VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY
INSTITUTE OF MATHEMATICS

DUONG THI VIET AN

SUBDIFFERENTIALS OF OPTIMAL VALUE FUNCTIONS IN PARAMETRIC CONVEX OPTIMIZATION PROBLEMS

DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY IN MATHEMATICS

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Supervisor: Prof. Dr.Sc. NGUYEN DONG YEN

HANOI - 2018

Confirmation

This dissertation was written on the basis of my research works carried out at Institute of Mathematics, Vietnam Academy of Science and Technology under the supervision of Prof. Nguyen Dong Yen. All the presented results have never been published by others.

May 27, 2018
The author

Duong Thi Viet An

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Contents

Table of Notations

$\mathbb{R}$	the set of real numbers
$\overline{\mathbb{R}}$	the extended real line
$\emptyset$	the empty set
$\|\|x\|\|$	the norm of a vector x
$\mathbb{B}_X$	the open unit ball of X
$\mathcal{N}(x)$	the set of all the neighborhoods of x
$\operatorname{int} A$	the topological interior of A
$\operatorname{cl} A$	the closure of a set A
$\operatorname{cl}^* A$	the closure of a set A in the weak* topology
$A^{\perp}$	the orthogonal complement of a set A
$\operatorname{cone} A$	the cone generated by A
$\operatorname{co} A$	the convex hull of A
$L^p([0,1],\mathbb{R}^n)$	the Banach space of Lebesgue measurable functions $x : [0,1] \to \mathbb{R}^n$ for which $\int_0^1 \|\|x(t)\|\|^p dt$ is finite
$W^{1,p}([0,1],\mathbb{R}^n)$	the Sobolev space consisting of absolutely continuous functions $x : [0,1] \to \mathbb{R}^n$ such that $\dot{x} \in L^p([0,1],\mathbb{R}^n)$
$M_{n,n}(\mathbb{R})$	the set of functions mapping $\mathbb{R}$ to the linear space of $n \times n$ real matrices
$\mathcal{L}_\alpha f = \{x \in X \mid f(x) \le \alpha\}$	a sublevel set of $f : X \to \overline{\mathbb{R}}$
$\sup\limits_{x \in K} f(x)$	the supremum of the set $\{f(x) \mid x \in K\}$
$\inf\limits_{x \in K} f(x)$	the infimum of the set $\{f(x) \mid x \in K\}$
$\operatorname{dom} f$	the effective domain of a function f
$\operatorname{epi} f$	the epigraph of f
$\partial f(x)$	the subdifferential of f at x
$\partial^\infty f(x)$	the singular subdifferential of f at x
$\nabla f(x)$	the Fréchet derivative of f at x

$\partial_x \varphi(\bar{x}, \bar{y})$	the partial subdifferential in x at $(\bar{x}, \bar{y})$
$N(\bar{x}; \Omega)$	the normal cone of Ω at $\bar{x}$
$F : X \rightrightarrows Y$	a set-valued map between X and Y
$\operatorname{dom} F$	the domain of F
$\operatorname{gph} F$	the graph of F
$D^*F(\bar{x}, \bar{y})(\cdot)$	the coderivative of F at $(\bar{x}, \bar{y})$
$M : X \to Y$	an operator from X to Y
$M^* : Y^* \to X^*$	the adjoint operator of M
$\ker M$	the null space
$\operatorname{rge} M$	the range of M
$\operatorname{span}\{(x_j^*, y_j^*) \mid j = 1, ..., m\}$	the linear subspace generated by the vectors $(x_j^*, y_j^*),\ j = 1, ..., m$
resp.	respectively
w.r.t.	with respect to
l.s.c.	lower semicontinuous
a.e.	almost everywhere

Introduction

If a mathematical programming problem depends on a parameter, that is, the objective function and the constraints depend on a certain parameter, then the optimal value is a function of the parameter, and the solution map is a set-valued map on the parameter of the problem. In general, the optimal value function is a fairly complicated function of the parameter; it is often nondifferentiable on the parameter, even if the functions defining the problem in question are smooth w.r.t. all the programming variables and the parameter. This is the reason of the great interest in having formulas for computing generalized directional derivatives (Dini directional derivative, Dini-Hadarmard directional derivative, Clarke generalized directional derivative,...) and formulas for evaluating subdifferentials (subdifferential in the sense of convex analysis, Clarke subdifferential, Fréchet subdifferential, limiting subdifferential – also called Mordukhovich subdifferential,...) of the optimal value function.

