VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY

INSTITUTE OF MATHEMATICS

DAO QUANG KHAI

SOME QUALITATIVE PROPERTIES OF SOLUTIONS TO NAVIER-STOKES EQUATIONS

DOCTORAL DISSERTATION IN MATHEMATICS

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Confirmation

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Aurhor: Dao Quang Khai

Tóm tắt

Trong luận án này, chúng tôi sử dụng những tiến bộ đạt được trong lĩnh vực giải tích điều hòa trong mười lăm năm gần đây để nghiên cứu phương trình Navier-Stokes. Chúng tôi muốn nói đến việc sử dụng biến đổi Fourier và các tính chất của nó, phù hợp hơn cho việc nghiên cứu các bài toán phi tuyến.
Chương 1 được dành cho việc nhắc lại một số kết quả đã biết về giải tích điều hòa. Trong Chương 2, chúng tôi xây dựng và nghiên cứu các không gian Sobolev sau (không gian Sobolev trên một không gian Banach bất biến với phép dịch chuyển):

- Không gian Sobolev không thuần nhất và không gian Sobolev thuần nhất trên các không gian Lebesgue.

- Không gian Sobolev thuần nhất trên các không gian Fourier-Lorentz.

- Không gian Sobolev thuần nhất trên các không gian Lorentz.

- Không gian Sobolev thuần nhất trên các không gian với chuẩn Lorentz hỗn hợp.

Trong các không gian này, chúng tôi chứng minh một số bất đẳng thức kiểu Young cho tích chập của hai hàm, một số bất đẳng thức kiểu Holder cho tích thông thường giữa hai hàm và một số bất đẳng thức kiểu Sobolev. Chúng tôi áp dụng những bất đẳng thức này để nghiên cứu bài toán Cauchy cho phương trình Navier-Stokes. Chúng tôi xây dựng nghiệm mềm cho phương trình Navier-Stokes trong những không gian này bằng nguyên lý ánh xạ co Picard và chỉ ra rằng phương trình Navier-Stokes được đặt chỉnh trong các không gian này theo nghĩa Hadarmard. Chúng tôi chứng minh sự tồn tại toàn cục và duy nhất của nghiệm mềm khi giá trị ban đầu đủ nhỏ và sự tồn tại địa phương của nghiệm mềm đối với giá trị ban đầu tùy ý. Những kết quả thu được có một quan hệ chặt chẽ giữa thời gian tồn tại và độ lớn của dữ liệu ban đầu: Thời gian lớn với dữ liệu ban đầu nhỏ hoặc dữ liệu ban đầu lớn với thời gian nhỏ.

Trong Chương 3, sử dụng phương pháp của Foias-Temam, chúng tôi nghiên cứu số chiều Hausdorff của tập hợp các điểm kỳ dị theo thời gian của nghiệm yếu của phương trình Navier-Stokes trên hình xuyến 3 chiều.

Abstract

In this thesis, we use the progress achieved in the field of harmonic analysis for the last fifteen years to study the Navier-Stokes equations. Namely, we use the tools of Fourier Analysis and properties of Fourier transform in order to study the Navier-Stokes equations.
Chapter 1 is devoted to the recalling of some well-known results of harmonic analysis.
In Chapter 2, we introduce and study the following Sobolev spaces (Sobolev spaces over a shift-invariant Banach space):

- Inhomogeneous Sobolev spaces and homogeneous Sobolev spaces over the Lebesgue spaces.

- Homogeneous Sobolev spaces over the Fourier-Lorentz spaces.

- Homogeneous Sobolev spaces over the Lorentz spaces.

- Homogeneous Sobolev spaces over the mixed-norm Lorentz spaces.

In these spaces, we prove some versions of Young's inequality type for convolutions of two functions, some versions of Holder's inequality type for point-wise product of two functions, and some versions of Sobolev's inequality. We apply these inequalities to study of the Cauchy problem for the Navier-Stokes equations. We construct mild solutions to the Navier-Stokes equations in these spaces by applying the Picard contraction principle and show that Navier-Stokes equations are well-posed in these spaces in the sense of Hadarmard. We prove the unique global existence of mild solutions when the the initial value is small enough and the local existence of mild solutions for an arbitrary initial value. The results have a standard relation between existence time and data size: large time with small data or large data with small time.

In Chapter 3, using the method of Foias-Temam, we investigate the Hausdorff dimension of the singular set in time of weak solutions to the Navier-Stokes equations on the 3D torus.

Contents

Function Spaces

$\mathbb{R}^d$	d-dimensional Euclidean space
H^s_q	Inhomogeneous Sobolev spaces
$\dot{H}^s_q$	Homogeneous Sobolev spaces
$\mathcal{S}'$	Tempered distribution
$\mathcal{S}$	Schwartz class
$B^{s,p}_q$	Inhomogeneous Besov spaces
$\dot{B}^{s,p}_q$	Homogeneous Besov spaces
$F^{s,p}_q$	Inhomogeneous Triebel-Lizorkin spaces
$\dot{F}^{s,p}_q$	Homogeneous Triebel-Lizorkin spaces
$M^{p,q}$	Inhomogeneous Morrey-Campanato spaces
$\dot{M}^{p,q}$	Homogeneous Morrey-Campanato spaces
$L^{p,q}$	Lorentz spaces
$\dot{H}^s_{L^{q,r}}$	Sobolev-Lorentz spaces
$\dot{H}^s_{\mathcal{L}^{p,r}}$	Sobolev-Fourier-Lorentz spaces
$\dot{H}^s_{L^{q,r}}$	Sobolev-Lorentz spaces

Notation

NSE	Navier-Stokes equations
$\|u\|_X$	Norm of u in the normed space X
$[x]$	Integer part of x
$\{x\}$	Fraction part of x
$\mathcal{L}^d$	Lebesgue measure in $\mathbb{R}^d$
μ_D	D-dimensional Hausdorff measures of a set in $\mathbb{R}^1$
X^*	Dual space of the normed space X
$\langle u^*, u\rangle_{(X^*,X)}$	Duality product $u^*(u)$ of $u \in X$ and $u^* \in X^*$
∇v	Gradient of the scalar function v
Δv	Laplacian of the scalar function v
$\mathrm{div}(u)$	Divergence of the vector-valued function u
$\mathbb{P}$	Leray projection operator
$\dot{\Lambda}$	Calderon homogeneous pseudo-differential operator
$e^{t\Delta}$	Heat kernel
$\hat{}$	Fourier transform
$\check{}$	Inverse Fourier transform
$\otimes$	Tensor product
R_j	Riesz transforms
$[\cdot,\cdot]$.	Complex interpolation spaces between two spaces

Introduction

Formulated and intensively studied at the beginning of the nineteenth century, the classical partial differential equations of mathematical physics represent the foundation of our knowledge of waves, heat conduction, hydrodynamics and other physical problems. Their study prompted further work by mathematical researchers and, in turn, benefited from the application of new methods in pure mathematics. It is a vast subject, intimately connected to various sciences such as Physics, Mechanics, Chemistry, Engineering Sciences, with a considerable number of applications to industrial problems. Although the theory of partial differential equations has undergone a great development in the twentieth century, some fundamental questions remain unresolved. They are essentially concerned with the global existence and uniqueness of solutions, as well as their asymptotic behavior.

From a mathematical viewpoint, one of the most intriguing unresolved questions concerning the Navier-Stokes equations and closely related to turbulence phenomena is the regularity and uniqueness of the solutions to the initial value problem. More precisely, given a smooth datum at time zero, will the solution of the Navier-Stokes equations continue to be smooth and unique for all time? This question was posed in 1934 by J. Leray [47, 49] and is still without answer, neither in the positive nor in the negative.

There is no uniqueness proof except for over small time intervals and it has been questioned whether the Navier-Stokes equations really describe general flows. But there is no proof for non-uniqueness either.

Uniqueness of the solutions of the equations of motion is the cornerstone of classical determinism [18]. If more than one solution were associated to the same initial data, the committed determinist will say that the space of the solutions is too large, beyond the real physical possibility, and that uniqueness can be restored if the unphysical solutions are excluded.

In the nineteenth century, the existence problems arising from mathematical physics were studied with the aim of finding exact solutions to the corresponding equations. This is only possible in particular cases. For instance, very few exact solutions of the Navier-Stokes equations were found and, except for some exact stationary solutions, almost all of them do not involve the specifically nonlinear aspects of the problem, since the corresponding nonlinear terms in the Navier-Stokes equations vanish. In the twentieth century, the strategy changed. Instead of explicit formulas in particular cases, the problems were studied in all their generality. This led to the concept of weak solutions. The price to pay is that only the existence of the solutions can be ensured. In fact, the construction of weak solutions as the limit of approximations solutions opens the possibility that there is more than one weak solutions.

A question intimately related to the uniqueness problem is the regularity of the solution. Do the solutions to the Navier-Stokes equations blow-up in finite time? The solution is initially regular and unique, but at the instant T when it ceases to be unique (if such an instant exists), the regularity could also be lost.

One may imagine that blow-up of initially regular solutions never happens, or that it becomes more likely as the initial norm increases, or that there is blow-up, but only on a very thin set of probability zero. The best result in this direction concerning the possible loss of smoothness for the Navier-Stokes equations was obtained by L. Caffarelli, R. Kohn and L. Nirenberg [9, 45], who proved that the one-dimensional Hausdorff measure of the singular set is zero.
We can say that if "some quantity" turns out to "be small", then the Navier-Stokes equations are well-posed in the sense of Hadamard (existence, uniqueness and stability of the corresponding solutions). For instance, the unique global solution exists when the initial value and the exterior force are small enough, and the solution is smooth depending on smoothness of these data. Another quantity that can be small is the dimension. If we are in dimension $n = 2$, the situation is easier than in dimension $n = 3$ and completely understood [41, 63]. Finally, if the domain $\Omega \subset \mathbb{R}^3$ is small, in the sense that $\Omega = \omega \times (0, \epsilon)$ is thin in one direction, say, then the question is also settled [66].
In this thesis, we study well-posedness for the Cauchy problem of incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \otimes u) - \nabla p, \\ \mathrm{div}(u) = 0, \\ u_0(0, x) = u_0, \end{cases} \tag{0.1}$$

where $t \in \mathbb{R}^+, x \in \mathbb{R}^d$ $(d \geq 2), u = (u_1, u_2, ..., u_d)$ denote the flow velocity vector and $p(t, x)$ describes the scalar pressure, $\nabla = (\partial_1, \partial_2, ..., \partial_d)$ is the gradient operator, $\Delta = \partial_1^2 + \partial_2^2 + ... + \partial_d^2$ is the Laplacian, $u_0(x) = (u_1^0, u_2^0, ..., u_d^0)$ is a given initial datum with $\mathrm{div}(u_0) = \partial_1 u_1^0 + \partial_2 u_2^0 + ... + \partial_d u_d^0 = 0$. For a tensor $F = (F_{ij})$ we define the vector $\nabla \cdot F$ by $(\nabla \cdot F)_i = \sum_{j=1}^d \partial_j F_{ij}$ and for two vectors u and v, we define their tensor product $(u \otimes v)_{ij} = u_i v_j$. It is to see that (0.1) can be rewritten in the following equivalent form:

$$\begin{cases} \partial_t u = \Delta u - \mathbb{P}\nabla \cdot (u \otimes u), \\ u_0(0, x) = u_0, \end{cases} \tag{0.2}$$

where the operator $\mathbb{P}$ is the Helmholtz-Leray projection onto the divergence-free fields. Let us recall that the Riesz transforms R_j are defined by $R_j = \frac{\partial_j}{\sqrt{-\Delta}}$, i. e., for $f \in L^2$ by $(R_j f)^\wedge = \frac{i\xi_j}{|\xi|}\hat{f}(\xi)$. Then $\mathbb{P}$ is defined on $(L^2)^d$ as $\mathbb{P} = Id + \mathcal{R} \otimes \mathcal{R}$ where $\mathcal{R}$ is the vector of the Riesz transformations: $(\mathbb{P}f)_j = f_j + \sum_{1 \leq k \leq d} R_j R_k f_k$,
It is known that (0.2) is essentially equivalent to the following integral equation:

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla \cdot (u \otimes u) \mathrm{d}\tau, \tag{0.3}$$

where the heat kernel $e^{t\Delta}$ is defined as

$$e^{t\Delta}u(x) = ((4\pi t)^{-d/2} e^{-|\cdot|^2/4t} * u)(x).$$

Note that (0.1) is scaling invariant in the following sense: if u solves (0.1), so does $u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$ and $p_\lambda(t, x) = \lambda^2 p(\lambda^2 t, \lambda x)$ with initial data $\lambda u_0(\lambda x)$. A function space X defined in $\mathbb{R}^d$ is said to be a critical space for (0.1) if its norm is invariant under the action of the scaling $f(x) \to \lambda f(\lambda x)$ for any $\lambda > 0$, i.e., $\|f(\cdot)\| = \|\lambda f(\lambda x)\|$. It is easy to see that the following spaces are critical spaces for the Navier-Stokes equations:

$$\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d) \hookrightarrow L^d(\mathbb{R}^d) \hookrightarrow \dot{B}_q^{\frac{d}{q}-1,\infty}(\mathbb{R}^d)_{(q<\infty)} \hookrightarrow BMO^{-1}(\mathbb{R}^d) \hookrightarrow \dot{B}_\infty^{-1,\infty}(\mathbb{R}^d). \tag{0.4}$$

It is remarkable feature that the Navier-Stokes equations are well-posed in the sense of Hadarmard (existence, uniqueness and continuous dependence on data) when the initial data are divergence-free and belong to the critical function spaces (except $\dot{B}_{\infty}^{-1,\infty}$) listed in (0.4) (see [11] for $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$, $L^d(\mathbb{R}^d)$, and $\dot{B}_q^{\frac{d}{q}-1,\infty}(\mathbb{R}^d)$, see [40] for $BMO^{-1}(\mathbb{R}^d)$, and the recent ill-posedness result in [3] for $\dot{B}_{\infty}^{-1,\infty}(\mathbb{R}^d)$). Very recently, ill-posedness of Navier-Stokes equations in critical Besov spaces $\dot{B}_{\infty,q}^{-1}$ was investigated in [68] and finite time blowup for an averaged three-dimensional Navier-Stokes equation was investigated in [65]. In the 1960s, mild solutions were first constructed by Kato and Fujita [34, 35] that are continuous in time and take values in the Sobolev spaces $H^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$, say $u \in C([0,T]; H^s(\mathbb{R}^d))$. In 1992, a modern treatment for mild solutions in $H^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$ was given by Chemin [16]. In 1995, using the simplified version of the bilinear operator, Cannone proved the existence of mild solutions in $\dot{H}^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$, see [11]. Results on the existence of mild solutions with value in $L^q(\mathbb{R}^d), (q > d)$ were established in the papers of Fabes, Jones and Rivière [19] and of Giga [27]. Concerning the initial data in the space L^∞, the existence of a mild solution was obtained by Cannone and Meyer in [11, 14]. Moreover, in [11, 14], they also obtained theorems on the existence of mild solutions with value in the Morrey-Campanato space $M_2^q(\mathbb{R}^d), (q > d)$ and the Sobolev space $H_q^s(\mathbb{R}^d), (q < d, \frac{1}{q} - \frac{s}{d} < \frac{1}{d})$, and in general in the so-called well-suited space $\mathcal{W}$ for the Navier-Stokes equations. The Navier-Stokes equations in the Morrey-Campanato spaces were also treated by Kato [38] and Taylor [62]. In 1981, Weissler [67] gave the first existence result for mild solutions in the half space $L^3(\mathbb{R}_+^3)$. Then Giga and Miyakawa [28] generalized the result to $L^3(\Omega)$, where Ω is an open bounded domain in $\mathbb{R}^3$. Finally, in 1984, Kato [37] obtained, by means of a purely analytical tool (involving only Hölder and Young inequalities and without using any estimate of fractional powers of the Stokes operator), an existence theorem in the whole space $L^3(\mathbb{R}^3)$. In [11, 12, 13], Cannone showed how to simplify Kato's proof. The idea is to take advantage of the structure of the bilinear operator in its scalar form. In particular, the divergence ∇ and heat $e^{t\Delta}$ operators can be treated as a single convolution operator. In 1994, Kato and Ponce [39] showed that the Navier-Stokes equations are well-posed when the initial data belong to the homogeneous Sobolev spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d), (d \leq q < \infty)$.

In this thesis, we use the progress achieved in the field of harmonic analysis for the last fifteen years to study the Navier-Stokes equations. Namely, we use the tools of Fourier Analysis and properties of Fourier transform in order to study the Navier-Stokes equations.

Chapter 1 is devoted to the recalling of some well-known results of harmonic analysis.

In Chapter 2, we apply these tools to the study of the Cauchy problem for the Navier-Stokes equations.

Section 2.1 presents the general shift-invariant space of distributions and some Sobolev spaces over a shift-invariant Banach space of distributions.
From Section 2.2 to Section 2.6, we construct mild solutions to (0.3), a natural approach is to iterate the transform $u \to e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)\mathrm{d}\tau$ and to find a fixed point u for this transform. This is the so-called Picard contraction method already in use by Oseen at the beginning of the 20th century to establish the local existence of a classical solution to the Navier-Stokes equations for a regular initial value, see Oseen [54]. We rewrite the equation (0.3) as follows

$$u = U_0 - B(u,u), \tag{0.5}$$

where

$$B(u,v)(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes v)\mathrm{d}s \tag{0.6}$$

and

$$U_0 = e^{t\Delta}u_0.$$

By Theorem 1.5.1 (see Section 1.5 of Chapter 1), to find a fixed point u for the equation (0.5), we need to try to find a Banach space $\mathcal{E}_T$ of functions defined on $(0,T)\times\mathbb{R}^d$ so that the bilinear operator B which is defined by (0.6) is bounded from $\mathcal{E}_T\times\mathcal{E}_T\to\mathcal{E}_T$. Section 2.2 to Section 2.6 are devoted to construct examples of such spaces $\mathcal{E}_T$. The obtained results have a standard relation between existence time and data size: large time with small data or large data with small time.

In Section 2.2, we study local and global well-posedness for the Navier-Stokes equations with initial data in homogeneous Sobolev spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ for $d\geq 2, 1<q\leq d$. The obtained result improves the known ones for $q=2$ and $q=d$. These cases were considered by many authors, see [11, 13, 16, 17, 34, 35, 37, 46, 57].

In Section 2.3, we study local well-posedness for the Navier-Stokes equations with arbitrary initial data in homogeneous Sobolev spaces $\dot{H}_p^s(\mathbb{R}^d)$ for $d\geq 2, p>\frac{d}{2}$, and $\frac{d}{p}-1\leq s<\frac{d}{2p}$. The obtained result improves the known ones for $p>d$ and $s=0$ (see [11, 14]). In the case of critical indexes $s=\frac{d}{p}-1$, we prove global well-posedness for Navier-Stokes equations when the norm of the initial value is small enough. This result is a generalization of the ones in [13] and [46] in which $(p=d, s=0)$ and $(p>d, s=\frac{d}{p}-1)$, respectively.

In Section 2.4, we introduce and study Sobolev-Fourier-Lorentz spaces $\dot{H}^s_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$. We then study local and global well-posedness for the Navier-Stokes equations with initial data in critical spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $d\geq 2, 1\leq p<\infty$, and $1\leq r<\infty$.

In Section 2.5, we study local well-posedness for the Navier-Stokes equations with the arbitrary initial value in homogeneous Sobolev-Lorentz spaces $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d):=(-\Delta)^{-s/2}L^{q,r}$ for $d\geq 2, q>1, s\geq 0$, $1\leq r\leq\infty$, and $\frac{d}{q}-1\leq s<\frac{d}{q}$. This result improves the known results for $q>d, r=q, s=0$ (see [11, 14]) and for $q=r=2, \frac{d}{2}-1<s<\frac{d}{2}$ (see [11, 16]). In the case of critical indexes $(s=\frac{d}{q}-1)$, we prove global well-posedness for the Navier-Stokes equations when the norm of the initial value is small enough. The result is a generalization of the result in [12] for $q=r=d, s=0$.

In Section 2.6, for $0\leq m<\infty$ and index vectors $\mathbf{q}=(q_1,q_2,...,q_d), \mathbf{r}=(r_1,r_2,...,r_d)$, where $1<q_i<\infty, 1\leq r_i\leq\infty$, and $1\leq i\leq d$, we introduce and study mixed-norm Sobolev-Lorentz spaces $\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}$. Then we investigate the existence and uniqueness of solutions to the Navier-Stokes equations in the spaces $\mathcal{Q}:=\mathcal{Q}_T=L^p([0,T];\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}})$ where $p>2$, $T>0$, and initial data is taken in the class $\mathcal{I}=\{u_0\in(\mathcal{S}'(\mathbb{R}^d))^d, \mathrm{div}(u_0)=0 : \|e^{\cdot\Delta}u_0\|_{\mathcal{Q}}<\infty\}$. In the case when $m=0$, $q_1=q_2=...=q_d=r_1=r_2=...=r_d$, our results recover those of Faber, Jones and Riviere [19].

In Chapter 3, using the method of Foias-Temam, we show the vanishing of Hausdorff measure of the singular set in time of weak solutions to the Navier-Stokes equations in 3D torus.

Chapter 1

Preliminaries

Throughout the thesis, a space of functions defined on $\mathbb{R}^d$, say $E(\mathbb{R}^d)$, will be abbreviated it as E. We do not distinguish between the vector-valued and scalar-valued spaces of functions. For any collection of Banach spaces $(X_m)_{m=1}^M$ and $X = X_1 \cap ... \cap X_m$, we set $\|g\|_X = \big(\sum_{m=1}^{m=M} \|g\|_{X_m}^2\big)^{\frac{1}{2}}$. Similarly, for a vector-valued function $f = (f_1, ..., f_M)$, we define $\|f\|_X = \big(\sum_{m=1}^{m=M} \|f_m\|_X^2\big)^{\frac{1}{2}}$. We sometimes use the notation $A \lesssim B$ as an equivalent to $A \leq CB$ with a uniform constant C. The notation $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$.

1.1 Some results of real harmonic analysis

This section is devoted to the recalling of some well-known results of harmonic analysis.

1.1.1. Littlewood-Paley decomposition

We take an arbitrary function φ in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ and whose Fourier transform $\hat{\varphi}$ is such that

$$0 \leq \hat{\varphi}(\xi) \leq 1,\ \hat{\varphi}(\xi) = 1 \text{ if } |\xi| \leq \frac{3}{4},\ \hat{\varphi}(\xi) = 0 \text{ if } |\xi| \geq \frac{3}{2},$$

and let

$$\psi(x) = 2^d \varphi(2x) - \varphi(x),$$
$$\varphi_j(x) = 2^{dj} \varphi(2^j x),\ j \in \mathbb{Z},$$
$$\psi_j(x) = 2^{dj} \psi(2^j x),\ j \in \mathbb{Z}.$$

We denote by S_j and Δ_j , respectively, the convolution operators with φ_j and ψ_j . Finally, the set $\{S_j, \Delta_j\}_{j\in\mathbb{Z}}$ is the Littlewood-Paley decomposition, so that

$$I = S_0 + \sum_{j\geq 0} \Delta_j. \tag{1.1}$$

To be more precise, we should say 'a decomposition', because there are different possible (equivalent) choices for the function φ. On the other hand, for an arbitrary tempered distribution f, the last identity gives

$$f = S_0 f + \sum_{j\geq 0} \Delta_j f. \tag{1.2}$$

The interest in decomposing a tempered distribution into a sum of dyadic blocks $\Delta_j f$, whose support in Fourier space is localized in a corona, comes from the nice behavior of these blocks with respect to differential operations. This fact is illustrated by the following celebrated Bernsteins lemma in $\mathbb{R}^d$, whose proof can be found in [44].

Lemma 1.1.1. *Let $1 \leq p \leq q \leq \infty$ and $k \in \mathbb{N}$, then one has*

$$\sup_{|\alpha|=k} \|\partial^\alpha f\|_p \simeq R^k \|f\|_p$$

and

$$\|f\|_q \lesssim R^{d(\frac{1}{p}-\frac{1}{q})} \|f\|_p$$

whenever f is a tempered distribution in $\mathcal{S}'$ whose Fourier transform $\hat{f}(\xi)$ is supported in the corona $|\xi| \simeq R$.

In the case of a function whose support is a ball (as, for instance, for $S_j f$) the lemma reads as follows

Lemma 1.1.2. *Let $1 \leq p \leq q \leq \infty$ and $k \in \mathbb{N}$, then one has*

$$\sup_{|\alpha|=k} \|\partial^\alpha f\|_p \lesssim R^k \|f\|_p$$

and

$$\|f\|_q \lesssim R^{d(\frac{1}{p}-\frac{1}{q})} \|f\|_p$$

whenever f is a tempered distribution in $\mathcal{S}'$ whose Fourier transform $\hat{f}(\xi)$ is supported in the ball $|\xi| \lesssim R$.

Let us go back to the decomposition of the unity equations (1.1) and (1.2). It was introduced in the early 1930s by Littlewood and Paley to estimate the L^p-norm of trigonometric Fourier series when $1 < p < \infty$. If we omit the trivial case $p = 2$, it is not possible to ensure the belonging of a generic Fourier series to the space L^p by simply using its Fourier coefficients, but this becomes true if we consider instead its dyadic blocks. In the case of a function f (not necessarily periodic), this property is given by the following equivalence

$$\text{if } 1 < p < \infty \text{ then } \|f\|_p \simeq \|S_0 f\|_p + \Big\| \Big(\sum_{j\geq 0} |\Delta_j f(\cdot)|^2 \Big)^{\frac{1}{2}} \Big\|_p. \tag{1.3}$$

It is even easier to prove that the classical Sobolev spaces $H^s = H^s_2, s \in \mathbb{R}$ can be characterized by the following equivalent norm

$$\|f\|_{H^s} \simeq \|S_0 f\|_2 + \Big(\sum_{j\geq 0} 2^{2js} \|\Delta_j f\|_2^2 \Big)^{\frac{1}{2}}. \tag{1.4}$$

As far as the more general norm $\|f\|_{H^s_p} := \big\|(I-\Delta)^{\frac{s}{2}} f\big\|_p, s \in \mathbb{R}, 1 < p < \infty$ corresponding to the Sobolev-Bessel spaces H^s_p (that, when s is an integer, reduce to the well-known Sobolev spaces W^s_p whose norms are given by $\|f\|_{W^s_p} = \sum_{|\alpha|\leq s} \|\partial^\alpha f\|_p$ we will see in the next section how the equation (1.4) has to be modified.

Before defining the Besov spaces that will play a key role in our study of the Navier-Stokes equations, let us recall the homogeneous decomposition of the unity, analogous to the equation (1.1), but containing also all the low frequencies ($j < 0$), say

$$I = \sum_{j \in \mathbb{Z}} \Delta_j.$$

If we apply this identity to an arbitrary tempered distribution f, we may be tempted to write

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f, \tag{1.5}$$

but, at variance with the equation (1.2), this identity has no meaning in $\mathcal{S}'$ for several reasons. First of all, the sum in the equation (1.5) does not necessarily converge in $\mathcal{S}'$ as we can see if we consider a test function $g \in \mathcal{S}$ whose Fourier transform is equal to 1 near the origin, because in this case the quantity $< \Delta_j f, g >$ is, for all $j \ll 0$, a positive constant not depending on j. And, even when the sum is convergent, the convergence has to be understood modulo polynomials, because, for these particular functions P, we have $\Delta_j P = 0$ for all $j \in \mathbb{Z}$. A way to restore the convergence is to "sufficiently" derive the formal series $\sum_{j \in \mathbb{Z}}$ as it stated in the following lemma (see [4, 5, 55] for a simple proof).

Lemma 1.1.3. *For any tempered distribution f there exists an integer m such that for any $\alpha, |\alpha| \geq m$ the series $\sum_{j<0} \partial^\alpha(\Delta_j f)$ converges in $\mathcal{S}'$.*

The following corollary, whose proof follows from the previous lemma, gives the correct meaning to the convergence in (1.5), that is modulo polynomials.

Corollary 1.1.4. *For any integer N, there exists a polynomial P_N of degree $< m$ such that the quantity $\sum_{j \geq -N} \Delta_j f - P_N$ converges in $\mathcal{S}'$when $N \to \infty$.*

In such a way, the series $\Delta_j f$ is always well-defined; furthermore, it is not difficult to prove that the difference $f - \sum_{j \in \mathbb{Z}} \Delta_j f$ has its spectrum reduced to zero; in other words, it is a polynomial. In this way, the convergence in (1.5), that fails to be valid in $\mathcal{S}'$, is ensured in the quotient space $\mathcal{S}'/\mathcal{P}$.

1.1.2. Besov spaces

The Littlewood-Paley decomposition is very useful because we can define (independently of the choice of the initial function φ) the following (inhomogeneous) Besov spaces [23, 56]

Definition 1.1.1. *Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Then a tempered distribution f belongs to the (inhomogeneous) Besov space $B_q^{s,p}$ if and only if*

$$\|S_0 f\|_q + \Big(\sum_{j \geq 0} (2^{sj}\|\Delta_j f\|_q)^p\Big)^{\frac{1}{p}} < \infty.$$

For the sake of completeness, we also define the (inhomogeneous) Triebel- Lizorkin spaces, even if we will not make a great use of them in the study of the Navier-Stokes equations.

Definition 1.1.2. *Let* $0 < p \leq \infty, 0 < q < \infty$, *and* $s \in \mathbb{R}$. *Then a tempered distribution* f *belongs to the (inhomogeneous) Triebel-Lizorkin space* $F_q^{s,p}$ *if and only if*

$$\|S_0 f\|_q + \Big\| \Big(\sum_{j \geq 0} \big(2^{sj} |\Delta_j f|\big)^p \Big)^{\frac{1}{p}} \Big\|_q < \infty.$$

It is easy to see that the above quantities define a norm if $p, q \geq 1$ and a quasi-norm in general, with the usual convention that $p = \infty$ in both cases corresponds to the usual L^∞ norm. On the other hand, we have not included the case $q = \infty$ in the second definition because the L^∞ norm has to be replaced here by a more complicated Carleson measure (see [23]). As we have already remarked before for some particular values of s, p, q, see the equations. (1.3) and (1.4), the Besov and Triebel-Lizorkin spaces generalize the usual Lebesgue ones, for instance

$$L^q = F_q^{0,2}, \; 1 < q < \infty,$$

and more generally, the Sobolev-Bessel spaces

$$H_q^s = F_q^{s,2}, \; s \in \mathbb{R}, 1 < q < \infty.$$

Let $1 < q < \infty$ and $s < d/q$. We define the homogeneous Sobolev space $\dot{H}_q^s$ as the closure of the space

$$S_0 = \big\{ f \in \mathcal{S} : \; 0 \notin \mathrm{Supp} \hat{f} \big\}$$

in the norm

$$\|f\|_{\dot{H}_q^s} = \|\dot{\Lambda}^s f\|_q$$

where, as usual, $\dot{\Lambda} = \sqrt{-\Delta}$ denotes the homogeneous Calderón pseudodifferential operator. Finally, when $d/q + m \leq s < d/q + m + 1$ and m is an integer, $\dot{H}_q^s$ is a space of distributions modulo polynomials of degree $\leq m$.

We are now ready to define the homogeneous version of the Besov and Triebel-Lizorkin spaces [4, 5, 23, 56]. If $m \in \mathbb{Z}$, we denote by $\mathcal{P}_m$ the set of polynomials of degree $\leq m$ with the convention that $\mathcal{P}_m = \emptyset$ if $m < 0$. If $p = 1$ and $s - d/q \in \mathbb{Z}$, we put $m = s - d/q - 1$; if not, we put $m = [s - d/q]$, with the brackets denoting the integer part function.

Definition 1.1.3. *Let* $0 < p, q \leq \infty$ *and* $s \in \mathbb{R}$. *Then a tempered distribution* f *belongs to the (homogeneous) Besov space* $\dot{B}_q^{s,p}$ *if and only if*

$$\Big(\sum_{j \in \mathbb{Z}} \big(2^{sj} \|\Delta_j f\|_q\big)^p \Big)^{\frac{1}{p}} < \infty \text{ and } f = \sum_{j \in \mathbb{Z}} \Delta_j f \text{ in } \mathcal{S}'/\mathcal{P}_m.$$

Definition 1.1.4. *Let* $0 < p \leq \infty, 0 < q < \infty$, *and* $s \in \mathbb{R}$. *Then a tempered distribution* f *belongs to the (homogeneous) Triebel-Lizorkin space* $\dot{F}_q^{s,p}$ *if and only if*

$$\Big\| \Big(\sum_{j \in \mathbb{Z}} \big(2^{sj} |\Delta_j f|\big)^p \Big)^{\frac{1}{p}} \Big\|_q < \infty \text{ and } f = \sum_{j \in \mathbb{Z}} \Delta_j f \text{ in } \mathcal{S}'/\mathcal{P}_m,$$

with an analogous modification as in the inhomogeneous case when $q = \infty$.

As expected, we have the following identifications:

$$L^q = \dot{F}_q^{0,2}, \; 1 < q < \infty,$$

and, more generally

$$\dot{H}_q^s = \dot{F}_q^{s,2}, \; s \in \mathbb{R}, 1 < q < \infty,$$

$$BMO = \dot{F}^{0,2}_{\infty},$$

and

$$BMO^{-1} = \dot{F}^{-1,2}_{\infty}.$$

Moreover, we have the following embeddings (see [1, 2, 8]).

Lemma 1.1.5. *For $1 \leq p, q, r \leq \infty$ and $s \in \mathbb{R}$, we have the following embedding mappings.*
(a) *If $1 < q \leq 2$ then*

$$\dot{B}^{s,q}_{q} \hookrightarrow \dot{H}^{s}_{q} \hookrightarrow \dot{B}^{s,2}_{q},\ B^{s,q}_{q} \hookrightarrow H^{s}_{q} \hookrightarrow B^{s,2}_{q}.$$

(b) *If $2 \leq q < \infty$ then*

$$\dot{B}^{s,2}_{q} \hookrightarrow \dot{H}^{s}_{q} \hookrightarrow \dot{B}^{s,q}_{q},\ B^{s,2}_{q} \hookrightarrow H^{s}_{q} \hookrightarrow B^{s,q}_{q}.$$

(c) *If $1 \leq p_1 < p_2 \leq \infty$ then*

$$\dot{B}^{s,p_1}_{q} \hookrightarrow \dot{B}^{s,p_2}_{q}, B^{s,p_1}_{q} \hookrightarrow B^{s,p_2}_{q},\ \dot{F}^{s,p_1}_{q} \hookrightarrow \dot{F}^{s,p_2}_{q}, F^{s,p_1}_{q} \hookrightarrow F^{s,p_2}_{q}.$$

(d) *If $s_1 > s_2,\ 1 \leq q_1,\ q_2 \leq \infty$, and $s_1 - \frac{d}{q_1} = s_2 - \frac{d}{q_2}$ then*

$$\dot{B}^{s_1,p}_{q_1} \hookrightarrow \dot{B}^{s_2,p}_{q_2}, B^{s_1,p}_{q_1} \hookrightarrow B^{s_2,p}_{q_2}, \dot{F}^{s_1,p}_{q_1} \hookrightarrow \dot{F}^{s_2,r}_{q_2},\ F^{s_1,p}_{q_1} \hookrightarrow F^{s_2,r}_{q_2}.$$

(e) *If $p \leq q$ then*

$$B^{s,p}_{q} \hookrightarrow F^{s,p}_{q},\ \dot{B}^{s,p}_{q} \hookrightarrow \dot{F}^{s,p}_{q}.$$

(f) *If $q \leq p$ then*

$$F^{s,p}_{q} \hookrightarrow B^{s,p}_{q},\ \dot{F}^{s,p}_{q} \hookrightarrow \dot{B}^{s,p}_{q}.$$

(g)

$$F^{s,q}_{q} = B^{s,q}_{q},\ \dot{F}^{s,q}_{q} = \dot{B}^{s,q}_{q}.$$

(h) *If $1 < q < \infty$*

$$H^{s}_{q} = F^{s,2}_{q},\ \dot{H}^{s}_{q} = \dot{F}^{s,2}_{q}.$$

We recall the following results

Lemma 1.1.6. *Let $1 \leq p, q \leq \infty$ and $s < 0$. Then the two quantities*

$$\Big(\int_0^\infty (t^{-\frac{s}{2}}\big\|e^{t\Delta}f\big\|_{L^q})^p\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{p}} \text{ and } \big\|f\big\|_{\dot{B}^{s,p}_{q}} \text{ are equivalent.}$$

Proof. See Proposition 3 in ([15], p. 182), or see Theorem 5.4 in ([46], p. 45). □

Lemma 1.1.7. *Let $1 \leq p, q \leq \infty$ and $s < 0$. Then the two quantities*

$$\Big\|\Big(\int_0^\infty (t^{-\frac{s}{2}}\big|e^{t\Delta}f\big|)^p\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{p}}\Big\|_{L^q} \text{ and } \big\|f\big\|_{\dot{F}^{s,p}_{q}} \text{ are equivalent.}$$

Proof. See Proposition 4 in ([15], p. 183). □

The following lemma is a generalization of Lemma 1.1.6.

Lemma 1.1.8. *Let $1 \leq p, q \leq \infty,\ \alpha \geq 0$, and $s < \alpha$. Then the two quantities*

$$\Big(\int_0^\infty (t^{-\frac{s}{2}}\big\|e^{t\Delta}t^{\frac{\alpha}{2}}\dot{\Lambda}^{\alpha}f\big\|_{L^q})^p\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{p}} \text{ and } \big\|f\big\|_{\dot{B}^{s,p}_{q}} \text{ are equivalent.}$$

Proof. Note that $\dot{\Lambda}^{s_0}$ is an isomorphism from $\dot{B}^{s,p}_{q}$ onto $\dot{B}^{s-s_0,p}_{q}$, see [8], then we can easily prove the lemma. □

1.1.3. Other useful function spaces

In this section we present other functional spaces, that will be useful in the following chapters.

1.1.4. Morrey-Campanato spaces

For $1 \leq p \leq q \leq \infty$, the inhomogeneous Morrey-Campanato space $M^{p,q}$ is defined as the space of functions f which are locally in L^p and such that

$$\sup_{x\in\mathbb{R}^d} \sup_{0<R<1} R^{d/q-d/p}\Big(\int_{|y-x|<R} |f(y)|^p \mathrm{d}y\Big)^{\frac{1}{p}} < \infty,$$

where the left-hand side of this inequality is the norm of f in $M^{p,q}$. The homogeneous Morrey-Campanato space $\dot{M}^{p,q}$ is defined in the same way, by taking the supremum over all $R \in (0,\infty)$ instead $R \in (0,1]$.

1.1.5. Lorentz spaces

For $1 \leq p, q \leq \infty$, the Lorentz space $L^{p,q}$ is defined as follows.
A measurable function $f \in L^{p,q}$ if and only if

$$\|f\|_{L^{p,q}} = \big(\frac{q}{p}\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{\mathrm{d}t}{t}\big)^{\frac{1}{q}} < \infty,$$

of course, if $q = \infty$ this means

$$\|f\|_{L^{p,\infty}} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty,$$

where f^* is the decreasing rearrangement of f:

$$f^*(t) = \inf\big\{s \geq 0 : \big|\{x : |f(x)| > s\}\big| \leq t\big\}, t \geq 0.$$

We recall the some results in [46].

Theorem 1.1.9. (Pointwise product in the Lorentz spaces).
Let $1 < p < \infty$ and $1 \leq q \leq \infty, 1/p' + 1/p = 1$ and $1/q' + 1/q = 1$. Then pointwise multiplication is a bounded bilinear operator:
(a) *from $L^{p,q} \times L^\infty$ to $L^{p,q}$,*
(b) *from $L^{p,q} \times L^{p',q'}$ to L^1,*
(c) *from $L^{p,q} \times L^{p_1,q_1}$ to L^{p_2,q_2}, for $1 < p, p_1, p_2 < \infty, 1 \leq q, q_1, q_2 \leq \infty$, $1/p_2 = 1/p + 1/p_1, 1/q_2 = 1/q + 1/q_1$.*

Theorem 1.1.10. (Convolution of the Lorentz spaces).
Let $1 < p < \infty$ and $1 \leq q \leq \infty, 1/p' + 1/p = 1$ and $1/q' + 1/q = 1$. Then convolution is a bounded bilinear operator:
(a) *from $L^{p,q} \times L^1$ to $L^{p,q}$,*
(b) *from $L^{p,q} \times L^{p',q'}$ to L^∞,*
(c) *from $L^{p,q} \times L^{p_1,q_1}$ to L^{p_2,q_2}, for $1 < p, p_1, p_2 < \infty, 1 \leq q, q_1, q_2 \leq \infty, 1/p_2 + 1 = 1/p + 1/p_1$, $1/q_2 = 1/q + 1/q_1$.*

1.2 Navier-Stokes equations

We consider the Navier-Stokes equations (NSE) in d dimensions in the special setting of a viscous, homogeneous, incompressible fluid that fills the entire space and is not submitted to external forces. Thus, the equations we consider are the system:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \otimes u) - \nabla p, \\ \operatorname{div}(u) = 0, \end{cases} \tag{1.6}$$

which is a condensed form of

$$\begin{cases} \text{For } 1 \leq k \leq d, \quad \partial_t u_k = \Delta u_k - \sum_{l=1}^d \partial_l (u_l u_k) - \partial_k p, \\ \sum_{l=1}^d \partial_l u_l = 0. \end{cases} \tag{1.7}$$

The unknown quantities are the velocity $u(t,x)$ of the fluid element at time t and position x and the pressure p. Taking the divergence of (1.6), we obtain:

$$\Delta p = -\nabla \otimes \nabla \cdot (u \otimes u) = -\sum_{k=1}^d \sum_{l=1}^d \partial_k \partial_l (u_k u_l). \tag{1.8}$$

Thus, we formally get the equations

$$\begin{cases} \partial_t u = \Delta u - \mathbb{P}\nabla \cdot (u \otimes u), \\ \operatorname{div}(u) = 0, \end{cases} \tag{1.9}$$

where $\mathbb{P}$ is the Helmholtz-Leray projection operator defined as

$$\mathbb{P} f := f - \nabla \frac{1}{\Delta}(\nabla \cdot f) = (I - \frac{\nabla \otimes \nabla}{\Delta}) f. \tag{1.10}$$

We shall study the Cauchy problem for the equation (1.9) (looking for a solution on $(0,T) \times \mathbb{R}^d$ with the initial value u_0), and transform (1.9) into the integral equation

$$\begin{cases} u = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u \otimes u) \mathrm{d}s, \\ \operatorname{div}(u_0) = 0. \end{cases} \tag{1.11}$$

The main property we use throughout this thesis is that the operator $e^{t\Delta}\mathbb{P}\nabla$ is a matrix of convolution operators with bounded integrable kernels.

Lemma 1.2.1. (The Oseen kernel).
For $1 \leq j,k \leq d$ and $t > 0$, the operator $O_{j,k,t} = \frac{1}{\Delta}\partial_j \partial_k e^{t\Delta}$ is a convolution operator $O_{j,k,t} f = K_{j,k,t} * f$, where the kernel $K_{j,k,t}$ satisfies $K_{j,k,t}(x) = \frac{1}{t^{d/2}} K_{j,k}(\frac{x}{\sqrt{t}})$ for a smooth function $K_{j,k}$ such that

$$(1+|x|)^{d+|\alpha|} \partial^\alpha K_{j,k} \in L^\infty(\mathbb{R}^d), \ \textit{for all} \ \ \alpha \in \mathbb{N}^d,$$

$$(1+|x|)^{d+m} \dot{\Lambda}^m K_{j,k} \in L^\infty(\mathbb{R}^d), \ \textit{for all} \ m \geq 0,$$

where

$$|x| = \Big(\sum_{i=1}^d x_i^2\Big)^{1/2}, x = (x_1, x_2, ..., x_d) \ \textit{and} \ D_x^\alpha \ \textit{denotes} \ \partial_x^{|\alpha|} = \partial^{|\alpha|}/\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} ... \partial_{x_d}^{\alpha_d}.$$

Proof. See Proposition 11.1 in [46], p. 107. □

1.3 Elimination of the pressure and integral formulation for the Navier-Stokes equations

We will focus on the invariance of the equation (1.7) under spatial translations and dilations, as we consider the problem on the whole space $\mathbb{R}^d$. We begin by defining what we call a weak solution for the Navier-Stokes equations.

Definition 1.3.1. (Weak solutions).
A weak solution of the Navier-Stokes equations on $(0,T)\times\mathbb{R}^d$ *is a distribution vector field* $u(t,x)$ *in* $(\mathcal{D}'((0,T)\times\mathbb{R}^d))^d$ *so that:*
(a) *u is locally square integrable on* $(0,T)\times\mathbb{R}^d$,
(b) $\operatorname{div}(u)=0$,
(c) $\exists p\in\mathcal{D}'((0,T)\times\mathbb{R}^d),\quad \partial_t u=\Delta u-\nabla\cdot(u\otimes u)-\nabla p.$

Notice that this is not the usual definition for weak solutions (as given in the book of Temam [63] and in Chapter 3). Next, we recall some results in [46].

Theorem 1.3.1. (Elimination of the pressure).
(a) *If u is uniformly locally square - integrable on* $(0,T)\times\mathbb{R}^d$ *(in the sense of the following definition:* $U_{t_0,t_1}(x)=(\int_{t_0}^{t_1}|u(t,x)|^2dt)^{1/2}$ *belongs to the Morrey space* L^2_{uloc} *for all* $0<t_0<t_1<T$, *where* $\|u\|_{L^p_{uloc}}=\sup_{x_0\in\mathbb{R}^d}(\int_{|x-x_0|<1}|f|^p dx)^{1/p}$), *then* $\mathbb{P}\nabla\cdot(u\otimes u)$ *is well defined in* $(\mathcal{D}'((0,T)\times\mathbb{R}^d))^d$, *and there exists a distribution* $p\in\mathcal{D}'((0,T)\times\mathbb{R}^d)$ *so that* $\mathbb{P}\nabla\cdot(u\otimes u)=\nabla\cdot(u\otimes u)+\nabla p$. *Thus, if u is a solution for* (1.9), *then it is also a solution for* (1.6).
(b) *Conversely, if u is a uniform weak solution for* (1.6), *and if u vanishes at infinity in the sense that for all* $t_0<t_1\in(0,T)$ *we have*

$$\lim_{R\to\infty}\sup_{x_0\in\mathbb{R}^d}\frac{1}{R^d}\int_{t_0}^{t_1}\int_{|x-x_0|<R}|u|^2\mathrm{d}x\mathrm{d}t=0,$$

then u is a solution for (1.9).

Theorem 1.3.2. (The equivalence theorem).
Let $u\in\cap_{t_1<T}(L^2_{uloc,x}L^2_t((0,t_1)\times\mathbb{R}^d))^d$. *Then, the following assertions are equivalent:*
(a) *u is a weak solution of the differential Navier-Stokes equations*

$$\begin{cases}\partial_t u=\Delta u-\mathbb{P}\nabla\cdot(u\otimes u),\\ \operatorname{div}(u)=0.\end{cases}$$

(b) *u is a solution of the integral Navier-Stokes equations*

$$\exists u_0\in(\mathcal{S}'(\mathbb{R}^d))^d\quad\begin{cases}u=e^{t\Delta}u_0-\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)\mathrm{d}s,\\ \operatorname{div}(u_0)=0.\end{cases}\tag{1.12}$$

1.4 Scaling invariance

The Navier-Stokes equations are invariant under a particular change of time and space scaling. More exactly, assume that, in $\mathbb{R}^d\times(0,\infty)$, $u(t,x)$ and $p(t,x)$ solve the system

$$\begin{cases}\partial_t u=\Delta u-\nabla\cdot(u\otimes u)-\nabla p,\\ \operatorname{div}(u)=0,\end{cases}$$

then the same is true for the rescaled functions

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x),\ p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, x).$$

The above scaling invariance leads to the following definition.

Definition 1.4.1. *A translation invariant Banach space of tempered distributions X is called a critical space for the Navier-Stokes equations if its norm is invariant under the action of the scaling $f(x) \to \lambda f(\lambda x)$ for any $\lambda > 0$. In other words, we require that*

$$X \hookrightarrow \mathcal{S}'$$

and that for any $f \in X$

$$\|f(\cdot)\| \simeq \|\lambda f(\lambda \cdot - x_0)\|,\ \forall \lambda > 0, \forall x_0 \in \mathbb{R}^d.$$

For example, in the Lebesgue space family $L^p = L^p(\mathbb{R}^d)$ the critical invariant space corresponds to the value $p = d$, and in the Sobolev space family $\dot{H}^s = \dot{H}^s(\mathbb{R}^d)$ the critical invariant space corresponds to the value $s = \frac{d}{2} - 1$.

Proposition 1.4.1. *If X is a critical space, then X is continuously embedded in the Besov space $\dot{B}^{-1,\infty}_{\infty}$.*

1.5 Outline of the dissertation

The idea is to construct the solution u for NSE as a solution for the integral equation (1.11). Let a Banach space $\mathcal{E}_T$ of functions defined on $(0,T) \times \mathbb{R}^d$ and such that $\mathcal{E}_T \subseteq \cap_{t_1<T}\big(L^2_{uloc,x}L^2_t((0,t_{t_1}) \times \mathbb{R}^d)\big)^d$, then we consider the space $E_T \subseteq \big(\mathcal{S}'(\mathbb{R}^d)\big)^d$ defined by $f \in E_T$ iff $f \in \big(\mathcal{S}'(\mathbb{R}^d)\big)^d$ and $(e^{t\Delta}f)_{0\le t\le T} \in \mathcal{E}_T$. If u is a solution of (1.11), $u \in \mathcal{E}_T$, $u_0 \in E_T$ then applying Theorems 1.3.1 and 1.3.2, we imply that u is also a weak solution of (1.6) and (1.9), we rewrite the equation (1.11) as follows

$$u = U_0 - B(u,u), \tag{1.13}$$

where

$$B(u,v)(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes v)\mathrm{d}s, \tag{1.14}$$

and

$$U_0 = e^{t\Delta}u_0.$$

Then we will find a fixed point u for the equation (1.13). This is the so-called Picard contraction method.

