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SUBDIFFERENTIALS OF OPTIMAL VALUE FUNCTIONS IN PARAMETRIC CONVEX OPTIMIZATION PROBLEMS

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SUMMARY

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Introduction

If a mathematical programming problem depends on a parameter, that is, the objective function and the constraints depend on a certain parameter, then the optimal value is a function of the parameter, and the solution map is a set-valued map on the parameter of the problem. In general, the optimal value function is a fairly complicated function of the parameter; it is often nondifferentiable on the parameter, even if the functions defining the problem in question are smooth w.r.t. all the programming variables and the parameter. This is the reason of the great interest in having formulas for computing generalized directional derivatives (Dini directional derivative, Dini-Hadarmard directional derivative, Clarke generalized directional derivative,...) and formulas for evaluating subdifferentials (subdifferential in the sense of convex analysis, Clarke subdifferential, Fréchet subdifferential, limiting subdifferential - also called Mordukhovich subdifferential,...) of the optimal value function.

Studies on differentiability properties of the optimal value function and of the solution map in parametric mathematical programming are usually classified as studies on *differential stability* of optimization problems.

For differentiable nonconvex programs, pioneering works are due to J. Gauvin and W.J. Tolle (1977), J. Gauvin and F. Dubeau (1982). The authors obtained formulas for computing and estimating Dini directional derivatives and Clarke generalized gradients of the optimal value function when the problem data undergoes smooth perturbations. A. Auslender (1979), R.T. Rockafellar (1982), B. Golan (1984), L. Thibault (1991), and many other authors, have shown that similar results can be obtained for nondifferentiable nonconvex programs. For optimization problems with inclusion constraints on Banach spaces, differentiability properties of the optimal value function have been established via the dual-space approach by B.S. Mordukhovich, N.M. Nam, and N.D. Yen (2009), where it is shown that the new general results imply several fundamental results which were obtained by the primal-space approach.

Differential stability for convex programs has been studied intensively in the last five decades. A formula for computing the subdifferential of the optimal value function of a standard convex mathematical programming problem with right-hand-side perturbations, called the *perturbation function*, via the set of *Kuhn-Tucker vectors* (i.e., the vectors of Kuhn-Tucker coefficients) was given by R.T. Rockafellar (1970). Until now, many analogues and extensions of this classical result have been given in the literature.

Besides the investigations on differential stability of parametric mathematical programming problems, the study on differential stability of optimal control problems is also an issue of importance.

According to A.E. Bryson (1996), optimal control had its origins in the calculus of variations in the 17th century. The calculus of variations was developed further in the 18th by L. Euler and J.L. Lagrange and in the 19th century by A.M. Legendre, C.G.J. Jacobi, W.R. Hamilton, and K.T.W. Weierstrass. In 1957, R.E. Bellman gave a new view of Hamilton-Jacobi theory which he called *dynamic programming*, essentially a nonlinear feedback control scheme. E.J. McShane (1939) and L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, and E.F. Mishchenko (1962) extended the calculus of variations to handle control variable inequality constraints. *The Maximum Principle* was enunciated by Pontryagin.

As noted by P.N.V. Tu (1984), although much pioneering work had been carried out by other authors, Pontryagin and his associates are the first ones to develop and present the Maximum Principle in unified manner. Their work attracted great attention among mathematicians, engineers, economists, and spurred wide research activities in the area.

Motivated by the recent work of B.S. Mordukhovich, N.M. Nam, and N.D. Yen (Math. Program., 2009) on the optimal value function in parametric programming under inclusion constraints, this dissertation focuses on differential stability of convex optimization problems. In other words, we study differential properties of the optimal value function. Namely, we obtain some formulas for computing the subdifferential and the singular subdifferential of the optimal value function of infinite-dimensional convex optimization problems under inclusion constraints and of infinite-dimensional convex optimization problems under geometrical and functional constraints. Our main tool is Moreau–Rockafellar Theorem and appropriate regularity conditions. By virtue of the convexity, several assumptions used in the just cited work, like the nonemptyness of the Fréchet upper subdifferential of the objective function, the existence of a local upper Lipschitzian selection of the solution map, as well as the μ-inner semicontinuity or the μ-inner semicompactness of the solution map, are no longer needed. We also discuss the connection between the subdifferentials of the optimal value function and certain multiplier sets. Applied to parametric optimal control problems, with convex objective functions and linear dynamical systems, either discrete or continuous, our results can lead to some rules for computing the subdifferential and the singular subdifferential of the optimal value function via the data of the given problem.

The dissertation has six chapters, a list of the related papers of the author, a section of general conclusions, and a list of references. The first four chapters, where some preliminaries and a series of new results on sensitivity analysis of parametric convex programming problems under inclusion constraints are given, constitute the first part of the dissertation. The second part is formed by the last two chapters, where applications of the just mentioned results to parametric convex control problems under linear constraints are carried on.

Chapter 1 collects some basic concepts from convex analysis, variational analysis, and functional analysis needed for subsequent chapters.

Chapter 2 presents some new results on differential stability of convex optimization problems under inclusion constraints in Hausdorff locally convex topological vector spaces. The main tool is the Moreau-Rockafellar Theorem, which can be viewed as a well-known result in convex analysis, and some appropriate regularity conditions. The results obtained here lead to new facts on differential stability of convex optimization problems under geometrical and functional constraints.

In Chapter 3 we first establish formulas for computing the subdifferentials of the optimal value function for parametric convex programs under three assumptions: the objective function is closed, the constraint multifunction has closed graph, and Aubin's regularity condition is satisfied. Then, we derive relationships between regularity conditions. Our investigations have revealed that one cannot use Aubin's regularity assumption in a Hausdorff locally convex topological vector space setting, because the related sum rule is established via the Banach open mapping theorem.

Chapter 4 discusses differential stability of convex programming problems in Hausdorff locally convex topological vector spaces. Optimality conditions for convex optimization problems under inclusion constraints and for convex optimization problems under geometrical and functional constraints are formulated here too. After establishing an upper estimate for the subdifferentials via the Lagrange multiplier sets, we give an example to show that the upper estimate can be strict. Then, by defining a satisfactory multiplier set, we obtain formulas for computing the subdifferential and the singular subdifferential of the optimal value function.

In Chapter 5 we first derive an upper estimate for the subdifferential of the optimal value function of convex discrete optimal control problems in Banach spaces. Then we present new calculus rules for computing the subdifferential if the objective function is differentiable. The main tools of our analysis are the formulas for computing subdifferentials of the optimal value function from Chapter 2. We also show that the singular subdifferential of the just mention optimal value function always consists of the origin of the dual space.

Finally, in Chapter 6, we focus on differential stability of convex continuous optimal control problems. Namely, based on the results of Chapter 5 about differential stability of parametric convex mathematical programming problems, we get new formulas for computing the subdifferential and the singular subdifferential of the optimal value function. Moreover, we also describe in details the process of finding vectors belonging to the subdifferential (resp., the singular subdifferential) of the optimal value function. Meaningful examples, which have the origin in the book of Pontryagin *et al.* (1962), are designed to illustrate our results.

Chapter 1

Preliminaries

Several concepts and results from convex analysis, variational analysis, and functional analysis are recalled in this chapter. Two types of parametric optimization problems to be considered in the subsequent three chapters are also presented in this chapter.

1.1 Subdifferentials

Let X, Y be Hausdorff locally convex topological vector spaces with the topological duals denoted respectively by X^* and Y^*.

Definition 1.1 For a convex set $\Omega \subset X$, the *normal cone* of Ω at $\bar{x} \in \Omega$ is given by

$$N(\bar{x}; \Omega) = \{x^* \in X^* \mid \langle x^*, x - \bar{x} \rangle \leq 0, \ \ \forall x \in \Omega\}.$$

Consider a function $f : X \to \overline{\mathbb{R}} = [-\infty, +\infty] := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ having values in the extended real line. One says that f is *proper* if $f(x) > -\infty$ for all $x \in X$, and the *domain* $\operatorname{dom} f := \{x \in X \mid f(x) < \infty\}$ is nonempty.

The *epigraph* of f is defined by $\operatorname{epi} f := \{(x, \alpha) \in X \times \mathbb{R} \mid \alpha \geq f(x)\}$. If $\operatorname{epi} f$ is a convex set, then f is said to be a convex function.

Definition 1.2 Let $f : X \to \overline{\mathbb{R}}$ be a convex function. Suppose that $\bar{x} \in X$ and $|f(\bar{x})| < \infty$.

(i) The set

$$\partial f(\bar{x}) = \{x^* \in X^* \mid \langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}), \ \forall x \in X\}$$

is called the *subdifferential* of f at $\bar{x}$.

(ii) The set

$$\partial^\infty f(\bar{x}) = \{x^* \in X^* \mid (x^*, 0) \in N((\bar{x}, f(\bar{x})); \operatorname{epi} f)\}$$

is called the *singular subdifferential* of f at $\bar{x}$.

In the case where $|f(\bar{x})| = \infty$, one lets $\partial f(\bar{x})$ and $\partial^\infty f(\bar{x})$ to be empty sets.

