alpha_1 - \langle x_1^*, x\rangle\right),$$
and
$$x_0 = x - \left(u + \mu_1 v_1 + \mu_2 v_2\right).$$
Note that $\mu_1 \geq 0$, $\mu_2 \geq 0$ and $x = u + \mu_1 v_1 + \mu_2 v_2 + x_0$. Since $\langle x_i^*, x_0\rangle = 0$ for $i = 1, 2$, we see that $x_0 \in X_0$. The formula (1.9) has been proved.

Based on the preceding example, we can easily construct an illustrative example for polyhedral convex sets in locally convex Hausdorff topological vector spaces.

Example 1.2 Keeping all the notations of Example 1.1, let us consider $X = C[a,b]$ with the weak topology. Then X is a locally convex Hausdorff topological vector space whose topology is not a norm topology. The analysis given above shows that the set D in (1.8) admits the representation (1.6).

From Theorem 1.2 we can obtain a representation formula for generalized polyhedral convex cones.

Theorem 1.4 *A nonempty set $K \subset X$ is a generalized polyhedral convex cone if and only if there exist $v_j \in K, j = 1, \ldots, \ell$, and a closed linear subspace X_0 such that*

$$K = \left\{ \sum_{j=1}^{\ell} \mu_j v_j \mid \mu_j \geq 0, \ \forall j = 1, \ldots, \ell \right\} + X_0. \tag{1.10}$$

Proof. *Necessity:* If K is a generalized polyhedral convex cone, then by Theorem 1.2 we can find $u_i \in K$, $i = 1, \ldots, k$, $v_j \in X$, $j = 1, \ldots, \ell$, and a closed linear subspace X_0 such that

$$\begin{aligned} K = \Bigg\{ \sum_{i=1}^{k} \lambda_i u_i + \sum_{j=1}^{\ell} \mu_j v_j \mid \lambda_i \geq 0, \forall i = 1, \ldots, k, \\ \sum_{i=1}^{k} \lambda_i = 1, \ \mu_j \geq 0, \ \forall j = 1, \ldots, \ell \Bigg\} + X_0. \end{aligned} \tag{1.11}$$

To show that $v_j \in K$ for $j = 1, \ldots, \ell$, it suffices to observe by (1.11) that

$$\frac{1}{t}(u_1 + t v_j) = \frac{1}{t} u_1 + v_j$$

belongs to K for all $t > 0$, because K is a cone. Letting $t \to \infty$, by the closedness of K, we get $v_j \in K$. Since $v_j \in K$ for $j = 1, \ldots, \ell$, and since $t_i u_i \in K$ for all $i = 1, \ldots, k$, and $t_i \geq 0$, by choosing $v_{\ell+i} = u_i$ for $i = 1, \ldots, k$, by (1.11) we see that K admits the representation (1.10) where ℓ is replaced by $\ell + k$.

Sufficiency: If K has the form (1.10) then it is a cone. In addition, K is a generalized polyhedral convex set by Theorem 1.2. □

Combining Theorem 1.3 with Theorem 1.4, we obtain a representation formula for polyhedral convex cones.

Theorem 1.5 *A nonempty set $K \subset X$ is a polyhedral convex cone if and only if there exist $v_j \in K, j = 1, \ldots, \ell$, and a closed linear subspace $X_0 \subset X$ of finite codimension such that* (1.10) *is valid.*

1.3 Characterizations via the Finiteness of the Faces

In this section, we show how a generalized polyhedral convex set can be characterized via the finiteness of the number of its faces. In order to obtain the desired results, we first recall some definitions.

Definition 1.2 (See [14, p. 20]) The *relative interior* $\operatorname{ri} C$ of a convex subset $C \subset X$ is the interior of C in the induced topology of the closed affine hull $\overline{\operatorname{aff}}\, C$ of C.

By [63, Theorem 6.4] and Lemma 1.3, if X is a finite-dimensional Hausdorff topological vector space, and $C \subset X$ is a nonempty convex set, then $u \in \operatorname{ri} C$ if and only if, for every $x \in C$, there exists $\varepsilon > 0$ such that $u - \varepsilon(x - u)$ belongs to C.

Remark 1.1 If X is finite-dimensional and $C \subset X$ is a nonempty convex subset, $\operatorname{ri} C$ is nonempty by [63, Theorem 6.2]. If X is infinite-dimensional, it may happen that $\operatorname{ri} C = \emptyset$ for certain nonempty convex subsets $C \subset X$. To justify the claim, it suffices to choose $X = \ell_2$ – the Hilbert space of all real sequences $x = (x_k)_{k=1}^{\infty}$ such that $\sum\limits_{k=1}^{\infty} x_k^2 < +\infty$ with the scalar product $\langle x, y\rangle = \sum\limits_{k=1}^{\infty} x_k y_k$. Put

$$C = \{x \in \ell_2 \mid x_k \geq 0,\ k = 1, 2, \dots\},$$

and observe that $\operatorname{ri} C = \operatorname{int} C = \emptyset$.

If $C \subset X$ is a nonempty generalized polyhedral convex set, then by [14, Proposition 2.197] we know that $\operatorname{ri} C \neq \emptyset$. The latter fact shows that generalized polyhedral convex sets have a nice topological structure.

Definition 1.3 (See [63, p. 162]) A convex subset F of a convex set $C \subset X$ is said to be a *face* of C if for every x^1, x^2 in C satisfying $(1-\lambda)x^1 + \lambda x^2 \in F$ with $\lambda \in (0, 1)$ one has $x^1 \in F$ and $x^2 \in F$.

Definition 1.4 (See [63, p. 162]) A convex subset F of a convex set $C \subset X$ is said to be an *exposed face* of C if there exists $x^* \in X^*$ such that

$$F = \big\{u \in C \mid \langle x^*, u\rangle = \inf_{x \in C} \langle x^*, x\rangle\big\}.$$

From the above definitions it is immediate that if F is an exposed face of a convex set C, then F is a face of C. To see that the converse may not true in general, it suffices to choose

$$C = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid -1 \leq x_1 \leq 1, x_2 \geq -\sqrt{1 - x_1^2}\right\}$$

and $F = \{(1, 0)\}$.

Clearly, a convex set $C \subset X$ itself is not only a face, but also an exposed face of it. The emptyset is a face of C, but it is not necessarily an exposed face of C. For example, a nonempty compact convex C does not have the emptyset as an exposed face of it.

In the spirit of Theorem 1.1, for a nonempty convex subset $D \subset X$, we are interested in establishment of relations between the following properties:

(a) *D is a generalized polyhedral convex set*;

(b) *There exist $u_1, \dots, u_k \in X$, $v_1, \dots, v_\ell \in X$, and a closed linear subspace $X_0 \subset X$ such that*

$$D = \left\{ \sum_{i=1}^{k} \lambda_i u_i + \sum_{j=1}^{\ell} \mu_j v_j \mid \lambda_i \geq 0,\ \forall i = 1, \dots, k, \right.$$
$$\left. \sum_{i=1}^{k} \lambda_i = 1,\ \mu_j \geq 0,\ \forall j = 1, \dots, \ell \right\} + X_0;$$

(c) *D is closed and has only a finite number of faces.*

As shown in Theorem 1.2, (a) and (b) are equivalent. Now, let us prove that (a) implies (c).

Theorem 1.6 *Every generalized polyhedral convex set has a finite number of faces and all the nonempty faces are exposed.*

Proof. Let D be a generalized polyhedral convex set given by (1.1). For any subset $J \subset I$, using the definition of face and formula (1.1), it is not difficult to show that

$$F_J := \left\{x \in D \mid \langle x_i^*, x\rangle = \alpha_i,\ \forall i \in J\right\}$$

is a face of D.

Claim 1. *Let $x, x' \in D$ and F be a face of D. If $x \in F$ and $I(x) \subset I(x')$, then $x' \in F$.*

Indeed, put $x_t := x - t(x' - x)$ where $t > 0$ and observe that $x_t \in L$, because $x_t = (1+t)x + (-t)x'$ and x, x' belong to the closed affine subspace L. For each $i \in I(x) \subset I(x')$, we have

$$\langle x_i^*, x_t \rangle = \langle x_i^*, x \rangle - t\langle x_i^*, x' - x \rangle = \alpha_i.$$

Since $\langle x_j^*, x \rangle < \alpha_j$ for all $j \in I \setminus I(x)$, we can find $t > 0$ such that $\langle x_j^*, x_t \rangle < \alpha_j$ for every $j \in I \setminus I(x)$. Hence, for the chosen t, we have $x_t \in D$. As $x \in F$ and $x = \frac{1}{1+t}x_t + \frac{t}{1+t}x'$, we must have $x' \in F$.

Claim 2. *If F is a nonempty face of D, then there exists $J \subset I$ such that $F = F_J$. Hence, the number of faces of D is finite. Moreover, F is an exposed face.*

Indeed, given a nonempty face F of D, we define $J = \bigcap\limits_{x \in F} I(x)$. It is clear that $F \subset F_J$. To have the inclusion $F_J \subset F$, we select a point $x_0 \in F$ such that the number of elements of $I(x_0)$ is the minimal one among the numbers of elements of $I(x)$, $x \in F$. Let us show that $I(x_0) = J$. Suppose, on the contrary, that $I(x_0) \neq J$. Then there must exist a point $x_1 \in F$ and an index $i_0 \in I(x_0) \setminus I(x_1)$. By the convexity of F, $\bar{x} := \frac{1}{2}x_0 + \frac{1}{2}x_1$ belongs to F. Since $\langle x_{i_0}^*, x_1 \rangle < \alpha_{i_0}$, we have $\langle x_{i_0}^*, \bar{x} \rangle < \alpha_{i_0}$, that is, $i_0 \notin I(\bar{x})$. If $j \notin I(x_0)$, i.e., $\langle x_j^*, x_0 \rangle < \alpha_j$, then $\langle x_j^*, \bar{x} \rangle < \alpha_j$; so $j \notin I(\bar{x})$. Thus, $I(\bar{x}) \subset I(x_0)$ and $I(\bar{x}) \neq I(x_0)$. This contradicts the minimality of $I(x_0)$. For any $x \in F_J$, it is clear that $J \subset I(x)$. Since $x_0 \in F$ and $I(x_0) = J \subset I(x)$, by Claim 1 we can assert that $x \in F$. The inclusion $F_J \subset F$ has been proved. Thus $F = F_J$.

As $J \subset I$ and I is finite, the above obtained result shows that the number of faces of D is finite.

If $J = \emptyset$, then $F_J = D$. For $x^* := 0$, one has $D = \mathrm{argmin}\big\{\langle x^*, x \rangle \mid x \in D\big\}$; hence D is an exposed face of it. It follows that $F_\emptyset$ is an exposed face. Now, suppose that $J \neq \emptyset$. Let k denote the number of elements of J. Setting $x_J^* = \frac{1}{k}\sum\limits_{j \in J}(-x_j^*)$, we have $F_J = \mathrm{argmin}\big\{\langle x_J^*, x \rangle \mid x \in D\big\}$. To prove this equality, it suffices to observe that

$$\langle x_J^*, x \rangle = -\frac{1}{k}\sum_{j \in J} \alpha_j, \ \ \forall x \in F_J$$

and

$$\langle x_J^*, x \rangle > -\frac{1}{k}\sum_{j \in J} \alpha_j, \ \ \forall x \in D \setminus F_J.$$

(The last strict inequality holds because, for any $x \in D \setminus F_J$, there exists

$j_0 \in J$ with $\langle -x^*_{j_0}, x\rangle > -\alpha_{j_0}$, while $\langle -x^*_j, x\rangle \geq -\alpha_j$ for all $j \in J$.) Hence, $F = F_J$ is an exposed face. $\square$

Remark 1.2 The point x_0, constructed in the proof of Theorem 1.6, belongs to $\mathrm{ri}\, F$. Conversely, for any $\bar{x} \in \mathrm{ri}\, F$, $I(\bar{x})$ has the minimality property of $I(x_0)$. The proof of these claims is omitted.

Theorem 1.7 *Let $D \subset X$ be a closed convex set with nonempty relative interior. If D has finitely many faces, then D is a generalized polyhedral convex set.*

Proof. By our assumption $\mathrm{ri}\, D \neq \emptyset$. We, first, consider the case, where $\mathrm{int}\, D \neq \emptyset$. We have $D = \mathrm{int}\, D \cup \partial D$, where $\partial D = D \setminus \mathrm{int}\, D$ is the boundary of D. If $\partial D = \emptyset$, then $D = X$ because D is both open and closed in X, which is a connected topological space. So D is a convex polyhedron. If $\partial D \neq \emptyset$, we pick a point $\bar{x} \in \partial D$. As $\{\bar{x}\} \cap \mathrm{int}\, D = \emptyset$ and since $\{\bar{x}\}$ and $\mathrm{int}\, D$ are convex sets, by the separation theorem [65, Theorem 3.4(a)], there exists $\varphi_{\bar{x}} \in X^* \setminus \{0\}$ such that $\langle \varphi_{\bar{x}}, \bar{x}\rangle \geq \langle \varphi_{\bar{x}}, x\rangle$ for all $x \in \mathrm{int}\, D$. Since D is convex and $\mathrm{int}\, D \neq \emptyset$, it follows that

$$\langle \varphi_{\bar{x}}, \bar{x}\rangle \geq \langle \varphi_{\bar{x}}, x\rangle, \ \ \forall x \in D. \tag{1.12}$$

Let $\alpha_{\bar{x}} := \langle \varphi_{\bar{x}}, \bar{x}\rangle$ and $F_{\bar{x},\varphi_{\bar{x}}} := \{x \in D \mid \langle \varphi_{\bar{x}}, x\rangle = \alpha_{\bar{x}}\}$. It is easy to show that $F_{\bar{x},\varphi_{\bar{x}}}$ is a face of D and $\bar{x} \in F_{\bar{x},\varphi_{\bar{x}}}$. As D has finitely many faces, we can find a finite sequence of points $x_1, \ldots, x_k$ in ∂D such that, for every $u \in \partial D$, there exists $i \in \{1, \ldots, k\}$ with $F_{u,\varphi_u} = F_{x_i,\varphi_{x_i}}$. Let

$$D' := \big\{x \in X \mid \langle \varphi_{x_i}, x\rangle \leq \alpha_{x_i},\ i = 1 \ldots, k\big\}. \tag{1.13}$$

By the construction of φ_{x_i}, $i = 1, \ldots, k$, and by (1.12), we have $D \subset D'$. To show that $D' = D$, suppose the contrary: There exists $u_1 \in D' \setminus D$. Select a point $u_0 \in \mathrm{int} D$. Let $[u_0, u_1] := \big\{(1 - t)u_0 + tu_1 \mid t \in [0, 1]\big\}$ denote the segment joining u_0 and u_1. Since $[u_0, u_1] \cap D$ is a nonempty closed convex set,

$$T := \{t \in [0, 1] \mid u_t := (1 - t)u_0 + tu_1 \in D\}$$

is a closed convex subset of $[0, 1]$. Note that $0 \in T$, but $1 \notin T$. Hence, $T = [0, \bar{t}]$ for some $\bar{t} \in [0, 1)$. As $u_0 \in \mathrm{int} D$, we must have $\bar{t} > 0$. It is easy to show that $\bar{u} := (1 - \bar{t})\, u_0 + \bar{t}u_1$ belongs to ∂D. Hence, $F_{\bar{u},\varphi_{\bar{u}}} = F_{x_i,\varphi_{x_i}}$ for some $i \in \{1, \ldots, k\}$. Since $u_0 \in \mathrm{int} D$ and $\varphi_{x_i} \neq 0$, from (1.12) it follows that

$$\langle \varphi_{x_i}, u_0\rangle < \alpha_{x_i}. \tag{1.14}$$

As $\bar{u} \in F_{x_i, \varphi_{x_i}}$, one has

$$\langle \varphi_{x_i}, \bar{u} \rangle = \alpha_{x_i}. \tag{1.15}$$

From the equality $\bar{u} = (1 - \bar{t})\, u_0 + \bar{t} u_1$ we can deduce that $u_1 = \frac{1}{\bar{t}} \bar{u} + \left(1 - \frac{1}{\bar{t}}\right) u_0$. Since $1 - \frac{1}{\bar{t}} < 0$, by (1.14) and (1.15) we have

$$\begin{aligned}\langle \varphi_{x_i}, u_1 \rangle &= \frac{1}{\bar{t}} \langle \varphi_{x_i}, \bar{u} \rangle + \left(1 - \frac{1}{\bar{t}}\right) \langle \varphi_{x_i}, u_0 \rangle \\ &> \frac{1}{\bar{t}} \alpha_{x_i} + \left(1 - \frac{1}{\bar{t}}\right) \alpha_{x_i} = \alpha_{x_i}.\end{aligned}$$

Then we obtain $\langle \varphi_{x_i}, u_1 \rangle > \alpha_{x_i}$, contradicting the assumption $u_1 \in D'$. We have thus proved that $D' = D$. Therefore, by (1.13) we can conclude that D is a polyhedral convex set.

Now, let us consider the case $\mathrm{int} D = \emptyset$. As $\mathrm{ri} D \neq \emptyset$, the interior of D in the induced topology of $\overline{\mathrm{aff}} D$ is nonempty. Take any $x_0 \in D$. Applying the above result for the closed convex subset $D_0 := D - x_0$ of the locally convex Hausdorff topological vector space $X_0 := \overline{\mathrm{aff}} D - x_0$, we find $x_i^* \in X_0^*$ and $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, m$, such that

$$D_0 = \big\{x \in X_0 \mid \langle x_i^*, x \rangle \leq \alpha_i, \;\; i = 1, \ldots, m\big\}. \tag{1.16}$$

By the Hahn-Banach extension theorem (see Lemma 1.2), there exist some $\widetilde{x}_i^* \in X^*$, for $i = 1, \ldots, m$, such that $\langle \widetilde{x}_i^*, x \rangle = \langle x_i^*, x \rangle$ for all $x \in X_0$. Then from (1.16) it follows that

$$D_0 = \big\{x \in X_0 \mid \langle \widetilde{x}_i^*, x \rangle \leq \alpha_i, \;\; i = 1, \ldots, m\big\}.$$

As $D = D_0 + x_0$, this implies that

$$D = \big\{u \in X_0 + x_0 \mid \langle \widetilde{x}_i^*, u \rangle \leq \alpha_i + \langle \widetilde{x}_i^*, x_0 \rangle, \;\; i = 1, \ldots, m\big\}.$$

Thus D is a generalized polyhedral convex set. □

Remark 1.3 Note that Maserick [54] introduced the concept of *convex polytope*, which is very different from the notion of generalized polyhedral convex set in [14, Definition 2.195]. On one hand, any convex polytope in the sense of Maserick must have nonempty interior, while a generalized polyhedral convex set in the sense of Bonnans and Shapiro may have empty interior (so it *is not a convex polytope in general*). On the other hand, *there exist convex polytopes in the sense of Maserick which cannot be represented as intersections of finitely many closed half-spaces and a closed affine subspace* of that topological vector space. For example, the closed unit ball $\bar{B}$ of c_0 – the Banach space of the real sequences $x = (x_1, x_2, \dots)$, $x_i \in \mathbb{R}$ for all i, $\lim_{i\to\infty} x_i = 0$, with the norm $\|x\| = \sup\{|x_i| \mid i = 1, 2, \dots\}$ – is a convex polytope in the sense of Maserick (see Theorem 4.1 on page 632 in [54]). However, since $\bar{B}$ has an infinite number of faces, it cannot be a generalized polyhedral convex set in the sense of Bonnans and Shapiro (see Theorem 1.6). Subsequently, the concept of convex polytope of [54] has been studied by Maserick and other authors (see, e.g., Durier and Papini [20], Fonf and Vesely [27]). However, after consulting many relevant research works which are available to us, we do hope that the results obtained herein are new.

1.4 Images via Linear Mappings and Sums of Generalized Polyhedral Convex Sets

Let us consider the following question: *Given locally convex Hausdorff topological vector spaces X and Y, whether the image of a generalized polyhedral convex set via a linear mapping from X to Y is a generalized polyhedral convex set, or not?* The answers in the affirmative are given in [63, Theorem 19.3] for the case where X and Y are finite-dimensional, in [82, Lemma 3.2] for the case where X is a Banach space and Y is finite-dimensional.

We are now in a position to extend Lemma 3.2 from the paper of Zheng and Yang [82], which was given in a normed space setting, to the case of convex polyhedra in locally convex Hausdorff topological vector spaces.

Proposition 1.2 *If $T : X \to Y$ is a linear mapping between locally convex Hausdorff topological vector spaces with Y being a space of finite dimension and if $D \subset X$ is a generalized polyhedral convex set, then $T(D)$ is a convex polyhedron of Y.*

Proof. Suppose that D is of the form (1.6). We have

$$T(D) = \Big\{ \sum_{i=1}^{k} \lambda_i(T(u_i)) + \sum_{j=1}^{\ell} \mu_j(T(v_j)) \mid \lambda_i \geq 0,\ \forall i = 1, \dots, k, \\ \sum_{i=1}^{k} \lambda_i = 1,\ \mu_j \geq 0,\ \forall j = 1, \dots, \ell \Big\} + T(X_0).$$

As $T(X_0)$ is a linear subspace of the finite dimensional space Y, $T(X_0)$ is a closed linear subspace. Hence, by Theorem 1.2, $T(D)$ is a polyhedral convex set of Y. □

One may wonder: *Whether the assumption on the finite dimensionality of Y can be removed from Proposition 1.2, or not?* Let us solve this question by an example.

Example 1.3 Let $X = C[0,1]$ be the linear space of continuous real valued functions on the interval $[0,1]$ with the norm defined by $||x|| = \max_{t\in[0,1]} |x(t)|$. Let

$$Y = C_0[0,1] := \{y \in C[0,1] \mid y(0) = 0\}$$

and let $T : X \to Y$ be the bounded linear operator given by

$$(T(x))(t) = \int_0^t x(\tau)d\tau,$$

where integral is Riemannian. Clearly, X is a generalized polyhedral convex set in X and

$$T(X) = \Big\{y \in C_0[0,1] \mid y \text{ is continuously differentiable on } (0,1)\Big\}.$$

To show that $T(X)$ is dense in Y, we take any $y \in Y$. By the Stone-Weierstrass Theorem (see, e.g., [48, Theorem 1.1, p. 52] and [48, Corollary 1.3, p. 54]), there exists a sequence of polynomial functions in one variable $\{p_k\}$ converging uniformly to y in Y. Put $q_k(t) = p_k(t) - p_k(0)$ for all $t \in [0,1]$. It is easily seen that $\{q_k\}$ converges uniformly to y in Y and $\{q_k\} \subset T(X)$. As $T(X) \neq Y$, we see that $T(X)$ is a non-closed linear subspace set of Y. Hence, $T(X)$ cannot be a generalized polyhedral convex set.

A careful analysis of Example 1.3 leads us to the following question: *Whether the image of a generalized polyhedral convex set via a surjective linear operator from a Banach space to another Banach space is a generalized polyhedral convex set, or not?*

Example 1.4 Let $X = C[0,1] \times C[0,1]$ with the norm defined by

$$||(x,u)|| = \max_{t\in[0,1]} |x(t)| + \max_{t\in[0,1]} |u(t)|,$$

$Y = C[0,1]$ and a linear mapping $T : X \to Y$ be defined by

$$T(x,u)(t) = \int_0^t x(\tau)d\tau + u(t),$$

where integral is Riemannian. Clearly, T is a surjective continuous linear mapping from X to Y. Note that $D := C[0,1] \times \{0\}$ is a generalized polyhedral convex set of X, but

$$T(D) = \Big\{ y \in C_0[0,1] \mid y \text{ is continuously differentiable on } (0,1) \Big\}$$

is not a generalized polyhedral convex set of Y.

In the above mentioned example, one sees that the image of a closed linear subspace of X via a continuous surjective linear operator may be not closed; hence it can be not a generalized polyhedral convex set.

The above results motivate the following proposition.

Proposition 1.3 *Suppose that $T : X \to Y$ is a linear mapping between locally convex Hausdorff topological vector spaces and $D \subset X$, $Q \subset Y$ are nonempty generalized polyhedral convex sets. Then, $\overline{T(D)}$ is a generalized polyhedral convex set. If T is continuous, then $T^{-1}(Q)$ is a generalized polyhedral convex set.*

Proof. Suppose that D is of the form (1.6). Then $T(D) = D' + T(X_0)$, where

$$D' := \text{conv}\,\{T(u_i) \mid i = 1, \ldots, k\} + \text{cone}\,\{T(v_j) \mid i = 1, \ldots, \ell\}.$$

Since $T(X_0) \subset Y$ is a linear subspace, $\overline{T(X_0)}$ is a closed linear subspace of Y by [65, Theorem 1.13(c)]; so $D' + \overline{T(X_0)}$ is a generalized polyhedral convex set by Theorem 1.2. In particular, $D' + \overline{T(X_0)}$ is closed. Hence, the inclusion $T(D) \subset D' + \overline{T(X_0)}$ yields

$$\overline{T(D)} \subset D' + \overline{T(X_0)}. \tag{1.17}$$

According to [65, Theorem 1.13(b)], we have

$$D' + \overline{T(X_0)} \subset \overline{T(D)}. \tag{1.18}$$

Combining (1.17) with (1.18) implies that $\overline{T(D)} = D' + \overline{T(X_0)}$. Therefore $\overline{T(D)}$ is a generalized polyhedral convex set.

Now, suppose that $Q \subset Y$ is a generalized polyhedral convex set given by

$$Q = \{y \in Y \mid B(y) = z,\ \langle y_j^*, y\rangle \le \beta_j,\ j = 1, \ldots, q\},$$

where $B : Y \to Z$ is a continuous linear mapping between two locally convex Hausdorff topological vector spaces, $z \in Z$ and $y_j^* \in Y^*$, $\beta_j \in \mathbb{R}$, $j = 1, \ldots, q$.

Then we have

$$\begin{aligned} T^{-1}(Q) =& \{x \in X \mid B(T(x)) = z,\ \langle y_j^*, T(x)\rangle \le \beta_j,\ j = 1, \dots, q\} \\ =& \{x \in X \mid (B \circ T)(x) = z,\ \langle T^*(y_j^*), x\rangle \le \beta_j,\ j = 1, \dots, q\}, \end{aligned}$$

where $T^* : Y^* \to X^*$ is the adjoint operator of T. Since $T : X \to Y$ and $B : Y \to Z$ are linear continuous mappings, $B \circ T : X \to Z$ is a continuous linear mapping. Hence, the above expression for $T^{-1}(Q)$ shows that the set is generalized polyhedral convex. □

Proposition 1.4 *If $D_1, \dots, D_m$ are nonempty generalized polyhedral convex sets in X, so is $\overline{D_1 + \dots + D_m}$.*

Proof. Consider the linear mapping $T : X^m \to X$ given by

$$T(x_1, \dots, x_m) = x_1 + \dots + x_m \quad \forall (x_1, \dots, x_m) \in X^m,$$

and observe that $T(D_1 \times \dots \times D_m) = D_1 + \dots + D_m$. Since D_k is a generalized polyhedral convex set in X for $k = 1, \dots, m$, using Definition 1.1, one can show that $D_1 \times \dots \times D_m$ is a generalized polyhedral convex set in X^m. Then, $\overline{T(D_1 \times \dots \times D_m)}$ is a generalized polyhedral convex set by Proposition 1.3. Hence, $\overline{D_1 + \dots + D_m}$ is a generalized polyhedral convex set in X. □

Remark 1.4 One may ask: *Whether the statement of Corollary 1.4 is valid also for the sum of the sets D_i, $i = 1, \dots, m$, without the closure operation.* When X is a finite-dimensional space, the sum of finitely many polyhedral convex sets in X is a polyhedral convex set (see, e.g., [46, Corollary 2.16], [63, Corollary 19.3.2]). However, when X is an infinite-dimensional space, the sum of a finite number of generalized polyhedral convex sets may be not a generalized polyhedral convex set. To see this, one can choose a suitable space X and closed linear subspaces X_1, X_2 of X satisfying $\overline{X_1 + X_2} = X$ and $X_1 + X_2 \ne X$ (see [9, Example 3.34] for an example of subspaces in any infinite-dimensional Hilbert space, [16, Exercise 1.14] for an example in ℓ^1, and [65, Exercise 20, p. 40] for an example in $L^2(-\pi, \pi)$). Clearly, X_1, X_2 are generalized polyhedral convex sets in X. Since $X_1 + X_2$ is non-closed, it cannot be a generalized polyhedral convex set.

Concerning the question stated in Remark 1.4, in the two following propositions we shall describe some situations where the closure sign can be dropped.

Proposition 1.5 *If D_1, D_2 are generalized polyhedral convex sets of X and* $\text{aff}D_1$ *is finite-dimensional, then $D_1 + D_2$ is a generalized polyhedral convex set.*

Proof. According to Theorem 1.2, for each $m \in \{1, 2\}$, we can represent D_m as $D_m = D'_m + X_{m,0}$ with $X_{m,0}$ being a closed linear subspace of X,

$$D'_m = \text{conv}\,\{u_{m,1}, \ldots, u_{m,k_m}\} + \text{cone}\,\{v_{m,1}, \ldots, v_{m,\ell_m}\}$$

for some $u_{m,1}, \ldots, u_{m,k_m}, v_{m,1}, \ldots, v_{m,\ell_m}$ in X. Since $\text{aff}D_1$ is finite-dimensional, we must have $\text{dim}X_{1,0} < \infty$. By Lemma 1.1, $X_{1,0} + X_{2,0}$ is a closed linear subspace of X. Let W be the finite-dimensional linear subspace generated by the vectors $u_{m,1}, \ldots, u_{m,k_m}$, $v_{m,1}, \ldots, v_{m,\ell_m}$, for $m = 1, 2$. Since D'_1 and D'_2 are polyhedral convex sets in W due to Theorem 1.1, $D'_1 + D'_2$ is a polyhedral convex set in W by [63, Corollary 19.3.2]. On account of Theorem 1.1, one can choose $u_1, \ldots, u_k$ in W, $v_1, \ldots, v_\ell$ in W such that

$$D'_1 + D'_2 = \text{conv}\,\{u_i \mid i = 1, \ldots, k\} + \text{cone}\,\{v_j \mid j = 1, \ldots, \ell\}\,.$$

It follows that

$$D_1 + D_2 = \text{conv}\,\{u_i \mid i = 1, \ldots, k\} + \text{cone}\,\{v_j \mid j = 1, \ldots, \ell\} + X_{1,0} + X_{2,0}.$$

Recalling that the linear subspace $X_{1,0}+X_{2,0}$ is closed, we can use Theorem 1.2 to assert that $D_1 + D_2$ is a generalized polyhedral convex set. $\square$

Before going further, let us present a useful lemma.

Lemma 1.4 *If X_1 and X_2 are linear subspaces of X with X_1 being closed and finite-codimensional, then $X_1 + X_2$ is closed and* $\text{codim}(X_1 + X_2) < \infty$*.*

Proof. Since $X_1 \subset X$ is finite-codimensional, there exists a finite-dimensional linear subspace $X'_1 \subset X$ such that $X = X_1 \cup X'_1$ and $X_1 \cap X'_1 = \{0\}$. Let $\pi_1 : X \to X/X_1$, $\pi_1(x) = x + X_1$ for every $x \in X$, be the canonical projection from X on the quotient space X/X_1. It is clear that the operator $\Phi_1 : X/X_1 \to X'_1$, $x' + X_1 \mapsto x'$ for all $x' \in X'_1$, is a linear bijective mapping. On one hand, by [65, Theorem 1.41(a)], π_1 is a linear continuous mapping. On the other hand, Φ_1 is a homeomorphism by Lemma 1.3. So, the operator $\pi := \Phi_1 \circ \pi_1 : X \to X'_1$ is linear and continuous. Note that $\pi(X_2)$ is closed, because it is a linear subspace of X'_1, which is finite-dimensional. Since π is continuous and $X_1 + X_2 = \pi^{-1}(\pi(X_2))$, we see that $X_1 + X_2$ is closed. The $\text{codim}X_1 < \infty$ clearly forces $\text{codim}(X_1 + X_2) < \infty$. $\square$

Proposition 1.6 *If $D_1 \subset X$ is a polyhedral convex set and $D_2 \subset X$ is a generalized polyhedral convex set, then $D_1 + D_2$ is a polyhedral convex set.*

Proof. By Theorem 1.3, there exist $u_{1,1}, \ldots, u_{1,k_1}$ in X, $v_{1,1}, \ldots, v_{1,\ell_1}$ in X and a closed finite-codimensional linear subspace $X_{1,0} \subset X$ such that

$$D_1 = D_1' + X_{1,0}$$

with $D_1' = \text{conv}\{u_{1,1}, \ldots, u_{1,k_1}\} + \text{cone}\{v_{1,1}, \ldots, v_{1,\ell_1}\}$. According to Theorem 1.2, there exist $u_{2,1}, \ldots, u_{2,k_2}$ in X, $v_{2,1}, \ldots, v_{2,\ell_2}$ in X and a closed linear subspace $X_{2,0}$ of X satisfying $D_2 = D_2' + X_{2,0}$ with

$$D_2' = \text{conv}\{u_{2,1}, \ldots, u_{2,k_2}\} + \text{cone}\{v_{2,1}, \ldots, v_{2,\ell_2}\}.$$

Let W be the finite-dimensional linear subspace generated by the vectors $u_{1,1}, \ldots, u_{1,k_1}, v_{1,1}, \ldots, v_{1,\ell_1}, u_{2,1}, \ldots, u_{2,k_2}, v_{2,1}, \ldots, v_{2,\ell_2}$. Since D_1' and D_2' are polyhedral convex sets in W by Theorem 1.1, Corollary 19.3.2 of [63] implies that $D_1' + D_2'$ is a polyhedral convex set. Applying Theorem 1.1 for the polyhedral convex set $D_1' + D_2'$ of W, one can find $u_1, \ldots, u_k$ and $v_1, \ldots, v_\ell$ in W such that

$$D_1' + D_2' = \text{conv}\{u_i \mid i = 1, \ldots, k\} + \text{cone}\{v_j \mid j = 1, \ldots, \ell\}.$$

Thus,

$$D_1 + D_2 = \text{conv}\{u_i \mid i = 1, \ldots, k\} + \text{cone}\{v_j \mid j = 1, \ldots, \ell\} + X_{1,0} + X_{2,0}. \quad (1.19)$$

In accordance with Lemma 1.4, $X_{1,0} + X_{2,0}$ is a closed finite-codimensional linear subspace. Hence, by Theorem 1.3 and formula (1.19) we conclude that $D_1 + D_2$ is a polyhedral convex set. □

The next result is an extension of [63, Corollary 19.3.2] to an infinite-dimensional setting.

Corollary 1.1 *Suppose that $D_1 \subset X$ is a polyhedral convex set and $D_2 \subset X$ is a generalized polyhedral convex set. If $D_1 \cap D_2 = \emptyset$, then there exists $x^* \in X^*$ such that*

$$\sup\{\langle x^*, u\rangle \mid u \in D_1\} < \inf\{\langle x^*, v\rangle \mid v \in D_2\}. \quad (1.20)$$

Proof. By Proposition 1.6, $D_2 - D_1 = D_2 + (-D_1)$ is a polyhedral convex set in X; hence it is closed. Since $D_2 - D_1$ is a closed convex set and $0 \notin D_2 - D_1$, by the strongly separation theorem [65, Theorem 3.4(b)] there exist $x^* \in X^*$ and $\gamma \in \mathbb{R}$ such that

$$\langle x^*, 0\rangle < \gamma \le \langle x^*, x\rangle, \quad \forall x \in D_2 - D_1.$$

This implies that

$$\sup\{\langle x^*, u\rangle \mid u \in D_1\} + \gamma \le \inf\{\langle x^*, v\rangle \mid v \in D_2\};$$

hence the strict inequality (1.20) is valid. □

The assertion of Corollary 1.1 would be false if D_1 is only assumed to be a generalized polyhedral convex set. Indeed, an answer in the negative for the question in [16, Exercise 1.14] assures us that there exist closed affine subspaces D_1 and D_2 in $X = \ell^1$ such that one cannot find any $x^* \in X^* \setminus \{0\}$ satisfying

$$\sup\{\langle x^*, u\rangle \mid u \in D_1\} \le \inf\{\langle x^*, v\rangle \mid v \in D_2\}.$$

So, with the chosen generalized polyhedral convex sets D_1 and D_2, one cannot have (1.20) for any $x^* \in X^* = \ell^\infty$.

1.5 Convex Hulls and Conic Hulls

As in [63, p. 61], the *recession cone* 0^+C of a convex set $C \subset X$ is given by

$$0^+C = \big\{v \in X \mid x + tv \in C,\ \forall x \in C,\ \forall t \ge 0\big\}.$$

If C is nonempty and closed, then 0^+C is a closed convex cone, and $v \in X$ belongs to 0^+C if and only if there exists $x \in C$ such that $x + tv \in C$ for all $t \ge 0$. These facts are well known [63, Theorems 8.2 and 8.3] for closed convex sets in $\mathbb{R}^n$. For the general case where X is a locally convex Hausdorff topological vector space, the results can be found in [14, p. 33].

Clearly, if D is represented in the form (1.6), then

$$0^+D = \text{cone}\{v_1, \ldots, v_\ell\} + X_0.$$

We are now in a position to extend Theorem 19.6 from the book of Rockafellar [63], which was given in $\mathbb{R}^n$, to the case of generalized polyhedral convex in locally convex Hausdorff topological vector spaces.

Theorem 1.8 *Suppose that $D_1, \dots, D_m$ are generalized polyhedral convex sets in X. Let D be the smallest closed convex subset of X that contains D_i for all $i = 1, \dots, m$. Then D is a generalized polyhedral convex set. If at least one of the sets $D_1, \dots, D_m$ is polyhedral convex, then D is a polyhedral convex set.*

Proof. By removing all the empty sets from the system $D_1, \dots, D_m$, we may assume that $D_i \neq \emptyset$ for all $i \in I := \{1, \dots, m\}$. Due to Theorem 1.2, for each $i \in I$, one can find $u_{i,1}, \dots, u_{i,k_i}$ and $v_{i,1}, \dots, v_{i,\ell_i}$ in X and a closed linear subspace $X_{i,0} \subset X$ such that

$$D_i = \operatorname{conv}\{u_{i,1}, \dots, u_{i,k_i}\} + \operatorname{cone}\{v_{i,1}, \dots, v_{i,\ell_i}\} + X_{i,0}. \tag{1.21}$$

Since $X_{1,0} + \dots + X_{m,0} \subset X$ is a linear subspace, $X_0 := \overline{X_{1,0} + \dots + X_{m,0}}$ is a closed linear subspace of X by [65, Theorem 1.13(c)]. Let

$$\begin{aligned} D' : = \operatorname{conv}\{u_{i,j} \mid i \in I, j = 1, \dots, k_i\} \\ + \operatorname{cone}\{v_{i,j} \mid i \in I, j = 1, \dots, \ell_i\} + X_0. \end{aligned} \tag{1.22}$$

On account of Theorem 1.2, D' is a generalized polyhedral convex set. In particular, D' is convex and closed. From (1.21) and (1.22) it follows that $D_i \subset D'$ for every $i \in I$. Hence, by the definition of D, we must have $D \subset D'$. Let us show that $D' \subset D$. Since $u_{i,j}$ belongs to $D_i \subset D$ for $i \in I$ and $j \in \{1, \dots, k_i\}$, and since D is convex,

$$\operatorname{conv}\{u_{i,j} \mid i \in I, j = 1, \dots, k_i\} \subset D. \tag{1.23}$$

It is clear that $0^+D_i = \operatorname{cone}\{v_{i,1}, \dots, v_{i,\ell_i}\} + X_{i,0}$ for every $i \in I$. As D is the smallest closed convex set containing $\bigcup_{i=1}^{m} D_i$, we have

$$\operatorname{cone}\{v_{i,j} \mid i \in I, j = 1, \dots, \ell_i\} \subset 0^+D$$

and $X_{1,0} + \dots + X_{m,0} \subset 0^+D$. Since the cone 0^+D is closed, $X_0 \subset 0^+D$. Thus

$$\operatorname{cone}\{v_{i,j} \mid i \in I, j = 1, \dots, \ell_i\} + X_0 \subset 0^+D. \tag{1.24}$$

Combining (1.22), (1.23) with (1.24) yields $D' \subset D$. Thus we have proved that $D' = D$. Since D' is a generalized polyhedral convex set, D is also a generalized polyhedral convex set.