Theorem 1.5.1. *Let X be a Banach space, and let $B : X \times X \to X$ be a continuous bilinear form such that there exists η so that*

$$\|B(x,y)\| \le \eta\|x\|\|y\|,$$

for any x and $y \in X$. Then for any fixed $y \in X$ such that $\|y\| < 1/(4\eta)$, the equation $x = y - B(x,x)$ has a unique solution $\overline{x} \in X$ satisfying $\|\overline{x}\| \le R$, with

$$R = \frac{1 - \sqrt{1 - 4\eta\|y\|}}{2\eta}.$$

Proof. See Theorem 22.4 ([46], p. 227). □

By the above Theorem, we need to try to find a Banach space $\mathcal{E}_T$ so that the bilinear operator B which is defined by (1.14) is bounded from $\mathcal{E}_T \times \mathcal{E}_T \to \mathcal{E}_T$.
Chapter 2 is devoted to construct examples of such spaces $\mathcal{E}_T$. The solutions that we obtain through the Picard contraction principle are called mild solutions. We call a space $\mathcal{E}_T$ if we may apply the Picard contraction principle as an admissible path space for the Navier-Stokes equations, and the associated space E_T as an adapted value space.
Let us review some results. We will indicate what are the admissible path space $\mathcal{E}_T$ and the associated adapted space E_T.

• Classical admissible spaces are provided by the L^p theory of Kato [37]:
- For $d < p < \infty, C([0;T];L^p)$ is admissible with the associated adapted space $L^p(\mathbb{R}^d)$.
- For $p = d$, the space

$$\{f \in C([0;T];L^d) : \sup_{0<t<T} \sqrt{t}\|f\|_{L^\infty(\mathrm{d}x)} < \infty \text{ and } \lim_{t\to 0}\sqrt{t}\|f\|_{L^\infty(\mathrm{d}x)} = 0\}$$

is admissible with the associated adapted space $L^d(\mathbb{R}^d)$.

• Prodi [52] gave the following admissible spaces, plus the corresponding the associated adapted space

$$\mathcal{E}_T = L^q([0,T], L^p),\ E_T = \dot{B}_p^{\frac{d}{p}-1,q} \text{ with } \frac{d}{p} + \frac{2}{q} = 1 \text{ and } d < p < \infty.$$

• Cannone [12] studied the space

$$\{f \in C([0;T];L^d) : \sup_{0<t<T} t^{\frac{\alpha}{2}}\|f\|_{L^q(\mathrm{d}x)} < \infty \text{ and } \lim_{t\to 0} t^{\frac{\alpha}{2}}\|f\|_{L^q(\mathrm{d}x)} = 0\}, \tag{1.15}$$

$$\text{with } q > d \text{ and } \alpha = 1 - \frac{d}{q}, \tag{1.16}$$

which is admissible with the associated adapted space $L^d(\mathbb{R}^d)$.

• In [25, 26], Gallagher and Planchon studied a Besov spaces scale

$$\mathcal{E}_T = L^q\dot{B}_p^{\frac{2}{q}+\frac{d}{p}-1,q},\ E_T = \dot{B}_p^{\frac{d}{p}-1,q} \text{ with } \frac{d}{p} + \frac{2}{q} > 1.$$

In Chapter 2 of this thesis we study some other admissible spaces with other associated adapted spaces.

In Section 2.2 of Chapter 2:
- For $2 < q \le d$ and p be such that $q < p < \min\Big\{\frac{(d-2)q}{d-q},\ d+2\Big\}$, we consider the admissible space

$$L^p\big([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\big) \cap L^\infty\big([0,T];\dot{H}_q^{d/q-1}\big)$$

which is admissible with the associated adapted space $\dot{H}_q^{d/q-1}(\mathbb{R}^d)$.
- For $1 < q \le 2$ we consider the admissible space

$$L^{2q}\big([0,T];\dot{H}_{\frac{dq}{d+1-q}}^{\frac{d+2-2q}{q}}\big) \cap L^\infty\big([0,T];\dot{H}_q^{d/q-1}\big)$$

which is admissible with the associated adapted space $\dot{H}_q^{d/q-1}(\mathbb{R}^d)$.

In Section 2.3 of Chapter 2:
- For $p > \frac{d}{2}$, $\frac{d}{p} - 1 \le s < \frac{d}{2p}$, $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$, and $r > \max\{p, q\}$, we consider the admissible space $\mathcal{K}^r_{q,T} \cap L^\infty([0,T]; \dot{H}^s_p)$ is admissible with the associated adapted space $\dot{H}^s_p(\mathbb{R}^d)$, where space $\mathcal{K}^r_{q,T}$ is made up of the functions $u(t,x)$ such that $\sup\limits_{0<t<T} t^{\frac{\alpha}{2}} \big\|u(t,x)\big\|_{L^r} < \infty$ and $\lim\limits_{t\to 0} t^{\frac{\alpha}{2}} \big\|u(t,x)\big\|_{L^r} = 0$ with $\alpha = d(\frac{1}{q} - \frac{1}{r})$.

In Section 2.4 of Chapter 2: For $s \in \mathbb{R}$ and $1 \le p, r \le \infty$ we introduce and study the Sobolev-Fourier-Lorentz spaces $\dot{H}^s_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$.
- For $1 < p \le d, 1 \le r < \infty$, and $\tilde{p}$ be such that $\frac{1}{2p} + \frac{[\frac{d}{p}]-1}{2d} < \frac{1}{\tilde{p}} < \min\Big\{\frac{[\frac{d}{p}]}{d}, \frac{1}{2} + \frac{[\frac{d}{p}]-1}{2d}\Big\}$, we consider the admissible space $\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T} \cap L^\infty\big([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}\big)$ which is admissible with the associated adapted space $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$, where the space $\mathcal{K}^{\tilde{p}}_{p,r,T}$ is made up by the functions $u(t,x)$ such that $\sup\limits_{0<t<T} t^{\frac{\alpha}{2}} \big\|u(t,x)\big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{\tilde{p},r}}} < \infty$ and $\lim\limits_{t\to 0} t^{\frac{\alpha}{2}} \big\|u(t,x)\big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{\tilde{p},r}}} = 0$ with $\alpha = d\big(\frac{1}{p} - \frac{1}{\tilde{p}}\big)$.
- For $p \ge d, r \ge 1$, and $q > p$, we consider the admissible space $\mathcal{K}^q_{d,1,T} \cap L^\infty([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}})$ is admissible with the associated adapted space $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$.
- For $d - 1 < s < d$ and $r \ge 1$, we consider the admissible space $\mathcal{K}_{s,r,T} \cap L^\infty([0,T]; \dot{H}^{d-1}_{\mathcal{L}^{1,r}})$ which is admissible with the associated adapted space $\dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$, where the space $\mathcal{K}_{s,r,T}$ is made up by the functions $u(t,x)$ such that $\sup\limits_{0<t<T} t^{\frac{\alpha}{2}} \big\|u(t,x)\big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} < \infty$ and $\lim\limits_{t\to 0} t^{\frac{\alpha}{2}} \big\|u(t,x)\big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} = 0$ with $\alpha = s + 1 - d$.

In Section 2.5 of Chapter 2: For $q > 1, 1 \le r \le \infty$ and $0 \le s < \frac{d}{q}$, we introduce and study the Sobolev-Lorentz spaces $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d)$, which are generalizations of the classical Sobolev spaces $\dot{H}^s_q(\mathbb{R}^d)$. For $s \ge 0, q > 1, r \ge 1, \frac{s}{d} < \frac{1}{q} \le \frac{s+1}{d}$, and $\tilde{q}$ be such that $\frac{1}{2}\Big(\frac{1}{q} + \frac{s}{d}\Big) < \frac{1}{\tilde{q}} < \min\Big\{\frac{1}{2} + \frac{s}{2d}, \frac{1}{q}\Big\}$, we consider the admissible space $\mathcal{K}^{s,\tilde{q}}_{q,1,T} \cap L^\infty([0,T]; \dot{H}^s_{L^{q,r}})$ which is admissible with the associated adapted space $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d)$, where space $\mathcal{K}^{s,\tilde{q}}_{q,r,T}$ is made up by the functions $u(t,x)$ such that $\sup\limits_{0<t<T} t^{\frac{\alpha}{2}} \big\|u(t,x)\big\|_{\dot{H}^s_{L^{\tilde{q},r}}} < \infty$ and $\lim\limits_{t\to 0} t^{\frac{\alpha}{2}} \big\|u(t,x)\big\|_{\dot{H}^s_{L^{\tilde{q},r}}} = 0$ with $\alpha = d(\frac{1}{q} - \frac{1}{\tilde{q}})$.

In Section 2.6 of Chapter 2: For $0 \le m < \infty$ and index vectors $\mathbf{q} = (q_1, q_2, ..., q_d)$ and $\mathbf{r} = (r_1, r_2, ..., r_d)$, where $1 < q_i < \infty, 1 \le r_i \le \infty$ for $i = 1, 2, .., d$, we introduce and study mixed-norm Sobolev-Lorentz spaces $\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}$. For $\mathbf{q} > \mathbf{1}$, $\mathbf{r} \ge \mathbf{1}, 2 < p < \infty$, and $m \ge 0$ be such that $m < \frac{1}{2}\sum_{i=1}^d \frac{1}{q_i}$, $\frac{2}{p} - m + \sum_{i=1}^d \frac{1}{q_i} \le 1$, $2 < \frac{q_i}{\Big(1 - \frac{m}{\sum_{i=1}^d \frac{1}{q_i}}\Big)} < \infty$, $i = 1, 2, .., d$, we consider the admissible space $L^p([0,T]; \dot{H}^m_{L^{\mathbf{q},\mathbf{r}}})$ which is admissible with the associated adapted space $B^{m - \frac{2}{p}, p}_{L^{\mathbf{q},\mathbf{r}}}$ (a Besov space).

In Chapter 3: Using the method of Foias-Temam, we investigate the Hausdorff dimension of the singular set in time of weak solutions to the Navier-Stokes equations.

Chapter 2

Mild solutions in some Sobolev spaces over a shift-invariant Banach space

In this chapter we investigate mild solutions to the Navier-Stokes equations in some Sobolev spaces over a shift-invariant Banach space of distributions.

2.1 Sobolev spaces over a shift-invariant Banach space of distributions

We shall often use Banach spaces of distributions whose norms are invariant under translations $\|f\|_E = \|f(x - x_0)\|_E$ and on which dilations operate boundedly.

Definition 2.1.1. (Shift-invariant Banach spaces of distributions.)
A shift-invariant Banach space of test functions is a Banach space E such that we have the continuous embeddings $\mathcal{S}(\mathbb{R}^d) \hookrightarrow E \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ and so that:
(a) *for all $x_0 \in \mathbb{R}^d$ and for all $f \in E, f(x - x_0) \in E$ and $\|f\|_E = \|f(x - x_0)\|_E$,*
(b) *for all $\lambda > 0$ there exists $C_\lambda > 0$ so that for all $f \in E$ $f(\lambda x) \in E$ and $\|f(\lambda x)\|_E \leq C_\lambda \|f\|_E$,*
(c) *$\mathcal{D}(\mathbb{R}^d)$ is dense in E.*

In the following definitions, we introduce the Sobolev spaces and their homogeneous spaces over a shift-invariant Banach space of distributions. Before proceeding to the definition of the Sobolev spaces, let us introduce several necessary notations. For a real number s, the operators $\dot{\Lambda}^s$ and $(Id - \Delta)^{s/2}$ are defined through the Fourier transform by

$$\big(\dot{\Lambda}^s f\big)^\wedge(\xi) = |\xi|^s \hat{f}(\xi) \text{ and } \big((Id - \Delta)^{s/2} f\big)^\wedge(\xi) = \big(1 + |\xi|^2\big)^{s/2} \hat{f}(\xi).$$

Definition 2.1.2. (Sobolev spaces.)
Let E be a shift-invariant Banach space of distributions. Then, for $s \in \mathbb{R}$, the space H^s_E is defined as the space $(Id - \Delta)^{-s/2} E$, equipped with the norm $\big\|f\big\|_{H^s_E} = \big\|(Id - \Delta)^{s/2} f\big\|_E$.

We now define homogeneous Sobolev spaces over a shift-invariant Banach space of distributions, in a very similar way as for the Sobolev spaces $\dot{H}_p^s$ based on the Lebesgue spaces L^p.

Definition 2.1.3. (Homogeneous Sobolev spaces.)
Let E be a shift-invariant Banach space of distributions. Then, for $s \in \mathbb{R}$, the space $\dot{H}_E^s$ is defined as the closure of the space $S_0 = \big\{f \in \mathcal{S} : \ 0 \notin \mathrm{Supp}\hat{f}\big\}$ in the norm $\|f\|_{\dot{H}_E^s} = \big\|\dot{\Lambda}^s f\big\|_E$.

In the rest of this chapter, we investigate mild solutions to NSE in the following Sobolev spaces over a shift-invariant Banach space of distributions:

- Inhomogeneous Sobolev spaces and homogeneous Sobolev spaces over the Lebesgue spaces, (Sections 2.2 and 2.3).

- Homogeneous Sobolev spaces over the Fourier-Lorentz spaces, (Section 2.4).

- Homogeneous Sobolev spaces over the Lorentz spaces, (Section 2.5).

- Homogeneous Sobolev spaces over the mixed-norm Lorentz spaces, (Section 2.6).

2.2 Mild solutions in critical Sobolev spaces

In this section, we investigate mild solutions to NSE in the spaces $L^\infty\big([0,T];\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)\big)$ when the initial data belong to the homogeneous Sobolev spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d), (d \geq 2, 1 < q \leq d)$. We obtain the existence of local mild solutions with arbitrary initial value and existence of global mild solutions when the norm of the initial value in the Besov spaces $\dot{B}_p^{\frac{d}{p}-1,\infty}(\mathbb{R}^d)$ is small enough, where p may take some suitable values.

2.2.1. Some auxiliary results

Lemma 2.2.1. *Let $p > 1$ and $s \in \mathbb{R}$. Then the following statements hold*
(1) *Assume that $u_0 \in H_p^s$. Then*
$e^{\cdot\Delta}u_0 \in L^\infty([0,\infty); H_p^s)$ *and* $\big\|e^{\cdot\Delta}u_0\big\|_{L^\infty([0,\infty);H_p^s)} \leq \big\|u_0\big\|_{H_p^s}$.
(2) *Assume that $u_0 \in \dot{H}_p^s$. Then*
$e^{\cdot\Delta}u_0 \in L^\infty([0,\infty); \dot{H}_p^s)$ *and* $\big\|e^{\cdot\Delta}u_0\big\|_{L^\infty([0,\infty);\dot{H}_p^s)} \leq \big\|u_0\big\|_{\dot{H}_p^s}$.

Proof. (1)

$$\big\|e^{t\Delta}u_0\big\|_{H_p^s} = \big\|e^{t\Delta}(Id-\Delta)^{s/2}u_0\big\|_{L^p} =$$
$$\frac{1}{(4\pi t)^{d/2}}\Big\|\int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}}\big((Id-\Delta)^{s/2}u_0\big)(\cdot-\xi)\mathrm{d}\xi\Big\|_{L^p}$$
$$\leq \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}}\big\|\big((Id-\Delta)^{s/2}u_0\big)(\cdot-\xi)\big\|_{L^p}\mathrm{d}\xi$$
$$= \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}}\big\|u_0\big\|_{H_p^s}\mathrm{d}\xi = \big\|u_0\big\|_{H_p^s},\ t \geq 0.$$

(2) The proof of (2) is similar to the proof of (1). □

In the following lemmas, we estimate the pointwise product of two functions in $\dot{H}^s_p(\mathbb{R}^d)$. These lemmas are generalizations of the Hölder inequality. In the case when $s = 0, p \geq 2$, we get back to the usual Hölder inequality.

Lemma 2.2.2. *Assume that*

$$1 < p, q < d \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{d}.$$

Then the following inequality holds

$$\big\|uv\big\|_{\dot{H}^1_r} \lesssim \big\|u\big\|_{\dot{H}^1_p}\big\|v\big\|_{\dot{H}^1_q}, \ \forall u \in \dot{H}^1_p, v \in \dot{H}^1_q,$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{1}{d}$.

Proof. By applying the Leibniz formula for the derivatives of a product of two functions, we have

$$\big\|uv\big\|_{\dot{H}^1_r} \simeq \sum_{|\alpha|=1} \big\|\partial^\alpha(uv)\big\|_{L^r} \leq \sum_{|\alpha|=1} \big\|(\partial^\alpha u)v\big\|_{L^r} + \sum_{|\alpha|=1} \big\|u(\partial^\alpha v)\big\|_{L^r}.$$

From the Hölder and Sobolev inequalities it follows that

$$\sum_{|\alpha|=1} \big\|(\partial^\alpha u)v\big\|_{L^r} \leq \sum_{|\alpha|=1} \big\|\partial^\alpha u\big\|_{L^p}\big\|v\big\|_{L^{q_1}} \lesssim \big\|u\big\|_{\dot{H}^1_p}\big\|v\big\|_{\dot{H}^1_q},$$

where

$$\frac{1}{q_1} = \frac{1}{q} - \frac{1}{d}.$$

Similar to the above proof, we have

$$\sum_{|\alpha|=1} \big\|u(\partial^\alpha v)\big\|_{L^r} \lesssim \big\|u\big\|_{\dot{H}^1_p}\big\|v\big\|_{\dot{H}^1_q}.$$

This gives the desired result

$$\big\|uv\big\|_{\dot{H}^1_r} \lesssim \big\|u\big\|_{\dot{H}^1_p}\big\|v\big\|_{\dot{H}^1_q}.$$

□

Lemma 2.2.3. *Assume that*

$$0 \leq s \leq 1, \frac{1}{p} > \frac{s}{d}, \frac{1}{q} > \frac{s}{d}, \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \frac{s}{d}. \tag{2.1}$$

Then the following inequality holds

$$\big\|uv\big\|_{\dot{H}^s_r} \lesssim \big\|u\big\|_{\dot{H}^s_p}\big\|v\big\|_{\dot{H}^s_q}, \ \forall u \in \dot{H}^s_p, v \in \dot{H}^s_q,$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{s}{d}$.

Proof. It is not difficult to show that if $p, q,$ and s satisfy (2.1) then there exist numbers $p_1, p_2, q_1, q_2 \in (1, +\infty)$ (may be many of them) such that

$$\frac{1}{p} = \frac{1-s}{p_1} + \frac{s}{p_2}, \frac{1}{q} = \frac{1-s}{q_1} + \frac{s}{q_2}, \frac{1}{p_1} + \frac{1}{q_1} < 1,$$
$$p_2 < d, q_2 < d, \text{ and } \frac{1}{p_2} + \frac{1}{q_2} < 1 + \frac{1}{d}.$$

Setting

$$\frac{1}{r_1} = \frac{1}{p_1} + \frac{1}{q_1}, \frac{1}{r_2} = \frac{1}{p_2} + \frac{1}{q_2} - \frac{1}{d},$$

we have

$$\frac{1}{r} = \frac{1-s}{r_1} + \frac{s}{r_2}.$$

Therefore, applying Theorem 6.4.5 (p. 152) of [2] (see also [33] for $\dot{H}^s_p$), we get

$$\dot{H}^s_p = [L^{p_1}, \dot{H}^1_{p_2}]_s, \dot{H}^s_q = [L^{q_1}, \dot{H}^1_{q_2}]_s, \dot{H}^s_r = [L^{r_1}, \dot{H}^1_{r_2}]_s.$$

Applying the Hölder inequality and Lemma 2.2.2 in order to obtain

$$\|uv\|_{L^{r_1}} \lesssim \|u\|_{L^{p_1}} \|v\|_{L^{q_1}}, \; \forall u \in L^{p_1}, v \in L^{q_1},$$
$$\|uv\|_{\dot{H}^1_{r_2}} \lesssim \|u\|_{\dot{H}^1_{p_2}} \|v\|_{\dot{H}^1_{q_2}}, \; \forall u \in \dot{H}^1_{p_2}, v \in \dot{H}^1_{q_2}.$$

From Theorem 4.4.1 (p. 96) of [2] we get

$$\|uv\|_{\dot{H}^s_r} \lesssim \|u\|_{\dot{H}^s_p} \|v\|_{\dot{H}^s_q}.$$

□

Lemma 2.2.4. *Assume that*

$$0 \le s < d, \frac{s}{d} < \frac{1}{p}, \frac{s}{d} < \frac{1}{q}, \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \frac{s}{d}. \tag{2.2}$$

Then we have the inequality

$$\|uv\|_{\dot{H}^s_r} \lesssim \|u\|_{\dot{H}^s_p} \|v\|_{\dot{H}^s_q}, \; \forall u \in \dot{H}^s_p, v \in \dot{H}^s_q,$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{s}{d}$.

Proof. Denote by $[s]$ the integer part of s and by $\{s\}$ the fraction part of s. Using the formula for the derivatives of a product of two functions, we have

$$\|uv\|_{\dot{H}^s_r} = \|\dot{\Lambda}^s(uv)\|_{L^r} = \|\dot{\Lambda}^{\{s\}}(uv)\|_{\dot{H}^{[s]}_r} \simeq$$
$$\sum_{|\alpha|=[s]} \|\partial^\alpha \dot{\Lambda}^{\{s\}}(uv)\|_{L^r} = \sum_{|\alpha|=[s]} \|\dot{\Lambda}^{\{s\}} \partial^\alpha (uv)\|_{L^r}$$
$$= \sum_{|\alpha|=[s]} \|\partial^\alpha (uv)\|_{\dot{H}^{\{s\}}_r} \lesssim \sum_{|\gamma|+|\beta|=[s]} \|\partial^\gamma u \partial^\beta v\|_{\dot{H}^{\{s\}}_r}.$$

Set

$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{s-|\gamma|-\{s\}}{d}, \frac{1}{\tilde{q}} = \frac{1}{q} - \frac{s-|\beta|-\{s\}}{d}.$$

Applying Lemma 2.2.3 and the Sobolev inequality in order to obtain

$$\|\partial^\gamma u \partial^\beta v\|_{\dot{H}^{\{s\}}_r} \lesssim \|\partial^\gamma u\|_{\dot{H}^{\{s\}}_{\tilde{p}}} \|\partial^\beta v\|_{\dot{H}^{\{s\}}_{\tilde{q}}}$$
$$\lesssim \|u\|_{\dot{H}^{|\gamma|+\{s\}}_{\tilde{p}}} \|v\|_{\dot{H}^{|\beta|+\{s\}}_{\tilde{q}}} \lesssim \|u\|_{\dot{H}^s_p} \|v\|_{\dot{H}^s_q}.$$

This gives the desired result

$$\|uv\|_{\dot{H}^s_r} \lesssim \|u\|_{\dot{H}^s_p} \|v\|_{\dot{H}^s_q}.$$

□

Remark 2.2.1. Lemmas 2.2.2, 2.2.3, and 2.2.4 are still valid when the homogeneous space $\dot{H}^s_p$ is replaced by the inhomogeneous space H^s_p.

2.2.2. On the continuity and regularity of the bilinear operator B

In this subsection a particular attention will be devoted to the study of the bilinear operator $B(u,v)(t)$ defined by (1.14) in Sobolev spaces.

Lemma 2.2.5. *Let*

$$d \geq 3,\ s \geq 0,\ p > 1,\ r > 2, \textit{and } T > 0 \tag{2.3}$$

be such that

$$\frac{s}{d} < \frac{1}{p} < \frac{1}{2} + \frac{s}{2d} \textit{ and } \frac{2}{r} + \frac{d}{p} - s \leq 1. \tag{2.4}$$

Then the bilinear operator $B(u,v)(t)$ is continuous from

$$L^r([0,T];H^s_p) \times L^r([0,T];H^s_p)$$

into

$$L^r([0,T];H^s_p),$$

and the following inequality holds

$$\big\|B(u,v)\big\|_{L^r([0,T];H^s_p)} \leq CT^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})}\big\|u\big\|_{L^r([0,T];H^s_p)}\big\|v\big\|_{L^r([0,T];H^s_p)}, \tag{2.5}$$

where C is a positive constant independent of T.

Proof. We have

$$\big\|B(u,v)(t)\big\|_{H^s_p} \leq \int_0^t \Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{H^s_p} \mathrm{d}\tau =$$
$$\int_0^t \Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(Id-\Delta)^{s/2}\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{L^p} \mathrm{d}\tau. \tag{2.6}$$

We have

$$\Big(e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(Id-\Delta)^{s/2}\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big)_j =$$
$$e^{(t-\tau)\Delta}\sum_{l,k=1}^{d}\Big(\delta_{jk}-\frac{\partial_j\partial_k}{\Delta}\Big)\partial_l(Id-\Delta)^{s/2}\big(u_l(\tau,\cdot)v_k(\tau,\cdot)\big).$$

From the property of the Fourier transform we have

$$\Big(e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(Id-\Delta)^{s/2}\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big)^{\wedge}_j(\xi) =$$
$$e^{-(t-\tau)|\xi|^2}\sum_{l,k=1}^{d}\Big(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l)\Big((Id-\Delta)^{s/2}\big(u_l(\tau,\cdot)v_k(\tau,\cdot)\big)\Big)^{\wedge}(\xi),$$

and therefore

$$\Big(e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(Id-\Delta)^{s/2}\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big)_j =$$
$$\frac{1}{(t-\tau)^{\frac{d+1}{2}}}\sum_{l,k=1}^{d}K_{l,k,j}\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big) * \Big((Id-\Delta)^{s/2}\big(u_l(\tau,\cdot)v_k(\tau,\cdot)\big)\Big), \tag{2.7}$$

where

$$\widehat{K_{l,k,j}}(\xi) = \frac{1}{(2\pi)^{d/2}}e^{-|\xi|^2}\Big(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l).$$

Applying Lemma 1.2.1 with $|\alpha|=1$ we obtain

$$|K_{l,k,j}(x)| \lesssim \frac{1}{(1+|x|)^{d+1}}.$$

Thus, the tensor $K(x)=\{K_{l,k,j}(x)\}$ satisfies

$$|K(x)| \lesssim \frac{1}{(1+|x|)^{d+1}}. \tag{2.8}$$

So, we can rewrite the equality (2.7) in the tensor form

$$e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(Id-\Delta)^{s/2}\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big) =$$
$$\frac{1}{(t-\tau)^{\frac{d+1}{2}}}K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)*\Big((Id-\Delta)^{s/2}\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big).$$

Set

$$\frac{1}{\tilde{p}}=\frac{2}{p}-\frac{s}{d},\ \frac{1}{h}=\frac{s}{d}-\frac{1}{p}+1. \tag{2.9}$$

Note that from the inequalities (2.3) and (2.4), we can check that the following relations are satisfied

$$1<h,\tilde{p}<\infty \text{ and } \frac{1}{p}+1=\frac{1}{h}+\frac{1}{\tilde{p}}.$$

Applying the Young inequality for the the convolution we obtain

$$\Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(Id-\Delta)^{s/2}\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{L^p} \lesssim$$
$$\frac{1}{(t-\tau)^{\frac{d+1}{2}}}\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{L^h}\Big\|(Id-\Delta)^{s/2}\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{L^{\tilde{p}}}. \tag{2.10}$$

Applying Lemma 2.2.4

$$\Big\|(Id-\Delta)^{s/2}\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{L^{\tilde{p}}} = \big\|u(\tau,\cdot)\otimes v(\tau,\cdot)\big\|_{H^s_{\tilde{p}}}$$
$$\lesssim \big\|u(\tau,\cdot)\big\|_{H^s_p}\big\|v(\tau,\cdot)\big\|_{H^s_p}. \tag{2.11}$$

From the estimate (2.8) and the equality (2.9), we have

$$\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{L^h} = (t-\tau)^{\frac{d}{2h}}\big\|K\big\|_{L^h} \simeq (t-\tau)^{\frac{s}{2}-\frac{d}{2p}+\frac{d}{2}}. \tag{2.12}$$

The inequalities (2.10), (2.11), and (2.12) imply that

$$\Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(Id-\Delta)^{s/2}\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{L^p} \lesssim$$
$$(t-\tau)^{\frac{s}{2}-\frac{d}{2p}-\frac{1}{2}}\big\|u(\tau,\cdot)\big\|_{H^s_p}\big\|v(\tau,\cdot)\big\|_{H^s_p}. \tag{2.13}$$

From the inequalities (2.6) and (2.13), we get

$$\big\|B(u,v)(t)\big\|_{H^s_p} \lesssim \int_0^t (t-\tau)^{\frac{s}{2}-\frac{d}{2p}-\frac{1}{2}}\big\|u(\tau,\cdot)\big\|_{H^s_p}\big\|v(\tau,\cdot)\big\|_{H^s_p}\mathrm{d}\tau.$$

Applying of Theorem 1.1.10 (c) for the convolution in the Lorentz spaces, we have the following estimates

$$\Big\|\big\|B(u,v)(t)\big\|_{H^s_p}\Big\|_{L^r_t(0,T)} = \Big\|\big\|B(u,v)(t)\big\|_{H^s_p}\Big\|_{L^{r,r}_t(0,T)} \leq \Big\|\big\|B(u,v)(t)\big\|_{H^s_p}\Big\|_{L^{r,\frac{r}{2}}_t(0,T)}$$
$$\lesssim \big\|1_{[0,T]}t^{\frac{s}{2}-\frac{d}{2p}-\frac{1}{2}}\big\|_{L^{r',\infty}}\Big\|\big\|u(t,\cdot)\big\|_{H^s_p}\big\|v(t,\cdot)\big\|_{H^s_p}\Big\|_{L^{\frac{r}{2},\frac{r}{2}}_t(0,T)}, \tag{2.14}$$

where $\frac{1}{r'}+\frac{1}{r}=1$ and $1_{[0,T]}$ is the indicator function of set $[0,T]$ on $\mathbb{R}$.
By applying the Hölder inequality we get

$$\begin{aligned}\Big\|\big\|u(t,\cdot)\big\|_{H^s_p}\big\|v(t,\cdot)\big\|_{H^s_p}\Big\|_{L_t^{\frac{r}{2},\frac{r}{2}}(0,T)} &= \Big\|\big\|u(t,\cdot)\big\|_{H^s_p}\big\|v(t,\cdot)\big\|_{H^s_p}\Big\|_{L_t^{\frac{r}{2}}(0,T)}\\ &\leq \Big\|\big\|u(t,\cdot)\big\|_{H^s_p}\Big\|_{L_t^r(0,T)}\Big\|\big\|v(t,\cdot)\big\|_{H^s_p}\Big\|_{L_t^r(0,T)}. \qquad (2.15)\end{aligned}$$

Note that

$$\Big\|1_{[0,T]}t^{\frac{s}{2}-\frac{d}{2p}-\frac{1}{2}}\Big\|_{L^{r',\infty}} \simeq T^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})}. \qquad (2.16)$$

Therefore the inequality (2.5) can be deduced from the inequalities (2.14), (2.15), and (2.16). $\square$

Remark 2.2.2. Lemma 2.2.5 is still valid when the inhomogeneous space H^s_p is replaced by the homogeneous space $\dot{H}^s_p$.

Lemma 2.2.6. *Let*

$$d\geq 3,\ 0\leq s<d,\ p>1,\ r>2, \textit{and } T>0$$

be such that

$$\frac{1}{p}<\frac{1}{2}+\frac{s}{2d}, \frac{2}{p}\geq\frac{s+1}{d},\ \textit{and}\ \frac{2}{r}+\frac{d}{p}-s=1.$$

Then the bilinear operator $B(u,v)(t)$ is continuous from

$$L^r\big([0,T];\dot{H}^s_p\big)\times L^r\big([0,T];\dot{H}^s_p\big)$$

into

$$L^\infty\Big([0,T];\dot{B}_{\tilde{p}}^{\frac{d}{\tilde{p}}-1,\frac{r}{2}}\Big),$$

where

$$\frac{1}{\tilde{p}}=\frac{2}{p}-\frac{s}{d},$$

and we have the inequality

$$\big\|B(u,v)\big\|_{L^\infty\big([0,T];\dot{B}_{\tilde{p}}^{\frac{d}{\tilde{p}}-1,\frac{r}{2}}\big)} \leq C\big\|u\big\|_{L^r([0,T];\dot{H}^s_p)}\big\|v\big\|_{L^r([0,T];\dot{H}^s_p)}, \qquad (2.17)$$

where C is a positive constant independent of T.

Proof. To prove this lemma by duality (in the x-variable), (see Proposition 3.9 in ([46], p. 29)), let us consider an arbitrary test function $h(x)\in\mathcal{S}(\mathbb{R}^d)$ and evaluate the quantity

$$I_t=\big\langle B(u,v)(t),h\big\rangle=\int_{\mathbb{R}^d}\big(B(u,v)(t)\big)(x)h(x)\mathrm{d}x. \qquad (2.18)$$

We have

$$\big\langle B(u,v)(t),h\big\rangle=\int_0^t\big\langle e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big),h\big\rangle\mathrm{d}\tau=$$

$$\int_0^t\Big\langle e^{(t-\tau)\Delta}\dot{\Lambda}\mathbb{P}\frac{\nabla}{\dot{\Lambda}}\cdot\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big),h\Big\rangle\mathrm{d}\tau=\int_0^t\Big\langle\mathbb{P}\frac{\nabla}{\dot{\Lambda}}\cdot\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big),e^{(t-\tau)\Delta}\dot{\Lambda}h\Big\rangle\mathrm{d}\tau$$

$$=\int_0^t\Big\langle\mathbb{P}\frac{\nabla}{\dot{\Lambda}}\cdot\dot{\Lambda}^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big),e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\Big\rangle\mathrm{d}\tau. \qquad (2.19)$$

By applying the Hölder inequality in the x-variable, from the equality (2.19) and the fact that (see [46])

$$\mathbb{P} \text{ and } \frac{\nabla}{\dot{\Lambda}} \text{ are continuous from } L^p \text{ into } L^p, 1 < p < \infty,$$

we get

$$\begin{aligned} |I_t| &\leq \int_0^t \Big\| \mathbb{P}\frac{\nabla}{\dot{\Lambda}} \cdot \dot{\Lambda}^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{L^{\tilde{p}}} \big\| e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}} \mathrm{d}\tau \\ &\lesssim \int_0^t \big\|\dot{\Lambda}^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\big\|_{L^{\tilde{p}}} \big\| e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}} \mathrm{d}\tau, \end{aligned} \tag{2.20}$$

where

$$\frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'} = 1.$$

Applying Lemma 2.2.4, we have

$$\big\|\dot{\Lambda}^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\big\|_{L^{\tilde{p}}} = \big\|u(\tau,\cdot)\otimes v(\tau,\cdot)\big\|_{\dot{H}^s_{\tilde{p}}} \lesssim \big\|u(\tau,\cdot)\big\|_{\dot{H}^s_p}\big\|v(\tau,\cdot)\big\|_{\dot{H}^s_p}. \tag{2.21}$$

From the inequalities (2.20) and (2.21), applying the Hölder inequality in the t-variable, we deduce that

$$\begin{aligned} |I_t| &\lesssim \int_0^t \big\|u(\tau,\cdot)\big\|_{\dot{H}^s_p}\big\|v(\tau,\cdot)\big\|_{\dot{H}^s_p}\big\| e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}} \mathrm{d}\tau \leq \\ &\Big(\int_0^t \big(\big\|u(\tau,\cdot)\big\|_{\dot{H}^s_p}\big\|v(\tau,\cdot)\big\|_{\dot{H}^s_p}\big)^{\frac{r}{2}}\mathrm{d}\tau\Big)^{\frac{2}{r}}\Big(\int_0^t \big(\big\| e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\big)^{\frac{r}{r-2}}\mathrm{d}\tau\Big)^{\frac{r-2}{r}} \\ &\leq \big\|u\big\|_{L^r([0,T];\dot{H}^s_p)}\big\|v\big\|_{L^r([0,T];\dot{H}^s_p)}\Big(\int_0^t \big(\big\| e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\big)^{\frac{r}{r-2}}\mathrm{d}\tau\Big)^{\frac{r-2}{r}}. \end{aligned} \tag{2.22}$$

From Lemma 1.1.8 and note that $\dot{\Lambda}^{s_0}$ is an isomorphism from $\dot{B}^{s,p}_q$ to $\dot{B}^{s-s_0,p}_q$ (see [8]), we have the following estimates

$$\begin{aligned} \Big(\int_0^t \big(\big\| e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\big)^{\frac{r}{r-2}}\mathrm{d}\tau\Big)^{\frac{r-2}{r}} &\leq \Big(\int_0^\infty \big(\big\| e^{t\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\big)^{\frac{r}{r-2}}\mathrm{d}t\Big)^{\frac{r-2}{r}} \\ = \Big(\int_0^\infty \big(t^{\frac{r-4}{2r}}\big\| e^{t\Delta}t^{\frac{1}{2}}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\big)^{\frac{r}{r-2}}\frac{\mathrm{d}t}{t}\Big)^{\frac{r-2}{r}} &\simeq \big\|\dot{\Lambda}^{-s}h\big\|_{\dot{B}^{\frac{4-r}{r},\frac{r}{r-2}}_{\tilde{p}'}} \\ \simeq \big\|h\big\|_{\dot{B}^{\frac{4-r}{r}-s,\frac{r}{r-2}}_{\tilde{p}'}} &= \big\|h\big\|_{\dot{B}^{1-\frac{d}{\tilde{p}},\frac{r}{r-2}}_{\tilde{p}'}}. \end{aligned} \tag{2.23}$$

From the equality (2.18) and the inequalities (2.22) and (2.23), we get

$$\big|\big\langle B(u,v)(t),h\big\rangle\big| \lesssim \big\|u\big\|_{L^r([0,T];\dot{H}^s_p)}\big\|v\big\|_{L^r([0,T];\dot{H}^s_p)}\Big\|h\Big\|_{\dot{B}^{1-\frac{d}{\tilde{p}},\frac{r}{r-2}}_{\tilde{p}'}}.$$

However, $\dot{B}^{1-\frac{d}{\tilde{p}},\frac{r}{r-2}}_{\tilde{p}'}$ is exactly the dual of $\dot{B}^{\frac{d}{\tilde{p}}-1,\frac{r}{2}}_{\tilde{p}}$, (the restriction $\frac{2}{p} \geq \frac{s+1}{d}$ is mainly because we are interested in non-negative indexes), therefore we conclude that

$$\Big\|B(u,v)(t)\Big\|_{\dot{B}^{\frac{d}{\tilde{p}}-1,\frac{r}{2}}_{\tilde{p}}} \lesssim \big\|u\big\|_{L^r([0,T];\dot{H}^s_p)}\big\|v\big\|_{L^r([0,T];\dot{H}^s_p)},\ 0 \leq t \leq T. \tag{2.24}$$

Finally, the estimate (2.17) can be deduced from the inequality (2.24). □

Combining Theorem 1.5.1 with Lemma 2.2.5, we get the following existence results, the particular case of which, when $s=0$, was obtained in [46].

Theorem 2.2.7. *Let*

$$d \geq 3, s \geq 0,\ p > 1,\ \textit{and}\ r > 2,$$

be such that

$$\frac{s}{d} < \frac{1}{p} < \frac{1}{2} + \frac{s}{2d}\ \textit{and}\ \frac{2}{r} + \frac{d}{p} - s \leq 1.$$

(a) *There exists a positive constant $\delta_{s,p,r,d}$ such that for all $T>0$ and for all $u_0 \in \mathcal{S}'(\mathbb{R}^d)$ with* $\operatorname{div}(u_0) = 0$ *satisfying*

$$T^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})} \big\| e^{\cdot\Delta} u_0 \big\|_{L^r([0,T];\dot{H}^s_p)} \leq \delta_{s,p,r,d}, \tag{2.25}$$

there is a unique mild solution $u \in L^r\big([0,T];\dot{H}^s_p)\big)$ for NSE.
If

$$e^{\cdot\Delta} u_0 \in L^r\big([0,1];\dot{H}^s_p\big),$$

then the inequality (2.25) *holds when $T(u_0)$ is small enough.*
(b) *If $\frac{2}{r}+\frac{d}{p}-s=1$ then there exists a positive constant $\delta_{s,p,d}$ such that we can take $T=\infty$ whenever $\big\|e^{\cdot\Delta}u_0\big\|_{L^r([0,\infty];\dot{H}^s_p)} \leq \delta_{s,p,d}$.*

Proof. (a) From Lemma 2.2.5, we have the estimate

$$\big\| B \big\|_{L^r([0,T];\dot{H}^s_p)} \leq C_{s,p,r,d} T^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})},$$

where $C_{s,p,r,d}$ is a positive constant independent of T. From Theorem 1.5.1 and the above inequality, we deduce the existence of a solution to the Navier-Stokes equations on the interval $(0,T)$ with

$$4C_{s,p,r,d} T^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})} \big\| e^{\cdot\Delta} u_0 \big\|_{L^r([0,T];\dot{H}^s_p)} \leq 1.$$

If $e^{\cdot\Delta}u_0 \in L^r([0,1];\dot{H}^s_p)$ then this condition is fulfilled for $T=T(u_0)$ small enough, this is obvious for the case when $\frac{2}{r}+\frac{d}{p}-s<1$ since $\lim\limits_{T\to 0} T^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})} = 0$. For the case when $\frac{2}{r}+\frac{d}{p}-s=1$, the condition is fulfilled since we have $\lim\limits_{T\to 0} \big\|e^{\cdot\Delta}u_0\big\|_{L^r([0,T];\dot{H}^s_p)} = 0$.
(b) This is obvious. □

Remark 2.2.3. From Theorem 5.3 ([46], p. 44), if $u_0 \in B_p^{s-\frac{2}{r},r} \cap \mathcal{S}'(\mathbb{R}^d)$ then $e^{\cdot\Delta}u_0 \in L^r([0,1];\dot{H}^s_p)$. From Lemma 1.1.8, if $u_0 \in \mathcal{S}'(\mathbb{R}^d)$ the two quantities $\big\|u_0\big\|_{\dot{B}_p^{s-\frac{2}{r},r}}$ and $\big\|e^{\cdot\Delta}u_0\big\|_{L^r([0,\infty];\dot{H}^s_p)}$ are equivalent.

2.2.3. Solutions to the Navier-Stokes equations with initial value in the critical spaces $H_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ and $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ for $3 \leq d \leq 4,\ 2 \leq q \leq d$

Lemma 2.2.8. *Let $d \geq 3$ and $2 \leq q \leq d$. Then the bilinear operator $B(u,v)(t)$ is continuous from*

$$L^4\Big([0,T];\dot{H}^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\Big) \times L^4\Big([0,T];\dot{H}^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\Big)$$

into

$$L^\infty\Big([0,T];\dot{B}_q^{\frac{d}{q}-1,2}\Big),$$

and we have the inequality

$$\begin{aligned}\big\|B(u,v)\big\|_{L^\infty([0,T];\dot H_q^{\frac{d}{q}-1})} &\lesssim \big\|B(u,v)\big\|_{L^\infty\left([0,T];\dot B_q^{\frac{d}{q}-1,2}\right)}\\ &\leq C\big\|u\big\|_{L^4\left([0,T];\dot H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\right)}\big\|v\big\|_{L^4\left([0,T];\dot H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\right)},\end{aligned} \tag{2.26}$$

where C is a positive constant and independent of T.

Proof. Applying Lemma 2.2.6 with $r=4, p=\frac{2dq}{2d-q}$, and $s=\frac{d}{q}-1$, we get

$$\frac{1}{\tilde p}=\frac{2}{p}-\frac{s}{d}=\frac{2d-q}{dq}-\frac{\frac{d}{q}-1}{d}=\frac{1}{q},$$

$$\big\|B(u,v)\big\|_{L^\infty\left([0,T];\dot B_q^{\frac{d}{q}-1,2}\right)} \lesssim \big\|u\big\|_{L^4\left([0,T];\dot H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\right)}\big\|v\big\|_{L^4\left([0,T];\dot H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\right)}. \tag{2.27}$$

From (b) of Lemma 1.1.5, we have

$$\dot B_q^{\frac{d}{q}-1,2}\hookrightarrow \dot H_q^{\frac{d}{q}-1}. \tag{2.28}$$

Finally, the estimate (2.26) can be deduced from the inequality (2.27) and the imbedding (2.28) □

Lemma 2.2.9. *Let $d\geq 3$ and $2\leq q\leq d$. Then the bilinear operator $B(u,v)(t)$ is continuous from*

$$L^4\Big([0,T];H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\Big)\times L^4\Big([0,T];H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\Big)$$

into

$$L^\infty\Big([0,T];H_q^{\frac{d}{q}-1}\Big),$$

and we have the inequality

$$\big\|B(u,v)\big\|_{L^\infty\left([0,T];H_q^{\frac{d}{q}-1}\right)} \leq C\big\|u\big\|_{L^4\left([0,T];H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\right)}\big\|v\big\|_{L^4\left([0,T];H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\right)}, \tag{2.29}$$

where C is a positive constant and independent of T.

Proof. To prove this lemma by duality (in the x-variable), let us consider an arbitrary test function $h(x)\in\mathcal{S}(\mathbb{R}^d)$. Similar to the proof of Lemma 2.2.6, we have

$$\Big|\big\langle(\sqrt{Id-\Delta})^{\frac{d}{q}-1}B(u,v)(t),h\big\rangle\Big| \lesssim \big\|u\big\|_{L^4\left([0,T];H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\right)}\big\|v\big\|_{L^4\left([0,T];H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\right)}\big\|h\big\|_{\dot B^{0,2}_{q'}},$$

where

$$\frac{1}{q}+\frac{1}{q'}=1.$$

However the dual space of $\dot B^{0,2}_{q'}$ is $\dot B^{0,2}_{q}$, therefore we get

$$\Big\|(\sqrt{Id-\Delta})^{\frac{d}{q}-1}B(u,v)(t)\Big\|_{\dot B_q^{0,2}} \lesssim \big\|u\big\|_{L^4\left([0,T];H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\right)}\big\|v\big\|_{L^4\left([0,T];H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\right)}. \tag{2.30}$$

From (b) of Lemma 1.1.5 and the estimate (2.30), we have

$$\Big\|B(u,v)(t)\Big\|_{H_q^{\frac{d}{q}-1}} = \Big\|(\sqrt{Id-\Delta})^{\frac{d}{q}-1}B(u,v)(t)\Big\|_{L^q} =$$
$$\Big\|(\sqrt{Id-\Delta})^{\frac{d}{q}-1}B(u,v)(t)\Big\|_{\dot{H}_q^0} \lesssim \Big\|(\sqrt{Id-\Delta})^{\frac{d}{q}-1}B(u,v)(t)\Big\|_{\dot{B}_q^{0,2}}$$
$$\lesssim \big\|u\big\|_{L^4\big([0,T];H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\big)}\big\|v\big\|_{L^4\big([0,T];H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\big)},\ 0\le t\le T. \tag{2.31}$$

Finally, the estimate (2.29) can be deduced from the inequality (2.31). □

Lemma 2.2.10. *Let $d\ge 3$ and $2\le q\le 4$.*

(a) *If* $u_0\in H_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ *then*

$$\big\|e^{\cdot\Delta}u_0\big\|_{L^4\big([0,\infty);H^{d/q-1}_{2dq/(2d-q)}\big)} \lesssim \big\|u_0\big\|_{H_q^{d/q-1}}.$$

(b) *If* $u_0\in \dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ *then*

$$\Big\|e^{\cdot\Delta}u_0\Big\|_{L^4\big([0,\infty);\dot{H}^{d/q-1}_{2dq/(2d-q)}\big)} \simeq \big\|u_0\big\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \lesssim \big\|u_0\big\|_{\dot{H}_q^{d/q-1}}.$$

Proof. (a) From Lemma 1.1.8, we have the estimates

$$\Big\|e^{\cdot\Delta}u_0\Big\|_{L^4\big([0,\infty);H^{d/q-1}_{2dq/(2d-q)}\big)} = \Big(\int_0^\infty \Big\|e^{t\Delta}(\sqrt{Id-\Delta})^{d/q-1}u_0\Big\|^4_{L^{2dq/(2d-q)}}\mathrm{d}t\Big)^{1/4}$$
$$= \Big(\int_0^\infty \Big(t^{\frac14}\Big\|e^{t\Delta}(\sqrt{Id-\Delta})^{d/q-1}u_0\Big\|_{L^{2dq/(2d-q)}}\Big)^4\frac{\mathrm{d}t}{t}\Big)^{1/4}$$
$$\simeq \Big\|(\sqrt{Id-\Delta})^{d/q-1}u_0\Big\|_{\dot{B}^{-1/2,4}_{2dq/(2d-q)}}. \tag{2.32}$$

Applying (b), (c), and (d) of Lemma 1.1.5 in order to obtain

$$L^q = \dot{H}_q^0 \hookrightarrow \dot{B}_q^{0,q} \hookrightarrow \dot{B}_q^{0,4} \hookrightarrow \dot{B}^{-1/2,4}_{2dq/(2d-q)}. \tag{2.33}$$

From the inequality (2.32) and the imbedding (2.33), we get

$$\Big\|e^{\cdot\Delta}u_0\Big\|_{L^4\big([0,\infty);H^{d/q-1}_{2dq/(2d-q)}\big)} \simeq \Big\|(\sqrt{Id-\Delta})^{d/q-1}u_0\Big\|_{\dot{B}^{-1/2,4}_{2dq/(2d-q)}}$$
$$\lesssim \big\|(\sqrt{Id-\Delta})^{d/q-1}u_0\big\|_{L^q} = \big\|u_0\big\|_{H_q^{d/q-1}}.$$

(b) Similar to the proof of (a) we have

$$\big\|e^{\cdot\Delta}u_0\big\|_{L^4\big([0,\infty);\dot{H}^{d/q-1}_{2dq/(2d-q)}\big)} \simeq \big\|\dot{\Lambda}^{\frac{d}{q}-1}u_0\big\|_{\dot{B}^{-1/2,4}_{2dq/(2d-q)}} \lesssim \big\|\dot{\Lambda}^{\frac{d}{q}-1}u_0\big\|_{L^q} = \big\|u_0\big\|_{\dot{H}_q^{d/q-1}},$$
$$\text{and}\quad \Big\|\dot{\Lambda}^{\frac{d}{q}-1}u_0\Big\|_{\dot{B}^{-1/2,4}_{2dq/(2d-q)}} \simeq \big\|u_0\big\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}}.$$

□

Combining Theorem 1.5.1 with Lemmas 2.2.1, 2.2.5, 2.2.8, and 2.2.10 we obtain the following existence result.