1.2 Coderivatives

Let $F : X \rightrightarrows Y$ be a convex set-valued map. The *graph* and the *domain* of F are given, respectively, by the formulas

$$\operatorname{gph} F := \{(x, y) \in X \times Y \mid y \in F(x)\},$$
$$\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}.$$

Definition 1.3 The *coderivative* of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is the multifunction $D^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ defined by

$$D^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{gph} F)\}, \ \forall y^* \in Y^*.$$

If $(\bar{x}, \bar{y}) \notin \operatorname{gph} F$, then we accept the convention that the set $D^*F(\bar{x}, \bar{y})(y^*)$ is empty for any $y^* \in Y^*$.

1.3 Optimal Value Function

Consider a function $\varphi : X \times Y \to \overline{\mathbb{R}}$, a set-valued map $G : X \rightrightarrows Y$ between Banach spaces. The *optimal value function* (or *the marginal function*) of the *parametric optimization problem under an inclusion constraint*, defined by G and φ, is the function $\mu : X \to \overline{\mathbb{R}}$, with

$$\mu(x) := \inf \{\varphi(x, y) \mid y \in G(x)\}. \tag{1.1}$$

By the convention $\inf \emptyset = +\infty$, we have $\mu(x) = +\infty$ for any $x \notin \operatorname{dom} G$.

The set-valued map G (resp., the function φ) is called the *map describing the constraint set* (resp., the *objective function*) of the optimization problem on the right-hand-side of (1.1).

Corresponding to each data pair $\{G, \varphi\}$ we have one *optimization problem depending on a parameter* x:

$$\min\{\varphi(x, y) \mid y \in G(x)\}. \tag{1.2}$$

Formulas for computing or estimating the subdifferentials (the Fréchet subdifferential, the Mordukhovich subdifferential, the singular subdifferential, and the subdifferential in the sense of convex analysis) of the optimal value function $\mu(.)$ are tightly connected with the *solution map* of (1.2). The just mentioned solution map, denoted by $M : \operatorname{dom} G \rightrightarrows Y$, is given by

$$M(x) := \{y \in G(x) \mid \mu(x) = \varphi(x, y)\} \quad (\forall x \in \operatorname{dom} G).$$

By imposing the convexity requirement on (1.2), in next Chapters 2 and 3, we need not to rely on the assumption $\widehat{\partial}^+\varphi(\bar{x}, \bar{y}) \neq \emptyset$ in Theorem 1 of the paper by B.S. Mordukhovich, N.M. Nam, and N.D. Yen (Math. Program., 2009), the condition saying that the solution map $M : \operatorname{dom} G \rightrightarrows Y$ has a local upper Lipschitzian selection at $(\bar{x}, \bar{y})$ in Theorem 2 of the just cited paper, as well as the sequentially normally compact property of φ, the μ-inner semicontinuity or the μ-inner semicompactness conditions on the solution map $M(\cdot)$ in Theorem 7 of the same article.

1.4 Problems under the Convexity

Let X and Y be Hausdorff locally convex topological vector spaces. Let $\varphi : X \times Y \to \overline{\mathbb{R}}$ be a proper convex extended-real-valued function. Given a convex set-valued map $G : X \rightrightarrows Y$, we consider the *parametric convex optimization problem under an inclusion constraint*

$$\min\{\varphi(x, y) \mid y \in G(x)\}. \tag{1.3}$$

depending on the parameter x. The *optimal value function* of problem (1.3), is the function $\mu : X \to \overline{\mathbb{R}}$, with

$$\mu(x) := \inf\left\{\varphi(x, y) \mid y \in G(x)\right\}. \tag{1.4}$$

The *solution map* $M : \operatorname{dom} G \rightrightarrows Y$ of that problem is defined by

$$M(x) := \{y \in G(x) \mid \mu(x) = \varphi(x, y)\} \quad (\forall x \in \operatorname{dom} G).$$

Proposition 1.1 *Let $G : X \rightrightarrows Y$ be a convex set-valued map, $\varphi : X \times Y \to \overline{\mathbb{R}}$ a convex function. Then, the function $\mu(.)$ is defined by* (1.4) *is convex.*

In next two chapters, to obtain formulas for computing/estimating the subdifferential of the optimal value function μ via the subdifferential of φ and the coderivative of G, we will apply the following scheme, which has been formulated clearly by Professor Truong Xuan Duc Ha in her review on this dissertation.

Step 1. Consider the unconstrained optimization problem

$$\mu(x) := \inf\left\{\varphi(x, y) + \delta((x, y); \operatorname{gph} G)\right\},$$

where $\delta(\cdot; \operatorname{gph} G)$ is the indicator function of $\operatorname{gph} G$.

Step 2. Apply some known results to show that

$$(x^*, 0) \in \partial\Big(\varphi + \delta(\cdot; \operatorname{gph} G)\Big)(\bar{x}, \bar{y})$$

for every $x^* \in \partial\mu(\bar{x})$ and for some $\bar{y} \in M(\bar{x})$.

Step 3. Employ the sum rule for subdifferentials to get

$$(x^*, 0) \in \partial\varphi(\bar{x}, \bar{y}) + \partial\delta((\bar{x}, \bar{y}); \operatorname{gph} G).$$

Step 4. Use the relationships between $\partial\delta((\bar{x}, \bar{y}); \operatorname{gph} G)$, $N((\bar{x}, \bar{y}); \operatorname{gph} G)$ and the definition of the coderivative in question.

1.5 Some Facts from Functional Analysis and Convex Analysis

Consider a continuous linear operator $A : X \to Y$ from a Banach space X to another Banach space Y with the adjoint $A^* : Y^* \to X^*$. The null space and the range of A are defined, respectively, by $\ker A = \{x \in X \mid Ax = 0\}$ and $\mathrm{rge}\, A = \{y \in Y \mid y = Ax,\, x \in X\}$.

Proposition 1.2 (See J.F. Bonnans and A. Shapiro (2000)) *The next properties are valid:*
(i) $(\ker \mathrm{A})^{\perp} = \mathrm{cl}^*(\mathrm{rge}\,(\mathrm{A}^*))$, *where* $\mathrm{cl}^*(\mathrm{rge}\,(A^*))$ *denotes the closure of the set* $\mathrm{rge}\,(A^*)$ *in the weak* topology of* X^*, *and*

$$(\ker A)^{\perp} = \{x^* \in X^* \mid \langle x^*, x\rangle = 0 \ \ \forall x \in \ker A\}$$

stands for the orthogonal complement of the set $\ker A$.
(ii) *If* $\mathrm{rge}\, A$ *is closed, then* $(\ker A)^{\perp} = \mathrm{rge}\,(A^*)$, *and there is* $c > 0$ *such that for every* $x^* \in \mathrm{rge}\,(A^*)$ *there exists* $y^* \in Y^*$ *with* $||y^*|| \le c||x^*||$ *and* $x^* = A^*y^*$.
(iii) *If, in addition,* $\mathrm{rge}\, A = Y$, *i.e.,* A *is onto, then* A^* *is one-to-one and there exists* $c > 0$ *such that* $||y^*|| \le c||A^*y^*||$, *for all* $y^* \in Y^*$.
(iv) $(\ker A^*)^{\perp} = \mathrm{cl}(\mathrm{rge}\, A)$.

Suppose that $A_0, A_1, \ldots, A_n$ are convex subsets of a Hausdorff locally convex topological vector space X and $A = A_0 \cap A_1 \cap \cdots \cap A_n$. By $\mathrm{int}\, A_i$, for $i = 1, \ldots, n$, we denote the interior of A_i. The following two propositions and one theorem can be found in the book *"Theory of Extremal Problems"* of A.D. Ioffe and V.M. Tihomirov (1979).

Proposition 1.3 *If one has*

$$A_0 \cap (\mathrm{int}\, A_1) \cap \cdots \cap (\mathrm{int}\, A_n) \neq \emptyset,$$

then $N(x; A) = N(x; A_0) + N(x; A_1) + \cdots + N(x; A_n)$ *for any point* $x \in A$.

Proposition 1.4 *If one has* $\mathrm{int}\, A_i \neq \emptyset$ *for* $i = 1, 2, \ldots, n$ *then, for any* $x_0 \in A$, *the following statements are equivalent:*
(a) $A_0 \cap (\mathrm{int}\, A_1) \cap \cdots \cap (\mathrm{int}\, A_n) = \emptyset$;
(b) *There exist* $x_i^* \in N(x_0; A_i)$ *for* $i = 0, 1, \ldots, n$, *not all zero, such that*

$$x_0^* + x_1^* + \cdots + x_n^* = 0.$$

Theorem 1.1 (The Moreau-Rockafellar Theorem) *Let* $f_1, \ldots, f_m$ *be proper convex functions on* X. *Then*

$$\partial(f_1 + \cdots + f_m)(x) \supset \partial f_1(x) + \cdots + \partial f_m(x)$$

for all $x \in X$. *If, at a point* $x^0 \in \mathrm{dom}\, f_1 \cap \cdots \cap \mathrm{dom}\, f_m$, *all the functions* $f_1, \ldots, f_m$, *except, possibly, one are continuous, then*

$$\partial(f_1 + \cdots + f_m)(x) = \partial f_1(x) + \cdots + \partial f_m(x)$$

for all $x \in X$.