Now, suppose that at least one of the set $D_1, \dots, D_m$ is polyhedral convex. Then, by Theorem 1.3, in the representation (1.21) for $D_1, \dots, D_m$ we may assume that at least one of the sets $X_{1,0}, \dots, X_{m,0}$ is finite-codimensional.

According to Lemma 1.4, $X_{1,0}+\cdots+X_{m,0}$ is a closed linear subspace of finite codimension in X; hence $\mathrm{codim} X_0 < \infty$. Due to (1.22), D' is a polyhedral convex set by Theorem 1.3. Since $D = D'$, we see that D is a polyhedral convex set. □

From Theorem 1.8 we obtain the following corollary.

Corollary 1.2 *If a convex subset $D \subset X$ is the union of a finite number of generalized polyhedral convex sets (resp., of polyhedral convex sets) in X, then D is a generalized polyhedral convex (resp., polyhedral convex) set.*

The reader is referred to [64, Lemma 2.50] for a different proof of Corollary 1.2 in the case where $X = \mathbb{R}^n$.

It turns out that the closure of the cone generated by a generalized polyhedral convex set is a generalized polyhedral convex cone. Hence, next proposition extends [63, Theorem 19.7] to a locally convex Hausdorff topological vector spaces setting.

Proposition 1.7 *If a nonempty subset $D \subset X$ is generalized polyhedral convex, then $\overline{\mathrm{cone}\, D}$ is a generalized polyhedral convex cone. In addition, if $0 \in D$ then $\mathrm{cone}\, D$ is a generalized polyhedral convex cone; hence $\mathrm{cone}\, D$ is closed.*

Proof. Suppose that D is of the form (1.6). According to Theorem 1.4, the set

$$C := \mathrm{cone}\,\{u_i, v_j \mid i = 1,\dots,k,\ j = 1,\dots,\ell\} + X_0 \tag{1.25}$$

where $u_1 \dots, u_k$, $v_1,\dots,v_\ell$, and X_0 are given by (1.6), is a generalized polyhedral convex cone. In particular, C is closed. Since C contains D, we must have $\overline{\mathrm{cone}\, D} \subset C$. From (1.6) we see that $0^+D = \mathrm{cone}\{v_j \mid j = 1,\dots,\ell\} + X_0$ and $u_i \in \overline{\mathrm{cone}\, D}$ for all $i = 1,\dots,k$. As $\overline{\mathrm{cone}\, D}$ is a closed convex cone, from (1.25) it follows that $C \subset \overline{\mathrm{cone}\, D}$. Thus we have shown that $\overline{\mathrm{cone}\, D} = C$. In particular, $\overline{\mathrm{cone}\, D}$ is a generalized polyhedral convex cone.

Now, suppose that $0 \in D$. To get the equality $\mathrm{cone}\, D = C$ with C being given by (1.25), we first observe that $\mathrm{cone}\, D \subset C$, because $C = \overline{\mathrm{cone}\, D}$. To verify that $\mathrm{cone} D \supset C$, take any $x \in C$. According to (1.25), one can find nonnegative numbers $\lambda_1,\dots,\lambda_k$, $\mu_1,\dots,\mu_\ell$, and a vector $x_0 \in X_0$, such that

$$x = \sum_{i=1}^{k} \lambda_i u_i + \sum_{j=1}^{\ell} \mu_j v_j + x_0.$$

If $\lambda := \sum_{i=1}^{k} \lambda_i$ is positive, then $\frac{1}{\lambda}x$ belongs to D; so $x \in \operatorname{cone} D$. If $\lambda = 0$, then $\lambda_1 = \cdots = \lambda_k = 0$ and $x = \sum_{j=1}^{\ell} \mu_j v_j + x_0$; hence $x \in 0^+D$. Since $0 \in D$, this implies that $0 + x$ is contained in D. The inclusion $\operatorname{cone} D \supset C$ has been proved. So we have $\operatorname{cone} D = C$. In particular, $\operatorname{cone} D$ is a generalized polyhedral convex cone. $\square$

An analogue of Proposition 1.7 for polyhedral convex sets can be formulated as follows.

Proposition 1.8 *If a nonempty subset $D \subset X$ is polyhedral convex, then $\overline{\operatorname{cone} D}$ is a polyhedral convex cone. In addition, if $0 \in D$ then $\operatorname{cone} D$ is a polyhedral convex cone; hence $\operatorname{cone} D$ is closed.*

Proof. Similar to the proof of Proposition 1.7. $\square$

In convex analysis, to every convex set and a point belonging to it, one associates a tangent cone. Let us complete this section by showing that the tangent cone to a generalized polyhedral convex set at a given point is a generalized polyhedral convex cone. By definition, the (Bouligand-Severi) *tangent cone* [5] $T_D(x)$ to a closed subset $D \subset X$ at $x \in D$ is the set of all $v \in X$ such that there exist sequences $t_k \to 0^+$ and $v_k \to v$ such that $x + t_k v_k \in D$ for every k. If D is convex, then

$$T_D(x) = \overline{\operatorname{cone}(D - x)}. \tag{1.26}$$

If D is a generalized polyhedral convex set and $x \in D$, then $D - x$ is a generalized polyhedral convex set containing 0. Therefore, according to Proposition 1.7, $\operatorname{cone}(D - x)$ is a generalized polyhedral convex cone, that is closed. So the closure sign in the right-hand side of (1.26) can be omitted. Similarly, according to Proposition 1.8, if D is a polyhedral convex set and $x \in D$, then $\operatorname{cone}(D - x)$ is a polyhedral convex cone and the closure sign in the right-hand side of (1.26) can be also omitted. Thus we have obtained the following result.

Proposition 1.9 *If $D \subset X$ is a generalized polyhedral convex set (resp., a polyhedral convex set) and if $x \in D$, then $T_D(x)$ is a generalized polyhedral convex cone (resp., a polyhedral convex cone) and one has*

$$T_D(x) = \operatorname{cone}(D - x).$$

1.6 Relative Interiors of Polyhedral Convex Cones

In this section, we obtain a formula for the relative interiors of a generalized polyhedral convex cone and the dual cone of a polyhedral convex cone.

Theorem 1.9 *Suppose that $C \subset X$ is a generalized polyhedral convex cone in a locally convex Hausdorff topological vector space. If*

$$C = \left\{ \sum_{i=1}^{p} \lambda_i u_i \mid \lambda_i \geq 0,\ i = 1, \ldots, p \right\},$$

where $u_i \in X$ for $i = 1, \ldots, p$, then

$$\operatorname{ri} C = \left\{ \sum_{i=1}^{p} \lambda_i u_i \mid \lambda_i > 0,\ i = 1, \ldots, p \right\}. \tag{1.27}$$

Proof. Let $X_0 := \operatorname{span}\{u_1, \ldots, u_p\}$ be the linear subspace generated by the vectors $u_1, \ldots, u_p$. As C is a convex cone of X_0 which is a space of finite dimension, $u \in \operatorname{ri} C$ if and only if, for every $x \in C$, there exists $\varepsilon > 0$ such that $u - \varepsilon(x - u) \in C$. Given any $u \in \operatorname{ri} C$, we will show that u belongs to the right-hand side of (1.27). Let $x = \sum\limits_{i=1}^{p} u_i \in C$ and let $\varepsilon > 0$ be such that $v := u - \varepsilon(x - u)$ belongs to C. Suppose that $v = \sum\limits_{i=1}^{p} \alpha_i u_i$, where $\alpha_i \geq 0$ for $i = 1, \ldots, p$. It is clear that

$$u = \frac{1}{1+\varepsilon} v + \frac{\varepsilon}{1+\varepsilon} x = \sum_{i=1}^{p} \left(\frac{\alpha_i}{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon} \right) u_i = \sum_{i=1}^{p} \lambda_i u_i,$$

where $\lambda_i := \frac{\alpha_i}{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon} > 0$, $i = 1, \ldots, p$. This establishes the inclusion "$\subset$" in (1.27).

Now, let u be an arbitrary element from the set on the right-hand side of (1.27). Suppose that $u = \sum\limits_{i=1}^{p} \lambda_i u_i$, where $\lambda_i > 0$ for all $i = 1, \ldots, p$. For any $x \in C$, one can choose $\alpha_1 \geq 0, \ldots, \alpha_p \geq 0$, satisfying $x = \sum\limits_{i=1}^{p} \alpha_i u_i$. Put

$$v_\varepsilon = u - \varepsilon(x - u) = \sum_{i=1}^{p} \big(\lambda_i - \varepsilon(\alpha_i - \lambda_i) \big) u_i,$$

where $\varepsilon > 0$. As $\lambda_i > 0$ for all $i = 1, \ldots, p$, one can find an $\varepsilon > 0$ satisfying $\lambda_i - \varepsilon(\alpha_i - \lambda_i) \geq 0$ for every $i = 1, \ldots, p$. Hence, for this ε, we have $v_\varepsilon \in C$. The inclusion "$\supset$" in (1.27) has been proved. $\square$

Let Y be a locally convex Hausdorff topological vector space. The positive dual cone of a cone $K \subset Y$ is given by

$$K^* := \{y^* \in Y^* \mid \langle y^*, y\rangle \geq 0 \ \ \forall y \in K\}.$$

Suppose that $K \subset Y$ is a polyhedral convex cone defined by

$$K = \Big\{y \in Y \mid \langle y_j^*, y\rangle \leq 0, \ j = 1, \ldots, q\Big\}, \tag{1.28}$$

where $y_j^* \in Y^* \setminus \{0\}$ for all $j = 1, \ldots, q$. It is clear that

$$\ell(K) = \{y \in Y \mid \langle y_j^*, y\rangle = 0, \ j = 1, \ldots, q\}.$$

To proceed furthermore, we put $Y_0 = \{y \in Y \mid \langle y_j^*, y\rangle = 0, \ j = 1, \ldots, q\}$ and note that $\ell(K) = Y_0$. Because Y_0 is a closed linear subspace of finite codimension of Y, there exists a finite-dimensional linear subspace Y_1 of Y, such that $Y = Y_0 + Y_1$ and $Y_0 \cap Y_1 = \{0\}$. By [65, Theorem 1.21(b)], Y_1 is closed. Clearly,

$$K_1 := \{y \in Y_1 \mid \langle y_j^*, y\rangle \leq 0, \ j = 1, \ldots, q\}$$

is a pointed polyhedral convex cone in Y_1 and

$$K = Y_0 + K_1. \tag{1.29}$$

For the case $Y = \mathbb{R}^n$, a result similar to the forthcoming one was given in [50, Lemma 2.6, p. 89].

Lemma 1.5 *It holds that*

$$K \setminus \ell(K) = Y_0 + K_1 \setminus \{0\}. \tag{1.30}$$

Proof. The inclusion "$\supset$" is obvious. For every $y \in K \setminus \ell(K)$, one has $y = y_0 + y_1$ for some $y_0 \in Y_0$ and $y_1 \in K_1$. If $y_1 = 0$, then $y = y_0 \in \ell(K)$. This contradicts the condition $y \notin \ell(K)$. Thus, $y \in Y_0 + K_1 \setminus \{0\}$. The equality (1.30) has been proved. □

The first assertion of the following proposition describes the interior of a polyhedral convex cone.

Proposition 1.10 *Let $K \subset Y$ be a polyhedral convex cone of the form* (1.28). *The following are valid:*

(a) *The interior of K has the representation*

$$\operatorname{int} K = \{y \in Y \mid \langle y_j^*, y\rangle < 0, \ j = 1, \ldots, q\}. \tag{1.31}$$

(b) *The set* $K \setminus \ell(K)$ *is a convex cone and*

$$K \setminus \ell(K) = \Big\{ y \in K \mid \textit{there exists } j \in \{1, \ldots, q\} \textit{ such that } \langle y_j^*, y \rangle < 0 \Big\}. \tag{1.32}$$

Proof. (a) As $\{y \in Y \mid \langle y_j^*, y\rangle < 0,\ j = 1, \ldots, q\}$ is an open subset of K, we have the inclusion "$\supset$" in (1.31). Now, to obtain the reverse inclusion, arguing by contradiction, we suppose that there exists $\bar{y} \in \operatorname{int} K$ for which there is $j_1 \in \{1, \ldots, q\}$ such that $\langle y_{j_1}^*, \bar{y}\rangle = 0$. Since $\bar{y} \in \operatorname{int} K$, one can find a balanced neighborhood $V \subset Y$ of 0 satisfying $\bar{y} + V \subset K$. Then we have

$$0 \geq \langle y_{j_1}^*, \bar{y} + v\rangle = \langle y_{j_1}^*, v\rangle$$

for all $v \in V$. It follows that $\langle y_{j_1}^*, v\rangle = 0$ for every $v \in V$. As $y_{j_1}^* \neq 0$, there exists $y \in Y$ with $\langle y_{j_1}^*, y\rangle \neq 0$. Since $ty \in V$ for sufficiently small $t > 0$, we get $\langle y_{j_1}^*, y\rangle = 0$, a contradiction. Thus, we have proved the inclusion "$\subset$" in (1.31).

(b) By (1.30) and the fact that $K_1 \setminus \{0\}$ is a convex cone, $K \setminus \ell(K)$ is a convex cone. Clearly, $y \in K \setminus \ell(K)$ if and only if $\langle y_j^*, y\rangle \leq 0$ for all $j = 1, \ldots, q$, and there exists $j \in \{1, \ldots, q\}$ satisfying $\langle y_j^*, y\rangle < 0$. Therefore, (1.32) holds true. □

Remark 1.5 From Proposition 1.10 it follows that $\operatorname{int} K \subset K \setminus \ell(K)$. The last inclusion can be strict. To see this, choose $Y = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$.

By [14, Proposition 2.42], we can represent the positive dual cone of K as

$$K^* = \left\{ \sum_{j=1}^{q} \lambda_j (-y_j^*) \mid \lambda_j \geq 0, j = 1, \ldots, q \right\}. \tag{1.33}$$

Lemma 1.6 *If* $y^* \in K^*$*, then* $\langle y^*, y\rangle = 0$ *for all* $y \in Y_0$.

Proof. If $y^* \in K^*$ then, for any $y \in Y_0$, one has $\langle y^*, y\rangle \geq 0$ and $\langle y^*, -y\rangle \geq 0$; hence $\langle y^*, y\rangle = 0$. □

Now we are in a position to describe the relative interior of the dual cone K^* by using the set $K \setminus \ell(K)$, which can be computed by (1.32).

Theorem 1.10 *If* K *is not a linear subspace of* Y*, then a vector* $y^* \in Y^*$ *belongs to* $\operatorname{ri} K^*$ *if and only if* $\langle y^*, y\rangle > 0$ *for all* $y \in K \setminus \ell(K)$.

Proof. *Necessity:* Suppose that $y^* \in \operatorname{ri} K^*$. By Theorem 1.9 and formula (1.33), there exist $\lambda_j > 0$ for $j = 1, \ldots, q$, such that $y^* = \sum_{j=1}^{q} \lambda_j(-y_j^*)$. For any $y \in K \setminus \ell(K)$, by Proposition 1.10 one can find $j_0 \in \{1, \ldots, q\}$ satisfying $\langle y_{j_0}^*, y\rangle < 0$. Then we have

$$\begin{aligned}\langle y^*, y\rangle &= (-\lambda_{j_0})\langle y_{j_0}^*, y\rangle + \sum_{j=1, j\neq j_0}^{q} (-\lambda_j)\langle y_j^*, y\rangle \\ &\geq (-\lambda_{j_0})\langle y_{j_0}^*, y\rangle > 0,\end{aligned}$$

as desired.

Sufficiency: Suppose that $y^* \in Y^*$ and $\langle y^*, y\rangle > 0$ for all $y \in K \setminus \ell(K)$. By (1.30) we have $y^* \in Y_0^* \cap K_1^*$, hence $y^* \in K^*$. Since K is not a linear subspace of Y, the cone K_1 is nontrivial. By [63, Theorem 19.1], one can find $v_i \in Y_1 \setminus \{0\}$, $i = 1, \ldots, \ell$, such that

$$K_1 = \left\{ \sum_{i=1}^{\ell} \mu_i v_i \mid \mu_i \geq 0, \quad i = 1, \ldots, \ell \right\}.$$

Since $v_i \in K \setminus \ell(K)$ for $i = 1, \ldots, \ell$, by (1.30), it follows that

$$\langle y^*, v_i\rangle > 0, \ i = 1, \ldots, \ell. \tag{1.34}$$

Take any $\widetilde{y}^* \in K^*$ and put $v_\varepsilon^* = y^* - \varepsilon(\widetilde{y}^* - y^*)$ with $\varepsilon > 0$. By (1.34), there exists $\varepsilon > 0$ such that

$$\langle v_\varepsilon^*, v_i\rangle > 0, \quad i = 1, \ldots, \ell.$$

As $K = Y_0 + K_1$, for every $y \in K$ one can find $y_0 \in Y_0$ and $\mu_i \geq 0$, $i = 1, \ldots, \ell$, such that $y = y_0 + \sum_{i=1}^{\ell} \mu_i v_i$. Because $y^*, \widetilde{y}^* \in K^*$, by Lemma 1.6 one has $\langle y^*, y_0\rangle = 0$ and $\langle \widetilde{y}^*, y_0\rangle = 0$. Hence, from (1.34) it follows that

$$\begin{aligned}\langle v_\varepsilon^*, y\rangle &= \left\langle v_\varepsilon^*, y_0 + \sum_{i=1}^{\ell} \mu_i v_i \right\rangle \\ &= \sum_{i=1}^{\ell} \mu_i \langle v_\varepsilon^*, v_i\rangle \geq 0.\end{aligned}$$

So we have $\langle v_\varepsilon^*, y\rangle \geq 0$ for every $y \in K$. This means that $v_\varepsilon^* \in K^*$. We have thus proved that, for any $\widetilde{y}^* \in K^*$, there exists a real number $\varepsilon > 0$ such that $y^* - \varepsilon(\widetilde{y}^* - y^*) \in K^*$. Since K^* is a convex cone in the finite dimensional space $\operatorname{span}\{y_1^*, \ldots, y_q^*\}$, by [63, Theorem 6.4], we can infer that $y^* \in \operatorname{ri} K^*$. □

Remark 1.6 If $K = Y_0$, then $K^* = Y_0^\perp$ where

$$Y_0^\perp := \{y^* \in Y^* \mid \langle y^*, y\rangle = 0, \ \forall y \in Y_0\}$$

is the annihilator of Y_0. So we have $\operatorname{ri} K^* = K^*$.

1.7 Solution Existence in Linear Optimization

Our aim in this section is to apply the representation formula for generalized polyhedral convex sets to proving solution existence theorems for generalized linear programming problems.

Consider a *generalized linear programming problem*

$$\text{(LP)} \qquad \min\{\langle x^*, x\rangle \mid x \in D\}$$

where, as before, X is a locally convex Hausdorff topological vector space, $D \subset X$ is a generalized polyhedral convex set, $x^* \in X^*$.

The following two existence theorems for (LP) are known results. Actually, in combination, they express the contents of Theorem 2.199 from [14]. The latter, in its turn, is a special case of Theorem 2.198 from [14]. The simple proofs given below show how Theorem 1.2 can be used to study the solution existence of generalized linear programs.

Theorem 1.11 (The Eaves-type existence theorem; see [14, Theorem 2.199]) *If D is nonempty, then* (LP) *has a solution if and only if $\langle x^*, v\rangle \geq 0$ for every $v \in 0^+D$.*

Proof. If (LP) has a solution $\overline{x}$, then for each $v \in 0^+D$ it holds that

$$\langle x^*, \overline{x}\rangle \leq \langle x^*, \overline{x} + tv\rangle = \langle x^*, \overline{x}\rangle + t\langle x^*, v\rangle \ \ \forall t \in \mathbb{R}_+,$$

because $\overline{x} + tv \in D$ for every $t \geq 0$. Hence $\langle x^*, v\rangle \geq 0$.

Conversely, suppose that $\langle x^*, v\rangle \geq 0$ for every $v \in 0^+D$. Let us represent D in the form (1.6). Select an element $u_{i_0} \in \{u_1, \dots, u_k\}$ such that

$$\langle x^*, u_{i_0}\rangle = \min\{\langle x^*, u_i\rangle \mid i = 1, \dots, k\}.$$

By (1.6), for every $x \in D$ there exist $\lambda_i \in \mathbb{R}_+$, $i = 1, \dots, k$, $\sum_{i=1}^{k} \lambda_i = 1$, and

$v \in 0^+D$ such that $x = \sum_{i=1}^{k} \lambda_i u_i + v$. Then we have

$$\langle x^*, x \rangle = \sum_{i=1}^{k} \lambda_i \langle x^*, u_i \rangle + \langle x^*, v \rangle \geq \sum_{i=1}^{k} \lambda_i \langle x^*, u_{i_0} \rangle = \langle x^*, u_{i_0} \rangle.$$

Since $x \in D$ can be chosen arbitrarily, u_{i_0} must be a solution of (LP). □

Remark 1.7 If $\langle x^*, v \rangle \geq 0$ for every $v \in 0^+D$, then one says that the functional x^* is *copositive* on the recession 0^+D. We called Theorem 1.11 the Eaves-type existence theorem in linear optimization to trace back Eaves' idea [21, Theorem 3 and Corollary 4, p. 702] (see also [49, Theorem 2.2]) in using recession cones for existence theorems in quadratic programming.

Theorem 1.12 (The Frank–Wolfe-type existence theorem; see [14, Theorem 2.199]) *If D is nonempty, then* (LP) *has a solution if and only if there is a real number γ such that $\langle x^*, x \rangle \geq \gamma$ for every $x \in D$.*

Proof. The necessity is obvious. To prove the sufficiency, suppose that there is a $\gamma \in \mathbb{R}$ such that $\langle x^*, x \rangle \geq \gamma$ for all $x \in D$. Then, for any $v \in 0^+D$ and $x \in D$ we have

$$\gamma \leq \langle x^*, x + tv \rangle = \langle x^*, x \rangle + t\langle x^*, v \rangle$$

for every $t > 0$. It follows that $\langle x^*, v \rangle \geq 0$ for any $v \in 0^+D$. So, by Theorem 1.11, we can assert that (LP) has a solution. □

Remark 1.8 Due to the formulation of the existence theorem in quadratic programming of Frank and Wolfe [28, p. 158] (see also [49, Theorem 2.1]), we called Theorem 1.12 the Frank–Wolfe-type existence theorem in linear optimization.

We are interested in studying the region G of all x^* for which (LP) has a nonempty solution set, assuming that the constraint set D is nonempty and fixed.

Proposition 1.11 *If D has the form* (1.6)*, then G is a generalized polyhedral convex cone of X^* which has the representation*

$$G = X_0^\perp \cap \{x^* \in X^* \mid \langle x^*, v_j \rangle \geq 0,\ j = 1, \ldots, \ell\}. \tag{1.35}$$

Proof. By Theorem 1.11,

$$G = \{x^* \in X^* \mid \langle x^*, v \rangle \geq 0,\ \forall v \in 0^+D\}.$$

Therefore, given any $x^* \in G$, we have $\langle x^*, v\rangle \geq 0$ for all $v \in 0^+D$. Hence, for every $x_0 \in X_0 \subset 0^+D$ one has $\langle x^*, x_0\rangle = 0$ because $-x_0 \in 0^+D$. Thus $x^* \in X_0^{\perp}$. In addition, for each $j = 1, \ldots, \ell$, one has $\langle x^*, v_j\rangle \geq 0$ as $v_j \in 0^+D$. This establishes the inclusion "$\subset$" in (1.35).

Conversely, suppose that $x^* \in X_0^{\perp} \cap \{x^* \in X^* \mid \langle x^*, v_j\rangle \geq 0,\ j = 1, \ldots, \ell\}$. Since $0^+D = \text{cone}\{v_1, \ldots, v_\ell\} + X_0$, the last inclusion implies that $\langle x^*, v\rangle \geq 0$, for every $v \in 0^+D$. Hence, by Theorem 1.11 we can conclude that $x^* \in G$. The inclusion "$\supset$" in (1.35) has been proved.

From (1.35) it follows that G is a generalized polyhedral convex set. The fact that G is a cone is obvious. □

Next, for each $x^* \in G$, we want to describe the solution set of (LP), which is denoted by $S(x^*)$. For doing so, let us suppose that D is given by (1.6) and consider the index sets

$$I(x^*) := \{i_0 \in \{1, \ldots, k\} \mid \langle x^*, u_{i_0}\rangle \leq \langle x^*, u_i\rangle\ \ \forall i = 1, \ldots, k\},$$

and

$$J(x^*) := \{j_0 \in \{1, \ldots, \ell\} \mid \langle x^*, v_{j_0}\rangle = 0\}.$$

Note that $I(x^*)$ is nonempty, but it may happen that $J(x^*)$ is empty.

Proposition 1.12 *If $x^* \in G$ and D is given by (1.6), then*

$$S(x^*) = \Bigg\{ \sum_{i \in I(x^*)} \lambda_i u_i + \sum_{j \in J(x^*)} \mu_j v_j \mid\ \lambda_i \geq 0\ \ \forall i \in I(x^*), \quad \sum_{i \in I(x^*)} \lambda_i = 1,\ \mu_j \geq 0\ \ \forall j \in J(x^*) \Bigg\} + X_0. \tag{1.36}$$

In particular, $S(x^*)$ is a generalized polyhedral convex set.

Proof. First, take an arbitrary element $\bar{x}$ from the set on the right-hand side of (1.36). Let

$$\bar{x} = \sum_{i \in I(x^*)} \bar{\lambda}_i u_i + \sum_{j \in J(x^*)} \bar{\mu}_j v_j + \bar{x}_0,$$

where $\bar{\lambda}_i \geq 0$ for all $i \in I(x^*)$, $\sum\limits_{i \in I(x^*)} \bar{\lambda}_i = 1$, $\bar{\mu}_j \geq 0$ for all $j \in J(x^*)$, and $\bar{x}_0 \in X_0$. By (1.6), for each $x \in D$ one can find $\lambda_i \geq 0$ for $i = 1, \ldots, k$, $\sum\limits_{i=1}^{k} \lambda_i = 1$, $\mu_j \geq 0$ for $j = 1, \ldots, \ell$, and $x_0 \in X_0$ such that

$$x = \sum_{i=1}^{k} \lambda_i u_i + \sum_{j=1}^{\ell} \mu_j v_j + x_0.$$

By Proposition 1.11, $\langle x^*, x_0 \rangle = \langle x^*, \bar{x}_0 \rangle = 0$. If $J(x^*) \neq \emptyset$, then using Theorem 1.11 and formula $0^+D = \text{cone}\{v_1, \ldots, v_\ell\} + X_0$ we get

$$\left\langle x^*, \sum_{j=1}^{\ell} \mu_j v_j \right\rangle \geq 0 = \left\langle x^*, \sum_{j \in J(x^*)} \bar{\mu}_j v_j \right\rangle.$$

Now, selecting an index $i_0 \in I(x^*)$ and recalling the definition $I(x^*)$, we get

$$\begin{aligned}\left\langle x^*, \sum_{i=1}^{k} \lambda_i u_i \right\rangle \geq \sum_{i=1}^{k} \lambda_i \langle x^*, u_{i_0} \rangle &= \langle x^*, u_{i_0} \rangle \\ &= \left\langle x^*, \sum_{i \in I(x^*)} \bar{\lambda}_i u_i \right\rangle.\end{aligned}$$

It follows that $\langle x^*, x \rangle \geq \langle x^*, \bar{x} \rangle$. We have shown that $\bar{x} \in S(x^*)$.

Second, take any vector $\bar{x} \in S(x^*)$ and represent it in the form

$$\bar{x} = \sum_{i=1}^{k} \bar{\lambda}_i u_i + \sum_{j=1}^{\ell} \bar{\mu}_j v_j + \bar{x}_0,$$

where $\bar{\lambda}_i \geq 0$ for $i = 1, \ldots, k$, $\sum_{i=1}^{k} \bar{\lambda}_i = 1$, $\bar{\mu}_j \geq 0$ for $j = 1, \ldots, \ell$, and $\bar{x}_0 \in X_0$. It is easy to show that $\bar{\lambda}_i = 0$ for all $i \notin I(x^*)$ and $\bar{\mu}_j = 0$ for all $j \notin J(x^*)$. This implies that $\bar{x}$ belongs to the set on the right-hand side of (1.36).

The proof is complete. □

1.8 Conclusions

We have studied basic properties of generalized polyhedral convex sets in locally convex Hausdorff topological vector spaces. Adopting an approach very different from that of Zheng, we have obtained a representation formula for generalized polyhedral convex sets in locally convex Hausdorff topological vector spaces, which is a comprehensive infinite-dimensional analogue of the celebrated theorem of Minkowski and Weyl. In this chapter, the formula has been used for proving solution existence theorems in generalized linear programming. We have shown that a generalized polyhedral convex set can be characterized via the finiteness of the number of its faces. Our results can be considered as adequate extensions of the corresponding classical results on polyhedral convex sets in [63, Section 19].

Chapter 2

Generalized Polyhedral Convex Functions

As the title indicates, the present chapter deals with the concept of generalized polyhedral convex functions. The latter is based on the notion of generalized polyhedral convex set, which has been considered in details in Chapter 1.

This chapter is written on the basis of the paper [A3] in the List of Author's Related Papers on page 113.

2.1 Generalized Polyhedral Convex Function as a Maximum of Finitely Many Affine Functions

Let f be a function from a locally convex Hausdorff topological vector space X to $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. The *effective domain* and the *epigraph* of f are defined, respectively, by setting

$$\operatorname{dom} f = \{x \in X \mid f(x) < +\infty\}$$

and $\operatorname{epi} f = \big\{(x, \alpha) \in X \times \mathbb{R} \mid x \in \operatorname{dom} f,\ f(x) \leq \alpha\big\}$. If $\operatorname{dom} f$ is nonempty and $f(x) > -\infty$ for all $x \in X$, then f is said to be *proper*. We say that f is *convex* if $\operatorname{epi} f$ is a convex set in $X \times \mathbb{R}$. It is easily verified that f is convex if and only if $\operatorname{dom} f$ is convex and the Jensen inequality

$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2)$$

is valid for any x_1, x_2 in $\operatorname{dom} f$ and $t \in (0, 1)$.

According to Rockafellar [63, p. 172], a real-valued function defined on $\mathbb{R}^n$ is called polyhedral convex if its epigraph is a polyhedral convex set in $\mathbb{R}^{n+1}$. The following notion of generalized polyhedral convex function appears naturally in that spirit.

Definition 2.1 Let X be a locally convex Hausdorff topological vector space. A function $f : X \to \bar{\mathbb{R}}$ is called *generalized polyhedral convex* (resp., *polyhedral convex*) if its epigraph is a generalized polyhedral convex set (resp., a polyhedral convex set) in $X \times \mathbb{R}$. If $-f$ is a generalized polyhedral convex function (resp., a polyhedral convex function), then f is said to be a *generalized polyhedral concave function* (resp., a *polyhedral concave function*).

The notion of generalized polyhedral convex function was introduced by Bonnans and Shapiro (see [14, p. 144]).

Complete characterizations of a generalized polyhedral convex function (resp., a polyhedral convex function) in the form of the maximum of a finite family of continuous affine functions over a certain generalized polyhedral convex set (resp., a polyhedral convex set) are given in next theorem.

Theorem 2.1 *Suppose that $f : X \to \bar{\mathbb{R}}$ is a proper function. Then f is generalized polyhedral convex (resp., polyhedral convex) if and only if* $\operatorname{dom} f$ *is a generalized polyhedral convex set (resp., a polyhedral convex set) in X and there exist $v_k^* \in X^*$, $\beta_k \in \mathbb{R}$, for $k = 1, \ldots, m$, such that*

$$f(x) = \begin{cases} \max\left\{\langle v_k^*, x\rangle + \beta_k \mid k = 1, \ldots, m\right\} & \text{if } x \in \operatorname{dom} f, \\ +\infty & \text{if } x \notin \operatorname{dom} f. \end{cases} \tag{2.1}$$

Proof. Let $f : X \to \bar{\mathbb{R}}$ be a proper function.

($\Longrightarrow$) Suppose that f is a generalized polyhedral convex function. Then one can find a closed affine subspace $L \subset X \times \mathbb{R}$, $u_i^* \in X^*$, $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}$, for $i = 1, \ldots, p$, such that

$$\operatorname{epi} f = \left\{(x,t) \in L \mid \langle u_i^*, x\rangle + a_i t \le b_i,\ i = 1, \ldots, p\right\}. \tag{2.2}$$

By [14, Remark 2.196], one can find a continuous linear mapping $\widetilde{A}$ from $X \times \mathbb{R}$ to a locally convex Hausdorff topological vector space Y and $y \in Y$ so that

$$L = \left\{(x,t) \in X \times \mathbb{R} \mid \widetilde{A}(x,t) = y\right\}.$$

Let the continuous linear mapping $A : X \to Y$ be defined by $A(x) = \widetilde{A}(x, 0)$. For $y_0 := \widetilde{A}(0, 1)$, we see that

$$\widetilde{A}(x,t) = \widetilde{A}(x,0) + t\widetilde{A}(0,1) = A(x) + ty_0 \quad (x \in X,\ t \in \mathbb{R}). \tag{2.3}$$

Given any $(\bar{x}, \bar{t}) \in \operatorname{epi} f$, it holds $(\bar{x}, \bar{t}+\gamma) \in \operatorname{epi} f$ for all $\gamma \geq 0$. In particular, $(\bar{x}, \bar{t} + \gamma) \in L$ for all $\gamma \geq 0$. So we have

$$y = A(\bar{x}) + (\bar{t} + \gamma)y_0 = (A(\bar{x}) + \bar{t}y_0) + \gamma y_0 = y + \gamma y_0$$

for all $\gamma \geq 0$. It follows that $y_0 = 0$. Substituting $(x, t) = (\bar{x}, \bar{t} + \gamma)$ into the inequalities in (2.2) yields $a_i \leq 0$ for $i = 1, \ldots, p$. There exists an index $i \in \{1, \ldots, p\}$ satisfying $a_i < 0$. Indeed, suppose on the contrary that $a_i = 0$ for $i = 1, \ldots, p$. By the properness of f, there exists $\bar{x} \in X$ with $|f(\bar{x})| < \infty$. Then $(\bar{x}, f(\bar{x})) \in \operatorname{epi} f \subset L$. As $y_0 = 0$, from (2.3) it follows that

$$y = \widetilde{A}(\bar{x}, f(\bar{x})) = A(\bar{x}).$$

Moreover, for any $t \in \mathbb{R}$, combining this with (2.3) one has $\widetilde{A}(\bar{x}, t) = y$. Hence $(\bar{x}, t) \in L$ for all $t \in \mathbb{R}$. Since

$$\langle u_i^*, \bar{x} \rangle + a_i t = \langle u_i^*, \bar{x} \rangle + a_i f(\bar{x}) \leq b_i$$

for $i = 1, \ldots, p$, we see that $(\bar{x}, t) \in \operatorname{epi} f$ for all $t \in \mathbb{R}$. Then $f(\bar{x}) = -\infty$. We have thus arrived at a contradiction.

For each $i \in \{1, \ldots, p\}$ with $a_i < 0$, we replace the inequality

$$\langle u_i^*, x \rangle + a_i t \leq b_i$$

by the following equivalent one:

$$\left\langle \frac{1}{|a_i|} u_i^*, x \right\rangle - t \leq \frac{b_i}{|a_i|}.$$

Then, reordering the family $\{a_1, \ldots, a_p\}$ (if necessary), we may assume that $a_k = -1$ for $k = 1, \ldots, m$, with $m \leq p$, and $a_i = 0$ for $i = m + 1, \ldots, p$. It follows that

$$\begin{aligned} \operatorname{epi} f = \big\{(x,t) \in X \times \mathbb{R} \mid A(x) = y,\ \langle u_k^*, x \rangle - b_k \leq t,\ k = 1, \ldots, m, \\ \langle u_i^*, x \rangle \leq b_i,\ i = m + 1, \ldots, p\big\}. \end{aligned} \tag{2.4}$$

This implies that

$$\operatorname{dom} f = \{x \in X \mid A(x) = y, \langle u_i^*, x \rangle \leq b_i,\ i = m + 1, \ldots, p\}. \tag{2.5}$$

In particular, $\operatorname{dom} f$ is a generalized polyhedral convex set in X. Combining (2.4) with (2.5) gives

$$\operatorname{epi} f = \big\{(x,t) \in X \times \mathbb{R} \mid x \in \operatorname{dom} f,\ \langle u_k^*, x\rangle - b_k \le t,\ k = 1,\dots,m\big\}.$$

So, f can be represented in the form (2.1) with $v_k^* := u_k^*$, $\beta_k := -b_k$ for $k = 1,\dots,m$.

In addition, if f is a polyhedral convex function on X, then we may assume that $\operatorname{epi} f$ is of the form (2.2), where $L = X \times \mathbb{R}$. In this case, we can repeat the above proof with $Y := \{0\}$ (the trivial space), $\widetilde{A}(x,t) \equiv 0$ and $y = 0$. Hence, it follows from (2.4) that f admits the representation (2.1) with $\operatorname{dom} f$ being a polyhedral convex set in X.