Studies on differentiability properties of the optimal value function and of the solution map in parametric mathematical programming are usually classified as studies on *differential stability* of optimization problems. Some results in this direction can be found in [2, 4, 6, 16, 18, 27] and the references therein.

For differentiable nonconvex programs, pioneering works are due to Gauvin and Tolle [19], Gauvin and Dubeau [17]. The authors obtained formulas for computing and estimating Dini directional derivatives and Clarke generalized gradients of the optimal value function when the problem data undergoes smooth perturbations. Auslender [8], Rockafellar [36], Golan [20], Thibault [42], Ioffe and Penot [21], and many other authors, have shown that similar results can be obtained for nondifferentiable nonconvex programs. In particular, the connections between subdifferential of the optimal value function in the Dini-Hadamard sense and in the Fréchet sense with the corresponding subdifferential of the objective function were pointed in [21]. For optimization problems with inclusion constraints on Banach spaces, differentiability properties of the optimal value function have been established via the dual-space approach by Mordukhovich *et al.* in [29], where it is shown that the new general results imply several fundamental results which were obtained by the primal-space approach.

Differential stability for convex programs has been studied intensively in the last five decades. A formula for computing the subdifferential of the optimal value function of a standard convex mathematical programming problem with right-hand-side perturbations, called the *perturbation function*, via the set of *Kuhn-Tucker vectors* (i.e., the vectors of Kuhn-Tucker coefficients; see [35, p. 274]) was given by Rockafellar [35, Theorem 29.1]. Until now, many analogues and extensions of this classical result have been given in the literature (see, e.g., [33, Theorem 3.85]).

Besides the investigations on differential stability of parametric mathematical programming problems, the study on differential stability of optimal control problems is also an issue of importance (see, e.g., [13–15, 23, 32, 37–39, 41, 43–46] and the references therein).

According to Bryson [12, p. 27, p. 32], optimal control had its origins in the calculus of variations in the 17th century. The calculus of variations was developed further in the 18th by L. Euler and J.L. Lagrange and in the 19th century by A.M. Legendre, C.G.J. Jacobi, W.R. Hamilton, and K.T.W. Weierstrass. In 1957, R.E. Bellman gave a new view of Hamilton-Jacobi theory which he called *dynamic programming*, essentially a nonlinear feedback control scheme. McShane [26] and Pontryagin *et al.* [34] extended the calculus of variations to handle control variable inequality constraints. *The Maximum Principle* was enunciated by Pontryagin.

As noted by Tu [47, p. 110], although much pioneering work had been carried out by other authors, Pontryagin and his associates are the first ones to develop and present the Maximum Principle in unified manner. Their work attracted great attention among mathematicians, engineers, economists, and spurred wide research activities in the area (see [28, Chapter 6], [47, 48], and the references therein).

Motivated by the recent work of Mordukhovich *et al.* [29] on the optimal value function in parametric programming under inclusion constraints, this dissertation focuses on differential stability of convex optimization problems. In other words, we study differential properties of the optimal value function. Namely, we obtain some formulas for computing the subdifferential and the singular subdifferential of the optimal value function of infinite-dimensional convex optimization problems under inclusion constraints and of infinite-dimensional convex optimization problems under geometrical and functional constraints. Our main tool is the Moreau–Rockafellar Theorem (see, e.g., [22, p. 48]) and appropriate regularity conditions. By virtue of the convexity, several assumptions used in the above paper by Mordukhovich *et al.*, like the nonemptyness of the Fréchet upper subdifferential of the objective function, the existence of a local upper Lipschitzian selection of the solution map, as well as the μ-inner semicontinuity and the μ-inner semicompactness of the solution map, are no longer needed. We also discuss the connection between the subdifferentials of the optimal value function and certain multiplier sets. Applied to parametric optimal control problems, with convex objective functions and linear dynamical systems, either discrete or continuous, our results can lead to some rules for computing the subdifferential and the singular subdifferential of the optimal value function via the data of the given problem.