Chapter 2

Differential Stability in Parametric Convex Programming Problems

This chapter establishes some new results about differential stability of convex optimization problems under inclusion constraints and functional constraints. By using a version of the Moreau-Rockafellar Theorem, which has been recalled in Theorem 1.1, and appropriate regularity conditions, we obtain formulas for computing the subdifferential and the singular subdifferential of the optimal value function.

2.1 Differential Stability of Convex Optimization Problems under Inclusion Constraints

The next theorem provides us with formulas for computing the subdifferential and the singular subdifferential of μ given in (1.4).

Theorem 2.1 *Let $G : X \rightrightarrows Y$ be a convex set-valued mapping and $\varphi : X \times Y \to \overline{\mathbb{R}}$ a proper convex function. If at least one of the following regularity conditions is satisfied:*

(a) $\operatorname{int}(\operatorname{gph} G) \cap \operatorname{dom} \varphi \neq \emptyset$,

(b) *φ is continuous at a point $(x^0, y^0) \in \operatorname{gph} G$,*

then for any $\bar{x} \in \operatorname{dom} \mu$, with $\mu(\bar{x}) \neq -\infty$, and for any $\bar{y} \in M(\bar{x})$ we have

$$\partial \mu(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial \varphi(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\}$$

and

$$\partial^\infty \mu(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial^\infty \varphi(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\}.$$

2.2 Convex Programming Problems under Functional Constraints

Consider the problem

$$\min\{\varphi(x,y) \mid (x,y)\in C,\ \ g_i(x,y)\le 0,\ i\in I,\ \ h_j(x,y)=0,\ j\in J\},\quad (2.1)$$

in which $\varphi : X\times Y\to\overline{\mathbb{R}}$ is a convex function, $C\subset X\times Y$ is a convex set, $I=\{1,\dots,m\}$, $J=\{1,\dots,k\}$, $g_i : X\times Y\to\mathbb{R}$ $(i\in I)$ are continuous convex functions, and $h_j : X\times Y\to\mathbb{R}$ $(j\in J)$ are continuous affine functions. For each $x\in X$, we put

$$G(x)=\{y\in Y \mid (x,y)\in C,\ g_i(x,y)\le 0,\ i\in I,\ h_j(x,y)=0,\ j\in J\}.\quad (2.2)$$

It clear that the set-valued map $G(\cdot)$ given by (2.2) is convex and

$$\operatorname{gph} G = C\cap\left(\bigcap_{i\in I}\Omega_i\right)\cap\left(\bigcap_{j\in J}Q_j\right),$$

where $\Omega_i := \{(x,y)\mid g_i(x,y)\le 0\}$ $(i\in I)$ and $Q_j := \{(x,y)\mid h_j(x,y)=0\}$ $(j\in J)$ are convex sets.

Theorem 2.2 *Suppose that the equality constraints $h_j(x,y)=0$ $(j\in J)$ are absent in (2.1). If at least one of the following regularity conditions*

(a1) *There exists a point $(u^0,v^0)\in\operatorname{dom}\varphi$ such that $(u^0,v^0)\in\operatorname{int}C$ and $g_i(u^0,v^0)<0$ for all $i\in I$,*

(b1) *φ is continuous at a point $(x^0,y^0)\in C$ where $g_i(x^0,y^0)<0$ for all $i\in I$,*

is satisfied, then for any $\bar x\in\operatorname{dom}\mu$, with $\mu(\bar x)\ne-\infty$, and for any $\bar y\in M(\bar x)$ we have

$$\partial\mu(\bar x)=\bigcup_{(x^*,y^*)\in\partial\varphi(\bar x,\bar y)}\{x^*+Q_0^*\}$$

and

$$\partial^\infty\mu(\bar x)=\bigcup_{(x^*,y^*)\in\partial^\infty\varphi(\bar x,\bar y)}\{x^*+Q_0^*\},$$

where

$$Q_0^* := \left\{u^*\in X^* \mid (u^*,-y^*)\in N((\bar x,\bar y);C)+\sum_{i\in I(\bar x,\bar y)}\operatorname{cone}\partial g_i(\bar x,\bar y)\right\}$$

with $I(\bar x,\bar y):=\{i\mid g_i(\bar x,\bar y)=0\}$ and $\operatorname{cone}M:=\{tz\mid t\ge 0, z\in M\}$ denoting the cone generated by M.

Theorem 2.3 *For every $j\in J$, suppose that*

$$h_j(x,y)=\langle(x_j^*,y_j^*),(x,y)\rangle-\alpha_j,\ \ \alpha_j\in\mathbb{R}.$$

If φ is continuous at a point (x^0, y^0) with $(x^0, y^0) \in \operatorname{int} C$, $g_i(x^0, y^0) < 0$, for all $i \in I$ and $h_j(x^0, y^0) = 0$, for all $j \in J$, then for any $\bar{x} \in \operatorname{dom} \mu$, with $\mu(\bar{x}) \neq -\infty$, and for any $\bar{y} \in M(\bar{x})$ we have

$$\partial \mu(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial \varphi(\bar{x}, \bar{y})} \{x^* + \tilde{Q}^*\}$$

and

$$\partial^\infty \mu(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial^\infty \varphi(\bar{x}, \bar{y})} \{x^* + \tilde{Q}^*\},$$

where

$$\tilde{Q}^* := \Big\{ u^* \in X^* \mid (u^*, -y^*) \in A + N((\bar{x}, \bar{y}); C) \Big\}$$

with $A := \sum\limits_{i \in I(\bar{x}, \bar{y})} \operatorname{cone} \partial g_i(\bar{x}, \bar{y}) + \operatorname{span}\{(x_j^*, y_j^*),\ j \in J\}$.

Chapter 3

Stability Analysis using Aubin's Regularity Condition

In this chapter, we obtain formulas for computing the subdifferentials of the optimal value function for parametric convex programs under three assumptions: the objective function is closed; the constraint multifunction has closed graph; and an interior regularity condition (we will call it *Aubin's regularity condition*) is satisfied.

3.1 Differential Stability under Aubin's Regularity Condition

Let $G : X \rightrightarrows Y$ be a convex multifunction between *Banach spaces*, whose graph is *closed*. Let $\varphi : X \times Y \to \overline{\mathbb{R}}$ be a proper, *closed*, convex function. Consider the *parametric optimization problem under an inclusion constraint*

$$\min \{\varphi(x, y) \mid y \in G(x)\}. \tag{3.1}$$

Using the regularity condition

$$(0, 0) \in \operatorname{int}(\operatorname{dom} \varphi - \operatorname{gph} G), \tag{3.2}$$

we will derive formulas for computing the subdifferential and the singular subdifferential of the *optimal value function* $\mu : X \to \overline{\mathbb{R}}$ of (3.1), which is given by

$$\mu(x) = \inf\{\varphi(x, y) \mid y \in G(x)\}. \tag{3.3}$$

Theorem 3.1 *If the regularity condition* (3.2) *is satisfied, then for every* $\bar{x} \in \operatorname{dom} \mu$ *with* $\mu(\bar{x}) \neq -\infty$*, and for every* $\bar{y} \in M(\bar{x})$*, we have*

$$\partial \mu(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial \varphi(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\}. \tag{3.4}$$

Theorem 3.2 *In addition to the assumption of Theorem 3.1, suppose that the set* $\operatorname{dom} \varphi$ *is closed. Then*

$$\partial^\infty \mu(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial^\infty \varphi(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\}.$$

3.2 An Analysis of the Regularity Conditions

Consider an example satisfying Aubin's regularity condition (3.2), but both regularity conditions (a) and (b) in Theorem 2.1 are not fulfilled, whereas the conclusion of the Theorem 3.1 holds true.