($\Longleftarrow$) Suppose that $\operatorname{dom} f$ is a generalized polyhedral convex set in X and f is given by (2.1). Then there exist $x_i^* \in X^*$, $\alpha_i \in \mathbb{R}$, $i = 1,\dots,p$, a continuous linear mapping B from X to a locally convex Hausdorff topological vector space Z, and a vector $z \in Z$ such that

$$\operatorname{dom} f = \big\{x \in X \mid B(x) = z,\ \langle x_i^*, x\rangle \le \alpha_i,\ i = 1,\dots,p\big\}. \tag{2.6}$$

Combining this with (2.1), we obtain

$$\begin{aligned}\operatorname{epi} f &= \big\{(x,t) \in X \times \mathbb{R} \mid B(x) = z,\ \langle x_i^*, x\rangle \le \alpha_i,\ i = 1,\dots,p,\\ &\qquad\qquad\qquad\qquad \langle v_k^*, x\rangle + \beta_k \le t,\ k = 1,\dots,m\big\}\\ &= \big\{(x,t) \in X \times \mathbb{R} \mid B(x) + 0t = z, \langle x_i^*, x\rangle + 0t \le \alpha_i,\ i = 1,\dots,p,\\ &\qquad\qquad\qquad\qquad \langle v_k^*, x\rangle - t \le -\beta_k,\ k = 1,\dots,m\big\}.\end{aligned} \tag{2.7}$$

This clearly shows that $\operatorname{epi} f$ is a generalized polyhedral convex set in $X \times \mathbb{R}$; hence f is a generalized polyhedral convex function.

Finally, let us assume that $\operatorname{dom} f$ is a polyhedral convex set in X and f is given by (2.1). Then, in the formula (2.6) for $\operatorname{dom} f$, we can choose $Z = \{0\}$, $B \equiv 0$, and $z = 0$. Clearly, with the chosen B and z, (2.7) implies that $\operatorname{epi} f$ is a polyhedral convex set; so f is polyhedral convex. □

Remark 2.1 For the case $X = \mathbb{R}^n$, the result in Theorem 2.1 is a known one (see [63, p. 172], [64, Theorem 2.49], [12, Proposition 3.2.3]). In the above proof, we have used some ideas of the proof of [12, Proposition 3.2.3].

2.2 Piecewise Linearity of Generalized Polyhedral Convex Functions and an Application

We will need the following infinite-dimensional generalization of the concept of piecewise linear function on $\mathbb{R}^n$ of [64].

Definition 2.2 A proper function $f : X \to \bar{\mathbb{R}}$, which is defined on a locally convex Hausdorff topological vector space, is said to be *generalized piecewise linear* (resp., *piecewise linear*) if there exist generalized polyhedral convex sets (resp., polyhedral convex sets) $D_1, \ldots, D_m$ in X, $v_1^*, \ldots, v_m^* \in X^*$, and $\beta_1, \ldots, \beta_m \in \mathbb{R}$ such that $\operatorname{dom} f = \bigcup_{k=1}^{m} D_k$ and $f(x) = \langle v_k^*, x\rangle + \beta_k$ for all $x \in D_k$, $k = 1, \ldots, m$.

Remark 2.2 By using the definition, it is not difficult to show that the sum a finite family of generalized piecewise linear functions (resp., a finite family of piecewise linear functions) is a generalized piecewise linear function (resp., a piecewise linear function).

Theorem 2.1 provides us with a general formula for any generalized polyhedral convex function on a locally convex Hausdorff topological vector space. For polyhedral convex functions on $\mathbb{R}^n$, there is another important characterization [64, Theorem 2.49]: *A proper convex function f is polyhedral convex if and only if f is piecewise linear.* It is of interest to obtain analogous results for generalized polyhedral convex functions and polyhedral convex functions on a locally convex Hausdorff topological vector space.

The forthcoming theorem clarifies the relationships between generalized polyhedral convex functions and generalized piecewise linear functions.

Theorem 2.2 *Suppose that $f : X \to \bar{\mathbb{R}}$ is a proper convex function. Then the function f is generalized polyhedral convex (resp., polyhedral convex) if and only if f is generalized piecewise linear (resp., piecewise linear).*

Proof. First, suppose that f is generalized polyhedral convex (resp., polyhedral convex). By Theorem 2.1, $\operatorname{dom} f$ is a generalized polyhedral convex set (resp., a polyhedral convex set) and there exist $v_k^* \in X^*$, $\beta_k \in \mathbb{R}$, $k = 1, \ldots, m$, such that

$$f(x) = \max\{\langle v_k^*, x\rangle + \beta_k \mid k = 1, \ldots, m\}$$

for all $x \in \operatorname{dom} f$. For each $k = 1, \ldots, m$, set

$$\begin{aligned} D_k &= \operatorname{dom} f \cap \big\{x \in X \mid \langle v_i^*, x\rangle + \beta_i \le \langle v_k^*, x\rangle + \beta_k,\ \forall i = 1, \ldots, m\big\} \\ &= \operatorname{dom} f \cap \big\{x \in X \mid \langle v_i^* - v_k^*, x\rangle \le \beta_k - \beta_i,\ \forall i = 1, \ldots, m\big\}. \end{aligned}$$

Observe that $\operatorname{dom} f = \bigcup\limits_{k=1}^{m} D_k$ and $f(x) = \langle v_k^*, x\rangle + \beta_k$ for every $x \in D_k$, $k = 1, \ldots, m$. Since $\operatorname{dom} f$ is a generalized polyhedral convex set (resp., a polyhedral convex set), D_k is also a generalized polyhedral convex set (resp., a polyhedral convex set). It follows that f is a generalized piecewise linear function (resp., a piecewise linear function).

Now, suppose that f is generalized piecewise linear (resp., piecewise linear). Then, one can find generalized polyhedral convex sets (resp., polyhedral convex sets) $D_1, \ldots, D_m$ in X, $v_1^*, \ldots, v_m^* \in X^*$, and $\beta_1, \ldots, \beta_m \in \mathbb{R}$ such that $\operatorname{dom} f = \bigcup\limits_{k=1}^{m} D_k$ and $f(x) = \langle v_k^*, x\rangle + \beta_k$ for all $x \in D_k$, $k = 1, \ldots, m$. It follows that $\operatorname{epi} f = \bigcup\limits_{k=1}^{m} E_k$, where

$$E_k := \big\{(x, t) \in X \times \mathbb{R} \mid x \in D_k,\ \langle v_k^*, x\rangle + \beta_k \le t\big\} \quad (k = 1, \ldots, m).$$

So, for each $k = 1, \ldots, m$, E_k is the intersection of the generalized polyhedral convex set (resp., the polyhedral convex set) $D_k \times \mathbb{R}$ and the polyhedral convex set $\big\{(x, t) \in X \times \mathbb{R} \mid \langle v_k^*, x\rangle + \beta_k \le t\big\}$. In particular, E_k is a generalized polyhedral convex set (resp., a polyhedral convex set). The convexity of f shows that $\operatorname{epi} f$ is convex. Combining this with the fact that $\operatorname{epi} f$ is the union of the generalized polyhedral convex sets (resp., the polyhedral convex sets) $E_1, \ldots, E_m$, we conclude by Corollary 1.2 that the set $\operatorname{epi} f$ is generalized polyhedral convex (resp., polyhedral convex). Thus f is a generalized polyhedral convex function (resp., a polyhedral convex function).

The proof is complete. □

Based on Theorem 2.2, we can prove that the class of generalized polyhedral convex functions (resp., the class of polyhedral convex functions) is invariant w.r.t. the addition of functions.

Theorem 2.3 *Let f_1, f_2 be two proper functions on X. If f_1, f_2 are generalized polyhedral convex (resp., polyhedral convex) and $(\operatorname{dom} f_1) \cap (\operatorname{dom} f_2)$ is nonempty, then $f_1 + f_2$ is a proper generalized polyhedral convex function (resp., a polyhedral convex function).*

Proof. Suppose that f_1, f_2 are proper generalized polyhedral convex functions (resp., proper polyhedral convex functions) defined on the space X with $(\mathrm{dom}\, f_1) \cap (\mathrm{dom}\, f_2) \neq \emptyset$. Then, $f_1 + f_2$ is a proper convex function. Due to Theorem 2.2, f_1 and f_2 are generalized piecewise linear (resp., piecewise linear); hence $f_1 + f_2$ is generalized piecewise linear (resp., piecewise linear) by Remark 2.2. So, according to Theorem 2.2, the function $f_1 + f_2$ is generalized polyhedral convex (resp., polyhedral convex). □

Remark 2.3 For the case $X = \mathbb{R}^n$, the result in Theorem 2.3 is a known one (see [63, Theorem 19.4]).

2.3 Directional Derivatives

In convex analysis, it is well known [40, 63] that the concept of directional derivative has an important role. We are going to discuss a property of the directional derivative mapping of a generalized polyhedral convex function (resp., a polyhedral convex function) at a given point.

If $f : X \to \bar{\mathbb{R}}$ is a proper convex function and $x \in X$ is such that $f(x)$ is finite, the *directional derivative*

$$f'(x; h) := \lim_{t \to 0^+} \frac{f(x + th) - f(x)}{t}$$

of f at x w.r.t. a direction $h \in X$, always exists (it can take values $-\infty$ or $+\infty$), and one has

$$f'(x; h) = \inf_{t > 0} \frac{f(x + th) - f(x)}{t}. \tag{2.8}$$

(The situation $f'(x; h) = -\infty$ may occur. To have a simple example, one can choose $f(x) = -\sqrt{2x - x^2}$ for $x \in [0, 2]$, $f(x) = +\infty$ otherwise, and note that $f'(0; 1) = -\infty$.) For the case where $X = \mathbb{R}^n$, the proof can be found in [63, Theorem 23.1]. For the case where X is a locally convex Hausdorff topological vector space, the fact has been discussed, e.g., in [14, pp. 48–49], [40, Proposition 3, p. 194] and [78, Theorem 2.1.13]. According to [14, Proposition 2.60], the closure of the epigraph of $f'(x; \cdot)$ coincides with the tangent cone to $\mathrm{epi}\, f$ at $(x, f(x))$, i.e.,

$$\overline{\mathrm{epi}\, f'(x; \cdot)} = T_{\mathrm{epi}\, f}(x, f(x)). \tag{2.9}$$

For proving the generalized polyhedral convexity of the directional derivative mapping of a proper generalized polyhedral convex function at a reference point, we need next lemma, which can be proved by using definition of tangent cone. When X is a Banach space, the forthcoming formula for $T_D(x)$ has been given in [7, Proposition 3.1].

Lemma 2.1 *Suppose that a subset $D \subset X$ is the union of nonempty closed convex sets $D_1, \ldots, D_m$. Then, one has $T_D(x) = \bigcup_{k \in J(x)} T_{D_k}(x)$ for every $x \in D$, where $J(x) := \{j \in \{1, \ldots, m\} \mid x \in D_j\}$.*

By [63, Theorem 23.10] we know that if $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is proper polyhedral convex, then the closure sign in (2.9) can be omitted and $f'(x; \cdot)$ is a proper polyhedral convex function. The last two facts can be extended to polyhedral convex functions on locally convex Hausdorff topological vector spaces and generalized polyhedral convex functions as follows.

Theorem 2.4 *Let f be a proper generalized polyhedral convex function (resp., a proper polyhedral convex function) on a locally convex Hausdorff topological vector space X. For any $x \in \operatorname{dom} f$, $f'(x; \cdot)$ is a proper generalized polyhedral convex function (resp., a proper polyhedral convex function). In particular,* $\operatorname{epi} f'(x; \cdot)$ *is closed and, by* (2.9) *one has*

$$\operatorname{epi} f'(x; \cdot) = T_{\operatorname{epi} f}(x, f(x)). \tag{2.10}$$

Proof. Let f be given as in the formulation of the theorem. Due to the "only if" part of Theorem 2.2, one can find generalized polyhedral convex sets (resp., polyhedral convex sets) $D_1, \ldots, D_m$ in X, $v_1^*, \ldots, v_m^* \in X^*$, and $\beta_1, \ldots, \beta_m \in \mathbb{R}$ such that $\operatorname{dom} f = \bigcup_{k=1}^{m} D_k$ and $f(x) = \langle v_k^*, x \rangle + \beta_k$ for all $x \in D_k$, $k = 1, \ldots, m$. We may assume that $D_k \neq \emptyset$ for all $k \in \{1, \ldots, m\}$. Given any $x \in \operatorname{dom} f$, $f(x)$ is finite because f is proper. By [40, p. 195], $f'(x; \cdot)$ is convex.

If $h \notin T_{\operatorname{dom} f}(x)$, then $x + th \notin \operatorname{dom} f$ for every $t > 0$; so $f'(x, h) = +\infty$.

If $h \in T_{\operatorname{dom} f}(x)$, then by Lemma 2.1 one can find D_k such that $x \in D_k$ and $h \in T_{D_k}(x)$. In addition, since D_k is a generalized polyhedral convex set (resp., a polyhedral convex set), $T_{D_k}(x) = \operatorname{cone}(D_k - x)$ by Proposition 1.9. Hence, there exists $\delta > 0$ satisfying $x + \delta h \in D_k$. As D_k is convex, one has

$x + th \in D_k$ for all $t \in [0, \delta]$. It follows that

$$\frac{f(x+th) - f(x)}{t} = \frac{(\langle v_k^*, x+th\rangle + \beta_k) - (\langle v_k^*, x\rangle + \beta_k)}{t} = \langle v_k^*, h\rangle$$

for every $t \in (0, \delta]$; so $f'(x; h) = \langle v_k^*, h\rangle$. Note that, if $h \in T_{D_{k_1}}(x) \cap T_{D_{k_2}}(x)$, then $\langle v_{k_1}^*, h\rangle = \langle v_{k_2}^*, h\rangle$. Indeed, one can find positive numbers δ_1, δ_2 such that $x + th \in D_{k_1}$ for all $t \in [0, \delta_1]$ and $x + th \in D_{k_2}$ for all $t \in [0, \delta_2]$. Setting $\delta = \min\{\delta_1, \delta_2\}$, we have $x + th \in D_{k_1} \cap D_{k_2}$ for every $t \in [0, \delta]$. Then

$$f(x+th) = \langle v_{k_1}^*, x+th\rangle + \beta_{k_1} = \langle v_{k_2}^*, x+th\rangle + \beta_{k_2} \quad (t \in [0, \delta]). \qquad (2.11)$$

In particular, $\langle v_{k_1}^*, x\rangle + \beta_{k_1} = \langle v_{k_2}^*, x\rangle + \beta_{k_2}$. So, (2.11) implies $\langle v_{k_1}^*, h\rangle = \langle v_{k_2}^*, h\rangle$.

We have shown that $f'(x; h)$ is well defined and finite for all $h \in T_{\mathrm{dom}\, f}(x)$. Consequently, the function $f'(x; \cdot)$ is proper and $\mathrm{dom}\, f'(x; \cdot) = T_{\mathrm{dom}\, f}(x)$. Moreover, applying Lemma 2.1 for $D = \mathrm{dom}\, f$, we can assert that $T_{\mathrm{dom}\, f}(x)$ is the union of the generalized polyhedral convex cones (resp., the union of the polyhedral convex cones) $T_{D_k}(x)$, $k \in J(x)$, and $f'(x; h) = \langle v_k^*, h\rangle$ if $h \in T_{D_k}(x)$ with $k \in J(x)$. This implies that the proper convex function $f'(x; \cdot)$ is generalized piecewise linear (resp., piecewise linear). Hence, by the "if" part of Theorem 2.2, $f'(x; \cdot)$ is proper generalized polyhedral convex (resp., proper polyhedral convex). Therefore, $\mathrm{epi}\, f'(x; \cdot)$ is closed and (2.10) follows from (2.9). □

2.4 Infimal Convolutions

In this section, we are interested in the concept of infimal convolution function, which was introduced by Fenchel [26] and discussed by many other authors (see, e.g., Rockafellar [63], Ioffe and Tihomirov [40], Attouch and Wets [4], Strömberg [68, 69]). According to Rockafellar [63, p. 34], the infimal convolution operation is analogous to the classical formula for integral convolution and, in a sense, is dual to the operation of addition of convex functions.

As noted by Nam [56, p. 2215] and Nam and Cuong [57, pp. 333–334], a large spectrum of known nonsmooth functions can be interpreted as infimal convolutions. In the above cited papers, the authors have obtained some upper estimates for three types of subdifferentials of a class of nonconvex infimal convolutions.

Although the infimal convolution of a finite family of functions can be defined [40, 63], for simplicity, we will only consider the infimal convolution of two functions. By induction, one can easily extends the result obtained in Proposition 2.1 below to infimal convolutions of finite families of generalized polyhedral convex functions, provided that one of them is polyhedral convex.

Definition 2.3 (See [40, p. 168] and [63, p. 34]) Let f_1, f_2 be two proper functions on a locally convex Hausdorff topological vector space X. The *infimal convolution* of f_1, f_2 is the function defined by

$$(f_1 \Box f_2)(x) := \inf \{ f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x \} . \tag{2.12}$$

If f_1, f_2 are proper convex, then $f_1 \Box f_2$ is convex (see, e.g., [78, p. 43]). However, if f_1, f_2 are proper, $f_1 \Box f_2$ may be not proper. For instance, choosing $X = \mathbb{R}$, $f_1(x) = x$ and $f_2(x) = 2x$, one has $(f_1 \Box f_2)(x) = -\infty$ for all $x \in X$.

The infimal convolution operation in (2.12) corresponds to the addition of the epigraphs of f_1 and f_2 as sets in $X \times \mathbb{R}$. Namely, as noted in [63, p. 34],

$$(f_1 \Box f_2)(x) = \inf \{ \alpha \mid (x, \alpha) \in \text{epi}\, f_1 + \text{epi}\, f_2 \} .$$

According to Proposition 1.6, the sum of a polyhedral convex set and a generalized polyhedral convex set is a polyhedral convex set. We will use this fact to prove the following proposition.

Proposition 2.1 *Let f_1, f_2 be two proper functions. If f_1 is polyhedral convex and f_2 is generalized polyhedral convex, then $f_1 \Box f_2$ is a polyhedral convex function.*

Proof. First, let us verify the inclusion $\text{epi}\, f_1 + \text{epi}\, f_2 \subset \text{epi}\, (f_1 \Box f_2)$. Pick any $(x_i, \alpha_i) \in \text{epi}\, f_i$, $i = 1, 2$. Then we have $f_1(x_1) + f_2(x_2) \leq \alpha_1 + \alpha_2$. Combining this with (2.12), we get $(f_1 \Box f_2)(x_1 + x_2) \leq \alpha_1 + \alpha_2$; hence

$$(x_1 + x_2, \alpha_1 + \alpha_2) \in \text{epi}\, (f_1 \Box f_2).$$

Now, to show that the inclusion $\text{epi}\, (f_1 \Box f_2) \subset \text{epi}\, f_1 + \text{epi}\, f_2$, select a point $(x, \alpha) \in \text{epi}\, (f_1 \Box f_2)$. For any $\varepsilon > 0$, since $(f_1 \Box f_2)(x) \leq \alpha$, there exist $x_1 \in \text{dom}\, f_1$ and $x_2 \in \text{dom}\, f_2$ such that $x_1 + x_2 = x$ and $f_1(x_1) + f_2(x_2) \leq \alpha + \varepsilon$. As $f_2(x_2) \leq \alpha + \varepsilon - f_1(x_1)$, one has

$$(x, \alpha + \varepsilon) = (x_1, f_1(x_1)) + (x_2, \alpha + \varepsilon - f_1(x_1)) \in \text{epi}\, f_1 + \text{epi}\, f_2.$$

So, letting $\varepsilon \to 0^+$ yields $(x, \alpha) \in \overline{\text{epi}\, f_1 + \text{epi}\, f_2}$. Since epi f_1 is a polyhedral convex set and epi f_2 is a generalized polyhedral convex set, epi f_1 + epi f_2 is a polyhedral convex set by Proposition 1.6. In particular, epi f_1 + epi f_2 is closed. Hence, $(x, \alpha) \in \text{epi}\, f_1 + \text{epi}\, f_2$.

We have thus proved that $\text{epi}\,(f_1 \square f_2) = \text{epi}\, f_1 + \text{epi}\, f_2$, where the set on the right-hand side is polyhedral convex. This means that $f_1 \square f_2$ is a polyhedral convex function. □

Remark 2.4 Proposition 2.1 is a generalization of [63, Corollary 19.3.4]), where the case $X = \mathbb{R}^n$ was treated. If X is a general locally convex Hausdorff topological vector space and f_1, f_2 are generalized polyhedral convex functions, $f_1 \square f_2$ may not be a generalized polyhedral convex function. To see this, one can choose a suitable space X and closed linear subspaces X_1, X_2 of X such that $\overline{X_1 + X_2} = X$ and $X_1 + X_2 \neq X$ (see Remark 1.4 for details). Let $f_i := \delta(\cdot, X_i)$ $(i = 1, 2)$ be the *indicator function* of X_i, i.e., $f_i(x) = 0$ for $x \in X_i$ and $f_i(x) = +\infty$ for $x \notin X_i$. Clearly, f_1 and f_2 are proper generalized polyhedral convex functions. An easy computation shows that $(f_1 \square f_2)(\cdot) = \delta(\cdot, X_1 + X_2)$ and $\text{epi}\,(f_1 \square f_2) = (X_1 + X_2) \times [0, +\infty)$. Since $X_1 + X_2$ is non-closed, $\text{epi}\,(f_1 \square f_2)$ is non-closed; hence $f_1 \square f_2$ cannot be a generalized polyhedral convex function.

2.5 Conclusions

We have studied basic properties of generalized polyhedral convex functions on locally convex Hausdorff topological vector spaces, and the related constructions such as sum of functions, directional derivative, infimal convolution. It is also proved that the infimal convolution of a generalized polyhedral convex function and a polyhedral convex function is a polyhedral convex function. Our results can be considered as adequate extensions of the corresponding classical results on polyhedral convex functions in [63, Section 19].

Chapter 3

Dual Constructions

Various properties of normal cones to and polars of generalized polyhedral convex sets, conjugates of generalized polyhedral convex functions, and subdifferentials of generalized polyhedral convex functions will be studied in this chapter, which is written on the basis of the paper [A3] in the List of Author's Related Papers on page 113.

3.1 Normal Cones

As before, X is a locally convex Hausdorff topological vector space and X^* is the dual space of X. According to [65, Theorem 3.10] (see also the property of the dual space described in [65, p. 65]), the weak*–topology turns X^* into a locally convex Hausdorff topological vector space whose dual space is X.

Now, suppose that $C \subset X$ is a nonempty convex set. The *normal cone* [40, p. 205] to C at $x \in C$ is the set

$$N_C(x) := \big\{x^* \in X^* \mid \langle x^*, u - x\rangle \le 0, \ \forall u \in C\big\}.$$

The formula $C^\perp := \{x^* \in X^* \mid \langle x^*, u\rangle = 0, \ \forall u \in C\}$ defines the *annihilator* [53, p. 117] of C. Note that $N_C(x)$ is a closed convex cone in X^*, while $C^\perp$ is a closed linear subspace of X^*. If C is a linear subspace of X, then $N_C(x)$ does not depend on x. Moreover, $N_C(x) = C^\perp$ for all $x \in C$.

In this chapter, if not otherwise stated, $D \subset X$ is a nonempty generalized polyhedral convex set given by (1.2). Let $I = \{1, \dots, p\}$. For every $x \in D$, we define the active index set $I(x) := \{i \in I \mid \langle x_i^*, x\rangle = \alpha_i\}$. If D is a polyhedral convex set, then one can choose $Y = \{0\}$, $A \equiv 0$, and $y = 0$.

Normal cones to a generalized polyhedral convex set also share the polyhedral structure.

Theorem 3.1 *If $D \subset X$ is a generalized polyhedral convex set and if $x \in D$, then $N_D(x)$ is a generalized polyhedral convex cone.*

Proof. Since $(\ker A)^\perp \subset X^*$ is a closed linear subspace, by using Theorem 1.4 we can assert that

$$Q_x := \text{cone}\,\{x_i^* \mid i \in I(x)\} + (\ker A)^\perp$$

is a generalized polyhedral convex cone of X^*. In particular, Q_x is convex and closed. To show that $Q_x \subset N_D(x)$, take any $x^* \in Q_x$. Then there exist $\lambda_i \geq 0$ for $i \in I(x)$ and $u^* \in (\ker A)^\perp$ such that $x^* = \sum\limits_{i \in I(x)} \lambda_i x_i^* + u^*$. For any $u \in D$, from (1.2) it follows that $\langle u^*, u - x\rangle = 0$, because $u - x \in \ker A$. Hence

$$\begin{aligned} \langle x^*, u - x\rangle &= \sum_{i \in I(x)} \lambda_i\big(\langle x_i^*, u\rangle - \langle x_i^*, x\rangle\big) + \langle u^*, u - x\rangle \\ &= \sum_{i \in I(x)} \lambda_i\big(\langle x_i^*, u\rangle - \alpha_i\big) \leq 0. \end{aligned}$$

The last inequality is clear as $\lambda_i \geq 0$ for all $i \in I(x)$ and $u \in D$. Hence $\langle x^*, u - x\rangle \leq 0$ for every $u \in D$; so $x^* \in N_D(x)$. We have thus proved that $Q_x \subset N_D(x)$. To obtain the opposite inclusion, take any $v^* \in X^* \setminus Q_x$. Since $\{v^*\} \cap Q_x = \emptyset$, by the strong separation theorem [65, Theorem 3.4(b)] we can find $v \in X$ such that

$$\langle v^*, v\rangle > \sup\{\langle x^*, v\rangle \mid x^* \in Q_x\}. \tag{3.1}$$

As the linear functional $\langle \cdot, v\rangle$ is bounded from the above on the generalized polyhedral convex set Q_x, by Theorem 1.12 we know that the linear programming problem $\max\{\langle x^*, v\rangle \mid x^* \in Q_x\}$ has a solution. Therefore, invoking Proposition 1.11, we have

$$v \in ((\ker A)^\perp)^\perp \cap \big\{x \in X \mid \langle x_i^*, x\rangle \leq 0,\, i \in I(x)\big\}.$$

Since the linear subspace $\ker A \subset X$ is closed, by [14, Proposition 2.40] one gets $((\ker A)^\perp)^\perp = \ker A$. Thus, $v \in \ker A$ and $\langle x_i^*, v\rangle \leq 0$ for all $i \in I(x)$. Since $0 \in Q_x$, (3.1) implies that $\langle v^*, v\rangle > 0$. Put $x_t = x + tv$, where $t > 0$, and note that

$$A(x_t) = A(x) + tA(v) = y.$$

If $i \in I(x)$, then for every $t > 0$,

$$\langle x_i^*, x_t \rangle = \langle x_i^*, x \rangle + t\langle x_i^*, v \rangle = \alpha_i + t\langle x_i^*, v \rangle \leq \alpha_i.$$

Since $\langle x_j^*, x \rangle < \alpha_j$ for any $j \in I \setminus I(x)$, one can find $t > 0$ such that

$$\langle x_j^*, x_t \rangle = \langle x_j^*, x \rangle + t\langle x_j^*, v \rangle < \alpha_j, \ \ \forall j \in I \setminus I(x).$$

Hence, for the chosen t, we have $x_t \in D$. Since $\langle v^*, x_t - x \rangle = t\langle v^*, v \rangle > 0$, it follows that $v^* \notin N_D(x)$. The inclusion $N_D(x) \subset Q_x$ has been proved. Thus $N_D(x) = Q_x$; so $N_D(x)$ is a generalized polyhedral convex cone. □

During the course of the proof of Theorem 3.1, we have obtained the following result.

Proposition 3.1 *Suppose that $D \subset X$ is a generalized polyhedral convex set given by* (1.2)*. Then, for any $x \in D$,*

$$N_D(x) = \text{cone}\,\{x_i^* \mid i \in I(x)\} + (\ker A)^{\perp}. \tag{3.2}$$

Remark 3.1 In a Banach space setting, formula (3.2) has been given in the proof of [7, Proposition 3.2].

In connection with Theorem 3.1, one may ask: *Given a polyhedral convex set $D \subset X$ and $x \in D$, whether $N_D(x)$ is a polyhedral convex set, or not?* An answer for that question is given in next statement.

Proposition 3.2 *Suppose that $D \subset X$ is a polyhedral convex set and $x \in D$. Then, $N_D(x)$ is a polyhedral convex cone in X^* if and only if X is finite-dimensional.*

Proof. Since D is a polyhedral convex set, D can be represented in the form (1.2) with $Y = \{0\}$, $A \equiv 0$, and $y = 0$. As $\ker A = X$ one has $(\ker A)^{\perp} = \{0\}$. So, by Proposition 3.1, $N_D(x) = \text{cone}\,\{x_i^* \mid i \in I(x)\}$. Applying Theorem 1.3 to $N_D(x)$ in X^*, we can assert that $N_D(x)$ is a polyhedral convex cone if and only if the linear subspace $\{0\}$ is of finite codimension in X^*, i.e., X^* is finite-dimensional. Since the dual space of any locally convex Hausdorff topological vector space of finite dimension is finite-dimensional [61, pp. 36–37], we have thus completed the proof. □

One has the following analogue of Proposition 3.1 for polyhedral convex sets.

Proposition 3.3 *Let $D \subset X$ be a polyhedral convex set of the form* (1.2)*, where $Y = \{0\}$, $A \equiv 0$, and $y = 0$. Then, for every $x \in D$,*

$$N_D(x) = \text{cone}\,\{x_i^* \mid i \in I(x)\}.$$

By the definition of normal cone, we have

$$N_{C_1}(x) + N_{C_2}(x) \subset N_{C_1 \cap C_2}(x), \tag{3.3}$$

for any $x \in C_1 \cap C_2$, where C_1, C_2 are convex subsets of X. The inclusion (3.3) holds with equality if $X = \mathbb{R}^n$, $\text{ri}\, C_1 \cap C_2 \neq \emptyset$ and C_2 is polyhedral convex (see [12, p. 267]), or X is a locally convex Hausdorff topological vector space and $\text{int}\, C_1 \cap C_2 \neq \emptyset$ (see [40, Proposition 1, p. 205]).

Before proving a property of normal cones to the intersection of generalized polyhedral convex set, we have to establish two lemmas.

Lemma 3.1 *Let C_1, C_2 be two subsets of a Hausdorff topological vector space Z. If $C_1 + \overline{C_2}$ is closed, then $C_1 + \overline{C_2} = \overline{C_1 + C_2}$.*

Proof. Since $C_1 + C_2 \subset C_1 + \overline{C_2}$ and since $C_1 + \overline{C_2}$ is a closed set, we see that $\overline{C_1 + C_2} \subset C_1 + \overline{C_2}$. Theorem 1.13(b) from [65] tells us that $\overline{C_1} + \overline{C_2} \subset \overline{C_1 + C_2}$; hence $C_1 + \overline{C_2} \subset \overline{C_1 + C_2}$. We have thus shown that $C_1 + \overline{C_2} = \overline{C_1 + C_2}$. □

Lemma 3.2 *Let M_1, M_2 be two closed linear subspaces of a locally convex Hausdorff topological vector space Z, whose dual space is Z^*. Then we have*

$$(M_1 \cap M_2)^\perp = \overline{M_1^\perp + M_2^\perp}. \tag{3.4}$$

Proof. Applying [14, formula (2.32), p. 32] with the closed convex cones being replaced by the closed linear subspaces M_1 and M_2 gives (3.4). □

Theorem 3.2 *Let D_1 and D_2 be two generalized polyhedral convex sets of X. For every $x \in D_1 \cap D_2$,*

$$N_{D_1 \cap D_2}(x) = \overline{N_{D_1}(x) + N_{D_2}(x)}. \tag{3.5}$$

Proof. For each $k \in \{1, 2\}$, since D_k is a generalized polyhedral convex set, there exist a continuous linear mapping A_k from X to a locally convex Hausdorff topological vector space Y_k, a point $y_k \in Y_k$, a finite index set I_k, $x_i^* \in X^*$ and $\alpha_i \in \mathbb{R}$ for $i \in I_k$, such that

$$D_k = \{x \in X \mid A_k(x) = y_k, \langle x_i^*, x\rangle \leq \alpha_i, \, i \in I_k\}.$$

(We assume that $I_1 \cap I_2 = \emptyset$.) For $I := I_1 \cup I_2$, one has

$$D_1 \cap D_2 = \{x \in X \mid A_k(x) = y_k, k = 1, 2, \langle x_i^*, x\rangle \le \alpha_i, \, i \in I\}.$$

For each $x \in D_1 \cap D_2$, put $I_k(x) = \{i \in I_k \mid \langle x_i^*, x\rangle = \alpha_i\}$ for $k = 1, 2$,

$$I(x) = \{i \in I \mid \langle x_i^*, x\rangle = \alpha_i\},$$

and note that $I(x) = I_1(x) \cup I_2(x)$. On one hand, by (3.2) we have

$$N_{D_1 \cap D_2}(x) = \text{cone}\,\{x_i^* \mid i \in I(x)\} + (\ker A_1 \cap \ker A_2)^\perp.$$

Since $(\ker A_1 \cap \ker A_2)^\perp = \overline{(\ker A_1)^\perp + (\ker A_2)^\perp}$ by Lemma 3.2, this implies that

$$N_{D_1 \cap D_2}(x) = \text{cone}\,\{x_i^* \mid i \in I(x)\} + \overline{(\ker A_1)^\perp + (\ker A_2)^\perp}. \tag{3.6}$$

On the other hand, applying Proposition 3.1 to both sets D_1 and D_2 for $x \in D_1 \cap D_2$, we get

$$\begin{aligned} N_{D_1}(x) + N_{D_2}(x) &= \big(\text{cone}\,\{x_i^* \mid i \in I_1(x)\} + (\ker A_1)^\perp\big) \\ &\quad + \big(\text{cone}\,\{x_i^* \mid i \in I_2(x)\} + (\ker A_2)^\perp\big) \\ &= \text{cone}\,\{x_i^* \mid i \in I(x)\} + (\ker A_1)^\perp + (\ker A_2)^\perp. \end{aligned} \tag{3.7}$$

Let $C_1 := \text{cone}\,\{x_i^* \mid i \in I(x)\}$ and $C_2 := (\ker A_1)^\perp + (\ker A_2)^\perp$ and observe that $C_1 + \overline{C_2}$ is a generalized polyhedral convex cone in X^* by Theorem 1.4. In particular, $C_1 + \overline{C_2}$ is closed. In accordance with Lemma 3.1

$$C_1 + \overline{C_2} = \overline{C_1 + C_2}.$$

In combination with (3.6) and (3.7), this equality justifies (3.5). □

Remark 3.2 One may ask: *Whether the closure sign in* (3.5) *can be omitted, or not?* If X is a finite-dimensional space, then $N_{D_1}(x)$ and $N_{D_2}(x)$ are polyhedral convex cones in the finite-dimensional X^*; hence $N_{D_1}(x) + N_{D_2}(x)$ is polyhedral convex by [63, Corollary 19.3.2]. Since $N_{D_1}(x) + N_{D_2}(x)$ is closed, the closure sign in (3.5) is superfluous. However, when X is an infinite-dimensional space, $N_{D_1}(x) + N_{D_2}(x)$ may be non-closed. To see this, one can choose an infinite-dimensional Hilbert space X and two suitable closed linear subspaces X_1, X_2 of X so that $\overline{X_1 + X_2} = X$ and $X_1 + X_2 \neq X$ (see [9, Example 3.34] for details). Let D_i be the orthogonal complement of X_i, i.e., $D_i = \{x \in X \mid \langle x, u\rangle = 0 \ \forall u \in X_i\}$, for $i = 1, 2$. It is clear that D_1, D_2 are generalized polyhedral convex sets in X and $D_1 \cap D_2 = \{0\}$. Since $N_{D_1}(0) = X_1$ and $N_{D_2}(0) = X_2$, we can assert that $N_{D_1}(0) + N_{D_2}(0)$ is non-closed.

In the proof of Theorem 3.2, if D_1 is a polyhedral convex set, then we can choose $Y_1 = \{0\}$, $A_1 \equiv 0$, and $y_1 = 0$. Since $(\ker A_1)^\perp = \{0\}$, one has $\overline{(\ker A_1)^\perp + (\ker A_2)^\perp} = (\ker A_2)^\perp$. Hence, (3.6) and (3.7) imply that $N_{D_1 \cap D_2}(x) = N_{D_1}(x) + N_{D_2}(x)$. Thus we have obtained the following result.

Theorem 3.3 *Suppose that $D_1 \subset X$ is a polyhedral convex set and $D_2 \subset X$ is a generalized polyhedral convex set. Then, for every $x \in D_1 \cap D_2$,*

$$N_{D_1 \cap D_2}(x) = N_{D_1}(x) + N_{D_2}(x).$$

3.2 Polars

Following [61, p. 34], we define the *polar* of a nonempty set C by

$$C^o := \{x^* \in X^* \mid \langle x^*, x\rangle \le 1, \ \forall x \in C\}.$$

Evidently, C^o is a weakly*-closed convex set containing 0. If C is a cone, then one has $C^o = N_C(0)$.

The forthcoming proposition extends [63, Corollary 19.2.2] to a locally convex Hausdorff topological vector spaces setting.