The dissertation has six chapters, a list of the related papers of the author, a section of general conclusions, and a list of references. The first four chapters, where some preliminaries and a series of new results on sensitivity analysis of parametric convex programming problems under inclusion constraints are given, constitute the first part of the dissertation. The second part is formed by the last two chapters, where applications of the just mentioned results to parametric convex control problems under linear constraints are carried on.

Chapter 1 collects some basic concepts from convex analysis, variational analysis, and functional analysis needed for subsequent chapters.

Chapter 2 presents some new results on differential stability of convex optimization problems under inclusion constraints in Hausdorff locally convex topological vector spaces. The main tool is the Moreau-Rockafellar Theorem, which can be viewed as a well-known result in convex analysis, and some appropriate regularity conditions. The results obtained here lead to new facts on differential stability of convex optimization problems under geometrical and functional constraints.

In Chapter 3 we first establish formulas for computing the subdifferentials of the optimal value function for parametric convex programs under three assumptions: the objective function is closed, the constraint multifunction has closed graph, and Aubin's regularity condition is satisfied. Then, we derive relationships between regularity conditions. Our investigations have revealed that one cannot use Aubin's regularity assumption in a Hausdorff locally convex topological vector space setting, because the related sum rule is established via the Banach open mapping theorem.

Chapter 4 discusses differential stability of convex programming problems in Hausdorff locally convex topological vector spaces. Optimality conditions for convex optimization problems under inclusion constraints and for convex optimization problems under geometrical and functional constraints are formulated here too. After establishing an upper estimate for the subdifferentials via the Lagrange multiplier sets, we give an example to show that the upper estimate can be strict. Then, by defining a satisfactory multiplier set, we obtain formulas for computing the subdifferential and the singular subdifferential of the optimal value function

In Chapter 5 we first derive an upper estimate for the subdifferential of the optimal value function of convex discrete optimal control problems in Banach spaces. Then we present new calculus rules for computing the subdifferential if the objective function is differentiable. The main tools of our analysis are the formulas for computing subdifferentials of the optimal value function from Chapter 2. We also show that the singular subdifferential of the just mention optimal value function always consists of the origin of the dual space.

Finally, in Chapter 6, we focus on differential stability of convex continuous optimal control problems. Namely, based on the results of Chapter 5 about differential stability of parametric convex mathematical programming problems, we get new formulas for computing the subdifferential and the singular subdifferential of the optimal value function. Moreover, we also describe in details the process of finding vectors belonging to the subdifferential (resp., the singular subdifferential) of the optimal value function. Meaningful examples, which have the origin in [34, Example 1, p. 23], are designed to illustrate our results.

The dissertation is written on the basis of 5 published papers: An and Yao [2] in *Journal of Optimization Theory and Applications*, An, Yao, and Yen [3] in *Applied Mathematics and Optimization* (FirstOnline), An and Yen [4] in *Applicable Analysis*, An and Toan [1] in *Acta Mathematica Vietnamica*, An and Yen [5] in *Vietnam Journal of Mathematics*.

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- "International Conference on Analysis and its Application" (December 20–22, 2017, Aligarh Muslim University, Aligarh, India);

- International Workshop "Mathematical Optimization Theory and Applications" (January 18–20, 2018, Vietnam Institute for Advanced Study in Mathematics, Hanoi, Vietnam);

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Chapter 1

Preliminaries

Several concepts and results from convex analysis, variational analysis, and functional analysis are recalled in this chapter. Two types of parametric optimization problems to be considered in the subsequent three chapters are also presented in this chapter.

The present chapter is written on the basis of the books of Bonnans and Shapiro [11], Ioffe and Tihomirov [22], and the paper by Mordukhovich, Nam, and Yen [29].

1.1 Subdifferentials

Let X, Y be Hausdorff locally convex topological vector spaces with the topological duals denoted respectively by X^* and Y^*.