Example 3.1 Let $X = Y = \mathbb{R}^2$ and $(\bar{x}, \bar{y}) = (0, 0)$. Consider the optimal value function $\mu(x)$ defined by (3.3) with $\varphi_0(y) = 0$ if $y_1 = 0$ and $\varphi_0(y) = +\infty$ if $y_1 \neq 0$, for every $y = (y_1, y_2) \in Y$, and

$$G(x) = \begin{cases} \mathbb{R} \times \{0\} & \text{if } x = 0, \\ \emptyset & \text{if } x \neq 0, \end{cases}$$

for every $x = (x_1, x_2) \in X$. Clearly, φ_0 is a proper, closed, convex function with $\operatorname{dom} \varphi_0$ being closed. In addition, G is a convex multifunction of closed graph. Setting $\varphi(x, y) = \varphi_0(y)$ for all $(x, y) \in X \times Y$, we have $\operatorname{gph} G = \{0_{\mathbb{R}^2}\} \times \mathbb{R} \times \{0\}$ and $\operatorname{dom} \varphi = \mathbb{R}^2 \times \{0\} \times \mathbb{R}$. Since $\operatorname{int}(\operatorname{gph} G) = \emptyset$, the regularity condition $\operatorname{int}(\operatorname{gph} G) \cap \operatorname{dom} \varphi \neq \emptyset$ fails to hold. Obviously, φ is discontinuous at any point $(x^0, y^0) \in \operatorname{gph} G$. Meanwhile, $\operatorname{dom} \varphi - \operatorname{gph} G = X \times Y$, so (3.2) is satisfied. It is easy to see that

$$\mu(x) = \inf\{\varphi_0(y) \mid y \in G(x)\} = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{if } x \neq 0. \end{cases}$$

A simple calculation shows that $\partial \mu(\bar{x}) = \mathbb{R}^2$ and $\partial \varphi(\bar{x}, \bar{y}) = \{0_{\mathbb{R}^2}\} \times \mathbb{R} \times \{0\}$. For any $y^* = (y_1^*, 0) \in \mathbb{R} \times \{0\}$, we have

$$D^*G(\bar{x}, \bar{y})(y^*) = \begin{cases} \mathbb{R}^2 & \text{if } y_1^* = 0, \\ \emptyset & \text{if } y_1^* \neq 0. \end{cases}$$

Hence the equality (3.4) is valid.

Proposition 3.1 *If the assumption* $\mathrm{int}(\mathrm{gph}\, G) \neq \emptyset$ *is fulfilled, then the regularity condition* (*a*) *in Theorem 2.1 is equivalent to Aubin's regularity condition* (3.2).

Proposition 3.2 *If the assumption* $\mathrm{int}(\mathrm{dom}\, \varphi) \neq \emptyset$ *is satisfied, then the regularity condition* (*b*) *in Theorem 2.1 and the condition* (3.2) *are equivalent.*

Chapter 4

Subdifferential Formulas Based on Multiplier Sets

This chapter discusses the connection between the subdifferentials of the optimal value function of parametric convex mathematical programming problems under geometrical and/or functional constraints and certain multiplier sets. Optimality conditions for convex optimization problems under inclusion constraints and functional constraints are formulated too.

4.1 Optimality Conditions for Convex Optimization

Optimality conditions for convex optimization problems, which can be derived from the calculus rules of convex analysis, have been presented in many books and research papers. We now present some optimality conditions for convex programs under inclusion constraints and for convex optimization problems under geometrical and functional constraints. These conditions lead to certain Lagrange multiplier sets which are used in our subsequent differential stability analysis of parametric convex programs. Note that Theorems 4.1 - 4.3 below are consequences of Proposition 1 on p. 81 in the book of A.D. Ioffe and V.M. Tihomirov (1979), and the Moreau-Rockafellar Theorem (see Theorem 1.1).

Let X and Y be Hausdorff locally convex topological vector spaces. Given a convex function $\varphi : X \times Y \to \overline{\mathbb{R}}$, we denote by $\partial_x \varphi(\bar{x}, \bar{y})$ (resp., $\partial_y \varphi(\bar{x}, \bar{y})$) its *partial subdifferential* in the first variable (resp., in the second variable) at $(\bar{x}, \bar{y})$. Thus, $\partial_x \varphi(\bar{x}, \bar{y}) = \partial \varphi(., \bar{y})(\bar{x})$ and $\partial_y \varphi(\bar{x}, \bar{y}) = \partial \varphi(\bar{x}, .)(\bar{y})$, provided that the expressions on the right-hand-sides are well defined.

4.1.1 Problems under Inclusion Constraints

Let $\varphi : X \times Y \to \overline{\mathbb{R}}$ be a proper convex function, $G : X \rightrightarrows Y$ a convex multifunction between Hausdorff locally convex topological vector spaces. Consider the *parametric optimization problem under an inclusion constraint*

$$(P_x) \qquad\qquad \min\{\varphi(x, y) \mid y \in G(x)\}$$

depending on the parameter x. The *optimal value function* $\mu : X \to \overline{\mathbb{R}}$ of problem (P_x) is

$$\mu(x) := \inf \{\varphi(x, y) \mid y \in G(x)\} .$$

The usual convention $\inf \emptyset = +\infty$ forces $\mu(x) = +\infty$ for every $x \notin \operatorname{dom} G$. The *solution map* $M : \operatorname{dom} G \rightrightarrows Y$ of that problem is defined by

$$M(x) := \{y \in G(x) \mid \mu(x) = \varphi(x, y)\}.$$

The next theorems describe some necessary and sufficient optimality conditions for (P_x) at a given parameter $\bar{x} \in X$.

Theorem 4.1 *Let $\bar{x} \in X$. Suppose that at least one of the following regularity conditions is satisfied:*
(a) $\operatorname{int} G(\bar{x}) \cap \operatorname{dom} \varphi(\bar{x}, .) \neq \emptyset$,
(b) $\varphi(\bar{x}, .)$ *is continuous at a point belonging to* $G(\bar{x})$.
Then, one has $\bar{y} \in M(\bar{x})$ if and only if

$$0 \in \partial_y \varphi(\bar{x}, \bar{y}) + N(\bar{y}; G(\bar{x})).$$

Theorem 4.2 *Let X, Y be Banach spaces, $\varphi : X \times Y \to \overline{\mathbb{R}}$ a proper, closed, convex function. Suppose that $G : X \rightrightarrows Y$ is a convex multifunction, whose graph is closed. Let $\bar{x} \in X$ be such that the regularity condition*

$$0 \in \operatorname{int} \big(\operatorname{dom} \varphi(\bar{x}, .) - G(\bar{x})\big)$$

is satisfied. Then, $\bar{y} \in M(\bar{x})$ if and only if

$$0 \in \partial_y \varphi(\bar{x}, \bar{y}) + N(\bar{y}; G(\bar{x})).$$

4.1.2 Problems under Geometrical and Functional Constraints

Consider the program

$$(\widetilde{P}_x) \qquad \min \{\varphi(x, y) \mid (x, y) \in C,\ g_i(x, y) \leq 0,\ i \in I,\ h_j(x, y) = 0,\ j \in J\}$$

depending on the parameter x, where $C \subset X \times Y$ is a convex set, the functions $g_i : X \times Y \to \mathbb{R}$ $(i \in I)$, with $I := \{1, \ldots, m\}$, are continuous convex, $h_j : X \times Y \to \mathbb{R}$ $(j \in J)$, with $J := \{1, \ldots, k\}$, are continuous affine. For each $x \in X$, we put

$$G(x) = \{y \in Y \mid (x, y) \in C,\ g(x, y) \leq 0,\ h(x, y) = 0\}, \tag{4.1}$$

where

$$g(x,y) := (g_1(x,y),\ldots,g_m(x,y))^T, \;\; h(x,y) := (h_1(x,y),\ldots,h_k(x,y))^T.$$

Fix a point $\bar{x} \in X$ and put

$$C_{\bar{x}} := \{y \in Y \mid (\bar{x},y) \in C\}. \tag{4.2}$$

Theorem 4.3 *If $\varphi(\bar{x},.)$ is continuous at a point $y^0 \in \operatorname{int} C_{\bar{x}}$, $g_i(\bar{x},y^0) < 0$ for all $i \in I$ and $h_j(\bar{x},y^0) = 0$ for all $j \in J$, then for a point $\bar{y} \in G(\bar{x})$ to be a solution of $(\widetilde{P}_{\bar{x}})$, it is necessary and sufficient that there exist $\lambda_i \geq 0$, $i \in I$, and $\mu_j \in \mathbb{R}$, $j \in J$, such that*
(a) $0 \in \partial_y \varphi(\bar{x},\bar{y}) + \sum_{i\in I} \lambda_i \partial_y g_i(\bar{x},\bar{y}) + \sum_{j\in J} \mu_j \partial_y h_j(\bar{x},\bar{y}) + N(\bar{y};C_{\bar{x}})$;
(b) $\lambda_i g_i(\bar{x},\bar{y}) = 0, \; i \in I$.

4.2 Subdifferential Estimates via Multiplier Sets

The *Lagrangian function* corresponding to the parametric problem $(\widetilde{P}_x)$ is

$$L(x,y,\lambda,\mu) := \varphi(x,y) + \lambda^T g(x,y) + \mu^T h(x,y) + \delta((x,y);C),$$

where $\lambda = (\lambda_1,\lambda_2,...,\lambda_m) \in \mathbb{R}^m$ and $\mu = (\mu_1,\mu_2,...,\mu_k) \in \mathbb{R}^k$. For each pair $(x,y) \in X \times Y$, by $\Lambda_0(x,y)$ we denote the set of all the multipliers $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^k$ with $\lambda_i \geq 0$ for all $i \in I$ and $\lambda_i = 0$ for every $i \in I \setminus I(x,y)$, where $I(x,y) = \{i \in I \mid g_i(x,y) = 0\}$.