Proposition 3.4 *The polar of a nonempty generalized polyhedral convex set is a generalized polyhedral convex set.*

Proof. Suppose that $D \subset X$ is given by (1.6). Then we have

$$D^o = \{x^* \in X_0^\perp \mid \langle x^*, u_i\rangle \le 1, \ i = 1, \ldots, k, \ \langle x^*, v_j\rangle \le 0, \ j = 1, \ldots, \ell\}. \quad (3.8)$$

Indeed, take any $x^* \in D^o$. The inequalities $\langle x^*, u_i\rangle \le 1$, $i = 1, \ldots, k$, are valid, because $u_i \in D$. As the linear functional $\langle x^*, \cdot\rangle$ is bounded from the above on D, Theorem 1.12 shows that the linear programming problem

$$\max\{\langle x^*, x\rangle \mid x \in D\}$$

has a solution. Therefore, by Proposition 1.11, we have $x^* \in X_0^\perp$ and $\langle x^*, v_j\rangle \le 0$ for all $j = 1, \ldots, \ell$. The inclusion "$\subset$" in (3.8) has been proved. To obtain the opposite inclusion, take any x^* from the set on the right-hand side of (3.8). By (1.6), for each $x \in D$, there exist nonnegative numbers $\lambda_1, \ldots, \lambda_k$, $\mu_1, \ldots, \mu_\ell$, and a vector $x_0 \in X_0$, such that $\sum_{i=1}^{k} \lambda_i = 1$ and

$$x = \sum_{i=1}^{k} \lambda_i u_i + \sum_{j=1}^{\ell} \mu_j v_j + x_0.$$

Since

$$\begin{aligned}\langle x^*, x\rangle &= \sum_{i=1}^{k} \lambda_i \langle x^*, u_i\rangle + \sum_{j=1}^{\ell} \mu_j \langle x^*, v_j\rangle + \langle x^*, x_0\rangle \\ &\le \sum_{i=1}^{k} \lambda_i \langle x^*, u_i\rangle \le \sum_{i=1}^{k} \lambda_i = 1,\end{aligned}$$

we see that $\langle x^*, x\rangle \le 1$ for any $x \in D$; hence $x^* \in D^o$. This completes the proof of (3.8). The fact that D^o is a generalized polyhedral convex set in X^* follows from (3.8) and Definition 1.1. □

3.3 Conjugate Functions

According to [40, p. 172], the *conjugate function* (or the *Young-Fenchel transform function*) of a function $f : X \to \bar{\mathbb{R}}$ is the function $f^* : X^* \to \bar{\mathbb{R}}$ given by

$$f^*(x^*) = \sup\big\{\langle x^*, x\rangle - f(x) \mid x \in X\big\}.$$

It is well known [40, Proposition 3, p. 174] that if f is proper convex and lower semicontinuous (i.e., $\operatorname{epi} f$ is a closed set), then f^* is also a proper convex lower semicontinuous function. It is clear that

$$f^*(x^*) = \sup\big\{\langle x^*, x\rangle - f(x) \mid x \in \operatorname{dom} f\big\}, \quad \forall x^* \in X^*.$$

Theorem 3.4 *The conjugate function of a proper generalized polyhedral convex function is a proper generalized polyhedral convex function.*

Proof. Suppose that $f : X \to \bar{\mathbb{R}}$ is a proper generalized polyhedral convex function. Then f^* is a proper convex function. Moreover, due to Theorem 2.2, f is a generalized piecewise linear function. So, there exist nonempty generalized polyhedral convex sets $D_1, \dots, D_m$ in X, $v_k^* \in X^*$, $\beta_k \in \mathbb{R}$, $k = 1, \dots, m$, such that $\operatorname{dom} f = \bigcup\limits_{k=1}^{m} D_k$ and $f(x) = \langle v_k^*, x\rangle + \beta_k$ for all $x \in D_k$, $k = 1, \dots, m$. For each k, by Theorem 1.2 we can find finite index sets I_k and J_k, points $u_i \in X$ with $i \in I_k$, vectors $v_j \in X$ with $j \in J_k$, and a closed linear subspace $X_{0,k}$ in X, such that

$$D_k = \operatorname{conv}\{u_i \mid i \in I_k\} + \operatorname{cone}\{v_j \mid j \in J_k\} + X_{0,k}.$$

(We assume that $I_k \cap I_\ell = \emptyset$ and $J_k \cap J_\ell = \emptyset$ whenever $k \ne \ell$.) For every $k \in \{1, \dots, m\}$, consider the function

$$\varphi_k(x^*) = \sup\big\{\langle x^*, x\rangle - \langle v_k^*, x\rangle - \beta_k \mid x \in D_k\big\}$$

defined on X^* and observe that $x^* \in \operatorname{dom} \varphi_k$ if and only if the linear functional $\langle x^*, \cdot\rangle - \langle v_k^*, \cdot\rangle - \beta_k$ is bounded from the above on D_k. The latter is equivalent to the property that the linear programming problem

$$\max\{\langle x^* - v_k^*, x\rangle - \beta_k \mid x \in D_k\}$$

has a solution (see Theorem 1.12). Therefore, by Proposition 1.11 we get

$$\operatorname{dom} \varphi_k = \left\{x^* \in X^* \mid x^* - v_k^* \in X_{0,k}^{\perp},\ \langle x^* - v_k^*, v_j\rangle \le 0,\ j \in J_k\right\}.$$

As $f^*(\cdot) = \max\left\{\varphi_k(\cdot) \mid k = 1, \ldots, m\right\}$, one has $\operatorname{dom} f^* = \bigcap_{k=1}^{m} \operatorname{dom} \varphi_k$; hence

$$\operatorname{dom} f^* = \left\{x^* \in \bigcap_{k=1}^{m} \left(v_k^* + X_{0,k}^{\perp}\right) \mid \langle x^*, v_j\rangle \le \langle v_k^*, v_j\rangle, k = 1, \ldots, m,\ j \in J_k\right\}.$$

Since $\bigcap_{k=1}^{m} \left(v_k^* + X_{0,k}^{\perp}\right)$ is a closed affine subspace of X^*, we can assert that $\operatorname{dom} f^*$ is a generalized polyhedral convex set. For every $x^* \in \operatorname{dom}\varphi_k$, it is a plain matter to show that

$$\varphi_k(x^*) = \max\{\langle x^* - v_k^*, u_i\rangle - \beta_k \mid i \in I_k\}.$$

Therefore,

$$f^*(x^*) = \begin{cases} \max\left\{\langle x^*, u_i\rangle - f(u_i) \mid k = 1, \ldots, m,\ i \in I_k\right\} & \text{if } x^* \in \operatorname{dom} f^*, \\ +\infty & \text{if } x^* \notin \operatorname{dom} f^*. \end{cases} \tag{3.9}$$

Since f^* is a proper function with $\operatorname{dom} f^*$ being a generalized polyhedral convex set, using Theorem 2.1 and (3.9) we can conclude that f^* is a generalized polyhedral convex function. □

Remark 3.3 Theorem 3.4 is a generalization of Theorem 19.2 from [63], where the case $X = \mathbb{R}^n$ was treated.

3.4 Subdifferentials

In this section, we will study subdifferentials of generalized polyhedral convex functions. It is well known that the subdifferential of a convex function is the basis for optimality conditions and other issues in convex programming.

On account of [40, p. 46], a linear functional $x^* \in X^*$ is said to be a *subgradient* of a proper convex function f at $x \in \operatorname{dom} f$ if

$$\langle x^*, u - x\rangle \leq f(u) - f(x) \quad (u \in X).$$

This condition is equivalent to the simple geometric property that the graph of the affine function $h(u) = f(x) + \langle x^*, u - x\rangle$ forms a non-vertical supporting hyperplane to $\operatorname{epi} f$ at the point $(x, f(x))$; see [63, pp. 214–215]. The *subdifferential* of f at x, denoted by $\partial f(x)$, is the set of all the subgradients of f at x. From the definition it follows that $\partial f(x)$ is a weakly*-closed convex set (see [14, p. 81]). Moreover, by [40, Propostion 1, p. 197], $x^* \in \partial f(x)$ if and only if $f(x) + f^*(x^*) = \langle x^*, x\rangle$. If C is a nonempty convex subset of X then, for any $x \in C$, one has $\partial\delta(x, C) = N_C(x)$, where $\delta(\cdot, C)$ is the indicator function of C.

The next theorem provides us with a formula for the subdifferential of a generalized polyhedral convex function.

Theorem 3.5 *Suppose that f is a proper generalized polyhedral convex function with* $\operatorname{dom} f = \{x \in X \mid A(x) = y,\ \langle x_i^*, x\rangle \leq \alpha_i,\ i = 1, \ldots, p\}$ *and*

$$f(x) = \max\left\{\langle v_j^*, x\rangle + \beta_j \mid j = 1, \ldots, m\right\} \quad (x \in \operatorname{dom} f),$$

where A is a continuous linear mapping from the space X to a locally convex Hausdorff topological vector space Y, $y \in Y$, $x_i^* \in X^*, \alpha_i \in \mathbb{R}$, $i = 1, \ldots, p$, $v_j^* \in X^*, \beta_j \in \mathbb{R}$, $j = 1, \ldots, m$. Then, for every $x \in \operatorname{dom} f$,

$$\partial f(x) = \operatorname{conv}\{v_j^* \mid j \in J(x)\} + \operatorname{cone}\{x_i^* \mid i \in I(x)\} + (\ker A)^{\perp}, \qquad (3.10)$$

where $I(x) = \left\{i \in \{1, \ldots, p\} \mid \langle x_i^*, x\rangle = \alpha_i\right\}$ and

$$J(x) = \left\{j \in \{1, \ldots, m\} \mid \langle v_j^*, x\rangle + \beta_j = f(x)\right\}.$$

In particular, if $Y = \{0\}$, $A \equiv 0$ and $y = 0$ (the case where $\operatorname{dom} f$ is a polyhedral convex set) then, for any $x \in \operatorname{dom} f$,

$$\partial f(x) = \operatorname{conv}\{v_j^* \mid j \in J(x)\} + \operatorname{cone}\{x_i^* \mid i \in I(x)\}.$$

Proof. Fix any $x \in D := \operatorname{dom} f$. As the functions $f_j(\cdot) := \langle v_j^*, \cdot\rangle + \beta_j$ is continuous at x for all $j = 1, \ldots, m$, the function

$$\widetilde{f}(\cdot) := \max\{f_j(\cdot) \mid j = 1, \ldots, m\}$$

is also continuous at x. Hence, applying the Moreau–Rockafellar theorem [40, Theorem 1, p. 200] to the sum $\widetilde{f}(\cdot) + \delta(\cdot, D) = f(\cdot)$, one gets

$$\partial f(x) = \partial \widetilde{f}(x) + \partial \delta(x, D) = \partial \widetilde{f}(x) + N_D(x). \tag{3.11}$$

Since $\partial f_j(\cdot) \equiv \{v_j^*\}$ for all $j = 1, \ldots, m$, by [40, Theorem 3, p. 201] we obtain

$$\partial \widetilde{f}(x) = \overline{\text{conv}}\Big(\bigcup_{j \in J(x)} \partial f_j(x) \Big) = \overline{\text{conv}}\,\{v_j^* \mid j \in J(x)\}.$$

On one hand, according to Theorem 1.2, $\text{conv}\,\{v_j^* \mid j \in J(x)\}$ is a generalized polyhedral convex set (hence it is closed). So, $\partial \widetilde{f}(x) = \text{conv}\,\{v_j^* \mid j \in J(x)\}$. On the other hand, in accordance with Proposition 3.1,

$$N_D(x) = \text{cone}\,\{x_i^* \mid i \in I(x)\} + (\ker A)^\perp.$$

Therefore, from (3.11) one obtains (3.10). □

From (3.10) and Theorem 1.2 it follows that $\partial f(x)$ is a generalized polyhedral convex set in X^*. Thus we have proved the following result, which is a known one [63, Theorem 23.10] in the case where $X = \mathbb{R}^n$.

Proposition 3.5 *If f is a proper generalized polyhedral convex function on X and if $x \in \text{dom}\, f$, then $\partial f(x)$ is a generalized polyhedral convex set.*

By the definition of subdifferential, if $f_1, \ldots, f_m$ are proper convex functions on X then, for every $x \in \bigcap_{i=1}^m \text{dom}\, f_i$,

$$\partial f_1(x) + \cdots + \partial f_m(x) \subset \partial(f_1 + \cdots + f_m)(x). \tag{3.12}$$

Since the set on the right-hand side of (3.12) is weakly*-closed, one has

$$\overline{\partial f_1(x) + \cdots + \partial f_m(x)} \subset \partial(f_1 + \cdots + f_m)(x).$$

The above-cited Moreau–Rockafellar theorem tells us that (3.12) holds with equality if there exists $x_0 \in \bigcap_{i=1}^m \text{dom}\, f_i$ such that all the functions $f_1, \ldots, f_m$ except, possibly, one are continuous at x_0. The specific structure of generalized polyhedral convex functions allows one to have a subdifferential sum rule without the continuity assumption.

Theorem 3.6 *Let $f_1, \ldots, f_m$ be proper generalized polyhedral convex functions. Then, for any $x \in \bigcap_{i=1}^m \text{dom}\, f_i$,*

$$\partial(f_1 + f_2 + \cdots + f_m)(x) = \overline{\partial f_1(x) + \partial f_2(x) + \cdots + \partial f_m(x)}. \tag{3.13}$$

Proof. According to Theorem 2.3, the sum of two proper generalized polyhedral convex functions whose effective domains have at least one common point, is again a proper generalized polyhedral convex function. So, to obtain the desired result, it suffices to prove (3.13) for $m = 2$ and then proceed by induction.

For each $i = 1, 2$, by Theorem 2.1, $D_i := \mathrm{dom}\, f_i$ is a generalized polyhedral convex set and there exist $v^*_{i,j} \in X^*$, $\beta_{i,j} \in \mathbb{R}$, $j = 1, \ldots, k_i$, such that

$$f_i(x) = \widetilde{f}_i(x) + \delta(x, D_i),$$

where

$$\widetilde{f}_i(x) = \max\left\{\langle v^*_{i,j}, x\rangle + \beta_{i,j} \mid j = 1, \ldots, k_i\right\} \quad (x \in X).$$

Clearly, $\widetilde{f}_1$ and $\widetilde{f}_2$ are proper and continuous on X. Let $x \in D_1 \cap D_2$. On one hand, by the formula $f_1 + f_2 = \widetilde{f}_1 + \widetilde{f}_2 + \delta(\cdot, D_1 \cap D_2)$ and by the Moreau–Rockafellar theorem,

$$\begin{aligned} \partial(f_1 + f_2)(x) &= \partial\widetilde{f}_1(x) + \partial\widetilde{f}_2(x) + \partial\delta(x, D_1 \cap D_2) \\ &= \partial\widetilde{f}_1(x) + \partial\widetilde{f}_2(x) + N_{D_1\cap D_2}(x). \end{aligned} \tag{3.14}$$

Since $N_{D_1\cap D_2}(x) = \overline{N_{D_1}(x) + N_{D_2}(x)}$ by Theorem 3.2, this implies that

$$\partial(f_1 + f_2)(x) = \partial\widetilde{f}_1(x) + \partial\widetilde{f}_2(x) + \overline{N_{D_1}(x) + N_{D_2}(x)}. \tag{3.15}$$

On the other hand, applying the Moreau–Rockafellar theorem to the proper convex functions $f_1 = \widetilde{f}_1 + \delta(\cdot, D_1)$ and $f_2 = \widetilde{f}_2 + \delta(\cdot, D_2)$, we obtain

$$\partial f_1(x) + \partial f_2(x) = \partial\widetilde{f}_1(x) + N_{D_1}(x) + \partial\widetilde{f}_2(x) + N_{D_2}(x). \tag{3.16}$$

Put $C_1 = \partial\widetilde{f}_1(x) + \partial\widetilde{f}_2(x)$ and $C_2 = N_{D_1}(x) + N_{D_2}(x)$. From (3.15) and the closedness of $\partial(f_1 + f_2)(x)$, it follows that $C_1 + \overline{C_2}$ is closed. Then, according to Lemma 3.1, $C_1 + \overline{C_2} = \overline{C_1 + C_2}$. Combining this equality with (3.15) and (3.16) yields (3.13). □

In the last proof, if f_1 is a polyhedral convex function, then D_1 is a polyhedral convex set by virtue of Theorem 2.1. Hence, by Theorem 3.3 we have

$$N_{D_1\cap D_2}(x) = N_{D_1}(x) + N_{D_2}(x).$$

Therefore, using (3.14) and (3.16), we can obtain formula (3.13) in the case $m = 2$ with no closure sign on the right-hand side. Thus, the following result is valid.

Theorem 3.7 *Suppose that f_1 is a proper polyhedral convex function and f_2 is a proper generalized polyhedral convex function. Then*

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$$

for any $x \in (\text{dom}\, f_1) \cap (\text{dom}\, f_2)$.

3.5 Conclusions

We have studied several dual constructions including the concepts normal cone, conjugate function, and subdifferential. Among other things, we obtain a formula to compute the normal cone of generalized polyhedral convex sets, the subdifferential of a generalized polyhedral convex function at a point. Moreover, we have shown that the specific structure of generalized polyhedral convex functions allows one to have a subdifferential sum rule without any assumption on continuity.

Chapter 4

Generalized Polyhedral Convex Optimization Problems

This chapter, which is written on the basis of the paper [A4] in the List of Author's Related Papers on page 113, is devoted to studying generalized polyhedral convex optimization problems.

4.1 Motivations

Let X be a locally convex Hausdorff topological vector space, $\varphi : X \to \bar{\mathbb{R}}$ a proper convex function, and $C \subset X$ a nonempty convex set. For each index $i \in \{1, \ldots, m\}$, where m is a positive integer, select some $x_i \in \operatorname{dom} \varphi$ and suppose that there exists some $x_i^* \in \partial\varphi(x_i)$. Then we have

$$\varphi(x) \geq \varphi(x_i) + \langle x_i^*, x - x_i \rangle, \quad \forall x \in X.$$

So,

$$\varphi(x) \geq \psi(x) := \max\{\varphi(x_i) + \langle x_i^*, x - x_i \rangle \mid i = 1, \ldots, m\}, \quad \forall x \in X. \quad (4.1)$$

Therefore, the polyhedral convex function ψ defined in (4.1) is a *lower piecewise convex approximation* of φ. Recall that the *relative interior* $\operatorname{ri} C$ of C is the interior of C in the induced topology of the closed affine hull $\overline{\operatorname{aff}}\, C$ of C. Select some points $u_1, \ldots, u_k$ in the boundary of C in the induced topology of $\overline{\operatorname{aff}}\, C$. Suppose that $\operatorname{ri} C$ is nonempty. Then, by the separation theorem [65, Theorem 3.4] and the Hahn-Banach extension theorem [65, Theorem 3.6] one can find $u_j^* \in N_C(u_j) \setminus \{0\}$ for $j = 1, \ldots, k$. Since

$$\langle u_j^*, x - u_j \rangle \leq 0, \quad \forall j = 1, \ldots, k, \ \forall x \in C,$$

one has

$$C \subset \widetilde{C} := \{x \in X \mid x \in \overline{\mathrm{aff}}\, C,\ \langle u_j^*, x\rangle \le \langle u_j^*, u_j\rangle,\ \ j = 1, \ldots, k\}. \tag{4.2}$$

In other words, *the generalized polyhedral convex set* $\widetilde{C}$ *defined in* (4.2) *is an outer approximation of* C. This outer approximation of C and the above construction of a lower convex approximation of φ allow us to consider the *generalized polyhedral convex optimization problem*

$$\min\{\psi(x) \mid x \in \widetilde{C}\} \tag{4.3}$$

an approximation of the convex optimization problem

$$\text{(P)} \qquad \min\{\varphi(x) \mid x \in C\}.$$

Since the optimal value of (4.3) is smaller than that of (P), it can serve as a lower bound for the latter. By increasing m and k, one attains tighter approximations of (P) in the form (4.3). Therefore, in this sense, *one can approximate any convex optimization problem by a generalized polyhedral convex optimization problem.* Similarly, one can approximate a differentiable nonlinear programming problem by a generalized polyhedral convex optimization problem, provided that the space of programming variables is Banach and the Fréchet derivatives of the objective function at some chosen points are used instead of the above subgradients. As concerning the constraint set defined by differentiable functions, one can approximate a nonlinear equality (resp., inequality) by a linear equality (resp., inequality) if one uses first-order Taylor expansions of the functions at the chosen points of the constraint set. Linearization techniques in optimization theory are the subjects of many books and research papers (see, e.g., [11, 13, 59, 60]). Note that since Pshenichnyj [59,60] used a penalty function to prevent the next current iteration point of going far away from the current iteration point, the objective function in his minimization problems in Euclidean spaces is a sum of a polyhedral convex function $\psi(x)$ and a differentiable strongly convex function of the form $\gamma\|x - x_k\|^2$, where $\gamma > 0$ is a penalty coefficient and x_k is the current iteration point.

One referee of the paper [A1] in the List of Author's Related Papers on page 113 raised the following questions: 1) *"How to select the points* $u_1, u_2, \ldots, u_k$ *on the boundary of* C*?"*, 2) *"Why the proposed approach of solving* (P) *by reduction to* (4.3) *is more efficient than many others?"*. The referee also observed that *"To provide the adequate approximation, the number of the selected points x_i, $i = 1, \ldots, m$, from* $\operatorname{dom} \varphi$ *should be also increased."* To answer these questions and to see how one can select the points x_i, $i = 1, \ldots, m$, from $\operatorname{dom} \varphi$ in the practical computation, we refer to the wonderful paper of Bertsekas and Yu [13] on a unified polyhedral approximation framework for convex optimization. Although the authors considered finite-dimensional convex optimization problems, their ideas can be applied to our infinite dimensional setting. Given a convex optimization problem of the form (P), we define

$$S = \{(z_1, z_2) \in X \times X \mid z_2 - z_1 = 0\},$$

$f_1(z_1) = \varphi(z_1)$ and $f_2(z_2) = \delta(z_2, C)$ for any $(z_1, z_2) \in X \times X$. Then, (P) is equivalent to the problem

$$\text{Minimize } f_1(z_1) + f_2(z_2) \quad \text{subject to} \quad (z_1, z_2) \in S, \tag{4.4}$$

which is a special case of the problem (1.1) in [13]. Namely, if $(\bar{z}_1, \bar{z}_2)$ is a solution of (4.4), then $\bar{x} := \bar{z}_1$ is a solution of (P). Conversely, if $\bar{x}$ is a solution of (P), then $(\bar{z}_1, \bar{z}_2) := (\bar{x}, \bar{x})$ is a solution of (4.4). In the terminology of [13], the function $\psi(.)$ defined in (4.1) is an outer linearization of $f_1(.)$, and $\delta(x, \widetilde{C})$ is an outer linearization of $f_2(.)$. Therefore, the results of Subsection 5.1 on asymptotic convergence analysis in pure cases in [13] are applicable to problem (4.4), provided that X is a finite-dimensional Euclidean space. Polyhedral approximation methods, including the outer linearization method, also have been studied in [11, Chapter 4 and Section 4.1]. To solve problem (4.3), where ψ is given in (4.1) and $\widetilde{C}$ is defined in (4.2), one can represent it equivalently as a linear program:

$$\begin{array}{ll} \min t & \\ \text{subject to} & \varphi(x_i) + \langle x_i^*, x - x_i \rangle - t \leq 0, \quad i = 1, \ldots, m, \\ & x \in \widetilde{C}, \ t \in \mathbb{R}. \end{array} \tag{4.5}$$

Clearly, if $(\bar{x}, \bar{t})$ is a solution of (4.5), then $\bar{x}$ is a solution of (4.3). Conversely, if $\bar{x}$ is a solution of (4.3), then $(\bar{x}, \bar{t}) := (\bar{x}, \psi(\bar{x}))$ is a solution of (4.5). Thus, in a sense, one may consider generalized polyhedral convex optimization problems as "easy" problems. To save the storage memory of the computer, one has to limit the number $m + k$ of the approximation points x_i and u_j by requiring that $m + k \leq q$, where q is a fixed integer. For this purpose, one can use the so-called *aggregation techniques* (see, e.g., [43–45]) The main idea of those is to *remove the unused approximation indexes* (and the corresponding points) which have not appeared as active indexes after several steps of computation. To be more precise, if $\bar{x}$ is a solution of (4.3), then we say that i is an active index at $\bar{x}$ (resp., j is an active index at $\bar{x}$) if it belongs to the set of those $i \in \{1, \ldots, m\}$ satisfying $\psi(\bar{x}) = \varphi(x_i) + \langle x_i^*, \bar{x} - x_i \rangle$ (resp., the set of those $j \in \{1, \ldots, k\}$ satisfying $\langle u_j^*, \bar{x} \rangle = \langle u_j^*, u_j \rangle$).

4.2 Solution Existence Theorems

Let $D \subset X$ be a nonempty generalized polyhedral convex set of the form (1.2). Set $I = \{1, \ldots, p\}$ and $I(x) = \{i \in I \mid \langle x_i^*, x \rangle = \alpha_i\}$ for $x \in D$. If D is a polyhedral convex set, then one can choose $Y = \{0\}$, $A \equiv 0$, and $y = 0$.

Consider a *generalized polyhedral convex optimization problem*

$$(\mathcal{P}) \qquad \min \{f(x) \mid x \in D\}$$

where, as before, X is a locally convex Hausdorff topological vector space, $D \subset X$ a nonempty generalized polyhedral convex set, and $f : X \to \bar{\mathbb{R}}$ a proper generalized polyhedral convex function. We say that $u \in D$ is a solution of $(\mathcal{P})$ if $f(u)$ is finite and $f(u) \leq f(x)$ for all $x \in D$. The solution set of $(\mathcal{P})$ is denoted by $\mathrm{Sol}(\mathcal{P})$.

From now on, if not otherwise stated, the constraint set D is given by (1.2), *and the objective function f is defined by* (2.1).

Since $\operatorname{dom} f$ is a generalized polyhedral convex set, it admits the representation

$$\operatorname{dom} f = \left\{x \in X \mid B(x) = z,\ \langle u_j^*, x \rangle \leq \gamma_j,\ j = 1, \ldots, q\right\}, \tag{4.6}$$

where B is a continuous linear mapping from X to a locally convex Hausdorff topological vector space Z, $z \in Z$, $u_j^* \in X^*$, $\gamma_j \in \mathbb{R}$, $j = 1, \ldots, q$. Set $J = \{1, \ldots, q\}$. For each $x \in \operatorname{dom} f$, let $J(x) = \left\{j \in J \mid \langle u_j^*, x \rangle = \gamma_j\right\}$ and

$$\Theta(x) = \{k \in \{1, \ldots, m\} \mid \langle v_k^*, x \rangle + \beta_k = f(x)\}.$$

If f is a polyhedral convex function, then $\operatorname{dom} f$ is polyhedral convex by Theorem 2.1; hence, we can choose $Z = \{0\}$, $B \equiv 0$, and $z = 0$.

Remark 4.1 On account of Theorem 3.5, if f is defined by (2.1) with $\operatorname{dom} f$ being given by (4.6) then, for any $x \in \operatorname{dom} f$, we have

$$\partial f(x) = \operatorname{conv}\,\{v_k^* \mid k \in \Theta(x)\} + \operatorname{cone}\,\{u_j^* \mid j \in J(x)\} + (\ker B)^{\perp} \qquad (4.7)$$

where $\operatorname{conv} \Omega$ denotes the convex hull of a subset $\Omega \subset X^*$.

Remark 4.2 If a nonempty generalized polyhedral convex set D is given by (1.2), then its recession cone can be computed by the formula

$$0^+D = \{v \in X \mid A(v) = 0,\ \langle x_i^*, v\rangle \le 0,\ i = 1, \ldots, p\}\,.$$

It follows that 0^+D is a generalized polyhedral convex cone.

Following [63, p. 66], we define the *recession function* $f0^+$ of a proper convex function $f : X \to \bar{\mathbb{R}}$ by the formula

$$f0^+(v) = \inf\{\mu \in \mathbb{R} \mid (v, \mu) \in 0^+(\operatorname{epi} f)\} \quad (v \in X). \qquad (4.8)$$

Remark 4.3 If f is nonconvex, similar notions bearing the names of *asymptotic function* [6, p. 48] and *horizon function* [64, p. 86] have been defined. It is not difficult to show that if f is proper convex and lower semicontinuous (i.e., $\operatorname{epi} f$ is a closed convex set), these notions coincide with that of recession function.

Several solution existence theorems for generalized polyhedral convex optimization problems will be obtained in this section.

Theorem 4.1 (A Frank–Wolfe-type existence theorem) *If $D \cap \operatorname{dom} f$ is nonempty then, $(\mathcal{P})$ has a solution if and only if there is a real value γ such that $f(x) \ge \gamma$ for every $x \in D$.*

Proof. The necessity is obvious. To prove the sufficiency, suppose that there exists a constant $\gamma \in \mathbb{R}$ such that $f(x) \ge \gamma$ for all $x \in D$. Clearly,

$$\Phi : X \times \mathbb{R} \to \mathbb{R},\ \ (x, \alpha) \mapsto \alpha$$

for all $(x, \alpha) \in X \times \mathbb{R}$, is a linear mapping. Since $\operatorname{epi} f \cap (D \times \mathbb{R})$ is a nonempty generalized polyhedral convex set in $X \times \mathbb{R}$, by Proposition 1.2 we can assert that $T := \Phi\,(\operatorname{epi} f \cap (D \times \mathbb{R}))$ is a nonempty polyhedral convex set in $\mathbb{R}$. Hence, T is convex and closed. For every $t \in T$, there exists an $x \in D$ satisfying $(x, t) \in \operatorname{epi} f$, i.e., $t \ge f(x)$; hence $t \ge f(x) \ge \gamma$. In addition, for every $t' \geq t$, since $(x,t') \in \text{epi} f \cap (D \times \mathbb{R})$, one has $t' \in T$. So, we must have $T = [\bar{\gamma}, +\infty)$ for some $\bar{\gamma} \geq \gamma$. On one hand, for every $x \in D$, the inclusion $(x, f(x)) \in \text{epi} f \cap (D \times \mathbb{R})$ yields $f(x) \in T$; hence $f(x) \geq \bar{\gamma}$. On the other hand, since $\bar{\gamma} \in T$, we can find $\bar{x} \in D$ such that $(\bar{x}, \bar{\gamma}) \in \text{epi} f \cap (D \times \mathbb{R})$. Then we have $\bar{\gamma} \geq f(\bar{x})$ and $f(x) \geq \bar{\gamma} \geq f(\bar{x})$ for every $x \in D$. Thus, $\bar{x}$ is a solution of $(\mathcal{P})$. □

Remark 4.4 Due to the similarity of the formulations of Theorem 4.1 and the solution existence theorem in quadratic programming in [28, p. 158] (see also [49, Theorem 2.1]), we call the above result a Frank–Wolfe-type existence theorem in generalized polyhedral convex optimization. If the function f is linear, Theorem 4.1 expresses a recent result in Theorem 1.12. For the case $X = \mathbb{R}^n$ and $D = X$, the result in Theorem 4.1 is a known one (see [12, p. 215]).

Theorem 4.2 (An Eaves-type existence theorem) *Suppose that $D \cap \operatorname{dom} f$ is nonempty. Then $(\mathcal{P})$ has a solution if and only if $f0^+(v) \geq 0$ for every $v \in 0^+D$.*

For proving this theorem, we need a lemma.

Lemma 4.1 *If f is a proper generalized polyhedral convex function given by* (2.1)*, then*

$$f0^+(v) = \begin{cases} \max\{\langle v_k^*, v\rangle \mid k = 1, \dots, m\} & \text{if } v \in 0^+(\operatorname{dom} f) \\ +\infty & \text{if } v \notin 0^+(\operatorname{dom} f). \end{cases} \tag{4.9}$$

In particular, $f0^+$ is a proper generalized polyhedral convex function.

Proof. Suppose that $\operatorname{dom} f$ is of the form (4.6). Then one gets

$$\begin{aligned} \operatorname{epi} f &= \big\{(x,t) \in X \times \mathbb{R} \mid B(x) = z,\ \langle u_j^*, x\rangle \leq \gamma_j,\ j = 1, \dots, q, \\ &\qquad\qquad \langle v_k^*, x\rangle + \beta_k \leq t,\ k = 1, \dots, m\big\} \\ &= \big\{(x,t) \in X \times \mathbb{R} \mid B(x) + 0t = z,\ \langle u_j^*, x\rangle + 0t \leq \gamma_j,\ j = 1, \dots, q, \\ &\qquad\qquad \langle v_k^*, x\rangle - t \leq -\beta_k,\ k = 1, \dots, m\big\}. \end{aligned}$$

Hence, applying Remark 4.2 to $\operatorname{epi} f$ gives

$$\begin{aligned} 0^+(\operatorname{epi} f) &= \big\{(v,\mu) \in X \times \mathbb{R} \mid B(v) = 0,\ \langle u_j^*, v\rangle \leq 0,\ j = 1, \dots, q, \\ &\qquad\qquad \langle v_k^*, v\rangle - \mu \leq 0,\ k = 1, \dots, m\big\} \\ &= \{(v,\mu) \in X \times \mathbb{R} \mid v \in 0^+(\operatorname{dom} f),\ \langle v_k^*, v\rangle \leq \mu,\ k = 1, \dots, m\}. \end{aligned}$$

From this and (4.8) we obtain (4.9). □

Proof of Theorem 4.2. First, suppose that $(\mathcal{P})$ has a solution x_0. Let $v \in 0^+D$ be given arbitrarily. If $v \notin 0^+(\operatorname{dom} f)$, then $f0^+(v) = +\infty$ by Lemma 4.1. If $v \in 0^+(\operatorname{dom} f)$, then $f0^+(v) = \max\{\langle v_k^*, v\rangle \mid k = 1, \ldots, m\}$ by Lemma 4.1. Select any $t > 0$. Since $x_0 + tv \in D \cap \operatorname{dom} f$, one has

$$\begin{aligned} f(x_0) \le f(x_0 + tv) &= \max\{\langle v_k^*, x_0\rangle + \beta_k + t\langle v_k^*, v\rangle \mid k = 1, \ldots, m\} \\ &\le \max\{\langle v_k^*, x_0\rangle + \beta_k \mid k = 1, \ldots, m\} + \max\{t\langle v_k^*, v\rangle \mid k = 1, \ldots, m\} \\ &= f(x_0) + tf0^+(v). \end{aligned}$$

It follows that $f0^+(v) \ge 0$.

Conversely, suppose that $f0^+(v) \ge 0$ for every $v \in 0^+D$. Since $D \cap \operatorname{dom} f$ is a nonempty generalized polyhedral convex set, by the representation theorem for generalized polyhedral convex sets Theorem 1.2, one can find $u_1, \ldots, u_d$ in $D \cap \operatorname{dom} f$, $v_1, \ldots, v_\ell$ in X, and a closed linear subspace $X_0 \subset X$ such that

$$D \cap \operatorname{dom} f = \operatorname{conv}\{u_1, \ldots, u_d\} + \operatorname{cone}\{v_1, \ldots, v_\ell\} + X_0. \tag{4.10}$$

Then, $0^+(D \cap \operatorname{dom} f) = \operatorname{cone}\{v_1, \ldots, v_\ell\} + X_0$. Put

$$\gamma = \min\{\langle v_k^*, u_i\rangle + \beta_k \mid k = 1, \ldots, m,\ i = 1, \ldots, d\}.$$

One has $f(x) \ge \gamma$ for every $x \in D$. Indeed, if $x \notin \operatorname{dom} f$, then the inequality is obvious, because $f(x) = +\infty$. Now, suppose that $x \in D \cap \operatorname{dom} f$. According to (4.10), there exist $\lambda_1 \ge 0, \ldots, \lambda_d \ge 0$, and $v \in 0^+(D \cap \operatorname{dom} f)$ satisfying $\sum_{i=1}^{d} \lambda_i = 1$ and $x = \sum_{i=1}^{d} \lambda_i u_i + v$. For each $k = 1, \ldots, m$, one has

$$\begin{aligned} \langle v_k^*, x\rangle + \beta_k &= \sum_{i=1}^{d} \lambda_i \langle v_k^*, u_i\rangle + \langle v_k^*, v\rangle + \beta_k \\ &= \sum_{i=1}^{d} \lambda_i \big(\langle v_k^*, u_i\rangle + \beta_k\big) + \langle v_k^*, v\rangle \\ &\ge \sum_{i=1}^{d} \lambda_i \gamma + \langle v_k^*, v\rangle = \gamma + \langle v_k^*, v\rangle. \end{aligned}$$

Consequently,

$$\max\{\langle v_k^*, x\rangle + \beta_k \mid k = 1, \ldots, m\} \ge \max\{\gamma + \langle v_k^*, v\rangle \mid k = 1, \ldots, m\}.$$

Combining this with (2.1), we obtain

$$f(x) \ge \max\{\gamma + \langle v_k^*, v\rangle \mid k = 1, \ldots, m\} = \gamma + \max\{\langle v_k^*, v\rangle \mid k = 1, \ldots, m\}.$$

Since $v \in 0^+(D \cap \operatorname{dom} f)$, one has $v \in 0^+(\operatorname{dom} f)$. So, by Lemma 4.1,

$$\max \{\langle v_k^*, v\rangle \mid k = 1, \ldots, m\} = f0^+(v).$$

Then, for every $x \in D \cap \operatorname{dom} f$ we have $f(x) \geq \gamma + f0^+(v) \geq \gamma$, where the last inequality holds because $v \in 0^+(D)$. Thus, by Theorem 4.1, $(\mathcal{P})$ has a solution. □

Remark 4.5 If $f0^+(v) \geq 0$ for every $v \in 0^+D$, then one says that the functional $f0^+$ is *copositive* on the recession cone 0^+D. We call Theorem 4.2 an Eaves-type existence theorem in generalized polyhedral convex optimization to trace back Eaves' idea [21, p. 702] (see also [49, Theorem 2.2]) in using recession cones for a solution existence theorem in quadratic programming. In the special case where f is linear on X, the result in Theorem 4.2 has been obtained in Theorem 1.11.

We now give an explicit criterion for $(\mathcal{P})$ to have a solution.

Theorem 4.3 *Let D be given by* (1.2)*, the function f be defined by* (2.1) *with* $\operatorname{dom} f$ *be given by* (4.6)*. Suppose that $D \cap \operatorname{dom} f$ is nonempty. Then $(\mathcal{P})$ has a solution if and only if*

$$\begin{aligned} 0 \in \operatorname{conv}\{v_k^* \mid k = 1, \ldots, m\} &+ \operatorname{cone}\{u_j^* \mid j = 1, \ldots, q\} \\ &+ \operatorname{cone}\{x_i^* \mid i = 1, \ldots, p\} + (\ker A \cap \ker B)^{\perp}. \end{aligned} \tag{4.11}$$

Proof. First, suppose that inclusion (4.11) is fulfilled. Then, there exist nonnegative numbers $\lambda_1, \ldots, \lambda_m$, $\mu_{1,1}, \ldots, \mu_{1,p}$, $\mu_{2,1}, \ldots, \mu_{2,q}$, and an element $u^* \in (\ker A \cap \ker B)^{\perp}$ such that $\sum\limits_{k=1}^{m} \lambda_k = 1$ and

$$\sum_{k=1}^{m} \lambda_k v_k^* + \sum_{i=1}^{p} \mu_{1,i} x_i^* + \sum_{j=1}^{q} \mu_{2,j} u_j^* + u^* = 0.$$

Select any x_0 from $D \cap \operatorname{dom} f$. For every $x \in D \cap \operatorname{dom} f$, one has

$$
\begin{aligned}
f(x) &= \max\{\langle v_\ell^*, x\rangle + \beta_\ell \mid \ell = 1,\dots,m\} \\
&= \left(\sum_{k=1}^m \lambda_k\right) \max\{\langle v_\ell^*, x\rangle + \beta_\ell \mid \ell = 1,\dots,m\} \\
&= \sum_{k=1}^m \left(\lambda_k \max\{\langle v_\ell^*, x\rangle + \beta_\ell \mid \ell = 1,\dots,m\}\right) \\
&\geq \sum_{k=1}^m \lambda_k[\langle v_k^*, x\rangle + \beta_k] = \left\langle \sum_{k=1}^m \lambda_k v_k^*, x\right\rangle + \sum_{k=1}^m \lambda_k\beta_k \\
&= \left\langle -\left(\sum_{i=1}^p \mu_{1,i}x_i^* + \sum_{j=1}^q \mu_{2,j}u_j^* + u^*\right), x\right\rangle + \sum_{k=1}^m \lambda_k\beta_k \\
&= -\sum_{i=1}^p \mu_{1,i}\langle x_i^*, x\rangle - \sum_{j=1}^q \mu_{2,j}\langle u_j^*, x\rangle - \langle u^*, x_0\rangle + \langle u^*, x_0 - x\rangle + \sum_{k=1}^m \lambda_k\beta_k \\
&\geq -\sum_{i=1}^p \mu_{1,i}\alpha_i - \sum_{j=1}^q \mu_{2,j}\gamma_j - \langle u^*, x_0\rangle + 0 + \sum_{k=1}^m \lambda_k\beta_k.
\end{aligned}
$$

Hence, f is bounded from below on D. Invoking Theorem 4.1, we conclude that $(\mathcal{P})$ has a solution. Thus, (4.11) implies the solution existence of $(\mathcal{P})$.