Theorem 2.2.11. *Let $3 \le d \le 4$ and $2 \le q \le d$. There exists a positive constant $\delta_{q,d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ with $\operatorname{div}(u_0) = 0$ satisfying*

$$\left\| e^{\cdot\Delta} u_0 \right\|_{L^4\left([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \le \delta_{q,d}, \tag{2.34}$$

NSE has a unique mild solution $u \in L^4\big([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\big) \cap L^\infty\big([0,T];\dot{H}_q^{d/q-1}\big)$. Denoting $w = u - e^{\cdot\Delta}u_0$, then we have

$$w \in L^4\big([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\big) \cap L^\infty\big([0,T];\dot{B}_q^{d/q-1,2}\big).$$

Finally, we have

$$\left\| e^{\cdot\Delta} u_0 \right\|_{L^4\left([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \lesssim \left\| u_0 \right\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \lesssim \left\| u_0 \right\|_{\dot{H}_q^{d/q-1}},$$

in particular, for arbitrary $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ the inequality (2.34) holds when $T(u_0)$ is small enough; and there exists a positive constant $\sigma_{q,d}$ such that for all $\left\| u_0 \right\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \le \sigma_{q,d}$ we can take $T = \infty$.

Proof. By applying Lemma 2.2.5 with $r = 4$, $p = \frac{2dq}{2d-q}$, $s = \frac{d}{q} - 1$, and notice that $1 + s - \frac{2}{r} - \frac{d}{p} = 0$ we have

$$\left\| B \right\|_{L^4\left([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \le C_{q,d},$$

where $C_{q,d}$ is a positive constant independent of T. From Theorem 1.5.1 and the above inequality, we deduce that for any $u_0 \in \dot{H}_q^{\frac{d}{q}-1}$ such that

$$\operatorname{div}(u_0) = 0, \quad \left\| e^{\cdot\Delta} u_0 \right\|_{L^4\left([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \le \frac{1}{4C_{q,d}},$$

NSE has a mild solution u on the interval $(0,T)$ so that

$$u \in L^4\big([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\big). \tag{2.35}$$

From the Lemma 2.2.8 and (2.35), we have $B(u,u) \in L^\infty\big([0,T];\dot{H}_q^{d/q-1}\big)$. From (2) of Lemma 2.2.1, we have $e^{\cdot\Delta}u_0 \in L^\infty\big([0,T];\dot{H}_q^{d/q-1}\big)$. Therefore

$$u = e^{\cdot\Delta}u_0 - B(u,u) \in L^\infty([0,T];\dot{H}_q^{d/q-1}).$$

From (b) of Lemma 2.2.10, we have

$$\left\| e^{\cdot\Delta} u_0 \right\|_{L^4\left([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \lesssim \left\| e^{\cdot\Delta} u_0 \right\|_{L^4\left([0,\infty);\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \simeq \left\| u_0 \right\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \lesssim \left\| u_0 \right\|_{\dot{H}_q^{d/q-1}} < \infty.$$

Hence, the left-hand side of the inequality (2.34) converges to 0 when T tends to 0. Therefore, for arbitrary $u_0 \in \dot{H}_q^{\frac{d}{q}-1}$ there is $T(u_0)$ small enough such that the inequality (2.34) holds. Also, there exists a positive constants $\sigma_{q,d}$ such that for all $\left\| u_0 \right\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \le \sigma_{q,d}$ and $T = \infty$ the inequality (2.34) holds. □

Remark 2.2.4. Theorem 2.2.11 in the particular case $q = d$ is Proposition 20.1 in [46].

Theorem 2.2.12. *Let $3 \le d \le 4$ and $2 \le q \le d$. There exists a positive constant $\delta_{q,d}$ such that for all $T > 0$ and for all $u_0 \in H_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ satisfying*

$$\left\|e^{\cdot\Delta}u_0\right\|_{L^4\left([0,T];H^{d/q-1}_{2dq/(2d-q)}\right)} \le \delta_{q,d}, \tag{2.36}$$

NSE has a unique mild solution $u \in L^4\big([0,T];H^{d/q-1}_{2dq/(2d-q)}\big) \cap L^\infty\big([0,T];H^{d/q-1}_q\big)$. Finally, we have

$$\left\|e^{\cdot\Delta}u_0\right\|_{L^4\left([0,T];H^{d/q-1}_{2dq/(2d-q)}\right)} \le \left\|e^{\cdot\Delta}u_0\right\|_{L^4\left([0,\infty);H^{d/q-1}_{2dq/(2d-q)}\right)} \lesssim \left\|u_0\right\|_{H^{d/q-1}_q},$$

in particular, for arbitrary $u_0 \in H_q^{\frac{d}{q}-1}$ the inequality (2.36) holds when $T(u_0)$ is small enough;

Proof. The proof of Theorem 2.2.12 is similar to the one of Theorem 2.2.11, by combining Theorem 1.5.1 with Lemmas 2.2.1, 2.2.5, 2.2.9, and 2.2.10. □

In the rest of this section, we investigate mild solutions to the Navier-Stokes equations with initial value in the critical spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ for $d \ge 3$ and $1 < q \le d$. We consider two cases $2 < q \le d$ and $1 < q \le 2$ separately.

2.2.4. Solutions to the Navier-Stokes equations with initial value in the critical spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ for $d \ge 3$ and $2 < q \le d$

Lemma 2.2.13. *Let $d \ge 3$ and $2 < q \le d$. Then for all p such that*

$$2 < p < \min\Big\{\frac{(d-2)q}{d-q},\ d+2\Big\}, \Big(\textit{if } q = d \textit{ then } \frac{(d-2)q}{d-q} = +\infty\Big),$$

the bilinear operator $B(u,v)(t)$ is continuous from

$$L^p([0,T];\dot{H}_p^{\frac{2+d-p}{p}}) \times L^p([0,T];\dot{H}_p^{\frac{2+d-p}{p}})$$

into

$$L^\infty\big([0,T];\dot{B}^{\frac{d+p-2}{p}-1,\frac{p}{2}}_{\frac{dp}{d+p-2}}\big),$$

and we have the inequality

$$\begin{aligned}\left\|B(u,v)\right\|_{L^\infty\left([0,T];\dot{H}_q^{\frac{d}{q}-1}\right)} &\lesssim \left\|B(u,v)\right\|_{L^\infty\left([0,T];\dot{B}^{\frac{d+p-2}{p}-1,\frac{p}{2}}_{\frac{dp}{d+p-2}}\right)} \\ &\le C\left\|u\right\|_{L^p\left([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\right)}\left\|v\right\|_{L^p\left([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\right)},\end{aligned} \tag{2.37}$$

where C is a positive constant independent of T.

Proof. Applying Lemma 2.2.6 with $r = p$ and $s = \frac{2+d-p}{p}$, we get

$$\frac{1}{\tilde{p}} = \frac{2}{p} - \frac{s}{d} = \frac{d+p-2}{dp},$$

$$\left\|B(u,v)\right\|_{L^\infty\left([0,T];\dot{B}^{\frac{d+p-2}{p}-1,\frac{p}{2}}_{\frac{dp}{d+p-2}}\right)} \lesssim \left\|u\right\|_{L^p\left([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\right)}\left\|v\right\|_{L^p\left([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\right)}. \tag{2.38}$$

Applying (e), (d), and (h) of Lemma 1.1.5 in order to obtain

$$\dot{B}^{\frac{d+p-2}{p}-1,\frac{p}{2}}_{\frac{dp}{d+p-2}} \hookrightarrow \dot{F}^{\frac{d+p-2}{p}-1,\frac{p}{2}}_{\frac{dp}{d+p-2}} \hookrightarrow \dot{F}^{\frac{d}{q}-1,2}_{q} = \dot{H}^{\frac{d}{q}-1}_{q}. \tag{2.39}$$

Therefore the estimate (2.37) is deduced from the inequality (2.38) and the imbedding (2.39). □

Lemma 2.2.14. *Let $2 < q < p < +\infty$. Then for all $u_0 \in \dot{H}^{\frac{d}{q}-1}_q$ we have the estimates*

$$\big\|e^{\cdot\Delta}u_0\big\|_{L^p\left([0,\infty);\dot{H}_p^{\frac{2+d-p}{p}}\right)} \simeq \big\|u_0\big\|_{\dot{B}^{\frac{d}{p}-1,p}_p} \lesssim \big\|u_0\big\|_{\dot{H}^{\frac{d}{q}-1}_q}.$$

Proof. From Lemma 1.1.6, we have the estimates

$$\big\|e^{\cdot\Delta}u_0\big\|_{L^p\left([0,\infty);\dot{H}_p^{\frac{2+d-p}{p}}\right)} = \Big(\int_0^\infty \Big\|e^{t\Delta}\dot{\Lambda}^{\frac{2+d-p}{p}}u_0\Big\|^p_{L^p}\mathrm{d}t\Big)^{1/p} =$$

$$\Big(\int_0^\infty \Big\|t^{-\frac{-2/p}{2}}e^{t\Delta}\dot{\Lambda}^{\frac{2+d-p}{p}}u_0\Big\|^p_{L^p}\frac{\mathrm{d}t}{t}\Big)^{1/p} \simeq \big\|\dot{\Lambda}^{\frac{2+d-p}{p}}u_0\big\|_{\dot{B}^{-2/p,p}_p} \simeq \big\|u_0\big\|_{\dot{B}^{\frac{d}{p}-1,p}_p}. \tag{2.40}$$

Applying (b), (d), and (c) of Lemma 1.1.5 in order to obtain

$$\dot{H}^{\frac{d}{q}-1}_q \hookrightarrow \dot{B}^{\frac{d}{q}-1,q}_q \hookrightarrow \dot{B}^{\frac{d}{p}-1,q}_p \hookrightarrow \dot{B}^{\frac{d}{p}-1,p}_p. \tag{2.41}$$

From the estimate (2.40) and the imbedding (2.41), we have

$$\Big\|e^{\cdot\Delta}u_0\Big\|_{L^p\left([0,\infty);\dot{H}_p^{\frac{2+d-p}{p}}\right)} \simeq \big\|u_0\big\|_{\dot{B}^{\frac{d}{p}-1,p}_p} \lesssim \big\|u_0\big\|_{\dot{H}^{\frac{d}{q}-1}_q}.$$

□

Theorem 2.2.15. *Let $d \geq 3$ and $2 < q \leq d$. Then for any p be such that*

$$q < p < \min\Big\{\frac{(d-2)q}{d-q},\ d+2\Big\},$$

there exists a constant $\delta_{q,p,d} > 0$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^{d/q-1}_q(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ *satisfying*

$$\big\|e^{\cdot\Delta}u_0\big\|_{L^p\left([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\right)} \leq \delta_{q,p,d}, \tag{2.42}$$

NSE has a unique mild solution $u \in L^p\big([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\big) \cap L^\infty\big([0,T];\dot{H}^{d/q-1}_q\big)$. Denoting $w = u - e^{\cdot\Delta}u_0$, then we have

$$w \in L^p\big([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\big) \cap L^\infty\Big([0,T];\dot{B}^{\frac{d+p-2}{p}-1,\frac{p}{2}}_{\frac{dp}{d+p-2}}\Big).$$

Finally, we have

$$\Big\|e^{\cdot\Delta}u_0\Big\|_{L^p\left([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\right)} \leq \Big\|e^{\cdot\Delta}u_0\Big\|_{L^p\left([0,\infty);\dot{H}_p^{\frac{2+d-p}{p}}\right)} \simeq \big\|u_0\big\|_{\dot{B}^{\frac{d}{p}-1,p}_p} \lesssim \big\|u_0\big\|_{\dot{H}^{\frac{d}{q}-1}_q},$$

in particular, for arbitrary $u_0 \in \dot{H}^{d/q-1}_q$ the inequality (2.42) holds when $T(u_0)$ is small enough; and there exists a positive constant $\sigma_{q,p,d}$ such that for all $\big\|u_0\big\|_{\dot{B}^{\frac{d}{p}-1,p}_p} \leq \sigma_{q,p,d}$ we can take $T = \infty$.

Proof. The proof of Theorem 2.2.15 is similar to the one of Theorem 2.2.11, by combining Theorem 1.5.1 with Lemmas 2.2.1, 2.2.5 (for $r = p,\ s = \frac{2+d-p}{p}$), 2.2.13, and 2.2.14. □

Remark 2.2.5. The case $q = d$ was treated by several authors, see for example [11, 17, 37]. However their results are different from ours.

2.2.5. Solutions to the Navier-Stokes equations with initial value in the critical spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ for $d \geq 3$ and $1 < q \leq 2$

Lemma 2.2.16. *Let $d \geq 3$ and $1 < q \leq 2$. Then the bilinear operator $B(u,v)(t)$ is continuous from*

$$L^{2q}\Big([0,T]; \dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\Big) \times L^{2q}\Big([0,T]; \dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\Big)$$

into

$$L^\infty\big([0,T]; \dot{B}_q^{\frac{d}{q}-1,q}\big),$$

and we have the inequality

$$\begin{aligned}\big\|B(u,v)\big\|_{L^\infty\big([0,T];\dot{H}_q^{\frac{d}{q}-1}\big)} &\lesssim \big\|B(u,v)\big\|_{L^\infty\big([0,T];\dot{B}_q^{\frac{d}{q}-1,q}\big)} \\ &\leq C\Big\|u\Big\|_{L^{2q}\big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)}\Big\|v\Big\|_{L^{2q}\big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)},\end{aligned}$$

where C is a positive constant independent of T.

Proof. Applying Lemma 2.2.6 with $r = 2q$, $p = \frac{dq}{d+1-q}$, and $s = \frac{d+2-2q}{q}$, we get

$$\frac{1}{\tilde{p}} = \frac{2}{p} - \frac{s}{d} = \frac{1}{q},$$

and from (a) of Lemma 1.1.5, we have

$$\begin{aligned}\big\|B(u,v)\big\|_{L^\infty\big([0,T];\dot{H}_q^{\frac{d}{q}-1}\big)} &\lesssim \big\|B(u,v)\big\|_{L^\infty\big([0,T];\dot{B}_q^{\frac{d}{q}-1,q}\big)} \\ &\lesssim \Big\|u\Big\|_{L^{2q}\big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)}\Big\|v\Big\|_{L^{2q}\big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)}.\end{aligned}$$

□

Lemma 2.2.17. *Assume that $u_0 \in \dot{H}_q^{\frac{d}{q}-1}$ with $d \geq 3$ and $1 < q \leq 2$. Then*

$$\Big\|e^{\cdot\Delta}u_0\Big\|_{L^{2q}\big([0,\infty);\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)} \simeq \big\|u_0\big\|_{\dot{B}^{(d+1)/q-2,2q}_{dq/(d+1-q)}} \lesssim \big\|u_0\big\|_{\dot{H}_q^{d/q-1}}.$$

Proof. By using (a), (c), and (d) of Lemma 1.1.5 in order to obtain

$$\dot{H}_q^{\frac{d}{q}-1} \hookrightarrow \dot{B}_q^{\frac{d}{q}-1,2} \hookrightarrow \dot{B}_q^{\frac{d}{q}-1,2q} \hookrightarrow \dot{B}^{(d+1)/q-2,2q}_{dq/(d+1-q)}. \tag{2.43}$$

Applying Lemma 1.1.6 and from the imbedding (2.43) we have the estimates

$$\begin{aligned}\Big\|e^{\cdot\Delta}u_0\Big\|_{L^{2q}\big([0,\infty);\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)} &= \Big(\int_0^\infty \Big\|e^{t\Delta}\dot{\Lambda}^{\frac{d+2-2q}{q}}u_0\Big\|^{2q}_{L^{dq/(d+1-q)}}\mathrm{d}t\Big)^{1/2q} \\ &= \Big(\int_0^\infty \Big\|t^{-\frac{-1/q}{2}}e^{t\Delta}\dot{\Lambda}^{\frac{d+2-2q}{q}}u_0\Big\|^{2q}_{L^{dq/(d+1-q)}}\frac{\mathrm{d}t}{t}\Big)^{1/2q} \\ &\simeq \big\|\dot{\Lambda}^{\frac{d+2-2q}{q}}u_0\big\|_{\dot{B}^{-1/q,2q}_{dq/(d+1-q)}} \simeq \big\|u_0\big\|_{\dot{B}^{(d+1)/q-2,2q}_{dq/(d+1-q)}} \lesssim \big\|u_0\big\|_{\dot{H}_q^{d/q-1}}.\end{aligned}$$

□

Theorem 2.2.18. *Let $d \geq 3$ and $1 < q \leq 2$. There exists a positive constant $\delta_{q,d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ with $\operatorname{div}(u_0) = 0$ satisfying*

$$\left\| e^{\cdot\Delta} u_0 \right\|_{L^{2q}\left([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\right)} \leq \delta_{q,d}, \tag{2.44}$$

NSE has a unique mild solution $u \in L^{2q}\big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big) \cap L^\infty\big([0,T];\dot{H}_q^{d/q-1}\big)$. Denoting $w = u - e^{\cdot\Delta}u_0$, we have

$$w \in L^{2q}\big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big) \cap L^\infty\big([0,T];\dot{B}_q^{\frac{d}{q}-1,q}\big).$$

Finally, we have

$$\left\| e^{\cdot\Delta} u_0 \right\|_{L^{2q}\left([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\right)} \leq \left\| e^{\cdot\Delta} u_0 \right\|_{L^{2q}\left([0,\infty);\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\right)} \simeq \left\| u_0 \right\|_{\dot{B}^{(d+1)/q-2,2q}_{dq/(d+1-q)}} \lesssim \left\| u_0 \right\|_{\dot{H}_q^{d/q-1}},$$

in particular, for arbitrary $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ the inequality (2.44) holds when $T(u_0)$ is small enough; and there exists a positive constant $\sigma_{q,d}$ such that for all $\|u_0\|_{\dot{B}^{(d+1)/q-2,2q}_{dq/(d+1-q)}} \leq \sigma_{q,d}$ we can take $T = \infty$.

Proof. The proof of Theorem 2.2.18 is similar to the one of Theorem 2.2.11, by combining Theorem 1.5.1 with Lemmas 2.2.1, 2.2.5 (for $r = 2q, p = \frac{dq}{d+1-q}, s = \frac{d+2-2q}{q}$), 2.2.16, and 2.2.17. ☐

Remark 2.2.6. The case $q = 2$ was treated by several authors, see for example [11, 35, 46]. However their results are different from ours.

2.2.6. Conclusions

In this section, for $d \geq 3, s \geq 0, p > 1,$ and $r > 2$ be such that $\frac{s}{d} < \frac{1}{p} < \frac{1}{2} + \frac{s}{2d}$ and $\frac{2}{r} + \frac{d}{p} - s \leq 1$, we investigate mild solutions to NSE in the spaces $L^r\big([0,T];\dot{H}_p^s(\mathbb{R}^d)\big)$. First we obtain the existence of local mild solutions with an arbitrary initial tempered distribution datum in the Besov spaces $B_p^{s-\frac{2}{r},r}$. In the case of critical indexes $\frac{2}{r} - s + \frac{d}{p} = 1$, we obtain the existence of global mild solutions when the norm of the initial tempered distribution datum in the Besov space $\dot{B}_p^{s-\frac{2}{r},r}$ is small enough. The particular case of the above result, when $s = 0$, was presented in the book by Lemarie-Rieusset [46]. Next, we present two different algorithms for constructing mild solutions in $L^\infty([0,T];\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d))$ or $L^\infty([0,T];H_q^{\frac{d}{q}-1}(\mathbb{R}^d))$ to the Cauchy problem for the Navier-Stokes equations when the initial datum belongs to Sobolev spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ $\big($or $H_q^{\frac{d}{q}-1}(\mathbb{R}^d)\big)$. We use the first algorithm to consider the case when the initial datum belongs to $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ or $H_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ with $3 \leq d \leq 4$ and $2 \leq q \leq d$. Our results, when $q = d$, are generalizations of the ones obtained in [46]. With the second algorithm, we can treat the case when the initial datum belongs to the critical spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ with $d \geq 3$ and $1 < q \leq d$. The cases $q = 2$ and $q = d$ were considered by many authors, see [11, 13, 16, 17, 34, 35, 37, 46, 57]. We also note that the Cauchy problem for an incompressible magneto-hydrodynamics system with positive viscosity and magnetic resistivity, in the framework of Besov spaces was considered in [53].

2.3 Mild solutions in Sobolev spaces of negative order

The main purpose of this section is to prove that NSE are well-posed when the initial datum belongs to the Sobolev spaces $\dot{H}^s_p(\mathbb{R}^d)$ with super-critical-negative-regular indexes $(p > d, \frac{d}{p} - 1 < s < 0)$.

2.3.1. Solutions to the Navier-Stokes equations with the initial value in the Sobolev spaces $\dot{H}^s_p(\mathbb{R}^d)$ for $d \geq 2, p > \frac{d}{2}$, and $\frac{d}{p} - 1 \leq s < \frac{d}{2p}$

We define the space $\mathcal{N}^s_{p,T}$ which is made up of the functions $u(t,x)$ such that

$$\big\|u\big\|_{\mathcal{N}^s_{p,T}} := \sup_{0<t<T} \big\|u(t,x)\big\|_{\dot{H}^s_p} < \infty,$$

and

$$\lim_{t\to 0} \big\|u(t,x)\big\|_{\dot{H}^s_p} = 0, \tag{2.45}$$

with $p > 1$ and $s \geq \frac{d}{p} - 1$.
Next we define the auxiliary space $\mathcal{K}^{\tilde{q}}_{q,T}$ which consists of functions $u(t,x)$ such that

$$\big\|u\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}} := \sup_{0<t<T} t^{\frac{\alpha}{2}} \big\|u(t,x)\big\|_{L^{\tilde{q}}} < \infty,$$

and

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \big\|u(t,x)\big\|_{L^{\tilde{q}}} = 0, \tag{2.46}$$

with $\tilde{q} \geq q \geq d$ and $\alpha = d(\frac{1}{q} - \frac{1}{\tilde{q}})$.

Remark 2.3.1. The auxiliary space $\mathcal{K}_{\tilde{q}} := \mathcal{K}^{\tilde{q}}_{d,T}$ $(\tilde{q} \geq d)$ was introduced by Weissler and systematically used by Kato [37] and Cannone [12].

Lemma 2.3.1. *Suppose that $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ with $p > 1$ and $\frac{d}{p} - 1 \leq s < \frac{d}{p}$. Then for all $\tilde{q}$ satisfying*

$$\tilde{q} > \max\{p, q\},$$

where

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{d},$$

we have

$$e^{\cdot\Delta} u_0 \in \mathcal{K}^{\tilde{q}}_{q,\infty}.$$

Proof. First, we consider the case $p \leq q$. In this case $s \geq 0$, applying a Sobolev's embedding theorem, we have $u_0 \in L^q$. We will prove that

$$\sup_{0<t<\infty} t^{\frac{\alpha}{2}} \big\|e^{t\Delta} u_0\big\|_{L^{\tilde{q}}} \lesssim \big\|u_0\big\|_{L^q} \quad \text{for all } \tilde{q} \geq q.$$

Set

$$\frac{1}{h} = 1 + \frac{1}{\tilde{q}} - \frac{1}{q}.$$

Applying Young's inequality we obtain

$$\begin{aligned} \big\|e^{t\Delta} u_0\big\|_{L^{\tilde{q}}} &= \frac{1}{(4\pi t)^{d/2}} \big\|e^{\frac{-|\cdot|^2}{4t}} * u_0\big\|_{L^{\tilde{q}}} \lesssim \frac{1}{t^{d/2}} \big\|e^{\frac{-|\cdot|^2}{4t}}\big\|_{L^h} \big\|u_0\big\|_{L^q} \\ &= t^{-\frac{\alpha}{2}} \big\|e^{\frac{-|\cdot|^2}{4}}\big\|_{L^h} \big\|u_0\big\|_{L^q} \simeq t^{-\frac{\alpha}{2}} \big\|u_0\big\|_{L^q}. \end{aligned} \tag{2.47}$$

We prove now that
$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|e^{t\Delta}u_0\big\|_{L^{\tilde q}}=0,\ \text{ for all } \tilde q>q.$$
Set $\mathcal{X}_n(x)=0$ for $x\in\{x\in\mathbb{R}^d:|x|<n\}\cap\{x\in\mathbb{R}^d:|u_0(x)|<n\}$ and $\mathcal{X}_n(x)=1$ otherwise. From the inequality (2.47) we have

$$t^{\frac{\alpha}{2}}\big\|e^{t\Delta}u_0\big\|_{L^{\tilde q}}\leq C\big(t^{\frac{\alpha-d}{2}}\big\|e^{\frac{-|\cdot|^2}{4t}}*(\mathcal{X}_nu_0)\big\|_{L^{\tilde q}}+t^{\frac{\alpha-d}{2}}\big\|e^{\frac{-|\cdot|^2}{4t}}*((1-\mathcal{X}_n)u_0)\big\|_{L^{\tilde q}}\big).$$

Applying Young's inequality for convolution, we have

$$Ct^{\frac{\alpha-d}{2}}\big\|e^{\frac{-|\cdot|^2}{4t}}*(\mathcal{X}_nu_0)\big\|_{L^{\tilde q}}\leq C_1\big\|e^{\frac{-|\cdot|^2}{4}}\big\|_{L^h}\big\|\mathcal{X}_nu_0\big\|_{L^q}\leq C_2\big\|\mathcal{X}_nu_0\big\|_{L^q}. \tag{2.48}$$

For any $\epsilon>0$, we can take n large enough such that $C_2\big\|\mathcal{X}_nu_0\big\|_{L^q}<\frac{\epsilon}{2}$.
Fixe one of such n and applying Young's inequality for convolution, we have

$$\begin{aligned}Ct^{\frac{\alpha-d}{2}}\big\|e^{\frac{-|\cdot|^2}{4t}}*((1-\mathcal{X}_n)u_0)\big\|_{L^{\tilde q}}&\leq C_3t^{\frac{\alpha-d}{2}}\big\|e^{\frac{-|\cdot|^2}{4t}}\big\|_{L^1}\big\|(1-\mathcal{X}_n)u_0\big\|_{L^{\tilde q}}\\ \leq C_3t^{\frac{\alpha}{2}}\big\|e^{\frac{-|\cdot|^2}{4}}\big\|_{L^1}\big\|n(1-\mathcal{X}_n)\big\|_{L^{\tilde q}}&=C_4nt^{\frac{\alpha}{2}}\big\|1-\mathcal{X}_n\big\|_{L^{\tilde q}}=C_5(n)t^{\frac{\alpha}{2}}<\frac{\epsilon}{2}\end{aligned} \tag{2.49}$$

for small enough $t\leq t(n)$. From the estimates (2.48) and (2.49), we have

$$t^{\frac{\alpha}{2}}\big\|e^{t\Delta}u_0\big\|_{L^{\tilde q}}\leq C_2\big\|\mathcal{X}_nu_0\big\|_{L^q}+C_5(n)t^{\frac{\alpha}{2}}<\epsilon.$$

Next, we consider the case $p>q$. In this case $s<0$, we will prove that

$$\sup_{0<t<\infty} t^{\frac{\alpha}{2}}\big\|e^{t\Delta}u_0\big\|_{L^{\tilde q}}\lesssim\big\|u_0\big\|_{\dot H^s_p}\ \text{ for all } \tilde q\geq p.$$

We have

$$e^{t\Delta}u_0=e^{t\Delta}\dot\Lambda^{-s}\dot\Lambda^su_0=\frac{1}{t^{\frac{d-s}{2}}}K\Big(\frac{\cdot}{\sqrt t}\Big)*(\dot\Lambda^su_0), \tag{2.50}$$

where

$$\hat K(\xi)=\frac{1}{(2\pi)^{\frac{d}{2}}}e^{-|\xi|^2}|\xi|^{-s}.$$

Applying Lemma 1.2.1 with $|\alpha|=-s$, we obtain

$$|K(x)|\lesssim\frac{1}{(1+|x|)^{d-s}}.$$

Set

$$\frac{1}{h}=1+\frac{1}{\tilde q}-\frac{1}{p}.$$

Applying Young's inequality for convolution to obtain

$$\big\|e^{t\Delta}u_0\big\|_{L^{\tilde q}}\lesssim t^{-\frac{\alpha}{2}}\big\|K\big\|_{L^h}\big\|\dot\Lambda^su_0\big\|_{L^p}\simeq t^{-\frac{\alpha}{2}}\big\|u_0\big\|_{\dot H^s_p}.$$

We claim now that

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|e^{t\Delta}u_0\big\|_{L^{\tilde q}}=0,\ \ \text{for all } \tilde q>p.$$

Set $\mathcal{X}_{n,s}(x)=0$ for $x\in\{x\in\mathbb{R}^d:|x|<n\}\cap\{x\in\mathbb{R}^d:|\dot\Lambda^su_0(x)|<n\}$ and $\mathcal{X}_{n,s}(x)=1$ otherwise. From the above proof we deduce that, for any $\epsilon>0$

$$t^{\frac{\alpha}{2}}\big\|e^{t\Delta}u_0\big\|_{L^{\tilde q}}\leq C_1\big\|K\big\|_{L^h}\big\|\mathcal{X}_{n,s}\dot\Lambda^su_0\big\|_{L^p}+C_2nt^{\frac{d}{2}(\frac{1}{p}-\frac{1}{\tilde q})}\big\|K\big\|_{L^1}\big\|1-\mathcal{X}_{n,s}\big\|_{L^{\tilde q}}<\epsilon$$

for large enough n and for small enough $t\leq t(n)$. $\square$

Remark 2.3.2. The particular case of Lemma 2.3.1, when $s = 0$ and $p = d$ is Lemma 9 in ([15], p. 196).

In the following lemmas a particular attention will be devoted to the study of the bilinear operator $B(u,v)(t)$ defined in (1.14) in Sobolev spaces.

Lemma 2.3.2. *Let*

$$p > \frac{d}{2} \text{ and } \frac{d}{p} - 1 \leq s < \frac{d}{2p}. \tag{2.51}$$

Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{q}}_{q,T} \times \mathcal{K}^{\tilde{q}}_{q,T}$ into $\mathcal{N}^s_{p,T}$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{d} \text{ and } q < \tilde{q} \leq 2p, \tag{2.52}$$

and the following inequality holds

$$\big\|B(u,v)\big\|_{\mathcal{N}^s_{p,T}} \leq CT^{\frac{1}{2}(1+s-\frac{d}{p})}\big\|u\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}}\big\|v\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}}, \tag{2.53}$$

where C is a positive constant and independent of T.

Proof. First note that

$$\begin{aligned}\big\|B(u,v)(t)\big\|_{\dot{H}^s_p} &\leq \int_0^t \big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\big\|_{\dot{H}^s_p}\mathrm{d}\tau \\ &= \int_0^t \big\|\dot{\Lambda}^s e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\big\|_{L^p}\mathrm{d}\tau. \end{aligned}\tag{2.54}$$

We have

$$\Big(\dot{\Lambda}^s e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\Big)_j = \dot{\Lambda}^s e^{(t-\tau)\Delta}\sum_{l,k=1}^d\Big(\delta_{jk} - \frac{\partial_j\partial_k}{\Delta}\Big)\partial_l\big(u_l(\tau)v_k(\tau)\big).$$

From the properties of the Fourier transform we have

$$\Big(\dot{\Lambda}^s e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\Big)^\wedge_j(\xi) = |\xi|^s e^{-(t-\tau)|\xi|^2}\sum_{l,k=1}^d\Big(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l)\big(u_l(\tau)v_k(\tau)\big)^\wedge(\xi),$$

and then

$$\Big(\dot{\Lambda}^s e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\Big)_j = \frac{1}{(t-\tau)^{\frac{d+1+s}{2}}}\sum_{l,k=1}^d K_{l,k,j}\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)*\big(u_l(\tau)v_k(\tau)\big), \tag{2.55}$$

Applying Lemma 1.2.1 with $|\alpha| = 1+s \geq \frac{d}{p}$, we obtain

$$|K_{l,k,j}(x)| \lesssim \frac{1}{(1+|x|)^{d+1+s}} \leq \frac{1}{(1+|x|)^{d(1+\frac{1}{p})}}.$$

Thus, the tensor $K(x) = \{K_{l,k,j}(x)\}$ satisfies

$$|K(x)| \lesssim \frac{1}{(1+|x|)^{d(1+\frac{1}{p})}}. \tag{2.56}$$

So, we can rewrite the equality (2.55) in the tensor form

$$\dot{\Lambda}^s e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)=\frac{1}{(t-\tau)^{\frac{d+1+s}{2}}}K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)*\big(u(\tau)\otimes v(\tau)\big). \tag{2.57}$$

Applying Young's inequality for convolution, we have

$$\big\|\dot{\Lambda}^s e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\big\|_{L^p}\lesssim\frac{1}{(t-\tau)^{\frac{d+1+s}{2}}}\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{L^r}\big\|u(\tau)\otimes v(\tau)\big\|_{L^{\frac{\tilde{q}}{2}}}, \tag{2.58}$$

where

$$\frac{1}{r}=1+\frac{1}{p}-\frac{2}{\tilde{q}}. \tag{2.59}$$

Note that from (2.51) and (2.52), we can check that r be such that $1\le r\le\infty$. Applying Hölder's inequality, we have

$$\big\|u(\tau)\otimes v(\tau)\big\|_{L^{\frac{\tilde{q}}{2}}}\le\big\|u(\tau)\big\|_{L^{\tilde{q}}}\big\|v(\tau)\big\|_{L^{\tilde{q}}}. \tag{2.60}$$

From the inequalities (2.56) and (2.59) it follows that

$$\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{L^r}=(t-\tau)^{\frac{d}{2r}}\big\|K\big\|_{L^r}\simeq(t-\tau)^{\frac{d}{2}(1+\frac{1}{p}-\frac{2}{\tilde{q}})}. \tag{2.61}$$

From the inequalities (2.58), (2.60), and (2.61) we deduce that

$$\big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\big\|_{\dot{H}^s_p}\lesssim(t-\tau)^{\frac{d}{2p}-\frac{d}{\tilde{q}}-\frac{s+1}{2}}\big\|u(\tau)\big\|_{L^{\tilde{q}}}\big\|v(\tau)\big\|_{L^{\tilde{q}}}. \tag{2.62}$$

Note that from (2.51) and (2.52), we can check that $\frac{d}{2p}-\frac{d}{\tilde{q}}-\frac{s+1}{2}>-1$ and $\alpha=d(\frac{1}{q}-\frac{1}{\tilde{q}})<1$. This gives the desired result

$$\begin{aligned}\big\|B(u,v)(t)\big\|_{\dot{H}^s_p}&\lesssim\int_0^t(t-\tau)^{\frac{d}{2p}-\frac{d}{\tilde{q}}-\frac{s+1}{2}}\big\|u(\tau)\big\|_{L^{\tilde{q}}}\big\|v(\tau)\big\|_{L^{\tilde{q}}}\mathrm{d}\tau\\ &\le\int_0^t(t-\tau)^{\frac{d}{2p}-\frac{d}{\tilde{q}}-\frac{s+1}{2}}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{L^{\tilde{q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{L^{\tilde{q}}}\mathrm{d}\tau\\ &=\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{L^{\tilde{q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{L^{\tilde{q}}}\int_0^t(t-\tau)^{\frac{d}{2p}-\frac{d}{\tilde{q}}-\frac{s+1}{2}}\tau^{-\alpha}\mathrm{d}\tau\\ &\simeq t^{\frac{1}{2}(1+s-\frac{d}{p})}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{L^{\tilde{q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{L^{\tilde{q}}}.\end{aligned} \tag{2.63}$$

Let us now check the validity of the condition (2.45) for the bilinear term $B(u,v)(t)$. In fact, from (2.63) it follows that

$$\lim_{t\to0}\big\|B(u,v)(t)\big\|_{\dot{H}^s_p}=0,$$

whenever

$$\lim_{t\to0}t^{\frac{\alpha}{2}}\big\|u(t)\big\|_{L^{\tilde{q}}}=\lim_{t\to0}t^{\frac{\alpha}{2}}\big\|v(t)\big\|_{L^{\tilde{q}}}=0.$$

□

Lemma 2.3.3. *Let*

$$\tilde{q}>q\ge d. \tag{2.64}$$

Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{q}}_{q,T} \times \mathcal{K}^{\tilde{q}}_{q,T}$ into $\mathcal{K}^{\tilde{q}}_{q,T}$ and the following inequality holds

$$\big\|B(u,v)\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}} \le CT^{\frac{1}{2}(1-\frac{d}{q})}\big\|u\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}}\big\|v\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}}, \tag{2.65}$$

where C is a positive constant and independent of T.

Proof. Applying the estimate (2.62) for $s=0$ and $p=\tilde{q}$, we have

$$\big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\big\|_{L^{\tilde{q}}} \lesssim (t-\tau)^{-\frac{d}{2\tilde{q}}-\frac{1}{2}}\big\|u(\tau)\big\|_{L^{\tilde{q}}}\big\|v(\tau)\big\|_{L^{\tilde{q}}}.$$

Note that from the inequality (2.64), we can check that $-\frac{d}{2\tilde{q}}-\frac{1}{2} > -1$ and $\alpha = d(\frac{1}{q}-\frac{1}{\tilde{q}}) < 1$. This gives the desired result

$$\begin{aligned}\big\|B(u,v)(t)\big\|_{L^{\tilde{q}}} &\lesssim \int_0^t (t-\tau)^{-\frac{d}{2\tilde{q}}-\frac{1}{2}}\big\|u(\tau)\big\|_{L^{\tilde{q}}}\big\|v(\tau)\big\|_{L^{\tilde{q}}}\mathrm{d}\tau \\ \le \int_0^t (t-\tau)^{-\frac{d}{2\tilde{q}}-\frac{1}{2}}\tau^{-\alpha}&\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{L^{\tilde{q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{L^{\tilde{q}}}\mathrm{d}\tau \\ = \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{L^{\tilde{q}}}&\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{L^{\tilde{q}}}\int_0^t (t-\tau)^{-\frac{d}{2\tilde{q}}-\frac{1}{2}}\tau^{-\alpha}\mathrm{d}\tau \\ \simeq t^{-\frac{\alpha}{2}}t^{\frac{1}{2}(1-\frac{d}{q})}&\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{L^{\tilde{q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{L^{\tilde{q}}}.\end{aligned} \tag{2.66}$$

Thus

$$t^{\frac{\alpha}{2}}\big\|B(u,v)(t)\big\|_{L^{\tilde{q}}} \lesssim t^{\frac{1}{2}(1-\frac{d}{q})}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{L^{\tilde{q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{L^{\tilde{q}}}. \tag{2.67}$$

Now we check the validity of the condition (2.46) for the bilinear term $B(u,v)(t)$. From (2.67) it follows that

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|B(u,v)(t)\big\|_{L^{\tilde{q}}} = 0,$$

whenever

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|u(t)\big\|_{L^{\tilde{q}}} = \lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|v(t)\big\|_{L^{\tilde{q}}} = 0.$$

□

The following lemma, the proof of which is omitted, is a generalization of Lemma 2.3.3.

Lemma 2.3.4. *Let $d \le q \le \tilde{q}_2 < \infty$ and $q < \tilde{q}_1 < \infty$ be such that one of the following conditions*

$$q < \tilde{q}_1 < 2d, q \le \tilde{q}_2 < \frac{d\tilde{q}_1}{2d-\tilde{q}_1},$$

or

$$2d \le \tilde{q}_1 \le 2q, q \le \tilde{q}_2 < \infty,$$

or

$$2q < \tilde{q}_1 < \infty, \frac{\tilde{q}_1}{2} < \tilde{q}_2 < \infty,$$

is satisfied.
Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{q}_1}_{q,T} \times \mathcal{K}^{\tilde{q}_1}_{q,T}$ into $\mathcal{K}^{\tilde{q}_2}_{q,T}$, and we have the inequality

$$\big\|B(u,v)\big\|_{\mathcal{K}^{\tilde{q}_2}_{q,T}} \le CT^{\frac{1}{2}(1-\frac{d}{q})}\big\|u\big\|_{\mathcal{K}^{\tilde{q}_1}_{q,T}}\big\|v\big\|_{\mathcal{K}^{\tilde{q}_1}_{q,T}},$$

where C is a positive constant and independent of T.

Remark 2.3.3. The particular case of Lemma 2.3.4 when $q = d$ is Lemma 11 in ([15], p. 196).

Combining Theorem 1.5.1 with Lemmas 1.1.6, 2.2.1, 2.3.1, 2.3.2, 2.3.3, and 2.3.4 we obtain the following existence result.

Theorem 2.3.5. *Let*

$$p > \frac{d}{2} \text{ and } \frac{d}{p} - 1 \leq s < \frac{d}{2p}.$$

Set

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{d}.$$

(a) *For all $\tilde{q} > \max\{p, q\}$, there exists a positive constant $\delta_{q,\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ with $\operatorname{div}(u_0) = 0$ satisfying*

$$T^{\frac{1}{2}(1+s-\frac{d}{p})} \sup_{0<t<T} t^{\frac{d}{2}(\frac{1}{p}-\frac{s}{d}-\frac{1}{\tilde{q}})} \big\|e^{t\Delta}u_0\big\|_{L^{\tilde{q}}} \leq \delta_{q,\tilde{q},d}, \tag{2.68}$$

NSE has a unique mild solution $u \in \bigcap_{r>\max\{p,q\}} \mathcal{K}^r_{q,T} \cap L^\infty([0,T]; \dot{H}^s_p)$.
In particular, the inequality (2.68) holds for arbitrary $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ when $T(u_0)$ is small enough.
(b) *If $s = \frac{d}{p} - 1$ then for all $\tilde{q} > \max\{p, d\}$ there exists a constant $\sigma_{\tilde{q},d} > 0$ such that if $\big\|u_0\big\|_{\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}} \leq \sigma_{\tilde{q},d}$ and $T = +\infty$ then the inequality (2.68) holds.*

Proof. (a) From Lemma 2.3.3, B is continuous from $\mathcal{K}^{\tilde{q}}_{q,T} \times \mathcal{K}^{\tilde{q}}_{q,T}$ to $\mathcal{K}^{\tilde{q}}_{q,T}$ and we have the inequality

$$\big\|B(u,v)\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}} \leq C_{q,\tilde{q},d} T^{\frac{1}{2}(1-\frac{d}{q})} \big\|u\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}} \big\|v\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}} = C_{q,\tilde{q},d} T^{\frac{1}{2}(1+s-\frac{d}{p})} \big\|u\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}} \big\|v\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}},$$

where $C_{q,\tilde{q},d}$ is a positive constant independent of T. From Theorem 1.5.1 and the above inequality, we deduce that for any $u_0 \in \dot{H}^s_p$ satisfying

$$T^{\frac{1}{2}(1+s-\frac{d}{p})} \big\|e^{\cdot\Delta}u_0\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}} = T^{\frac{1}{2}(1+s-\frac{d}{p})} \sup_{0<t<T} t^{\frac{\alpha}{2}} \big\|e^{t\Delta}u_0\big\|_{L^{\tilde{q}}} \leq \frac{1}{4C_{q,\tilde{q},d}},$$

where

$$\alpha = d\Big(\frac{1}{q} - \frac{1}{\tilde{q}}\Big) = d\Big(\frac{1}{p} - \frac{s}{d} - \frac{1}{\tilde{q}}\Big),$$

NSE has a solution u on the interval $(0,T)$ so that $u \in \mathcal{K}^{\tilde{q}}_{q,T}$.
We prove that $u \in \bigcap_{r>\max\{p,q\}} \mathcal{K}^r_{q,T}$. We consider three cases $q < \tilde{q} < 2d$, $2d \leq \tilde{q} \leq 2q$, and $2q < \tilde{q} < \infty$ separately.
Note that if $\max\{p,q\} \geq 2d$ then there does not exist $\tilde{q}$ satisfying the condition of the first case, and if $p \geq 2q$ then there does not exist $\tilde{q}$ satisfying the condition of the second case. Therefore the number of cases that can occur depends on s and p.
First, we consider the case $q < \tilde{q} < 2d$. There are two possibilities $\tilde{q} > \frac{4d}{3}$ and $\tilde{q} \leq \frac{4d}{3}$. In the case $\tilde{q} > \frac{4d}{3}$, we apply Lemmas 2.3.1 and 2.3.4 to obtain $u \in \mathcal{K}^r_{q,T}$ for all r satisfying $\max\{p,q\} < r < \tilde{q}_1$ where $\tilde{q}_1 = \frac{d\tilde{q}}{2d-\tilde{q}} > 2d$. Thus, $u \in \mathcal{K}^{2d}_{q,T}$. Applying again Lemmas 2.3.1 and 2.3.4, we deduce that $u \in \mathcal{K}^r_{q,T}$ for all $r > \max\{p,q\}$. In the case $\tilde{q} \leq \frac{4d}{3}$, we set up the following series of numbers $\{\tilde{q}_i\}_{0\leq i\leq N}$ by induction. Set $\tilde{q}_0 = \tilde{q}$ and $\tilde{q}_1 = \frac{d\tilde{q}_0}{2d-\tilde{q}_0}$. We have

$\tilde{q}_1 > \tilde{q}_0$. If $\tilde{q}_1 > \frac{4d}{3}$ then set $N = 1$ and stop here. In the case $\tilde{q}_1 \le \frac{4d}{3}$ set $\tilde{q}_2 = \frac{d\tilde{q}_1}{2d-\tilde{q}_1}$. We have $\tilde{q}_2 > \tilde{q}_1$. If $\tilde{q}_2 > \frac{4d}{3}$ then set $N = 2$ and stop here. In the case $\tilde{q}_2 \le \frac{4d}{3}$, set $\tilde{q}_3 = \frac{d\tilde{q}_2}{2d-\tilde{q}_2}$. We have $\tilde{q}_3 > \tilde{q}_2$, and so on, there exists $k \ge 0$ such that $\tilde{q}_k \le \frac{4d}{3}, \tilde{q}_{k+1} = \frac{d\tilde{q}_k}{2d-\tilde{q}_k} > \frac{4d}{3}$. We set $N = k + 1$ and stop here, and we have

$$\tilde{q}_0 = \tilde{q}, \tilde{q}_i = \frac{d\tilde{q}_{i-1}}{2d - \tilde{q}_{i-1}}, \tilde{q}_i > \tilde{q}_{i-1} \text{ for } i = 1, 2, 3, .., N,$$

$$2d \ge \tilde{q}_N > \frac{4d}{3} \ge \tilde{q}_{N-1}.$$

From $u \in \mathcal{K}_{q,T}^{\tilde{q}_0}$, applying Lemmas 2.3.1 and 2.3.4 to obtain $u \in \mathcal{K}_{q,T}^{r}$ for all r satisfying $\max\{p, q\} < r < \tilde{q}_1$. Then applying again Lemmas 2.3.1 and 2.3.4 to get $u \in \mathcal{K}_{q,T}^{r}$ for all r satisfying $\max\{p, q\} < r < \tilde{q}_2$, and so on, finishing we have $u \in \mathcal{K}_{q,T}^{r}$ for all r satisfying $\max\{p, q\} < r < \tilde{q}_N$. Therefore $u \in \mathcal{K}_{q,T}^{r}$ for all r satisfying $\frac{4d}{3} < r < \tilde{q}_N$. From the proof of the case $\tilde{q} > \frac{4d}{3}$, we have $u \in \mathcal{K}_{q,T}^{r}$ for all $r > \max\{p, q\}$.
Next, we consider the case $2d \le \tilde{q} \le 2q$. We show that $u \in \mathcal{K}_{q,T}^{r}$ for all $r > \max\{p, q\}$. This is easily deduced by applying Lemmas 2.3.1 and 2.3.4.
Finally, we consider the case $2q < \tilde{q} < \infty$. Let $i \in \mathbb{N}$ be such that

$$\frac{\tilde{q}}{2^{i-1}} \ge \max\{2q, p\} > \frac{\tilde{q}}{2^i}.$$

From $\tilde{q} > \max\{p, q\}$, and $\tilde{q} > 2q$ we have $\tilde{q} > \max\{2q, p\}$, hence $i \ge 1$. Applying Lemmas 2.3.1 and 2.3.4 to obtain $u \in \mathcal{K}_{q,T}^{r}$ for all $r > \frac{\tilde{q}}{2}$. Applying again Lemmas 2.3.1 and 2.3.4 to get $u \in \mathcal{K}_{q,T}^{r}$ for all $r > \frac{\tilde{q}}{2^2}$, and so on, finishing we have $u \in \mathcal{K}_{q,T}^{r}$ for all $r > \frac{\tilde{q}}{2^{i-1}}$. Applying again Lemmas 2.3.1 and 2.3.4 to obtain $u \in \mathcal{K}_{q,T}^{r}$ for all $r > \max\{p, q, \frac{\tilde{q}}{2^i}\}$. If $\max\{p, q\} \ge \frac{\tilde{q}}{2^i}$ then we have $u \in \mathcal{K}_{q,T}^{r}$ for all $r > \max\{p, q\}$. If $\max\{p, q\} < \frac{\tilde{q}}{2^i}$ then $2q > \frac{\tilde{q}}{2^i}$. Thus $u \in \mathcal{K}_{q,T}^{r}$ for all r satisfying $\frac{\tilde{q}}{2^i} < r$, hence $u \in \mathcal{K}_{q,T}^{2q}$. Applying Lemmas 2.3.1 and 2.3.4 to obtain $u \in \mathcal{K}_{q,T}^{r}$ for all $r > \max\{p, q\}$.
We now prove that $u \in L^\infty([0, T]; \dot{H}_p^s)$. Indeed, from Lemma 2.3.2, we have $B(u, u) \in \mathcal{N}_{p,T}^{s} \subseteq L^\infty\big([0, T]; \dot{H}_p^s\big)$. On the other hand, from Lemma 2.2.1, we have $e^{\cdot\Delta}u_0 \in L^\infty\big([0, T]; \dot{H}_p^s\big)$. Therefore

$$u = e^{\cdot\Delta}u_0 - B(u, u) \in L^\infty\big([0, T]; \dot{H}_p^s\big).$$

From the definition of $\mathcal{K}_{q,T}^{\tilde{q}}$ and Lemma 2.3.1, we deduce that the left-hand side of the inequality (2.68) converges to 0 when T goes to 0. Therefore the inequality (2.68) holds for arbitrary $u_0 \in \dot{H}_p^s(\mathbb{R}^d)$ when $T(u_0)$ is small enough.
(b) From Lemma 1.1.6, the two quantities $\big\|u_0\big\|_{\dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1,\infty}}$ and $\sup_{0<t<\infty} t^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\big\|e^{t\Delta}u_0\big\|_{L^{\tilde{q}}}$ are equivalent. Thus, there exists a positive constant $\sigma_{\tilde{q},d}$ such that we can take $T = \infty$ and the inequality (2.68) holds whenever $\big\|u_0\big\|_{\dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1,\infty}} \le \sigma_{\tilde{q},d}$. □

In the case of critical indexes ($s = \frac{d}{p} - 1, p > \frac{d}{2}$), we obtain the following statement which is stronger than the Cannone theorem (see Theorem 3 in [15], p. 195). In particular, the case $p = d$ is the Cannone theorem.