For a parameter $\bar{x}$, the *Lagrangian function* corresponding to the unperturbed problem $(\widetilde{P}_{\bar{x}})$ is

$$L(\bar{x},y,\lambda,\mu) = \varphi(\bar{x},y) + \lambda^T g(\bar{x},y) + \mu^T h(\bar{x},y) + \delta((\bar{x},y);C). \tag{4.3}$$

Denote by $\Lambda(\bar{x},\bar{y})$ the *Lagrange multiplier set* corresponding to an optimal solution $\bar{y}$ of problem $(\widetilde{P}_{\bar{x}})$. Thus, $\Lambda(\bar{x},\bar{y})$ consists of the pairs $(\lambda,\mu) \in \mathbb{R}^m \times \mathbb{R}^k$ satisfying

$$\begin{cases} 0 \in \partial_y L(\bar{x},\bar{y},\lambda,\mu), \\ \lambda_i g_i(\bar{x},\bar{y}) = 0, \; i = 1,\ldots,m, \\ \lambda_i \geq 0, \; i = 1,\ldots,m, \end{cases}$$

where $\partial_y L(\bar{x},\bar{y},\lambda,\mu)$ is the subdifferential of the function $L(\bar{x},.,\lambda,\mu)$ defined by (4.3) at $\bar{y}$. It is clear that $\delta((\bar{x},y);C) = \delta(y;C_{\bar{x}})$, where $C_{\bar{x}}$ has been defined by (4.2).

Theorem 4.4 *Suppose that $h_j(x,y) = \langle (x_j^*,y_j^*),(x,y)\rangle - \alpha_j$, $\alpha_j \in \mathbb{R}$, $j \in J$, and $M(\bar{x})$ is nonempty for some $\bar{x} \in \operatorname{dom}\mu$. If φ is continuous at a point $(x^0,y^0) \in \operatorname{int} C$, $g_i(x^0,y^0) < 0$ for all $i \in I$ and $h_j(x^0,y^0) = 0$ for all $j \in J$*

then, for any $\bar{y} \in M(\bar{x})$, one has

$$\partial\mu(\bar{x}) = \left\{ \bigcup_{(\lambda,\mu)\in\Lambda_0(\bar{x},\bar{y})} \mathrm{pr}_{X^*}\Big(\partial L(\bar{x},\bar{y},\lambda,\mu) \cap \big(X^* \times \{0\}\big)\Big) \right\}, \qquad (4.4)$$

where $\partial L(\bar{x},\bar{y},\lambda,\mu)$ is the subdifferential of the function $L(.,.,\lambda,\mu)$ at $(\bar{x},\bar{y})$ and, for any $(x^*,y^*) \in X^* \times Y^*$, $\mathrm{pr}_{X^*}(x^*,y^*) := x^*$.

Example 4.1 Let $X = Y = \mathbb{R}$, $C = X \times Y$, $\varphi(x,y) = |x+y|$, $m = 1$, $k = 0$ (no equality functional constraint), $g_1(x,y) = y$ for all $(x,y) \in X \times Y$. Choose $\bar{x} = 0$, $\bar{y} = 0$, and note that $M(\bar{x}) = \{\bar{y}\}$. We have $\Lambda_0(\bar{x},\bar{y}) = [0,\infty)$ and $L(x,y,\lambda) = \varphi(x,y) + \lambda y$. We also have

$$\partial\varphi(\bar{x},\bar{y}) = \mathrm{co}\left\{(1,1)^T,\, (-1,-1)^T\right\}.$$

Since $\partial L(\bar{x},\bar{y},\lambda) = \partial\varphi(\bar{x},\bar{y}) + \{(0,\lambda)\}$, by (4.4) we can compute

$$\begin{aligned}
\partial\mu(\bar{x}) &= \left\{ \bigcup_{\lambda\in\Lambda_0(\bar{x},\bar{y})} \mathrm{pr}_{X^*}\Big(\partial L(\bar{x},\bar{y},\lambda) \cap \big(X^* \times \{0\}\big)\Big) \right\} \\
&= \mathrm{pr}_{X^*}\left[\Big(\bigcup_{\lambda\in\Lambda_0(\bar{x},\bar{y})} \partial L(\bar{x},\bar{y},\lambda)\Big) \cap \big(X^* \times \{0\}\big)\right] \\
&= \mathrm{pr}_{X^*}\left\{ \Big[\mathrm{co}\left\{(1,1)^T,\, (-1,-1)^T\right\} + \Big(\{0\} \times \mathbb{R}_+\Big)\Big] \cap \big(X^* \times \{0\}\big) \right\} \\
&= [-1,0].
\end{aligned}$$

To verify this result, observe that

$$\mu(x) = \inf\left\{|x+y| \mid y \le 0\right\} = \begin{cases} 0, & \text{if } x \ge 0, \\ -x, & \text{if } x < 0. \end{cases}$$

So we find $\partial\mu(\bar{x}) = [-1,0]$, justifying (4.4) for the problem under consideration.

Theorem 4.5 *Under the assumptions of Theorem 4.4, one has*

$$\partial\mu(\bar{x}) \subset \bigcup_{(\lambda,\mu)\in\Lambda(\bar{x},\bar{y})} \partial_x L(\bar{x},\bar{y},\lambda,\mu), \qquad (4.5)$$

where $\partial_x L(\bar{x},\bar{y},\lambda,\mu)$ stands for the subdifferential of $L(.,\bar{y},\lambda,\mu)$ at $\bar{x}$.

The next example shows that the inclusion in Theorem 4.5 can be strict.

Example 4.2 Let $X = Y = \mathbb{R}$, $C = X \times Y$, $\varphi(x,y) = |x+y|$, $m = 1$, $k = 0$ (no equality functional constraint), $g_1(x,y) = y$ for all $(x,y) \in X \times Y$. Choose $\bar{x} = 0$, $\bar{y} = 0$, and note that $M(\bar{x}) = \{\bar{y}\}$. We have $L(x,y,\lambda) = \varphi(x,y) + \lambda y$ and

$$\Lambda(\bar{x},\bar{y}) = \{\lambda \ge 0 \mid 0 \in \partial_y L(\bar{x},\bar{y},\lambda)\} = [0,1].$$

As in Example 4.1, one has $\partial\mu(\bar{x}) = [-1, 0]$. We now compute the right-hand-side of (4.5). By simple computation, we can easily obtain $\partial_x L(\bar{x}, \bar{y}, \lambda) = [-1, 1]$ for all $\lambda \in \Lambda(\bar{x}, \bar{y})$. Then $\bigcup\limits_{\lambda\in\Lambda(\bar{x},\bar{y})} \partial_x L(\bar{x}, \bar{y}, \lambda) = [-1, 1]$. Therefore, in this example, inclusion (4.5) is strict.

4.3 Computation of the Singular Subdifferential

First, we observe that $x \in \operatorname{dom}\mu$ if and only if $\mu(x) = \inf\{\varphi(x, y) \mid y \in G(x)\} < \infty$, with $G(x)$ being given by (4.1). Since the strict inequality holds if and only if there exists $y \in G(x)$ with $(x, y) \in \operatorname{dom}\varphi$, we have

$$\delta(x; \operatorname{dom}\mu) = \inf\{\delta((x, y); \operatorname{dom}\varphi) \mid y \in G(x)\}.$$

To compute the singular subdifferential of $\mu(.)$, let us consider the minimization problem

$$(\widehat{P}_x) \quad \begin{cases} \delta((x, y); \operatorname{dom}\varphi) \to \inf \\ \text{subject to } \ (x, y) \in C, \ g_i(x, y) \le 0, \ i \in I, \ h_j(x, y) = 0, \ j \in J. \end{cases}$$

The Lagrangian function corresponding to $(\widehat{P}_x)$ is

$$\widehat{L}(x, y, \lambda, \mu) = \delta((x, y); \operatorname{dom}\varphi) + \lambda^T g(x, y) + \mu^T h(x, y) + \delta((x, y); C),$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbb{R}^m$, $\mu = (\mu_1, \mu_2, ..., \mu_k) \in \mathbb{R}^k$.

Theorem 4.6 *Under the hypotheses of Theorem 4.4, for any $\bar{y} \in M(\bar{x})$, one has*

$$\partial^\infty\mu(\bar{x}) = \left\{ \bigcup_{(\lambda,\mu)\in\Lambda_0(\bar{x},\bar{y})} \operatorname{pr}_{X^*}\Big(\partial\widehat{L}(\bar{x}, \bar{y}, \lambda, \mu) \cap \big(X^* \times \{0\}\big)\Big) \right\},$$

where

$$\partial\widehat{L}(\bar{x}, \bar{y}, \lambda, \mu) = \partial^\infty\varphi(\bar{x}, \bar{y}) + \sum_{i\in I(\bar{x},\bar{y})} \lambda_i \partial g_i(\bar{x}, \bar{y}) + \sum_{j\in J} \mu_j \partial h_j(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}); C)$$

is the subdifferential of the function $\widehat{L}(., ., \lambda, \mu)$ at $(\bar{x}, \bar{y})$, provided that a pair $(\lambda, \mu) \in \Lambda_0(\bar{x}, \bar{y})$ has been chosen.