To complete the proof, it suffices to show that if (4.11) does not hold, then $(\mathcal{P})$ has no solutions. Suppose that $0 \notin Q$, where Q denotes the set on the right-hand side of (4.11). By Theorem 1.2, the nonempty set Q is generalized polyhedral convex. Hence, Q is convex and weakly*-closed. Since $\{0\} \cap Q = \emptyset$, by the strong separation theorem [65, Theorem 3.4(b)] one can find $v \in X$ and $\gamma \in \mathbb{R}$ such that

$$\sup\{\langle x^*, v\rangle \mid x^* \in Q\} < \gamma < \langle 0, v\rangle. \tag{4.12}$$

On one hand, (4.12) assures that the linear functional $\langle \cdot, v\rangle$ is bounded from above on Q. Hence, according to Theorem 1.12, the generalized linear programming problem $\max\{\langle x^*, v\rangle \mid x^* \in Q\}$ has a solution. Therefore, by Proposition 1.11, one has $\langle v^*, v\rangle \leq 0$ for every vector v^* from the recession cone 0^+Q of Q. As (4.11) yields

$$0^+Q = \operatorname{cone}\{u_j^* \mid j = 1,\dots,q\} + \operatorname{cone}\{x_i^* \mid i = 1,\dots,p\} + (\ker A \cap \ker B)^\perp,$$

one gets $\langle x_i^*, v\rangle \leq 0$ for all $i \in \{1,\dots,p\}$, $\langle u_j^*, v\rangle \leq 0$ for every $j \in \{1,\dots,q\}$, and $v \in ((\ker A \cap \ker B)^\perp)^\perp$. Since the linear subspace $\ker A \cap \ker B$ is closed, by using [14, Proposition 2.40] one has

$$((\ker A \cap \ker B)^\perp)^\perp = \ker A \cap \ker B.$$

Hence, applying Remark 4.2 simultaneously to generalized polyhedral convex sets D and $\operatorname{dom} f$, we obtain $v \in 0^+D$ and $v \in 0^+(\operatorname{dom} f)$. So, by the second inclusion and by Lemma 4.1,

$$f0^+(v) = \max\{\langle v_k^*, v\rangle \mid k = 1, \ldots, m\}. \tag{4.13}$$

On the other hand, for each $k = 1, \ldots, m$, since $v_k^* \in Q$, the inequalities in (4.12) yield $\langle v_k^*, v\rangle < \gamma < 0$. So, from (4.13) it follows that $f0^+(v) < 0$. Hence $\operatorname{Sol}(\mathcal{P}) = \emptyset$ by Theorem 4.2.

The proof is complete. □

Corollary 4.1 *In the notations of Theorem 4.3, suppose that* $\operatorname{dom} f \subset D$. *Then,* $(\mathcal{P})$ *has a solution if and only if*

$$0 \in \operatorname{conv}\{v_k^* \mid k = 1, \ldots, m\} + \operatorname{cone}\{u_j^* \mid j = 1, \ldots, q\} + (\ker B)^{\perp}.$$

Proof. As $\operatorname{dom} f \subset D$, $(\mathcal{P})$ is equivalent to the problem

$$\min\{f(x) \mid x \in \operatorname{dom} f\}.$$

Hence, applying Theorem 4.3 to the latter, we obtain the assertion. □

Corollary 4.2 *Suppose that* $D = X$ *and* f *is given by* (2.1) *with* $\operatorname{dom} f = X$. *Then* $(\mathcal{P})$ *has a solution if and only if* $0 \in \operatorname{conv}\{v_k^* \mid k = 1, \ldots, m\}$.

Proof. Since $\operatorname{dom} f = X$, we can choose $Z = \{0\}$, $B \equiv 0$, $z = 0$ and $q = 0$. Therefore, by using Corollary 4.1 we obtain the assertion. □

Next, we will describe the solution set of $(\mathcal{P})$.

Proposition 4.1 $\operatorname{Sol}(\mathcal{P})$ *is a generalized polyhedral convex set. If* D *and* $\operatorname{dom} f$ *are polyhedral convex, so is* $\operatorname{Sol}(\mathcal{P})$.

Proof. If $\operatorname{Sol}(\mathcal{P})$ is empty, then the claim is trivial. If $\operatorname{Sol}(\mathcal{P})$ is nonempty, select a point $\bar{x} \in \operatorname{Sol}(\mathcal{P})$ and put $\bar{\gamma} = f(\bar{x})$. Then, $f(x) \geq \bar{\gamma}$ for every $x \in D$. With f being given by (2.1), one has

$$\begin{aligned}\operatorname{Sol}(\mathcal{P}) &= \{x \in D \mid f(x) = \bar{\gamma}\} = \{x \in D \mid f(x) \leq \bar{\gamma}\} \\ &= \{x \in D \cap \operatorname{dom} f \mid f(x) \leq \bar{\gamma}\} \\ &= \{x \in D \cap \operatorname{dom} f \mid \langle v_k^*, x\rangle + \beta_k \leq \bar{\gamma},\ k = 1, \ldots, m\}.\end{aligned} \tag{4.14}$$

Since $\operatorname{dom} f$ is a generalized polyhedral convex set (see Theorem 2.1), this implies that $\operatorname{Sol}(\mathcal{P})$ is a generalized polyhedral convex set. In the case where D

and dom f are polyhedral convex, (4.14) shows that Sol($\mathcal{P}$) is a polyhedral convex set. $\square$

The following example is an illustration for our results in this section.

Example 4.1 (Cf. Examples 1.1 and 1.2) Let $X = C[-1,1]$ be the Banach space of continuous real-valued functions defined on $[-1,1]$ with the norm $||x|| = \max_{t\in[-1,1]} |x(t)|$. By the Riesz representation theorem (see, e.g., [47, Theorem 6, p. 374] and [53, Theorem 1, p. 113]), the dual space of X is $X^* = NBV[-1,1]$. To construct a generalized polyhedral convex optimization problem on X, we first define the elements x_1^*, $x_2^* \in X^*$ by setting

$$\langle x_i^*, x\rangle = \int_{-1}^{1} t^i x(t)dt \quad (i = 1,2), \tag{4.15}$$

where the integrals are Riemannian. For each index $i \in \{1,2\}$, the corresponding integral in (4.15) is equal to the Riemann-Stieltjes integral

$$\int_{-1}^{1} x(t)dy_i(t),$$

which is given by the C^1-smooth functions $y_i(t) = \int_{-1}^{t} \tau^i d\tau$ (see, e.g., [47, p. 367]). Consider X with the *weak topology*. Then X is a locally convex Hausdorff topological vector space whose topology is much weaker than the norm topology. Clearly, $X_0 := \ker x_1^* \cap \ker x_2^*$ is a closed linear subspace of X. Let

$$e_1(t) = \begin{cases} 0 & \text{if } t \in [-1,0] \\ -60t^2 + 48t & \text{if } t \in [0,1], \end{cases}$$

and

$$e_2(t) = \begin{cases} 0 & \text{if } t \in [-1,0] \\ 80t^2 - 60t & \text{if } t \in [0,1]. \end{cases}$$

We have $e_1, e_2 \in X$, $\langle x_1^*, e_1\rangle = \langle x_2^*, e_2\rangle = 1$, and $\langle x_1^*, e_2\rangle = \langle x_2^*, e_1\rangle = 0$. For any $x \in X$, put $t_i = \langle x_i^*, x\rangle$ for $i = 1,2$, and observe that the vector $x_0 := x - t_1e_1 - t_2e_2$ belongs to X_0. Conversely, if $x = x_0 + t_1e_1 + t_2e_2$, with $x_0 \in X_0$ and $t_1, t_2 \in \mathbb{R}$, then

$$\langle x_i^*, x\rangle = \langle x_i^*, x_0\rangle + t_1\langle x_i^*, e_1\rangle + t_2\langle x_i^*, e_2\rangle = t_i \quad (i = 1,2).$$

Therefore, for any $x \in X$, there exists a unique element $(x_0, t_1, t_2) \in X_0 \times \mathbb{R} \times \mathbb{R}$ such that $x = x_0 + t_1 e_1 + t_2 e_2$. Given any $e_0 \in X_0$ and put

$$L = \{x \in X \mid x(t) = e_0(t),\ t \in [-1, 0]\}.$$

Clearly, L is a closed affine subspace of X. Consider $(\mathcal{P})$ with the constraint set

$$D := \{x \in L \mid \langle x_1^*, x\rangle \le 1,\ \langle x_2^*, x\rangle \le 2\}.$$

Observe that D is a generalized polyhedral convex set, $e_0 + e_2 \in D$, and D is not a polyhedral convex set in X. To define the objective function, choose $v_1^* = x_1^* - x_2^*$, $v_2^* = -x_1^* - x_2^*$, and put

$$f(x) = \max\{\langle v_1^*, x\rangle + 1, \langle v_2^*, x\rangle\} \quad (x \in X).$$

For any $x \in D$, we have

$$\begin{aligned} f(x) &\ge \frac{1}{2}[\langle v_1^*, x\rangle + 1] + \frac{1}{2}\langle v_2^*, x\rangle \\ &= \frac{1}{2}\langle v_1^* + v_2^*, x\rangle + \frac{1}{2} \\ &= \langle -x_2^*, x\rangle + \frac{1}{2} \ge -\frac{3}{2}. \end{aligned}$$

Thus, by Theorem 4.1, $(\mathcal{P})$ has a solution.

4.3 Optimality Conditions

We now obtain some optimality conditions for $(\mathcal{P})$.

Theorem 4.4 (Optimality condition I) *A vector $x \in D \cap \operatorname{dom} f$ is a solution of $(\mathcal{P})$ if and only if*

$$0 \in \overline{\partial f(x) + N_D(x)}. \tag{4.16}$$

Proof. Clearly, $\mathrm{Sol}(\mathcal{P})$ coincides with the solution set of a problem

$$(\mathcal{P}') \quad \min\{f(x) + \delta(x, D) \mid x \in X\}.$$

Since the functions f, $\delta(\cdot, D)$ are proper generalized polyhedral convex and since $D \cap \operatorname{dom} f \neq \emptyset$, the function $\widetilde{f} := f + \delta(\cdot, D)$ is proper generalized polyhedral convex by Theorem 2.3. On one hand, by Theorem 3.6 we have

$$\partial \widetilde{f}(x) = \overline{\partial f(x) + N_D(x)} \quad (x \in D \cap \operatorname{dom} f).$$

On the other hand, since $\widetilde{f}$ is proper convex, a vector $x \in X$ belongs to $\mathrm{Sol}(\mathcal{P}')$ if and only if $0 \in \partial\widetilde{f}(x)$; see [40, Proposition 1, p. 81]. Therefore, vector $x \in D \cap \mathrm{dom}\, f$ is a solution of $(\mathcal{P})$ if and only if $0 \in \partial\widetilde{f}(x) = \overline{\partial f(x) + N_D(x)}$.

The proof is complete. □

One may ask: *The closure sign in* (4.16) *can be omitted, or not?* If X is finite-dimensional, then f and $\delta(\cdot, D)$ are polyhedral convex functions. So, by Theorem 3.7,

$$\partial(f + \delta(\cdot, D))(x) = \partial f(x) + N_D(x) \quad (x \in D \cap \mathrm{dom}\, f).$$

So, the closure sign in (4.16) is superfluous. However, as shown in next example, if X is infinite-dimensional, then the closure sign in (4.16) is essential.

Example 4.2 According to [9, Example 3.34], one can find an infinite-dimensional Hilbert space X and two suitable closed linear subspaces X_1, X_2 of X with $\overline{X_1 + X_2} = X$ and $X_1 + X_2 \neq X$. Let D_i be the *orthogonal complement* of X_i, i.e.,

$$D_i = \{x \in X \mid \langle x, u\rangle = 0,\ \forall u \in X_i\}, \quad (i = 1, 2).$$

It is clear that D_1, D_2 are generalized polyhedral convex sets in X and $D_1 \cap D_2 = \{0\}$. Since $X_1 + X_2 \neq X$, there exists $v^* \in X \setminus (X_1 + X_2)$. Hence, $-v^* \notin X_1 + X_2$ because $X_1 + X_2$ is a linear subspace. Consider a generalized polyhedral convex optimization problem $(\mathcal{P})$ with

$$f(x) = \begin{cases} \langle v^*, x\rangle & \text{if } x \in D_1 \\ +\infty & \text{if } x \notin D_1, \end{cases}$$

and $D = D_2$. Obviously, $\mathrm{Sol}(\mathcal{P}) = \{0\}$. Note that

$$\partial f(0) = v^* + N_{D_1}(0) = v^* + X_1.$$

Combining this with the equality $N_D(0) = X_2$, one has

$$\partial f(0) + N_D(0) = v^* + X_1 + X_2.$$

The inclusion $-v^* \notin X_1 + X_2$ yields $0 \notin v^* + X_1 + X_2$. It follows that $x = 0$ is a solution of $(\mathcal{P})$, but $0 \notin \partial f(0) + N_D(0)$.

Note that, if f is a polyhedral convex function or D is a polyhedral convex set, then $\partial(f + \delta(\cdot, D))(x) = \partial f(x) + N_D(x)$ for all $x \in D \cap \mathrm{dom}\, f$ by Theorem 3.7. Thus, the following statement holds.

Theorem 4.5 (Optimality condition II) *Assume that either f is a proper polyhedral convex function or D is polyhedral convex set. Then, $x \in D \cap \operatorname{dom} f$ is a solution of $(\mathcal{P})$ if and only if $0 \in \partial f(x) + N_D(x)$.*

The forthcoming example is designed as an illustration for Theorem 4.5.

Example 4.3 Let $(\mathcal{P})$ be the problem described in Example 4.1. To solve it, we first compute the set $\partial f(x) + N_D(x)$ for every $x \in D$. Clearly, f is a polyhedral convex function with $\operatorname{dom} f = X$. Hence, $\operatorname{dom} f$ can be given by (4.6), where $Z = \{0\}, B \equiv 0, z = 0$ and $q = 0$. Therefore, by (4.7) one gets

$$\partial f(x) = \operatorname{conv}\{v_k^* \mid k \in \Theta(x)\} \quad (x \in X).$$

Since L is a closed affine subspace of X, by [14, Remark 2.196], one can find a continuous surjective linear mapping A from X to a locally convex Hausdorff topological vector space Y and a vector $y \in Y$ satisfying

$$L = \{x \in X \mid A(x) = y\}.$$

It is easy to verify that $\ker A = L - e_0$. Combining this with (3.1), we obtain

$$N_D(x) = \operatorname{cone}\{x_i^* \mid i \in I(x)\} + (L - e_0)^\perp$$

for every $x \in D$. Hence,

$$\partial f(x) + N_D(x) = \operatorname{conv}\{v_k^* \mid k \in \Theta(x)\} + \operatorname{cone}\{x_i^* \mid i \in I(x)\} + (L - e_0)^\perp \tag{4.17}$$

for every $x \in D$. Now, suppose that x is a solution of $(\mathcal{P})$. The "only if" part of Theorem 4.5 tells us that $0 \in \partial f(x) + N_D(x)$. Then, due to (4.17), we have

$$0 = \lambda_1 v_1^* + \lambda_2 v_2^* + \mu_1 x_1^* + \mu_2 x_2^* + x_0^*,$$

where $\lambda_1 \geq 0, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, $\mu_1 \geq 0$, $\mu_2 \geq 0$, $x_0^* \in (L - e_0)^\perp$, with $\lambda_k = 0$ if $k \notin \Theta(x)$, and $\mu_i = 0$ if $i \notin I(x)$. Since $v_1^* = x_1^* - x_2^*$ and $v_2^* = -x_1^* - x_2^*$, one has

$$(\lambda_1 - \lambda_2 + \mu_1)x_1^* + (\mu_2 - 1)x_2^* + x_0^* = 0.$$

Therefore,

$$(\lambda_1 - \lambda_2 + \mu_1)\langle x_1^*, e_i\rangle + (\mu_2 - 1)\langle x_2^*, e_i\rangle + \langle x_0^*, e_i\rangle = 0$$

with $i = 1, 2$. For each index $i \in \{1, 2\}$, since $e_0 + e_i \in L$, we must have $\langle x_0^*, e_i\rangle = 0$. Consequently, $\lambda_1 - \lambda_2 + \mu_1 = 0$ and $\mu_2 - 1 = 0$. The latter fact yields $2 \in I(x)$, i.e., $\langle x_2^*, x\rangle = 2$. We observe that $1 \notin I(x)$. Indeed, on the contrary, suppose that $1 \in I(x)$, i.e., $\langle x_1^*, x\rangle = 1$. Then, $\langle v_1^*, x\rangle + 1 = 0$ and $\langle v_2^*, x\rangle = -3$; so $f(x) = 0$ and $\Theta(x) = \{1\}$. Thus $\lambda_2 = 0$, $\lambda_1 = 1$ and $\mu_1 = -1 < 0$, a contradiction. Since $1 \notin I(x)$, we must have $\mu_1 = 0$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$; hence $\Theta(x) = \{1, 2\}$. This means that

$$f(x) = \langle v_1^*, x\rangle + 1 = \langle v_2^*, x\rangle.$$

Of course, $\langle x_1^*, x\rangle = -\frac{1}{2}$. We have thus proved that if $x \in \mathrm{Sol}(\mathcal{P})$, then $\langle x_1^*, x\rangle = -\frac{1}{2}$ and $\langle x_2^*, x\rangle = 2$. Conversely, if $u \in D$ satisfies $\langle x_1^*, u\rangle = -\frac{1}{2}$ and $\langle x_2^*, u\rangle = 2$, then one has $\Theta(u) = \{1, 2\}$ and $I(u) = \{2\}$. By (4.17),

$$0 = \frac{1}{2}v_1^* + \frac{1}{2}v_2^* + x_2^* + 0 \in \partial f(u) + N_D(u).$$

Therefore, the "if" part in Theorem 4.5 shows that u is a solution of $(\mathcal{P})$. Thus,

$$\mathrm{Sol}(\mathcal{P}) = \left\{u \in L \mid \langle x_1^*, u\rangle = -\frac{1}{2},\ \langle x_2^*, u\rangle = 2\right\}.$$

Using this formula, one can verify that $e_0 - \frac{1}{2}e_1 + 2e_2 \in \mathrm{Sol}(\mathcal{P})$. Thus, thanks to Theorem 4.5, we have found the formula for the solution set of $(\mathcal{P})$ and showed that it is nonempty. The optimal value of $(\mathcal{P})$ is $-\frac{3}{2}$.

Under the assumptions of Theorem 4.5, by Proposition 1.6 we know that $D - \mathrm{dom}\, f$ is a polyhedral convex set in X. We want to have an analogue of Theorem 4.5 in a Banach space setting for the case $D - \mathrm{dom}\, f$ is a generalized polyhedral convex set. Next lemma is useful for the proof of the desired result.

Lemma 4.2 *Let L_1, L_2 be closed affine subspaces, P_1, P_2 polyhedral convex sets in X. Suppose that $D_1 := L_1 \cap P_1$ and $D_2 := L_2 \cap P_2$ are nonempty. If $D_1 - D_2$ is a generalized polyhedral convex set in X, so is $L_1 - L_2$.*

Proof. For any $i \in \{1, 2\}$ and $x_i \in D_i$, observe that $D_i' := D_i - x_i$ is a nonempty polyhedral convex set in the closed linear subspace $M_i := L_i - x_i$. By Theorem 1.3, there exist $u_{i,1}, \ldots, u_{i,k_i}$ in M_i, $v_{i,1}, \ldots, v_{i,\ell_i}$ in M_i, and a closed linear subspace $M_{i,0}$ of finite codimension of M_i such that

$$D_i' = \mathrm{conv}\, \{u_{i,1}, \ldots, u_{i,k_i}\} + \mathrm{cone}\, \{v_{i,1}, \ldots, v_{i,\ell_i}\} + M_{i,0}.$$

Due to the finite codimension property of $M_{i,0}$ in M_i, one can find $x_{i,1}, \ldots, x_{i,m_i}$

in M_i such that $M_i = M_{i,0} + \text{span}\{x_{i,1}, \ldots, x_{i,m_i}\}$. Hence,

$$\begin{aligned} M_i &\supset D_i' + \text{span}\{x_{i,1}, \ldots, x_{i,m_i}\} \\ &= \text{conv}\,\{u_{i,1}, \ldots, u_{i,k_i}\} + \text{cone}\,\{v_{i,1}, \ldots, v_{i,\ell_i}\} + M_{i,0} + \text{span}\{x_{i,1}, \ldots, x_{i,m_i}\} \\ &= \text{conv}\,\{u_{i,1}, \ldots, u_{i,k_i}\} + \text{cone}\,\{v_{i,1}, \ldots, v_{i,\ell_i}\} + M_i \\ &= M_i. \end{aligned}$$

This forces $M_i = D_i' + \text{span}\{x_{i,1}, \ldots, x_{i,m_i}\}$. Consequently,

$$\begin{aligned} L_i = x_i + M_i &= x_i + D_i' + \text{span}\{x_{i,1}, \ldots, x_{i,m_i}\} \\ &= D_i + \text{span}\{x_{i,1}, \ldots, x_{i,m_i}\}. \end{aligned}$$

It is clear that $-L_2 = (-D_2) + \text{span}\{x_{2,1}, \ldots, x_{2,m_2}\}$. Therefore,

$$L_1 - L_2 = (D_1 - D_2) + \text{span}\{x_{1,1}, \ldots, x_{1,m_1}, x_{2,1}, \ldots, x_{2,m_2}\}.$$

Since $D_1 - D_2$ is a generalized polyhedral convex set by our assumption and $\text{span}\{x_{1,1}, \ldots, x_{1,m_1}, x_{2,1}, \ldots, x_{2,m_2}\}$ is a finite-dimensional subspace, $L_1 - L_2$ is a generalized polyhedral convex set; see Proposition 1.5. □

Theorem 4.6 (Optimality condition III) *Suppose that X is a Banach space and the set $D - \text{dom}\, f$ is generalized polyhedral convex. Then, $x \in D \cap \text{dom}\, f$ is a solution of $(\mathcal{P})$ if and only if $0 \in \partial f(x) + N_D(x)$.*

Proof. Let $\text{dom}\, f$ be described by (4.6). Put

$$L_1 = \{x \in X \mid A(x) = y\},\ L_2 = \{x \in X \mid B(x) = z\}$$

and

$$\begin{aligned} P_1 &= \{x \in X \mid \langle x_i^*, x\rangle \le \alpha_i,\ i = 1, \ldots, p\}, \\ P_2 &= \{x \in X \mid \langle u_j^*, x\rangle \le \gamma_j,\ j = 1, \ldots, q\}. \end{aligned}$$

Clearly, L_1, L_2 are closed affine subspaces, and P_1, P_2 are polyhedral convex sets. One has $D = L_1 \cap P_1$ and $\text{dom}\, f = L_2 \cap P_2$. Since $D - \text{dom}\, f$ is a generalized polyhedral convex set, $L_1 - L_2$ is a generalized polyhedral convex set by Lemma 4.2. For every $i \in \{1, 2\}$, select a point $x_i \in L_i$. Obviously,

$$\begin{aligned} \ker A + \ker B &= \ker A - \ker B \\ &= (L_1 - x_1) - (L_2 - x_2) = (L_1 - L_2) - (x_1 - x_2). \end{aligned}$$

Since $L_1 - L_2$ is a generalized polyhedral convex set, $\ker A + \ker B$ is a generalized polyhedral convex set by Proposition 1.5. In particular, $\ker A + \ker B$ is closed. Hence, by [16, Theorem 2.16],

$$(\ker A)^{\perp} + (\ker B)^{\perp} = (\ker A \cap \ker B)^{\perp}. \tag{4.18}$$

Therefore, for every $x \in D \cap \operatorname{dom} f$, from (3.1), (4.7), and (4.18) we obtain

$$\begin{aligned}\partial f(x) + N_D(x) &= \operatorname{conv}\{v_k^* \mid k \in \Theta(x)\} + \operatorname{cone}\{u_j^* \mid j \in J(x)\} + (\ker B)^{\perp} \\ &\qquad + \operatorname{cone}\{x_i^* \mid i \in I(x)\} + (\ker A)^{\perp} \\ &= \operatorname{conv}\{v_k^* \mid k \in \Theta(x)\} + \operatorname{cone}\{x_i^* \mid i \in I(x)\} \\ &\qquad + \operatorname{cone}\{u_j^* \mid j \in J(x)\} + (\ker A \cap \ker B)^{\perp}.\end{aligned}$$

Then, by the representation theorem for generalized polyhedral convex sets Theorem 1.2, we conclude that $\partial f(x) + N_D(x)$ is a generalized polyhedral convex set. So, the latter is closed. Combining this with Theorem 4.4, we obtain the assertion. □

Turning back to the optimality condition given by Theorem 4.4, we observe that sometimes it is difficult to find the topological closure of $\partial f(x)+N_D(x)$. The forthcoming theorem gives a new optimality condition for $(\mathcal{P})$ in the general case, where no topological closure sign is needed.

Theorem 4.7 (Optimality condition IV) *A vector $x \in D \cap \operatorname{dom} f$ is a solution of $(\mathcal{P})$ if and only if*

$$\begin{aligned}0 \in\ &\operatorname{conv}\{v_k^* \mid k \in \Theta(x)\} + \operatorname{cone}\{x_i^* \mid i \in I(x)\} \\ &\qquad + \operatorname{cone}\{u_j^* \mid j \in J(x)\} + (\ker A \cap \ker B)^{\perp}.\end{aligned} \tag{4.19}$$

For proving this result, we will use the directional differentiability property of convex functions and a lemma.

According Theorem 2.4, if f is a proper generalized polyhedral convex function (resp., a proper polyhedral convex function), so is $f'(x;\cdot)$. In addition, from (2.8) it follows that $\bar{x} \in D$ is a solution of $(\mathcal{P})$ if and only if $f'(\bar{x};h) \geq 0$ for every $h \in T_D(\bar{x})$.

Lemma 4.3 *If $x \in \operatorname{dom} f$, then*

$$f'(x;h) = \begin{cases} \max\{\langle v_k^*, h\rangle \mid k \in \Theta(x)\} & \text{if } h \in T_{\operatorname{dom} f}(x) \\ +\infty & \text{if } h \notin T_{\operatorname{dom} f}(x). \end{cases} \tag{4.20}$$

Proof. Since $\operatorname{dom} f$ is a nonempty generalized polyhedral convex set, by Proposition 1.9 one has $T_{\operatorname{dom} f}(x) = \operatorname{cone}[(\operatorname{dom} f) - x]$.

If $h \notin T_{\operatorname{dom} f}(x)$, then $x + th \notin \operatorname{dom} f$ for every $t > 0$. So, $f'(x;h) = +\infty$.

If $h \in T_{\operatorname{dom} f}(x)$, then there exists $\delta_0 > 0$ such that $x + th \in \operatorname{dom} f$ for all $t \in [0, \delta_0]$. Therefore, for every $t \in [0, \delta_0]$, by (2.1) one has

$$f(x + th) = \max \{\langle v_k^*, x\rangle + \beta_k + t\langle v_k^*, h\rangle \mid k = 1, \ldots, m\}. \tag{4.21}$$

Select an index $k_0 \in \Theta(x)$. For any $\ell \notin \Theta(x)$, as

$$f(x) = \langle v_{k_0}^*, x\rangle + \beta_{k_0} > \langle v_\ell^*, x\rangle + \beta_\ell,$$

there must exists $\delta_\ell > 0$ satisfying

$$\langle v_{k_0}^*, x\rangle + \beta_{k_0} + t\langle v_{k_0}^*, h\rangle > \langle v_\ell^*, x\rangle + \beta_\ell + t\langle v_\ell^*, h\rangle \quad (t \in [0, \delta_\ell]). \tag{4.22}$$

Choose $\delta > 0$ such that $\delta \le \delta_0$ and $\delta \le \delta_\ell$ for all $\ell \notin \Theta(x)$. Then, for every $t \in [0, \delta]$, from (4.21) and (4.22) it follows that

$$\begin{aligned} f(x + th) &\ge \langle v_{k_0}^*, x\rangle + \beta_{k_0} + t\langle v_{k_0}^*, h\rangle \\ &> \max \{\langle v_\ell^*, x\rangle + \beta_\ell + t\langle v_\ell^*, h\rangle \mid \ell \notin \Theta(x)\}. \end{aligned} \tag{4.23}$$

Thus, combining (4.21) with (4.23), we have

$$\begin{aligned} f(x + th) &= \max \{\langle v_k^*, x\rangle + \beta_k + t\langle v_k^*, h\rangle \mid k \in \Theta(x)\} \\ &= \max \{f(x) + t\langle v_k^*, h\rangle \mid k \in \Theta(x)\} \\ &= f(x) + t \max \{\langle v_k^*, h\rangle \mid k \in \Theta(x)\} \end{aligned}$$

for all $t \in [0, \delta]$. It follows that $f'(x; h) = \max \{\langle v_k^*, h\rangle \mid k \in \Theta(x)\}$. We have thus proved formula (4.20). □

Proof of Theorem 4.7. Let $x \in D \cap \operatorname{dom} f$. First, to prove the sufficiency, suppose that (4.19) is fulfilled. Then there exist nonnegative numbers λ_k, $\mu_{1,i}$, $\mu_{2,j}$, for $k \in \Theta(x)$, $i \in I(x)$, $j \in J(x)$, and an element $u^* \in (\ker A \cap \ker B)^\perp$ such that $\sum_{k \in \Theta(x)} \lambda_k = 1$ and

$$\sum_{k \in \Theta(x)} \lambda_k v_k^* + \sum_{i \in I(x)} \mu_{1,i} x_i^* + \sum_{j \in J(x)} \mu_{2,j} u_j^* + u^* = 0. \tag{4.24}$$

For any $h \in T_D(x)$, we have $f'(x; h) \ge 0$. Indeed, if $h \notin T_{\operatorname{dom} f}(x)$, then $f'(x; h) = +\infty$ by (4.20). If $h \in T_{\operatorname{dom} f}(x)$, then

$$h \in \operatorname{cone}((D \cap \operatorname{dom} f) - x). \tag{4.25}$$

To prove (4.25), we can argue as follows. Since D and $\operatorname{dom} f$ are generalized polyhedral convex sets, thanks to Proposition 1.9, we have

$$T_D(x) = \operatorname{cone}(D - x), \;\; T_{\operatorname{dom} f}(x) = \operatorname{cone}[(\operatorname{dom} f) - x].$$

This implies that

$$h \in \operatorname{cone}(D - x) \cap \operatorname{cone}[(\operatorname{dom} f) - x] = \operatorname{cone}((D \cap \operatorname{dom} f) - x).$$

So, (4.25) is valid. To proceed furthermore, from

$$D \cap \operatorname{dom} f = \big\{x \in X \mid A(x) = y,\ B(x) = z, \\ \langle x_i^*, x\rangle \le \alpha_i,\ i \in I,\ \langle u_j^*, x\rangle \le \gamma_j,\ j \in J\big\},$$

we deduce that

$$\operatorname{cone}((D \cap \operatorname{dom} f) - x) = \big\{u \in X \mid A(u) = 0,\ \langle x_i^*, u\rangle \le 0,\ i \in I(x), \\ B(u) = 0,\ \langle u_j^*, u\rangle \le 0,\ j \in J(x)\big\}. \tag{4.26}$$

Thus, by (4.25) one has $\langle x_i^*, h\rangle \le 0$ for every $i \in I(x)$, $\langle u_j^*, h\rangle \le 0$ for all $j \in J(x)$, and $h \in \ker A \cap \ker B$. Since $h \in T_{\operatorname{dom} f}(x)$, one has

$$f'(x; h) = \max\{\langle v_k^*, h\rangle \mid k \in \Theta(x)\}$$

by Lemma 4.3. Therefore, using the equality $\sum\limits_{k\in\Theta(x)} \lambda_k = 1$ and (4.24), we have

$$\begin{aligned} f'(x; h) &\ge \sum_{k\in\Theta(x)} \lambda_k \langle v_k^*, h\rangle \\ &= \left\langle -\left(\sum_{i\in I(x)} \mu_{1,i} x_i^* + \sum_{j\in J(x)} \mu_{2,j} u_j^* + u^*\right), h\right\rangle \\ &= \sum_{i\in I(x)} \mu_{1,i}(-\langle x_i^*, h\rangle) + \sum_{j\in J(x)} \mu_{2,j}(-\langle u_j^*, h\rangle) + \langle u^*, h\rangle \ge 0. \end{aligned}$$

We have proved $f'(x; h) \ge 0$ for every $h \in T_D(x)$. Hence, x is a solution of $(\mathcal{P})$.

Now, to prove the necessity, denote the set on the right-hand side of (4.19) by Q and suppose that $0 \notin Q$. We need to show that $x \notin \operatorname{Sol}(\mathcal{P})$. Due to Theorem 1.2, Q is a nonempty generalized polyhedral convex set. In particular, Q is convex and weakly*-closed. Since $0 \notin Q$, by the strong separation theorem [65, Theorem 3.4(b)] we can find $v \in X$ and $\gamma \in \mathbb{R}$ such that

$$\sup_{x^*\in Q} \langle x^*, v\rangle < \gamma < \langle 0, v\rangle. \tag{4.27}$$

On one hand, the first inequality in (4.27) implies that the linear functional $\langle \cdot, v\rangle$ is bounded from above on Q. Hence, by Theorem 1.12, the generalized linear programming problem $\max\{\langle x^*, v\rangle \mid x^* \in Q\}$ has a solution. Then we have $\langle v^*, v\rangle \le 0$ for all $v^* \in 0^+Q$ (see Proposition 1.11). On the other hand, (4.19) yields

$$0^+Q = \operatorname{cone}\{x_i^* \mid i \in I(x)\} + \operatorname{cone}\{u_j^* \mid j \in J(x)\} + (\ker A \cap \ker B)^\perp.$$

Therefore, $\langle x_i^*, v\rangle \le 0$ for every $i \in I(x)$, $\langle u_j^*, v\rangle \le 0$ for all $j \in J(x)$, and v belongs to $((\ker A \cap \ker B)^\perp)^\perp$. Since $\ker A \cap \ker B$ is a closed linear subspace of X, applying [14, Proposition 2.40], one has

$$((\ker A \cap \ker B)^\perp)^\perp = \ker A \cap \ker B.$$

Consequently, formula (4.26) allows us to have $v \in \operatorname{cone}((D \cap \operatorname{dom} f) - x)$, i.e., $v \in T_D(x) \cap T_{\operatorname{dom} f}(x)$. Hence, by Lemma 4.3 one has

$$f'(x; v) = \max\{\langle v_k^*, v\rangle \mid k \in \Theta(x)\}.$$

For every $k \in \Theta(x)$, since $v_k^* \in Q$, the inequalities in (4.27) yield

$$\langle v_k^*, v\rangle < \gamma < 0.$$

Consequently, $f'(x; v) = \max\{\langle v_k^*, v\rangle \mid k \in \Theta(x)\} < \gamma < 0$. So, we have $x \notin \operatorname{Sol}(\mathcal{P})$.

The proof is complete. □

4.4 Duality

In this section, we will use the general conjugate duality scheme presented in [14, pp. 107–108] to construct a dual problem for $(\mathcal{P})$ and obtain several duality theorems.

If we define $F : X \to \bar{\mathbb{R}}$ and $G : X \to X$, respectively, by $F(\cdot) = \delta(\cdot, D)$ and $G(x) = x$, then problem $(\mathcal{P})$ can be rewritten as

$$(\widetilde{\mathcal{P}}) \qquad \min\{f(x) + F(G(x)) \mid x \in X\}.$$

By the conjugate duality scheme in [14, formulas (2.298) and (2.296)], we obtain the following *dual problem* of $(\widetilde{\mathcal{P}})$:

$$(\widetilde{\mathcal{D}}) \qquad \max\Big\{\inf_{x\in X} L(x, x^*) - F^*(x^*) \mid x^* \in X^*\Big\},$$

where $L(x, x^*) := f(x) + \langle x^*, G(x)\rangle$ is the *standard Lagrangian* of $(\widetilde{\mathcal{P}})$. On one hand, it holds that $F^*(x^*) = \delta^*(\cdot, D)(x^*)$, where $\delta^*(\cdot, D)(x^*) = \sup_{x \in D} \langle x^*, x\rangle$ is the *support function* of D. On the other hand,

$$\begin{aligned}\inf_{x\in X} L(x, x^*) &= \inf_{x\in X} \big(f(x) + \langle x^*, G(x)\rangle\big)\\ &= \inf_{x\in X} \big(f(x) + \langle x^*, x\rangle\big)\\ &= -\sup_{x\in X} \big(\langle -x^*, x\rangle - f(x)\big) = -f^*(-x^*).\end{aligned}$$

Therefore, $(\widetilde{\mathcal{D}})$ is nothing than the following problem

$$(\mathcal{D}) \qquad \max\{g(x^*) \mid x^* \in X^*\}$$

with

$$g(x^*) := \inf_{x\in X} L(x, x^*) - F^*(x^*) = -f^*(-x^*) - \delta^*(\cdot, D)(x^*).$$

Since f and $\delta(\cdot, D)$ are proper generalized polyhedral convex functions, by Theorem 3.4 we can assert that f^* and $\delta^*(\cdot, D)$ are proper generalized polyhedral convex functions. Hence, in particular, $x^* \mapsto f^*(-x^*)$ is a proper generalized polyhedral convex function. If $(-\operatorname{dom} f^*) \cap \operatorname{dom} \delta^*(\cdot, D) \neq \emptyset$, then $-g$ is a proper generalized polyhedral convex function by Theorem 2.3. So, the objective function of the maximization problem $(\mathcal{D})$ is generalized polyhedral concave. If $(-\operatorname{dom} f^*) \cap \operatorname{dom} \delta^*(\cdot, D) = \emptyset$, then $(-g)(x^*) = +\infty$ for all $x^* \in X^*$. In this case, the objective function of $(\mathcal{D})$ is also generalized polyhedral concave.

A *weak duality* relationship between $(\mathcal{P})$ and $(\mathcal{D})$ can be described as follows.

Theorem 4.8 (Weak duality theorem) *For every $u \in D$ and $u^* \in X^*$, the inequality $g(u^*) \leq f(u)$ holds. Hence, if $f(u) = g(u^*)$, then $u \in \mathrm{Sol}(\mathcal{P})$ and $u^* \in \mathrm{Sol}(\mathcal{D})$.*

Proof. Given any $u \in D$ and $u^* \in X^*$, it suffices to observe that

$$\begin{aligned}g(u^*) &= -f^*(-u^*) - \delta^*(\cdot, D)(u^*)\\ &= \inf_{x\in X} \big[\langle u^*, x\rangle + f(x)\big] - \sup_{x\in D}\langle u^*, x\rangle\\ &\leq \langle u^*, u\rangle + f(u) - \langle u^*, u\rangle = f(u).\end{aligned} \tag{4.28}$$

This justifies the assertions of the theorem. □

Remark 4.6 The result in Theorem 4.8 is a known one (see, e.g., [70, Theorem 5.7], [71, Theorem 5.1]).