Corollary 2.3.6. *Let $p > \frac{d}{2}$. Then for any $\tilde{q} > \max\{p, d\}$, there exists a positive constant $\delta_{\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d)$ with* $\mathrm{div}(u_0) = 0$ *satisfying*

$$\sup_{0<t<T} t^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\big\|e^{t\Delta}u_0\big\|_{L^{\tilde{q}}} \le \delta_{\tilde{q},d}, \tag{2.69}$$

NSE has a unique mild solution $u \in \bigcap_{r>\max\{p,d\}} \mathcal{K}^r_{d,T} \cap L^\infty([0,T]; \dot{H}_p^{\frac{d}{p}-1})$.

Denoting $w = u - e^{\cdot\Delta}u_0$ *then* $w \in \mathcal{N}_{\tilde{p},T}^{\frac{d}{\tilde{p}}-1}$ *for all* $\tilde{p} > \frac{1}{2}\max\{p,d\}$.

In particular, the inequality (2.69) *holds for arbitrary* $u_0 \in \dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d)$ *when* $T(u_0)$ *is small enough, and there exists a positive constant* $\sigma_{\tilde{q},d}$ *such that if*

$$\|u_0\|_{\dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1,\infty}} \leq \sigma_{\tilde{q},d} \text{ and } T = +\infty, \tag{2.70}$$

then the inequality (2.69) *holds.*

Proof. By Theorem 2.3.5, we only need to prove that $w \in \mathcal{N}_{\tilde{p},T}^{\frac{d}{\tilde{p}}-1}$ for all $\tilde{p} > \frac{1}{2}\max\{p,d\}$. Indeed, applying Lemma 2.3.2, we have for all $\tilde{p} > \frac{d}{2}$ and r satisfying $d < r \leq 2\tilde{p}$, the bilinear operator B is continuous from $\mathcal{K}^r_{d,T} \times \mathcal{K}^r_{d,T}$ into $\mathcal{N}_{\tilde{p},T}^{\frac{d}{\tilde{p}}-1}$; hence from $u \in \bigcap_{r>\max\{p,d\}} \mathcal{K}^r_{d,T}$ and $2\tilde{p} > \max\{p,d\}$, we have $w = -B(u,u) \in \mathcal{N}_{\tilde{p},T}^{\frac{d}{\tilde{p}}-1}$. □

Remark 2.3.4. We have the following imbeddings

$$\dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d)_{(\frac{d}{2}<p<d)} \hookrightarrow L^d(\mathbb{R}^d) \hookrightarrow \dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d)_{(p>d)} \hookrightarrow BMO^{-1}(\mathbb{R}^d).$$

In Chapter 3, we gave the existence and uniqueness of solutions to NSE when the initial datum belongs to Sobolev spaces $\dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d)$ with $1 < p \leq d$. Thus, NSE are well-posed when the initial datum belongs to the Sobolev spaces $\dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d)$ with $1 < p < \infty$.

Corollary 2.3.7. *Let* $p > \frac{d}{2}$ *and* $\frac{d}{p}-1 < s < \frac{d}{2p}$. *Then for any* $\tilde{q}$ *be such that* $\tilde{q} > \max\{p,q\}$, *where*

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{d},$$

there exists a positive constant $\delta_{q,\tilde{q},d}$ *such that for all* $T > 0$ *and for all* $u_0 \in \dot{H}_p^s(\mathbb{R}^d)$ *with* $\mathrm{div}(u_0) = 0$ *satisfying*

$$T^{\frac{1}{2}(1+s-\frac{d}{p})}\|u_0\|_{\dot{B}_{\tilde{q}}^{s-(\frac{d}{p}-\frac{d}{\tilde{q}}),\infty}} \leq \delta_{q,\tilde{q},d},$$

NSE has a unique mild solution $u \in \bigcap_{r>\max\{p,q\}} \mathcal{K}^r_{q,T} \cap L^\infty([0,T]; \dot{H}_p^s)$.

Proof. By Lemma 1.1.6, we deduce that the two quantities $\|u_0\|_{\dot{B}_{\tilde{q}}^{s-(\frac{d}{p}-\frac{d}{\tilde{q}}),\infty}}$ and $\sup_{0<t<\infty} t^{\frac{d}{2}(\frac{1}{p}-\frac{s}{d}-\frac{1}{\tilde{q}})}\|e^{t\Delta}u_0\|_{L^{\tilde{q}}}$ are equivalent. Therefore

$$\sup_{0<t<T} t^{\frac{d}{2}(\frac{1}{p}-\frac{s}{d}-\frac{1}{\tilde{q}})}\|e^{t\Delta}u_0\|_{L^{\tilde{q}}} \lesssim \|u_0\|_{\dot{B}_{\tilde{q}}^{s-(\frac{d}{p}-\frac{d}{\tilde{q}}),\infty}}.$$

The corollary is proved by applying the above inequality and Theorem 2.3.5. □

Remark 2.3.5. We have the following imbedding

$$\dot{H}_p^s(\mathbb{R}^d) \hookrightarrow \dot{H}_{\tilde{q}}^{s-(\frac{d}{p}-\frac{d}{\tilde{q}})}(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{s-(\frac{d}{p}-\frac{d}{\tilde{q}}),\tilde{q}}(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{s-(\frac{d}{p}-\frac{d}{\tilde{q}}),\infty}(\mathbb{R}^d).$$

On the other hand, a function in $\dot{H}_p^s(\mathbb{R}^d)$ can be arbitrarily large in the $\dot{H}_p^s(\mathbb{R}^d)$ norm but small in the $\dot{B}_{\tilde{q}}^{s-(\frac{d}{p}-\frac{d}{\tilde{q}}),\infty}(\mathbb{R}^d)$ norm.

Applying Corollary 2.3.7 for $s < 0$, we get the following result.

Corollary 2.3.8. *Let $p > d$ and $\frac{d}{p} - 1 < s < 0$. Then for any $\tilde{q} > p$, there exists a constant $\delta_{q,\tilde{q},d} > 0$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ with* $\mathrm{div}(u_0) = 0$ *satisfying*

$$T^{\frac{1}{2}(1+s-\frac{d}{p})}\big\|u_0\big\|_{\dot{B}^{s-(\frac{d}{p}-\frac{d}{\tilde{q}}),\infty}_{\tilde{q}}} \leq \delta_{q,\tilde{q},d},$$

NSE has a unique mild solution $u \in \bigcap\limits_{r>p} \mathcal{K}^r_{q,T} \cap L^\infty([0,T]; \dot{H}^s_p)$, where $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$.

Applying Corollary 2.3.7 for $p > d$ and $s = 0$, we get the following statement which is stronger than that of Cannone and Meyer [11, 14] but under a much weaker condition on the initial data.

Corollary 2.3.9. *Let $p > d$. Then for any $\tilde{q} > p$, there exists a constant $\delta_{p,\tilde{q},d} > 0$ such that for all $T > 0$ and for all $u_0 \in L^p(\mathbb{R}^d)$ with* $\mathrm{div}(u_0) = 0$ *satisfying*

$$T^{\frac{1}{2}(1-\frac{d}{p})}\big\|u_0\big\|_{\dot{B}^{\frac{d}{\tilde{q}}-\frac{d}{p},\infty}_{\tilde{q}}} \leq \delta_{p,\tilde{q},d}, \tag{2.71}$$

NSE has a unique mild solution $u \in \bigcap\limits_{r>p} \mathcal{K}^r_{p,T} \cap L^\infty([0,T]; L^p)$.

Remark 2.3.6. If in (2.71) we replace the $\dot{B}^{\frac{d}{\tilde{q}}-\frac{d}{p},\infty}_{\tilde{q}}$ norm by the L^p norm then we get the assumption made in [11, 14]. We show that the condition (2.71) is weaker than the condition in [11, 14]. In Remark 2.3.5 we have showed that

$$L^p(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{d}{\tilde{q}}-\frac{d}{p},\infty}_{\tilde{q}}(\mathbb{R}^d), (\tilde{q} > p \geq d),$$

but these two spaces are different. Indeed, we have $\big|x\big|^{-\frac{d}{p}} \notin L^p(\mathbb{R}^d)$. On the other hand by using Lemma 1.1.6, we can easily prove that $\big|x\big|^{-\frac{d}{p}} \in \dot{B}^{\frac{d}{\tilde{q}}-\frac{d}{p},\infty}_{\tilde{q}}(\mathbb{R}^d)$ for all $\tilde{q} > p$.

2.3.2. Conclusions

In this section, we present a different algorithm for constructing mild solutions in the spaces $L^\infty([0,T]; \dot{H}^s_p(\mathbb{R}^d))$ to the Cauchy problem for NSE when the initial datum belongs to the Sobolev spaces $\dot{H}^s_p(\mathbb{R}^d)$, with $d \geq 2, p > \frac{d}{2}$, and $\frac{d}{p} - 1 \leq s < \frac{d}{2p}$.
We obtain the existence of mild solutions with arbitrary initial value when T is small enough; and existence of mild solutions for any $T > 0$ when the norm of the initial value in the Besov spaces $\dot{B}^{s-d(\frac{1}{p}-\frac{1}{\tilde{q}}),\infty}_{\tilde{q}}$, ($\tilde{q} > \max\{p, q\}$, where $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$) is small enough. In the case $p > d$ and $s = 0$, in terms of the Besov spaces we get the result that is stronger than that of Cannone and Meyer [11, 14] but under a weaker condition on the initial data. In the case of critical indexes ($p > \frac{d}{2}, s = \frac{d}{p} - 1$), we can take $T = \infty$ when the norm of the initial value in the Besov spaces $\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}(\mathbb{R}^d), (\tilde{q} > \max\{d, p\})$ is small enough. This result when $s = 0$ and $p = d$ is one of the Cannone theorems (see Theorem 3 in [15], p. 195).

2.4 Mild solutions in the Sobolev-Fourier-Lorentz spaces

In this section, for $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, we introduce and study the Sobolev-Fourier-Lorentz spaces $\dot{H}^s_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$. In the family spaces $\dot{H}^s_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$, the critical invariant spaces for the Navier-Stokes equations correspond to the value $s = \frac{d}{p} - 1$. When the initial datum belongs to the critical spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $d \geq 2, 1 \leq p < \infty$, and $1 \leq r < \infty$, we establish the existence of local mild solutions to the Cauchy problem for the Navier-Stokes equations in spaces $L^\infty([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d))$ with arbitrary initial value, and existence of global mild solutions in spaces $L^\infty([0,\infty); \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d))$ when the norm of the initial value in the Besov spaces $\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}(\mathbb{R}^d)$ is small enough, where $\tilde{p}$ may take some suitable values.

2.4.1. Sobolev-Fourier-Lozentz Space

Definition 2.4.1. (Fourier-Lebesgue spaces). (See [30].)
For $1 \leq p \leq \infty$, the Fourier-Lebesgue spaces $\mathcal{L}^p(\mathbb{R}^d)$ are defined as the space $\mathcal{F}^{-1}(L^{p'}(\mathbb{R}^d))$, $(\frac{1}{p'} + \frac{1}{p} = 1)$, equipped with the norm

$$\big\|f\big\|_{\mathcal{L}^p(\mathbb{R}^d)} := \big\|\mathcal{F}(f)\big\|_{L^{p'}(\mathbb{R}^d)},$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse.

Definition 2.4.2. (Sobolev-Fourier-Lebesgue spaces).
For $s \in \mathbb{R}$, and $1 \leq p \leq \infty$, the Sobolev-Fourier-Lebesgue spaces $\dot{H}^s_{\mathcal{L}^p}(\mathbb{R}^d)$ are defined as the space $\dot{\Lambda}^{-s}\mathcal{L}^p(\mathbb{R}^d)$, equipped with the norm

$$\big\|u\big\|_{\dot{H}^s_{\mathcal{L}^p}} := \big\|\dot{\Lambda}^s u\big\|_{\mathcal{L}^p}.$$

Definition 2.4.3. (Fourier-Lorentz spaces). *For $1 \leq p, r \leq \infty$, the Fourier-Lorentz spaces $\mathcal{L}^{p,r}(\mathbb{R}^d)$ are defined as the space $\mathcal{F}^{-1}(L^{p',r}(\mathbb{R}^d)), (\frac{1}{p'} + \frac{1}{p} = 1)$, equipped with the norm*

$$\big\|f\big\|_{\mathcal{L}^{p,r}(\mathbb{R}^d)} := \big\|\mathcal{F}(f)\big\|_{L^{p',r}(\mathbb{R}^d)}.$$

Definition 2.4.4. (Sobolev-Fourier-Lorentz spaces).
For $s \in \mathbb{R}$ and $1 \leq r, p \leq \infty$, the Sobolev-Fourier-Lorentz spaces $\dot{H}^s_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ are defined as the space $\dot{\Lambda}^{-s}\mathcal{L}^{p,r}(\mathbb{R}^d)$, equipped with the norm

$$\big\|u\big\|_{\dot{H}^s_{\mathcal{L}^{p,r}}} := \big\|\dot{\Lambda}^s u\big\|_{\mathcal{L}^{p,r}}.$$

Theorem 2.4.1. (*Holder's inequality in Fourier-Lorentz spaces*).
Let $1 < r, q, \tilde{q} < \infty$ and $1 \leq h, \tilde{h}, \hat{h} \leq +\infty$ satisfy the relations

$$\frac{1}{r} = \frac{1}{q} + \frac{1}{\tilde{q}} \text{ and } \frac{1}{h} = \frac{1}{\tilde{h}} + \frac{1}{\hat{h}}.$$

Suppose that $u \in \mathcal{L}^{q,\tilde{h}}$ and $v \in \mathcal{L}^{\tilde{q},\hat{h}}$. Then $uv \in \mathcal{L}^{r,h}$ and we have the inequality

$$\big\|uv\big\|_{\mathcal{L}^{r,h}} \lesssim \big\|u\big\|_{\mathcal{L}^{q,\tilde{h}}}\big\|v\big\|_{\mathcal{L}^{\tilde{q},\hat{h}}}. \tag{2.72}$$

Proof. Let r', q', and $\tilde{q}'$ be such that

$$\frac{1}{r} + \frac{1}{r'} = 1, \frac{1}{q} + \frac{1}{q'} = 1, \text{ and } \frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1.$$

It is easily checked that the following conditions are satisfied

$$1 < r', q', \tilde{q}' < +\infty \text{ and } \frac{1}{r'} + 1 = \frac{1}{q'} + \frac{1}{\tilde{q}'}.$$

We have

$$\|uv\|_{\mathcal{L}^{r,h}} = \|\widehat{uv}\|_{L^{r',h}} = \frac{1}{(2\pi)^{d/2}}\|\hat{u} * \hat{v}\|_{L^{r',h}}. \tag{2.73}$$

Applying Theorem 1.1.10 (c), we have

$$\|\hat{u} * \hat{v}\|_{L^{r',h}} \lesssim \|\hat{u}\|_{L^{q',\tilde{h}}}\|\hat{v}\|_{L^{\tilde{q}',\hat{h}}} = \|u\|_{\mathcal{L}^{q,\tilde{h}}}\|v\|_{\mathcal{L}^{\tilde{q},\hat{h}}}. \tag{2.74}$$

Now, the estimate (2.72) follows from the equality (2.73) and the inequality (2.74). □

Theorem 2.4.2. (*Young's inequality for convolution in Fourier-Lorentz spaces*). *Let $1 < r, q, \tilde{q} < \infty$, and $1 \le h, \tilde{h}, \hat{h} \le \infty$ satisfy the relations*

$$\frac{1}{r} + 1 = \frac{1}{q} + \frac{1}{\tilde{q}} \text{ and } \frac{1}{h} = \frac{1}{\tilde{h}} + \frac{1}{\hat{h}}.$$

Suppose that $u \in L^{q,\tilde{h}}$ and $v \in L^{\tilde{q},\hat{h}}$. Then $u * v \in L^{r,h}$ and the following inequality holds

$$\|u * v\|_{\mathcal{L}^{r,h}} \lesssim \|u\|_{\mathcal{L}^{q,\tilde{h}}}\|v\|_{\mathcal{L}^{\tilde{q},\hat{h}}}. \tag{2.75}$$

Proof. Let r', q', and $\tilde{q}'$ be such that

$$\frac{1}{r} + \frac{1}{r'} = 1, \frac{1}{q} + \frac{1}{q'} = 1, \text{ and } \frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1.$$

By definition

$$\|u * v\|_{\mathcal{L}^{r,h}} = \|\widehat{u * v}\|_{L^{r',h}} = (2\pi)^{d/2}\|\hat{u}\hat{v}\|_{L^{r',h}}. \tag{2.76}$$

We can check that the following conditions are satisfied

$$1 < r', q', \tilde{q}' < +\infty \text{ and } \frac{1}{r'} = \frac{1}{q'} + \frac{1}{\tilde{q}'}.$$

Applying Theorem 1.1.9 (c), we have

$$\|\hat{u}\hat{v}\|_{L^{r',h}} \lesssim \|\hat{u}\|_{L^{q',\tilde{h}}}\|\hat{v}\|_{L^{\tilde{q}',\hat{h}}} = \|u\|_{\mathcal{L}^{q,\tilde{h}}}\|v\|_{\mathcal{L}^{\tilde{q},\hat{h}}}. \tag{2.77}$$

Now, the estimate (2.75) follows from the equality (2.76) and the inequality (2.77). □

Theorem 2.4.3. (*Sobolev inequality for Sobolev-Fourier-Lorentz spaces*). *Let $1 < q \le \tilde{q} < \infty, s, \tilde{s} \in \mathbb{R}, s - \frac{d}{q} = \tilde{s} - \frac{d}{\tilde{q}}$, and $1 \le r \le \infty$. Then*

$$\|u\|_{\dot{H}^{\tilde{s}}_{\mathcal{L}^{\tilde{q},r}}} \lesssim \|u\|_{\dot{H}^{s}_{\mathcal{L}^{q,r}}}, \forall u \in \dot{H}^{s}_{\mathcal{L}^{q,r}}. \tag{2.78}$$

Proof. We have

$$\|u\|_{\dot{H}^{\tilde{s}}_{\mathcal{L}^{\tilde{q},r}}} = \|\dot{\Lambda}^{\tilde{s}-s}\dot{\Lambda}^{s}u\|_{\mathcal{L}^{\tilde{q},r}} = \||\xi|^{\tilde{s}-s}\widehat{\dot{\Lambda}^{s}u}(\xi)\|_{L^{\tilde{q}',r}}, \tag{2.79}$$

where

$$\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1.$$

Note that

$$|\xi|^{-r} \in L^{\frac{d}{r},\infty}(\mathbb{R}^d) \text{ for all } r \text{ satisfying } 0 < r \le d.$$

Applying Theorem 1.1.9 (c), we have

$$\||\xi|^{\tilde{s}-s}\widehat{\dot{\Lambda}^{s}u}(\xi)\|_{L^{\tilde{q}',r}} \lesssim \||\xi|^{\tilde{s}-s}\|_{L^{\frac{d}{s-\tilde{s}},\infty}}\|\widehat{\dot{\Lambda}^{s}u}(\xi)\|_{L^{q',r}} \simeq \|u\|_{\dot{H}^{s}_{\mathcal{L}^{q,r}}}. \tag{2.80}$$

The estimate (2.78) follows from the equality (2.79) and the inequality (2.80). □

Lemma 2.4.4. *Let $s \in \mathbb{R}, 1 \le p \le \infty,$ and $1 \le r \le \tilde{r} \le \infty$.*
(a) *We have the following imbedding maps*

$$\mathcal{L}^{p,1} \hookrightarrow \mathcal{L}^{p,r} \hookrightarrow \mathcal{L}^{p,\tilde{r}} \hookrightarrow \mathcal{L}^{p,\infty},$$
$$\dot{H}^s_{\mathcal{L}^{p,1}} \hookrightarrow \dot{H}^s_{\mathcal{L}^{p,r}} \hookrightarrow \dot{H}^s_{\mathcal{L}^{p,\tilde{r}}} \hookrightarrow \dot{H}^s_{\mathcal{L}^{p,\infty}}.$$

(b) $\dot{H}^s_{\mathcal{L}^p} = \dot{H}^s_{\mathcal{L}^{p,p'}}$ *(equality of the norm), where* $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. It is easily deduced from the properties of the standard Lorentz spaces. □

Lemma 2.4.5. *Let $s \in \mathbb{R}$ and $1 < p < \infty$. We have*
(a) *If $1 < q \le 2$ then $\dot{H}^s_q \hookrightarrow \dot{H}^s_{\mathcal{L}^q}$.*
(b) *If $2 \le q < \infty$ then $\dot{H}^s_{\mathcal{L}^q} \hookrightarrow \dot{H}^s_q$.*

Proof. It is deduced from Theorem 1.2.1 ([2], p. 6). □

Lemma 2.4.6. *Assume that $1 \le r, p \le \infty$ and $k \in \mathbb{N}$, then the two quantities*

$$\big\|u\big\|_{\dot{H}^k_{\mathcal{L}^{p,r}}} \text{ and } \sum_{|\alpha|=k} \big\|\partial^\alpha u\big\|_{\mathcal{L}^{p,r}}$$

are equivalent.

Proof. First, we prove that

$$\sum_{|\alpha|=k} \big\|\partial^\alpha u\big\|_{\mathcal{L}^{p,r}} \lesssim \big\|u\big\|_{\dot{H}^k_{\mathcal{L}^{p,r}}}.$$

We have

$$\sum_{|\alpha|=k} \big\|\partial^\alpha u\big\|_{\mathcal{L}^{p,r}} = \sum_{|\alpha|=k} \big\|i^k \xi^\alpha \hat{u}(\xi)\big\|_{L^{p',r}} = \sum_{|\alpha|=k} \Big\|\frac{\xi^\alpha}{|\xi|^k}|\xi|^k \hat{u}(\xi)\Big\|_{L^{p',r}}$$
$$\le \sum_{|\alpha|=k} \big\||\xi|^k \hat{u}(\xi)\big\|_{L^{p',r}} \lesssim \big\|\widehat{\dot{\Lambda}^k u}(\xi)\big\|_{L^{p',r}} = \big\|u\big\|_{\dot{H}^k_{\mathcal{L}^{p,r}}}.$$

Next, we prove that

$$\big\|u\big\|_{\dot{H}^k_{\mathcal{L}^{p,r}}} \lesssim \sum_{|\alpha|=k} \big\|\partial^\alpha u\big\|_{\mathcal{L}^{p,r}}.$$

It is easy to see that for all $\xi \in \mathbb{R}^d$, we have

$$|\xi|^k \le d^{\frac{k}{2}} \sum_{|\alpha|=k} |\xi^\alpha|.$$

This gives the desired result

$$\big\|u\big\|_{\dot{H}^k_{\mathcal{L}^{p,r}}} = \big\||\xi|^k \hat{u}(\xi)\big\|_{L^{p',r}} \le d^{\frac{k}{2}} \Big\|\sum_{|\alpha|=k} |\xi^\alpha| \hat{u}(\xi)\Big\|_{L^{p',r}}$$
$$\le d^{\frac{k}{2}} \sum_{|\alpha|=k} \big\|\xi^\alpha \hat{u}(\xi)\big\|_{L^{p',r}} = d^{\frac{k}{2}} \sum_{|\alpha|=k} \big\|\partial^\alpha u\big\|_{\mathcal{L}^{p,r}}.$$

□

Lemma 2.4.7. *Let $k \in \mathbb{N}, p \in \mathbb{R}$, and $r \in \mathbb{R}$ be such that*

$$0 \le k \le d-1, \frac{k}{d} < \frac{1}{p} < \frac{1}{2} + \frac{k}{2d}, \text{ and } 1 \le r \le \infty.$$

Then the following inequality holds

$$\|uv\|_{\dot{H}^k_{\mathcal{L}^{q,r}}} \lesssim \|u\|_{\dot{H}^k_{\mathcal{L}^{p,r}}} \|v\|_{\dot{H}^k_{\mathcal{L}^{p,r}}}, \ \forall u, v \in \dot{H}^k_{\mathcal{L}^{p,r}},$$

where

$$\frac{1}{q} = \frac{2}{p} - \frac{k}{d}.$$

Proof. First, we estimate $\|\partial^\alpha(uv)\|_{\mathcal{L}^{q,r}}$, where

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{N}^d, \ |\alpha| = \sum_{i=1}^d \alpha_i = k.$$

By the general Leibniz rule, we have

$$\partial^\alpha(uv) = \sum_{\gamma+\beta=\alpha} \binom{\alpha}{\gamma} (\partial^\gamma u)(\partial^\beta v).$$

Set

$$\frac{1}{q_1} = \frac{1}{p} - \frac{k-|\gamma|}{d}, \frac{1}{q_2} = \frac{1}{p} - \frac{k-|\beta|}{d}.$$

We have

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{2}{p} - \frac{2k}{d} + \frac{|\gamma|+|\beta|}{d} = \frac{2}{p} - \frac{k}{d} = \frac{1}{q}.$$

Therefore applying Theorems 2.4.1, 2.4.3, and Lemma 2.4.4 (a) in order to obtain

$$\|(\partial^\gamma u)(\partial^\beta v)\|_{\mathcal{L}^{q,r}} \lesssim \|\partial^\gamma u\|_{\mathcal{L}^{q_1,r}} \|\partial^\beta v\|_{\mathcal{L}^{q_2,\infty}} \lesssim \|\partial^\gamma u\|_{\dot{H}^{k-|\gamma|}_{\mathcal{L}^{p,r}}} \|\partial^\beta v\|_{\dot{H}^{k-|\beta|}_{\mathcal{L}^{p,\infty}}}$$
$$\lesssim \|\partial^\gamma u\|_{\dot{H}^{k-|\gamma|}_{\mathcal{L}^{p,r}}} \|\partial^\beta v\|_{\dot{H}^{k-|\beta|}_{\mathcal{L}^{p,r}}} \lesssim \|u\|_{\dot{H}^k_{\mathcal{L}^{p,r}}} \|v\|_{\dot{H}^k_{\mathcal{L}^{p,r}}}.$$

Thus, for all $\alpha \in \mathbb{N}^d$ with $|\alpha| = k$, we have

$$\|\partial^\alpha(uv)\|_{\mathcal{L}^{q,r}} \lesssim \|u\|_{\dot{H}^k_{\mathcal{L}^{p,r}}} \|v\|_{\dot{H}^k_{\mathcal{L}^{p,r}}}.$$

Applying Lemma 2.4.6, we have

$$\|uv\|_{\dot{H}^k_{\mathcal{L}^{q,r}}} \lesssim \|u\|_{\dot{H}^k_{\mathcal{L}^{p,r}}} \|v\|_{\dot{H}^k_{\mathcal{L}^{p,r}}}, \ \forall u, v \in \dot{H}^k_{\mathcal{L}^{p,r}}.$$

□

Lemma 2.4.8. *Assume that $1 \le p, r \le \infty$ and $s \in \mathbb{R}$. If $u_0 \in \dot{H}^s_{\mathcal{L}^{p,r}}$ then $e^{\cdot\Delta}u_0 \in L^\infty([0,\infty); \dot{H}^s_{\mathcal{L}^{p,r}})$ and*

$$\|e^{\cdot\Delta}u_0\|_{L^\infty([0,\infty);\dot{H}^s_{\mathcal{L}^{p,r}})} \le \|u_0\|_{\dot{H}^s_{\mathcal{L}^{p,r}}}.$$

Proof. For $t \ge 0$, we have

$$\|e^{t\Delta}u_0\|_{\dot{H}^s_{\mathcal{L}^{p,r}}} = \|e^{t\Delta}\dot{\Lambda}^s u_0\|_{\mathcal{L}^{p,r}} = \|e^{-t|\xi|^2}|\xi|^s \hat{u}_0\|_{L^{p',r}} \le$$
$$\||\xi|^s \hat{u}_0\|_{L^{p',r}} = \|\widehat{\dot{\Lambda}^s u_0(\xi)}\|_{L^{p',r}} = \|\dot{\Lambda}^s u_0(\xi)\|_{\mathcal{L}^{p,r}} = \|u_0\|_{\dot{H}^s_{\mathcal{L}^{p,r}}}.$$

□

2.4.2. Solutions to the Navier-Stokes equations with the initial value in the critical spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $1 < p \le d$ and $1 \le r < \infty$

We define an auxiliary space $\mathcal{K}^{\tilde{p}}_{p,r,T}$ which is made up by the functions $u(t,x)$ such that

$$\big\|u\big\|_{\mathcal{K}^{\tilde{p}}_{p,r,T}} := \sup_{0<t<T} t^{\frac{\alpha}{2}} \Big\|u(t,x)\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{\tilde{p},r}}} < \infty,$$

and

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \Big\|u(t,x)\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{\tilde{p},r}}} = 0, \tag{2.81}$$

with

$$1 < p \le \tilde{p} < \infty, \frac{1}{p} - \frac{1}{d} < \frac{1}{\tilde{p}}, 1 \le r \le \infty, T > 0,$$

and

$$\alpha = \alpha(p,\tilde{p}) = d\Big(\frac{1}{p} - \frac{1}{\tilde{p}}\Big).$$

In the case $\tilde{p} = p$, it is also convenient to define the space $\mathcal{K}^{p}_{p,r,T}$ as the natural space $L^\infty\big([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}\big)$ with the additional condition that its elements $u(t,x)$ satisfy

$$\lim_{t\to 0} \Big\|u(t,x)\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}} = 0. \tag{2.82}$$

Lemma 2.4.9. *Let $1 \le r \le \tilde{r} \le \infty$. Then we have the following imbedding*

$$\mathcal{K}^{\tilde{p}}_{p,1,T} \hookrightarrow \mathcal{K}^{\tilde{p}}_{p,r,T} \hookrightarrow \mathcal{K}^{\tilde{p}}_{p,\tilde{r},T} \hookrightarrow \mathcal{K}^{\tilde{p}}_{p,\infty,T}.$$

Proof. It is easily deduced from Lemma 2.4.4 (a) and the definition of $\mathcal{K}^{\tilde{p}}_{p,r,T}$. □

Lemma 2.4.10. *Suppose that $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $1 < p \le d$ and $1 \le r < \infty$, then $e^{\cdot\Delta}u_0 \in \mathcal{K}^{\tilde{p}}_{p,1,\infty}$ with $\frac{1}{p} - \frac{1}{d} < \frac{1}{\tilde{p}} < \frac{1}{p}$.*

Proof. Before proving this lemma, we need to prove the following lemma.

Lemma 2.4.11. *Suppose that $u_0 \in L^{q,r}(\mathbb{R}^d)$ with $1 \le q \le \infty$ and $1 \le r < \infty$. Then $\lim\limits_{n\to\infty} \big\|1_{B_n^c} u_0\big\|_{L^{q,r}} = 0$, where $n \in \mathbb{N}, B_n = \{x \in \mathbb{R}^d : |x| < n\}, B_n^c = \mathbb{R}^d \backslash B_n$, and $1_{B_n^c}$ is the indicator function of the set B_n^c on $\mathbb{R}^d : 1_{B_n^c}(x) = 1$ for $x \in B_n^c$ and $1_{B_n^c}(x) = 0$ otherwise.*

Proof. With $\delta > 0$ being fixed, we have

$$\big\{x : |1_{B_n^c}u_0(x)| > \delta\big\} \supseteq \big\{x : |1_{B_{n+1}^c}u_0(x)| > \delta\big\}, \tag{2.83}$$

and

$$\bigcap_{n=0}^{\infty} \{x : |1_{B_n^c}u_0(x)| > \delta\} = \emptyset. \tag{2.84}$$

Note that

$$\mathcal{L}^d\big(\{x : |1_{B_0^c}u_0(x)| > \delta\}\big) = \mathcal{L}^d\big(\{x : |u_0(x)| > \delta\}\big),$$

where $\mathcal{L}^d$ being the Lebesgue measure in $\mathbb{R}^d$. We prove that

$$\mathcal{L}^d\big(\{x : |u_0(x)| > \delta\}\big) < \infty, \tag{2.85}$$

assuming on the contrary

$$\mathcal{L}^d\big(\{x : |u_0(x)| > \delta\}\big) = \infty.$$

Set
$$u_0^*(t) = \inf\big\{\tau : \mathcal{L}^d\big(\{x : |u_0(x)| > \tau\}\big) \le t\big\}.$$
We have $u_0^*(t) \ge \delta$ for all $t > 0$, from the definition of the Lorentz space, we get
$$\big\|u_0\big\|_{L^{q,r}} = \Big(\int_0^\infty (t^{\frac{1}{q}} u_0^*(t))^r \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{r}} \ge \Big(\int_0^\infty (t^{\frac{1}{q}}\delta)^r \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{r}} = \delta\Big(\int_0^\infty t^{\frac{r}{q}-1}\mathrm{d}t\Big)^{\frac{1}{r}} = \infty,$$
a contradiction.
From (2.83), (2.84), and (2.85), we have
$$\lim_{n\to\infty} \mathcal{L}^d\big(\{x : |1_{B_n^c} u_0(x)| > \delta\}\big) = 0. \tag{2.86}$$
Set
$$u_n^*(t) = \inf\big\{\tau : \mathcal{L}^d\big(\{x : |1_{B_n^c} u_0(x)| > \tau\}\big) \le t\big\}.$$
We have
$$u_n^*(t) \ge u_{n+1}^*(t). \tag{2.87}$$
Fix $t > 0$. For any $\epsilon > 0$, from (2.86) it follows that there exists $n_0 = n_0(t, \epsilon)$ large enough such that
$$\mathcal{L}^d\big(\{x : |1_{B_n^c} u_0(x)| > \epsilon\}\big) \le t, \forall n \ge n_0.$$
From this we deduce that
$$u_n^*(t) \le \epsilon, \forall n \ge n_0,$$
therefore
$$\lim_{n\to\infty} u_n^*(t) = 0. \tag{2.88}$$
From (2.87) and (2.88), we apply Lebesgue's monotone convergence theorem to get
$$\lim_{n\to\infty} \big\|1_{B_n^c} u_0\big\|_{L^{q,r}} = \lim_{n\to\infty} \Big(\int_0^\infty (t^{\frac{1}{q}} u_n^*(t))^r \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{r}} = 0.$$
□

Now we return to prove Lemma 2.4.10. We prove that
$$\sup_{0<t<\infty} t^{\frac{\alpha}{2}} \Big\|e^{t\Delta} u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{\tilde{p},1}}} \lesssim \Big\|u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}}. \tag{2.89}$$
Let p' and $\tilde{p}'$ be such that
$$\frac{1}{p} + \frac{1}{p'} = 1 \text{ and } \frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'} = 1.$$
We have
$$\Big\|e^{t\Delta} u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{\tilde{p},1}}} = \big\|e^{-t|\xi|^2}|\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L_\xi^{\tilde{p}',1}}. \tag{2.90}$$
Applying Theorem 1.1.9 (c), we have
$$\begin{aligned}\big\|e^{-t|\xi|^2}|\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L_\xi^{\tilde{p}',1}} &\lesssim \big\|e^{-t|\xi|^2}\big\|_{L_\xi^{\frac{p\tilde{p}}{\tilde{p}-p},1}} \big\||\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{p',\infty}} = \\ t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{\tilde{p}})}\big\|e^{-|\xi|^2}\big\|_{L^{\frac{p\tilde{p}}{\tilde{p}-p},1}} \big\||\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{p',\infty}} &\lesssim t^{-\frac{\alpha}{2}} \big\||\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{p',r}} \\ &= t^{-\frac{\alpha}{2}} \Big\|u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}}.\end{aligned} \tag{2.91}$$

The estimate (2.89) follows from the equality (2.90) and the estimate (2.91).
We claim now that

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\Big\|e^{t\Delta}u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{\tilde{p},1}}} = 0. \tag{2.92}$$

From the equality (2.90), we have

$$t^{\frac{\alpha}{2}}\Big\|e^{t\Delta}u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{\tilde{p},1}}} \le t^{\frac{\alpha}{2}}\big\|e^{-t|\xi|^2}1_{B_n^c}|\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L_\xi^{\tilde{p}',1}} + \; t^{\frac{\alpha}{2}}\big\|e^{-t|\xi|^2}1_{B_n}|\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L_\xi^{\tilde{p}',1}}.$$

For any $\epsilon > 0$, applying Holder's inequality in the Lorentz spaces and using Lemma 2.4.11, we have

$$t^{\frac{\alpha}{2}}\big\|e^{-t|\xi|^2}1_{B_n^c}|\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L_\xi^{\tilde{p}',1}} \le Ct^{\frac{\alpha}{2}}\big\|e^{-t|\xi|^2}\big\|_{L_\xi^{\frac{p\tilde{p}}{\tilde{p}-p},1}}\big\|1_{B_n^c}|\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{p',\infty}} =$$

$$C\big\|e^{-|\xi|^2}\big\|_{L^{\frac{p\tilde{p}}{\tilde{p}-p},1}}\big\|1_{B_n^c}|\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{p',\infty}} \le C'\big\|1_{B_n^c}|\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{p',r}} < \frac{\epsilon}{2} \tag{2.93}$$

for large enough n. Fix one of such n and applying Holder's inequality in the Lorentz spaces, we have

$$t^{\frac{\alpha}{2}}\big\|e^{-t|\xi|^2}1_{B_n}|\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L_\xi^{\tilde{p}',1}} \le Ct^{\frac{\alpha}{2}}\big\|1_{B_n}e^{-t|\xi|^2}\big\|_{L_\xi^{\frac{p\tilde{p}}{\tilde{p}-p},1}}\big\||\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{p',\infty}}$$

$$\le Ct^{\frac{\alpha}{2}}\big\|1_{B_n}\big\|_{L^{\frac{p\tilde{p}}{\tilde{p}-p},1}}\big\||\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{p',\infty}} \le C''(n)t^{\frac{\alpha}{2}}\big\||\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{p',r}}$$

$$= C''(n)t^{\frac{\alpha}{2}}\Big\|u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}} < \frac{\epsilon}{2} \tag{2.94}$$

for small enough $t = t(n) > 0$. From estimates (2.93) and (2.94), we have,

$$t^{\frac{\alpha}{2}}\Big\|e^{t\Delta}u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{\tilde{p},1}}} \le C'\big\|1_{B_n^c}|\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{p',r}} + C''(n)t^{\frac{\alpha}{2}}\Big\|u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}} < \epsilon.$$

□

In the following lemmas a particular attention will be devoted to the study of the bilinear operator $B(u,v)(t)$ defined by (1.14) in the Sobolev-Fourier-Lorentz spaces.

Lemma 2.4.12. *Let $1 < p \le d$. Then for all $\tilde{p}$ be such that*

$$\frac{1}{2p} + \frac{[\frac{d}{p}]-1}{2d} < \frac{1}{\tilde{p}} < \min\Big\{\frac{[\frac{d}{p}]}{d}, \frac{1}{2} + \frac{[\frac{d}{p}]-1}{2d}\Big\}, \tag{2.95}$$

the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T} \times \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}$ into $\mathcal{K}^p_{p,1,T}$ and the following inequality holds

$$\big\|B(u,v)\big\|_{\mathcal{K}^p_{p,1,T}} \le C\Big\|u\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}}\Big\|v\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}}, \tag{2.96}$$

where C is a positive constant and independent of T.

Proof. We have

$$\Big\|B(u,v)(t)\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,1}}} \le \int_0^t \Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(u(\tau)\otimes v(\tau))\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,1}}}\mathrm{d}\tau$$

$$= \int_0^t \Big\|\dot{\Lambda}^{\frac{d}{p}-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(u(\tau)\otimes v(\tau))\Big\|_{\mathcal{L}^{p,1}}\mathrm{d}\tau. \tag{2.97}$$

Note that

$$\Big(\dot{\Lambda}^{\frac{d}{p}-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\Big)^{\wedge}_j(\xi)=\Big(\dot{\Lambda}^{\{\frac{d}{p}\}}e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\dot{\Lambda}^{[\frac{d}{p}]-1}\big(u(\tau)\otimes v(\tau)\big)\Big)^{\wedge}_j(\xi)$$

$$=|\xi|^{\{\frac{d}{p}\}}e^{-(t-\tau)|\xi|^2}\sum_{l,k=1}^{d}\Big(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l)\Big(\dot{\Lambda}^{[\frac{d}{p}]-1}\big(u_l(\tau)v_k(\tau)\big)\Big)^{\wedge}(\xi).$$

Thus

$$\Big(\dot{\Lambda}^{\frac{d}{p}-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\Big)_j$$

$$=\frac{1}{(t-\tau)^{\frac{\{\frac{d}{p}\}+d+1}{2}}}\sum_{l,k=1}^{d}K_{l,k,j}\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)*\Big(\dot{\Lambda}^{[\frac{d}{p}]-1}\big(u_l(\tau)v_k(\tau)\big)\Big), \tag{2.98}$$

where

$$\widehat{K_{l,k,j}}(\xi)=\frac{1}{(2\pi)^{d/2}}|\xi|^{\{\frac{d}{p}\}}e^{-|\xi|^2}\Big(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l). \tag{2.99}$$

Setting the tensor $K(x)=\{K_{l,k,j}(x)\}$, we can rewrite the equality (2.98) in the tensor form

$$\dot{\Lambda}^{\frac{d}{p}-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)=\frac{1}{(t-\tau)^{\frac{\{\frac{d}{p}\}+d+1}{2}}}K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)*\Big(\dot{\Lambda}^{[\frac{d}{p}]-1}\big(u(\tau)\otimes v(\tau)\big)\Big).$$

Applying Theorem 2.4.2 for convolution in the Fourier-Lorentz spaces, we have

$$\Big\|\dot{\Lambda}^{\frac{d}{p}-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\Big\|_{\mathcal{L}^{p,1}}\lesssim$$

$$\frac{1}{(t-\tau)^{\frac{\{\frac{d}{p}\}+d+1}{2}}}\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{\mathcal{L}^{r,1}}\big\|\dot{\Lambda}^{[\frac{d}{p}]-1}\big(u(\tau)\otimes v(\tau)\big)\big\|_{\mathcal{L}^{q,\infty}}, \tag{2.100}$$

where

$$\frac{1}{q}=\frac{2}{\tilde{p}}-\frac{[\frac{d}{p}]-1}{d}\text{ and }\frac{1}{r}=1+\frac{1}{p}-\frac{2}{\tilde{p}}+\frac{[\frac{d}{p}]-1}{d}. \tag{2.101}$$

Note that from the inequality (2.95), we can check that r and q satisfy the relations

$$1<r,q<\infty,\frac{1}{p}+1=\frac{1}{q}+\frac{1}{r}.$$

Applying Lemma 2.4.7, we have

$$\big\|\dot{\Lambda}^{[\frac{d}{p}]-1}\big(u(\tau)\otimes v(\tau)\big)\big\|_{\mathcal{L}^{q,\infty}}\lesssim\big\|u(\tau)\big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\big\|v(\tau)\big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}. \tag{2.102}$$

From the equalities (2.99) and (2.101), we obtain

$$\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{\mathcal{L}^{r,1}}=(t-\tau)^{\frac{d}{2}}\Big\|\hat{K}\big(\sqrt{t-\tau}\ (\cdot)\big)\Big\|_{L^{r',1}}=$$

$$(t-\tau)^{\frac{d}{2}-\frac{d}{2.r'}}\big\|\hat{K}\big\|_{L^{r',1}}=(t-\tau)^{\frac{d}{2.r}}\big\|\hat{K}\big\|_{L^{r',1}}\simeq(t-\tau)^{\frac{d}{2}\big(1+\frac{1}{p}-\frac{2}{\tilde{p}}+\frac{[\frac{d}{p}]-1}{d}\big)}. \tag{2.103}$$

From the estimates (2.100), (2.102), and (2.103), we deduce that

$$\Big\|\dot{\Lambda}^{\frac{d}{p}-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\Big\|_{\mathcal{L}^{p,1}}\lesssim(t-\tau)^{[\frac{d}{p}]-\frac{d}{\tilde{p}}-1}\Big\|u(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\Big\|v(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}$$

$$=(t-\tau)^{\alpha-1}\Big\|u(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\Big\|v(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}},$$

where

$$\alpha = \alpha\Big(\frac{d}{[\frac{d}{p}]}, \tilde{p}\Big) = \Big[\frac{d}{p}\Big] - \frac{d}{\tilde{p}} \ ,$$

this gives the desired result

$$\begin{aligned}
\Big\|B(u,v)(t)\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,1}}} &\lesssim \int_0^t (t-\tau)^{\alpha-1}\Big\|u(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\Big\|v(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\mathrm{d}\tau \\
&\lesssim \int_0^t (t-\tau)^{\alpha-1}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\mathrm{d}\tau \\
&= \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\int_0^t (t-\tau)^{\alpha-1}\tau^{-\alpha}\mathrm{d}\tau \\
&\simeq \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}. \qquad (2.104)
\end{aligned}$$

Let us now check the validity of the condition (2.82) for the bilinear term $B(u,v)(t)$. Indeed, from (2.104)

$$\lim_{t\to 0}\Big\|B(u,v)(t)\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,1}}} = 0,$$

whenever

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\Big\|u(t)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}} = \lim_{t\to 0} t^{\frac{\alpha}{2}}\Big\|v(t)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}} = 0.$$

The estimate (2.96) is deduced from the inequality (2.104). □

Lemma 2.4.13. *Let $1 < p \le d$. Then for all $\tilde{p}$ be such that*

$$\frac{[\frac{d}{p}]-1}{d} < \frac{1}{\tilde{p}} < \min\Big\{\frac{[\frac{d}{p}]}{d}, \frac{1}{2} + \frac{[\frac{d}{p}]-1}{2d}\Big\}, \qquad (2.105)$$

the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T} \times \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}$ into $\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T}$ and the following inequality holds

$$\Big\|B(u,v)\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T}} \le C\Big\|u\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}}\Big\|v\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}}, \qquad (2.106)$$

where C is a positive constant and independent of T.