Definition 1.1 For a convex set $\Omega \subset X$, the *normal cone* of Ω at $\bar{x} \in \Omega$ is given by

$$N(\bar{x}; \Omega) = \{x^* \in X^* \mid \langle x^*, x - \bar{x}\rangle \leq 0, \ \ \forall x \in \Omega\}.$$

Consider a function $f : X \to \overline{\mathbb{R}} = [-\infty, +\infty] := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ having values in the extended real line. One says that f is *proper* if $f(x) > -\infty$ for all $x \in X$, and the *domain* $\operatorname{dom} f := \{x \in X \mid f(x) < \infty\}$ is nonempty. The *epigraph* of f is defined by

$$\operatorname{epi} f := \{(x, \alpha) \in X \times \mathbb{R} \mid \alpha \geq f(x)\}.$$

If $\operatorname{epi} f$ is a convex set, then f is said to be a convex function.

Definition 1.2 Let $f : X \to \overline{\mathbb{R}}$ be a convex function. Suppose that $\bar{x} \in X$ and $|f(\bar{x})| < \infty$.

(i) The set

$$\partial f(\bar{x}) = \{x^* \in X^* \mid \langle x^*, x - \bar{x}\rangle \le f(x) - f(\bar{x}),\ \forall x \in X\}$$

is called the *subdifferential* of f at $\bar{x}$.

(ii) The set

$$\partial^\infty f(\bar{x}) = \{x^* \in X^* \mid (x^*, 0) \in N((\bar{x}, f(\bar{x})); \operatorname{epi} f)\} \tag{1.1}$$

is called the *singular subdifferential* of f at $\bar{x}$.

In the case where $|f(\bar{x})| = \infty$, one lets $\partial f(\bar{x})$ and $\partial^\infty f(\bar{x})$ to be empty sets.

Given a convex subset $\Omega \subset X$, one defines the indicator function $\delta(\cdot;\Omega) : X \to \overline{\mathbb{R}}$ of Ω by setting

$$\delta(x;\Omega) := \begin{cases} 0 & \text{if } x \in \Omega, \\ +\infty & \text{if } x \notin \Omega. \end{cases}$$

For any $\bar{x} \in \Omega$, it is easy to see that

$$\partial^\infty \delta(\bar{x};\Omega) = \partial\delta(\bar{x};\Omega) = N(\bar{x};\Omega).$$

The following proposition give us a relationship between the singular subdifferential of the convex function and the normal cone of its domain.

Proposition 1.1 *If $f : X \to \overline{\mathbb{R}}$ is a convex function, then*

$$\partial^\infty f(x) = N(x; \operatorname{dom} f),\ \forall x \in X.$$

Proof. Indeed, since $\operatorname{epi} f$ is a convex set, by (1.1) we have

$$\begin{aligned} x^* \in \partial^\infty f(x) &\Leftrightarrow\ (x^*, 0) \in N((x, f(x)); \operatorname{epi} f) \\ &\Leftrightarrow\ \langle (x^*, 0), (u, \mu) - (x, f(x))\rangle \le 0,\ \ \forall (u,\mu) \in \operatorname{epi} f \\ &\Leftrightarrow\ \langle (x^*, 0), (u - x, \mu - f(x))\rangle \le 0,\ \ \forall (u,\mu) \in \operatorname{epi} f \\ &\Leftrightarrow\ \langle x^*, u - x\rangle \le 0,\ \ \forall u \in \operatorname{dom} f \\ &\Leftrightarrow\ x^* \in N(x; \operatorname{dom} f). \end{aligned}$$

This proves that $\partial^\infty f(x) = N(x; \operatorname{dom} f)$. □

In a Banach space setting, the singular subdifferential will be useful for the study of non-Lipschitzian functions. Because, if the function f is Lipschitz continuous around $\bar{x}$, then $\partial^\infty f(\bar{x}) = \{0\}$, see, e.g., [30, Theorem 3.1(ii)].

1.2 Coderivatives

Let $F : X \rightrightarrows Y$ be a convex set-valued map. The *graph* and the *domain* of F are given, respectively, by the formulas

$$\operatorname{gph} F := \{(x, y) \in X \times Y \mid y \in F(x)\},$$
$$\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}.$$

Equipping the product space $X \times Y$ with the norm $\|(x, y)\| := \|x\| + \|y\|$, by the above notions of normal cones, one can define the concept coderivative of convex set-valued maps as follows.