Since the existence of an element u^* satisfying $u^* \in N_D(u) \cap (-\partial f(u))$ is equivalent to the property $0 \in \partial f(u) + N_D(u)$, next statement can be interpreted as a sufficient optimality condition for $(\mathcal{P})$ and $(\mathcal{D})$.

Proposition 4.2 *If $u \in X$ and $u^* \in N_D(u) \cap (-\partial f(u))$, then $u \in \mathrm{Sol}(\mathcal{P})$ and $u^* \in \mathrm{Sol}(\mathcal{D})$. Moreover, the optimal values of $(\mathcal{P})$ and $(\mathcal{D})$ are equal.*

Proof. Suppose that $u \in X$ and $u^* \in N_D(u) \cap (-\partial f(u))$. Then one has $u \in D \cap \mathrm{dom}\, f$. On one hand, since $-u^* \in \partial f(u)$, by [78, Theorem 2.4.2(iii)] we can assert that

$$f(u) + f^*(-u^*) = \langle -u^*, u \rangle.$$

So, $-f^*(-u^*) = f(u) + \langle u^*, u \rangle$. On the other hand, the inclusion $u^* \in N_D(u)$ implies that $\sup\{\langle u^*, x \rangle \mid x \in D\} = \langle u^*, u \rangle$; hence $\delta^*(\cdot, D)(u^*) = \langle u^*, u \rangle$. Consequently,

$$g(u^*) = -f^*(-u^*) - \delta^*(\cdot, D)(u^*) = f(u).$$

Thus, the desired conclusions follow from Theorem 4.8. □

If the optimal value of $(\mathcal{D})$ equals to the optimal value of $(\mathcal{P})$, then one says that the *strong duality* relationship among the dual pair holds. We are going to show that if either f is polyhedral convex or D is polyhedral convex, then this property is available under a mild condition.

Theorem 4.9 (Strong duality theorem I) *Assume that either f is a proper polyhedral convex function and D is a nonempty generalized polyhedral convex set, or f is a proper generalized polyhedral convex function and D is a nonempty polyhedral convex set. If one of the two problems has a solution, then both of them have solutions and the optimal values are equal.*

Proof. Under the assumptions of the theorem, we suppose firstly that $(\mathcal{P})$ has a solution u. Then, according to Theorem 4.5, it holds that $0 \in \partial f(u) + N_D(u)$. Hence there exists $u^* \in N_D(u) \cap (-\partial f(u))$. Applying Proposition 4.2 yields the solution existence of $(\mathcal{D})$ and the equality of the optimal values.

Secondly, suppose that $(\mathcal{D})$ has a solution u^*. Since f is a proper generalized polyhedral convex function, $\mathrm{dom}\, f$ is a nonempty generalized polyhedral convex set by Theorem 2.1. If f is a proper polyhedral convex function then, also by Theorem 2.1, $\operatorname{dom} f$ is a nonempty polyhedral convex set. Thus, by the assumptions of the theorem, $\operatorname{dom} f$ and $-D$ are generalized polyhedral convex set, and one of them is polyhedral convex. Hence, in accordance with Proposition 1.6, the set $(\operatorname{dom} f) - D$ is polyhedral convex. In particular, $(\operatorname{dom} f) - D$ is a closed set. Let us show that $D \cap \operatorname{dom} f$ is nonempty. On the contrary, suppose that $D \cap \operatorname{dom} f = \emptyset$. Hence, $0 \notin (\operatorname{dom} f) - D$. Since the nonempty set $(\operatorname{dom} f) - D$ is closed, by the strong separation theorem [65, Theorem 3.4(b)] there exist $x^* \in X^*$ and real number ε such that $0 < \varepsilon < \langle x^*, x - u \rangle$ for all $x \in \operatorname{dom} f$ and $u \in D$. Consequently,

$$\varepsilon + \sup_{u \in D} \langle x^*, u \rangle \le \inf_{x \in \operatorname{dom} f} \langle x^*, x \rangle. \tag{4.29}$$

On one hand, for any $\lambda > 0$, using the equalities in (4.28) and the inequality (4.29) we have

$$\begin{aligned}
&g(u^* + \lambda x^*) \\
&= \inf_{x \in X} [\langle u^* + \lambda x^*, x \rangle + f(x)] - \sup_{x \in D} \langle u^* + \lambda x^*, x \rangle \\
&= \inf_{x \in \operatorname{dom} f} [\langle u^* + \lambda x^*, x \rangle + f(x)] - \sup_{u \in D} \langle u^* + \lambda x^*, u \rangle \\
&\ge \inf_{x \in \operatorname{dom} f} [\langle u^*, x \rangle + f(x)] + \lambda \inf_{x \in \operatorname{dom} f} \langle x^*, x \rangle - \sup_{u \in D} \langle u^*, u \rangle - \lambda \sup_{u \in D} \langle x^*, u \rangle \\
&= \left(\inf_{x \in \operatorname{dom} f} [\langle u^*, x \rangle + f(x)] - \sup_{u \in D} \langle u^*, u \rangle \right) + \lambda \left[\inf_{x \in \operatorname{dom} f} \langle x^*, x \rangle - \sup_{u \in D} \langle x^*, u \rangle \right] \\
&\ge g(u^*) + \lambda \varepsilon.
\end{aligned}$$

On the other hand, since u^* is a solution of $(\mathcal{D})$, the estimate

$$g(u^*) \ge g(u^* + \lambda x^*)$$

is valid. Hence, $g(u^*) \ge g(u^*) + \lambda \varepsilon$. This contradicts the fact that λ, ε are positive numbers. Thus, we have proved that $(\operatorname{dom} f) \cap D \neq \emptyset$. Setting $\gamma = g(u^*)$ and applying Theorem 4.8, we obtain $f(x) \ge \gamma$ for all $x \in D$. Therefore, on account of Theorem 4.1, we can assert that $(\mathcal{P})$ has a solution. Finally, to show that the optimal values of $(\mathcal{P})$ and $(\mathcal{D})$ are equal, it suffices to use the result already obtained in the first part of this proof. □

Example 4.4 Consider problem $(\mathcal{P})$ in the setting and notations of Example 4.1. To have a concrete form of the dual problem $(\mathcal{D})$, we have to find the function g. Suppose that $x^* \in X^*$ and $|g(x^*)| < \infty$. Since f is proper, for $\alpha := \inf_{x \in X} [f(x) + \langle x^*, x \rangle]$, we have $\alpha < +\infty$. In addition, as D is nonempty, the number $\beta := \sup_{x\in D} \langle x^*, x\rangle$ is greater than $-\infty$. Thus, the equalities in (4.28) yield $+\infty > \alpha = g(x^*) + \beta > -\infty$. Hence, both α and β are finite. In particular, the function $x \mapsto f(x) + \langle x^*, x\rangle$ is bounded from below on X. Since $f(x) = \max\{\langle v_1^*, x\rangle + 1, \langle v_2^*, x\rangle\}$, one see that

$$f(\cdot) + \langle x^*, \cdot\rangle = \max\{\langle v_1^* + x^*, \cdot\rangle + 1, \langle v_2^* + x^*, \cdot\rangle\} \tag{4.30}$$

is a polyhedral convex function. So, according to Theorem 4.1, the generalized polyhedral convex optimization problem $\min\{f(x) + \langle x^*, x\rangle \mid x \in X\}$ has a solution. Therefore, by (4.30) and Corollary 4.2, we must have

$$0 \in \operatorname{conv}\{v_1^* + x^*, v_2^* + x^*\}.$$

Let $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ be such that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1(v_1^* + x^*) + \lambda_2(v_2^* + x^*) = 0$. It is clear that

$$\begin{aligned} x^* &= -\lambda_1 v_1^* - \lambda_2 v_2^* = -\lambda_1(x_1^* - x_2^*) - \lambda_2(-x_1^* - x_2^*) \\ &= (1 - 2\lambda_1)x_1^* + x_2^*. \end{aligned}$$

Writing $\lambda = 1 - 2\lambda_1$, we obtain $x^* = \lambda x_1^* + x_2^*$ with $\lambda \in [-1, 1]$. It is a simple matter to verify that $e_0 + t_1 e_1 \in D$ for all $t_1 \leq 1$. Hence,

$$\sup_{x\in D} \langle x^*, x\rangle \geq \sup_{t_1\leq 1} \langle x^*, e_0 + t_1 e_1\rangle = \sup_{t_1\leq 1} \lambda t_1.$$

If $\lambda < 0$, then $\sup_{t_1\leq 1} \lambda t_1 = +\infty$. So, we get $\sup_{x\in D} \langle x^*, x\rangle = +\infty$, which contradicts the fact that $\beta \in \mathbb{R}$. Thus we have proved that if $g(x^*)$ is finite, then there exists $\lambda \in [0, 1]$ satisfying $x^* = \lambda x_1^* + x_2^*$. Let us compute the value $g(x^*)$ when $|g(x^*)| < \infty$. Suppose that $x^* = \lambda x_1^* + x_2^*$ with $\lambda \in [0, 1]$. For every $x \in D$, since $\langle x_1^*, x\rangle \leq 1$ and $\langle x_2^*, x\rangle \leq 2$, one has $\langle x^*, x\rangle = \lambda\langle x_1^*, x\rangle + \langle x_2^*, x\rangle \leq \lambda + 2$. In addition, as $e_0 + e_1 + 2e_2 \in D$ with $\langle x^*, e_0 + e_1 + 2e_2\rangle = \lambda + 2$, we must have $\sup_{x\in D} \langle x^*, x\rangle = \lambda + 2$. From (4.30) we see that

$$f(x) + \langle x^*, x\rangle = \max\{(\lambda + 1)\langle x_1^*, x\rangle + 1, (\lambda - 1)\langle x_1^*, x\rangle\}.$$

If $\langle x_1^*, x\rangle \geq -\frac{1}{2}$, then

$$\begin{aligned} f(x) + \langle x^*, x\rangle &= (\lambda + 1)\langle x_1^*, x\rangle + 1 \\ &\geq (\lambda + 1)\left(-\frac{1}{2}\right) + 1 = \frac{1}{2} - \frac{\lambda}{2}. \end{aligned}$$

If $\langle x_1^*, x\rangle < -\frac{1}{2}$, then

$$\begin{aligned} f(x) + \langle x^*, x\rangle &= (\lambda - 1)\langle x_1^*, x\rangle \\ &\geq (\lambda - 1)\left(-\frac{1}{2}\right) = \frac{1}{2} - \frac{\lambda}{2}. \end{aligned}$$

Since $\langle x_1^*, -\frac{1}{2}e_1\rangle = -\frac{1}{2}$, we have $f(-\frac{1}{2}e_1) + \langle x^*, -\frac{1}{2}e_1\rangle = \frac{1}{2} - \frac{\lambda}{2}$. Consequently, $\inf_{x\in X} [f(x) + \langle x^*, x\rangle] = \frac{1}{2} - \frac{\lambda}{2}$. Thus, the equalities in (4.28) imply that

$$g(x^*) = \begin{cases} -\frac{3}{2} - \frac{3}{2}\lambda & \text{if } x^* = \lambda x_1^* + x_2^* \text{ with } 0 \le \lambda \le 1 \\ -\infty & \text{otherwise.} \end{cases}$$

Using this formula for g, it is easy to check that x_2^* is a unique solution of $(\mathcal{D})$ with $g(x_2^*) = -\frac{3}{2}$. In Example 4.3, we have shown that $(\mathcal{P})$ has a nonempty solution set and the optimal value is $-\frac{3}{2}$. These facts justify the assertion of Theorem 4.9 for the problems $(\mathcal{P})$ and $(\mathcal{D})$ which we are dealing with.

The conclusion of Theorem 4.9 may not true in the general case, where one just assumes that f is a proper generalized polyhedral convex function and D is a nonempty generalized polyhedral convex set.

Example 4.5 Consider problem $(\mathcal{P})$ in the setting and notations of Example 4.2. We know that $(\mathcal{P})$ has a unique solution $x = 0$. Recall that $\operatorname{dom} f = D_1$ and $f(x) = \langle v^*, x\rangle$ for all $x \in D_1$. In addition, since D_1 is the orthogonal complement of X_1, we have

$$\inf_{x\in X} [f(x) + \langle x^*, x\rangle] = \inf_{x\in D_1} \langle v^* + x^*, x\rangle = \begin{cases} 0 & \text{if } x^* \in -v^* + X_1 \\ -\infty & \text{if } x^* \notin -v^* + X_1. \end{cases} \tag{4.31}$$

Similarly, since D is the orthogonal complement of X_2,

$$\sup_{x\in D} \langle x^*, x\rangle = \begin{cases} 0 & \text{if } x^* \in X_2 \\ +\infty & \text{if } x^* \notin X_2. \end{cases} \tag{4.32}$$

Combining (4.31), (4.32) with the equalities in (4.28) yields

$$g(x^*) = \begin{cases} 0 & \text{if } x^* \in (-v^* + X_1) \cap X_2 \\ -\infty & \text{otherwise.} \end{cases}$$

Since $(-v^* + X_1) \cap X_2 = \emptyset$ (see Example 4.2), we can assert that $g(x^*) = -\infty$ for all $x^* \in X^*$. Therefore, $(\mathcal{D})$ has no solution. Thus, it happens that $(\mathcal{P})$ has a solution, while $(\mathcal{D})$ has an empty solution set.

The assumption of Theorem 4.9 implies that $D - \operatorname{dom} f$ is a polyhedral convex set in X. In particular, $D - \operatorname{dom} f$ is closed. Interestingly, in a Banach space setting, the polyhedral convexity of $D - \operatorname{dom} f$ can be replaced by its closedness – a weaker property.

Theorem 4.10 (Strong duality theorem II) *Suppose that X is a Banach space and the set $D - \operatorname{dom} f$ is closed. If one of the two problems $(\mathcal{P})$ and $(\mathcal{D})$ has a solution, then both of them have solutions and the optimal values are equal.*

Proof. First, suppose that $(\mathcal{P})$ has a solution u. Then, by the closedness of $D - \operatorname{dom} f$ and Theorem 4.6, we have $0 \in \partial f(u) + N_D(u)$. Select any $u^* \in N_D(u) \cap (-\partial f(u))$. By Proposition 4.2, u^* is a solution of $(\mathcal{D})$. Moreover, the optimal values of $(\mathcal{P})$ and $(\mathcal{D})$ are equal.

Now, suppose that $(\mathcal{D})$ has a solution u^*. Arguing similarly as in the proof of Theorem 4.9 (the closedness of $D - \operatorname{dom} f$ allows us to apply the strong separation theorem), we can prove that $(\mathcal{P})$ has a solution and the optimal values of $(\mathcal{P})$ and $(\mathcal{D})$ are equal. □

In optimization theory, a strong duality theorem can be formulated as a combined statement about the solution existence of the primal and dual problems when they have feasible points where the objective functions are finite, and the equality of the optimal values. In that spirit, for generalized polyhedral convex optimization problems we have next result.

Theorem 4.11 (Strong duality theorem III) *Suppose that the problems $(\mathcal{P})$ and $(\mathcal{D})$ have feasible points, at which the values of the object functions are finite. Then both problems have solutions. In addition, if either f or D is polyhedral convex, then there is no duality gap between the problems.*

Proof. Let $u \in D$ and $u^* \in X^*$ be such that $f(u)$ and $g(u^*)$ are finite. By Theorem 4.8, we have $f(x) \geq g(u^*)$ for every $x \in D$. Thus, f is bounded from below on D and $D \cap \operatorname{dom} f \neq \emptyset$. Therefore, $(\mathcal{P})$ has a solution by Theorem 4.1. To show that $(\mathcal{D})$ possesses a solution, we first observe by Theorem 4.8 that $-g(x^*) \geq -f(u)$ for all $x^* \in X^*$. Hence, the proper generalized polyhedral convex function $(-g)$ is bounded from below on X^* by the finite value $(-f(u))$. Consequently, by Theorem 4.1, the problem $\min\{-g(x^*) \mid x^* \in X^*\}$ has a solution. Since $(\mathcal{D})$ is equivalent to the latter, the solution set of $(\mathcal{D})$ is nonempty.

Now, if either f or D is polyhedral convex, then by using Theorem 4.9 we can assert that the optimal values of $(\mathcal{P})$ and $(\mathcal{D})$ are equal. □

Concerning Theorem 4.11, the following question seems to be interesting: *Whether the conclusion "there is no duality gap between two problems" is still true, if one drops the assumption "either f or D is polyhedral convex"?* Our attempts in constructing a counterexample have not achieved the goal, so far.

4.5 Conclusions

Generalized polyhedral convex optimization problems in locally convex Hausdorff topological vector spaces have been studied systematically in this chapter. We have established solution existence theorems, necessary and sufficient optimality conditions, weak and strong duality theorems. In particular, we have shown that the dual problem has the same structure as the primal problem, and the strong duality relation holds under three different sets of conditions.

Chapter 5

Linear and Piecewise Linear Vector Optimization Problems

In this chapter, we study structure of efficient solutions sets of linear vector optimization problems and piecewise linear vector optimization problems.

The present chapter is written on the basis of the papers [A1], [A2], and [A5] in the List of Author's Related Papers on page 113.

5.1 Preliminaries

Given two locally convex Hausdorff topological vector spaces X and Y, a vector-valued function $f : X \to Y$, a generalized polyhedral convex set $D \subset X$, and a polyhedral convex cone $K \subset Y$ with $K \neq Y$, we consider a *vector optimization problem*

$$\text{(VP)} \qquad \text{Min}_K\{f(x) \mid x \in D\}.$$

A point y from a subset $Q \subset Y$ is called an *efficient point* (resp., a *weakly efficient point*) of Q if there is no $y' \in Q$ such that $y - y' \in K \setminus \ell(K)$ (resp., $y - y' \in \text{int}\, K$). The efficient point set and the weakly efficient point set of Q are denoted, respectively, by $E(Q|K)$ and $E^w(Q|K)$. Clearly, in the case where K is a pointed cone, a point $y \in Q$ belongs to $E(Q|K)$ if and only if $y - y' \notin K \setminus \{0\}$ for every $y' \in Q$.

A vector $u \in D$ is called an *efficient solution* (resp., a *weakly efficient solution*) of (VP) if $f(u) \in E(f(D)|K)$ (resp., $f(u) \in E^w(f(D)|K)$). The efficient solution set and the weakly efficient solution set of (VP) are denoted, respectively, by Sol(VP) and Sol^w(VP).

Since $\mathrm{int}\, K \subset K \setminus \ell(K)$ by Proposition 1.10, one gets $E(Q|K) \subset E^w(Q|K)$; hence, $\mathrm{Sol(VP)} \subset \mathrm{Sol}^w\mathrm{(VP)}$.

In the terminology of [31, p. 341], one says that f is a *K-function*[1] on D if

$$(1-\lambda)f(x_1) + \lambda f(x_2) - f((1-\lambda)x_1 + \lambda x_2) \in K$$

for any x_1, x_2 in D and $\lambda \in [0,1]$. It is clear that if f is linear, then it is a K-function on D. In the case where $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, f is a K-function on D if and only if f is convex on D. When f is a K-function on D, we say that (VP) is a *convex problem*.

Similarly as in [82], we say that a mapping $f : X \to Y$ is a *piecewise linear function* (or a piecewise affine function) if there exist polyhedral convex sets $P_1, \ldots, P_m$ in X, continuous linear mappings $T_1, \ldots, T_m$ from X to Y, and vectors $b_1, \ldots, b_m$ in Y such that $X = \bigcup\limits_{k=1}^{m} P_k$ and $f(x) = T_k(x) + b_k$ for all $x \in P_k$, $k = 1, \ldots, m$.

In the sequel, if not otherwise stated, f is a piecewise linear function.

5.2 The Weakly Efficient Solution Set in Linear Vector Optimization

Let us consider a special case of (VP). Namely, we suppose that $f(x) = M(x)$ with $M : X \to Y$ being a continuous linear mapping. Consider a *generalized linear vector optimization problem*

$$\text{(LVOP)} \qquad \mathrm{Min}_K \left\{ M(x) \mid x \in D \right\}.$$

The efficient solution set and the weakly efficient solution set of (LVOP) are denoted, respectively, by Sol(LVOP) and Sol^w(LVOP).

By a standard scalarization scheme in vector optimization, given any element $y^* \in Y^*$, we define the scalar problem

$$\text{(LP)}_{y^*} \qquad \min \left\{ \langle y^*, M(x) \rangle \mid x \in D \right\}.$$

[1]Some authors use the term "*K-convex function*" instead of K-function.

To make our presentation easier for reading, we give simple proof for the following known result.

Lemma 5.1 (See [50, Proposition 3.2, p. 95]) *Suppose that* int$K$ *is nonempty. Then* $u \in D$ *is a weakly efficient solution of* (LVOP) *if and only if there exists* $y^* \in K^* \setminus \{0\}$ *such that*

$$u \in \text{argmin}\left((\text{LP})_{y^*}\right). \tag{5.1}$$

Proof. First, suppose that $u \in \text{Sol}^w(\text{LVOP})$. Since

$$\big(M(u) - M(D)\big) \cap \text{int}K = \emptyset$$

and since $M(u) - M(D)$ and int$K$ are convex sets, by the separation theorem (see, e.g., [65, Theorem 3.4(a)]), there exists $y^* \in Y^* \setminus \{0\}$ such that

$$\langle y^*, M(u) - M(x)\rangle \leq \langle y^*, y\rangle$$

for all $x \in D$ and $y \in K$. Substituting $x = u$ to the above inequality yields $\langle y^*, y\rangle \geq 0$ for all $y \in K$. Hence $y^* \in K^*$. Choosing $y = 0$, one has

$$\langle y^*, M(u)\rangle \leq \langle y^*, M(x)\rangle$$

for every $x \in D$. This shows that the inclusion (5.1) is valid.

Now, suppose that $u \in D$ and there is $y^* \in K^* \setminus \{0\}$ such that (5.1) is satisfied. If $u \notin \text{Sol}^w(\text{LVOP})$, then there exist $x \in D$ and a balanced neighborhood V of 0 satisfying $M(u) - M(x) + V \subset K$. Hence, for each $v \in V$, one has $\langle y^*, M(u) - M(x) + v\rangle \geq 0$. In combination with the inequality

$$\langle y^*, M(u) - M(x)\rangle \leq 0$$

which is guaranteed by (5.1), this implies $\langle y^*, v\rangle \geq 0$. As V is a balanced neighborhood of 0, we can assert that $\langle y^*, v\rangle = 0$ for all $v \in V$. Let $y \in Y$ be such that $\langle y^*, y\rangle \neq 0$. Since there exists $t > 0$ with $ty \in V$, we get $\langle y^*, y\rangle = 0$, a contradiction. □

Remark 5.1 Looking back to the proof of Lemma 5.1, we can observe that it suffices to assume that $K \subset Y$ is a convex cone. In other words, the polyhedrality of K is superfluous for the assertion of the lemma.

We have $\langle y^*, M(x)\rangle = \langle M^*(y^*), x\rangle$ for all $x \in X$, where $M^* : Y^* \to X^*$ is the adjoint operator of M.

Theorem 5.1 *Problem* (LVOP) *has a weakly efficient solution if and only if*

$$M^*\left(K^* \setminus \{0\}\right) \cap \left(0^+D\right)^* \neq \emptyset. \tag{5.2}$$

In particular, if

$$M^*\left(K^*\right) \cap \left(0^+D\right)^* \neq \{0\}, \tag{5.3}$$

then (LVOP) *has a weakly efficient solution.*

Proof. By Lemma 5.1, (LVOP) has a weakly efficient solution if and only if there exists $y^* \in K^* \setminus \{0\}$ such that the solution set of $(\mathrm{LP})_{y^*}$ is non empty. According to Theorem 1.11, this solution set is non-void if and only if $M^*(y^*) \in \left(0^+D\right)^*$. Thus, we have shown that (LVOP) has a weakly efficient solution if and only if (5.2) is fulfilled.

Now, suppose that (5.3) is satisfied. Then we can find an element $y^* \in K^*$ such that $M^*(y^*) \in \left(0^+D\right)^*$ and $M^*(y^*) \neq 0$. Since the later obviously implies that $y^* \neq 0$, we have

$$y^* \in M^*\left(K^* \setminus \{0\}\right) \cap \left(0^+D\right)^*.$$

Hence (5.2) holds true, so (LVOP) has a weakly efficient solution. □

Remark 5.2 The assertions of Theorem 5.1 are valid for the case $K \subset Y$ is an arbitrary convex cone.

We conclude this section by a statement about the structure of Sol^w(LVOP) which is applicable to the case where $K \subset Y$ is an arbitrary convex cone.

Theorem 5.2 *The weakly efficient solution set of* (LVOP) *is the union of finitely many generalized polyhedral convex sets.*

Proof. Using Lemma 5.1, we can represent the weakly efficient solution set of (LVOP) as follows

$$\mathrm{Sol}^w(\mathrm{LVOP}) = \bigcup_{y^* \in K^* \setminus \{0\}} \operatorname{argmin}\left((\mathrm{LP})_{y^*}\right). \tag{5.4}$$

Setting $x^* = M^*(y^*)$, we can rewrite (5.4) as

$$\mathrm{Sol}^w(\mathrm{LVOP}) = \bigcup_{x^* \in M^*(K^* \setminus \{0\})} S(x^*), \tag{5.5}$$

where $S(x^*)$ is the solution set of the problem (LP) considered in Chapter 1. Invoking (1.36) and noting that the number of the index sets $I(x^*)$ (resp., the number of the index sets $J(x^*)$) is finite, from (5.5) we obtain the desired conclusion. □

5.3 The Efficient Solution Set in Linear Vector Optimization

In the preceding section, in a locally convex Hausdorff topological vector space setting, we have obtained a scalarization formula for the weakly efficient solution set of a generalized linear vector optimization problem, and proved that the latter is the union of finitely many generalized polyhedral convex sets. It is reasonable to look for similar results for the corresponding efficient solution set.

In the remaining part of this chapter, suppose that the polyhedral convex cone K is given by (1.28). Keeping the notations of Section 1.5, we let $\pi_0 : Y \to Y/Y_0$, $y \mapsto y + Y_0$ for all $y \in Y$, be the canonical projection from Y on the quotient space Y/Y_0. It is clear that the operator

$$\Phi_0 : Y/Y_0 \to Y_1,\ [y_1] := y_1 + Y_0 \mapsto y_1$$

for all $y_1 \in Y_1$, is a linear bijective mapping. By [65, Theorem 1.41(a)], π_0 is a linear continuous mapping. Moreover, Φ_0 is a homeomorphism by Lemma 1.3. Hence, the operator $\pi := \Phi_0 \circ \pi_0 : Y \to Y_1$ is linear and continuous. Therefore, by Proposition 1.2, $D_1 := (\pi \circ M)(D)$ is a convex polyhedron in Y_1.

We now show how the verification of the inclusion $u \in \mathrm{Sol(LVOP)}$, for every $u \in D$, can be reduced to the checking a relation in the finite dimensional space Y_1.

Proposition 5.1 *For any $u \in D$, one has $u \in \mathrm{Sol(LVOP)}$ if and only if*

$$\big((\pi \circ M)(u) - D_1\big) \cap \big(K_1 \setminus \{0\}\big) = \emptyset. \tag{5.6}$$

Proof. *Necessity:* Suppose the contrary that there is some $u \in \mathrm{Sol(LVOP)}$ with

$$(u_1 - D_1) \cap \big(K_1 \setminus \{0\}\big) \neq \emptyset,$$

where $u_1 := (\pi \circ M)(u) = \pi(M(u))$. Setting $u_0 = M(u) - u_1$, we have $M(u) = u_0 + u_1$ with $u_1 = \pi(M(u)) \in Y_1$. So $u_0 \in Y_0$. Select an element $v_1 \in K_1 \setminus \{0\}$ such that $v_1 \in u_1 - D_1$. As $u_1 - v_1 \in D_1 = (\pi \circ M)(D)$, there exist $y \in D$ and $y_0 \in Y_0$ satisfying $M(y) = y_0 + (u_1 - v_1)$. Since

$$\begin{aligned} M(u) - M(y) &= (u_0 + u_1) - \big(y_0 + (u_1 - v_1)\big) \\ &= (u_0 - y_0) + v_1 \in (Y_0 - Y_0) + K_1 \setminus \{0\} = Y_0 + K_1 \setminus \{0\}, \end{aligned}$$

by Lemma 1.5, we have $M(u) - M(y) \in K \setminus \ell(K)$. This contradicts the assumption $u \in \text{Sol(LVOP)}$. We have thus proved that if $u \in \text{Sol(LVOP)}$ then (5.6) holds.

Sufficiency: Ab absurdo, suppose that there exists $u \in D$ satisfying (5.6), but $u \notin \text{Sol(LVOP)}$. As $\big(M(u) - M(D)\big) \cap \big(K \setminus \ell(K)\big) \neq \emptyset$, one can find $y \in D$ and $v \in K \setminus \ell(K)$ satisfying $M(u) - M(y) = v$. Invoking Lemma 1.5, we can assert that $v \in Y_0 + K_1 \setminus \{0\}$, i.e., $v = v_0 + v_1$ for some $v_0 \in Y_0$ and $v_1 \in K_1 \setminus \{0\}$. Then, from the equality $M(u) - M(y) = v_0 + v_1$ we get

$$\begin{aligned}\pi(M(u)) - \pi(M(y)) &= \pi(M(u) - M(y)) \\ &= \pi(v_0 + v_1) = \pi(v_0) + \pi(v_1) = v_1.\end{aligned}$$

It follows that $v_1 \in \Big(\pi(M(u)) - \pi(M(D))\Big) \cap \Big(K_1 \setminus \{0\}\Big)$. This is incompatible with (5.6). The proof is complete. □

We will need the following technical lemma of [50].

Lemma 5.2 (See [50, Lemma 2.6, p. 89]) *Suppose that Z is a finite dimensional locally convex Hausdorff topological vector space. Let A be a convex polyhedron containing 0 and $K \subset Z$ be a pointed polyhedral convex cone. If $A \cap \big(K \setminus \{0\}\big) = \emptyset$, then there exists $z^* \in Z^*$ such that*

$$\langle z^*, z\rangle \leq 0 < \langle z^*, v\rangle \tag{5.7}$$

for all $z \in A$ and for any $v \in K \setminus \{0\}$.

Proof. See [34, Lemma 2.2 (i)]. □

Theorem 5.3 *If K is not a linear subspace of Y, then $u \in Y$ is an efficient solution of* (LVOP) *if and only if there exists $y^* \in \operatorname{ri} K^*$ satisfying $u \in \operatorname{argmin}\Big(\text{(LP)}_{y^*}\Big)$. In other words,*

$$\text{Sol(LVOP)} = \bigcup_{y^* \in \operatorname{ri} K^*} \operatorname{argmin}\Big(\text{(LP)}_{y^*}\Big). \tag{5.8}$$

Proof. If $u \in \text{Sol(LVOP)}$, then (5.6) holds by Proposition 5.1. According to Proposition 1.2 and [63, Corollary 19.3.2], $\pi(M(u)) - D_1$ is a polyhedral convex set in Y_1. Using Lemma 5.2 for the convex polyhedron $\pi(M(u)) - D_1$ corresponding the pointed polyhedral convex cone K_1 in Y_1, one can find $v^* \in Y_1^*$ such that

$$\langle v^*, w\rangle \leq 0 < \langle v^*, v\rangle, \quad \forall w \in \pi(M(u)) - D_1, \forall v \in K_1 \setminus \{0\}. \tag{5.9}$$

Set $y^* = v^* \circ \pi$ and note that $y^* \in Y^*$. For any $x \in D$, since

$$\pi(M(u)) - \pi(M(x)) \in \pi(M(u)) - D_1,$$

one has

$$\langle y^*, M(u) - M(x)\rangle = \langle v^*, \pi(M(u)) - \pi(M(x))\rangle \leq 0.$$

Hence, we obtain $\langle y^*, M(u)\rangle \leq \langle y^*, M(x)\rangle$ for all $x \in D$; so

$$u \in \operatorname{argmin}\left((\mathrm{LP})_{y^*}\right).$$

Let us show that $y^* \in \operatorname{ri} K^*$. Given any $y \in K \setminus \ell(K)$, by Lemma 1.5 one can find $y_0 \in Y_0$ and $y_1 \in K_1 \setminus \{0\}$ such that $y = y_0 + y_1$. Then, by (5.9) one has

$$\langle y^*, y\rangle = \langle v^*, \pi(y)\rangle = \langle v^*, y_1\rangle > 0.$$

So, according to Theorem 1.10, $y^* \in \operatorname{ri} K^*$. The inclusion

$$\mathrm{Sol(LVOP)} \subset \bigcup_{y^* \in \operatorname{ri} K^*} \operatorname{argmin}\left((\mathrm{LP})_{y^*}\right)$$

has been established.

Now, to obtain the reverse inclusion, suppose on contrary that there exists

$$u \in \operatorname{argmin}\left((\mathrm{LP})_{y^*}\right), \tag{5.10}$$

with $y^* \in \operatorname{ri} K^*$, but $u \notin \mathrm{Sol(LVOP)}$. Select an $x \in D$ such that

$$M(u) - M(x) \in K \setminus \ell(K).$$

Then, $\langle y^*, M(u) - M(x)\rangle > 0$ by Theorem 1.10. This contradicts the condition (5.10). The proof of (5.8) is thus complete. □

Remark 5.3 One of the referees of the paper [A2] in the List of Author's Related Papers on page 113 showed that Theorem 5.3 can be proved in a shorter way by using Theorem 5.4 from [35]. To make our presentation easier for reading, the original proof, which is elementary and direct, is given here.

The scalarization formula (5.8) allows us to obtain the following result on the structure of the efficient solution set of (LVOP).

Theorem 5.4 *The efficient solution set of* (LVOP) *is the union of finitely many generalized polyhedral convex sets.*

Proof. The conclusion follows from (5.8) and an argument similar to that of the proof of Theorem 5.2. □

If the spaces in question are finite dimensional, then the result in Theorem 5.4 expresses one assertion of the Arrow-Barankin-Blackwell Theorem. Another assertion of the latter says that Sol(LVOP) is connected by line segments. Recall that a subset $A \subset X$ is said to be *connected by line segments* if for any points u, v in A, there are some points $u_1, \ldots, u_r$ in A with $u_1 = u$ and $u_r = v$ such that $[u_i, u_{i+1}] \subset A$ for $i = 1, 2, \ldots, r-1$. A natural question arises: *Whether the efficient solution set of* (LVOP) *is connected by line segments, or not?*

According to [50], the connectedness by line segments of efficient solution set Sol(LVOP) in finite dimensional setting can be proved by a scheme suggested by Podinovski and Nogin [58]. As it will be seen below, an adaption of the scheme works for the locally convex Hausdorff topological vector spaces setting that we are interested in.

Theorem 5.5 *The efficient solution set* Sol(LVOP) *of* (LVOP) *is connected by line segments.*

Proof. According to Theorem 5.3, given any points u, v in Sol(LVOP), one can find $\xi_0^*, \xi_1^* \in \operatorname{ri} K^*$ such that

$$u \in \operatorname{argmin}\left((\mathrm{LP})_{\xi_0^*}\right), \quad v \in \operatorname{argmin}\left((\mathrm{LP})_{\xi_1^*}\right).$$

Since $\operatorname{ri} K^*$ is a convex set, $\xi_t^* := (1-t)\xi_0^* + t\,\xi_1^*$ belongs to $\operatorname{ri} K^*$ for every $t \in [0, 1]$. Noting that $\langle y^*, M(x)\rangle = \langle M^* y^*, x\rangle$, by Proposition 1.12, we can find finitely many nonempty generalized polyhedral convex sets $F_1, \ldots, F_q$, which are subsets of D such that, for any $y^* \in Y^*$ with $\operatorname{argmin}\left((\mathrm{LP})_{y^*}\right) \neq \emptyset$, the latter solution set coincides with one of the set F_i, $i = 1, \ldots, q$.

By remembering the family $\{F_1, \ldots, F_q\}$ we can assume that

$$\operatorname{argmin}\left((\mathrm{LP})_{\xi_0^*}\right) = F_1.$$

For each $i \in \{1, \ldots, q\}$, put

$$\Delta(i) = \left\{t \in [0, 1] \mid F_i \subset \operatorname{argmin}\left((\mathrm{LP})_{\xi_t^*}\right)\right\}.$$

To show that $\Delta(i)$ is a convex set, we take any $t_1, t_2 \in \Delta(i)$ and $\lambda \in (0, 1)$. For $\bar{t} := (1-\lambda)t_1 + \lambda t_2$ and for any $u \in F_i$, one has

$$\begin{aligned}&\langle \xi_{\bar{t}}^*, M(x) - M(u)\rangle \\ &\quad = (1-\lambda)\langle \xi_{t_1}^*, M(x) - M(u)\rangle + \lambda\langle \xi_{t_2}^*, M(x) - M(u)\rangle \geq 0, \ \ \forall x \in D.\end{aligned}$$

Consequently, $u \in \operatorname{argmin}\Big((\mathrm{LP})_{\xi^*_{\bar t}}\Big)$. It follows that $F_i \subset \operatorname{argmin}\Big((\mathrm{LP})_{\xi^*_{\bar t}}\Big)$; hence $\bar t \in \Delta(i)$. The convexity of $\Delta(i)$ has been proved.

If $\Delta(i)$ has only one element, it is closed. Now, suppose that $[t_1, t_2) \subset \Delta(i)$, $t_1 < t_2$. Since $\bar t := (1-\lambda)t_1 + \lambda t_2 \in \Delta(i)$ for all $\lambda \in (0,1)$, for any $u \in F_i$ and $x \in D$, one has

$$0 \le \langle \xi^*_{\bar t}, M(x) - M(u)\rangle = (1-\lambda)\langle \xi^*_{t_1}, M(x) - M(u)\rangle + \lambda\langle \xi^*_{t_2}, M(x) - M(u)\rangle.$$

Letting $\lambda \to 1$, we obtain $\langle \xi^*_{t_2}, M(x) - M(u)\rangle \ge 0$ for all $u \in F_i$ and $x \in D$. This implies that $F_i \subset \operatorname{argmin}\Big((\mathrm{LP})_{\xi^*_{t_2}}\Big)$, i.e., $t_2 \in \Delta(i)$. Similarly, one can show that if $(t_1, t_2] \subset \Delta(i)$ then $t_1 \in \Delta(i)$. We have thus proved that $\Delta(i)$ is a closed convex set for all $i = 1, \ldots, q$. Invoking Theorem 1.11, it is easy to show that the set of $y^* \in Y^*$ with $\operatorname{argmin}\Big((\mathrm{LP})_{y^*}\Big) \neq \emptyset$ is a convex cone. Hence, for any $t \in [0,1]$, $\operatorname{argmin}\Big((\mathrm{LP})_{\xi^*_t}\Big) \neq \emptyset$. It follows that $[0,1] = \bigcup_{i=1}^{q} \Delta(i)$. Consequently, there exist some numbers $t_1, \ldots, t_m$ from $[0,1]$ with $t_1 \le \cdots \le t_m$, $t_1 = 0$, $t_m = 1$, and $m-1$ indexes $i_1, \ldots, i_{m-1}$ such that $[t_j, t_{j+1}] \subset \Delta(i_j)$ for all $j = 1, \ldots, m-1$. Clearly, $u \in F_{i_0}$ and there exists i_r satisfying $v \in F_{i_r}$. Given $u_j \in F_{i_j}$ for $j = 1, \ldots, r$, where $u_1 = u$ and $u_r = v$. For each $j = 1, \ldots, r-1$, since $t_{j+1} \in \Delta(i_j) \cap \Delta(i_{j+1})$, it follows that $u_j \in \operatorname{argmin}\left((\mathrm{LP})_{\xi^*_{t_{j+1}}}\right)$ and $u_{j+1} \in \operatorname{argmin}\left((\mathrm{LP})_{\xi^*_{t_{j+1}}}\right)$. Hence,

$$[u_j, u_{j+1}] \subset \operatorname{argmin}\left((\mathrm{LP})_{\xi^*_{t_{j+1}}}\right) \subset \mathrm{Sol}(\mathrm{LVOP}).$$

We have already been proved that the line segments $[u_j, u_{j+1}]$, $j = 1, \ldots, r-1$, connect the vectors u, v in Sol(LVOP). The proof is complete. □

A similar result for the weakly efficient solution set of (LVOP) can be obtained.