Proof. First, arguing as in Lemma 2.4.12, we derive

$$\dot{\Lambda}^{[\frac{d}{p}]-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big) = \frac{1}{(t-\tau)^{\frac{d+1}{2}}}K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big) * \Big(\dot{\Lambda}^{[\frac{d}{p}]-1}\big(u(\tau)\otimes v(\tau)\big)\Big),$$

where

$$\widehat{K_{l,k,j}}(\xi) = \frac{1}{(2\pi)^{d/2}}e^{-|\xi|^2}\Big(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l). \qquad (2.107)$$

Applying Theorem 2.4.2 for the convolution in the Fourier-Lorentz spaces, we have

$$\begin{aligned}
&\Big\|\dot{\Lambda}^{[\frac{d}{p}]-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\Big\|_{\mathcal{L}^{\tilde{p},1}} \\
&\lesssim \frac{1}{(t-\tau)^{\frac{d+1}{2}}}\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{\mathcal{L}^{r,1}}\big\|\dot{\Lambda}^{[\frac{d}{p}]-1}\big(u(\tau)\otimes v(\tau)\big)\big\|_{\mathcal{L}^{q,\infty}}, \qquad (2.108)
\end{aligned}$$

where

$$\frac{1}{q}=\frac{2}{\tilde{p}}-\frac{[\frac{d}{p}]-1}{d}\text{ and }\frac{1}{r}=1-\frac{1}{\tilde{p}}+\frac{[\frac{d}{p}]-1}{d}. \tag{2.109}$$

Note that from the inequality (2.105), we can check that r and q satisfy the relations

$$1<r,q<\infty, \frac{1}{p}+1=\frac{1}{q}+\frac{1}{r}.$$

Applying Lemma 2.4.7, we have

$$\big\|\dot{\Lambda}^{[\frac{d}{p}]-1}\big(u(\tau)\otimes v(\tau)\big)\big\|_{\mathcal{L}^{q,\infty}}\lesssim\Big\|u(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\Big\|v(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}. \tag{2.110}$$

From the equalities (2.107) and (2.109), we obtain

$$\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{\mathcal{L}^{r,1}}=(t-\tau)^{\frac{d}{2.r}}\big\|\hat{K}\big\|_{L^{r',1}}\simeq(t-\tau)^{\frac{d}{2}\big(1-\frac{1}{\tilde{p}}+\frac{[\frac{d}{p}]-1}{d}\big)}. \tag{2.111}$$

From the estimates (2.108), (2.110), and (2.111), we deduce that

$$\Big\|\dot{\Lambda}^{[\frac{d}{p}]-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\Big\|_{\mathcal{L}^{\tilde{p},1}}\lesssim$$
$$(t-\tau)^{\frac{1}{2}([\frac{d}{p}]-\frac{d}{\tilde{p}})-1}\Big\|u(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\Big\|v(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}=(t-\tau)^{\frac{\alpha}{2}-1}\Big\|u(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\Big\|v(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}},$$

where

$$\alpha=\alpha\Big(\frac{d}{[\frac{d}{p}]},\tilde{p}\Big)=\Big[\frac{d}{p}\Big]-\frac{d}{\tilde{p}}\ ,$$

this gives the desired result

$$\begin{aligned}\Big\|B(u,v)(t)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},1}}}&\lesssim\int_0^t(t-\tau)^{\frac{\alpha}{2}-1}\Big\|u(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\Big\|v(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\mathrm{d}\tau\\ &\leq\int_0^t(t-\tau)^{\frac{\alpha}{2}-1}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\mathrm{d}\tau\\ &=\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\int_0^t(t-\tau)^{\frac{\alpha}{2}-1}\tau^{-\alpha}\mathrm{d}\tau\\ &\simeq t^{-\frac{\alpha}{2}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}.\end{aligned} \tag{2.112}$$

Now we check the validity of the condition (2.81) for the bilinear term $B(u,v)(t)$. From (2.112) we infer that

$$\lim_{t\to 0}t^{\frac{\alpha}{2}}\Big\|B(u,v)(t)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},1}}}=0,$$

whenever

$$\lim_{t\to 0}t^{\frac{\alpha}{2}}\Big\|u(t)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}=\lim_{t\to 0}t^{\frac{\alpha}{2}}\Big\|v(t)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}=0.$$

Finally, the estimate (2.106) can be deduced from the inequality (2.112). □

Theorem 2.4.14. *Let $1 < p \le d$ and $1 \le r < \infty$. Then for all $\tilde{p}$ be such that*

$$\frac{1}{2p} + \frac{[\frac{d}{p}] - 1}{2d} < \frac{1}{\tilde{p}} < \min\Big\{\frac{[\frac{d}{p}]}{d}, \frac{1}{2} + \frac{[\frac{d}{p}] - 1}{2d}\Big\},$$

there exists a positive constant $\delta_{p,\tilde{p},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $\operatorname{div}(u_0) = 0$ satisfying

$$\sup_{0<t<T} t^{\frac{1}{2}([\frac{d}{p}]-\frac{d}{\tilde{p}})} \Big\| e^{t\Delta} u_0 \Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}} \le \delta_{p,\tilde{p},d}, \tag{2.113}$$

NSE has a unique mild solution $u \in \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T} \cap L^\infty\big([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}\big)$.

In particular, the inequality (2.113) *holds for arbitrary $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ when $T(u_0)$ is small enough, and there exists a positive constant $\sigma_{p,\tilde{p},d}$ such that we can take $T = \infty$ whenever $\Big\| u_0 \Big\|_{\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}} \le \sigma_{p,\tilde{p},d}$.*

Proof. From Lemmas 2.4.9 and 2.4.13, the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T} \times \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}$ into $\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T}$ and we have the inequality

$$\Big\| B(u,v) \Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}} \le \Big\| B(u,v) \Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T}} \le C_{p,\tilde{p},d} \Big\| u \Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}} \Big\| v \Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}},$$

where $C_{p,\tilde{p},d}$ is positive constant independent of T. From Theorem 1.5.1 and the above inequality, we deduce that for any $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}$ such that

$$\Big\| e^{\cdot\Delta} u_0 \Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}} = \sup_{0<t<T} t^{\frac{1}{2}([\frac{d}{p}]-\frac{d}{\tilde{p}})} \Big\| e^{t\Delta} u_0 \Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}} \le \frac{1}{4C_{p,\tilde{p},d}},$$

the NSE has a solution u on the interval $(0,T)$ so that

$$u \in \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}. \tag{2.114}$$

From Lemmas 2.4.9 and 2.4.12, and (2.114), we have

$$B(u,u) \in \mathcal{K}^{p}_{p,1,T} \subseteq \mathcal{K}^{p}_{p,r,T} \subseteq L^\infty\big([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}\big).$$

From Lemma 2.4.8, we also have $e^{\cdot\Delta}u_0 \in L^\infty\big([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}\big)$. Therefore

$$u = e^{\cdot\Delta}u_0 - B(u,u) \in L^\infty\big([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}\big).$$

For all $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}$, applying Theorem 2.4.3, we deduce that

$$u_0 \in \dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{d/[\frac{d}{p}],r}}. \tag{2.115}$$

From (2.115), applying Lemma 2.4.10, we get $e^{\cdot\Delta}u_0 \in \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}$. From the definition of $\mathcal{K}^{\tilde{p}}_{p,r,T}$, we deduce that the left-hand side of the inequality (2.113) converges to 0 when T

tends to 0. Therefore the inequality (2.113) holds for arbitrary $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}$ when $T(u_0)$ is small enough. Applying Lemmas 2.4.10 and 2.4.13, we conclude that $u \in \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T}$.

Next, applying Theorem 5.4 ([46], p. 45), we deduce that the two quantities $\Big\|u_0\Big\|_{\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}}$ and $\sup_{0<t<\infty} t^{\frac{1}{2}([\frac{d}{p}]-\frac{d}{\tilde{p}})}\Big\|e^{t\Delta}u_0\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}$ are equivalent, then there exists a positive constant $\sigma_{p,\tilde{p},d}$ such that $T=\infty$ and (2.113) holds whenever $\Big\|u_0\Big\|_{\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}} \leq \sigma_{p,\tilde{p},d}$. □

Remark 2.4.5. From Theorem 2.4.3 and the proof of Lemma 2.4.10, and Theorem 5.4 ([46], p. 45), we have the following imbedding maps

$$\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d) \hookrightarrow \dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{d/[\frac{d}{p}],r}}(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},1}}(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}(\mathbb{R}^d).$$

On the other hand, a function in $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ can be arbitrarily large in the $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ norm but small in the $\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}(\mathbb{R}^d)$ norm.

2.4.3. Solutions to the Navier-Stokes equations with the initial value in the critical spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $d \leq p < \infty$ and $1 \leq r < \infty$

Lemma 2.4.15. *Suppose that $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}$ with $d \leq p < \infty$ and $1 \leq r < \infty$. Then $e^{\cdot\Delta}u_0 \in \mathcal{K}^{\tilde{p}}_{d,1,\infty}$ for all $\tilde{p} > p$.*

Proof. We prove that

$$\sup_{0<t<\infty} t^{\frac{\alpha}{2}}\big\|e^{t\Delta}u_0\big\|_{\mathcal{L}^{\tilde{p},1}} \lesssim \Big\|u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}},$$

where

$$\alpha = \alpha(d,\tilde{p}) = 1 - \frac{d}{\tilde{p}}.$$

Let p' and $\tilde{p}'$ be such that

$$\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'} = 1.$$

We have

$$\big\|e^{t\Delta}u_0\big\|_{\mathcal{L}^{\tilde{p},1}} = \big\|e^{-t|\xi|^2}\hat{u}_0(\xi)\big\|_{L^{\tilde{p}',1}_\xi} = \big\|e^{-t|\xi|^2}|\xi|^{1-\frac{d}{p}}|\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{\tilde{p}',1}_\xi}.$$

Applying Holder's inequality in the Lorentz spaces to obtain

$$\begin{aligned}\Big\|e^{-t|\xi|^2}|\xi|^{1-\frac{d}{p}}|\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\Big\|_{L^{\tilde{p}',1}_\xi} &= \Big\|e^{-t|\xi|^2}|\xi|^{1-\frac{d}{p}}\Big\|_{L^{\frac{p\tilde{p}}{\tilde{p}-p},1}_\xi}\big\||\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{p',\infty}}\\ &= t^{-\frac{1}{2}(1-\frac{d}{\tilde{p}})}\Big\|e^{-|\xi|^2}|\xi|^{1-\frac{d}{p}}\Big\|_{L^{\frac{p\tilde{p}}{\tilde{p}-p},1}}\big\||\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{p',\infty}}\\ &\simeq t^{-\frac{\alpha}{2}}\big\||\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{p',\infty}} \lesssim t^{-\frac{\alpha}{2}}\big\||\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{p',r}} = t^{-\frac{\alpha}{2}}\Big\|u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}}.\end{aligned}$$

Therefore this gives the desired result

$$\big\|e^{t\Delta}u_0\big\|_{\mathcal{L}^{\tilde{p},1}} \lesssim t^{-\frac{\alpha}{2}}\Big\|u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}}.$$

We claim now that

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|e^{t\Delta}u_0\big\|_{\mathcal{L}^{\tilde p,1}}=0.$$

For any $\epsilon>0$, applying Lemma 2.4.11 and from the above proof we deduce that

$$\begin{aligned}
t^{\frac{\alpha}{2}}\big\|e^{t\Delta}u_0\big\|_{\mathcal{L}^{\tilde p,1}} &\le t^{\frac{\alpha}{2}}\Big\|e^{-t|\xi|^2}|\xi|^{1-\frac{d}{p}}1_{B_n^c}|\xi|^{\frac{d}{p}-1}\hat u_0(\xi)\Big\|_{L_\xi^{\tilde p',1}}+t^{\frac{\alpha}{2}}\Big\|e^{-t|\xi|^2}|\xi|^{1-\frac{d}{p}}1_{B_n}|\xi|^{\frac{d}{p}-1}\hat u_0(\xi)\Big\|_{L_\xi^{\tilde p',1}}\\
&\le C_1\Big\|e^{-|\xi|^2}|\xi|^{1-\frac{d}{p}}\Big\|_{L^{\frac{p\tilde p}{\tilde p-p},1}}\Big\|1_{B_n^c}|\xi|^{\frac{d}{p}-1}\hat u_0(\xi)\Big\|_{L^{p',\infty}}+\ C_2t^{\frac{\alpha}{2}}\Big\|1_{B_n}|\xi|^{1-\frac{d}{p}}\Big\|_{L^{\frac{p\tilde p}{\tilde p-p},1}}\big\||\xi|^{\frac{d}{p}-1}\hat u_0(\xi)\big\|_{L^{p',\infty}}\\
&\le C_3\Big\|1_{B_n^c}|\xi|^{\frac{d}{p}-1}\hat u_0(\xi)\Big\|_{L^{p',r}}+C_4(n)t^{\frac{\alpha}{2}}\Big\|u_0\Big\|_{\dot H^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}}<\epsilon
\end{aligned}$$

for large enough n and small enough $t=t(n)>0$. □

Lemma 2.4.16. *Let*

$$p\ge d \text{ and } d<\tilde p<2p. \tag{2.116}$$

Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde p}_{d,\infty,T}\times\mathcal{K}^{\tilde p}_{d,\infty,T}$ into $\mathcal{K}^{p}_{p,1,T}$, and we have the inequality

$$\big\|B(u,v)\big\|_{\mathcal{K}^{p}_{p,1,T}}\le C\big\|u\big\|_{\mathcal{K}^{\tilde p}_{d,\infty,T}}\big\|v\big\|_{\mathcal{K}^{\tilde p}_{d,\infty,T}}, \tag{2.117}$$

where C is a positive constant and independent of T.

Proof. First, arguing as in Lemma 2.4.12, we derive

$$\dot\Lambda^{\frac{d}{p}-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)=\frac{1}{(t-\tau)^{\frac{d}{2}(\frac{1}{p}+1)}}K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)*\big(u(\tau)\otimes v(\tau)\big),$$

where the tensor $K(x)=\{K_{l,k,j}(x)\}$ is given by the formula

$$\widehat{K_{l,k,j}}(\xi)=\frac{1}{(2\pi)^{d/2}}|\xi|^{\frac{d}{p}-1}e^{-|\xi|^2}\Big(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l). \tag{2.118}$$

Applying Theorem 2.4.2 for the convolution in the Fourier-Lorentz spaces, we have

$$\begin{aligned}
&\Big\|\dot\Lambda^{\frac{d}{p}-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\Big\|_{\mathcal{L}^{p,1}}\\
&\lesssim\frac{1}{(t-\tau)^{\frac{d}{2}(\frac{1}{p}+1)}}\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{\mathcal{L}^{r,1}}\big\|\big(u(\tau)\otimes v(\tau)\big)\big\|_{\mathcal{L}^{\frac{\tilde p}{2},\infty}},
\end{aligned} \tag{2.119}$$

where

$$\frac{1}{r}=1+\frac{1}{p}-\frac{2}{\tilde p}. \tag{2.120}$$

Note that from the inequality (2.116), we can check that $1<r<\infty$. Applying Theorem 2.4.1, we have

$$\big\|u(\tau)\otimes v(\tau)\big\|_{\mathcal{L}^{\frac{\tilde p}{2},\infty}}\lesssim\big\|u(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\big\|v(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}. \tag{2.121}$$

From the equalities (2.118) and (2.120) it follows that

$$\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{\mathcal{L}^{r,1}}=(t-\tau)^{\frac{d}{2r}}\big\|\hat K\big\|_{L^{r',1}}\simeq(t-\tau)^{\frac{d}{2}(1+\frac{1}{p}-\frac{2}{\tilde p})}. \tag{2.122}$$

From the estimates (2.119), (2.121), and (2.122), we deduce that

$$\begin{aligned}\Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\Big\|_{\dot H^{\frac{d}{p}-1}_{\mathcal{L}^{p,1}}} &\lesssim (t-\tau)^{-\frac{d}{\tilde p}}\big\|u(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\big\|v(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\\ &= (t-\tau)^{\alpha-1}\big\|u(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\big\|v(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}},\end{aligned}$$

where

$$\alpha=\alpha(d,\tilde p)=1-\frac{d}{\tilde p}.$$

This gives the desired result

$$\begin{aligned}\Big\|B(u,v)(t)\Big\|_{\dot H^{\frac{d}{p}-1}_{\mathcal{L}^{p,1}}} &\lesssim \int_0^t (t-\tau)^{\alpha-1}\big\|u(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\big\|v(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\mathrm{d}\tau\\ &\le \int_0^t (t-\tau)^{\alpha-1}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{\mathcal{L}^{\tilde p,\infty}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{\mathcal{L}^{\tilde p,\infty}}\mathrm{d}\tau\\ &= \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{\mathcal{L}^{\tilde p,\infty}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{\mathcal{L}^{\tilde p,\infty}}\int_0^t (t-\tau)^{\alpha-1}\tau^{-\alpha}\mathrm{d}\tau\\ &\simeq \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{\mathcal{L}^{\tilde p,\infty}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{\mathcal{L}^{\tilde p,\infty}}.\end{aligned}\tag{2.123}$$

From (2.123) it follows the validity of (2.82) since

$$\lim_{t\to 0}\Big\|B(u,v)(t)\Big\|_{\dot H^{\frac{d}{p}-1}_{\mathcal{L}^{p,1}}}=0,$$

whenever

$$\lim_{t\to 0}t^{\frac{\alpha}{2}}\big\|u(t)\big\|_{\mathcal{L}^{\tilde p,\infty}}=\lim_{t\to 0}t^{\frac{\alpha}{2}}\big\|v(t)\big\|_{\mathcal{L}^{\tilde p,\infty}}=0.$$

The estimate (2.117) can be deduced from the inequality (2.123). □

Lemma 2.4.17. *Let $\tilde p>d$, then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde p}_{d,\infty,T}\times\mathcal{K}^{\tilde p}_{d,\infty,T}$ into $\mathcal{K}^{\tilde p}_{d,1,T}$, and we have the inequality*

$$\big\|B(u,v)\big\|_{\mathcal{K}^{\tilde p}_{d,1,T}}\le C\big\|u\big\|_{\mathcal{K}^{\tilde p}_{d,\infty,T}}\big\|v\big\|_{\mathcal{K}^{\tilde p}_{d,\infty,T}},\tag{2.124}$$

where C is a positive constant and independent of T.

Proof. First, arguing as in Lemma 2.4.12, we derive

$$e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)=\frac{1}{(t-\tau)^{\frac{d+1}{2}}}K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)*\big(u(\tau)\otimes v(\tau)\big),$$

where the tensor $K(x)=\{K_{l,k,j}(x)\}$ is given by the formula (2.107).
Applying Theorem 2.4.2 for the convolution in the Fourier-Lorentz spaces, we have

$$\begin{aligned}&\Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\Big\|_{\mathcal{L}^{\tilde p,1}}\\ &\lesssim \frac{1}{(t-\tau)^{\frac{d+1}{2}}}\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{\mathcal{L}^{r,1}}\big\|\big(u(\tau)\otimes v(\tau)\big)\big\|_{\mathcal{L}^{\frac{\tilde p}{2},\infty}},\end{aligned}\tag{2.125}$$

where

$$\frac{1}{r}=1-\frac{1}{\tilde p}.\tag{2.126}$$

Applying Theorem 2.4.1, we have

$$\big\|u(\tau)\otimes v(\tau)\big\|_{\mathcal{L}^{\frac{\tilde p}{2},\infty}} \lesssim \big\|u(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\big\|v(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}. \tag{2.127}$$

From the equalities (2.107) and (2.126) it follows that

$$\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{\mathcal{L}^{r,1}} = (t-\tau)^{\frac{d}{2r}}\big\|\hat K\big\|_{L^{r',1}} \simeq (t-\tau)^{\frac{d}{2}(1-\frac{1}{\tilde p})}. \tag{2.128}$$

From the estimates (2.125), (2.127), and (2.128), we deduce that

$$\begin{aligned}\Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\Big\|_{\mathcal{L}^{\tilde p,1}} &\lesssim (t-\tau)^{-\frac{1}{2}(\frac{d}{\tilde p}+1)}\big\|u(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\big\|v(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\\ &= (t-\tau)^{\frac{\alpha}{2}-1}\big\|u(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\big\|v(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}},\end{aligned}$$

where

$$\alpha=\alpha(d,\tilde p)=1-\frac{d}{\tilde p}.$$

This gives the desired result

$$\begin{aligned}\big\|B(u,v)(t)\big\|_{\mathcal{L}^{\tilde p,1}} &\lesssim \int_0^t (t-\tau)^{\frac{\alpha}{2}-1}\big\|u(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\big\|v(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\mathrm{d}\tau\\ &\le \int_0^t (t-\tau)^{\frac{\alpha}{2}-1}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{\mathcal{L}^{\tilde p,\infty}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{\mathcal{L}^{\tilde p,\infty}}\mathrm{d}\tau\\ &= \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{\mathcal{L}^{\tilde p,\infty}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{\mathcal{L}^{\tilde p,\infty}}\int_0^t (t-\tau)^{\frac{\alpha}{2}-1}\tau^{-\alpha}\mathrm{d}\tau\\ &\simeq t^{-\frac{\alpha}{2}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{\mathcal{L}^{\tilde p,\infty}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{\mathcal{L}^{\tilde p,\infty}}.\end{aligned} \tag{2.129}$$

From (2.129) it follows the validity of (2.81) since

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|B(u,v)(t)\big\|_{\mathcal{L}^{\tilde p,1}}=0,$$

whenever

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|u(t)\big\|_{\mathcal{L}^{\tilde p,\infty}}=\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|v(t)\big\|_{\mathcal{L}^{\tilde p,\infty}}=0.$$

Finally, the estimate (2.124) can be deduced from the inequality (2.129). □

The following lemma is a generalization of Lemma 2.4.17.

Lemma 2.4.18. *Let $d<\tilde p_1<\infty$ and $d\le\tilde p_2<\infty$ be such that one of the following conditions*

$$d<\tilde p_1<2d,\, d\le\tilde p_2<\frac{d\tilde p_1}{2d-\tilde p_1},$$

or

$$\tilde p_1=2d,\, d\le\tilde p_2<\infty,$$

or

$$2d<\tilde p_1<\infty,\, \frac{\tilde p_1}{2}<\tilde p_2<\infty$$

is satisfied.
Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde p_1}_{d,\infty,T}\times\mathcal{K}^{\tilde p_1}_{d,\infty,T}$ into $\mathcal{K}^{\tilde p_2}_{d,1,T}$, and we have the inequality

$$\big\|B(u,v)\big\|_{\mathcal{K}^{\tilde p_2}_{d,1,T}}\le C\big\|u\big\|_{\mathcal{K}^{\tilde p_1}_{d,\infty,T}}\big\|v\big\|_{\mathcal{K}^{\tilde p_1}_{d,\infty,T}},$$

where C is a positive constant and independent of T.

Theorem 2.4.19. *Let $p \geq d$ and $1 \leq r < \infty$. Then for any $\tilde{p}$ such that*

$$\tilde{p} > p, \tag{2.130}$$

there exists a positive constant $\delta_{\tilde{p},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$, with $\operatorname{div}(u_0) = 0$ *satisfying*

$$\sup_{0<t<T} t^{\frac{1}{2}(1-\frac{d}{\tilde{p}})} \big\| e^{t\Delta} u_0 \big\|_{\mathcal{L}^{\tilde{p},\infty}} \leq \delta_{\tilde{p},d}, \tag{2.131}$$

NSE has a unique mild solution $u \in \bigcap_{q>p} \mathcal{K}^q_{d,1,T} \cap L^\infty([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}})$.

In particular, the inequality (2.131) *holds for arbitrary $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $T(u_0)$ is small enough, and there exists a positive constant $\sigma_{\tilde{p},d}$ such that we can take $T = \infty$ whenever $\big\| u_0 \big\|_{\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}} \leq \sigma_{\tilde{p},d}$.*

Proof. The proof of Theorem 2.4.19 is similar to the one of Theorem 2.3.5 by combining Lemmas 2.4.8,2.4.15,2.4.16, 2.4.18, and Theorem 1.5.1. □

Remark 2.4.6. From the proof of Lemma 2.4.15 and Theorem 5.4 ([46], p. 45), we have the following imbedding maps

$$\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},1}}(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}(\mathbb{R}^d).$$

On the other hand, a function in $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ can be arbitrarily large in the $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ norm but small in the $\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}(\mathbb{R}^d)$ norm.

2.4.4. Solutions to the Navier-Stokes equations with the initial value in the critical spaces $\dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$ with $1 \leq r < \infty$

We define an auxiliary space $\mathcal{K}_{s,r,T}$ which consists of functions $u(t,x)$ such that

$$\big\| u \big\|_{\mathcal{K}_{s,r,T}} := \sup_{0<t<T} t^{\frac{\alpha}{2}} \Big\| u(t,x) \Big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} < \infty,$$

and

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \Big\| u(t,x) \Big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} = 0, \tag{2.132}$$

with

$$d-1 \leq s < d, 1 \leq r \leq \infty, T > 0,$$

and

$$\alpha = \alpha(s) = s + 1 - d.$$

In the case $s = d-1$, it is also convenient to define the space $\mathcal{K}_{d-1,r,T}$ as the natural space $L^\infty\big([0,T]; \dot{H}^{d-1}_{\mathcal{L}^{1,r}}\big)$ with the additional condition that its elements $u(t,x)$ satisfy

$$\lim_{t\to 0} \big\| u(t,x) \big\|_{\dot{H}^{d-1}_{\mathcal{L}^{1,r}}} = 0. \tag{2.133}$$

Lemma 2.4.20. *Let $1 \leq r \leq \tilde{r} \leq \infty$. Then we have the following imbeddings*

$$\mathcal{K}_{s,1,T} \hookrightarrow \mathcal{K}_{s,r,T} \hookrightarrow \mathcal{K}_{s,\tilde{r},T} \hookrightarrow \mathcal{K}_{s,\infty,T}.$$

Proof. It is deduced from Lemma 2.4.4 (a) and the definition of $\mathcal{K}_{s,r,T}$. □

Lemma 2.4.21. *Suppose that $u_0 \in \dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$ with $1 \le r < \infty$. Then $e^{\cdot\Delta}u_0 \in \mathcal{K}_{s,r,\infty}$ with $d-1 < s < d$.*

Proof. We prove that

$$\sup_{0<t<\infty} t^{\frac{\alpha}{2}} \Big\| e^{t\Delta}u_0 \Big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} \lesssim \Big\| u_0 \Big\|_{\dot{H}^{d-1}_{\mathcal{L}^{1,r}}} \text{ for } 1 \le r \le \infty. \tag{2.134}$$

We have

$$\begin{aligned} \Big\| e^{t\Delta}u_0 \Big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} &= \big\| e^{-t|\xi|^2}|\xi|^s \hat{u}_0(\xi) \big\|_{L^{\infty,r}_\xi} = \big\| |\xi|^{s+1-d} e^{-t|\xi|^2} |\xi|^{d-1}\hat{u}_0(\xi) \big\|_{L^{\infty,r}_\xi} \\ &\le t^{-\frac{s+1-d}{2}} \big\| |\xi|^{s+1-d} e^{-|\xi|^2} \big\|_{L^\infty} \big\| |\xi|^{d-1}\hat{u}_0(\xi) \big\|_{L^{\infty,r}} \simeq t^{-\frac{\alpha}{2}} \Big\| u_0 \Big\|_{\dot{H}^{d-1}_{\mathcal{L}^{1,r}}}. \end{aligned} \tag{2.135}$$

We claim now that

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \Big\| e^{t\Delta}u_0 \Big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} = 0 \text{ for } 1 \le r < \infty.$$

From the inequality (2.135), we have

$$\begin{aligned} &t^{\frac{\alpha}{2}} \Big\| e^{t\Delta}u_0 \Big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} \le \\ &t^{\frac{\alpha}{2}} \big\| |\xi|^{s+1-d} e^{-t|\xi|^2} 1_{B_n^c} |\xi|^{d-1}\hat{u}_0(\xi) \big\|_{L^{\infty,r}_\xi} + t^{\frac{\alpha}{2}} \big\| |\xi|^{s+1-d} e^{-t|\xi|^2} 1_{B_n} |\xi|^{d-1}\hat{u}_0(\xi) \big\|_{L^{\infty,r}_\xi}. \end{aligned}$$

For any $\epsilon > 0$, applying Lemma 2.4.11, we have

$$\begin{aligned} t^{\frac{\alpha}{2}} \big\| |\xi|^{s+1-d} e^{-t|\xi|^2} 1_{B_n^c} |\xi|^{d-1}\hat{u}_0(\xi) \big\|_{L^{\infty,r}_\xi} &\le \big\| |\xi|^{s+1-d} e^{-|\xi|^2} \big\|_{L^\infty} \big\| 1_{B_n^c} |\xi|^{d-1}\hat{u}_0(\xi) \big\|_{L^{\infty,r}} \\ &= C \big\| 1_{B_n^c} |\xi|^{d-1}\hat{u}_0(\xi) \big\|_{L^{\infty,r}} < \frac{\epsilon}{2}, \end{aligned} \tag{2.136}$$

for large enough n. Fixed one of such n, we have the following estimates

$$\begin{aligned} t^{\frac{\alpha}{2}} \big\| |\xi|^{s+1-d} e^{-t|\xi|^2} 1_{B_n} |\xi|^{d-1}\hat{u}_0(\xi) \big\|_{L^{\infty,r}_\xi} &\le t^{\frac{\alpha}{2}} \big\| 1_{B_n} |\xi|^{s+1-d} e^{-t|\xi|^2} \big\|_{L^\infty} \big\| |\xi|^{d-1}\hat{u}_0(\xi) \big\|_{L^{\infty,r}} \\ \le t^{\frac{\alpha}{2}} \big\| 1_{B_n} |\xi|^{s+1-d} \big\|_{L^\infty} \big\| |\xi|^{d-1}\hat{u}_0(\xi) \big\|_{L^{\infty,r}} &= t^{\frac{\alpha}{2}} n^{s+1-d} \big\| |\xi|^{d-1}\hat{u}_0(\xi) \big\|_{L^{\infty,r}} \\ &= t^{\frac{\alpha}{2}} n^{s+1-d} \Big\| u_0 \Big\|_{\dot{H}^{d-1}_{\mathcal{L}^{1,r}}} < \frac{\epsilon}{2} \end{aligned} \tag{2.137}$$

for small enough $t = t(n) > 0$. From the estimates (2.136) and (2.137), we have,

$$t^{\frac{\alpha}{2}} \Big\| e^{t\Delta}u_0 \Big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} \le C \big\| 1_{B_n^c} |\xi|^{d-1}\hat{u}_0(\xi) \big\|_{L^{\infty,r}} + t^{\frac{\alpha}{2}} n^{s+1-d} \Big\| u_0 \Big\|_{\dot{H}^{d-1}_{\mathcal{L}^{1,r}}} < \epsilon.$$

□

Lemma 2.4.22. *Let $d-1 < s < d$. Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}_{s,\infty,T} \times \mathcal{K}_{s,\infty,T}$ into $\mathcal{K}_{s,1,T}$ and we have the inequality*

$$\big\| B(u,v) \big\|_{\mathcal{K}_{s,1,T}} \le C \big\| u \big\|_{\mathcal{K}_{s,\infty,T}} \big\| v \big\|_{\mathcal{K}_{s,\infty,T}}, \tag{2.138}$$

where C is a positive constant and independent of T.

Proof. Using the Fourier transform we get

$$\mathcal{F}\big(B(u,v)_j(t)\big)(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_0^t e^{-(t-\tau)|\xi|^2}\sum_{l,k=1}^{d}\Big(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l)\big(\widehat{u_l(\tau)}*\widehat{v_k(\tau)}\big)(\xi)\mathrm{d}\tau.$$

Thus

$$\big||\xi|^s\mathcal{F}\big(B(u,v)(t)\big)(\xi)\big| \lesssim \int_0^t |\xi|^s e^{-(t-\tau)|\xi|^2}|\xi|\big(|\widehat{u(\tau)}|*|\widehat{v(\tau)}|\big)(\xi)\mathrm{d}\tau.$$

We have

$$|\xi|^s|\widehat{u(\tau)}(\xi)| \le \sup_{\xi\in\mathbb{R}^d}\big||\xi|^s\widehat{u(\tau)}(\xi)\big| = \big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}} \text{ and } |\xi|^s|\widehat{v(\tau)}(\xi)| \le \big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}},$$

therefore

$$|\widehat{u(\tau)}(\xi)| \le \frac{\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}}}{|\xi|^s},\ \ |\widehat{v(\tau)}(\xi)| \le \frac{\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}}}{|\xi|^s}.$$

A standard argument shows that

$$\frac{1}{|\xi|^s}*\frac{1}{|\xi|^s} = \frac{C}{|\xi|^{2s-d}}.$$

From the above estimates and Lemma 2.4.4 (b), we have

$$\big(|\widehat{u(\tau)}|*|\widehat{v(\tau)}|\big)(\xi) \le \frac{\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}}}{|\xi|^s}*\frac{\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}}}{|\xi|^s} \simeq$$

$$\frac{\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}}}{|\xi|^{2s-d}} = \frac{\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}}{|\xi|^{2s-d}},$$

this gives the desired result

$$\int_0^t |\xi|^s e^{-(t-\tau)|\xi|^2}|\xi|\big(|\widehat{u(\tau)}|*|\widehat{v(\tau)}|\big)(\xi)\mathrm{d}\tau \lesssim \int_0^t |\xi|^{d+1-s}e^{-(t-\tau)|\xi|^2}\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau.$$

Thus

$$\begin{aligned}
\Big\||\xi|^s\mathcal{F}\big(B(u,v)(t)\big)(\xi)\Big\|_{L^{\infty,1}_\xi} &\lesssim \int_0^t \Big\||\xi|^{d+1-s}e^{-(t-\tau)|\xi|^2}\Big\|_{L^{\infty,1}_\xi}\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau \\
&= \int_0^t (t-s)^{\frac{s-d-1}{2}}\Big\||\xi|^{d+1-s}e^{-|\xi|^2}\Big\|_{L^{\infty,1}}\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau \\
&\lesssim \int_0^t (t-\tau)^{\frac{\alpha}{2}-1}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau \\
&= \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\int_0^t (t-\tau)^{\frac{\alpha}{2}-1}\tau^{-\alpha}\mathrm{d}\tau \\
&\simeq t^{-\frac{\alpha}{2}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}.
\end{aligned} \tag{2.139}$$

Let us now check the validity of the condition (2.132) for the bilinear term $B(u,v)(t)$. Indeed, from (2.139)

$$\lim_{t\to0}t^{\frac{\alpha}{2}}\Big\|B(u,v)(t)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,1}}} = \lim_{t\to0}t^{\frac{\alpha}{2}}\Big\||\xi|^s\mathcal{F}\big(B(u,v)(t)\big)(\xi)\Big\|_{L^{\infty,1}_\xi} = 0,$$

whenever

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \Big\|u(t)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}} = \lim_{t\to 0} t^{\frac{\alpha}{2}} \Big\|v(t)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}} = 0.$$

The estimate (2.138) is deduced from the inequality (2.139). □

Lemma 2.4.23. *Let $d-1<s<d$. Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}_{s,\infty,T}\times\mathcal{K}_{s,\infty,T}$ into $\mathcal{K}_{d-1,1,T}$ and we have the inequality*

$$\big\|B(u,v)\big\|_{\mathcal{K}_{d-1,1,T}} \le C\big\|u\big\|_{\mathcal{K}_{s,\infty,T}}\big\|v\big\|_{\mathcal{K}_{s,\infty,T}}, \tag{2.140}$$

where C is a positive constant and independent of T.

Proof. First, arguing as in Lemma 2.4.22, we have the following estimates

$$\begin{aligned}\big||\xi|^{d-1}\mathcal{F}\big(B(u,v)(t)\big)(\xi)\big| &\lesssim \int_0^t |\xi|^{d-1}e^{-(t-\tau)|\xi|^2}|\xi|\big(|\widehat{u(\tau)}|*|\widehat{v(\tau)}|\big)(\xi)\mathrm{d}\tau\\ &\lesssim \int_0^t |\xi|^{2d-2s}e^{-(t-\tau)|\xi|^2}\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau,\end{aligned}$$

this gives the desired result

$$\begin{aligned}\Big\||\xi|^{d-1}\mathcal{F}\big(B(u,v)(t)\big)(\xi)\Big\|_{L^{\infty,1}_\xi} &\lesssim \int_0^t \Big\||\xi|^{2d-2s}e^{-(t-\tau)|\xi|^2}\Big\|_{L^{\infty,1}_\xi}\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau\\ &= \int_0^t (t-s)^{s-d}\Big\||\xi|^{2d-2s}e^{-|\xi|^2}\Big\|_{L^{\infty,1}}\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau\\ &\lesssim \int_0^t (t-\tau)^{\alpha-1}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau\\ &= \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\int_0^t (t-\tau)^{\alpha-1}\tau^{-\alpha}\mathrm{d}\tau\\ &\simeq \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}.\end{aligned} \tag{2.141}$$

From (2.141) it follows (2.133) since

$$\lim_{t\to 0}\Big\|B(u,v)(t)\Big\|_{\dot{H}^{d-1}_{\mathcal{L}^{1,1}}} = \lim_{t\to 0}\Big\||\xi|^{d-1}\mathcal{F}\big(B(u,v)(t)\big)(\xi)\Big\|_{L^{\infty,1}_\xi} = 0,$$

whenever

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \Big\|u(t)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}} = \lim_{t\to 0} t^{\frac{\alpha}{2}} \Big\|v(t)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}} = 0.$$

The estimate (2.140) can be deduced from the inequality (2.141). □

Theorem 2.4.24. *Let $d-1<s<d$ and $1\le r<\infty$. Then there exists a positive constant $\delta_{s,d}$ such that for all $T>0$ and for all $u_0\in\dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$, with $\mathrm{div}(u_0)=0$ satisfying*

$$\sup_{0<t<T} t^{\frac{1}{2}(s+1-d)}\big\|e^{t\Delta}u_0\big\|_{\dot{H}^s_{\mathcal{L}^1}} \le \delta_{s,d}, \tag{2.142}$$

NSE has a unique mild solution $u\in\mathcal{K}_{s,r,T}\cap L^\infty([0,T];\dot{H}^{d-1}_{\mathcal{L}^{1,r}})$.
In particular, the inequality (2.142) holds for arbitrary $u_0\in\dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$ when $T(u_0)$ is small enough, and there exists a positive constant $\sigma_{s,d}$ such that we can take $T=\infty$ whenever $\Big\|u_0\Big\|_{\dot{H}^{d-1}_{\mathcal{L}^1}}\le\sigma_{s,d}$.

Proof. The proof of Theorem 2.4.24 is similar to that of Theorem 2.4.14. Applying Lemma 2.4.22 and Theorem 1.5.1, we deduce that there exists a positive constant $\delta_{s,d}$ such that for any $u_0 \in \dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ such that

$$\sup_{0<t<T} t^{\frac{1}{2}(s+1-d)} \big\| e^{t\Delta} u_0 \big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}} = \sup_{0<t<T} t^{\frac{1}{2}(s+1-d)} \big\| e^{t\Delta} u_0 \big\|_{\dot{H}^s_{\mathcal{L}^1}} \leq \delta_{s,d},$$

NSE has a solution $u \in \mathcal{K}_{s,\infty,T}$. Applying Lemmas 2.4.8 and 2.4.23 we deduce that $u \in L^\infty([0,T]; \dot{H}^{d-1}_{\mathcal{L}^{1,r}})$. Applying Lemma 2.4.21, we get $e^{\cdot\Delta}u_0 \in \mathcal{K}_{s,r,T}$. From the definition of $\mathcal{K}_{s,r,T}$, we deduce that the left-hand side of the inequality (2.142) converges to 0 when T tends to 0. Therefore the inequality (2.142) holds for arbitrary $u_0 \in \dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$ when $T(u_0)$ is small enough.
Next, from the inequality (2.134) with $r = \infty$, we deduce that

$$\sup_{0<t<\infty} t^{\frac{1}{2}(s+1-d)} \big\| e^{t\Delta} u_0 \big\|_{\dot{H}^s_{\mathcal{L}^1}} \lesssim \big\| u_0 \big\|_{\dot{H}^{d-1}_{\mathcal{L}^1}},$$

then there exists a positive constant $\sigma_{s,d}$ such that $T = \infty$ and (2.142) holds whenever $\big\| u_0 \big\|_{\dot{H}^{d-1}_{\mathcal{L}^1}} \leq \sigma_{s,d}$. □

Remark 2.4.7. The case $r = \infty$ was studied by Le Jan and Sznitman in [7]. They showed that NSE are well-posed when the initial datum belongs to the space $\dot{H}^{d-1}_{\mathcal{L}^{1,\infty}}$. For $1 \leq r < \infty$ we have the following imbedding map

$$\dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d) \hookrightarrow \dot{H}^{d-1}_{\mathcal{L}^{1,\infty}}(\mathbb{R}^d) = \dot{H}^{d-1}_{\mathcal{L}^1}(\mathbb{R}^d).$$

2.4.5. Conclusions

In 1997, Le Jan and Sznitman [7] considered a very simple space convenient to the study of NSE, which is the space E of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ so that $\hat{f}(\xi)$ is a locally integrable function on $\mathbb{R}^d$ and $\sup_{\xi} |\xi|^{d-1} |\hat{f}(\xi)| < \infty$. This space can be defined as a Besov space based on the spaces PM of pseudomeasures (PM is the space of the image of the Fourier transforms of essentially bounded functions: $PM = \mathcal{F}L^\infty$). More precisely, $E = \dot{B}^{d-1,\infty}_{PM}(\mathbb{R}^d)$. They showed that the bilinear operator B is bicontinuous in $L^\infty([0,T]; \dot{B}^{d-1,\infty}_{PM})$ for all $0 < T \leq \infty$. Therefore they can easily deduce the existence of global mild solutions in spaces $L^\infty([0,\infty); \dot{B}^{d-1,\infty}_{PM})$ when the norm of the initial value in the spaces $\dot{B}^{d-1,\infty}_{PM}(\mathbb{R}^d)$ is small enough. From Definitions 2.4.1 and 2.4.2, we have

$$PM = \mathcal{L}^1, \dot{B}^{d-1,\infty}_{PM}(\mathbb{R}^d) = \dot{H}^{d-1}_{\mathcal{L}^1}(\mathbb{R}^d).$$

In 2011, Lei and Lin [50] showed that NSE are well-posed when the initial datum belongs to the spaces $\mathcal{X}^{-1}(\mathbb{R}^d)$ defined by

$$f \in \mathcal{X}^{-1}(\mathbb{R}^d) \text{ if and only if } \big\| \dot{\Lambda}^{-1} f \big\|_{\mathcal{X}} < \infty, \text{where } \big\| f \big\|_{\mathcal{X}} = \big\| \hat{f} \big\|_{L^1}.$$

They established the existence of global mild solutions in the space $L^\infty([0,\infty); \mathcal{X}^{-1})$ when the norm of the initial value in the spaces $\mathcal{X}^{-1}(\mathbb{R}^d)$ is small enough. From Definitions 2.4.1 and 2.4.2 we see that

$$\mathcal{X}^{-1}(\mathbb{R}^d) = \dot{H}^{-1}_{\mathcal{L}^\infty}(\mathbb{R}^d).$$

Thus, the spaces $\dot{B}^{d-1,\infty}_{PM}$ and $\mathcal{X}^{-1}$, studied in [7] and [50], are the special cases of the critical Sobolev-Fourier-Lebesgue spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^p}$ with $p = 1$ and $p = \infty$, respectively.

In this section, for $s \in \mathbb{R}$ and $1 \le p, r \le \infty$, we first recall the notion of the Fourier-Lebesgue spaces $\mathcal{L}^p(\mathbb{R}^d)$, introduced and investigated in [30]; then we introduce and study the Sobolev-Fourier-Lebesgue spaces $\dot{H}^s_{\mathcal{L}^p}(\mathbb{R}^d)$, and the Sobolev-Fourier-Lorentz spaces $\dot{H}^s_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$. After that we show that the Navier-Stokes equations are well-posed when the initial datum belongs to the critical Sobolev-Fourier-Lorentz spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $d \ge 2, 1 \le p < \infty$, and $1 \le r < \infty$. The spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ are more general than the spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p}}(\mathbb{R}^d)$. In particular, $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d) = \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p}}(\mathbb{R}^d)$ when $\frac{1}{p} + \frac{1}{r} = 1$.

2.5 Mild solutions in Sobolev-Lorentz spaces

In this section, for $q > 1, 1 \le r \le \infty$, and $0 \le s < \frac{d}{q}$, we introduce and study the Sobolev-Lorentz spaces $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d)$, which are generalizations of the classical Sobolev spaces $\dot{H}^s_q(\mathbb{R}^d)$. Then we investigate the existence and uniqueness of solutions to the Navier-Stokes equations in the spaces $L^\infty\big([0,T]; \dot{H}^s_{L^{q,r}}(\mathbb{R}^d)\big)$ when the initial data belong to the Sobolev-Lorentz spaces $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d), (d \ge 2, q > 1, r \ge 1, s \ge 0$, and $\frac{d}{q} - 1 \le s < \frac{d}{q})$. We obtain the local existence of mild solutions with arbitrary initial value, and existence of mild solutions for any $0 < T < \infty$ when the norm of the initial value in the Besov spaces $\dot{B}^{s-d(\frac{1}{q}-\frac{1}{\tilde{q}}),\infty}_{\tilde{q}}(\mathbb{R}^d)$, $\big(\frac{1}{2}(\frac{1}{q}+\frac{s}{d}) < \frac{1}{\tilde{q}} < \min\{\frac{1}{2}+\frac{s}{2d}, \frac{1}{q}\}\big)$ is small enough. In the case of critical indexes $(1 < q \le d, s = \frac{d}{q} - 1)$, we get the existence of global mild solutions in the spaces $L^\infty([0,\infty); \dot{H}^{\frac{d}{q}-1}_{L^{q,r}}(\mathbb{R}^d))$ when the norm of the initial value in the Besov spaces $\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}(\mathbb{R}^d)$, $\Big(\frac{1}{q} - \frac{1}{2d} < \frac{1}{\tilde{q}} < \min\Big\{\frac{1}{2} + \frac{1}{2q} - \frac{1}{2d}, \frac{1}{q}\Big\}\Big)$ is small enough.

2.5.1. Sobolev-Lorentz spaces

Before proceeding to the definition of the Sobolev-Lorentz spaces, let us introduce several necessary notations. For $0 < s < d$, the operator $\dot{\Lambda}^s$ can be viewed as the inverse of the Riesz potential I_s up to positive constant

$$I_s(f)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-s}}\, \mathrm{d}y \quad \text{for } x \in \mathbb{R}^d.$$

For $q > 1, r \ge 1$, and $0 \le s < \frac{d}{q}$, the operator I_s is continuous from $L^{q,r}$ to $L^{\tilde{q},r}$, where $\frac{1}{\tilde{q}} = \frac{1}{q} - \frac{s}{d}$, see ([46], Theorem 2.4 *iii*), pp. 20).

Definition 2.5.1. (Sobolev-Lorentz spaces). (See [29].)
For $q > 1, r \ge 1$, and $0 \le s < \frac{d}{q}$, the Sobolev-Lorentz space $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d)$ is defined as the space $I_s(L^{q,r}(\mathbb{R}^d))$, equipped with the norm

$$\big\|f\big\|_{\dot{H}^s_{L^{q,r}}} := \big\|\dot{\Lambda}^s f\big\|_{L^{q,r}}.$$

Lemma 2.5.1. *Let $q > 1, 1 \le r \le \tilde{r} \le \infty$, and $0 \le s < \frac{d}{q}$. Then we have the following imbedding maps*
(a)

$$\dot{H}^s_{L^{q,1}} \hookrightarrow \dot{H}^s_{L^{q,r}} \hookrightarrow \dot{H}^s_{L^{q,\tilde{r}}} \hookrightarrow \dot{H}^s_{L^{q,\infty}}.$$

(b) $\dot{H}^s_q = \dot{H}^s_{L^{q,q}}$ *(equality of the norm).*

Proof. It is easily deduced from the properties of the standard Lorentz spaces. □

Lemma 2.5.2. *Assume that $q > 1, 1 \le r \le \infty$, and $0 \le s < \frac{d}{q}$. The following statement is true: If $u_0 \in \dot{H}^s_{L^{q,r}}$ then $e^{\cdot\Delta}u_0 \in L^\infty([0,\infty); \dot{H}^s_{L^{q,r}})$ and $\big\|e^{\cdot\Delta}u_0\big\|_{L^\infty([0,\infty);\dot{H}^s_{L^{q,r}})} \le \big\|u_0\big\|_{\dot{H}^s_{L^{q,r}}}$.*

Proof. We have

$$\big\|e^{t\Delta}u_0\big\|_{\dot{H}^s_{L^{q,r}}} = \big\|e^{t\Delta}\dot{\Lambda}^s u_0\big\|_{L^{q,r}} = \frac{1}{(4\pi t)^{d/2}}\Big\|\int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}}\dot{\Lambda}^s u_0(\cdot - \xi)\mathrm{d}\xi\Big\|_{L^{q,r}} \le$$

$$\frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}}\big\|\dot{\Lambda}^s u_0(\cdot - \xi)\big\|_{L^{q,r}}\mathrm{d}\xi = \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}}\big\|u_0\big\|_{\dot{H}^s_{L^{q,r}}}\mathrm{d}\xi = \big\|u_0\big\|_{\dot{H}^s_{L^{q,r}}}.$$

□

2.5.2. Auxiliary spaces

We define the auxiliary space $\mathcal{K}^{s,\tilde{q}}_{q,r,T}$ which is made up by the functions $u(t,x)$ such that

$$\big\|u\big\|_{\mathcal{K}^{s,\tilde{q}}_{q,r,T}} := \sup_{0<t<T} t^{\frac{\alpha}{2}}\big\|u(t,\cdot)\big\|_{\dot{H}^s_{L^{\tilde{q},r}}} < \infty,$$

and

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|u(t,\cdot)\big\|_{\dot{H}^s_{L^{\tilde{q},r}}} = 0, \tag{2.143}$$

where $r, q, \tilde{q}, s$ being fixed constants satisfying

$$q, \tilde{q} \in (1, +\infty), r \ge 1, s \ge 0, \frac{s}{d} < \frac{1}{\tilde{q}} \le \frac{1}{q} \le \frac{s+1}{d},$$

and

$$\alpha = \alpha(q, \tilde{q}) = d\Big(\frac{1}{q} - \frac{1}{\tilde{q}}\Big).$$

In the case $\tilde{q} = q$, it is also convenient to define the space $\mathcal{K}^{s,\tilde{q}}_{q,r,T}$ as the natural space $L^\infty([0,T]; \dot{H}^s_{L^{q,r}}(\mathbb{R}^d))$ with the additional condition that its elements $u(t,x)$ satisfy

$$\lim_{t\to 0}\big\|u(t,\cdot)\big\|_{\dot{H}^s_{L^{q,r}}} = 0. \tag{2.144}$$

Remark 2.5.2. The auxiliary space $\mathcal{K}_{\tilde{q}} := \mathcal{K}^{0,\tilde{q}}_{d,\tilde{q},T}$ $(\tilde{q} \ge d)$ was introduced by Weissler and systematically used by Kato [37] and Cannone [12].