Definition 1.3 The *coderivative* of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is the multifunction $D^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ defined by

$$D^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{gph} F)\}, \ \forall y^* \in Y^*.$$

If $(\bar{x}, \bar{y}) \notin \operatorname{gph} F$, then we accept the convention that the set $D^*F(\bar{x}, \bar{y})(y^*)$ is empty for any $y^* \in Y^*$.

Note that, in a Banach space setting, the coderivative of the convex set-valued map has been defined in [7, Definition 1, p. 178] under the name *codifferential*.

1.3 Optimal Value Function

Consider a set-valued map $G : X \rightrightarrows Y$ between Banach spaces, a function $\varphi : X \times Y \to \overline{\mathbb{R}}$. The *optimal value function* (or *the marginal function*) of the *parametric optimization problem under an inclusion constraint*, defined by G and φ, is the function $\mu : X \to \overline{\mathbb{R}}$, with

$$\mu(x) := \inf \{\varphi(x, y) \mid y \in G(x)\}. \tag{1.2}$$

By the convention $\inf \emptyset = +\infty$, we have $\mu(x) = +\infty$ for any $x \notin \operatorname{dom} G$.

The set-valued map G (resp., the function φ) is called the *map describing the constraint set* (resp., the *objective function*) of the optimization problem on the right-hand-side of (1.2).

Corresponding to each data pair $\{G, \varphi\}$ we have one *optimization problem depending on a parameter* x:

$$\min\{\varphi(x, y) \mid y \in G(x)\}. \tag{1.3}$$

Formulas for computing or estimating the subdifferentials (the Fréchet subdifferential, the Mordukhovich subdifferential, the singular subdifferential, and the subdifferential in the sense of convex analysis) of the optimal value function $\mu(.)$ are tightly connected with the *solution map* of (1.3). The just mentioned solution map, denoted by $M : \operatorname{dom} G \rightrightarrows Y$, is given by

$$M(x) := \{y \in G(x) \mid \mu(x) = \varphi(x, y)\} \quad (\forall x \in \operatorname{dom} G). \tag{1.4}$$

Namely, in [29] the authors have obtained an upper estimate for the Fréchet subdifferential of the optimal value function in formula (1.2) at a given parameter $\bar{x}$. This estimate is established via the Fréchet coderivative of the map G describing the constraint set and the Fréchet upper subdifferential of the objective function φ. In addition, if φ is Fréchet differentiable at $(\bar{x}, \bar{y})$ and the solution map M given in (1.4) has a local upper Lipschitzian selection at $(\bar{x}, \bar{y})$, then the obtained upper estimate become an equality (see [29, Theorems 1 and 2] for details).

The assumption about the nonempty property of the Fréchet upper subdifferential of φ, i.e., $\widehat{\partial}^+\varphi(\bar{x}, \bar{y}) \neq \emptyset$, in [29, Theorem 1] is rather strict. For instance, it excludes from our consideration Lipschitzian convex functions of the type $\varphi(x, y) = |x| + y$, $(x, y) \in \mathbb{R} \times \mathbb{R}$, or $\varphi(x, y) = ||x|| + g(y)$, $(x, y) \in X \times Y$, where $g : Y \to \mathbb{R}$ is a given function, X and Y are Banach spaces with $\dim X \geq 1$. Indeed, for the first example, choosing $(\bar{x}, \bar{y}) = (0, 0)$ we have $\widehat{\partial}^+\varphi(\bar{x}, \bar{y}) = \emptyset$. For the second example, we have $\widehat{\partial}^+\varphi(\bar{x}, \bar{y}) = \emptyset$ for any $(\bar{x}, \bar{y}) = (0, v) \in X \times Y$.

Moreover, to obtain formulas for computing the Mordukhovich subdifferential of $\mu(.)$ in (1.2), Mordukhovich *et al.* need some assumptions about the sequentially normally compact property of φ, the existence of a local upper Lipschitzian selection of the solution map M, as well as the μ-inner semicontinuity or the μ-inner semicompactness of the solution map M (see [29, Theorem 7] for details).