Theorem 5.6 *If* $\operatorname{int} K \neq \emptyset$*, then the weakly efficient solution set* $\mathrm{Sol}^w(\mathrm{LVOP})$ *of* (LVOP) *is connected by line segments.*

Proof. Let us first prove that the cone $K^* \setminus \{0\}$ is convex. Assume by contradiction that there exist $y_1^*, y_2^* \in K^* \setminus \{0\}$ and $\lambda \in (0,1)$ satisfying

$$y^* := (1-\lambda)y_1^* + \lambda y_2^* \notin K^* \setminus \{0\}.$$

Since y_1^*, $y_2^* \in K^*$, which is a convex cone, $y^* \in K^*$; hence $y^* = 0$. This implies that $\langle y_1^*, y\rangle = 0$ for every $y \in K$. By $\operatorname{int} K \neq \emptyset$, it is not difficult to show that $\langle y_1^*, y \rangle = 0$ for all $y \in Y$, which contradicts the assumption $y_1^* \in K^* \setminus \{0\}$.

Now, by Theorem 5.2, we can apply the proof scheme of Theorem 5.5, with $\mathrm{ri}\, K^*$ being replaced by $K^* \setminus \{0\}$, to assert that $\mathrm{Sol}^w(\mathrm{LVOP})$ is connected by line segments. □

5.4 Structure of the Solution Sets in the Convex Case

In this section, we will study piecewise linear vector optimization problems whose object functions are convex.

The invariance of $K \setminus \ell(K)$ and $\mathrm{int}\, K$ w.r.t. a translation by a vector from K is described by the forthcoming lemma, which can be proved similarly as [50, Proposition 4.3, p. 19] (see also [17, Lemma 1.2 (iii)]).

Lemma 5.3 *We have*

(a) $K \setminus \ell(K) + K \subset K \setminus \ell(K)$;

(b) $\mathrm{int}\, K + K \subset \mathrm{int}\, K$.

The next result is an extension of [75, Theorem 2.2] and [82, Theorems 3.2 and 3.3] to the locally convex Hausdorff topological vector space setting.

Theorem 5.7 *If f is a K-function on D, the efficient solution set and the weakly efficient solution set of* (VP) *are the unions of finitely many generalized polyhedral convex sets and they are connected by line segments.*

Proof. Without loss of generality, we may assume that D is nonempty.

CLAIM 1. *The sum $f(D) + K$ is a polyhedral convex set.*

Following [50, p. 18], we define the epigraph of f by the formula

$$\mathrm{epi}\, f = \{(x, y) \in X \times Y \mid y \in f(x) + K\}.$$

Since f is a K-function, $\mathrm{epi}\, f$ is convex by [50, Proposition 6.2, p. 29]. As $f(D) + K$ is the projection of $\mathrm{epi}\, f \cap (D \times Y)$ on Y, it is convex.

For $k = 1, \ldots, m$, set $M_k := f(D \cap P_k)$ and note that

$$M_k = T_k(D \cap P_k) + b_k.$$

We will show that $M_k + K$, $k = 1, \ldots, m$, are polyhedral convex sets. Clearly, $D \cap P_k$ is a generalized polyhedral convex set in X. If $D \cap P_k$ is empty, then $M_k = \emptyset$; hence $M_k + K = \emptyset$ is a special polyhedral convex set. In the case where $D \cap P_k$ is nonempty, by the representation for generalized polyhedral convex sets (see Theorem 1.2) one can find $u_{k,1}, \ldots, u_{k,r_k}$, $v_{k,1}, \ldots, v_{k,s_k}$ in X and a closed linear subspace $X_{0,k} \subset X$ such that

$$D \cap P_k = \Big\{ \sum_{i=1}^{r_k} \lambda_i u_{k,i} + \sum_{j=1}^{s_k} \mu_j v_{k,j} \mid \lambda_i \geq 0,\ i = 1, \ldots, r_k, \\ \sum_{i=1}^{r_k} \lambda_i = 1,\ \mu_j \geq 0,\ j = 1, \ldots, s_k \Big\} + X_{0,k}.$$

It is not difficult to show that $M_k = M'_k + T_k(X_{0,k})$, where

$$M'_k = \Big\{ \sum_{i=1}^{r_k} \lambda_i (T u_{k,i} + b_k) + \sum_{j=1}^{s_k} \mu_j T v_{k,j} \mid \lambda_i \geq 0,\ i = 1, \ldots, r_k, \\ \sum_{i=1}^{r_k} \lambda_i = 1,\ \mu_j \geq 0,\ j = 1, \ldots, s_k \Big\}.$$

Combining this with the equality (1.29), one has

$$M_k + K = M'_k + T_k(X_{0,k}) + Y_0 + K_1.$$

Since $T_k(X_{0,k})$ is a linear subspace of Y and Y_0 is a closed finite-codimensional linear subspace of Y, Lemma 1.4 shows that $Y_0 + T_k(X_{0,k})$ is a closed linear subspace of Y and $\operatorname{codim}\big(Y_0 + T_k(X_{0,k})\big)$ is finite. Hence, by Theorem 1.3, $M'_k + \big(Y_0 + T_k(X_{0,k})\big)$ is a polyhedral convex set in Y. As K_1 is a generalized polyhedral convex set in Y, Proposition 1.6 yields $K_1 + M'_k + Y_0 + T_k(X_{0,k})$ is a polyhedral convex set. So, $M_k + K$ is a polyhedral convex set in Y.

In accordance with Corollary 1.2, since the convex set $f(D) + K$ is the union of polyhedral convex sets $M_1 + K, \ldots, M_m + K$, we may conclude that $f(D) + K$ is a polyhedral convex set.

Claim 2. *The sets $E(M + K|K)$ and $E^w(M + K|K)$, where $M := f(D)$, are the unions of finitely many generalized polyhedral convex sets and they are connected by line segments.*

It suffices to apply Theorems 5.2, 5.4, 5.5, and 5.6 to the problem

$$\mathrm{Min}_K \{G(y) \mid y \in M + K\}$$

with $G(y) := y$ for all $y \in Y$.

CLAIM 3. *A vector* $u \in D$ *belongs to* Sol(VP) *(resp., belongs to* $\mathrm{Sol}^w(\mathrm{VP})$*) if and only if* $f(u) \in E(M+K|K)$ *(resp.,* $f(u) \in E^w(M+K|K)$*).*

Arguing similarly as in the proof of [52, Proposition 7.10], we can show that

$$E(M|K) = M \cap E(M+K|K), \; E^w(M|K) = M \cap E^w(M+K|K). \quad (5.11)$$

For any $u \in D$, one has $f(u) \in M$. By definition, u belongs to Sol(VP) (resp., to $\mathrm{Sol}^w(\mathrm{VP})$) if and only if $f(u) \in E(M|K)$ (resp., $f(u) \in E^w(M|K)$). Hence, the assertions follow from (5.11).

CLAIM 4. *The efficient solution set and the weakly efficient solution set of* (VP) *are the unions of finitely many generalized polyhedral convex sets.*

According to Claim 2, one can find generalized polyhedral convex sets $Q_1, \ldots, Q_d$ in Y such that $E(M+K|K) = \bigcup\limits_{j=1}^{d} Q_j$. For each $k \in \{1, \ldots, m\}$, by using Claim 3, one has

$$\begin{aligned} \mathrm{Sol}(\mathrm{VP}) \cap P_k &= \big\{u \in D \cap P_k \mid f(u) \in E(M|K)\big\} \\ &= \big\{u \in D \cap P_k \mid f(u) \in E(M+K|K)\big\} \\ &= \bigcup_{j=1}^{d} \big\{u \in D \cap P_k \mid f(u) \in Q_j\big\} \\ &= \bigcup_{j=1}^{d} \big\{u \in D \cap P_k \mid T_k u + b_k \in Q_j\big\} \\ &= \bigcup_{j=1}^{d} \Big(D \cap P_k \cap T_k^{-1}(-b_k + Q_j)\Big). \end{aligned}$$

Since $\mathrm{Sol}(\mathrm{VP}) = \bigcup\limits_{k=1}^{m} \big(\mathrm{Sol}(\mathrm{VP}) \cap P_k\big)$, we obtain

$$\mathrm{Sol}(\mathrm{VP}) = \bigcup_{k=1}^{m} \bigcup_{j=1}^{d} \Big(D \cap P_k \cap T_k^{-1}(-b_k + Q_j)\Big). \quad (5.12)$$

For any $k \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, d\}$, the set $-b_k + Q_j$ is generalized polyhedral convex by Proposition 1.4. Hence, since T_k is a continuous linear mapping, by Proposition 1.3 we can assert that $T_k^{-1}(-b_k+Q_j)$ is a generalized polyhedral convex set. This implies that $D \cap P_k \cap T_k^{-1}(-b_k+Q_j)$ is a generalized polyhedral convex set. Then, formula (5.12) justifies the fact that Sol(VP) is the union of finitely many generalized polyhedral convex sets. The assertion concerning $\mathrm{Sol}^w(\mathrm{VP})$ can be proved similarly.

Claim 5. *The efficient solution set and the weakly efficient solution set of* (VP) *are connected by line segments.*

By analogy, it suffices to prove the assertion about the efficient solution set. Take any u, u' in Sol(VP). By Claim 3, $f(u)$ and $f(u')$ are contained in $E(M+K|K)$. Since $E(M+K|K)$ is connected by line segments (see Claim 2), there exist $y_1, \dots, y_r$ in Y such that $y_1 = f(u)$, $y_r = f(u')$ and

$$[y_i, y_{i+1}] \subset E(M+K|K) \quad (i = 1, \dots, r-1).$$

Let $u_i \in D$ and $w_i \in K$ be such that $y_i = f(u_i) + w_i$ for $i = 2, \dots, r-1$. Setting $u_1 = u$, $w_1 = 0$, $u_r = u'$, and $w_r = 0$, we have $y_i = f(u_i) + w_i$ for $i = 1, \dots, r$. Let $i \in \{1, \dots, r-1\}$ be chosen arbitrarily. To show that $[u_i, u_{i+1}] \subset$ Sol(VP), we suppose the contrary: There is a $\lambda \in [0,1]$ such that the vector $u_\lambda := (1-\lambda)u_i + \lambda u_{i+1}$ does not belong to Sol(VP). Then one can find $\bar{x} \in D$ and $\bar{w} \in K \setminus \ell(K)$ satisfying $f(u_\lambda) = f(\bar{x}) + \bar{w}$. As f is a K-function on D, there is $w \in K$ with

$$(1-\lambda)f(u_i) + \lambda f(u_{i+1}) = f((1-\lambda)u_i + \lambda u_{i+1}) + w. \tag{5.13}$$

Set $y_\lambda := (1-\lambda)y_i + \lambda y_{i+1}$. On one hand, the inclusion $[y_i, y_{i+1}] \subset E(M+K|K)$ gives $y_\lambda \in E(M+K|K)$. On the other hand,

$$\begin{aligned} y_\lambda &= (1-\lambda)(f(u_i) + w_i) + \lambda(f(u_{i+1}) + w_{i+1}) \\ &= [(1-\lambda)f(u_i) + \lambda f(u_{i+1})] + (1-\lambda)w_i + \lambda w_{i+1}. \end{aligned} \tag{5.14}$$

Combining (5.13), (5.14) with the equality $f(u_\lambda) = f(\bar{x}) + \bar{w}$, we obtain

$$\begin{aligned} y_\lambda &= [f((1-\lambda)u_i + \lambda u_{i+1}) + w] + (1-\lambda)w_i + \lambda w_{i+1}. \\ &= [f(u_\lambda) + w] + (1-\lambda)w_i + \lambda w_{i+1} \\ &= f(\bar{x}) + \bar{w} + (w + (1-\lambda)w_i + \lambda w_{i+1}). \end{aligned}$$

By the convexity of the cone K, one has $w + (1-\lambda)w_i + \lambda w_{i+1} \in K$. Therefore, as $\bar{w} \in K \setminus \ell(K)$, Lemma 5.3 shows that

$$\bar{w} + (w + (1-\lambda)w_i + \lambda w_{i+1}) \in K \setminus \ell(K).$$

It follows that $y_\lambda - f(\bar{x}) \in K \setminus \ell(K)$. Hence, $y_\lambda \notin E(M+K|K)$, a contradiction. We have proved that the line segments $[u_i, u_{i+1}], i = 1, \dots, r-1$, which connect u with u', lie wholly in Sol(VP).

The proof is complete. □

Remark 5.4 In the proof of Theorem 5.7, we have used some ideas of Yang and Yen [75]. For the case where X, Y are normed spaces, Y is finite-dimensional, $K \subset Y$ is a pointed cone, and D is a polyhedral convex set, the assertions of Theorem 5.7 about Sol(VP) recover [75, Theorem 2.2]. For the case where X, Y are normed spaces and D is a polyhedral convex set, the assertions of Theorem 5.7 about $\mathrm{Sol}^w(\mathrm{VP})$ have been established by Zheng and Yang [82, Theorems 3.2 and 3.3]. For the special case where f is linear, Theorem 5.7 expresses several recent results in Theorems 5.2, 5.4, 5.5 and 5.6.

The following example is designed as an illustration for Theorem 5.7.

Example 5.1 (Cf. Example 4.1) Keep the notations X, X_0, x_i^*, e_i for $i = 1, 2$, of Example 4.1.

Let $Y = X$, $y_i^* = x_i^*$ for $i = 1, 2$, and $K := \{y \in Y \mid \langle y_i^*, y\rangle \le 0,\ i = 1, 2\}$. Note that K is a polyhedral convex cone. Clearly,

$$K = \{x_0 + t_1e_1 + t_2e_2 \mid x_0 \in X_0,\ t_i \le 0,\ i = 1, 2\}. \tag{5.15}$$

By Lemma 1.10, $\mathrm{int}\, K = \{y \in Y \mid \langle y_i^*, y\rangle < 0,\ i = 1, 2\}$, and

$$K \setminus \ell(K) = \{y \in Y \mid \langle y_1^*, y\rangle \le 0,\ \langle y_2^*, y\rangle < 0\} \\ \cup \{y \in Y \mid \langle y_1^*, y\rangle < 0,\ \langle y_2^*, y\rangle \le 0\}.$$

An easy computation shows that

$$\mathrm{int}\, K = \{x_0 + t_1e_1 + t_2e_2 \mid x_0 \in X_0,\ t_1 < 0,\ t_2 < 0\} \tag{5.16}$$

and

$$K \setminus \ell(K) = \{x_0 + t_1e_1 + t_2e_2 \mid x_0 \in X_0,\ t_1 \le 0,\ t_2 \le 0,\ t_1 + t_2 < 0\}. \tag{5.17}$$

Given any $e_0 \in X_0$ and put $L = \{x \in X \mid x(t) = e_0(t),\ t \in [-1, 0]\}$. Clearly, L is a closed affine subspace of X. Therefore, the set

$$D := \{x \in L \mid \langle x_1^*, x\rangle \le 0,\ \langle x_2^*, x\rangle \le 1\}$$

is generalized polyhedral convex. Observe that $e_0 + e_2 \in D$ and D is not a polyhedral convex set in X.

Let X_2 be the linear subspace of X generated by e_2. It is clear that $\dim X_2 = 1$, $X = \ker x_2^* + X_2$, and $\ker x_2^* \cap X_2 = \{0\}$. Let

$$\pi : X \to X/\ker x_2^*, \quad \pi(x) = x + \ker x_2^* \quad (x \in X),$$

be the canonical projection from X on the quotient space $X/\ker x_2^*$. According to [65, Theorem 1.41(a)], the linear mapping π is continuous. Since the operator

$$\Phi : X/\ker x_2^* \to X_2, \quad x + \ker x_2^* \mapsto x \quad (x \in X_2),$$

is a linear bijective mapping, Φ is a homeomorphism by Lemma 1.3. So,

$$\Phi \circ \pi : X \to X_2$$

is a linear continuous mapping. Define the map $\varphi : X_2 \to Y$ by $\varphi(te_2) = te_1$ for all $t \in \mathbb{R}$. In accordance with Theorem 3.4 of [66, p. 22], since $\dim X_2 = 1$, the linear mapping φ is continuous. Set $T = \varphi \circ \Phi \circ \pi$, and observe that $T : X \to X$ is a linear continuous mapping. It is easy to check that if $x = x_0 + t_1e_1 + t_2e_2$ with $x_0 \in X_0$ and $t_1, t_2 \in \mathbb{R}$, then $T(x) = t_2e_1$. Moreover, $\ker T = \ker x_2^*$.

Obviously, the space X is the union of two polyhedral convex sets

$$P_1 := \{x \in X \mid \langle x_2^*, x\rangle \geq 0\}$$

and $P_2 := \{x \in X \mid \langle x_2^*, x\rangle \leq 0\}$. We see at once that if $x = x_0 + t_1e_1 + t_2e_2$, where $x_0 \in X_0$ and $t_1, t_2 \in \mathbb{R}$, then x belongs to P_1 (resp., belongs to P_2) if and only if $t_2 \geq 0$ (resp., $t_2 \leq 0$). Let $f : X \to Y$ be given by

$$f(x) = \begin{cases} x - T(x) & \text{if } x \in P_1 \\ x + T(x) & \text{if } x \in P_2. \end{cases}$$

Since $x - T(x) = x + T(x) = x$ for any $x \in P_1 \cap P_2 = \ker T$, the values of f is well defined. Moreover, f is a piecewise linear vector-valued function.

CLAIM 1. *The mapping f is a K-function on X.*

Indeed, take any vectors x_1, $x_2 \in X$, a number $\lambda \in [0, 1]$, and set

$$y := (1 - \lambda)f(x_1) + \lambda f(x_2) - f((1 - \lambda)x_1 + \lambda x_2).$$

If there exists an index $i \in \{1, 2\}$ such that $x_1, x_2 \in P_i$, then $y = 0$ because f is linear on P_i; so $y \in K$. If $x_1 \in P_1$ and $x_2 \in P_2$, then $x_i = x_{i,0} + t_{i,1}e_1 + t_{i,2}e_2$, $i = 1, 2$, with $x_{i,0} \in X_0$, $t_{i,1} \in \mathbb{R}$, $i = 1, 2$, and $t_{1,2} \geq 0 \geq t_{2,2}$. Therefore, $T(x_i) = t_{i,2}e_1$ for $i = 1, 2$. In the case where $(1 - \lambda)x_1 + \lambda x_2 \in P_1$, we obtain

$$\begin{aligned} y &= (1 - \lambda)(x_1 - T(x_1)) + \lambda(x_2 + T(x_2)) \\ &\qquad - ((1 - \lambda)x_1 + \lambda x_2) + T((1 - \lambda)x_1 + \lambda x_2) \\ &= -(1 - \lambda)T(x_1) + \lambda T(x_2) + (1 - \lambda)T(x_1) + \lambda T(x_2) \\ &= 2\lambda T(x_2) = 2\lambda t_{2,2}e_1. \end{aligned}$$

Since $2\lambda t_{2,2} \le 0$, one has $y \in K$ by (5.15). If $(1-\lambda)x_1 + \lambda x_2 \in P_2$, then

$$\begin{aligned} y &= (1-\lambda)(x_1 - T(x_1)) + \lambda(x_2 + T(x_2)) \\ &\qquad\qquad - ((1-\lambda)x_1 + \lambda x_2) - T((1-\lambda)x_1 + \lambda x_2) \\ &= -(1-\lambda)T(x_1) + \lambda T(x_2) - (1-\lambda)T(x_1) - \lambda T(x_2) \\ &= -2(1-\lambda)T(x_1) = -2(1-\lambda)t_{1,2}e_1. \end{aligned}$$

From (5.15), since $-2(1-\lambda)t_{1,2} \le 0$, one gets $y \in K$. With $x_1 \in P_2$ and $x_2 \in P_1$, a similar conclusion is obtained. It follows that f is a K-function on X.

Claim 2. *It holds that*

$$\mathrm{Sol(VP)} = \{u \in L \mid \langle x_1^*, u\rangle = 0,\ 0 \le \langle x_2^*, u\rangle \le 1\}. \tag{5.18}$$

First, take any $u \in \mathrm{Sol(VP)}$. Let $u_0 \in X_0$ and $t_1, t_2 \in \mathbb{R}$ be such that $u = u_0 + t_1e_1 + t_2e_2$. Since $u \in D$, $t_1 \le 0$ and $t_2 \le 1$. Moreover, $u_0(t) = e_0(t)$ for all $t \in [-1, 0]$; so, $u_0 \in D$. If $t_2 < 0$, then $u \in P_2$. Therefore,

$$f(u) = u + Tu = u_0 + (t_1 + t_2)e_1 + t_2e_2.$$

Since $f(u) - f(u_0) = (t_1 + t_2)e_1 + t_2e_2$ with $t_1 + t_2 < 0$ and $t_2 < 0$, by (5.17), $f(u) - f(u_0) \in K \setminus \ell(K)$. The fact contradicts the assumption $u \in \mathrm{Sol(VP)}$. This clearly forces $0 \le t_2 \le 1$; hence $u \in P_1$. If $t_1 < 0$, then we choose $x = u_0 + t_2e_2$. Since $x \in D \cap P_1$,

$$\begin{aligned} f(u) - f(x) &= (u - T(u)) - (x - T(x)) \\ &= (u_0 + t_1e_1 + t_2e_2 - t_2e_1) - (u_0 + t_2e_2 - t_2e_1) = t_1e_1. \end{aligned}$$

Since $t_1 < 0$, by (5.17), $f(u) - f(x) \in K \setminus \ell(K)$. This inclusion contradicts the assumption $u \in \mathrm{Sol(VP)}$. We thus get $t_1 = 0$ and $0 \le t_2 \le 1$. Consequently, $u \in S$, where S is the set on the right-hand side of (5.18). We have proved that $\mathrm{Sol(VP)} \subset S$. To obtain the opposite inclusion, take any $u \in S$. Let us show that $f(u) - f(x) \notin K \setminus \ell(K)$ for all $x \in D$. Suppose that

$$u = u_0 + t_1e_1 + t_2e_2$$

with $u_0 \in X_0$ and $t_1, t_2 \in \mathbb{R}$. Of course, $t_1 = 0$ and $0 \le t_2 \le 1$. Since $u \in P_1$,

$$f(u) = u - T(u) = u_0 - t_2e_1 + t_2e_2.$$

For any $x \in D$, there exist a vector $x_0 \in X_0$, numbers $\tau_1 \le 0$ and $\tau_2 \le 1$ satisfying $x = x_0 + \tau_1e_1 + \tau_2e_2$.

If $\tau_2 < 0$, then $x \in P_2$; so $f(x) = x+T(x) = x_0+(\tau_1+\tau_2)e_1+\tau_2 e_2$. According to (5.17), since $f(u) - f(x) = (u_0 - x_0) + (-t_2 - \tau_1 - \tau_2)e_1 + (t_2 - \tau_2)e_2$ with $t_2 - \tau_2 > 0$, we can assert that $f(u) - f(x) \notin K \setminus \ell(K)$.

If $\tau_2 \geq 0$ then $x \in P_1$. Since $f(x) = x - T(x) = x_0 + (\tau_1 - \tau_2)e_1 + \tau_2 e_2$,

$$f(u) - f(x) = (u_0 - x_0) + (-t_2 - \tau_1 + \tau_2)e_1 + (t_2 - \tau_2)e_2.$$

As $(-t_2 - \tau_1 + \tau_2) + (t_2 - \tau_2) = -\tau_1 \geq 0$, one gets $f(u) - f(x) \notin K \setminus \ell(K)$ by (5.17). From what that has already been said, we obtain $u \in \mathrm{Sol(VP)}$. We have proved that $\mathrm{Sol(VP)} = S$.

Claim 3. *It holds that*

$$\begin{aligned} \mathrm{Sol}^w(\mathrm{VP}) = \{u \in L \mid \langle x_1^*, u\rangle = 0,\ 0 \leq \langle x_2^*, u\rangle \leq 1\} \\ \cup \{u \in L \mid \langle x_1^*, u\rangle \leq 0,\ \langle x_2^*, u\rangle = 1\}. \end{aligned} \tag{5.19}$$

First, take any $u \in \mathrm{Sol}^w(\mathrm{VP})$. Suppose that $u = u_0+t_1e_1+t_2e_2$ with $u_0 \in X_0$ and $t_1, t_2 \in \mathbb{R}$. Since $u \in D$, $t_1 \leq 0$ and $t_2 \leq 1$. Moreover, $u_0(t) = e_0(t)$ for all $t \in [-1, 0]$. Therefore, $u_0 \in D$. If $t_2 < 0$ then $u \in P_2$; so

$$f(u) = u + T(u) = u_0 + (t_1 + t_2)e_1 + t_2 e_2.$$

Since $f(u) - f(u_0) = (t_1 + t_2)e_1 + t_2e_2$ with $t_1 + t_2 < 0$ and $t_2 < 0$, by (5.16), $f(u) - f(u_0) \in \mathrm{int}\, K$. This contradicts the assumption $u \in \mathrm{Sol}^w(\mathrm{VP})$. We thus get $0 \leq t_2 \leq 1$. Therefore, $u \in P_1$ and

$$f(u) = u - T(u) = u_0 + (t_1 - t_2)e_1 + t_2 e_2.$$

If $t_1 < 0$ and $t_2 < 1$, one can find a positive number ε such that $t_1 + \varepsilon < 0$ and $t_2 + \varepsilon < 1$. Take $x = u_0 + (t_2 + \varepsilon)e_2$, and observe that $x \in D \cap P_1$. Since $f(x) = x - T(x) = u_0 - (t_2 + \varepsilon)e_1 + (t_2 + \varepsilon)e_2$,

$$f(u) - f(x) = (t_1 + \varepsilon)e_1 + (-\varepsilon)e_2.$$

As $t_1 + \varepsilon < 0$ and $-\varepsilon < 0$, formula (5.16) shows that $f(u) - f(x) \in \mathrm{int}\, K$, which is impossible because $u \in \mathrm{Sol}^w(\mathrm{VP})$. We thus get $t_1 = 0$ or $t_2 = 1$. Consequently, u belongs to S^w, where S^w is the set on the right-hand side of (5.19). We have proved that $\mathrm{Sol}^w(\mathrm{VP}) \subset S^w$. To obtain the opposite inclusion, take any $u \in S^w$. Let $u_0 \in X_0$ and $t_1, t_2 \in \mathbb{R}$ be such that

$$u = u_0 + t_1 e_1 + t_2 e_2.$$

Case 1: $t_1 = 0$ and $0 \leq t_2 \leq 1$. Clearly, $\langle x_1^*, u\rangle = 0$ and $0 \leq \langle x_2^*, u\rangle \leq 1$. According to (5.18), $u \in \mathrm{Sol(VP)}$; hence $u \in \mathrm{Sol}^w(\mathrm{VP})$.

Case 2: $t_1 < 0$ and $t_2 = 1$. Since $u \in P_1$, one has

$$f(u) = u - T(u) = u_0 + (t_1 - 1)e_1 + e_2.$$

For any $x \in D$, there are $x_0 \in X_0$ and $\tau_1, \tau_2 \in \mathbb{R}$ satisfying $x = x_0 + \tau_1 e_1 + \tau_2 e_2$. Clearly, $\tau_1 \leq 0$ and $\tau_2 \leq 1$. If $\tau_2 < 0$, then $x \in P_2$ and

$$f(x) = x + T(x) = x_0 + (\tau_1 + \tau_2)e_1 + \tau_2 e_2.$$

According to (5.16), since $f(u) - f(x) = (u_0 - x_0) + (t_1 - 1 - \tau_1 - \tau_2)e_1 + (1 - \tau_2)e_2$ with $1 - \tau_2 > 0$, $f(u) - f(x) \notin \operatorname{int} K$. If $0 \leq \tau_2 \leq 1$, then $x \in P_1$ and

$$f(x) = x - T(x) = x_0 + (\tau_1 - \tau_2)e_1 + \tau_2 e_2.$$

Since $f(u) - f(x) = (u_0 - x_0) + (t_1 - 1 - \tau_1 + \tau_2)e_1 + (1 - \tau_2)e_2$ with $1 - \tau_2 \geq 0$, formula (5.16) gives $f(u) - f(x) \notin \operatorname{int} K$. It follows that $f(u) - f(x) \notin \operatorname{int} K$ for all $x \in D$. Hence, $u \in \mathrm{Sol}^w(\mathrm{VP})$. We have proved that $\mathrm{Sol}^w(\mathrm{VP}) = S^w$.

Observe that $\mathrm{Sol}^w(\mathrm{VP}) \neq \mathrm{Sol}(\mathrm{VP})$. Indeed, let e_3 in X be given by

$$e_3(t) = \begin{cases} 0 & \text{if } t \in [-1, 0] \\ 155t^2 - 120t & \text{if } t \in [0, 1]. \end{cases}$$

Since $e_0 + e_3 \in L$, $\langle x_1^*, e_0 + e_3 \rangle = -\frac{5}{4}$, and $\langle x_2^*, e_0 + e_3 \rangle = 1$, we can see that $e_0 + e_3$ belongs to $\mathrm{Sol}^w(\mathrm{VP})$ but $e_0 + e_3 \notin \mathrm{Sol}(\mathrm{VP})$.

5.5 Structure of the Solution Sets in the Nonconvex Case

In Theorem 5.7, the assumption f is a K-function on D cannot be dropped (see [75, Example 2.1] for an example about the efficient solution set, [82, p. 1252] for an example about the weakly efficient solution set). For the case where X, Y are normed spaces, Y is of finite dimension, $K \subset Y$ is a pointed cone, and $D \subset X$ is a polyhedral convex set, the efficient solution set of (VP) is shown to be the union of finitely many semi-closed polyhedral convex sets (see [75, Theorem 2.1]).

According to Yang and Yen [75], a subset of a normed space is called a semi-closed polyhedron if it is the intersection of a finite family of (closed or open) half-spaces. The following definition appears naturally in that spirit.

Definition 5.1 A subset $D \subset X$ is said to be a *semi-closed generalized polyhedral convex set*, or a *semi-closed generalized convex polyhedron*, if there exist $x_i^* \in X^*$, $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, q$, with a positive integer $p \leq q$, and a closed affine subspace $L \subset X$, such that

$$D=\big\{x \in L \mid \langle x_i^*, x\rangle \leq \alpha_i,\ i = 1, \ldots, p;\ \langle x_i^*, x\rangle < \alpha_i,\ i = p+1, \ldots, q\big\}. \quad (5.20)$$

If D can be represented in the form (5.20) with $L = X$, then we say that it is a *semi-closed polyhedral convex set*, or a *semi-closed convex polyhedron.*

Remark 5.5 From Definitions 1.1 and 5.1 it follows that every polyhedral convex set is a semi-closed polyhedral convex set. It is not difficult to show that, if D_1 and D_2 are semi-closed polyhedral convex sets, then $D_1 \setminus D_2$ is the union of finitely many semi-closed polyhedral convex sets.

Theorem 2.1 in [75] can be extended to the locally convex Hausdorff topological vector space setting, which we are considering, as follows.

Theorem 5.8 *The efficient solution set of* (VP) *is the union of finitely many semi-closed generalized polyhedral convex sets.*

Proof. We may assume, without loss of generality, that D is a nonempty set.

Claim 1. *There exists a closed linear subspace X_1 of X such that D is a polyhedral convex set in X_1.*

Suppose that D is given by (1.2). Set $X_0 := \ker A$ and observe that X_0 is a closed linear subspace of X.

If $z = 0$, then $D = \{x \in X_0 \mid \langle x_i^*, x\rangle \leq \alpha_i,\ i = 1, \ldots, p\}$. For $i = 1, \ldots, p$, let $x_{i,0}^*$ be the restriction to X_0 of the functional x_i^*. Clearly, $x_{i,0}^* \in X_0^*$ for $i = 1, \ldots, p$. Since

$$D = \{x \in X_0 \mid \langle x_{i,0}^*, x\rangle \leq \alpha_i,\ i = 1, \ldots, p\},$$

we can assert that D is a polyhedral convex set in X_0.

If $z \neq 0$, then fix any vector $\bar{x} \in D$. Clearly, $\bar{x}$ does not belong to X_0. Set

$$X_1 := X_0 + \{t\bar{x} \mid t \in \mathbb{R}\}.$$

Since $\{t\bar{x} \mid t \in \mathbb{R}\}$ is a linear subspace of one-dimension, the linear subspace X_1 is closed by Lemma 1.1. According to [65, Theorem 3.5], there exists $x^*_{1,0} \in X^*_1$ such that $\langle x^*_{1,0}, \bar{x}\rangle = 1$ and $\langle x^*_{1,0}, x\rangle = 0$ for all $x \in X_0$. It is not difficult to show that $X_0 = \ker x^*_{1,0}$. For $i = 1, \ldots, p$, let $x^*_{1,i}$ be the restriction to X_1 of the functional x^*_i. Of course, $x^*_{1,i} \in X^*_1$ for $i = 1, \ldots, p$. Set

$$D' := \{x \in X_1 \mid \langle x^*_{1,0}, x\rangle = 1, \langle x^*_{1,i}, x\rangle \le \alpha_i, i = 1, \ldots, p\},$$

and observe that D' is a polyhedral convex set in X_1. Let us show that $D = D'$. To obtain the inclusion $D \subset D'$, take any $x \in D$. As $x_0 := x - \bar{x}$ belongs to X_0, one has $x \in X_1$. Therefore,

$$\langle x^*_{1,0}, x\rangle = \langle x^*_{1,0}, \bar{x}\rangle + \langle x^*_{1,0}, x_0\rangle = \langle x^*_{1,0}, \bar{x}\rangle = 1.$$

For each $i = 1, \ldots, p$, since $\langle x^*_{1,i}, x\rangle = \langle x^*_i, x\rangle$, one gets $\langle x^*_{1,i}, x\rangle \le \alpha_i$. It follows that $x \in D'$. We have proved that $D \subset D'$. To obtain the opposite inclusion, take any $x \in D'$. Let $x_0 \in X_0$ and $t \in \mathbb{R}$ be such that $x = x_0 + t\bar{x}$. Then

$$\langle x^*_{1,0}, x\rangle = \langle x^*_{1,0}, x_0\rangle + t\langle x^*_{1,0}, \bar{x}\rangle = t.$$

Since $x \in D'$, $\langle x^*_{1,0}, x\rangle = 1$; so $t = 1$. Therefore,

$$A(x) = A(x_0 + \bar{x}) = A(x_0) + A\bar{x} = z.$$

For each $i = 1, \ldots, p$, the inequality $\langle x^*_{1,i}, x\rangle \le \alpha_i$ implies $\langle x^*_i, x\rangle \le \alpha_i$. The inclusion $D' \subset D$ has been proved. Thus $D = D'$; hence D is a polyhedral convex set in X_1.

By Claim 1, we may assume that D is a polyhedral convex set in X_1, where X_1 is a closed linear subspace of X. For each $k = 1, \ldots, m$, put $P_{1,k} = P_k \cap X_1$ and observe that $P_{1,k}$ is a polyhedral convex set in X_1. Of course, $X_1 = \bigcup\limits_{k=1}^{m} P_{1,k}$. Let

$$\pi_1 : Y \to Y/Y_0, \quad \pi_1(y) = y + Y_0 \quad (y \in Y),$$

be the canonical projection from Y on the quotient space Y/Y_0. By [65, Theorem 1.41(a)], π_1 is a linear continuous mapping. Since the operator

$$\Phi_1 : Y/Y_0 \to Y_1, \quad y + Y_0 \mapsto y \quad (y \in Y_1),$$

is a linear bijective mapping, Φ_1 is a homeomorphism by Lemma 1.3. So, the operator $\pi := \Phi_1 \circ \pi_1 : Y \to Y_1$ is linear and continuous.

Claim 2. *For any $y \in Y$, we have $\pi(y) \in K_1 \setminus \{0\}$ if and only if $y \in K \setminus \ell(K)$.*

Indeed, suppose that $y \in K \setminus \ell(K)$. By Lemma 1.5, one can find $y_0 \in Y_0$ and $y_1 \in K_1 \setminus \{0\}$ such that $y = y_0 + y_1$. Then $\pi(y) = y_1 \in K_1 \setminus \{0\}$.

Now, suppose that $\pi(y) \in K_1 \setminus \{0\}$, i.e., there exists $y_1 \in K_1 \setminus \{0\}$ satisfying $\pi(y) = y_1$. This implies that $y - y_1 \in Y_0$; hence $y \in y_1 + Y_0 \subset K_1 \setminus \{0\} + Y_0$. In accordance with Lemma 1.5, $y \in K \setminus \ell(K)$.

Let f_1 be the restriction to X_1 of the mapping $\pi \circ f$. Clearly, f_1 is a piecewise linear vector-valued function from X_1 to Y_1. Let us consider a piecewise linear vector optimization problem

$$(\mathrm{VP}_1) \qquad \mathrm{Min}_{K_1}\{f_1(x) \mid x \in D\}.$$

Claim 3. *It holds that* $\mathrm{Sol}(\mathrm{VP}) = \mathrm{Sol}(\mathrm{VP}_1)$.

First, to show that $\mathrm{Sol}(\mathrm{VP}) \subset \mathrm{Sol}(\mathrm{VP}_1)$, we suppose the contrary: There exists $u \in \mathrm{Sol}(\mathrm{VP})$ not belonging to $\mathrm{Sol}(\mathrm{VP}_1)$. Then one can find $x \in D$ such that $f_1(u) - f_1(x) \in K_1 \setminus \{0\}$. Since $\pi(f(u) - f(x)) \in K_1 \setminus \{0\}$, by Claim 2, one has $f(u) - f(x) \in K \setminus \ell(K)$. This contradicts the assumption $u \in \mathrm{Sol}(\mathrm{VP})$.