Lemma 2.5.3. *Let $1 \le r \le \tilde{r} \le \infty$. Then we have the following imbedding maps*

$$\mathcal{K}^{s,\tilde{q}}_{q,1,T} \hookrightarrow \mathcal{K}^{s,\tilde{q}}_{q,r,T} \hookrightarrow \mathcal{K}^{s,\tilde{q}}_{q,\tilde{r},T} \hookrightarrow \mathcal{K}^{s,\tilde{q}}_{q,\infty,T}.$$

Proof. It is easily deduced from Lemma 2.5.1 (a) and the definition of $\mathcal{K}^{s,\tilde{q}}_{q,r,T}$. □

Lemma 2.5.4. *If $u_0 \in \dot{H}^s_{L^{q,r}}(\mathbb{R}^d)$ with $q > 1, r \ge 1, s \ge 0$, and $\frac{s}{d} < \frac{1}{q} \le \frac{s+1}{d}$ then for all $\tilde{q}$ satisfying*

$$\frac{s}{d} < \frac{1}{\tilde{q}} < \frac{1}{q},$$

we have

$$e^{\cdot\Delta}u_0 \in \mathcal{K}^{s,\tilde{q}}_{q,1,\infty},$$

and the following imbedding map

$$\dot{H}^s_{L^{q,r}}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s-(\frac{d}{q}-\frac{d}{\tilde{q}}),\infty}_{\tilde{q}}(\mathbb{R}^d). \tag{2.145}$$

Proof. Before proving this lemma, we need to prove the following lemma.

Lemma 2.5.5. *Suppose that* $u_0 \in L^{q,r}(\mathbb{R}^d)$ *with* $1 \le q \le \infty$ *and* $1 \le r < \infty$. *Then* $\lim_{n\to\infty} \|\mathcal{X}_n u_0\|_{L^{q,r}} = 0$, *where* $n \in \mathbb{N}, \mathcal{X}_n(x) = 0$ *for* $x \in \{x : \ |x| < n\} \cap \{x : |u_0(x)| < n\}$ *and* $\mathcal{X}_n(x) = 1$ *otherwise.*

Proof. With $\delta > 0$ being fixed, we have

$$\big\{x : |\mathcal{X}_n u_0(x)| > \delta\big\} \supseteq \big\{x : |\mathcal{X}_{n+1} u_0(x)| > \delta\big\}, \tag{2.146}$$

and

$$\bigcap_{n=0}^{\infty} \{x : |\mathcal{X}_n u_0(x)| > \delta\} = \{x : |u_0(x)| = +\infty\}. \tag{2.147}$$

We prove that

$$\mathcal{L}^d(\{x : |u_0(x)| = +\infty\}) = 0, \tag{2.148}$$

Assume on the contrary

$$\mathcal{L}^d(\{x : |u_0(x)| = +\infty\}) > 0.$$

We have $u_0^*(t) := \inf\big\{\tau : \mathcal{L}^d\big(\{x : |u_0(x)| > \tau\}\big) \le t\big\} = +\infty$ for all t such that $0 < t < \mathcal{L}^d(\{x : |u_0(x)| = +\infty\})$ and then $\|u_0\|_{L^{q,r}} = +\infty$, a contradiction.
Note that

$$\mathcal{L}^d\big(\{x : |\mathcal{X}_0 u_0(x)| > \delta\}\big) = \mathcal{L}^d\big(\{x : |u_0(x)| > \delta\}\big).$$

We prove that

$$\mathcal{L}^d\big(\{x : |u_0(x)| > \delta\}\big) < \infty, \tag{2.149}$$

assuming on the contrary

$$\mathcal{L}^d\big(\{x : |u_0(x)| > \delta\}\big) = \infty.$$

We have $u_0^*(t) \ge \delta$ for all $t > 0$, from the definition of the Lorentz space, we get

$$\|u_0\|_{L^{q,r}} = \Big(\int_0^\infty (t^{\frac{1}{q}} u_0^*(t))^r \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{r}} \ge \Big(\int_0^\infty (t^{\frac{1}{q}} \delta)^r \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{r}} = \delta\Big(\int_0^\infty t^{\frac{r}{q}-1} \mathrm{d}t\Big)^{\frac{1}{r}} = \infty,$$

a contradiction.
From (2.146), (2.147), (2.148), and (2.149), we infer that

$$\lim_{n\to\infty} \mathcal{L}^d\big(\{x : |\mathcal{X}_n u_0(x)| > \delta\}\big) = \mathcal{L}^d(\{x : |u_0(x)| = +\infty\}) = 0. \tag{2.150}$$

Set

$$u_n^*(t) = \inf\big\{\tau : \mathcal{L}^d\big(\{x : |\mathcal{X}_n u_0(x)| > \tau\}\big) \le t\big\}.$$

We have

$$u_n^*(t) \ge u_{n+1}^*(t). \tag{2.151}$$

Fix $t > 0$. For any $\epsilon > 0$, from (2.150) it follows that there exists a number $n_0 = n_0(t, \epsilon)$ large enough such that

$$\mathcal{L}^d\big(\{x : |\mathcal{X}_n u_0(x)| > \epsilon\}\big) \le t, \forall n \ge n_0.$$

From this we deduce that

$$u_n^*(t) \le \epsilon, \forall n \ge n_0,$$

therefore

$$\lim_{n\to\infty} u_n^*(t) = 0. \tag{2.152}$$

From (2.151) and (2.152), we apply Lebesgue's monotone convergence theorem to get

$$\lim_{n\to\infty}\big\|\mathcal{X}_n u_0\big\|_{L^{q,r}} = \lim_{n\to\infty}\Big(\int_0^\infty (t^{\frac{1}{q}}u_n^*(t))^r\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{r}} = 0.$$

□

Now we return to prove Lemma 2.5.4. We prove that

$$\sup_{0<t<\infty} t^{\frac{\alpha}{2}}\big\|e^{t\Delta}u_0\big\|_{\dot{H}^s_{L^{\tilde{q},1}}} \lesssim \big\|u_0\big\|_{\dot{H}^s_{L^{q,r}}}. \tag{2.153}$$

Set

$$\frac{1}{h} = 1 + \frac{1}{\tilde{q}} - \frac{1}{q}.$$

Applying Theorem 1.1.10 (c), we have

$$\big\|e^{t\Delta}u_0\big\|_{\dot{H}^s_{L^{\tilde{q},1}}} = \Big\|e^{t\Delta}\dot{\Lambda}^s u_0\Big\|_{L^{\tilde{q},1}} = \frac{1}{(4\pi t)^{d/2}}\Big\|e^{-\frac{|\cdot|^2}{4t}} * \dot{\Lambda}^s u_0\Big\|_{L^{\tilde{q},1}} \lesssim$$
$$\frac{1}{t^{d/2}}\big\|e^{-\frac{|\cdot|^2}{4t}}\big\|_{L^{h,1}}\big\|\dot{\Lambda}^s u_0\big\|_{L^{q,\infty}} = t^{-\frac{\alpha}{2}}\big\|e^{-\frac{|\cdot|^2}{4}}\big\|_{L^{h,1}}\big\|u_0\big\|_{\dot{H}^s_{L^{q,\infty}}} \lesssim t^{-\frac{\alpha}{2}}\big\|u_0\big\|_{\dot{H}^s_{L^{q,r}}}.$$

We claim now that

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|e^{t\Delta}u_0\big\|_{\dot{H}^s_{L^{\tilde{q},1}}} = 0.$$

From Lemma 2.5.5, we have

$$\lim_{n\to\infty}\Big\|\mathcal{X}_{n,s}\dot{\Lambda}^s u_0\Big\|_{L^{q,r}} = 0, \tag{2.154}$$

where $\mathcal{X}_{n,s}(x) = 0$ for $x \in \{x : \ |x| < n\} \cap \{x : \big|\dot{\Lambda}^s u_0(x)\big| < n\}$ and $\mathcal{X}_{n,s}(x) = 1$ otherwise. We have

$$t^{\frac{\alpha}{2}}\big\|e^{t\Delta}u_0\big\|_{\dot{H}^s_{L^{\tilde{q},1}}} \leq$$
$$\frac{t^{\frac{\alpha}{2}-\frac{d}{2}}}{(4\pi)^{d/2}}\Big\|e^{-\frac{|\cdot|^2}{4t}} * (\mathcal{X}_{n,s}\dot{\Lambda}^s u_0)\Big\|_{L^{\tilde{q},1}} + \frac{t^{\frac{\alpha}{2}-\frac{d}{2}}}{(4\pi)^{d/2}}\Big\|e^{-\frac{|\cdot|^2}{4t}} * ((1-\mathcal{X}_{n,s})\dot{\Lambda}^s u_0)\Big\|_{L^{\tilde{q},1}}. \tag{2.155}$$

For any $\epsilon > 0$, applying Theorem 1.1.10 (c) and note that (2.154), we have

$$\frac{t^{\frac{\alpha}{2}-\frac{d}{2}}}{(4\pi)^{d/2}}\Big\|e^{-\frac{|\cdot|^2}{4t}} * (\mathcal{X}_{n,s}\dot{\Lambda}^s u_0)\Big\|_{L^{\tilde{q},1}}$$
$$\leq C_1\big\|e^{-\frac{|\cdot|^2}{4}}\big\|_{L^{h,1}}\Big\|\mathcal{X}_{n,s}\dot{\Lambda}^s u_0\Big\|_{L^{q,\infty}} \leq C_2\Big\|\mathcal{X}_{n,s}\dot{\Lambda}^s u_0\Big\|_{L^{q,r}} < \frac{\epsilon}{2}, \tag{2.156}$$

for large enough n. Fix one of such n, applying Theorem 1.1.10 (a), we conclude that

$$\frac{t^{\frac{\alpha}{2}-\frac{d}{2}}}{(4\pi)^{d/2}}\Big\|e^{-\frac{|\cdot|^2}{4t}} * ((1-\mathcal{X}_{n,s})\dot{\Lambda}^s u_0)\Big\|_{L^{\tilde{q},1}} \leq C_3 t^{\frac{\alpha}{2}-\frac{d}{2}}\big\|e^{-\frac{|\cdot|^2}{4t}}\big\|_{L^1}\Big\|(1-\mathcal{X}_{n,s})\dot{\Lambda}^s u_0\Big\|_{L^{\tilde{q},1}}$$
$$\leq C_4 t^{\frac{\alpha}{2}}\big\|e^{-\frac{|\cdot|^2}{4}}\big\|_{L^1}\big\|n(1-\mathcal{X}_{n,s})\big\|_{L^{\tilde{q},1}} = C_5 n t^{\frac{\alpha}{2}}\big\|(1-\mathcal{X}_{n,s})\big\|_{L^{\tilde{q},1}} = C_6(n)t^{\frac{\alpha}{2}} < \frac{\epsilon}{2}, \tag{2.157}$$

for small enough $t > 0$. From the estimates (2.155), (2.156), and (2.157) it follows that

$$t^{\frac{\alpha}{2}}\big\|e^{t\Delta}u_0\big\|_{\dot{H}^s_{L^{\tilde{q},1}}} \leq C_2\Big\|\mathcal{X}_{n,s}\dot{\Lambda}^s u_0\Big\|_{L^{q,r}} + C_6(n)t^{\frac{\alpha}{2}} < \epsilon.$$

Finally, the embedding (2.145) is derived from the inequality (2.153), Lemmas 2.5.1 and 1.1.8.

Remark 2.5.3. In the case $s = 0$ and $q = r = d$, Lemma 2.5.4 is a generalization of Lemma 9 in ([15], p. 196).

2.5.3. On the continuity and regularity of the bilinear operator

In this subsection a particular attention will be devoted to the study of the bilinear operator $B(u,v)(t)$ defined by (1.14) in Sobolev-Lorentz spaces.

Lemma 2.5.6. *Let $s, q \in \mathbb{R}$ be such that*

$$s \geq 0, q > 1, \text{ and } \frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}. \tag{2.158}$$

Then for all $\tilde{q}$ satisfying

$$\frac{s}{d} < \frac{1}{\tilde{q}} < \min\Big\{\frac{1}{2} + \frac{s}{2d}, \frac{1}{q}\Big\}, \tag{2.159}$$

the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T} \times \mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T}$ into $\mathcal{K}^{s,\tilde{q}}_{q,1,T}$ and the following inequality holds

$$\big\|B(u,v)\big\|_{\mathcal{K}^{s,\tilde{q}}_{q,1,T}} \leq CT^{\frac{1}{2}(1+s-\frac{d}{q})}\big\|u\big\|_{\mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T}}\big\|v\big\|_{\mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T}}, \tag{2.160}$$

where C is a positive constant independent of T.

Proof. We have

$$\big\|B(u,v)(t)\big\|_{\dot{H}^s_{L^{\tilde{q},1}}} \leq \int_0^t \Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{\dot{H}^s_{L^{\tilde{q},1}}}\mathrm{d}\tau =$$
$$\int_0^t \Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\dot{\Lambda}^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{L^{\tilde{q},1}}\mathrm{d}\tau. \tag{2.161}$$

By the properties of the Fourier transform

$$\Big(e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\dot{\Lambda}^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big)_j = \frac{1}{(t-\tau)^{\frac{d+1}{2}}}\sum_{l,k=1}^d K_{l,k,j}\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)*\Big(\dot{\Lambda}^s\big(u_l(\tau,\cdot)v_k(\tau,\cdot)\big)\Big). \tag{2.162}$$

Applying Lemma 1.2.1 with $|\alpha| = 1$ we see that the tensor $K(x) = \{K_{l,k,j}(x)\}$ satisfies

$$|K(x)| \lesssim \frac{1}{(1+|x|)^{d+1}}. \tag{2.163}$$

So, we can rewrite the equality (2.162) in the tensor form

$$e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\dot{\Lambda}^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big) = \frac{1}{(t-\tau)^{\frac{d+1}{2}}}K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)*\Big(\dot{\Lambda}^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big). \tag{2.164}$$

Set

$$\frac{1}{r} = \frac{2}{\tilde{q}} - \frac{s}{d}, \ \frac{1}{h} = \frac{s}{d} - \frac{1}{\tilde{q}} + 1. \tag{2.165}$$

From the inequalities (2.158) and (2.159), we can check that the following conditions are satisfied

$$1 < h, r < \infty \text{ and } \frac{1}{\tilde{q}} + 1 = \frac{1}{h} + \frac{1}{r}.$$

Applying Theorem 1.1.10 (c), we have

$$\Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\dot{\Lambda}^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{L^{\tilde{q},1}} \lesssim$$
$$\frac{1}{(t-\tau)^{\frac{d+1}{2}}}\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{L^{h,1}}\Big\|\dot{\Lambda}^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{L^{r,\infty}}. \tag{2.166}$$

Applying Lemma 2.2.4 we obtain

$$\Big\|\dot\Lambda^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{L^{r,\infty}} \le \Big\|\dot\Lambda^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{L^r} = \big\|u(\tau,\cdot)\otimes v(\tau,\cdot)\big\|_{\dot H^s_r} \lesssim \big\|u(\tau,\cdot)\big\|_{\dot H^s_{\tilde q}}\big\|v(\tau,\cdot)\big\|_{\dot H^s_{\tilde q}}. \tag{2.167}$$

Fom the inequalities (2.163) and (2.165) we infer that

$$\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{L^{h,1}} = (t-\tau)^{\frac{d}{2h}}\big\|K\big\|_{L^{h,1}} \simeq (t-\tau)^{\frac{s}{2}-\frac{d}{2\tilde q}+\frac{d}{2}}. \tag{2.168}$$

From the inequalities (2.166), (2.167), and (2.168) we deduce that

$$\Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\dot\Lambda^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{L^{\tilde q,1}} \lesssim (t-\tau)^{\frac{s}{2}-\frac{d}{2\tilde q}-\frac{1}{2}}\big\|u(\tau,\cdot)\big\|_{\dot H^s_{\tilde q}}\big\|v(\tau,\cdot)\big\|_{\dot H^s_{\tilde q}}. \tag{2.169}$$

From the estimates (2.161) and (2.169), and note that from the inequalities (2.158) and (2.159), we can check that $\frac{s}{2}-\frac{d}{2\tilde q}-\frac{1}{2} > -1$ and $\alpha = d(\frac{1}{q}-\frac{1}{\tilde q}) < 1$, this gives the desired result

$$\begin{aligned} \big\|B(u,v)(t)\big\|_{\dot H^s_{L^{\tilde q,1}}} &\lesssim \int_0^t (t-\tau)^{\frac{s}{2}-\frac{d}{2\tilde q}-\frac{1}{2}}\big\|u(\tau,\cdot)\big\|_{\dot H^s_{\tilde q}}\big\|v(\tau,\cdot)\big\|_{\dot H^s_{\tilde q}}\mathrm{d}\tau \lesssim \\ &\int_0^t (t-\tau)^{\frac{s}{2}-\frac{d}{2\tilde q}-\frac{1}{2}}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta,\cdot)\big\|_{\dot H^s_{\tilde q}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta,\cdot)\big\|_{\dot H^s_{\tilde q}}\mathrm{d}\tau = \\ &\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta,\cdot)\big\|_{\dot H^s_{\tilde q}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta,\cdot)\big\|_{\dot H^s_{\tilde q}}\int_0^t (t-\tau)^{\frac{s}{2}-\frac{d}{2\tilde q}-\frac{1}{2}}\tau^{-\alpha}\mathrm{d}\tau \simeq \\ &t^{-\frac{\alpha}{2}}t^{\frac{1}{2}(1+s-\frac{d}{q})}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta,\cdot)\big\|_{\dot H^s_{L^{\tilde q,\tilde q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta,\cdot)\big\|_{\dot H^s_{L^{\tilde q,\tilde q}}}. \end{aligned} \tag{2.170}$$

Let us now check the validity of the condition (2.143) for the bilinear term $B(u,v)(t)$. Indeed, we have

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|B(u,v)(t)\big\|_{\dot H^s_{L^{\tilde q,1}}} = 0,$$

whenever

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|u(t,\cdot)\big\|_{\dot H^s_{\tilde q}} = \lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|v(t,\cdot)\big\|_{\dot H^s_{\tilde q}} = 0.$$

The estimate (2.160) is now deduced from the inequality (2.170). □

Remark 2.5.4. In the case $s = 0$ and $q = d$, Lemma 2.5.4 is a generalization of Lemma 10 in ([15], p. 196).

Lemma 2.5.7. *Let $s, q \in \mathbb{R}$ be such that*

$$s \ge 0, q > 1, \text{ and } \frac{s}{d} < \frac{1}{q} \le \frac{s+1}{d}. \tag{2.171}$$

Then for all $\tilde q$ satisfying

$$\frac{1}{2}\Big(\frac{1}{q}+\frac{s}{d}\Big) < \frac{1}{\tilde q} < \min\Big\{\frac{1}{2}+\frac{s}{2d}, \frac{1}{q}\Big\}, \tag{2.172}$$

the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{s,\tilde q}_{q,\tilde q,T} \times \mathcal{K}^{s,\tilde q}_{q,\tilde q,T}$ into $\mathcal{K}^{s,q}_{q,1,T}$ and the following inequality holds

$$\big\|B(u,v)\big\|_{\mathcal{K}^{s,q}_{q,1,T}} \le CT^{\frac{1}{2}(1+s-\frac{d}{q})}\big\|u\big\|_{\mathcal{K}^{s,\tilde q}_{q,\tilde q,T}}\big\|v\big\|_{\mathcal{K}^{s,\tilde q}_{q,\tilde q,T}}, \tag{2.173}$$

where C is a positive constant independent of T.

Proof. Set

$$\frac{1}{r}=\frac{2}{\tilde{q}}-\frac{s}{d},\ \frac{1}{h}=1+\frac{1}{q}-\frac{2}{\tilde{q}}+\frac{s}{d}. \tag{2.174}$$

From the inequalities (2.171) and (2.172), we can check that h and r satisfy

$$1<h,r<\infty \text{ and } \frac{1}{q}+1=\frac{1}{h}+\frac{1}{r}.$$

From the equality (2.164), applying Theorem 1.1.10 (c), we obtain

$$\Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\dot{\Lambda}^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{L^{q,1}}\lesssim$$
$$\frac{1}{(t-\tau)^{\frac{d+1}{2}}}\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{L^{h,1}}\Big\|\dot{\Lambda}^s\Big(u(\tau,\cdot)\otimes v(\tau,\cdot)\Big)\Big\|_{L^{r,\infty}}. \tag{2.175}$$

Applying Lemma 2.2.4, we have

$$\Big\|\dot{\Lambda}^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{L^{r,\infty}}\leq\Big\|\dot{\Lambda}^s\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{L^r}\lesssim\big\|u(\tau,\cdot)\big\|_{\dot{H}^s_{\tilde{q}}}\big\|v(\tau,\cdot)\big\|_{\dot{H}^s_{\tilde{q}}}. \tag{2.176}$$

From the inequalities (2.163) and (2.174) it follows that

$$\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{L^{h,1}}=(t-\tau)^{\frac{d}{2h}}\big\|K\big\|_{L^{h,1}}\simeq(t-\tau)^{\frac{d}{2}+\frac{d}{2q}-\frac{d}{\tilde{q}}+\frac{s}{2}}. \tag{2.177}$$

From the estimates (2.175), (2.176), (2.177) we deduce that

$$\Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau,\cdot)\otimes v(\tau,\cdot)\big)\Big\|_{\dot{H}^s_{L^{q,1}}}\lesssim(t-\tau)^{\frac{d}{2q}-\frac{d}{\tilde{q}}+\frac{s}{2}-\frac{1}{2}}\big\|u(\tau,\cdot)\big\|_{\dot{H}^s_{\tilde{q}}}\big\|v(\tau,\cdot)\big\|_{\dot{H}^s_{\tilde{q}}}$$
$$=(t-\tau)^{\alpha+\frac{s}{2}-\frac{d}{2q}-\frac{1}{2}}\big\|u(\tau,\cdot)\big\|_{\dot{H}^s_{\tilde{q}}}\big\|v(\tau,\cdot)\big\|_{\dot{H}^s_{\tilde{q}}}.$$

From the inequalities (2.171) and (2.172), we can check that $\alpha+\frac{s}{2}-\frac{d}{2q}-\frac{1}{2}>-1$ and $\alpha=d(\frac{1}{q}-\frac{1}{\tilde{q}})<1$, this gives the desired result

$$\big\|B(u,v)(t)\big\|_{\dot{H}^s_{L^{q,1}}}\lesssim\int_0^t(t-\tau)^{\alpha+\frac{s}{2}-\frac{d}{2q}-\frac{1}{2}}\big\|u(\tau,\cdot)\big\|_{\dot{H}^s_{\tilde{q}}}\big\|v(\tau,\cdot)\big\|_{\dot{H}^s_{\tilde{q}}}\mathrm{d}\tau\lesssim$$
$$\int_0^t(t-\tau)^{\alpha+\frac{s}{2}-\frac{d}{2q}-\frac{1}{2}}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta,\cdot)\big\|_{\dot{H}^s_{\tilde{q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta,\cdot)\big\|_{\dot{H}^s_{\tilde{q}}}\mathrm{d}\tau=$$
$$\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta,\cdot)\big\|_{\dot{H}^s_{\tilde{q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta,\cdot)\big\|_{\dot{H}^s_{\tilde{q}}}\int_0^t(t-\tau)^{\alpha+\frac{s}{2}-\frac{d}{2q}-\frac{1}{2}}\tau^{-\alpha}\mathrm{d}\tau\simeq$$
$$t^{\frac{1}{2}(1+s-\frac{d}{q})}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta,\cdot)\big\|_{\dot{H}^s_{L^{\tilde{q},\tilde{q}}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta,\cdot)\big\|_{\dot{H}^s_{L^{\tilde{q},\tilde{q}}}}. \tag{2.178}$$

Let us now check the validity of the condition (2.144) for the bilinear term $B(u,v)(t)$. Indeed, we have

$$\lim_{t\to 0}\big\|B(u,v)(t)\big\|_{\dot{H}^s_{L^{q,1}}}=0$$

whenever

$$\lim_{t\to 0}t^{\frac{\alpha}{2}}\big\|u(t,\cdot)\big\|_{\dot{H}^s_{\tilde{q}}}=\lim_{t\to 0}t^{\frac{\alpha}{2}}\big\|v(t,\cdot)\big\|_{\dot{H}^s_{\tilde{q}}}=0.$$

The estimate (2.173) is now deduced from the inequality (2.178). □

2.5.4. Solutions to the Navier-Stokes equations with the initial value in the Sobolev-Lorentz spaces

Theorem 2.5.8. *Let $s, q,$ and $r \in \mathbb{R}$ be such that*

$$s \geq 0, q > 1, r \geq 1, \text{ and } \frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}. \tag{2.179}$$

(a) *For all $\tilde{q}$ satisfying*

$$\frac{1}{2}\Big(\frac{1}{q} + \frac{s}{d}\Big) < \frac{1}{\tilde{q}} < \min\Big\{\frac{1}{2} + \frac{s}{2d}, \frac{1}{q}\Big\}, \tag{2.180}$$

there exists a positive constant $\delta_{s,q,\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^s_{L^{q,r}}(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ *satisfying*

$$T^{\frac{1}{2}(1+s-\frac{d}{q})} \sup_{0<t<T} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})} \big\| e^{t\Delta} u_0 \big\|_{\dot{H}^s_{\tilde{q}}} \leq \delta_{s,q,\tilde{q},d}, \tag{2.181}$$

NSE has a unique mild solution $u \in \mathcal{K}^{s,\tilde{q}}_{q,1,T} \cap L^\infty\big([0,T]; \dot{H}^s_{L^{q,r}}\big)$. In particular, for arbitrary $u_0 \in \dot{H}^s_{L^{q,r}}$ with $\mathrm{div}(u_0) = 0$, *there exists $T(u_0)$ small enough such that the inequality* (2.181) *holds.*

(b) *If $1 < q \leq d$ and $s = \frac{d}{q} - 1$ then for any $\tilde{q}$ be such that*

$$\frac{1}{q} - \frac{1}{2d} < \frac{1}{\tilde{q}} < \min\Big\{\frac{1}{2} + \frac{1}{2q} - \frac{1}{2d}, \frac{1}{q}\Big\},$$

there exists a positive constant $\sigma_{q,\tilde{q},d}$ such that if $\big\|u_0\big\|_{\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}} \leq \sigma_{q,\tilde{q},d}$ and $T = \infty$ then the inequality (2.181) *holds.*

Proof. From Lemmas 2.5.6 and 2.5.3, the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T} \times \mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T}$ into $\mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T}$ and we have the inequality

$$\big\|B(u,v)\big\|_{\mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T}} \leq \big\|B(u,v)\big\|_{\mathcal{K}^{s,\tilde{q}}_{q,1,T}} \leq C_{s,q,\tilde{q},d} T^{\frac{1}{2}(1+s-\frac{d}{q})} \big\|u\big\|_{\mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T}} \big\|v\big\|_{\mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T}},$$

where $C_{s,q,\tilde{q},d}$ is a positive constant independent of T. From Theorem 1.5.1 and the above inequality, we deduce the following: for any $u_0 \in \dot{H}^s_{L^{q,r}}(\mathbb{R}^d)$ such that

$$\mathrm{div}(u_0) = 0,\ T^{\frac{1}{2}(1+s-\frac{d}{q})} \sup_{0<t<T} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})} \big\|e^{t\Delta}u_0\big\|_{\dot{H}^s_{\tilde{q}}} \leq \frac{1}{4C_{s,q,\tilde{q},d}},$$

NSE has a mild solution u on the interval $(0,T)$ so that

$$u \in \mathcal{K}^{s,\tilde{q}}_{q,\tilde{q},T}. \tag{2.182}$$

Lemma 2.5.7 and the relation (2.182) imply that

$$B(u,u) \in \mathcal{K}^{s,q}_{q,1,T} \subseteq \mathcal{K}^{s,q}_{q,r,T} \subseteq L^\infty\Big([0,T]; \dot{H}^s_{L^{q,r}}\Big).$$

On the other hand, from Lemma 2.5.2, we have $e^{\cdot\Delta}u_0 \in L^\infty\Big([0,T]; \dot{H}^s_{L^{q,r}}\Big)$.
Therefore

$$u = e^{\cdot\Delta}u_0 - B(u,u) \in L^\infty\Big([0,T]; \dot{H}^s_{L^{q,r}}\Big).$$

From Lemmas 2.5.4 and 2.5.6, we deduce that $u \in \mathcal{K}^{s,\tilde{q}}_{q,1,T}$.
From the definition of $\mathcal{K}^{s,\tilde{q}}_{q,r,T}$ and Lemma 2.5.4, we deduce that the left-hand side of the inequality (2.181) converges to 0 when T tends to 0. Therefore the inequality (2.181) holds for arbitrary $u_0 \in \dot{H}^s_{L^{q,r}}(\mathbb{R}^d)$ when $T(u_0)$ is small enough.
(b) From Lemma 1.1.8, the two quantities

$$\big\|u_0\big\|_{\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}} \quad \text{and} \quad \sup_{0<t<\infty} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})}\big\|e^{t\Delta}u_0\big\|_{\dot{H}^{\frac{d}{q}-1}_{\tilde{q}}}$$

are equivalent, then there exists a positive constant $\sigma_{q,\tilde{q},d}$ such that if $\big\|u_0\big\|_{\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}} \leq \sigma_{q,\tilde{q},d}$ and $T = \infty$ then the inequality (2.181) holds. □

Remark 2.5.5. In the case when the initial data belong to the critical Sobolev-Lorentz spaces $\dot{H}^{\frac{d}{q}-1}_{L^{q,r}}(\mathbb{R}^d)$, $(1 < q \leq d, r \geq 1)$, from Theorem 2.5.8 (b), we get the existence of global mild solutions in the spaces $L^\infty([0,\infty); \dot{H}^{\frac{d}{q}-1}_{L^{q,r}}(\mathbb{R}^d))$ when the norm of the initial value in the Besov spaces $\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}(\mathbb{R}^d)$ is small enough. Note that from Lemma 2.5.4 we have the following imbedding maps

$$\dot{H}^{\frac{d}{q}-1}_{L^{q,r}}(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}(\mathbb{R}^d), \Big(\frac{1}{q}-\frac{1}{d} < \frac{1}{\tilde{q}} < \frac{1}{q}\Big).$$

This result is stronger than that of Cannone. In particular, when $q = r = d, s = 0$, we get back the Cannone theorem (Theorem 1.1 in [12]).

Next, we consider the super-critical indexes $s > \frac{d}{q} - 1$.

Theorem 2.5.9. *Let*

$$s \geq 0, q > 1, r \geq 1, \text{ and } \frac{s}{d} < \frac{1}{q} < \frac{s+1}{d}.$$

Then for any $\tilde{q}$ be such that

$$\frac{1}{2}\Big(\frac{1}{q}+\frac{s}{d}\Big) < \frac{1}{\tilde{q}} < \min\Big\{\frac{1}{2}+\frac{s}{2d}, \frac{1}{q}\Big\},$$

there exists a positive constant $\delta_{s,q,\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^s_{L^{q,r}}(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ satisfying

$$T^{\frac{1}{2}(1+s-\frac{d}{q})}\big\|u_0\big\|_{\dot{B}^{s-(\frac{d}{q}-\frac{d}{\tilde{q}}),\infty}_{\tilde{q}}} \leq \delta_{s,q,\tilde{q},d},$$

NSE has a unique mild solution $u \in \mathcal{K}^{s,\tilde{q}}_{q,1,T} \cap L^\infty([0,T]; \dot{H}^s_{L^{q,r}})$.

Proof. Applying Lemma 1.1.8, the two quantities $\big\|u_0\big\|_{\dot{B}^{s-(\frac{d}{q}-\frac{d}{\tilde{q}}),\infty}_{\tilde{q}}}$ and $\sup_{0<t<\infty} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})}\big\|e^{t\Delta}u_0\big\|_{\dot{H}^s_{\tilde{q}}}$ are equivalent. Thus

$$\sup_{0<t<T} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})}\big\|e^{t\Delta}u_0\big\|_{\dot{H}^s_{\tilde{q}}} \lesssim \big\|u_0\big\|_{\dot{B}^{s-(\frac{d}{q}-\frac{d}{\tilde{q}}),\infty}_{\tilde{q}}},$$

the theorem is proved by applying the above inequality and Theorem 2.5.8. □

Remark 2.5.6. In the case when the initial data belong to the Sobolev-Lorentz spaces $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d), (q > 1, r \geq 1, s \geq 0,$ and $\frac{d}{q} - 1 < s < \frac{d}{q})$, we obtain the existence of mild solutions in the spaces $L^\infty([0,T]; \dot{H}^s_{L^{q,r}}(\mathbb{R}^d))$ for any $T > 0$ when the norm of the initial value in the Besov spaces $\dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{\tilde{q}}),\infty}(\mathbb{R}^d)$ is small enough. Note that from Lemma 2.5.4 we have the following imbedding maps (see Lemma 2.5.4)

$$\dot{H}^s_{L^{q,r}}(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{\tilde{q}}),\infty}(\mathbb{R}^d),\ (\frac{s}{d} < \frac{1}{\tilde{q}} < \frac{1}{q}).$$

Applying Theorem 2.5.9 for $q > d, r = q$ and $s = 0$, we get the following proposition which is stronger than that of Cannone and Meyer [11, 14]. In particular, we obtain a result that is stronger than that of Cannone and Meyer but under a much weaker condition on the initial data.

Proposition 2.5.10. *Let $q > d$. Then for any $\tilde{q}$ be such that $q < \tilde{q} < 2q$, there exists a positive constant $\delta_{q,\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in L^q(\mathbb{R}^d)$ with* $\mathrm{div}(u_0) = 0$ *satisfying*

$$T^{\frac{1}{2}(1-\frac{d}{q})}\big\|u_0\big\|_{\dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-\frac{d}{q},\infty}} \leq \delta_{q,\tilde{q},d}, \tag{2.183}$$

NSE has a unique mild solution $u \in \mathcal{K}^{0,\tilde{q}}_{q,1,T} \cap L^\infty([0,T]; L^q)$.

Remark 2.5.7. If in (2.183) we replace the $\dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-\frac{d}{q},\infty}$ norm by the L^q norm, then we get the assumption made in [11, 14]. We show that the condition (2.183) is weaker than the condition in [11, 14]. In Remark 2.5.6 we have showed that

$$L^q(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-\frac{d}{q},\infty}(\mathbb{R}^d), (\tilde{q} > q \geq d).$$

However these two spaces are different. Indeed, we have $\big|x\big|^{-\frac{d}{q}} \notin L^q(\mathbb{R}^d)$. On the other hand by using Lemma 1.1.8, we can easily prove that $\big|x\big|^{-\frac{d}{q}} \in \dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-\frac{d}{q},\infty}(\mathbb{R}^d)$ for all $\tilde{q} > q$.

Applying Theorem 2.5.9 for $q = r = 2, \frac{d}{2} - 1 < s < \frac{d}{2}$, we get the following proposition which is stronger than that of Chemin in [16] and Cannone in [11]. In particular, we obtain the result that is stronger than that of Chemin and Cannone but under a much weaker condition on the initial data.

Proposition 2.5.11. *Let $\frac{d}{2} - 1 < s < \frac{d}{2}$. Then for any $\tilde{q}$ be such that*

$$\frac{1}{2}\Big(\frac{1}{2} + \frac{s}{d}\Big) < \frac{1}{\tilde{q}} < \frac{1}{2},$$

there exists a positive constant $\delta_{s,\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^s(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ *satisfying*

$$T^{\frac{1}{2}(1+s-\frac{d}{2})}\big\|u_0\big\|_{\dot{B}_{\tilde{q}}^{s-(\frac{d}{2}-\frac{d}{\tilde{q}}),\infty}} \leq \delta_{s,\tilde{q},d}, \tag{2.184}$$

NSE has a unique mild solution $u \in \mathcal{K}^{s,\tilde{q}}_{2,1,T} \cap L^\infty([0,T]; \dot{H}^s)$.

Remark 2.5.8. If in (2.184) we replace the $\dot{B}_{\tilde{q}}^{s-(\frac{d}{2}-\frac{d}{\tilde{q}}),\infty}$ norm by the $\dot{H}^s(\mathbb{R}^d)$ norm, then we get the assumption made in [11, 16]. We show that the condition (2.184) is weaker than the condition in [11, 16]. In Remark 2.5.6 we showed that

$$\dot{H}^s(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{s-(\frac{d}{2}-\frac{d}{\tilde{q}}),\infty},\ \frac{1}{2}\Big(\frac{1}{2} + \frac{s}{d}\Big) < \frac{1}{\tilde{q}} < \frac{1}{2}.$$

However that these two spaces are different. Indeed, we have $\dot{\Lambda}^{-s}|\cdot|^{-\frac{d}{2}} \notin \dot{H}^s(\mathbb{R}^d)$, on the other hand by using Lemma 1.1.8, we easily prove that $\dot{\Lambda}^{-s}|\cdot|^{-\frac{d}{2}} \in \dot{B}_{\tilde{q}}^{s-(\frac{d}{2}-\frac{d}{\tilde{q}}),\infty}(\mathbb{R}^d)$ for all $\tilde{q} > 2$.

2.5.5. Conclusions

In this section, for $q > 1, 1 \leq r \leq \infty$, and $0 \leq s < \frac{d}{q}$, we introduce and study the Sobolev-Lorentz spaces $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d)$. The spaces $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d)$ are more general than the spaces $\dot{H}^s_q(\mathbb{R}^d)$, $(\dot{H}^s_q(\mathbb{R}^d) = \dot{H}^s_{L^{q,q}}(\mathbb{R}^d))$. Then we investigate mild solutions to NSE in the spaces $L^\infty\big([0,T];\dot{H}^s_{L^{q,r}}(\mathbb{R}^d)\big)$ when the initial data belong to the Sobolev-Lorentz spaces $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d), (d \geq 2, q > 1, r \geq 1, s \geq 0,$ and $\frac{d}{q} - 1 \leq s < \frac{d}{q})$. We obtain the existence of mild solutions with arbitrary initial value when T is small enough, and the existence of mild solutions for any $T > 0$ when the norm of the initial value in the Besov spaces $\dot{B}_{\tilde{q}}^{s-d(\frac{1}{q}-\frac{1}{\tilde{q}}),\infty}(\mathbb{R}^d)$, $\big(\frac{1}{2}(\frac{1}{q}+\frac{s}{d}) < \frac{1}{\tilde{q}} < \min\{\frac{1}{2}+\frac{s}{2d}, \frac{1}{q}\}\big)$ is small enough.
In the particular case $(q > d, r = q, s = 0)$, we get the result which is more general than that of Cannone and Meyer [11, 14]. Here we obtain a statement that is stronger than that Cannone and Meyer but under a much weaker condition on the initial data.
In the particular case $(q = r = 2, \frac{d}{2} - 1 < s < \frac{d}{2})$, we get the result which is more general than those of Chemin in [16] and Cannone in [11]. Here we obtain a statement that is stronger than those of Chemin in [16] and Cannone in [11] but under a much weaker condition on the initial data.
In the case of critical indexes $(1 < q \leq d, r \geq 1, s = \frac{d}{q} - 1)$, we get the existence of global mild solutions in spaces $L^\infty([0,\infty); \dot{H}^{\frac{d}{q}-1}_{L^{q,r}}(\mathbb{R}^d))$ when the norm of the initial value in the Besov spaces $\dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1,\infty}(\mathbb{R}^d)$, $\big(\frac{1}{q} - \frac{1}{2d} < \frac{1}{\tilde{q}} < \min\Big\{\frac{1}{2} + \frac{1}{2q} - \frac{1}{2d}, \frac{1}{q}\Big\}\big)$ is small enough. This result is a generalization of a result of Cannone [12]. In particular, when $q = r = d, s = 0$, we get back the Cannone theorem (Theorem 1.1 in [12]).

2.6 Mild solutions in mixed-norm Sobolev-Lorentz spaces

In this section, for $0 \leq m < \infty$ and index vectors $\mathbf{q} = (q_1, q_2, ..., q_d), \mathbf{r} = (r_1, r_2, ..., r_d)$, where $1 < q_i < \infty, 1 \leq r_i \leq \infty$, and $1 \leq i \leq d$, we introduce and study mixed-norm Sobolev-Lorentz spaces $\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}$, which are more general than the classical Sobolev spaces $\dot{H}^s_q$. Then we investigate the existence and uniqueness of solutions to the NSE in the spaces $\mathcal{Q} := \mathcal{Q}_T = L^p([0,T]; \dot{H}^m_{L^{\mathbf{q},\mathbf{r}}})$ where $p > 2,\ T > 0$, and initial data is taken in the class

$$\mathcal{I} = \{u_0 \in (\mathcal{S}'(\mathbb{R}^d))^d, \mathrm{div}(u_0) = 0 : \|e^{\cdot\Delta}u_0\|_{\mathcal{Q}} < \infty\}.$$

The results have a standard relation between existence time and data size: large time with small data or large data with small time. In the case with $T = \infty$ and critical indexes $\frac{2}{p} + \sum_{i=1}^d \frac{1}{q_i} - m = 1$, the space $\mathcal{I}$ coincides with the homogeneous Besov space $\dot{B}^{m-\frac{2}{p},p}_{L^{\mathbf{q},\mathbf{r}}}$.
In the case when

$$m = 0,\ q_1 = q_2 = ... = q_d = r_1 = r_2 = ... = r_d,$$

our results recover those of Faber, Jones and Riviere [19].

2.6.1. Mixed-norm Lorentz spaces

We consider $L^{\mathbf{q},\mathbf{r}}$ type norms of functions on $\mathbb{R}^d$ involving different exponents in different coordinate directions. Given a measurable function u on $\mathbb{R}^d$ and index vectors $\mathbf{q} = (q_1, q_2, ..., q_d)$, $\mathbf{r} = (r_1, r_2, ..., r_d)$, where $1 < q_i < \infty, 1 \leq r_i \leq \infty, 1 \leq i \leq d$, we can define the norm $\|u\|_{L^{\mathbf{q},\mathbf{r}}}$ by calculating first the L^{q_1,r_1} - Lorentz norm of $u(x_1, x_2, ..., x_d)$ with respect to the variable x_1, and then L^{q_2,r_2} - Lorentz norm of the resulting quantity with respect to the variable x_2, and so on, finishing with the L^{q_d,r_d} - Lorentz norm with respect to the variable x_d:

$$\|u\|_{L^{\mathbf{q},\mathbf{r}}} = \Big\| ...\big\| \|u\|_{L^{q_1,r_1}_{x_1}} \big\|_{L^{q_2,r_2}_{x_2}} ... \Big\|_{L^{q_d,r_d}_{x_d}}. \tag{2.185}$$

For a short notation, for a vector $\mathbf{q} = (q_1, q_2, ..., q_d)$ we will write $\frac{1}{\mathbf{q}}$ for the vector $\left(\frac{1}{q_1}, \ldots, \frac{1}{q_d}\right)$.

Lemma 2.6.1. *Let $\boldsymbol{q} = (q_1, q_2, ..., q_d)$, $\boldsymbol{r} = (r_1, r_2, ..., r_d)$, where $1 < q_i < \infty, 1 \leq r_i \leq \infty$ for $1 \leq i \leq d$.*
(1) Assume that $|f| \leq |g|$ on $\mathbb{R}^d$. Then $\|f\|_{L^{\boldsymbol{q},\boldsymbol{r}}} \leq \|g\|_{L^{\boldsymbol{q},\boldsymbol{r}}}$.
(2) If $\|f\|_{L^{\boldsymbol{q},\boldsymbol{r}}} < +\infty$ then $\mathcal{L}^d(\{x : |f(x)| = +\infty\}) = 0$.
(3) Assuming $\boldsymbol{1} \leq \boldsymbol{r} \leq \tilde{\boldsymbol{r}} \leq \infty$ we have the following imbedding $L^{\boldsymbol{q},\boldsymbol{1}} \hookrightarrow L^{\boldsymbol{q},\boldsymbol{r}} \hookrightarrow L^{\boldsymbol{q},\tilde{\boldsymbol{r}}} \hookrightarrow L^{\boldsymbol{q},\infty}$.