Next, denote by $\Lambda^\infty(\bar{x}, \bar{y})$ the *singular Lagrange multiplier set* corresponding to an optimal solution $\bar{y}$ of problem $(\widehat{P}_{\bar{x}})$, which consists of the pairs $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^k$ satisfying

$$\begin{cases} 0 \in \partial_y \widehat{L}(\bar{x}, \bar{y}, \lambda, \mu), \\ \lambda_i g_i(\bar{x}, \bar{y}) = 0, \ i = 1, \dots, m, \\ \lambda_i \ge 0, \ i = 1, \dots, m. \end{cases}$$

Theorem 4.7 *Under the assumptions of Theorem 4.4, for any $\bar{y} \in M(\bar{x})$, one has*

$$\partial^\infty \mu(\bar{x}) \subset \bigcup_{(\lambda,\mu)\in\Lambda^\infty(\bar{x},\bar{y})} \partial_x \widehat{L}(\bar{x},\bar{y},\lambda,\mu),$$

where $\partial_x \widehat{L}(\bar{x},\bar{y},\lambda,\mu)$ stands for the subdifferential of $\widehat{L}(.,\bar{y},\lambda,\mu)$ at $\bar{x}$.

Chapter 5

Stability Analysis of Convex Discrete Optimal Control Problems

In this chapter we present some new results on differential stability of convex discrete optimal control problems. The main tools of our analysis are the formulas for computing subdifferentials of the optimal value function from Chapter 2.

5.1 Control Problem

Let X_k, U_k, W_k, for $k = 0, 1, \ldots, N-1$, and X_N be Banach spaces, where N is a positive natural number. Let there be given

- convex sets $\Omega_0 \subset U_0, \ldots, \Omega_{N-1} \subset U_{N-1}$, and $C \subset X_0$;

- continuous linear operators $A_k : X_k \to X_{k+1},\ B_k : U_k \to X_{k+1},\ T_k : W_k \to X_{k+1}$, for $k = 0, 1, \ldots, N-1$;

- functions $h_k : X_k \times U_k \times W_k \to \mathbb{R}$, for $k = 0, 1, \ldots, N-1$, and $h_N : X_N \to \mathbb{R}$, which are convex.

Put $W = W_0 \times W_1 \times \cdots \times W_{N-1}$. For every vector $w = (w_0, w_1, \ldots, w_{N-1}) \in W$, consider the following *convex discrete optimal control problem*: Find a pair (x, u) where $x = (x_0, x_1, \ldots, x_N) \in X_0 \times X_1 \times \cdots \times X_N$ is a trajectory and $u = (u_0, u_1, \ldots, u_{N-1}) \in U_0 \times U_1 \times \cdots \times U_{N-1}$ is a control sequence, which minimizes the *objective function*

$$\sum_{k=0}^{N-1} h_k(x_k, u_k, w_k) + h_N(x_N) \tag{5.1}$$

and satisfies $x_{k+1} = A_k x_k + B_k u_k + T_k w_k,\ \ k = 0, 1, \ldots, N-1$, the *initial condition* $x_0 \in C$, and the *control constraints*

$$u_k \in \Omega_k \subset U_k, \quad k = 0, 1, \ldots, N-1. \tag{5.2}$$

Put $X = X_0 \times X_1 \times \cdots \times X_N$, $U = U_0 \times U_1 \times \cdots \times U_{N-1}$. For every parameter $w = (w_0, w_1, \ldots, w_{N-1}) \in W$, denote by $V(w)$ the optimal value of problem (5.1)–(5.2), and by $S(w)$ the solution set of that problem. The extended real-valued function $V : W \to \overline{\mathbb{R}}$ is called *the optimal value function* of problem (5.1)–(5.2). It is assumed that V is finite at a certain parameter $\bar{w} = (\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{N-1}) \in W$ and $(\bar{x}, \bar{u})$ is a solution of (5.1)–(5.2), that is $(\bar{x}, \bar{u}) \in S(\bar{w})$ where $\bar{x} = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_N), \bar{u} = (\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1})$.

For each $w = (w_0, w_1, \ldots, w_{N-1}) \in W$, let

$$f(x, u, w) = \sum_{k=0}^{N-1} h_k(x_k, u_k, w_k) + h_N(x_N).$$

Then, setting $\Omega = \Omega_0 \times \Omega_1 \times \cdots \times \Omega_{N-1}$, $\widetilde{X} = X_1 \times X_2 \times \cdots \times X_N$, and

$$G(w) = \{(x, u) \in X \times U \mid x_{k+1} = A_k x_k + B_k u_k + T_k w_k,\ k = 0, 1, \ldots, N-1\},$$

we have

$$V(w) = \inf_{(x,u) \in G(w) \cap (C \times \widetilde{X} \times \Omega)} f(x, u, w).$$

5.2 Differential Stability of the Parametric Mathematical Programming Problem

Suppose that X, W and Z are Banach spaces with the dual spaces X^*, W^* and Z^*, respectively. Assume that $M : Z \to X$ and $T : W \to X$ are continuous linear operators. Let $M^* : X^* \to Z^*$ and $T^* : X^* \to W^*$ be the adjoint operators of M and T, respectively. Let $f : W \times Z \to \overline{\mathbb{R}}$ be a convex function and Ω a convex subset of Z with nonempty interior. For each $w \in W$, put $H(w) = \big\{z \in Z \mid Mz = Tw\big\}$ and consider the optimization problem

$$\min\{f(z, w) \mid z \in H(w) \cap \Omega\}. \tag{5.3}$$

We want to compute the subdifferential and the singular subdifferential of the optimal value function

$$h(w) := \inf_{z \in H(w) \cap \Omega} f(z, w) \tag{5.4}$$

of the parametric problem (5.3). Denote by $\widehat{S}(w)$ the solution set of (5.3).

Define the linear operator $\Phi : W \times Z \to X$ by setting $\Phi(w, z) = -Tw + Mz$ for all $(w, z) \in W \times Z$.

Lemma 5.1 *For each* $(\bar{w}, \bar{z}) \in \operatorname{gph} H$, *one has*

$$N\big((\bar{w}, \bar{z}); \operatorname{gph} H\big) = \operatorname{cl}^*\big\{(-T^* x^*, M^* x^*) \mid x^* \in X^*\big\}.$$

Moreover, if Φ *has closed range, then*

$$N\big((\bar{w}, \bar{z}); \operatorname{gph} H\big) = \big\{(-T^* x^*, M^* x^*) \mid x^* \in X^*\big\}. \tag{5.5}$$

In particular, if Φ *is surjective, then* (5.5) *is valid.*

Lemma 5.2 *If* Φ *has closed range and* $\ker T^* \subset \ker M^*$, *then one has for each* $(\bar{w}, \bar{z}) \in \operatorname{gph} H$ *the equality*

$$N\big((\bar{w}, \bar{z}); (W \times \Omega) \cap \operatorname{gph} H\big) = \{0\} \times N(\bar{z}; \Omega) + N\big((\bar{w}, \bar{z}); \operatorname{gph} H\big).$$

Theorem 5.1 *Suppose that* Φ *has closed range and* $\ker T^* \subset \ker M^*$. *If the optimal value function* h *in* (5.4) *is finite at* $\bar{w} \in \operatorname{dom} \widehat{S}$ *and* f *is continuous at* $(\bar{w}, \bar{z}) \in (W \times \Omega) \cap \operatorname{gph} H$, *then*

$$\partial h(\bar{w}) = \bigcup_{(w^*, z^*) \in \partial f(\bar{z}, \bar{w})} \ \bigcup_{v^* \in N(\bar{z}; \Omega)} \big[w^* + T^*\big((M^*)^{-1}(z^* + v^*)\big)\big]$$

and

$$\partial^\infty h(\bar{w}) = \bigcup_{(w^*, z^*) \in \partial^\infty f(\bar{z}, \bar{w})} \ \bigcup_{v^* \in N(\bar{z}; \Omega)} \big[w^* + T^*\big((M^*)^{-1}(z^* + v^*)\big)\big],$$

where $\big(M^*\big)^{-1}(z^* + v^*\big) = \{x^* \in X^* \mid M^* x^* = z^* + v^*\}$.

Theorem 5.2 *Under the assumptions of Theorem 5.1, suppose additionally that the function* f *is Fréchet differentiable at* $(\bar{z}, \bar{w})$. *Then*

$$\partial h(\bar{w}) = \bigcup_{v^* \in N(\bar{z}; \Omega)} \big[\nabla_w f(\bar{z}, \bar{w}) + T^*\big((M^*)^{-1}(\nabla_z f(\bar{z}, \bar{w}) + v^*)\big)\big],$$

where $\nabla_z f(\bar{z}, \bar{w})$ *and* $\nabla_w f(\bar{z}, \bar{w})$, *respectively, stand for the Fréchet derivatives of* $f(\cdot, \bar{w})$ *at* $\bar{z}$ *and of* $f(\bar{z}, \cdot)$ *at* $\bar{w}$.