By imposing the convexity requirement on (1.3), in next Chapters 2 and 3, we need not to rely on the assumption $\widehat{\partial}^+\varphi(\bar{x}, \bar{y}) \neq \emptyset$ in [29, Theorem 1], the condition saying that the solution map $M : \operatorname{dom} G \rightrightarrows Y$ has a local upper Lipschitzian selection at $(\bar{x}, \bar{y})$ in [29, Theorem 2], as well as the sequentially normally compact property of φ, the μ-inner semicontinuity or the μ-inner semicompactness conditions on the solution map $M(\cdot)$ in [29, Theorem 7].

1.4 Problems under the Convexity

Let X and Y be Hausdorff locally convex topological vector spaces. Let $\varphi : X \times Y \to \overline{\mathbb{R}}$ be a proper convex extended-real-valued function. Given a convex set-valued map $G : X \rightrightarrows Y$, we consider the *parametric convex optimization problem under an inclusion constraint*

$$\min\{\varphi(x, y) \mid y \in G(x)\}. \tag{1.5}$$

depending on the parameter x. The *optimal value function* of problem (1.5), is the function $\mu : X \to \overline{\mathbb{R}}$, with

$$\mu(x) := \inf \{\varphi(x, y) \mid y \in G(x)\}. \tag{1.6}$$

The *solution map* $M : \operatorname{dom} G \rightrightarrows Y$ of that problem is defined by

$$M(x) := \{y \in G(x) \mid \mu(x) = \varphi(x, y)\} \quad (\forall x \in \operatorname{dom} G).$$

Proposition 1.2 *Let $G : X \rightrightarrows Y$ be a convex set-valued map, $\varphi : X \times Y \to \overline{\mathbb{R}}$ a convex function. Then, the function $\mu(.)$ is defined by (1.6) is convex.*

Proof. We will prove that $\operatorname{epi} \mu = \{(x, \alpha) \in X \times \mathbb{R} \mid \mu(x) \le \alpha\}$ is a convex subset of $X \times \mathbb{R}$. Taking any (x, α), $(x', \beta) \in \operatorname{epi} \mu$ and $\lambda \in (0, 1)$, we need to show that

$$\lambda(x, \alpha) + (1 - \lambda)(x', \beta) \in \operatorname{epi} \mu.$$

This is equivalent to

$$\inf\{\varphi(\lambda x + (1 - \lambda)x', z) \mid z \in G(\lambda x + (1 - \lambda)x')\} \le \lambda\alpha + (1 - \lambda)\beta.$$

For any $\varepsilon > 0$, since $(x, \alpha) \in \operatorname{epi} \mu$, one has

$$\alpha \ge \mu(x) = \inf\{\varphi(x, y) \mid y \in G(x)\}. \tag{1.7}$$

• If $\inf\{\varphi(x, y) \mid y \in G(x)\} = -\infty$, then for every $\alpha \in \mathbb{R}$, there exists $y_1 \in G(x)$ such that

$$\alpha \ge \varphi(x, y_1). \tag{1.8}$$

• If $\inf\{\varphi(x, y) \mid y \in G(x)\} > -\infty$, then from (1.7) we have

$$\alpha > \inf\{\varphi(x, y) \mid y \in G(x)\} - \varepsilon.$$

Thus, we can find $u \in G(x)$ satisfying

$$\alpha > \varphi(x, u) - \varepsilon. \tag{1.9}$$

Similarly, we consider the case $(x', \beta) \in \operatorname{epi} \mu$.

• If $\inf\{\varphi(x', y') \mid y' \in G(x')\} = -\infty$, then for any $\beta \in \mathbb{R}$, there exists $y_2 \in G(x')$ such that

$$\beta \geq \varphi(x', y_2). \tag{1.10}$$

• If $\inf\{\varphi(x', y') \mid y' \in G(x')\} > -\infty$, there exists $v \in G(x')$ such that

$$\beta > \varphi(x', v) - \varepsilon. \tag{1.11}$$

Moreover, since $(x, u) \in \operatorname{gph} G, (x', v) \in \operatorname{gph} G$, the convexity of $\operatorname{gph} G$ gives us

$$(\lambda x + (1-\lambda)x', \lambda u + (1-\lambda)v) \in \operatorname{gph} G.$$