Proof. (1) We can prove this inequality by induction. When $d = 1$, it is deduced from the definition of the Lorentz spaces. Assuming that the inequality is true for $d = n - 1$, we need to prove that the inequality is true for $d = n$. For a fixed point $x' = (x_2, ..., x_n)$ we have

$$\|f(x_1, x')\|_{L^{q_1,r_1}_{x_1}} \leq \|g(x_1, x')\|_{L^{q_1,r_1}_{x_1}}.$$

Denote by $\mathbf{q}' = (q_2, ..., q_n)$, $\mathbf{r}' = (r_2, ..., r_n)$. By the inductive assumption we get

$$\|f\|_{L^{\mathbf{q},\mathbf{r}}} = \big\| \|f(x_1, x')\|_{L^{q_1,r_1}_{x_1}} \big\|_{L^{\mathbf{q}',\mathbf{r}'}_{x'}} \leq \big\| \|g(x_1, x')\|_{L^{q_1,r_1}_{x_1}} \big\|_{L^{\mathbf{q}',\mathbf{r}'}_{x'}} = \|g\|_{L^{\mathbf{q},\mathbf{r}}}.$$

(2) We can prove this proposition by induction. When $d = 1$, we need to prove that:
If $\|f\|_{L^{q,r}} < +\infty$ then $\mathcal{L}^1(\{x \in \mathbb{R} : |f(x)| = +\infty\}) = 0$.
Assume

$$\mathcal{L}^1(\{x \in \mathbb{R} : |f(x)| = +\infty\}) > 0.$$

We have $f^*(t) = +\infty$ for all t such that $0 < t < \mathcal{L}^1(\{x \in \mathbb{R} : |f(x)| = +\infty\})$ and then $\|f\|_{L^{q,r}} = +\infty$, this contradicts the hypothesis.
Assuming that the proposition is true for $d = n - 1$, we need to prove that the proposition is true for $d = n$. Assume that

$$\|f\|_{L^{\mathbf{q},\mathbf{r}}} = \big\| \|f(x_1, x')\|_{L^{q_1,r_1}_{x_1}} \big\|_{L^{\mathbf{q}',\mathbf{r}'}_{x'}} < +\infty.$$

By the inductive assumption we get

$$\mathcal{L}^{n-1}(M) = 0,$$

where

$$M = \{x' \in \mathbb{R}^{n-1} : \|f(x_1, x')\|_{L^{q_1,r_1}_{x_1}} = +\infty\}.$$

Assuming $x' \notin M$, since the proposition is true for $d = 1$, we have

$$\mathcal{L}^1(\{x_1 \in \mathbb{R} : |f(x_1, x')| = +\infty\}) = 0,$$

and then

$$\mathcal{L}^n(\{x \in \mathbb{R}^d : |f(x)| = +\infty\}) = \int_{\mathbb{R}^{n-1}} \mathcal{L}^1(\{x_1 \in \mathbb{R} : |f(x_1, x')| = +\infty\})\ \mathrm{d}x' =$$

$$\int_{\mathbb{R}^{n-1}\setminus M} \mathcal{L}^1(\{x_1 \in \mathbb{R} : |f(x_1, x')| = +\infty\})\ \mathrm{d}x' = 0.$$

(3) We can prove this proposition by induction. When $d = 1$, it is deduced from the properties of the standard Lorentz spaces. Assuming that the inequality is true for $d = n-1$, we need to prove that the inequality is true for $d = n$. For a fixed point $x' = (x_2, ..., x_n)$ we have

$$\|f(x_1, x')\|_{L^{q_1,\tilde{r}_1}_{x_1}} \leq \|f(x_1, x')\|_{L^{q_1,r_1}_{x_1}}.$$

Denote by $\mathbf{q}' = (q_2, ..., q_n)$, $\mathbf{r}' = (r_2, ..., r_n)$, $\tilde{\mathbf{r}}' = (\tilde{r}_2, ..., \tilde{r}_n)$. From the inductive assumption and (1) we get

$$\|f\|_{L^{\mathbf{q},\tilde{\mathbf{r}}}} = \left\| \|f(x_1, x')\|_{L^{q_1,\tilde{r}_1}_{x_1}} \right\|_{L^{\mathbf{q}',\tilde{\mathbf{r}}'}_{x'}} \leq \left\| \|f(x_1, x')\|_{L^{q_1,r_1}_{x_1}} \right\|_{L^{\mathbf{q}',\mathbf{r}'}_{x'}} = \|f\|_{L^{\mathbf{q},\mathbf{r}}}.$$

□

Theorem 2.6.2. (Holder's inequality for mixed-norm Lorentz spaces).
Let $1 < q_i, \tilde{q}_i, r_i < \infty$ and $1 \leq h_i, \tilde{h}_i, \hat{h}_i \leq \infty$ $(1 \leq i \leq d)$ satisfy the relations

$$\frac{1}{\boldsymbol{r}} = \frac{1}{\boldsymbol{q}} + \frac{1}{\tilde{\boldsymbol{q}}},\ \frac{1}{\boldsymbol{h}} = \frac{1}{\tilde{\boldsymbol{h}}} + \frac{1}{\hat{\boldsymbol{h}}}. \tag{2.186}$$

Suppose that $u \in L^{\boldsymbol{q},\tilde{\boldsymbol{h}}}$ and $v \in L^{\tilde{\boldsymbol{q}},\hat{\boldsymbol{h}}}$. Then $uv \in L^{\boldsymbol{r},\boldsymbol{h}}$ and we have the inequality

$$\|uv\|_{L^{\boldsymbol{r},\boldsymbol{h}}} \leq \|u\|_{L^{\boldsymbol{q},\tilde{\boldsymbol{h}}}} \|v\|_{L^{\tilde{\boldsymbol{q}},\hat{\boldsymbol{h}}}}. \tag{2.187}$$

Proof. We can prove this inequality by induction. When $d = 1$, this follows by applying Theorem 1.1.9 (c). Assuming that the inequality is true for $d = n - 1$, we need prove the inequality is true for $d = n$. We estimate the quantity $\|uv\|_{L^{r_1,h_1}_{x_1}}$ by calculating first the L^{r_1,h_1} - norm of $(uv)(x_1, x_2, ..., x_n)$ with respect to the variable x_1, and then applying Theorem 1.1.9 (c)

$$\|uv\|_{L^{r_1,h_1}} \leq \|u\|_{L^{q_1,\tilde{h}_1}} \|v\|_{L^{\tilde{q}_1,\hat{h}_1}}.$$

By the inductive hypothesis, the inequality (2.187) is true for $d = n - 1$. Therefore we can apply it to the functions $\|u\|_{L^{q_1,\tilde{h}_1}}$ and $\|v\|_{L^{\tilde{q}_1,\hat{h}_1}}$ with $d = n-1$, $\mathbf{r'} = (r_2, r_3, ..., r_n)$, $\mathbf{h'} = (h_2, h_3, ..., h_n)$, $\mathbf{q}' = (q_2, q_3, ..., q_n)$, $\tilde{\mathbf{h}}' = (\tilde{h}_2, \tilde{h}_3, ..., \tilde{h}_n)$, $\tilde{\mathbf{q}}' = (\tilde{q}_2, \tilde{q}_3, ..., \tilde{q}_n)$, $\hat{\mathbf{h}}' = (\hat{h}_2, \hat{h}_3, ..., \hat{h}_n)$ and use (1) of Lemma 2.6.1 in order to obtain

$$\begin{aligned}\|uv\|_{L^{\mathbf{r},\mathbf{h}}} &= \left\| \|uv\|_{L^{r_1,h_1}} \right\|_{L^{\mathbf{r}',\mathbf{h}'}} \leq \left\| \|u\|_{L^{q_1,\tilde{h}_1}} \|v\|_{L^{\tilde{q}_1,\hat{h}_1}} \right\|_{L^{\mathbf{r}',\mathbf{h}'}} \\ &\leq \left\| \|u\|_{L^{q_1,\tilde{h}_1}} \right\|_{L^{\mathbf{q}',\tilde{\mathbf{h}}'}} \left\| \|v\|_{L^{\tilde{q}_1,\hat{h}_1}} \right\|_{L^{\tilde{\mathbf{q}}',\hat{\mathbf{h}}'}} = \|u\|_{L^{\mathbf{q},\tilde{\mathbf{h}}}} \|v\|_{L^{\tilde{\mathbf{q}},\hat{\mathbf{h}}}}.\end{aligned}$$

□

Theorem 2.6.3. (Convolution in mixed-norm Lorentz spaces).
Let $1 < q_i, \tilde{q}_i, r_i < \infty$ and $1 \leq h_i, \tilde{h}_i, \hat{h}_i \leq \infty$ $(1 \leq i \leq d)$ satisfy the relations

$$\frac{1}{\boldsymbol{r}} + \boldsymbol{1} = \frac{1}{\boldsymbol{q}} + \frac{1}{\tilde{\boldsymbol{q}}},\ \frac{1}{\boldsymbol{h}} = \frac{1}{\tilde{\boldsymbol{h}}} + \frac{1}{\hat{\boldsymbol{h}}}, \tag{2.188}$$

where $\boldsymbol{1} = (1, 1, ..., 1) \in \mathbb{R}^d$. Suppose that $u \in L^{\boldsymbol{q},\tilde{\boldsymbol{h}}}$ and $v \in L^{\tilde{\boldsymbol{q}},\hat{\boldsymbol{h}}}$. Then $u * v \in L^{\boldsymbol{r},\boldsymbol{h}}$ and we have the inequality

$$\|u * v\|_{L^{\boldsymbol{r},\boldsymbol{h}}} \leq \|u\|_{L^{\boldsymbol{q},\tilde{\boldsymbol{h}}}} \|v\|_{L^{\tilde{\boldsymbol{q}},\hat{\boldsymbol{h}}}}. \tag{2.189}$$

Proof. We can prove this inequality by induction. When $d = 1$, this follows by applying Theorem 1.1.10 (c). Assuming that the inequality is true for $d = n - 1$, we need prove it for $d = n$. We have

$$\int_{\mathbb{R}^n} u(x-y).v(y)\mathrm{d}y = \int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} u(x_1 - y_1, x' - y').v(y_1, y')\mathrm{d}y_1 \Big)\mathrm{d}y', \tag{2.190}$$

where $x' = (x_2, x_3, ..., x_n),\ y' = (y_2, y_3, ..., y_n)$.
We calculate the quantity $\|u * v\|_{L^{r_1,h_1}_{x_1}}$ by calculating first the L^{r_1,h_1} - norm of $(u*v)(x_1, x_2, ..., x_n)$ with respect to the variable x_1, and assuming $x' \in \mathbb{R}^{n-1}$ such that

$$\int_{\mathbb{R}^{n-1}} \big\|u(x_1, x'-y')\big\|_{L^{q_1,\tilde{h}_1}_{x_1}} \big\|v(x_1, y')\big\|_{L^{\tilde{q}_1,\hat{h}_1}_{x_1}} \mathrm{d}y' < +\infty. \tag{2.191}$$

Then applying Theorem 1.1.10 (c) and using the inequality

$$\Big\| \int_{\mathbb{R}^{n-1}} f(x)\mathrm{d}x \Big\| \le \int_{\mathbb{R}^{n-1}} \|f(x)\|\mathrm{d}x,$$

we get

$$\begin{aligned} &\Big\| \int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} u(x_1 - y_1, x'-y')v(y_1,y')\mathrm{d}y_1 \Big)\mathrm{d}y' \Big\|_{L^{r_1,h_1}_{x_1}} \le \\ &\int_{\mathbb{R}^{n-1}} \Big\| \int_{\mathbb{R}} u(x_1 - y_1, x'-y')v(y_1,y')\mathrm{d}y_1 \Big\|_{L^{r_1,h_1}_{x_1}} \mathrm{d}y' \le \\ &\int_{\mathbb{R}^{n-1}} \big\|u(x_1, x'-y')\big\|_{L^{q_1,\tilde{h}_1}_{x_1}} \big\|v(x_1, y')\big\|_{L^{\tilde{q}_1,\hat{h}_1}_{x_1}} \mathrm{d}y'. \end{aligned} \tag{2.192}$$

By the inductive hypothesis, the inequality (2.189) is true for $d = n - 1$. So, we may apply it to the functions $\|u\|_{L^{q_1,\tilde{h}_1}}$ and $\|v\|_{L^{\tilde{q}_1,\hat{h}_1}}$ with $d = n-1$, $\mathbf{r'} = (r_2, r_3, ..., r_n)$, $\mathbf{h'} = (h_2, h_3, ..., h_n)$, $\mathbf{q}' = (q_2, q_3, ..., q_n)$, $\tilde{\mathbf{h}}' = (\tilde{h}_2, \tilde{h}_3, ..., \tilde{h}_n)$, $\tilde{\mathbf{q}}' = (\tilde{q}_2, \tilde{q}_3, ..., \tilde{q}_n)$, $\hat{\mathbf{h}}' = (\hat{h}_2, \hat{h}_3, ..., \hat{h}_n)$ in order to obtain

$$\begin{aligned} &\Big\| \int_{\mathbb{R}^{n-1}} \big\|u(x_1, x'-y')\big\|_{L^{q_1,\tilde{h}_1}_{x_1}} \big\|v(x_1, y')\big\|_{L^{\tilde{q}_1,\hat{h}_1}_{x_1}} \mathrm{d}y' \Big\|_{L^{\mathbf{r}',\mathbf{h}'}_{x'}} \le \\ &\big\| \|u(x_1,x')\|_{L^{q_1,\tilde{h}_1}_{x_1}} \big\|_{L^{\mathbf{q}',\tilde{\mathbf{h}}'}_{x'}} \big\| \|v(x_1,x')\|_{L^{\tilde{q}_1,\hat{h}_1}_{x_1}} \big\|_{L^{\tilde{\mathbf{q}}',\hat{\mathbf{h}}'}_{x'}} = \|u\|_{L^{\mathbf{q},\tilde{\mathbf{h}}}} \|v\|_{L^{\tilde{\mathbf{q}},\hat{\mathbf{h}}}}. \end{aligned} \tag{2.193}$$

From (2.193) and (2) of Lemma 2.6.1 we have (2.191) for almost every $x' \in \mathbb{R}^{n-1}$. From (2.192) and (2.193) with (1) of Lemma 2.6.1 we have

$$\begin{aligned} \Big\| \int_{\mathbb{R}^n} u(x-y)v(y)\mathrm{d}y \Big\|_{L^{\mathbf{r},\mathbf{h}}} &= \Big\| \Big\| \int_{\mathbb{R}^n} u(x-y)v(y)\mathrm{d}y \Big\|_{L^{r_1,h_1}_{x_1}} \Big\|_{L^{\mathbf{r}',\mathbf{h}'}_{x'}} \\ \le \Big\| \int_{\mathbb{R}^{n-1}} \big\|u(x_1, x'-y')\big\|_{L^{q_1,\tilde{h}_1}_{x_1}} \big\|v(x_1, y')\big\|_{L^{\tilde{q}_1,\hat{h}_1}_{x_1}} \mathrm{d}y' \Big\|_{L^{\mathbf{r}',\mathbf{h}'}_{x'}} &\le \|u\|_{L^{\mathbf{q},\tilde{\mathbf{h}}}} \|v\|_{L^{\tilde{\mathbf{q}},\hat{\mathbf{h}}}}. \end{aligned}$$

□

2.6.2. Mixed-norm Sobolev-Lorentz spaces

Definition 2.6.1. *For $m \in \mathbb{R}$ and $\boldsymbol{q}, \boldsymbol{r} \in \mathbb{R}^d$, $\boldsymbol{1} < \boldsymbol{q} < \infty, \boldsymbol{1} \le \boldsymbol{r} \le \infty$, the space $\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}$ is defined as the space $\dot{\Lambda}^{-m}L^{\boldsymbol{q},\boldsymbol{r}}$, equipped with the norm $\|u\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}} = \big\|\dot{\Lambda}^m u\big\|_{L^{\mathbf{q},\mathbf{r}}}$.*

Theorem 2.6.4. (Sobolev inequality for mixed-norm Sobolev-Lorentz spaces).
Let $\tilde{\boldsymbol{q}}, \boldsymbol{q}, \boldsymbol{r} \in \mathbb{R}^d$, $\boldsymbol{1} < \boldsymbol{q} < \tilde{\boldsymbol{q}} < \infty$, *and* $\boldsymbol{1} \leq \boldsymbol{r} \leq \infty$. *Then*

$$\|u\|_{L^{\tilde{\mathbf{q}},\boldsymbol{r}}} \lesssim \|u\|_{\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}}}, \tag{2.194}$$

where

$$m = \sum_{i=1}^{d} \Big(\frac{1}{q_i} - \frac{1}{\tilde{q}_i}\Big), \boldsymbol{q} = (q_1, q_2, ..., q_d), \tilde{\boldsymbol{q}} = (\tilde{q}_1, \tilde{q}_2, ..., \tilde{q}_d).$$

Proof. Note that the operator $\frac{1}{\dot{\Lambda}^m}$ is a convolution with kernel $\frac{c_{m,d}}{|x|^{d-m}} \in L^{\frac{d}{d-m},\infty}$. Let $m_i = \frac{1}{q_i} - \frac{1}{\tilde{q}_i} > 0$, we have

$$\|u\|_{L^{\tilde{\mathbf{q}},\mathbf{r}}} = \Big\|\frac{1}{\dot{\Lambda}^m}\dot{\Lambda}^m u\Big\|_{L^{\tilde{\mathbf{q}},\mathbf{r}}} \simeq \Big\|\int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-m}}\big(\dot{\Lambda}^m u\big)(y)\mathrm{d}y\Big\|_{L_x^{\tilde{\mathbf{q}},\mathbf{r}}} =$$

$$\Big\|\int_{\mathbb{R}^d} \prod_{i=1}^{d} \frac{1}{|x-y|^{1-m_i}}\big(\dot{\Lambda}^m u\big)(y)\mathrm{d}y\Big\|_{L_x^{\tilde{\mathbf{q}},\mathbf{r}}} =$$

$$\Big\|\int_{\mathbb{R}^d} \prod_{i=1}^{d} \frac{1}{\Big(\sqrt{\sum_{k=1}^d (x_k-y_k)^2}\Big)^{1-m_i}}\big(\dot{\Lambda}^m u\big)(y)\mathrm{d}y\Big\|_{L_x^{\tilde{\mathbf{q}},\mathbf{r}}} \lesssim$$

$$\Big\|\int_{\mathbb{R}^d} \prod_{i=1}^{d} \frac{1}{|x_i-y_i|^{1-m_i}}\big|(\dot{\Lambda}^m u)(y)\big|\mathrm{d}y\Big\|_{L_x^{\tilde{\mathbf{q}},\mathbf{r}}}. \tag{2.195}$$

For

$$\mathbf{h} = (h_1, h_2, ..., h_d),\ h_i = \frac{1}{1-m_i}, 1 \leq i \leq d,$$

we get

$$\frac{1}{|x_i|^{1-m_i}} \in L_{x_i}^{h_i,\infty}(\mathbb{R}),\ 1 \leq i \leq d.$$

Now, note that if

$$f(x) = \prod_{i=1}^{d} f_i(x_i),\ f_i \in L^{q_i,r_i}(\mathbb{R}),\ 1 < q_i < \infty,\ 1 \leq r_i \leq \infty,\ i = 1, 2, ..., d$$

then

$$f \in L^{\mathbf{q},\mathbf{r}}(\mathbb{R}^d), \mathbf{q} = (q_1, q_2, ..., q_d),\ \mathbf{r} = (r_1, r_2, ..., r_d),$$

$$\big\|f(x)\big\|_{L_x^{\mathbf{q},\mathbf{r}}(\mathbb{R}^d)} = \prod_{i=1}^{d} \big\|f_i(x_i)\big\|_{L_{x_i}^{q_i,r_i}(\mathbb{R})}.$$

The last equality can be proved easily from the definition of $L^{\mathbf{q},\mathbf{r}}$. It follows that

$$\Big\|\prod_{i=1}^{d} \frac{1}{|x_i|^{1-m_i}}\Big\|_{L_x^{\mathbf{h},\infty}} = \prod_{i=1}^{d} \Big\|\frac{1}{|x_i|^{1-m_i}}\Big\|_{L_{x_i}^{h_i,\infty}} < +\infty.$$

Applying Theorem 2.6.3, we obtain

$$\Big\|\int_{\mathbb{R}^d} \prod_{i=1}^{d} \frac{1}{|x_i-y_i|^{1-m_i}}\big|(\dot{\Lambda}^m u)(y)\big|\mathrm{d}y\Big\|_{L_x^{\tilde{\mathbf{q}},\mathbf{r}}} \lesssim$$

$$\Big\|\prod_{i=1}^{d} \frac{1}{|x_i|^{1-m_i}}\Big\|_{L_x^{\mathbf{h},\infty}} \big\|\dot{\Lambda}^m u\big\|_{L^{\mathbf{q},\mathbf{r}}} \lesssim \|u\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}}. \tag{2.196}$$

Combining (2.195), (2.196) we obtain (2.194). □

2.6.3. $L^pL^{\mathbf{q},\mathbf{r}}$ solutions of the Navier-Stokes equations

Lemma 2.6.5. *Let*

$$\boldsymbol{q} = (q_1, q_2, ..., q_d),\ \ \boldsymbol{r} = (r_1, r_2, ..., r_d), 2 < p < \infty,\ \ m \geq 0,\ \ \text{and } 0 < T < \infty, \tag{2.197}$$

be such that

$$m < \frac{1}{2}\sum_{i=1}^{d} \frac{1}{q_i},\ \ \frac{2}{p} - m + \sum_{i=1}^{d} \frac{1}{q_i} \leq 1, \tag{2.198}$$

$$1 \leq r_i \leq \infty,\ \ 2 < \frac{q_i}{\left(1 - \frac{m}{\sum_{i=1}^{d} \frac{1}{q_i}}\right)} < \infty,\ \ i = 1, 2, .., d. \tag{2.199}$$

Then the bilinear operator $B(u, v)(t)$ is continuous from $L^p([0, T]; \dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}}) \times L^p([0, T]; \dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})$ to $L^p([0, T]; \dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})$ and we have the inequality

$$\big\|B(u, v)\big\|_{L^p([0,T];\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})} \lesssim T^{\frac{1}{2}(1+m-\frac{2}{p}-\sum_{i=1}^{d}\frac{1}{q_i})}\big\|u\big\|_{L^p([0,T];\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})}\big\|v\big\|_{L^p([0,T];\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})}. \tag{2.200}$$

Proof. Let us estimate

$$\big\|B(u, v)(t)\big\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}} \leq \int_0^t \Big\|e^{(t-s)\Delta}\mathbb{P}\nabla \cdot \big(u(s, \cdot) \otimes v(s, \cdot)\big)\Big\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}} \mathrm{d}s =$$
$$\int_0^t \Big\|\dot{\Lambda}^m e^{(t-s)\Delta}\mathbb{P}\nabla \cdot \big(u(s, \cdot) \otimes v(s, \cdot)\big)\Big\|_{L^{\mathbf{q},\mathbf{r}}} \mathrm{d}s, \tag{2.201}$$

We use the Fourier transform to get

$$\Big(\dot{\Lambda}^m e^{(t-s)\Delta}\mathbb{P}\nabla \cdot \big(u(s, \cdot) \otimes v(s, \cdot)\big)\Big)_j = \frac{1}{(t-s)^{\frac{d+m+1}{2}}} \sum_{l,k=1}^{d} K_{l,k,j}\Big(\frac{\cdot}{\sqrt{t-s}}\Big) * \big(u_l(s, \cdot)v_k(s, \cdot)\big). \tag{2.202}$$

Applying Lemma 1.2.1 with $|\alpha| = 1 + m$ we obtain

$$|K_{l,k,j}(x)| \lesssim \frac{1}{(1 + |x|)^{d+m+1}} \leq \frac{1}{(1 + |x|)^{d+1}}.$$

Thus, the tensor $K(x) = \{K_{l,k,j}(x)\}$ satisfies

$$|K(x)| \lesssim \frac{1}{(1 + |x|)^{d+1}}. \tag{2.203}$$

So, we can rewrite (2.202) in the tensor form

$$\dot{\Lambda}^m e^{(t-s)\Delta}\mathbb{P}\nabla \cdot \big(u(s, \cdot) \otimes v(s, \cdot)\big) = \frac{1}{(t-s)^{\frac{d+m+1}{2}}} K\big(\frac{\cdot}{\sqrt{t-s}}\big) * \big(u(s, \cdot) \otimes v(s, \cdot)\big).$$

Applying Theorem 2.6.3, we have that

$$\Big\|\dot{\Lambda}^m e^{(t-s)\Delta}\mathbb{P}\nabla \cdot \big(u(s, \cdot) \otimes v(s, \cdot)\big)\Big\|_{L^{\mathbf{q},\mathbf{r}}} \leq \Big\|\dot{\Lambda}^m e^{(t-s)\Delta}\mathbb{P}\nabla \cdot \big(u(s, \cdot) \otimes v(s, \cdot)\big)\Big\|_{L^{\mathbf{q},\mathbf{1}}}$$
$$\lesssim \frac{1}{(t-s)^{\frac{d+m+1}{2}}} \Big\|K\big(\frac{\cdot}{\sqrt{t-s}}\big)\Big\|_{L^{\mathbf{q}_1,\mathbf{1}}} \|u(s, \cdot) \otimes v(s, \cdot)\|_{L^{\mathbf{q}_2,\infty}}, \tag{2.204}$$

with

$$\frac{1}{\mathbf{q}_1} = \mathbf{1} - \Big(1 - \frac{2m}{\sum_{i=1}^d \frac{1}{q_i}}\Big)\frac{1}{\mathbf{q}},\ \ \frac{1}{\mathbf{q}_2} = 2\Big(1 - \frac{m}{\sum_{i=1}^d \frac{1}{q_i}}\Big)\frac{1}{\mathbf{q}} \tag{2.205}$$

satisfying

$$\frac{1}{\mathbf{q}} + \mathbf{1} = \frac{1}{\mathbf{q}_1} + \frac{1}{\mathbf{q}_2},\ \ \mathbf{1} < \mathbf{q}, \mathbf{q}_1, \mathbf{q}_2 < \infty. \tag{2.206}$$

Notice that from (2.197), (2.198) and (2.199), we can check that the condition (2.206) is satisfied. Applying Theorems 2.6.2 and 2.6.4 we have

$$\begin{gathered}\big\|u(s,\cdot)\otimes v(s,\cdot)\big\|_{L^{\mathbf{q}_2,\infty}} \leq \|u(s,\cdot)\|_{L^{\mathbf{q}_3,\infty}}\|v(s..)\|_{L^{\mathbf{q}_3,\infty}} \lesssim \\ \|u(s,\cdot)\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}}\|v(s,\cdot)\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}},\ \text{where}\ \frac{1}{\mathbf{q}_3} = \frac{1}{2\mathbf{q}_2} = \Big(1 - \frac{m}{\sum_{i=1}^d \frac{1}{q_i}}\Big)\frac{1}{\mathbf{q}}.\end{gathered} \tag{2.207}$$

From (2.203) we have

$$\begin{gathered}\Big\|K\big(\frac{x}{\sqrt{t}}\big)\Big\|_{L_x^{\mathbf{q}_1,\mathbf{1}}} \lesssim \Big\|\Big(1+\frac{|x|}{\sqrt{t}}\Big)^{-(d+1)}\Big\|_{L_x^{\mathbf{q}_1,\mathbf{1}}} = \Big\|\Big(1+\frac{(\sum_{k=1}^d x_k^2)^{1/2}}{\sqrt{t}}\Big)^{-(d+1)}\Big\|_{L_x^{\mathbf{q}_1,\mathbf{1}}} \\ = \Big\|\prod_{i=1}^d\Big(1+\frac{(\sum_{k=1}^d x_k^2)^{1/2}}{\sqrt{t}}\Big)^{-\frac{d+1}{d}}\Big\|_{L_x^{\mathbf{q}_1,\mathbf{1}}} \leq \Big\|\prod_{i=1}^d\Big(1+\frac{|x_i|}{\sqrt{t}}\Big)^{-\frac{d+1}{d}}\Big\|_{L_x^{\mathbf{q}_1,\mathbf{1}}} = \\ \prod_{i=1}^d\Big\|\Big(1+\frac{|x_i|}{\sqrt{t}}\Big)^{-\frac{d+1}{d}}\Big\|_{L_{x_i}^{q_{1,i},1}},\end{gathered} \tag{2.208}$$

where

$$\mathbf{q}_1 = (q_{1,1}, q_{1,2}, ..., q_{1,d}).$$

Using the interpolation inequality

$$L^{p,q} = [L^1, L^\infty]_{1-\frac{1}{p},q},\ \ \|u\|_{L^{p,q}} \lesssim \|u\|_{L^1}^{1/p}\|u\|_{L^\infty}^{1-1/p}, 1 < p < \infty, 1 \leq q \leq \infty,$$

we get

$$\begin{gathered}\Big\|\Big(1+\frac{|x_i|}{\sqrt{t}}\Big)^{-\frac{d+1}{d}}\Big\|_{L_{x_i}^{q_{1,i},1}} \lesssim \Big\|\Big(1+\frac{|x_i|}{\sqrt{t}}\Big)^{-\frac{d+1}{d}}\Big\|_{L^1_{x_i}}^{\frac{1}{q_{1,i}}}\Big\|\Big(1+\frac{|x_i|}{\sqrt{t}}\Big)^{-\frac{d+1}{d}}\Big\|_{L^\infty_{x_i}}^{1-\frac{1}{q_{1,i}}} \\ \lesssim \Big\|\Big(1+\frac{|x_i|}{\sqrt{t}}\Big)^{-\frac{d+1}{d}}\Big\|_{L^1_{x_i}}^{\frac{1}{q_{1,i}}} \simeq t^{\frac{1}{2q_{1,i}}},\ \ i = 1, 2, ..., d.\end{gathered} \tag{2.209}$$

From (2.208), (2.209) and (2.205) we obtain

$$\Big\|K\Big(\frac{x}{\sqrt{t}}\Big)\Big\|_{L_x^{\mathbf{q}_1,1}} \lesssim t^{\frac{1}{2}\sum_{i=1}^d \frac{1}{q_{1,i}}} = t^{\frac{(d+2m-\sum_{i=1}^d \frac{1}{q_i})}{2}}. \tag{2.210}$$

From (2.204), (2.207), (2.210) we deduce that

$$\begin{gathered}\Big\|\dot{\Lambda}^m e^{(t-s)\Delta}\mathbb{P}\nabla\cdot\big(u(s,\cdot)\otimes v(s,\cdot)\big)\Big\|_{L^{\mathbf{q},\mathbf{r}}} \lesssim \\ (t-s)^{\frac{(m-1-\sum_{i=1}^d \frac{1}{q_i})}{2}}\big\|u(s,\cdot)\big\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}}\big\|v(s,\cdot)\big\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}}.\end{gathered} \tag{2.211}$$

From (2.201) and (2.211) we have

$$\big\|B(u,v)(t)\big\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}} \lesssim \int_0^t (t-s)^{\frac{(m-1-\sum_{i=1}^d \frac{1}{q_i})}{2}}\big\|u(s,\cdot)\big\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}}\big\|v(s,\cdot)\big\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}}\mathrm{d}s.$$

Applying Theorem 1.1.10 (c) we have

$$\left\|\|B(u,v)(t)\|_{\dot H^m_{L\mathbf{q},\mathbf{r}}}\right\|_{L^p_t} = \left\|\|B(u,v)(t)\|_{\dot H^m_{L\mathbf{q},\mathbf{r}}}\right\|_{L^{p,p}_t} \leq \left\|\|B(u,v)(t)\|_{\dot H^m_{L\mathbf{q},\mathbf{r}}}\right\|_{L^{p,p/2}_t}$$

$$\lesssim \left\|1_{[0,T]}t^{\frac{(m-1-\sum_{i=1}^d \frac{1}{q_i})}{2}}\right\|_{L^{p',\infty}} \left\|\|u(t,\cdot)\|_{\dot H^m_{L\mathbf{q},\mathbf{r}}}\|v(t,\cdot)\|_{\dot H^m_{L\mathbf{q},\mathbf{r}}}\right\|_{L^{p/2,p/2}_t}, \tag{2.212}$$

where $\frac{1}{p'}+\frac{1}{p}=1$, and $1_{[0,T]}$ is the indicator function of the set $[0,T]$ on $\mathbb{R}$. By applying the Holder inequality we get

$$\left\|\|u(t,\cdot)\|_{\dot H^m_{L\mathbf{q},\mathbf{r}}}\|v(t,\cdot)\|_{\dot H^m_{L\mathbf{q},\mathbf{r}}}\right\|_{L^{p/2,p/2}_t} = \left\|\|u(t,\cdot)\|_{\dot H^m_{L\mathbf{q},\mathbf{r}}}\|v(t,\cdot)\|_{\dot H^m_{L\mathbf{q},\mathbf{r}}}\right\|_{L^{p/2}_t}$$

$$\leq \left\|\|u(t,\cdot)\|_{\dot H^m_{L\mathbf{q},\mathbf{r}}}\right\|_{L^p_t}\left\|\|v(t,\cdot)\|_{\dot H^m_{L\mathbf{q},\mathbf{r}}}\right\|_{L^p_t}. \tag{2.213}$$

We deduce that

$$\left\|1_{[0,T]}t^{\frac{(m-1-\sum_{i=1}^d \frac{1}{q_i})}{2}}\right\|_{L^{p',\infty}} \simeq T^{\frac{1}{2}(1+m-\frac{2}{p}-\sum_{i=1}^d \frac{1}{q_i})}. \tag{2.214}$$

The estimate (2.200) follows from (2.212), (2.213), (2.214). □

Combining Lemma 2.6.5 with Theorem 1.5.1 we obtain the following existence result.

Theorem 2.6.6. *Let*

$$\boldsymbol{q}=(q_1,q_2,...,q_d),\ \ \boldsymbol{r}=(r_1,r_2,...,r_d), 2<p<\infty, \textit{and } m\geq 0$$

be such that

$$m<\frac{1}{2}\sum_{i=1}^d\frac{1}{q_i},\ \ \frac{2}{p}-m+\sum_{i=1}^d\frac{1}{q_i}\leq 1,$$

$$1\leq r_i\leq\infty,\ \ 2<\frac{q_i}{\left(1-\frac{m}{\sum_{i=1}^d\frac{1}{q_i}}\right)}<\infty,\ \ i=1,2,..,d.$$

(a) *There exists a positive constant $\delta_{(m,\boldsymbol{q},\boldsymbol{r},p)}>0$ such that for all $T>0$ and for all $u_0\in\mathcal{S}'(\mathbb{R}^d)$ with* $\operatorname{div}(u_0)=0$ *satisfying*

$$T^{\frac{1}{2}(1+m-\frac{2}{p}-\sum_{i=1}^d \frac{1}{q_i})}\left\|e^{\cdot\Delta}u_0\right\|_{L^p([0,T];\dot H^m_{L\boldsymbol{q},\boldsymbol{r}})}\leq\delta_{(m,\boldsymbol{q},\boldsymbol{r},p)}, \tag{2.215}$$

there is a unique mild solution $u\in L^p([0,T];\dot H^m_{L\boldsymbol{q},\boldsymbol{r}})$ for NSE.
If

$$e^{\cdot\Delta}u_0\in L^p([0,1];\dot H^m_{L\boldsymbol{q},\boldsymbol{r}}),$$

then the inequality (2.215) *holds when $T(u_0)$ is small enough.*
(b) *If $\frac{2}{p}+\sum_{i=1}^d\frac{1}{q_i}-m=1$ then there exists a positive $\delta_{(m,\boldsymbol{q},\boldsymbol{r},p)}>0$ such that we can take $T=\infty$ whenever $\left\|e^{\cdot\Delta}u_0\right\|_{L^p([0,\infty];\dot H^m_{L\boldsymbol{q},\boldsymbol{r}})}\leq\delta_{(m,\boldsymbol{q},\boldsymbol{r},p)}$.*

Proof. In order to prove (a), from Lemma 2.6.5, we use the estimate

$$\|B\|_{L^p([0,T];\dot H^m_{L\mathbf{q},\mathbf{r}})}\leq CT^{\frac{1}{2}(1+m-\frac{2}{p}-\sum_{i=1}^d \frac{1}{q_i})}.$$

From Theorem 1.5.1 and the above inequality, we deduce the existence of a solution to the Navier-Stokes equations on the interval $(0,T)$ with

$$4CT^{\frac{1}{2}(1+m-\frac{2}{p}-\sum_{i=1}^d \frac{1}{q_i})}\left\|e^{\cdot\Delta}u_0\right\|_{L^p([0,T];\dot H^m_{L\mathbf{q},\mathbf{r}})}\leq 1.$$

If $e^{\cdot\Delta}u_0 \in L^p([0,1];\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}})$ then this condition is fulfilled for $T = T(u_0)$ small enough. This is obvious for the case when $\frac{2}{p} + \sum_{i=1}^d \frac{1}{q_i} - m < 1$ since $\lim_{T\to 0} T^{\frac{1}{2}(1+m-\frac{2}{p}-\sum_{i=1}^d \frac{1}{q_i})} = 0$. For the case when $\frac{2}{p} + \sum_{i=1}^d \frac{1}{q_i} - m = 1$, the condition is fulfilled since we have $\lim_{T\to 0} \|e^{\cdot\Delta}u_0\|_{L^p([0,T];\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}})} = 0$.
(b) This is obvious. □

Remark 2.6.2. From Theorem 5.3 ([46], p. 44), if $u_0 \in B^{m-\frac{2}{p},p}_{L^{\mathbf{q},\mathbf{r}}}$ then $e^{\cdot\Delta}u_0 \in L^p([0,1];\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}})$. From Theorem 5.4 ([46], p. 45), $u_0 \in \dot{B}^{m-\frac{2}{p},p}_{L^{\mathbf{q},\mathbf{r}}}$ is equivalent to $e^{\cdot\Delta}u_0 \in L^p([0,\infty];\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}})$.

In the case $m = 0$ we have the following consequences.

Theorem 2.6.7. *Let*

$$\boldsymbol{q} = (q_1, q_2, ..., q_d),\ \boldsymbol{r} = (r_1, r_2, ..., r_d),\ \text{and } 2 < p < \infty$$

be such that

$$\frac{2}{p} + \sum_{i=1}^d \frac{1}{q_i} \le 1,\ 2 < q_i < \infty,\ 1 \le r_i \le \infty,\ i = 1, 2, ..., d.$$

(a) *There exists a positive constant $\delta_{(\boldsymbol{q},\boldsymbol{r},p)} > 0$ such that for all $T > 0$ and for all $u_0 \in (\mathcal{S}'(\mathbb{R}^d))^d$ with $\operatorname{div}(u_0) = 0$ satisfying*

$$T^{\frac{1}{2}(1-\frac{2}{p}-\sum_{i=1}^d \frac{1}{q_i})} \big\|e^{\cdot\Delta}u_0\big\|_{L^p([0,T];L^{\boldsymbol{q},\boldsymbol{r}})} \le \delta_{(\boldsymbol{q},\boldsymbol{r},p)}, \tag{2.216}$$

there is a unique mild solution $u \in L^p([0,T];L^{\boldsymbol{q},\boldsymbol{r}})$ for NSE.
If

$$e^{\cdot\Delta}u_0 \in L^p([0,1];L^{\boldsymbol{q},\boldsymbol{r}}),$$

then the inequality (2.216) *holds when $T(u_0)$ is small enough.*
(b) *If $\frac{2}{p} + \sum_{i=1}^d \frac{1}{q_i} = 1$ then there exists a positive $\delta_{(\boldsymbol{q},\boldsymbol{r},p)} > 0$ such that we can take $T = \infty$ whenever $\|e^{\cdot\Delta}u_0\|_{L^p([0,\infty];L^{\boldsymbol{q},\boldsymbol{r}})} \le \delta_{(\boldsymbol{q},\boldsymbol{r},p)}$.*

Remark 2.6.3. If $u_0 \in B^{-\frac{2}{p},p}_{L^{\mathbf{q},\mathbf{r}}}$ then $e^{\cdot\Delta}u_0 \in L^p([0,1];L^{\mathbf{q},\mathbf{r}})$, and $u_0 \in \dot{B}^{-\frac{2}{p},p}_{L^{\mathbf{q},\mathbf{r}}}$ is equivalent to $e^{\cdot\Delta}u_0 \in L^p([0,\infty];L^{\mathbf{q},\mathbf{r}})$.

In what follows we consider the case $p = \infty$.

Lemma 2.6.8. *Let*

$$\boldsymbol{q} = (q_1, q_2, ..., q_d),\ \boldsymbol{r} = (r_1, r_2, ..., r_d),\ m \ge 0,\ \text{and }\ 0 < T < \infty$$

be such that

$$m < \frac{1}{2}\sum_{i=1}^d \frac{1}{q_i},\ \sum_{i=1}^d \frac{1}{q_i} - m < 1,$$
$$1 \le r_i \le \infty,\ 2 < \frac{q_i}{\left(1 - \frac{m}{\sum_{i=1}^d \frac{1}{q_i}}\right)} < \infty,\ i = 1, 2, .., d.$$

Then the bilinear operator $B(u,v)(t)$ is continuous from $L^\infty([0,T];\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}}) \times L^\infty([0,T];\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})$ to $L^\infty([0,T];\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})$ and we have the inequality

$$\big\|B(u,v)\big\|_{L^\infty([0,T];\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})} \lesssim T^{\frac{1}{2}(1+m-\sum_{i=1}^d \frac{1}{q_i})} \big\|u\big\|_{L^\infty([0,T];\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})} \big\|v\big\|_{L^\infty([0,T];\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})}.$$

Proof. Lemma 2.6.8 can be obtained from the proof of Lemma 2.6.5 with a slight change. We omit the details. □

Lemma 2.6.9. *If* $u \in \dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}$ *then* $\|e^{\cdot\Delta}u\|_{L^\infty([0,\infty];\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}})} \le \|u\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}}$.

Proof. We have

$$\|e^{t\Delta}u\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}} = \|e^{t\Delta}\dot{\Lambda}^m u\|_{L^{\mathbf{q},\mathbf{r}}} = \frac{1}{(4\pi t)^{d/2}}\Big\|\int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}}(\dot{\Lambda}^m u)(\cdot-\xi)\mathrm{d}\xi\Big\|_{L^{\mathbf{q},\mathbf{r}}} \le$$

$$\frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}}\Big\|(\dot{\Lambda}^m u)(\cdot-\xi)\Big\|_{L^{\mathbf{q},\mathbf{r}}}\mathrm{d}\xi = \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}}\|u\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}}\mathrm{d}\xi = \|u\|_{\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}}.$$

□

Combining Lemmas 2.6.8 and 2.6.9 with Theorem 1.5.1 we obtain the following existence result.

Theorem 2.6.10. *Let*

$$\boldsymbol{q} = (q_1, q_2, ..., q_d),\ \ \boldsymbol{r} = (r_1, r_2, ..., r_d), \text{and } m \ge 0$$

be such that

$$m < \frac{1}{2}\sum_{i=1}^d \frac{1}{q_i},\ \sum_{i=1}^d \frac{1}{q_i} - m < 1,$$
$$1 \le r_i \le \infty, 2 < \frac{q_i}{\left(1 - \frac{m}{\sum_{i=1}^d \frac{1}{q_i}}\right)} < \infty,\ \ i = 1, 2, .., d.$$

There exists a positive constant $\delta_{(m,\boldsymbol{q},\boldsymbol{r})} > 0$ *such that for all* $T > 0$ *and for all* $u_0 \in \dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}}$ *with* $\mathrm{div}(u_0) = 0$ *satisfying*

$$T^{\frac{1}{2}(1+m-\sum_{i=1}^d \frac{1}{q_i})}\|u_0\|_{\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}}} \le \delta_{(m,\boldsymbol{q},\boldsymbol{r})}. \tag{2.217}$$

Then is a unique mild solution $u \in L^\infty([0,T];\dot{H}^m_{L^{\boldsymbol{q},\boldsymbol{r}}})$ *for NSE and the inequality* (2.217) *holds when* $T(u_0)$ *is small enough.*

2.6.4. Uniqueness theorems

In this subsection, we give a theorem on the uniqueness of solutions. The result obtained here is more general than the classical theorem of Serrin (see [60]).

Definition 2.6.4. (Pointwise multipliers of negative order, see [46]).
For $0 \le r < d/2$, *we define the space* $X_r(\mathbb{R}^d)$ *as the space of functions, which are locally square-integrable on* R^d *and such that pointwise multiplication with these functions maps boundedly* $H^r(\mathbb{R}^d)$ *to* $L^2(\mathbb{R}^d)$. *The norm of* X_r *is given by the operator norm of pointwise multiplication:*

$$\|f\|_{X_r} = \sup_{\|g\|_{H^r}\le 1}\{\|fg\|_2\}. \tag{2.218}$$

Lemma 2.6.11. *Let* $\boldsymbol{q} \in \mathbb{R}^d, \boldsymbol{2} < \boldsymbol{q} < \infty, \sum_{i=1}^d \frac{1}{q_i} = r$. *Then we have the imbedding map* $L^{\boldsymbol{q},\infty} \hookrightarrow X_r$.

Proof. Let
$$\frac{1}{\tilde{q}_i} = \frac{1}{2} - \frac{1}{q_i} \text{ and } 1 \le i \le d.$$
We have that
$$2 < \tilde{q}_i < \infty, 1 \le i \le d, \text{ and } \sum_{i=1}^{d} \Big(\frac{1}{2} - \frac{1}{\tilde{q}_i}\Big) = r.$$
Assuming $f \in L^{\mathbf{q},\infty}$ and $g \in H^r$, we can apply Theorems 2.6.2 and 2.6.4 in order to obtain
$$\|fg\|_{L^2} \le \|f\|_{L^{\mathbf{q},\infty}} \|g\|_{L^{\tilde{\mathbf{q}},2}} \lesssim \|f\|_{L^{\mathbf{q},\infty}} \big\|\dot{\Lambda}^r g\big\|_{L^2} \lesssim \|f\|_{L^{\mathbf{q},\infty}} \|g\|_{H^r},$$
and then
$$\|f\|_{X_r} \lesssim \|f\|_{L^{\mathbf{q},\infty}(\mathbb{R}^d)}.$$
□

Theorem 2.6.12. *If $u \in L^p\big((0,T),(L^{\boldsymbol{q},\infty}(\mathbb{R}^d))^d\big)$ is a Leray weak solution associated with u_0, where $p \in \mathbb{R}, \boldsymbol{q} \in \mathbb{R}^d$, $\boldsymbol{2} < \boldsymbol{q} < \infty$, $2 < p < \infty$ and $\frac{2}{p} + \sum_{i=1}^{d} \frac{1}{q_i} = 1$, then the condition iii) of Theorem* 21.2 ([46], p. 212) *is satisfied, and $u \in L^p\big((0,T),(X_r)^d\big)$ where $r = \sum_{i=1}^{d} \frac{1}{q_i} \in (0,1), \frac{2}{p} + r = 1$, and u is the unique Leray solution associated with u_0 on $(0,T)$.*

Proof. By using Lemma 2.6.11 we see that the condition *iii*) of Theorem 21.2 ([46], p. 212) is satisfied. Therefore the uniqueness follows. □

2.6.5. Conclusions

L^pL^q solutions of NSE have been considered by many authors in the 60's (see [60] and the reference therein), and continued by others in the 70's (see [19] and therein references). In the 80's, they have been thoroughly investigated in the paper [27]. L^pL^q spaces are defined as follows
$$L^pL^q = \{f \in L^p([0,T], L^q(\mathbb{R}^d))\}.$$
In the 90's (see for instance [11, 20, 21, 38, 62, 36]), those results have been extend to spaces based on Morrey-Campanato spaces instead of the Lebesgue spaces. In this section we investigate solutions of the NSE in the mixed norm Sobolev-Lorentz spaces and obtain some results which are more general than those in some of the cited papers.

This chapter was written on the basis of the papers

[1] D. Q. Khai and N. M. Tri, *Well-posedness for the Navier-Stokes equations with data in Sobolev-Lorentz spaces*, Nonlinear Analysis, **149** (2017), 130-145.

[2] D. Q. Khai, *Well-posedness for the Navier-Stokes equations with datum in the Sobolev spaces*, Acta Math Vietnam (2016). doi:10.1007/s40306-016-0192-x.

[3] D. Q. Khai and N. M. Tri, *Well-posedness for the Navier-Stokes equations with datum in Sobolev-Fourier-Lorentz spaces*, Journal of Mathematical Analysis and Applications, **437** (2016), 854-781.

[4] D. Q. Khai and N. M. Tri, *On the initial value problem for the Navier-Stokes equations with the initial datum in critical Sobolev and Besov spaces*, Journal of Mathematical Sciences, the University of Tokyo, **23** (2016), 499-528.

[5] D. Q. Khai and N. M. Tri, *Solutions in mixed-norm Sobolev-Lorentz spaces to the initial value problem for the Navier-Stokes equations*, Journal of Mathematical Analysis and Applications, **417** (2014), 819-833.

Chapter 3

Hausdorff dimension of the set of singularities for weak solutions

In this chapter we investigate the Hausdorff dimension of the possible singular set in time of weak solutions to the Navier-Stokes equation on the three dimensional torus under some regularity conditions of Serrin's type. The results in the paper relate the regularity conditions of Serrin's type to the Hausdorff dimension of the singular set in time. More precisely, we prove that if a weak solution u belongs to $L^r(0,T;V_\alpha)$ then the $(1-\frac{r(2\alpha-1)}{4})$ - dimensional Hausdorff measure of the singular set in time of u is zero. Here r is just assumed to be possitive. We also establish that if a weak solution u belongs to $L^r(0,T;W^{1,q})$ then the $(1-\frac{r(2q-3)}{2q})$ - dimensional Hausdorff measure of the singular set in time of u is zero. When $r=2,\alpha=1$ or $r=2,q=2$ we recover a result of Leray and Scheffer (see [49, 58, 64]).

3.1 Functional setting of the equations

In this chapter, we consider the initial value problem for the non stationary Navier-Stokes equations on the torus $\mathbb{T}^3=\mathbb{R}^3/\mathbb{Z}^3$, or in other words in $\mathbb{R}^3$ with periodic boundary conditions

$$\frac{\partial u_i}{\partial t}+\sum_{j=1}^{3}u_j\frac{\partial u_i}{\partial x_j}-\Delta u_i+\frac{\partial p}{\partial x_i}-f_i=0 \text{ on } \mathbb{T}^3_T:=\mathbb{T}^3\times(0,T), i=\overline{1,3} \tag{3.1}$$

$$\operatorname{div}(u)=\sum_{i=0}^{3}\frac{\partial u_i}{\partial x_i}=0 \text{ on } \mathbb{T}^3_T, \tag{3.2}$$

$$u(x,0)=u_0(x) \text{ in } \mathbb{T}^3\times\{0\}, \tag{3.3}$$

where $f_i(x,t)=(f_1(x,t),f_2(x,t),f_3(x,t)),u_0(x)$ are given functions with $u_0(x)$ satisfying the condition $\operatorname{div}(u_0)=0$. Denote by $\dot{\mathcal{V}}(\mathbb{T}^3)$ the space of all infinitely differentiable solenoidal vector fields with zero averaging on $\mathbb{T}^3$; by $\dot{\mathcal{V}}(\mathbb{T}^3_T)$ the space of all compactly supported in $\mathbb{T}^3_T$ infinitely differentiable solenoidal vector fields with zero averaging on $\mathbb{T}^3$ for each $t\in[0,T]$; H,V are the closures of the set $\dot{\mathcal{V}}(\mathbb{T}^3)$ in the spaces $L^2(\mathbb{T}^3),H^1(\mathbb{T}^3)$, respectively. Assume that $f\in L^\infty(0,T;V'),u_0\in H$, where V' is the dual space of V. A weak solution of the problem (3.1) - (3.3) in $\mathbb{T}^3_T$ is a vector field such that

$$u \in L^2((0,T);V) \cap L^\infty((0,T);H) \cap C([0,T];L^2_{\mathrm{w}});$$

$$\int_{\mathbb{T}^3_T} \Big(-\sum_{i=1}^3 u_i \frac{\partial v_i}{\partial t} + \sum_{i,j=1}^3 \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} + u_i u_j \frac{\partial v_i}{\partial x_j}\Big) \mathrm{d}x\mathrm{d}t = < f, v >, \forall v \in \dot{C}^\infty(\mathbb{T}^3_T);$$

$$\frac{1}{2}\int_{\mathbb{T}^3} \sum_{i=1}^3 |u_i(x,t_1)|^2 \mathrm{d}x + \int_{\mathbb{T}^3 \times (t_0,t_1)} \sum_{i,j=1}^3 \Big|\frac{\partial u_i}{\partial x_j}\Big|^2 \mathrm{d}x\mathrm{d}t \le \frac{1}{2}\int_{\mathbb{T}^3} \sum_{i=1}^3 |u_i(x,t_0)|^2 \mathrm{d}x;$$

$$\forall t_0 \in [0,T]\backslash\Sigma,\ t_1 \in [t_0,T], \text{ where } \Sigma \text{ has Lebesgue measure zero and } 0 \notin \Sigma;$$

$$\|u(x,t) - u_0(x)\|_{L^2(\mathbb{T}^3)} \to 0 \text{ as } t \to 0,$$

where $< .,. >$ is the pairing between V and V'. It was proved by Leray that there exists at least one weak solution of the problem (3.1) - (3.3).