5.3 Differential Stability of the Control Problem

In the notation of Section 5.1, put $Z = X \times U$ and $K = C \times \widetilde{X} \times \Omega$ and note that $V(w)$ can be expressed as $V(w) = \inf\limits_{z \in G(w) \cap K} f(z, w)$, where $G(w) = \big\{z = (x, u) \in Z \,|\, Mz = Tw\big\}$ with $M : Z \to \widetilde{X}$ and $T : W \to \widetilde{X}$ being defined, respectively, by

$$Mz = \begin{pmatrix} -A_0 & I & 0 & 0 & \dots & 0 & 0 & -B_0 & 0 & 0 & \dots & 0 \\ 0 & -A_1 & I & 0 & \dots & 0 & 0 & 0 & -B_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -A_{N-1} & I & 0 & 0 & 0 & \dots & -B_{N-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \\ u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix},$$

$$Tw = \begin{pmatrix} T_0 w_0 \\ T_1 w_1 \\ \vdots \\ T_{N-1} w_{N-1} \end{pmatrix}.$$

Then problem (5.1)–(5.2) reduces to the mathematical programming problem (5.3). For every $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, ..., \tilde{x}_N^*) \in \widetilde{X}^*$, one has

$$T^*\tilde{x}^* = \big(T_0^*\tilde{x}_1^*, T_1^*\tilde{x}_2^*, \cdots, T_{N-1}^*\tilde{x}_N^*\big) \in W^* = W_0^* \times W_1^* \times \cdots \times W_{N-1}^*$$

and

$$M^*\tilde{x}^* = \begin{pmatrix} -A_0^* & 0 & 0 & \ldots & 0 \\ I & -A_1^* & 0 & \ldots & 0 \\ 0 & I & & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -A_{N-1}^* \\ 0 & 0 & 0 & \ldots & I \\ -B_0^* & 0 & 0 & \ldots & 0 \\ 0 & -B_1^* & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -B_{N-1}^* \end{pmatrix} \begin{pmatrix} \tilde{x}_1^* \\ \tilde{x}_2^* \\ \vdots \\ \tilde{x}_N^* \end{pmatrix}.$$

Theorem 5.3 *Suppose that h_k, $k = 0, 1, \ldots, N$, are continuous and the interiors of Ω_k, for $k = 0, 1, \ldots, N-1$, are nonempty. Suppose in addition that the following conditions are satisfied:*

(i) $\ker T^* \subset \ker M^*$;

(ii) *The operator $\Phi : W \times Z \to \widetilde{X}$ defined by $\Phi(w, z) = -Tw + Mz$ has closed range.*

If a vector $\tilde{w}^* = (\tilde{w}_0^*, \tilde{w}_1^*, \ldots, \tilde{w}_{N-1}^*) \in \partial V(\bar{w})$ then there exist vectors $x_0^* \in N(\bar{x}_0; C)$, $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, \ldots, \tilde{x}_N^*) \in \widetilde{X}^*$, and $u^* = (u_0^*, u_1^*, \ldots, u_{N-1}^*) \in N(\bar{u}; \Omega)$, such that

$$\begin{cases} \tilde{x}_N^* \in \partial h_N(\bar{x}_N), \\ \tilde{x}_k^* \in \partial_{x_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) + A_k^*\tilde{x}_{k+1}^*, \;\; k = 1, 2, ..., N-1, \\ x_0^* \in -\partial_{x_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) - A_0^*\tilde{x}_1^*, \\ u_k^* \in -\partial_{u_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) - B_k^*\tilde{x}_{k+1}^*, \;\; k = 0, 1, ..., N-1, \\ \tilde{w}_k^* \in \partial_{w_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) + T_k^*\tilde{x}_{k+1}^*, \;\; k = 0, 1, ..., N-1. \end{cases}$$

Theorem 5.4 *Under the assumptions of Theorem 5.3, suppose additionally that the functions h_k, for $k = 0, 1, \ldots, N$, are Fréchet differentiable. Then, $\tilde{w}^* = (\tilde{w}_0^*, \tilde{w}_1^*, \ldots, \tilde{w}_{N-1}^*) \in W^*$ belongs to $\partial V(\bar{w})$ if and only if there exist $x_0^* \in N(\bar{x}_0; C)$, $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, \ldots, \tilde{x}_N^*) \in \widetilde{X}^*$, and $u^* = (u_0^*, u_1^*, \ldots, u_{N-1}^*) \in N(\bar{u}; \Omega)$ such that*

$$\begin{cases} \tilde{x}_N^* = \nabla h_N(\bar{x}_N), \\ \tilde{x}_k^* = \nabla_{x_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) + A_k^*\tilde{x}_{k+1}^*, \;\; k = 1, 2, \ldots, N-1, \\ x_0^* = -\nabla_{x_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) - A_0^*\tilde{x}_1^*, \\ u_k^* = -\nabla_{u_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) - B_k^*\tilde{x}_{k+1}^*, \;\; k = 0, 1, \ldots, N-1, \\ \tilde{w}_k^* = \nabla_{w_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) + T_k^*\tilde{x}_{k+1}^*, \;\; k = 0, 1, \ldots, N-1. \end{cases}$$

Theorem 5.5 *Under the assumptions of Theorem 5.3, we have*

$$\partial^{\infty} V(\bar{w}) = \{0_{W^*}\}.$$

Chapter 6

Stability Analysis of Convex Continuous Optimal Control Problems

In this chapter we develop the approach of N.T. Toan and L.Q. Thuy (2016) to deal with *constrained control problems*. Namely, based on the result of Chapter 5 about differential stability of parametric convex mathematical programming problems, we will get new formulas for computing the subdifferential and the singular subdifferential of the optimal value function. The computation procedures and illustrative examples are presented in the dissertation.

6.1 Problem Setting and Auxiliary Results

Let $W^{1,p}([0,1],\mathbb{R}^n)$, $1 \le p < \infty$, be the Sobolev space consisting of absolutely continuous functions $x : [0,1] \to \mathbb{R}^n$ such that $\dot{x} \in L^p([0,1],\mathbb{R}^n)$. Let there be given

- matrix-valued functions $A(t) = (a_{ij}(t))_{n\times n}$, $B(t) = (b_{ij}(t))_{n\times m}$, and $C(t) = (c_{ij}(t))_{n\times k}$;

- real-valued functions $g : \mathbb{R}^n \to \mathbb{R}$ and $L : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$;

- a convex set $\mathcal{U} \subset L^p([0,1],\mathbb{R}^m)$;

- a pair of parameters $(\alpha,\theta) \in \mathbb{R}^n \times L^p([0,1],\mathbb{R}^k)$.

Put

$$\begin{aligned} &X = W^{1,p}([0,1],\mathbb{R}^n),\ U = L^p([0,1],\mathbb{R}^m),\ Z = X \times U,\\ &\Theta = L^p([0,1],\mathbb{R}^k),\ W = \mathbb{R}^n \times \Theta. \end{aligned}$$

Consider the constrained fixed time optimal control problem which depends on a pair of parameters (α,θ): *Find a pair (x,u), where $x \in W^{1,p}([0,1],\mathbb{R}^n)$ is a trajectory and $u \in L^p([0,1],\mathbb{R}^m)$ is a control function, which minimizes the objective function*

$$g(x(1)) + \int_0^1 L(t,x(t),u(t),\theta(t))dt \tag{6.1}$$

and satisfies the linear ordinary differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)\theta(t) \ \ a.e.\ t \in [0,1], \tag{6.2}$$

the initial value

$$x(0) = \alpha, \tag{6.3}$$

and the control constraint

$$u \in \mathcal{U}. \tag{6.4}$$

It is well known that X, U, Z, and Θ are Banach spaces. For each $w = (\alpha, \theta) \in W$, denote by $V(w)$ and $S(w)$, respectively, the optimal value and the solution set of (6.1)–(6.4). We call $V : W \to \overline{\mathbb{R}}$ the *optimal value function* of problem in question. If for each $w = (\alpha, \theta) \in W$ we put

$$J(x, u, w) = g(x(1)) + \int_0^1 L(t, x(t), u(t), \theta(t))dt,$$

$$G(w) = \{z = (x, u) \in X \times U \mid (6.2) \text{ and } (6.3) \text{ are satisfied}\},$$

and

$$K = X \times \mathcal{U},$$

then problem (6.1)–(6.4) can be written formally as $\min\{J(z, w) \mid z \in G(w) \cap K\}$, and

$$V(w) = \inf\{J(z, w) \mid z = (x, u) \in G(w) \cap K\}. \tag{6.5}$$

It is assumed that V is finite at $\bar{w} = (\bar{\alpha}, \bar{\theta}) \in W$ and $(\bar{x}, \bar{u})$ is a solution of the corresponding problem, that is $(\bar{x}, \bar{u}) \in S(\bar{w})$.

Consider the following assumptions:

(A1) The matrix-valued functions $A : [0, 1] \to M_{n,n}(\mathbb{R})$, $B : [0, 1] \to M_{n,m}(\mathbb{R})$, and $C : [0, 1] \to M_{n,k}(\mathbb{R})$, are measurable and essentially bounded.