Now, to obtain the inclusion $\mathrm{Sol}(\mathrm{VP}_1) \subset \mathrm{Sol}(\mathrm{VP})$, take any $u \notin \mathrm{Sol}(\mathrm{VP})$. Then there exists $x \in D$ such that $f(u) - f(x) \in K \setminus \ell(K)$. Combining this with Claim 2, one gets $\pi(f(u) - f(x)) \in K_1 \setminus \{0\}$, i.e., $f_1(u) - f_1(x) \in K_1 \setminus \{0\}$. Therefore, $u \notin \mathrm{Sol}(\mathrm{VP}_1)$. We have thus proved that $\mathrm{Sol}(\mathrm{VP}) = \mathrm{Sol}(\mathrm{VP}_1)$.

Since Y_1 is finite-dimensional, K_1 is a pointed cone, D is a polyhedral convex set in X_1, arguing similarly as in the proof of [75, Theorem 2.1], we can assert that $\mathrm{Sol}(\mathrm{VP}_1)$ is the union of finitely many semi-closed polyhedral convex sets in X_1. As $\mathrm{Sol}(\mathrm{VP}) = \mathrm{Sol}(\mathrm{VP}_1)$ by Claim 3, the assertion of the theorem has been proved. □

The next result is a generalization of [82, Theorem 3.1].

Theorem 5.9 *If* $\mathrm{int}\, K$ *is nonempty, then the weakly efficient solution set of* (VP) *is the union of finitely many generalized polyhedral convex sets.*

Proof. We may assume, without loss of generality, that D is nonempty. Set

$$H_j := \{y \in Y \mid \langle y_j^*, y\rangle \le 0\} \quad (j = 1, \ldots, q).$$

Invoking Lemma 1.10, we have $\mathrm{int}\, H_j = \{y \in Y \mid \langle y_j^*, y\rangle < 0\}$ for $j = 1, \ldots, q$, and $\mathrm{int}\, K = \bigcap_{j=1}^{q} \mathrm{int}\, H_j$. By Claim 1 in the proof of Theorem 5.7, one sees $M_k + K$ is a polyhedral convex set in Y, where $M_k := f(D \cap P_k)$ for all $k = 1, \ldots, m$. Then, one can find $y_{k,1}^*, \ldots, y_{k,\ell_k}^*$ in Y^*, $\beta_{k,1}, \ldots, \beta_{k,\ell_k}$ in $\mathbb{R}$ such that

$$M_k + K = \bigcap_{i=1}^{\ell_k} H_{k,i},$$

with $H_{k,i} := \{y \in Y \mid \langle y^*_{k,i}, y\rangle \le \beta_{k,i}\}$. Put $Q = f(D) + K$ and observe that $Q = \bigcup\limits_{k=1}^{m} (M_k + K)$. Since $E^w(Q|K) = Q \setminus (Q + \operatorname{int} K)$, one has

$$\begin{aligned} E^w(Q|K) &= Q \setminus \Big(\bigcup_{k=1}^{m} (M_k + K) + \operatorname{int} K \Big) \\ &= \bigcap_{k=1}^{m} \Big(Q \setminus (M_k + K + \operatorname{int} K) \Big) \\ &= \bigcap_{k=1}^{m} \Big(Q \setminus (\bigcap_{i=1}^{\ell_k} H_{k,i} + \operatorname{int} K) \Big) \\ &= \bigcap_{k=1}^{m} \bigcup_{i=1}^{\ell_k} \Big(Q \setminus (H_{k,i} + \operatorname{int} K) \Big). \end{aligned} \tag{5.21}$$

For any $k \in \{1, \dots, m\}$ and $i \in \{1, \dots, \ell_k\}$,

$$\begin{aligned} Q \setminus \big(H_{k,i} + \operatorname{int} K\big) &= Q \setminus \left(\bigcap_{j=1}^{q} (H_{k,i} + \operatorname{int} H_j) \right) \\ &= \bigcup_{j=1}^{q} \Big(Q \setminus (H_{k,i} + \operatorname{int} H_j) \Big) \\ &= \bigcup_{j=1}^{q} \bigcup_{k_1=1}^{m} \Big((M_{k_1} + K) \setminus \big(H_{k,i} + \operatorname{int} H_j\big) \Big). \end{aligned} \tag{5.22}$$

One can assert that, *for any* $k, k_1 \in \{1, \dots, m\}$, $i \in \{1, \dots, \ell_k\}$, *and* $j \in \{1, \dots, q\}$, *the set* $(M_{k_1} + K) \setminus \big(H_{k,i} + \operatorname{int} H_j\big)$ *is polyhedral convex.*

Indeed, let us show that there exist $y^*_{k,i,j} \in Y^*$ and $\beta_{k,i,j} \in \mathbb{R}$ such that

$$H_{k,i} + \operatorname{int} H_j = \{y \in Y \mid \langle y^*_{k,i,j}, y\rangle < \beta_{k,i,j}\}. \tag{5.23}$$

First, consider the case where $y^*_{k,i} = 0$. If $\beta_{k,i} < 0$, then $H_{k,i}$ is empty. So, one can choose $y^*_{k,i,j} = 0$ and $\beta_{k,i,j} = \beta_{k,i}$ because $H_{k,i} + \operatorname{int} H_j = \emptyset$. If $\beta_{k,i} \ge 0$, then $H_{k,i} = Y$. Since $H_{k,i} + \operatorname{int} H_j = Y$, (5.23) is fulfilled with $y^*_{k,i,j} = 0$ and $\beta_{k,i,j} = \beta_{k,i}$. Now, we consider the case where $y^*_{k,i} \neq 0$. One can find $\bar{y}_{k,i}, w_{k,i}$ in Y satisfying $\langle y^*_{k,i}, \bar{y}_{k,i}\rangle = \beta_{k,i}$ and $\langle y^*_{k,i}, w_{k,i}\rangle = 1$. It is not difficult to show that

$$H_{k,i} = \bar{y}_{k,i} + \{t_{k,i} w_{k,i} \mid t_{k,i} \le 0\} + \ker y^*_{k,i}.$$

Since $y_j^* \neq 0$, there exists $w_j \in Y$ such that $\langle y_j^*, w_j \rangle = 1$. We see at once that $H_j = \{t_j w_j \mid t_j \leq 0\} + \ker y_j^*$ and $\operatorname{int} H_j = \{t_j w_j \mid t_j < 0\} + \ker y_j^*$. It follows that

$$H_{k,i} + \operatorname{int} H_j = \bar{y}_{k,i} + \{t_{k,i} w_{k,i} \mid t_{k,i} \leq 0\} + \{t_j w_j \mid t_j < 0\} \\ + \ker y_{k,i}^* + \ker y_j^*. \quad (5.24)$$

If $\ker y_{k,i}^* \neq \ker y_j^*$, then $\ker y_{k,i}^* + \ker y_j^* = Y$ by $\operatorname{codim}(\ker y_{k,i}^*) = 1$. Therefore, (5.24) shows that $H_{k,i} + \operatorname{int} H_j = Y$. So, one can choose $y_{k,i,j}^* = 0$ and $\beta_{k,i,j} = 1$.

If $\ker y_{k,i}^* = \ker y_j^*$, then we take $\lambda_{k,i,j} = \langle y_j^*, w_{k,i} \rangle$. For every $y \in Y$, put $t_{k,i} = \langle y_{k,i}^*, y \rangle$. Clearly, the vector $y_0 := y - t_{k,i} w_{k,i}$ belongs to $\ker y_{k,i}^*$. Therefore,

$$\begin{aligned} \langle y_j^* - \lambda_{k,i,j} y_{k,i}^*, y \rangle &= \langle y_j^* - \lambda_{k,i,j} y_{k,i}^*, y_0 \rangle + \langle y_j^* - \lambda_{k,i,j} y_{k,i}^*, t_{k,i} w_{k,i} \rangle \\ &= \langle y_j^*, y_0 \rangle - \lambda_{k,i,j} \langle y_{k,i}^*, y_0 \rangle + t_{k,i} \langle y_j^* - \lambda_{k,i,j} y_{k,i}^*, w_{k,i} \rangle \\ &= t_{k,i} \left(\langle y_j^*, w_{k,i} \rangle - \lambda_{k,i,j} \langle y_{k,i}^*, w_{k,i} \rangle \right) \\ &= t_{k,i} \big(\lambda_{k,i,j} - \lambda_{k,i,j} \big) = 0. \end{aligned}$$

We thus get $y_j^* = \lambda_{k,i,j} y_{k,i,j}^*$. Since $y_j^* \neq 0$, one has $\lambda_{k,i,j} \neq 0$. If $\lambda_{k,i,j} > 0$, then $\operatorname{int} H_j = \{y \in Y \mid \langle y_{k,i}^*, y \rangle < 0\}$. So,

$$H_{k,i} + \operatorname{int} H_j = \{y \in Y \mid \langle y_{k,i}^*, y \rangle < \beta_{k,i}\}.$$

Of course, the formula (5.23) is fulfilled with $y_{k,i,j}^* = y_{k,i}^*$ and $\beta_{k,i,j} = \beta_{k,i}$. If $\lambda_{k,i,j} < 0$, then $\operatorname{int} H_j = \{y \in Y \mid \langle y_{k,i}^*, y \rangle > 0\}$. Therefore, $H_{k,i} + \operatorname{int} H_j = Y$. Hence, one can choose $y_{k,i,j}^* = 0$ and $\beta_{k,i,j} = 1$.

From (5.23) we see that

$$(M_{k_1} + K) \setminus \big(H_{k,i} + \operatorname{int} H_j\big) = (M_{k_1} + K) \cap \{y \in Y \mid \langle y_{k,i,j}^*, y \rangle \geq \beta_{k,i,j}\}.$$

Therefore, $(M_{k_1} + K) \setminus \big(H_{k,i} + \operatorname{int} H_j\big)$ is a polyhedral convex set in Y.

From (5.22) it follows that, for all $k \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, \ell_k\}$, $Q \setminus \big(H_{k,i} + \operatorname{int} K\big)$ is the union of finitely many generalized polyhedral convex sets. Therefore, by (5.21), $E^w(Q|K)$ is the union of finitely many generalized polyhedral convex sets. Hence, using the same argument for getting Claim 4 in the proof of Theorem 5.7, we can prove that $\mathrm{Sol}^w(\mathrm{VP})$ is the union of finitely many generalized polyhedral convex sets. □

Remark 5.6 For the case where X, Y are normed spaces and D is a polyhedral convex set, the result in Theorem 5.9 is due to Zheng and Yang [82, Theorem 3.1].

Let us consider an illustrative example for Theorems 5.8 and 5.9.

Example 5.2 Keeping the notations of Example 5.1, we redefine the piecewise linear function f by

$$f(x) = \begin{cases} x + T(x) & \text{if } x \in P_1 \\ x - T(x) & \text{if } x \in P_2. \end{cases}$$

CLAIM 1. *It holds that*

$$\begin{aligned} \text{Sol(VP)} = \{u \in L \mid \langle x_1^*, u\rangle = 0,\ \langle x_2^*, u\rangle = 1\} \\ \cup \{u \in L \mid \langle x_1^*, u\rangle = 0,\ \langle x_2^*, u\rangle < -1\}. \end{aligned} \tag{5.25}$$

First, to show that $\text{Sol(VP)} \subset S$, where S is the set on the right-hand side of (5.25), take any $u \in \text{Sol(VP)}$. Then one can find a vector $u_0 \in X_0$ and numbers t_1, t_2 satisfying $u = u_0 + t_1 e_1 + t_2 e_2$. Since $u \in D$, we have $t_1 \le 0$ and $t_2 \le 1$. Moreover, $u_0(t) = e_0(t)$ for all $t \in [-1, 0]$; so, $u_0 \in D$. The condition $u \in \text{Sol(VP)}$ yields $f(u) - f(x) \notin K \setminus \ell(K)$ for every $x \in D$.

If $t_2 \ge 0$, then $u \in P_1$; so $f(u) = u + T(u) = u_0 + (t_1 + t_2)e_1 + t_2 e_2$. Observe that $x := u_0 + e_2$ belongs to $D \cap P_1$. It is clear that

$$f(x) = x + T(x) = u_0 + e_1 + e_2;$$

hence

$$f(u) - f(x) = (t_1 + t_2 - 1)e_1 + (t_2 - 1)e_2.$$

Since $t_1 + t_2 - 1 \le 0$ and $t_2 - 1 \le 0$, (5.15) yields $f(u) - f(x) \in K$. Combining this with the inclusion $f(u) - f(x) \notin K \setminus \ell(K)$, one gets $f(u) - f(x) \in \ell(K)$. This implies that $t_1 + t_2 - 1 = 0$ and $t_2 - 1 = 0$, i.e., $t_1 = 0$ and $t_2 = 1$. Therefore, $u \in S$.

If $t_2 < 0$, then $u \in P_2$ and $f(u) = u - Tu = u_0 + (t_1 - t_2)e_1 + t_2 e_2$. If we take $x = u_0 + e_2$, then $x \in D \cap P_1$. As $f(x) = x + T(x) = u_0 + e_1 + e_2$, one has

$$f(u) - f(x) = (t_1 - t_2 - 1)e_1 + (t_2 - 1)e_2.$$

Since $f(u) - f(x) \notin K \setminus \ell(K)$, (5.15) shows that $t_1 - t_2 - 1 > 0$, by $t_2 - 1 < 0$. Therefore, $t_2 < -1$ as $t_1 \le 0$. If $t_1 < 0$, then one can find a positive number ε such that $t_1 + \varepsilon < 0$ and $t_2 + \varepsilon < -1$. Set $x = u_0 + (t_2 + \varepsilon)e_2$, and observe that $x \in D \cap P_2$. Since $f(x) = x - T(x) = u_0 - (t_2 + \varepsilon)e_1 + (t_2 + \varepsilon)e_2$,

$$f(u) - f(x) = (t_1 + \varepsilon)e_1 + (-\varepsilon)e_2.$$

As $t_1+\varepsilon<0$ and $-\varepsilon<0$, (5.17) yields $f(u)-f(x)\in K\setminus\ell(K)$. This contradicts the assumption $u\in \mathrm{Sol(VP)}$. We thus get $t_1=0$. Consequently, $u\in S$.

We have proved that $\mathrm{Sol(VP)}\subset S$. To obtain the opposite inclusion, take any $u\in S$. Let $u_0\in X_0$ and $t_1,t_2\in\mathbb{R}$ be such that $u=u_0+t_1e_1+t_2e_2$. Of course, $t_1=0$. Given any $x\in D$, one can find a vector $x_0\in X_0$, numbers $\tau_1\le 0$ and $\tau_2\le 1$ such that $x=x_0+\tau_1e_1+\tau_2e_2$.

If $t_2=1$, then $u\in P_1$ and $f(u)=u+T(u)=u_0+e_1+e_2$. If $0\le\tau_2\le 1$, then $x\in P_1$ and $f(x)=x_0+(\tau_1+\tau_2)e_1+\tau_2e_2$. Since

$$f(u)-f(x)=(u_0-x_0)+(1-\tau_1-\tau_2)e_1+(1-\tau_2)e_2,$$

with $1-\tau_1-\tau_2\ge 0$ and $1-\tau_2\ge 0$, (5.17) shows that $f(u)-f(x)\notin K\setminus\ell(K)$. If $\tau_2<0$, then $x\in P_2$; so $f(x)=x_0+(\tau_1-\tau_2)e_1+\tau_2e_2$. According to (5.17), since $f(u)-f(x)=(u_0-x_0)+(1-\tau_1+\tau_2)e_1+(1-\tau_2)e_2$ with

$$(1-\tau_1+\tau_2)+(1-\tau_2)=2-\tau_1>0,$$

one gets $f(u)-f(x)\notin K\setminus\ell(K)$. It follows that $f(u)-f(x)\notin K\setminus\ell(K)$ for all $x\in D$; hence, $u\in\mathrm{Sol(VP)}$.

If $t_2<-1$, then $u\in P_2$. Therefore, $f(u)=u-T(u)=u_0-t_2e_1+t_2e_2$. If $0\le\tau_2\le 1$, then $x\in P_1$ and $f(x)=x_0+(\tau_1+\tau_2)e_1+\tau_2e_2$. As

$$f(u)-f(x)=(u_0-x_0)+(-t_2-\tau_1-\tau_2)e_1+(t_2-\tau_2)e_2$$

with $-t_2-\tau_1-\tau_2>1-\tau_1-\tau_2\ge 0$, one has $f(u)-f(x)\notin K\setminus\ell(K)$ by (5.17). If $\tau_2<0$, then $x\in P_2$ and $f(x)=x_0+(\tau_1-\tau_2)e_1+\tau_2e_2$. Observe that

$$f(u)-f(x)=(u_0-x_0)+(-t_2-\tau_1+\tau_2)e_1+(t_2-\tau_2)e_2$$

with $(-t_2-\tau_1+\tau_2)+(t_2-\tau_2)=-\tau_1\ge 0$. So, $f(u)-f(x)\notin K\setminus\ell(K)$ by (5.17). Therefore, $f(u)-f(x)\notin K\setminus\ell(K)$ for all $x\in D$. Hence, $u\in\mathrm{Sol(VP)}$. We have proved that $\mathrm{Sol(VP)}=S$.

Observe that Sol(VP) is the union of two semi-closed generalized polyhedral convex sets. Furthermore, Sol(VP) is *disconnected* and *non-closed.*

CLAIM 2. *It holds that*

$$\begin{aligned}\mathrm{Sol}^w(\mathrm{VP})=\{u\in L\mid\langle x_1^*,u\rangle\le 0,\ \langle x_2^*,u\rangle=1\}&\\ \cup\{u\in L\mid\langle x_1^*,u\rangle=0,\ \langle x_2^*,u\rangle\le -1\}&.\end{aligned}\tag{5.26}$$

First, to clear that $\mathrm{Sol}^w(\mathrm{VP})\subset S^w$, where S^w is the set on the right-hand side of (5.26), take any $u\in\mathrm{Sol}^w(\mathrm{VP})$. Suppose that $u=u_0+t_1e_1+t_2e_2$

with $u_0 \in X_0$ and $t_1, t_2 \in \mathbb{R}$. Since $u \in D$, $t_1 \le 0$ and $t_2 \le 1$. Observe that $u_0(t) = e_0(t)$ for all $t \in [-1, 0]$; so, $u_0 \in D$. The inclusion $u \in \mathrm{Sol}^w(\mathrm{VP})$ implies that $f(u) - f(x) \notin \operatorname{int} K$ for all $x \in D$.

If $t_2 \ge 0$, then $u \in P_1$ and $f(u) = u + T(u) = u_0 + (t_1 + t_2)e_1 + t_2 e_2$. Since the vector $x := u_0 + e_2$ belongs to $D \cap P_1$, $f(x) = x + T(x) = u_0 + e_1 + e_2$. Therefore,

$$f(u) - f(x) = (t_1 + t_2 - 1)e_1 + (t_2 - 1)e_2.$$

If $t_2 < 1$, then $t_1 + t_2 - 1 < 0$ and $t_2 - 1 < 0$. So, $f(u) - f(x) \in \operatorname{int} K$ by (5.16). This contradicts the assumption $u \in \mathrm{Sol}^w(\mathrm{VP})$. We thus get $t_2 = 1$. Consequently, $u \in S^w$.

If $t_2 < 0$, then $u \in P_2$ and $f(u) = u - T(u) = u_0 + (t_1 - t_2)e_1 + t_2 e_2$. Clearly, the vector $x := u_0 + e_2$ belongs to $D \cap P_1$. Since $f(x) = x + T(x) = u_0 + e_1 + e_2$,

$$f(u) - f(x) = (t_1 - t_2 - 1)e_1 + (t_2 - 1)e_2.$$

If $t_1 - t_2 - 1 < 0$, then $f(u) - f(x) \in \operatorname{int} K$ by (5.16). This contradicts the assumption $u \in \mathrm{Sol}^w(\mathrm{VP})$. It follows that $t_1 - t_2 - 1 \ge 0$. Hence, $t_2 \le -1$ as $t_1 \le 0$. Therefore, $u \in S^w$.

We have proved that $\mathrm{Sol}^w(\mathrm{VP}) \subset S^w$. To obtain the opposite inclusion, take any $u \in S^w$. Let $u_0 \in X_0$ and $t_1, t_2 \in \mathbb{R}$ be such that $u = u_0 + t_1 e_1 + t_2 e_2$. Given any $x \in D$, one can find a vector $x_0 \in X_0$, numbers $\tau_1 \le 0$ and $\tau_2 \le 1$ such that $x = x_0 + \tau_1 e_1 + \tau_2 e_2$.

If $t_2 = 1$ and $t_1 \le 0$, then $f(u) = u + T(u) = u_0 + (t_1 + 1)e_1 + e_2$ by $u \in P_1$. If $0 \le \tau_2 \le 1$, then $x \in P_1$ and $f(x) = x_0 + (\tau_1 + \tau_2)e_1 + \tau_2 e_2$. Since

$$f(u) - f(x) = (u_0 - x_0) + (t_1 + 1 - \tau_1 - \tau_2)e_1 + (1 - \tau_2)e_2$$

with $1 - \tau_2 \ge 0$, one gets $f(u) - f(x) \notin \operatorname{int} K$ by (5.16). If $\tau_2 < 0$, then $x \in P_2$ and $f(x) = x_0 + (\tau_1 - \tau_2)e_1 + \tau_2 e_2$. In accordance with (5.16), since

$$f(u) - f(x) = (u_0 - x_0) + (t_1 + 1 - \tau_1 + \tau_2)e_1 + (1 - \tau_2)e_2$$

with $1 - \tau_2 \ge 0$, one gets $f(u) - f(x) \notin \operatorname{int} K$. It follows that

$$f(u) - f(x) \notin \operatorname{int} K$$

for all $x \in D$. Hence, $u \in \mathrm{Sol}^w(\mathrm{VP})$.

If $t_2 \le -1$ and $t_1 = 0$, then $f(u) = u - T(u) = u_0 - t_2 e_1 + t_2 e_2$ by $u \in P_2$. If $0 \le \tau_2 \le 1$, then $x \in P_1$ and $f(x) = x_0 + (\tau_1 + \tau_2)e_1 + \tau_2 e_2$. Therefore,

$$f(u) - f(x) = (u_0 - x_0) + (-t_2 - \tau_1 - \tau_2)e_1 + (t_2 - \tau_2)e_2$$

By (5.16), since $-t_2 - \tau_1 - \tau_2 \geq 1 - \tau_1 - \tau_2 \geq 0$, one has $f(u) - f(x) \notin \operatorname{int} K$. If $\tau_2 < 0$, then $x \in P_2$ and $f(x) = x_0 + (\tau_1 - \tau_2)e_1 + \tau_2 e_2$. As

$$f(u) - f(x) = (u_0 - x_0) + (-t_2 - \tau_1 + \tau_2)e_1 + (t_2 - \tau_2)e_2$$

with $(-t_2 - \tau_1 + \tau_2) + (t_2 - \tau_2) = -\tau_1 \geq 0$, by (5.16), $f(u) - f(x) \notin \operatorname{int} K$. It follows that $f(u) - f(x) \notin \operatorname{int} K$ for all $x \in D$. Hence, $u \in \mathrm{Sol}^w(\mathrm{VP})$. We have proved that $\mathrm{Sol}^w(\mathrm{VP}) = S^w$.

Clearly, $\mathrm{Sol}^w(\mathrm{VP})$ is *disconnected* and it is the union of two generalized polyhedral convex sets.

5.6 Conclusions

Linear and piecewise linear vector optimization problems in a locally convex Hausdorff topological vector spaces setting have been considered in this chapter. The efficient solution set of these problems are shown to be the unions of finitely many semi-closed generalized polyhedral convex sets. If, in addition, the problem is convex, then the efficient solution set and the weakly efficient solution set are the unions of finitely many generalized polyhedral convex sets and they are connected by line segments. Our results develop the preceding ones of Zheng and Yang [82], and Yang and Yen [75], which were established in a normed spaces setting.

General Conclusions

This dissertation has applied different tools from functional analysis, convex analysis, variational analysis, and optimization theory, to study generalized polyhedral convex structure on locally convex Hausdorff topological vector spaces setting.

The main results of the dissertation include:

1) A representation formula for generalized polyhedral convex sets and polyhedral convex sets in locally convex Hausdorff topological vector spaces.

2) A number of basic properties of generalized polyhedral convex sets in locally convex Hausdorff topological vector spaces.

3) Fundamental properties of generalized polyhedral convex functions on locally convex Hausdorff topological vector spaces.

4) Various properties of normal cones to and polars of generalized polyhedral convex sets, conjugates of generalized polyhedral convex functions, and subdifferentials of generalized polyhedral convex functions.

5) Solution existence theorems, necessary and sufficient optimality conditions, weak and strong duality theorems for generalized polyhedral convex optimization problems in locally convex Hausdorff topological vector spaces.

6) Several theorems describing the structures of the efficient and weakly efficient solutions sets of linear and piecewise linear vector optimization problems.

Developing a concept studied by Zheng [80], we say that a multifunction between two locally convex Hausdorff topological vector spaces is *generalized polyhedral* if its graph is a union of finitely many generalized polyhedral convex sets. In the light of the theory of set-valued optimization [42], we think that generalized polyhedral multifunctions and optimization problems with such multifunctions as the objective functions deserve a careful study.

List of Author's Related Papers

[A1] N.N. Luan and N.D. Yen, *A representation of generalized convex polyhedra and applications*, Optimization, DOI: 10.1080/02331934.2019.16/-14179

[A2] N.N. Luan, *Efficient solutions in generalized linear vector optimization*, Applicable Analysis **98** (2019), 1694–1704.

[A3] N.N. Luan, J.-C. Yao, and N.D. Yen, *On some generalized polyhedral convex constructions*, Numerical Functional Analysis and Optimization **39** (2018), 537–570.

[A4] N.N. Luan and J.-C. Yao, *Generalized polyhedral convex optimization problems*, Journal of Global Optimization, DOI: 10.1007/s10898-019-00763-4.

[A5] N.N. Luan, *Piecewise linear vector optimization problems on locally convex Hausdorff topological vector spaces*, Acta Mathematica Vietnamica **43** (2018), 289–308.

References

[1] E.J. Anderson and P. Nash, *Linear Programming in Infinite Dimensional Spaces: Theory and Applications*, John Wiley & Sons, Chichester, 1987.

[2] Q.H. Ansari, E. Köbis, and J.-C. Yao, *Vector Variational Inequalities and Vector Optimization. Theory and Applications*, Springer, Cham, 2018.

[3] K.J. Arrow, E.W. Barankin, and D. Blackwell, *Admissible points of convex sets*. In: H.W. Kuhn and A.W. Tucker (Eds.) "Contributions to The Theory of Games **2**", Annals of Mathematics Studies **28**, 87–91. Princeton University Press, Princeton, NJ, 1953.

[4] H. Attouch and R.J.-B. Wets, *Epigraphical analysis*, Ann. Inst. H. Poincaré Anal. Non-Linéaire **6** (1989), 73–100.

[5] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Reprint of the 1990 edition, Birkhäuser, Boston, 2009.

[6] A. Auslender and M. Teboulle, *Asymptotic Cones and Functions in Optimization and Variational Inequalities*, Springer-Verlag, New York, 2003.

[7] L. Ban, B.S. Mordukhovich, and W. Song, *Lipschitzian stability of the parameterized variational inequalities over generalized polyhedron in reflexive Banach spaces*, Nonlinear Anal. **74** (2011), 441–461.

[8] L. Ban and W. Song, *Linearly perturbed generalized polyhedral normal cone mappings and applications*, Optimization **65** (2016), 9–34.

[9] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, 2011.

[10] D.P. Bertsekas, *Convex Optimization Theory*, Athena Scientific, Belmont, MA, 2009.

[11] D.P. Bertsekas, *Convex Optimization Algorithms*, Athena Scientific, Belmont, MA, 2015.

[12] D.P. Bertsekas, A. Nedíc, and A.E. Ozdaglar, *Convex Analysis and Optimization*, Athena Scientific, Belmont, MA, 2003.

[13] D.P. Bertsekas and H. Yu, *A unifying polyhedral approximation framework for convex optimization*, SIAM J. Optim. **21** (2011), 333–360.

[14] J.F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer-Verlag, New York, 2000.

[15] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, 2004.

[16] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011.

[17] G.Y. Chen, X.X. Huang, and X.Q. Yang, *Vector Optimization. Set-Valued and Variational Analysis*, Lecture Notes in Economics and Mathematical Systems **541**, Springer-Verlag, Berlin, 2005.

[18] L. Contesse, *Une caractérisation complète des minima locaux en programmation quadratique*, Numer. Math. **34** (1980), 315–332.

[19] V.V. Dong and N.N. Tam, *On the solution existence of convex quadratic programming problems in Hilbert spaces*, Taiwanese J. Math. **20** (2016), 1417–1436.

[20] R. Durier and P.L. Papini, *Polyhedral norms in an infinite-dimensional space*, Rocky Mountain J. Math. **23** (1993), 863–875.

[21] B.C. Eaves, *On quadratic programming*, Management Sci. **17** (1971), 698–711.

[22] G. Eichfelder, *Adaptive Scalarization Methods in Multiobjective Optimization*, Springer-Verlag, Berlin, 2008.

[23] Y.P. Fang, N.J. Huang, and X.Q. Yang, *Local smooth representations of parametric semiclosed polyhedra with applications to sensitivity in piecewise linear programs*, J. Optim. Theory Appl. **155** (2012), 810–839.

[24] Y.P. Fang, K.W. Meng, and X.Q. Yang, *Piecewise linear multicriteria programs: the continuous case and its discontinuous generalization*, Oper. Res. **60** (2012), 398–409.

[25] Y.P. Fang, K.W. Meng, and X.Q. Yang, *On minimal generators for semi-closed polyhedra*, Optimization **64** (2015), 761–770.

[26] W. Fenchel, *Convex Cone, Sets, and Functions*, Lecture Notes, Princeton University, NJ, 1953.

[27] V. Fonf and L. Vesely, *Infinite-dimensional polyhedrality*, Canad. J. Math. **56** (2004), 472–494.

[28] M. Frank and P. Wolfe, *An algorithm for quadratic programming*, Naval Res. Logist. Quart. **3** (1956), 95–110.

[29] H. Gfrerer,*On directional metric subregularity and second-order optimality conditions for a class of nonsmooth mathematical programs.* SIAM J. Optim. **23** (2013), 632–665.

[30] H. Gfrerer, *On metric pseudo-(sub)regularity of multifunctions and optimality conditions for degenerated mathematical programs*, Set-Valued Var. Anal. **22** (2014), 79–115.

[31] F. Giannessi, *Theorem of the alternative and optimality conditions*, J. Optim. Theory Appl. **42** (1984), 331–365.

[32] F. Giannessi, *Theorems of the alternative for multifunctions with applications to optimization: General results*, J. Optim. Theory Appl. **55** (1987), 233–256.

[33] B. Grünbaum, *Convex Polytopes*, Springer-Verlag, New York, 2003.

[34] A. Hamel, F. Heyde, A. Löhne, C. Tammer, and K. Winkler, *Closing the duality gap in linear vector optimization*, J. Convex Anal. **11** (2004), 163–178.

[35] R. Hartley, *On cone–efficiency, cone–convexity and cone–compactness*, SIAM J. Appl. Math. **34** (1978), 211–222.

[36] R. Henrion, B.S. Mordukhovich, and N.M. Nam, *Second-order analysis of polyhedral systems in finite and infinite dimensions with applications to robust stability of variational inequalities*, SIAM J. Optim. **20** (2010), 2199–2227.

[37] T.N. Hoa, T.D. Phuong, and N.D. Yen, *Linear fractional vector optimization problems with many components in the solution sets*, J. Ind. Manag. Optim. **1** (2005), 477–486.

[38] N.T.T. Huong and N.D. Yen, *The Pascoletti–Serafini scalarization scheme and linear vector optimization*, J. Optim. Theory Appl. **162** (2014), 559–576.

[39] N.T.T. Huong and N.D. Yen, *The adaptive parameter control method and linear vector optimization*, Vietnam J. Math. **43** (2015), 471–486.

[40] A.D. Ioffe and V.M. Tihomirov, *Theory of Extremal Problems*, North-Holland Publishing Company, Amsterdam-New York, 1979.

[41] J. Jahn, *Vector Optimization. Theory, Application, and Extensions*, 2nd ed., Springer-Verlag, Berlin-Heidelberg, 2011

[42] A.A. Khan, C. Tammer, and C. Zălinescu, *Set-valued Optimization. An Introduction with Applications*, Springer, Heidelberg, 2015.

[43] K.C. Kiwiel, *An aggregate subgradient method for nonsmooth convex minimization*, Math Program. **27** (1983), 320–341.

[44] K.C. Kiwiel, *Methods of Descent for Nondifferentiable Optimization*, Springer, Berlin, 1985.

[45] K.C. Kiwiel, *An aggregate subgradient method for nonsmooth and nonconvex minimization*, J. Comput. Appl. Math. **14** (1986), 391–400.

[46] V. Klee, *Some characterizations of convex polyhedra*, Acta Math. **102** (1959), 79–107.

[47] A.N. Kolmogorov and S.V. Fomin, *Introductory Real Analysis,* Dover Publications, Inc., New York, 1975.

[48] S. Lang, *Real and Functional Analysis*, 3rd ed., Springer-Verlag, New York, 1993.

[49] G.M. Lee, N.N. Tam, and N.D. Yen, *Quadratic Programming and Affine Variational Inequalities: A Qualitative Study* (Series: Nonconvex Optimization and Its Applications **78**), Springer-Verlag, New York, 2005.

[50] D.T. Luc, *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems **319**, Springer-Verlag, Berlin-Heidelberg, 1989.

[51] D.T. Luc, *Multiobjective Linear Programming. An Introduction*, Springer, Cham, 2016.

[52] D.T. Luc and A. Raţiu, *Vector optimization: basic concepts and solution methods.* In: S.A.R. Al-Mezel, F.R.M. Al-Solamy, and Q.H. Ansari (Eds.) "Fixed Point Theory, Variational Analysis, and Optimization", 249–305, CRC Press, Boca Raton, FL, 2014.

[53] D.G. Luenberger, *Optimization by Vector Space Methods*, John Wiley and Sons, Inc., New York-London-Sydney, 1969.

[54] P.H. Maserick, *Convex polytopes in linear spaces*, Illinois J. Math. **9** (1965), 623–635.

[55] H. Minkowski, *Geometrie der Zahlen*, Teubner, Leipzig, 1910.

[56] N.M. Nam, *Subdifferential formulas for a class of non-convex infimal convolutions*, Optimization **64** (2015), 2213–2222.

[57] N.M. Nam and D.V Cuong, *Generalized differentiation and characterizations for differentiability of infimal convolutions*, Set-Valued Var. Anal. **23** (2015), 333–353.

[58] V.V. Podinovskji and V.D. Nogin, *Pareto Optimal Solutions in Multicriteria Optimization Problems*, Nauka, Moscow, 1982.

[59] B.N. Pshenichnyj, *The linearization method*, Optimization **18** (1987), 179–196.

[60] B.N. Pshenichnyj, *The Linearization Method for Constrained Optimization.* Translated from the 1983 Russian original by Stephen S. Wilson. Springer Series in Computational Mathematics, 22. Springer-Verlag, Berlin, 1994.

[61] A.P. Robertson and W.J. Robertson, *Topological Vector Spaces*, Cambridge University Press, Cambridge, 1973.

[62] S.M. Robinson, *Some continuity properties of polyhedral multifunctions*, Math. Programming Stud. **14** (1981), 206–214.

[63] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[64] R.T. Rockafellar and R.J.-B. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.

[65] W. Rudin, *Functional Analysis*, 2nd ed., McGraw-Hill, Inc., New York, 1991.

[66] H.H. Schaefer, *Topological Vector Spaces*, Graduate Texts in Mathematics **3**, Springer-Verlag, New York-Berlin, 1971.

[67] R.E. Steuer, *Multiple Criteria Optimization: Theory, Computation and Application*, John Wiley & Sons, New York, 1986.

[68] T. Strömberg, *Representation formulae for infimal convolution with applications*. In: M. Gyllenberg and L.E. Persson (Eds.) "Analysis, Algebra, and Computers in Mathematical Research" (Luleå, 1992), Lecture Notes in Pure and Appl. Math. **156**, 319–334, Dekker, New York, 1994.

[69] T. Strömberg, *The operation of infimal convolution*, Diss. Math. **352** (1996), 1–61.

[70] X.K. Sun, *Regularity conditions characterizing Fenchel-Lagrange duality and Farkas-type results in DC infinite programming*, J. Math. Anal. Appl. **414** (2014), 590–611.

[71] X.K. Sun, X.-J. Long, and M. Li, *Some characterizations of duality for DC optimization with composite functions*, Optimization **66** (2017), 1425–1443.

[72] D.W. Walkup and R.J.-B. Wets, *A Lipschitzian characterization of convex polyhedra*, Proc. Amer. Math. Soc. **23** (1969), 167–173.

[73] H. Weyl, *Elementare theorie der konvexen polyeder*, Comment. Math. Helv. **7** (1934), 290–306.

[74] H. Weyl, *The elementary theory of convex polyhedra*. In: H.W. Kuhn and A.W. Tucker (Eds.) "Contributions to The Theory of Games", Annals of Mathematics Studies **24**, 3–18, Princeton University Press, Princeton, NJ, 1950.

[75] X.Q. Yang and N.D. Yen, *Structure and weak sharp minimum of the Pareto solution set for piecewise linear multiobjective optimization*, J. Optim. Theory Appl. **147** (2010), 113–124.

[76] N.D. Yen and T.D. Phuong, *Connectedness and stability of the solution sets in linear fractional vector optimization problems*. In: F. Giannessi (Eds.) "Vector Variational Inequalities and Vector Equilibria", 479–489, Kluwer Academic Publishers, Dordrecht (Holland), 2000.

[77] N.D. Yen and X.Q. Yang, *Affine variational inequalities on normed spaces*, J. Optim. Theory Appl. **178** (2018), 36–55.

[78] C. Zălinescu, *Convex Analysis in General Vector Spaces*, Word Scientific, River Edge, NJ, 2002.

[79] X.Y. Zheng, *Generalizations of a theorem of Arrow, Barankin, and Blackwell in topological vector spaces*, J. Optim. Theory Appl. **96** (1998), 221–233.

[80] X.Y. Zheng, *Pareto solutions of polyhedral-valued vector optimization problems in Banach spaces*, Set-Valued Anal. **17** (2009), 389–408.

[81] X.Y. Zheng and K.F. Ng, *Metric subregularity of piecewise linear multifunctions and applications to piecewise linear multiobjective optimization*, SIAM J. Optim. **24** (2014), 154–174.

[82] X.Y. Zheng and X.Q. Yang, *The structure of weak Pareto solution sets in piecewise linear multiobjective optimization in normed spaces*, Sci. China Ser. A. **51** (2008), 1243–1256.