The classical results on the local existence of strong solutions and global existence of weak solutions to the initial boundary value problems for the Navier-Stokes equations were obtained in [31, 49, 42] (see also the monographs [43, 63]). The study of Navier -Stokes equations on the torus is overviewed in [64]. The Hausdorff measures of singular sets of weak (or suitably weak) solutions to the Navier-Stokes equations were investigated in [9, 24, 59, 58] (see also the references therein). The uniqueness and regularity of weak solutions with some additional assumptions, say the Serrin conditions L^rL^q or $L^rW^{1,q}$ were proved in numerous papers (see [6, 61] and the references therein). In this chapter we study the Hausdorff dimension of the possible singular set in time of weak solutions under additional assumptions that the solutions belong to L^rH^α or $L^rW^{1,q}$. The chapter is organized as follows. In Section 3.2 we consider weak solutions, which belong to L^rH^α. In Section 3.3 we consider weak solutions, which belong to $L^rW^{1,q}$. Throughout the chapter we denote by C a general constant which may vary from place to place and it can take different value even in one line.

3.2 Weak solutions in L^rH^α

Let $A = -\Delta u$ with $D(A) = \{u \in H,\ \ \Delta u \in H\}$ and G be the orthogonal complement of H in $L^2(\mathbb{T}^3)$. The operator A can be seen as an unbounded positive linear selfadjoint operator on H, and we can define the powers $A^\alpha, \alpha \in \mathbb{R}$, with domain $D(A^\alpha)$. Denote $V_\alpha = D(A^{\alpha/2})$. Then A is an isomorphism from $V_{\alpha+2}$ onto V_α. The norm of an element $u \in V_\alpha$ will be denoted by $|u|_\alpha$.

For $u, v, w \in \dot{\mathcal{V}}(\mathbb{T}^3)$, we set

$$b(u,v,w) = \sum_{i,j=1}^3 \int_{\mathbb{T}^3} u_i D_i v_j w_j \mathrm{d}x.$$

Lemma 3.2.1. *Suppose that $\alpha \in [\frac{1}{2}, 2]$. Then there exists a constant C such that*

$$|b(u,v,w)| \le C|u|_\alpha |v|_{\alpha+1} |w|_{1-\alpha} \tag{3.4}$$

for all $u, v, w \in \dot{\mathcal{V}}(\mathbb{T}^3)$.

Proof. First we consider the case $\frac{1}{2} \le \alpha \le 1$. By applying Lemma 2.1 of [63] with $m_1 = \alpha, m_2 = \alpha, m_3 = 1-\alpha$ we get (3.4).

Now we consider the case $1 < \alpha \le 2$. By integration by parts, using the Stokes formula we get

$$b(u,v,w) = b(u,v,AA^{-1}w) = \sum_{i,j,k=1}^{3} \int_{\mathbb{T}^3} u_i D_{ik} v_j D_k (A^{-1}w)_j \mathrm{d}x + \sum_{i,j,k=1}^{3} \int_{\mathbb{T}^3} D_k u_i D_i v_j D_k (A^{-1}w)_j \mathrm{d}x. \tag{3.5}$$

Again by applying Lemma 2.1 of [63] with $m_1 = \alpha, m_2 = \alpha - 1, m_3 = 2 - \alpha$ for the first term on the right-hand side of (3.5); with $m_1 = \alpha - 1, m_2 = \alpha, m_3 = 2 - \alpha$ for the second term on the right-hand side of (3.5) we get

$$|b(u,v,w)| \le C|u|_\alpha |v|_{\alpha+1} |A^{-1}w|_{3-\alpha} \le C|u|_\alpha |v|_{\alpha+1} |w|_{1-\alpha}.$$

This proves the lemma. □

Using the property of trilinearity of the form b, from Lemma 3.2.1 for $\alpha \in [\frac{1}{2}, 2]$ we can extend b from $\dot{\mathcal{V}}^3(\mathbb{T}^3)$ to $V_\alpha \times V_{\alpha+1} \times V_{1-\alpha}$ satisfying

$$|b(u,v,w)| \le C|u|_\alpha |v|_{\alpha+1} |w|_{1-\alpha}$$

for all $(u,v,w) \in V_\alpha \times V_{\alpha+1} \times V_{1-\alpha}$.

For $u \in \dot{\mathcal{V}}(\mathbb{T}^3)$ put $b_{1,\alpha}(u) = b(u,u,A^\alpha u)$.

Lemma 3.2.2. *Suppose that $\alpha \in [\frac{1}{2}, \infty)$. Then*

$$\begin{aligned} |b_{1,\alpha}(u)| &\le C|u|_\alpha^{\frac{1}{2}+\alpha} |u|_{\alpha+1}^{\frac{5}{2}-\alpha} \text{ if } \alpha \in [\frac{1}{2}, \frac{3}{2}), \\ |b_{1,\frac{3}{2}}(u)| &\le C(\varepsilon)|u|_{\frac{3}{2}}^{2-\varepsilon} |u|_{\frac{5}{2}}^{1+\varepsilon} \text{ for } 0 < \varepsilon < \frac{1}{2}, \\ |b_{1,\alpha}(u)| &\le C|u|_\alpha^2 |u|_{\alpha+1} \text{ if } \alpha > \frac{3}{2} \end{aligned} \tag{3.6}$$

for all $u \in \dot{\mathcal{V}}(\mathbb{T}^3)$.

Proof. First we consider the case $\frac{1}{2} \le \alpha < 1$. By applying Lemma 2.1 of [63] with $m_1 = \alpha + \frac{1}{2},\ m_2 = 1 - \alpha,\ m_3 = 0$ and then using an interpolation inequality for the Sobolev norms we get

$$|b_{1,\alpha}(u)| \le C|u|_{\alpha+\frac{1}{2}} |u|_{2-\alpha} |u|_{2\alpha} \le C|u|_\alpha^{\frac{1}{2}+\alpha} |u|_{\alpha+1}^{\frac{5}{2}-\alpha}.$$

Now we consider the case $1 \le \alpha < \frac{3}{2}$. By integration by parts, using the Stokes formula we get

$$b_{1,\alpha}(u) = b(u,u,AA^{\alpha-1}u) = \sum_{i,j,k=1}^{3} \int_{\mathbb{T}^3} u_i D_{ik} u_j D_k (A^{\alpha-1}u)_j \mathrm{d}x + \sum_{i,j,k=1}^{3} \int_{\mathbb{T}^3} D_k u_i D_i u_j D_k (A^{\alpha-1}u)_j \mathrm{d}x. \tag{3.7}$$

Again by applying Lemma 2.1 of [63] with $m_1 = \alpha, m_2 = 0, m_3 = \frac{3}{2} - \alpha$ for the first term on the right-hand side of (3.7); with $m_1 = \alpha - \frac{3}{4}, m_2 = \alpha - \frac{3}{4}, m_3 = 3 - 2\alpha$ for the second term on the right-hand side of (3.7) and then using an interpolation inequality for Sobolev norms we get

$$|b_{1,\alpha}(u)| \leq C(|u|_\alpha |u|_2 |u|_{\alpha+\frac{1}{2}} + |u|_{\alpha+\frac{1}{4}} |u|_{\alpha+\frac{1}{4}} |u|_2) \leq C|u|_\alpha^{\frac{1}{2}+\alpha} |u|_{\alpha+1}^{\frac{5}{2}-\alpha}.$$

Finally we consider case $\alpha \geq \frac{3}{2}$. By integration by parts, using the Stokes formula we get

$$b_{1,\alpha}(u) = b(u, u, A^{[\alpha]} A^{\{\alpha\}} u) \tag{3.8}$$
$$= \sum_{(\alpha_1,\alpha_2,\alpha_3)\in D} c(\alpha_1, \alpha_2, \alpha_3) \sum_{i,j,k=1}^{3} \int_{\mathbb{T}^3} D^{\alpha_1} u_i D^{\alpha_2} D_i u_j D^{\alpha_3} (A^{\{\alpha\}} u)_j \mathrm{d}x,$$

where D is a subset of $\{(\alpha_1, \alpha_2, \alpha_3) : |\alpha_1| + |\alpha_2| = [\alpha], |\alpha_3| = [\alpha]\}$. Applying Lemma 3.2.1 for each term of the right hand side of (3.8) with $m_1 = \alpha - [\alpha_1], m_2 = [\alpha] - [\alpha_2], m_3 = 0$ if $\alpha > \frac{3}{2}$, $m_3 = \varepsilon$, $0 < \varepsilon < \frac{1}{2}$ if $\alpha = \frac{3}{2}$, we get

$$|b_{1,\alpha}(u)| \leq C|u|_\alpha |u|_{[\alpha]+1} |u|_{\alpha+\{\alpha\}} \text{ if } \alpha > \frac{3}{2},$$
$$|b_{1,\alpha}(u)| \leq C(\varepsilon)|u|_{\frac{3}{2}} |u|_2 |u|_{2+\varepsilon} \text{ if } \alpha = \frac{3}{2},$$

and then using an interpolation inequality for the Sobolev norms we get the desireble inequalities. This proves the lemma. □

From Lemma 3.2.2, for $\alpha \in [\frac{1}{2}, \infty)$ we can extend b_1 from $\dot{\mathcal{V}}(\mathbb{T}^3)$ to $V_{\alpha+1}$ satisfying the inequality (3.6).

Lemma 3.2.3. *Assume that* $\alpha \in (\frac{1}{2}, \frac{3}{2}), u \in L^2(0, T; V_{\alpha+1}) \cap L^\infty(0, T; V_\alpha)$, *the function* $\{t \mapsto |u(t)|_\alpha^2\}$ *is absolutely continuous and almost everywhere satisfies the equality*

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |u(t)|_\alpha^2 + |u(t)|_{\alpha+1}^2 + b_{1,\alpha}(u(t)) = (f(t), A^\alpha u(t)), \tag{3.9}$$

where $f \in L^\infty(0, T; V_{\alpha-1})$. *Then there exist constants* T^*, K, L *depending only on* $|u(0)|_\alpha, \alpha, \sup_{0\leq t\leq T} |f(t)|_{\alpha-1}$ *such that*

$$\sup_{0\leq t\leq T^*} |u(t)|_\alpha \leq K, \tag{3.10}$$
$$\int_0^{T^*} |u(t)|_{\alpha+1}^2 \mathrm{d}t \leq L. \tag{3.11}$$

Proof. By applying the Holder and Young inequalities we get

$$|(f(t), A^\alpha u(t))| \leq |f(t)|_{\alpha-1} |A^\alpha u(t)|_{1-\alpha} \leq C|f(t)|_{\alpha-1} |A^{\frac{\alpha+1}{2}} u(t)|$$
$$\leq \frac{1}{4} |u(t)|_{\alpha+1}^2 + C|f(t)|_{\alpha-1}^2 \leq \frac{1}{4} |u(t)|_{\alpha+1}^2 + C(f, \alpha), \tag{3.12}$$

where $C(f, \alpha)$ is a constant depending only on $\sup_{0\leq t\leq T} |f(t)|_{\alpha-1}$.

From Lemma 3.2.2, by applying Young's inequality we have

$$|b_{1,\alpha}(u(t))| \leq C|u(t)|_\alpha^{\frac{1}{2}+\alpha} |u(t)|_{\alpha+1}^{\frac{5}{2}-\alpha} \leq \frac{1}{4} |u(t)|_{\alpha+1}^2 + C(\alpha)|u(t)|_{\alpha+1}^{\frac{2(1+2\alpha)}{2\alpha-1}}. \tag{3.13}$$

Since $\frac{2(1+2\alpha)}{2\alpha-1} > 2$ from (3.9), (3.12) and (3.13) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|_\alpha^2 + \frac{1}{2}|u(t)|_{\alpha+1}^2$$
$$\leq C(\alpha)|u(t)|_\alpha^{\frac{2(1+2\alpha)}{2\alpha-1}} + C(f,\alpha) \leq C(f,\alpha)(|u(t)|_\alpha^2+1)^{\frac{(1+2\alpha)}{2\alpha-1}}. \tag{3.14}$$

If we set

$$y(t) = |u(t)|_\alpha^2 + 1,$$

from (3.14) it follows that

$$y'(t) \leq 2C(N,\alpha)y^{\frac{(1+2\alpha)}{2\alpha-1}} = Cy^{\frac{(1+2\alpha)}{2\alpha-1}}, \tag{3.15}$$

where $C = 2C(N,\alpha)$ is a constant. By integrating (3.15) we obtain

$$y(t) \leq (Ca)^{-\frac{1}{a}}\left(\frac{1}{Cay^a(0)} - t\right)^{-\frac{1}{a}},\ 0 \leq t < \frac{1}{Cay^a(0)}, \tag{3.16}$$

where $a = \frac{2}{2\alpha-1}$. Now we define

$$T^* = \frac{1-2^{-a}}{aCy^a(0)} = \frac{C(\alpha,N)}{(|u(0)|_\alpha^2+1)^{\frac{2}{2\alpha-1}}}. \tag{3.17}$$

When t equals T^*, then the right side of (3.16) is equal to $2y(0)$. From (3.16) and (3.17) we have

$$y(t) \leq 2y(0),\ \forall t \in [0,T^*]. \tag{3.18}$$

Using (3.14) and (3.18) we get

$$|u(t)|_\alpha^2 \leq 2|u(0)|_\alpha^2 + 1,\ \forall t \in [0,T^*]. \tag{3.19}$$

Integrating both sides of the equation (3.14) and using (3.19), we get

$$\int_0^{T^*} |u(t)|_{\alpha+1}^2\mathrm{d}t \leq 2T^*C(N,\alpha)(2|u(0)|_\alpha^2+2)^{\frac{(1+2\alpha)}{2\alpha-1}} + |u(0)|_\alpha^2. \tag{3.20}$$

The lemma is proven by (3.20) and (3.19). □

Theorem 3.2.4. *Assume that*

$$f \in L^\infty(0,T;V_{\alpha-1}),\ u_0 \in V_\alpha,\ \frac{1}{2} < \alpha < \frac{3}{2}. \tag{3.21}$$

Then there exists a unique strong solution to NSE satisfying

$$u \in L^2(0,T^{**};V_{\alpha+1}) \cap C(0,T^{**};V_\alpha), \tag{3.22}$$

where $T^{**} = \min(T,T^*)$, T^* *given by* (3.17).

Proof. To prove the existence of a strong solution we use the standard Galerkin method (see for example [64]) combining with estimates proven in Lemmas 3.2.1-3.2.3.
To prove the uniqueness we note that

$$|u(t)|_{\alpha+\frac{1}{4}} \leq |u(t)|_\alpha^{\frac{3}{4}}|u(t)|_{\alpha+1}^{\frac{1}{4}}. \tag{3.23}$$

From (3.22), (3.23) we get $u \in L^8(0,T^{**};V_{\alpha+\frac{1}{4}})$. Since $\alpha > \frac{1}{2}$ by Sobolev's embedding theorem we have $L^8(0,T^{**};V_{\alpha+\frac{1}{4}}) \subset L^8(0,T^{**};L^4)$. Therefore $u \in L^8(0,T^{**};L^4)$, u satisfies Serrin's uniqueness condition. This completes the proof of the theorem. □

Let $\alpha \in (\frac{1}{2}, \frac{3}{2})$. We say that a weak solution u is $H^\alpha(\mathbb{T}^3)$ - regular on (t_1, t_2) if $u \in C((t_1, t_2), H^\alpha(\mathbb{T}^3))$. We say that an H^α regularity interval (t_1, t_2) is maximal for u if there does not exist any interval of H^α regularity strictly containing (t_1, t_2). From Theorem 3.2.4 we can easily prove that if (t_1, t_2) is a H^α - maximal interval of solution u, then

$$\lim_{t \mapsto t_2 - 0} \sup |u(t)|_\alpha = +\infty.$$

Theorem 3.2.5. *Let $\alpha \in (\frac{1}{2}, \frac{3}{2}), u_0 \in H,\ f \in L^\infty(0, T; V_{\alpha-1})$. Assume that u is a weak solution of NSE. Then u is $H^\alpha(\mathbb{T}^3)$ - regular on an open set of $(0, T)$ whose complement has Lebesgue measure 0.*

Proof. Since u is weakly continuous from $[0, T]$ into H, $u(t)$ is well defined for every t and we can define

$$\begin{aligned}
\textstyle\sum_\alpha &= \{t \in [0, T], u(t) \notin V_\alpha\}, \\
\Omega_\alpha &= \{t \in [0, T], u(t) \in V_\alpha\}, \\
\mathcal{O}_\alpha &= \{t \in (0, T), \exists \varepsilon > 0,\ u(t) \in C((t - \varepsilon, t + \varepsilon), V_\alpha)\}.
\end{aligned}$$

It is clear $\mathcal{O}_\alpha$ is open. First we claim that $u \in L^1(0, T; V_\alpha)$. Indeed, if $\alpha \in (\frac{1}{2}, 1]$ the claim follows from the very definition of the weak solution. If $\alpha \in (1, \frac{3}{2})$ from the proof of Theorem 4.2 in [64] we note that $u \in L^1(0, T; V_{\frac{3}{2}})$, hence the claim follows. From the just proved statement we have $\mathcal{L}^1(\sum_\alpha) = 0$. Thus $\mathcal{L}^1(\sum_\alpha \cup \Sigma) = 0$ (Σ was introduced in connection with the energy inequality in the definition of weak solutions). If $t_0 \in \Omega_\alpha \backslash (\mathcal{O}_\alpha \cup \Sigma)$ then, according to Theorem 3.2.4 and the uniqueness theorem of Sather and Serrin (see [63]), t_0 is the left end of an interval of H^α regularity, i.e., one of connected components of $\mathcal{O}_\alpha$. Thus $\Omega_\alpha \backslash (\mathcal{O}_\alpha \cup \Sigma)$ is a set no more than countable and therefore $\mathcal{L}^1(\Omega_\alpha \backslash (\mathcal{O}_\alpha \cup \Sigma)) = 0$. By noting that $[0, T] \setminus \mathcal{O}_\alpha \subset (\Omega_\alpha \backslash (\mathcal{O}_\alpha \cup \Sigma)) \cup \Sigma_\alpha \cup \Sigma$ we get $\mathcal{L}^1([0, T] \setminus \mathcal{O}_\alpha) = 0$. The theorem is proved. □

Lemma 3.2.6. *Let $T > 0$. Assume that a set $O \subset [0, T]$ is open. Denote by S the complement of O in $[0, T]$ and assume that $\mathcal{L}^1(S) = 0$. Suppose that there exists a function $y(t) \in L^s[0, T]$ satisfying the following conditions*

$$\begin{cases} y(t) \geq 0, & \forall t \in O, \\ \left(t, \min\left\{t + \frac{M}{y^a(t)}, T\right\}\right) \subset O, & \forall t \in O, \end{cases} \tag{3.24}$$

(*our convention is $\frac{1}{0} = +\infty$*), *where M and a are constants satisfying the condition*

$$M > 0, a > s.$$

Then we have $\mu_{1-\frac{s}{a}}(S) = 0$.

Proof. We have

$$O = \cup_{i \in I}(c_i, d_i),$$

where I is an index set no more than countable. The set (c_i, d_i) are connected components of O. Now take an arbitrary index $i \in I$ and a point $\tau \in (c_i, d_i)$. From (3.24) it follows that

$$d_i \geq \min\left\{\tau + \frac{M}{y^a(\tau)}, T\right\}. \tag{3.25}$$

From (3.25) we deduce that

$$\frac{1}{(d_i-\tau)^{\frac{s}{a}}} \le \max\left\{\frac{y^s(\tau)}{M^{\frac{q}{a}}}, \frac{1}{(T-\tau)^{\frac{s}{a}}}\right\} \le \frac{y^s(\tau)}{M^{\frac{s}{a}}} + \frac{1}{(T-\tau)^{\frac{s}{a}}}$$

for every $\tau \in (c_i, d_i)$. Integrating the above inequality from c_i to d_i we obtain

$$(d_i-c_i)^{1-\frac{s}{a}} \le \Big(1-\frac{s}{a}\Big)\Big(M^{-\frac{s}{a}}\int_{c_i}^{d_i} y^s(\tau)\mathrm{d}\tau + \int_{c_i}^{d_i}\frac{\mathrm{d}\tau}{(T-\tau)^{\frac{s}{a}}}\Big).$$

Summing in $i \in I$ we have

$$\sum_{i\in I}(d_i-c_i)^{1-\frac{s}{a}} \le \Big(1-\frac{s}{a}\Big)\Big(M^{-\frac{s}{a}}\int_0^T y^s(\tau)\mathrm{d}\tau + \frac{T^{1-\frac{s}{a}}}{1-\frac{s}{a}}\Big) < +\infty. \tag{3.26}$$

For any $\varepsilon > 0$ from (3.26) and the assumption that $\mathcal{L}^1(S) = 0$ it follows that there exists a finite set $I_\varepsilon \subset I$ such that

$$\sum_{i\in I\backslash I_\varepsilon}(d_i-c_i)^{1-\frac{s}{a}} < \varepsilon, \ \sum_{i\in I\backslash I_\varepsilon}(d_i-c_i) < \varepsilon.$$

It is easily seen that there exists a natural number m such that

$$[0,T]\backslash \cup_{i\in I_\varepsilon}(c_i,d_i) = \cup_{j=1}^m [a_j,b_j] = \cup_{j=1}^m B_j,$$

where $B_j \cup B_{j'} = \emptyset$ if $j \neq j'$. By denoting I_j the set of $i \in I\backslash I_\varepsilon$ such that $(c_i,d_i) \subset B_j$ we have for every j

$$B_j = \bigcup_{i\in I_j}(c_i,d_i)\bigcup(B_j\cap S). \tag{3.27}$$

From (3.27) we deduce that

$$\mathrm{diam}(B_j) = b_j - a_j = \sum_{i\in I_j}(d_i-c_i) \le \sum_{i\in I\backslash I_\varepsilon}(d_i-c_i) < \varepsilon.$$

By using the inequality

$$\Big(\sum_{i=1}^n a_i\Big)^\gamma \le \sum_{i=1}^n a_i^\gamma, \ \ (a_i \ge 0, i=\overline{1,n}), \ \ 0<\gamma\le 1,$$

we have

$$\begin{aligned}\mu_{1-\frac{s}{a},\varepsilon} &\le \sum_{j=1}^m \mathrm{diam}(B_j)^{1-\frac{s}{a}} = \sum_{j=1}^m\Big(\sum_{i\in I_j}(d_i-c_i)\Big)^{1-\frac{s}{a}} \\ &\le \sum_{j=1}^m\sum_{i\in I_j}(d_i-c_i)^{1-\frac{s}{a}} = \sum_{i\in I\backslash I_\varepsilon}(d_i-c_i)^{1-\frac{s}{a}} < \varepsilon.\end{aligned}$$

By letting $\varepsilon \to 0$ we conclude that $\mu_{1-\frac{s}{a}}(S) = 0$. □

Theorem 3.2.7. *Assume that $\alpha \in (\frac{1}{2},\frac{3}{2}), u_0 \in H, f \in L^\infty(0,T;V_{\alpha-1})$ and u is a weak solution of NSE and satisfies the following condition*

$$u \in L^r(0,T;V_\alpha), r>0, r(2\alpha-1)<4. \tag{3.28}$$

Then there exists a closed set $S_\alpha \subset [0,T]$ such that $u \in C([0,T]\setminus S_\alpha;V_\alpha)$ and $\mu_{1-\frac{r(2\alpha-1)}{4}}(S_\alpha) = 0$.

Proof. Denote $S_\alpha = [0,T] \setminus \mathcal{O}_\alpha$, where $\mathcal{O}_\alpha$ was introduced in the proof of Theorem 3.2.5. By Theorem 3.2.5 we get $\mathcal{L}^1(S_\alpha) = 0$. From (3.28) we have $y_\alpha(t) \in L^{\frac{r}{2}}(0,T)$, where $y_\alpha(t) = |u(t)|_\alpha^2 + 1$. By applying Lemma 3.2.6 with $O = \mathcal{O}_\alpha, y_\alpha(t) = |u(t)|_\alpha^2 + 1, a = \frac{2}{2\alpha-1}, s = \frac{r}{2}$, we see that $\mu_{1-\frac{r(2\alpha-1)}{4}}(S_\alpha) = 0$. This proves the theorem. □

Remark 3.2.1. The condition $r(2\alpha - 1) < 4$ in Theorem 3.2.7 is not essential since if $r(2\alpha - 1) \geq 4$ then u satisfies the Serrin condition, hence u is smooth if f is smooth. In this case $\mu_0(S_\alpha) = 0$.

Remark 3.2.2. The set Σ where the energy inequality in the definition of weak solutions may fail is a subset of S_α. Therefore under the hypotheses of Theorem 3.2.7, the $\left(1 - \frac{r(2\alpha-1)}{4}\right)$ - dimensional Hausdorff measure of Σ is equal to zero.

Remark 3.2.3. If $r = 2, \alpha = 1$, then the condition $u \in L^2(0,T;V_1)$ in Theorem 3.2.7 is redundant since it is assumed in the definition of weak solutions. In this case we recover the early result of Scheffer and Leray (see [49, 58, 64]). When $2 \geq r(2\alpha - 1), \alpha \in (\frac{1}{2}, 1]$ and the force is smooth, the condition $u \in L^r(0,T;V_\alpha)$ is redundant, and in this case our result is not new. However, even if $2 \geq r(2\alpha - 1), \alpha \in (\frac{1}{2}, 1]$ our result is new with respect to non regular force $f \in L^\infty(0,T;V_{\alpha-1})$.

Remark 3.2.4. Under the conditions of Theorem 3.2.7 with $\alpha \in (\frac{1}{2}, \frac{3}{2})$ replaced by $\alpha \in [\frac{3}{2}, \infty)$ the conclusion of Theorem 3.2.7 should be replaced by

$$\text{for} \quad \alpha = \frac{3}{2} : \mu_{1-\frac{r}{2}+\varepsilon}(S_\alpha) = 0 \quad \text{for any } \varepsilon > 0,$$
$$\text{for} \quad \alpha > \frac{3}{2} : \mu_{1-\frac{r}{2}}(S_\alpha) = 0.$$

To prove it we use the estimate (3.6) in Lemma 3.2.2 for $b_{1,\alpha}(u)$ with $\alpha \in [\frac{3}{2}, \infty)$ and then exactly follow the proof of Theorem 3.2.7.

3.3 Weak solutions in $L^r W^{1,q}$

In this section, for $q \geq 2$ we use the following notations

$$\|\nabla u\|^q_{L^q(\mathbb{T}^3)} = \sum_{i,l=1}^{3} \int_{\mathbb{T}^3} \Big|\frac{\partial u_i}{\partial x_l}\Big|^q \mathrm{d}x,$$
$$\Big\|\nabla\big(|\nabla u|^{\frac{q}{2}}\big)\Big\|^2_{L^2(\mathbb{T}^3)} = \frac{q^2}{4}\Big\|\nabla\Big(\sum_{i,j=1}\Big|\frac{\partial u_i}{\partial x_j}\Big|^{\frac{q}{2}}\Big)\Big\|^2_{L^2(\mathbb{T}^3)}$$
$$= \frac{q^2}{4}\sum_{i,l,j=1}^{3}\int_{\mathbb{T}^3}\Big(\frac{\partial^2 u_i}{\partial x_l \partial x_j}\Big)^2\Big|\frac{\partial u_i}{\partial x_j}\Big|^{q-2}\mathrm{d}x.$$

Lemma 3.3.1. *If the force and solution of the Navier-Stokes equations are smooth enough then we have*

$$
\begin{aligned}
&-\frac{1}{q(q-1)}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^q_{L^q(\mathbb{T}^3)} - \frac{4}{q^2}\Big\|\nabla\big(|\nabla u|^{\frac{q}{2}}\big)\Big\|^2_{L^2(\mathbb{T}^3)} \\
&+\sum_{l,j=1}^{3}\int_{\mathbb{T}^3} u_l\frac{\partial u_i}{\partial x_l}\frac{\partial^2 u_i}{\partial x_j^2}\Big|\frac{\partial u_i}{\partial x_j}\Big|^{q-2}\mathrm{d}x + \sum_{j=1}^{3}\int_{\mathbb{T}^3}\frac{\partial p}{\partial x_i}\frac{\partial^2 u_i}{\partial x_j^2}\Big|\frac{\partial u_i}{\partial x_j}\Big|^{q-2}\mathrm{d}x \\
&= \sum_{j=1}^{3}\int_{\mathbb{T}^3} f_i\frac{\partial^2 u_i}{\partial x_j^2}\Big|\frac{\partial u_i}{\partial x_j}\Big|^{q-2}\mathrm{d}x.
\end{aligned}
$$

Proof. First by integrating by parts, we note that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^q_{L^q(\mathbb{T}^3)} &= q\sum_{i,l=1}^{3}\int_{\mathbb{T}^3}\Big|\frac{\partial u_i}{\partial x_l}\Big|^{q-1}\frac{\partial^2 u_i}{\partial x_l \partial t}\mathrm{sign}\Big(\frac{\partial u_i}{\partial x_j}\Big)\mathrm{d}x \\
&= -q(q-1)\sum_{i,l=1}^{3}\int_{\mathbb{T}^3}\Big|\frac{\partial u_i}{\partial x_l}\Big|^{q-2}\frac{\partial^2 u_i}{\partial x_l^2}\frac{\partial u_i}{\partial t}\mathrm{d}x, \qquad (3.29)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i,j,l=1}^{3}\int_{\mathbb{T}^3}\frac{\partial^2 u_i}{\partial x_l^2}\Big|\frac{\partial u_i}{\partial x_j}\Big|^{q-2}\frac{\partial^2 u_i}{\partial x_j^2}\mathrm{d}x \\
&= \frac{1}{q-1}\sum_{i,j=1}^{3}\int_{\mathbb{T}^3}\frac{\partial^2 u_i}{\partial x_l^2}\frac{\partial}{\partial x_j}\Big(\Big|\frac{\partial u_i}{\partial x_j}\Big|^{q-1}\Big)\mathrm{sign}\Big(\frac{\partial u_i}{\partial x_j}\Big)\mathrm{d}x \\
&= -\frac{1}{q-1}\sum_{i,j=1}^{3}\int_{\mathbb{T}^3}\frac{\partial^3 u_i}{\partial x_l^2 \partial x_j}\Big|\frac{\partial u_i}{\partial x_j}\Big|^{q-1}\mathrm{sign}\Big(\frac{\partial u_i}{\partial x_j}\Big)\mathrm{d}x \\
&= \sum_{i,j=1}^{3}\int_{\mathbb{T}^3}\Big(\frac{\partial^2 u_i}{\partial x_l \partial x_j}\Big)^2\Big|\frac{\partial u_i}{\partial x_j}\Big|^{q-2}\mathrm{d}x = \frac{4}{q^2}\Big\|\nabla\big(|\nabla u|^{\frac{q}{2}}\big)\Big\|^2_{L^2(\mathbb{T}^3)}. \qquad (3.30)
\end{aligned}
$$

Now by multiplying the i-equation by $\sum_{l=1}^{3}\big|\frac{\partial u_i}{\partial x_l}\big|^{q-2}\frac{\partial^2 u_i}{\partial x_l^2}$, summing them all together and integrating over $\mathbb{T}^3$ using the formulas (3.29), (3.30), we obtain the desired result. ☐

Lemma 3.3.2. *Assume that* $q \in [2,3), u \in L^2(0,T;\tilde{W}^{2,q}) \cap L^\infty(0,T;W^{1,q})$, *the function* $\Big\{t \mapsto \|\nabla u(t)\|^p_{L^q(\mathbb{T}^3)}\Big\}$ *is absolutely continuous and almost everywhere satisfies the equality*

$$
\begin{aligned}
&-\frac{1}{q(q-1)}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|^q_{L^p(\mathbb{T}^3)} - \Big\|\nabla\big(|\nabla u(t)|^{\frac{q}{2}}\big)\Big\|^2_{L^2(\mathbb{T}^3)} \\
&+\frac{4}{q^2}\sum_{l,j=1}^{3}\int_{\mathbb{T}^3} u_l\frac{\partial u_i}{\partial x_l}\frac{\partial^2 u_i}{\partial x_j^2}\Big|\frac{\partial u_i}{\partial x_j}\Big|^{q-2}\mathrm{d}x + \sum_{j=1}^{3}\int_{\mathbb{T}^3}\frac{\partial p}{\partial x_i}\frac{\partial^2 u_i}{\partial x_j^2}\Big|\frac{\partial u_i}{\partial x_j}\Big|^{q-2}\mathrm{d}x \\
&= \sum_{j=1}^{3}\int_{\mathbb{T}^3} f_i\frac{\partial^2 u_i}{\partial x_j^2}\Big|\frac{\partial u_i}{\partial x_j}\Big|^{q-2}\mathrm{d}x, \qquad (3.31)
\end{aligned}
$$

where $f \in L^\infty(0,T;L^q(\mathbb{T}^3))$ *and*

$$
\|\nabla p(t)\|_{L^q(\mathbb{T}^3)} \le C\|(u,\nabla)u(t)\|_{L^q(\mathbb{T}^3)}. \qquad (3.32)
$$

Then there exist constants T^*, K, L depending only on $\left\|\nabla u(0)\right\|^q_{L^q(\mathbb{T}^3)}$, $q, \sup_{0\le t\le T}\|f(t)\|_{L^q(\mathbb{T}^3)}$ such that

$$\sup_{0\le t\le T^*}\left\|\nabla u(t)\right\|^q_{L^q(\mathbb{T}^3)}\le K,$$
$$\int_0^{T^*}\left\|\nabla\big(|\nabla u(t)|^{\frac{q}{2}}\big)\right\|^2_{L^2(\mathbb{T}^3)}\mathrm{d}t\le L.$$

Proof. By the Sobolev inequality

$$\|\nabla u(t)\|^{\frac{q}{2}}_{L^{3q}(\mathbb{T}^3)}=\left\||\nabla u(t)|^{\frac{q}{2}}\right\|_{L^6(\mathbb{T}^3)}\le C\left\|\nabla\big(|\nabla u(t)|^{\frac{q}{2}}\big)\right\|_{L^2(\mathbb{T}^3)}.$$

For a function $g\in L^q(\mathbb{T}^3)$ by the Holder inequality we have

$$\Big|\int_{\mathbb{T}^3}g(x)\frac{\partial^2u_i}{\partial x_j^2}\Big|\frac{\partial u_i}{\partial x_j}\Big|^{q-2}\mathrm{d}x\Big|\le\Big\|\frac{\partial^2u_i}{\partial x_j^2}\Big|\frac{\partial u_i}{\partial x_j}\Big|^{\frac{q-2}{2}}\Big\|_{L^2(\mathbb{T}^3)}\|\nabla u(t)\|^{\frac{q-2}{2}}_{L^q(\mathbb{T}^3)}\|g\|_{L^p(\mathbb{T}^3)}.$$

Therefore

$$\begin{aligned}\Big|\int_{\mathbb{T}^3}f_i\frac{\partial^2u_i}{\partial x_j^2}\Big|\frac{\partial u_i}{\partial x_j}\Big|^{q-2}\mathrm{d}x\Big|&\le\Big\|\frac{\partial^2u_i}{\partial x_j^2}\Big|\frac{\partial u_i}{\partial x_j}\Big|^{\frac{p-2}{2}}\Big\|_{L^2(\mathbb{T}^3)}\|\nabla u(t)\|^{\frac{q-2}{2}}_{L^q(\mathbb{T}^3)}\|f(t)\|_{L^p(\mathbb{T}^3)}\\&\le\left\|\nabla u(t)\right\|^{\frac{q(2q-1)}{2q-3}}_{L^q(\mathbb{T}^3)}+\frac{\left\|\nabla\big(|\nabla u(t)|^{\frac{q}{2}}\big)\right\|^2_{L^2(\mathbb{T}^3)}}{4}+C\|f(t)\|^{\frac{q(2q-1)}{3(q-1)}}_{L^q(\mathbb{T}^3)}.\end{aligned}\tag{3.33}$$

Furthermore

$$\begin{aligned}\Big\|u_l\frac{\partial u_i}{\partial x_l}\Big\|_{L^q(\mathbb{T}^3)}&\le\|u(t)\|_{L^\infty(\mathbb{T}^3)}\|\nabla u(t)\|_{L^q(\mathbb{T}^3)}\le\|\nabla u(t)\|^{\frac{q+1}{2}}_{L^q(\mathbb{T}^3)}\left\|\nabla u(t)\right\|^{\frac{3-q}{2}}_{L^{3q}(\mathbb{T}^3)}\\&\le\left\|\nabla u(t)\right\|^{\frac{q+1}{2}}_{L^q(\mathbb{T}^3)}\left\|\nabla\big(|\nabla u(t)|^{\frac{q}{2}}\big)\right\|^{\frac{3-q}{q}}_{L^2(\mathbb{T}^3)}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{l,j=1}^3\Big|\int_{\mathbb{T}^3}u_l\frac{\partial u_i}{\partial x_l}\frac{\partial^2u_i}{\partial x_j^2}\Big|\frac{\partial u_i}{\partial x_j}\Big|^{q-2}\mathrm{d}x\Big|&\le\|\nabla u(t)\|^{\frac{2q-1}{2}}_{L^q(\mathbb{T}^3)}\left\|\nabla\big(|\nabla u(t)|^{\frac{q}{2}}\big)\right\|^{\frac{3}{q}}_{L^2(\mathbb{T}^3)}\\&\le\left\|\nabla u(t)\right\|^{\frac{q(2q-1)}{2q-3}}_{L^q(\mathbb{T}^3)}+\frac{\left\|\nabla\big(|\nabla u(t)|^{\frac{q}{2}}\big)\right\|^2_{L^2(\mathbb{T}^3)}}{4}.\end{aligned}\tag{3.34}$$

From (3.32) we also have

$$\|\nabla p(t)\|_{L^q(\mathbb{T}^3)}\le\left\|\nabla u(t)\right\|^{\frac{q+1}{2}}_{L^q(\mathbb{T}^3)}\left\|\nabla\big(|\nabla u(t)|^{\frac{q}{2}}\big)\right\|^{\frac{3-q}{q}}_{L^2(\mathbb{T}^3)}.$$

Thus

$$\begin{aligned}\sum_{l,j=1}^3\Big|\int_{\mathbb{T}^3}\frac{\partial p}{\partial x_i}\frac{\partial^2u_i}{\partial x_j^2}\Big|\frac{\partial u_i}{\partial x_j}\Big|^{q-2}\mathrm{d}x\Big|&\le\|\nabla u(t)\|^{\frac{2q-1}{2}}_{L^p(\mathbb{T}^3)}\left\|\nabla\big(|\nabla u(t)|^{\frac{q}{2}}\big)\right\|^{\frac{3}{q}}_{L^2(\mathbb{T}^3)}\\&\le\left\|\nabla u(t)\right\|^{\frac{q(2q-1)}{2q-3}}_{L^q(\mathbb{T}^3)}+\frac{\left\|\nabla\big(|\nabla u(t)|^{\frac{q}{2}}\big)\right\|^2_{L^2(\mathbb{T}^3)}}{4}+C\|f(t)\|^{\frac{q(2q-1)}{3(q-1)}}_{L^q(\mathbb{T}^3)}.\end{aligned}\tag{3.35}$$

Summing inequalities (3.33), (3.34), (3.35), using (3.31) and reducing similar terms we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\big\|\nabla u(t)\big\|^q_{L^q(\mathbb{T}^3)}\Big)+\frac{\Big\|\nabla\big(|\nabla u(t)|^{\frac{q}{2}}\big)\Big\|^2_{L^2(\mathbb{T}^3)}}{4}\le\big\|\nabla u(t)\big\|^{\frac{q(2q-1)}{2q-3}}_{L^q(\mathbb{T}^3)}+C\|f(t)\|^{\frac{q(2q-1)}{3(q-1)}}_{L^q(\mathbb{T}^3)}$$
$$\le C\Big(\big\|\nabla u(t)\big\|^q_{L^q(\mathbb{T}^3)}+1\Big)^{\frac{(2q-1)}{2q-3}}. \qquad (3.36)$$

As in the proof of Lemma 3.2.3, by putting

$$y(t)=\big\|\nabla u(t)\big\|^q_{L^q(\mathbb{T}^3)}+1$$

from (3.36) we can get the conclusion of Lemma 3.3.2. □

Theorem 3.3.3. *Assume that*

$$f\in L^\infty(0,T;L^q(\mathbb{T}^3)),\ u_0\in W^{1,q}(\mathbb{T}^3),\ q\in[2,3). \qquad (3.37)$$

Then there exists a constant T^{**} *depending only on* $\big\|\nabla u(0)\big\|^q_{L^q(\mathbb{T}^3)}, q, \sup_{0\le t\le T}\|f(t)\|_{L^q(\mathbb{T}^3)}$ *and a unique strong solution to NSE, satisfying*

$$u\in L^2(0,T^{**};\tilde{W}^{2,q})\cap C(0,T^{**};W^{1,q}). \qquad (3.38)$$

Proof. The proof of this theorem is similar to the one of Theorem 3.2.4 by using Lemma 3.3.2. We omit the details. □

Let $q\in[2,3)$. We say that a weak solution u is $W^{1,q}(\mathbb{T}^3)$ - regular on (t_1,t_2) if $u\in C((t_1,t_2),W^{1,q}(\mathbb{T}^3))$. We say that an $W^{1,q}$ regularity interval (t_1,t_2) is maximal for u if there does not exist any interval of $W^{1,q}$ regularity strictly containing (t_1,t_2). From Theorem 3.3.3 we can easily prove that if (t_1,t_2) is a $W^{1,q}$ - maximal interval of solution u, then

$$\lim_{t\mapsto t_2-0}\sup\|u(t)\|_{W^{1,q}(\mathbb{T}^3)}=+\infty. \qquad (3.39)$$

Theorem 3.3.4. *Let* $q\in[2,3), u_0\in H,\ f\in L^\infty(0,T;L^p(\mathbb{T}^3))$. *Assume that* u *is a weak solution of the NSE belonging to* $L^1(0,T;W^{1,q})$. *Then* u *is* $W^{1,q}$ *- regular on an open set of* $(0,T)$ *whose complement has Lebesgue measure 0.*

Theorem 3.3.5. *Assume that* $q\in[2,3), u_0\in H,\ f\in L^\infty(0,T;L^p(\mathbb{T}^3))$ *and* u *is a weak solution of the NSE and satisfies the following condition*

$$u\in L^r(0,T;W^{1,q}),\ \frac{r(2q-3)}{2q}<1. \qquad (3.40)$$

Then there exists a closed set $S_q\subset[0,T]$ *such that* $u\in C([0,T]\setminus S_q;W^{1,q})$ *and* $\mu_{1-\frac{r(2q-3)}{2q}}(S_q)=0$.

Proof. By the hypothesis of the theorem $y(t)\in L^{\frac{r}{q}}(0,T)$, where $y(t)=\big\|\nabla u(t)\big\|^q_{L^q(\mathbb{T}^3)}+1$. Now apply Lemma 3.2.6 with $s=\frac{r}{q}, a=\frac{2}{2q-3}$ we get the desired result. □

Remark 3.3.1. The condition $\frac{r(2q-3)}{2q}<1$ in Theorem 3.3.5 is not essential since if $\frac{r(2q-3)}{2q}\ge 1$ then u satisfies the Serrin condition, hence u is smooth if f is smooth. In this case $\mu_0(S_q)=0$.

Remark 3.3.2. Under the hypotheses of Theorem 3.3.5, the $\left(\frac{r(2q-3)}{2q}\right)$ - dimensional Hausdorff measure of Σ is equal to zero.

Remark 3.3.3. If $r = 2, q = 1$, then the condition $u \in L^2(0,T;V_1)$ in Theorem 3.3.5 is redundant since it is assumed in the definition of weak solutions. In this case we recover the early result of Scheffer and Leray (see [49, 58, 64]). When $r \le 2, q = 1$, the condition $u \in L^r(0,T;V_\alpha)$ is also redundant since it is weaker the condition $u \in L^2(0,T;V_1)$. When $q \ge r(2q-3)$ our result is not new.

Remark 3.3.4. From the proofs of Theorems 3.2.7, 3.3.5 we see that the condition $u \in L^r(0,T;V_\alpha)$ (with $\alpha = 1$) or $u \in L^r(0,T;W^{1,q})$ may be weaken by $\nabla u \in L^r(0,T;H)$ or $\nabla u \in L^r(0,T;L^q)$, respectively. This kind of requirements was studied earlier in the note [6] and now in numerous papers (see [32, 51] and the references therein).

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D. Q. Khai and N. M. Tri, *On the Hausdorff dimension of the singular set in time for weak solutions to the nonstationary Navier-Stokes equations on torus*,Vietnam Journal of Mathematics, **43** (2015), 283-295.

General Conclusions

In this thesis, we construct mild solutions to the Navier-Stokes equations by applying the Picard contraction principle. For the Sobolev spaces $\dot{H}^s_q$ $(q > 1, \frac{d}{q} - 1 \leq s < \frac{d}{q})$, we obtain the local existence of mild solutions in the spaces $L^\infty\big([0,T]; \dot{H}^s_q(\mathbb{R}^d)\big)$ with arbitrary initial value in $\dot{H}^s_q(\mathbb{R}^d)$, in the case of critical indexes $(q > 1, s = \frac{d}{q} - 1)$ we get the existence of global mild solutions in the spaces $L^\infty([0,\infty); \dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d))$ when the norm of the initial value is small enough. The same argument is applied to following spaces:

- Critical Sobolev-Fourier-Lorentz spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d), (r \geq 1, 1 \leq p < \infty)$.

- Sobolev-Lorentz spaces $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d), (s \geq 0, q > 1, r \geq 1, \frac{d}{q} - 1 \leq s < \frac{d}{q})$ with critical indexes $s = \frac{d}{q} - 1$.

- For $0 \leq m < \infty$ and index vectors $\mathbf{q} = (q_1, q_2, ..., q_d), \mathbf{r} = (r_1, r_2, ..., r_d)$, where $1 < q_i < \infty, 1 \leq r_i \leq \infty$, and $1 \leq i \leq d$, we introduce and study mixed-norm Sobolev-Lorentz spaces $\dot{H}^m_{L^{\mathbf{q},\mathbf{r}}}$. Then we investigate the existence and uniqueness of solutions to the Navier-Stokes equations in the spaces $\mathcal{Q} := \mathcal{Q}_T = L^p([0,T]; \dot{H}^m_{L^{\mathbf{q},\mathbf{r}}})$ where $p > 2$, $T > 0$, and initial data is taken in the class $\mathcal{I} = \{u_0 \in (\mathcal{S}'(\mathbb{R}^d))^d, \mathrm{div}(u_0) = 0 : \|e^{\cdot\Delta}u_0\|_{\mathcal{Q}} < \infty\}$. The results have a standard relation between existence time and data size: large time with small data or large data with small time. In the case with $T = \infty$ and critical indexes $\frac{2}{p} + \sum_{i=1}^d \frac{1}{q_i} - m = 1$, the space $\mathcal{I}$ coincides with the homogeneous Besov space $\dot{B}^{m-\frac{2}{p},p}_{L^{\mathbf{q},\mathbf{r}}}$.

Finally, we investigate the Hausdorff dimension of the possible singular set in time of weak solutions to the Navier-Stokes equations on the three dimensional torus under some regularity conditions of Serrin's type. The results in the chapter relate the regularity conditions of Serrin's type to the Hausdorff dimension of the singular set in time.

List of the author's publications related to the dissertation

[1] D. Q. Khai and N. M. Tri, *Well-posedness for the Navier-Stokes equations with data in Sobolev-Lorentz spaces*, Nonlinear Analysis, **149** (2017), 130-145.

[2] D. Q. Khai, *Well-posedness for the Navier-Stokes equations with datum in the Sobolev spaces*, Acta Math Vietnam (2016). doi:10.1007/s40306-016-0192-x.

[3] D. Q. Khai and N. M. Tri, *Well-posedness for the Navier-Stokes equations with datum in Sobolev-Fourier-Lorentz spaces*, Journal of Mathematical Analysis and Applications, **437** (2016), 854-781.

[4] D. Q. Khai and N. M. Tri, *On the initial value problem for the Navier-Stokes equations with the initial datum in critical Sobolev and Besov spaces*, Journal of Mathematical Sciences the University of Tokyo, **23** (2016), 499-528.

[5] D. Q. Khai and N. M. Tri, *On the Hausdorff dimension of the singular set in time for weak solutions to the nonstationary Navier-Stokes equations on torus*,Vietnam Journal of Mathematics, **43** (2015), 283-295.

[6] D. Q. Khai and N. M. Tri, *Solutions in mixed-norm Sobolev-Lorentz spaces to the initial value problem for the Navier-Stokes equations*, Journal of Mathematical Analysis and Applications, **417** (2014), 819-833.

Author's other relevant papers

[7] D. Q.Khai, N.M. Tri, *On general axisymmetric explicit solutions for the Navier-Stokes equations*, International Journal of Evolution Equations **6** (2013), 325-336.

[8] D. Q. Khai and V. T. T. Duong, *On the initial value problem for the Navier-Stokes equations with the initial datum in the Sobolev spaces*, preprint arXiv:1603.04219.

[9] D. Q. Khai and N. M. Tri, *The existence and decay rates of strong solutions for Navier-Stokes Equations in Bessel-potential spaces*, preprint, arXiv:1603.01896.

[10] D. Q. Khai and N. M. Tri *The existence and space-time decay rates of strong solutions to Navier-Stokes Equations in weighed* $L^\infty(|x|^\gamma \mathrm{dx}) \cap L^\infty(|x|^\beta \mathrm{dx})$ *spaces*, preprint, arXiv:1601.01441.

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2) Ph.D. Students Conference, Hanoi Institute of Mathematics, Oct 25, 2013.

3) Ph.D. Students Conference, Hanoi Institute of Mathematics, Oct 30, 2014.

4) Seminar on Differential equations and its application, Hanoi Institute of Mathematics.

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