(A2) The functions $g : \mathbb{R}^n \to \mathbb{R}$ and $L : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ are such that $g(\cdot)$ is convex and continuously differentiable on $\mathbb{R}^n$, $L(\cdot, x, u, v)$ is measurable for all $(x, u, v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$, $L(t, \cdot, \cdot, \cdot)$ is convex and continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$ for almost every $t \in [0, 1]$, and there exist constants $c_1 > 0$, $c_2 > 0$, $r \geq 0$, $p \geq p_1 \geq 0$, $p - 1 \geq p_2 \geq 0$, and a nonnegative function $w_1 \in L^p([0, 1], \mathbb{R})$, such that

$$|L(t, x, u, v)| \leq c_1\big(w_1(t) + ||x||^{p_1} + ||u||^{p_1} + ||v||^{p_1}\big),$$

$$\max\{|L_x(t, x, u, v)|, |L_u(t, x, u, v)|, |L_v(t, x, u, v)|\} \leq c_2\big(||x||^{p_2} + ||u||^{p_2} + ||v||^{p_2}\big) + r,$$

for all $(t, x, u, v) \in [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$.

6.2 Differential Stability of the Control Problem

Let

$$\Psi_A : L^q([0,1],\mathbb{R}^n) \to \mathbb{R}, \quad \Psi_B : L^q([0,1],\mathbb{R}^n) \to L^q([0,1],\mathbb{R}^m),$$
$$\Psi_C : L^q([0,1],\mathbb{R}^n) \to L^q([0,1],\mathbb{R}^k), \quad \Psi : L^q([0,1],\mathbb{R}^n) \to L^q([0,1],\mathbb{R}^n)$$

be defined by

$$\Psi_A(v) = \int_0^1 A^T(t)v(t)dt, \quad \Psi_B(v)(t) = -B^T(t)v(t) \text{ a.e. } t \in [0,1],$$
$$\Psi_C(v)(t) = C^T(t)v(t) \text{ a.e. } t \in [0,1], \quad \Psi(v) = -\int_0^{(.)} A^T(\tau)v(\tau)d\tau.$$

We will employ the following two assumptions.

(A3) Suppose that

$$\ker \Psi_C \subset \big(\ker \Psi_A \cap \ker \Psi_B \cap \mathrm{Fix}\,\Psi\big),$$

where $\mathrm{Fix}\,\Psi := \{x \in X \mid \Psi(x) = x\}$ is the set of the fixed points of Ψ, and $\ker \Psi_A$ (resp., $\ker \Psi_B$, $\ker \Psi_C$) denotes the kernel of Ψ_A (resp., Ψ_B, Ψ_C).

(A4) The operator $\Phi : W \times Z \to X$, which is given by

$$\Phi(w,z) = x - \int_0^{(.)} A(\tau)x(\tau)d\tau - \int_0^{(.)} B(\tau)v(\tau)d\tau - \alpha - \int_0^{(.)} C(\tau)\theta(\tau)d\tau$$

for every $w = (\alpha,\theta) \in W$ and $z = (x,v) \in Z$, has closed range.

The assumption (H_3) in Toan and Thuy (2016) can be stated as follows

(A5) There exists a constant $c_3 > 0$ such that, for every $v \in \mathbb{R}^n$,

$$||C^T(t)v|| \geq c_3||v|| \text{ a.e. } t \in [0,1].$$

Proposition 6.1 *If* (**A5**) *is satisfied, then* (**A3**) *and* (**A4**) *are fulfilled.*

Theorem 6.1 *Suppose that the optimal value function V in* (6.5) *is finite at $\bar{w} = (\bar{\alpha},\bar{\theta})$, $\mathrm{int}\,\mathcal{U} \neq \emptyset$, and* (**A1**) $-$ (**A4**) *are fulfilled. In addition, suppose that problem* (6.1)–(6.4)*, with $\bar{w} = (\bar{\alpha},\bar{\theta})$ playing the role of $w = (\alpha,\theta)$, has a solution $(\bar{x},\bar{u})$. Then, a vector $(\alpha^*,\theta^*) \in \mathbb{R}^n \times L^q([0,1],\mathbb{R}^k)$ belongs to $\partial V(\bar{\alpha},\bar{\theta})$ if and only if*

$$\alpha^* = g'(\bar{x}(1)) + \int_0^1 L_x(t,\bar{x}(t),\bar{u}(t),\bar{\theta}(t))dt - \int_0^1 A^T(t)y(t)dt,$$

$$\theta^*(t) = -C^T(t)y(t) + L_\theta(t,\bar{x}(t),\bar{u}(t),\bar{\theta}(t)) \;\; a.e. \; t \in [0,1],$$

where $y \in W^{1,q}([0,1],\mathbb{R}^n)$ *is the unique solution of the system*

$$\begin{cases} \dot{y}(t) + A^T(t)y(t) = L_x(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)) \ \ a.e. \ \mathrm{t} \in [0,1], \\ y(1) = -g'(\bar{x}(1)), \end{cases}$$

such that the function $u^* \in L^q([0,1],\mathbb{R}^m)$ *defined by*

$$u^*(t) = B^T(t)y(t) - L_u(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)) \ \ a.e. \ t \in [0,1]$$

satisfies the condition $u^* \in N(\bar{u};\mathcal{U})$.

Theorem 6.2 *Suppose that all the assumptions of Theorem 6.1 are satisfied. Then, a vector* $(\alpha^*, \theta^*) \in \mathbb{R}^n \times L^q([0,1],\mathbb{R}^k)$ *belongs to* $\partial^\infty V(\bar{w})$ *if and only if* $\alpha^* = \int_0^1 A^T(t)v(t)dt$, $\theta^*(t) = C^T(t)v(t)$ *a.e.* $t \in [0,1]$, *where* $v \in W^{1,q}([0,1],\mathbb{R}^n)$ *is the unique solution of the system*

$$\begin{cases} \dot{v}(t) = -A^T(t)v(t) \ \ a.e. \ t \in [0,1], \\ v(0) = \alpha^*, \end{cases}$$

such that the function $u^* \in L^q([0,1],\mathbb{R}^m)$ *given by* $u^*(t) = -B^T(t)v(t)$ *a.e.* $t \in [0,1]$ *belongs to* $N(\bar{u},\mathcal{U})$.

General Conclusions

The main results of this dissertation include:

1) Formulas for computing or estimating the subdifferential and the singular subdifferential of the optimal value function of parametric convex mathematical programming problems under inclusion constraints;

2) Formulas showing the connection between the subdifferentials of the optimal value function of parametric convex mathematical programming problems under geometrical and/or functional constraints and certain multiplier sets;

3) Formulas for computing the subdifferential and the singular subdifferential of the optimal value function of convex optimal control problems under linear constraints via the problem data.

List of Author's Related Papers

1. D.T.V. An and N.D. Yen, *Differential stability of convex optimization problems under inclusion constraints*, Applicable Analysis **94** (2015), 108–128. (SCIE)
2. D.T.V. An and J.-C. Yao, *Further results on differential stability of convex optimization problems*, Journal of Optimization Theory and Applications **170** (2016), 28–42. (SCI)
3. D.T.V. An and N.T. Toan, *Differential stability of convex discrete optimal control problems,* Acta Mathematica Vietnamica **43** (2018), 201–217. (Scopus, ESCI)
4. D.T.V. An, J.-C. Yao, and N.D. Yen, *Differential stability of a class of convex optimal control problems,* Applied Mathematics and Optimization (2017), DOI 10.1007/s00245-017-9475-4. (SCI)
5. D.T.V. An and N.D. Yen, *Subdifferential stability analysis for convex optimization problems via multiplier sets*, Vietnam Journal of Mathematics **46** (2018), 365–379. (Scopus, ESCI)

The results of this dissertation have been presented at

- The weekly seminar of the Department of Numerical Analysis and Scientific Computing, Institute of Mathematics, Vietnam Academy of Science and Technology;

- The 10th Workshop on "Optimization and Scientific Computing" (April 18–21, 2012, Ba Vi, Hanoi);

- "Taiwan-Vietnam 2015 Winter Mini-Workshop on Optimization" (November 17, 2015, National Cheng Kung University, Tainan, Taiwan);

- The 14th Workshop on "Optimization and Scientific Computing" (April 21–23, 2016, Ba Vi, Hanoi);

- International Conference "New Trends in Optimization and Variational Analysis for Applications" (December 7–10, 2016, Quy Nhon, Vietnam);

- "Vietnam-Korea Workshop on Selected Topics in Mathematics" (February 20–24, 2017, Danang, Vietnam);

- "International Conference on Analysis and its Application" (December 20–22, 2017, Aligarh Muslim University, Aligarh, India);

- International Workshop "Mathematical Optimization Theory and Applications" (January 18–20, 2018, Vietnam Institute for Advanced Study in Mathematics, Hanoi, Vietnam);

- The 7th International Conference "High Performance Scientific Computing" (March 19–23, 2018, Hanoi, Vietnam);

- The 16th Workshop on "Optimization and Scientific Computing" (April 19–21, 2018, Ba Vi, Hanoi).