This means that $\lambda u + (1-\lambda)v \in G(\lambda x + (1-\lambda)x')$. Therefore,

$$\begin{aligned}
&\inf\{\varphi(\lambda x + (1-\lambda)x', z) \mid z \in G(\lambda x + (1-\lambda)x')\} \\
&\quad \leq \varphi(\lambda x + (1-\lambda)x', \lambda u + (1-\lambda)v) \leq \varphi(\lambda(x, u)) + \varphi((1-\lambda)(x', v)) \\
&\quad \leq \lambda\varphi(x, u) + (1-\lambda)\varphi(x', v).
\end{aligned}$$

By (1.8), (1.9), (1.10), and (1.11), the last equality is equivalent to

$$\begin{aligned}
&\inf\{\varphi(\lambda x + (1-\lambda)x', z) \mid z \in G(\lambda x + (1-\lambda)x')\} \\
&\quad \leq \lambda(\alpha + \varepsilon) + (1-\lambda)(\beta + \varepsilon) \\
&\quad = \lambda\alpha + (1-\lambda)\beta + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0^+$, we obtain the convexity of the optimal value function μ. □

In next two chapters, to obtain formulas for computing/estimating the subdifferential of the optimal value function μ via the subdifferential of φ and the coderivative of G, we will apply the following scheme, which has been formulated clearly by Professor Truong Xuan Duc Ha in her review on this dissertation.

Step 1. Consider the unconstrained optimization problem

$$\mu(x) := \inf\big\{\varphi(x, y) + \delta((x, y); \operatorname{gph} G)\big\},$$

where $\delta(\cdot; \operatorname{gph} G)$ is the indicator function of $\operatorname{gph} G$.

Step 2. Apply some known results to show that

$$(x^*, 0) \in \partial\Big(\varphi + \delta(\cdot; \operatorname{gph} G)\Big)(\bar{x}, \bar{y})$$

for every $x^* \in \partial\mu(\bar{x})$ and for some $\bar{y} \in M(\bar{x})$.

Step 3. Employ the sum rule for subdifferentials to get

$$(x^*, 0) \in \partial\varphi(\bar{x}, \bar{y}) + \partial\delta((\bar{x}, \bar{y}); \operatorname{gph} G).$$

Step 4. Use the relationships between $\partial\delta((\bar{x}, \bar{y}); \operatorname{gph} G)$, $N((\bar{x}, \bar{y}); \operatorname{gph} G)$ and the definition of the coderivative in question.

Note that, in Step 2, one can apply the results like Propositions 3.1 and 3.2 from [21], and Proposition 3.37 from [33]. In particular, Mordukhovich and Shao [31, pp. 1256–1257] used this scheme. However, following the above scheme is not a must. Namely, in some cases, one can get the desired estimates by direct proofs; see, e.g., [29, Theorems 1 and 2].

Thanks to some regularity conditions on the function φ and the mapping G, the result of Steps 2 and 3 is an *upper estimate* for $\partial\mu(\bar{x})$. In the sequel, we will see that the *inner estimate* (that is the reverse inclusion of the upper estimate) is valid for convex optimization problems without any regularity condition.

1.5 Some Facts from Functional Analysis and Convex Analysis

First, we recall a result related to continuous linear operators. Consider a continuous linear operator $A : X \to Y$ from a Banach space X to another Banach space Y with the adjoint $A^* : Y^* \to X^*$. The null space and the range of A are defined, respectively, by $\ker A = \{x \in X \mid Ax = 0\}$ and

$$\operatorname{rge} A = \{y \in Y \mid y = Ax,\, x \in X\}.$$

Proposition 1.3 (See [11, Proposition 2.173]) *The following properties are valid:*
(i) $(\ker \mathrm{A})^\perp = \mathrm{cl}^*(\operatorname{rge}(\mathrm{A}^*))$, *where* $\mathrm{cl}^*(\operatorname{rge}(A^*))$ *denotes the closure of the set* $\operatorname{rge}(A^*)$ *in the weak** *topology of* X^*, *and*

$$(\ker A)^\perp = \{x^* \in X^* \mid \langle x^*, x\rangle = 0 \ \ \forall x \in \ker A\}$$

stands for the orthogonal complement of the set $\ker A$.
(ii) *If* $\operatorname{rge} A$ *is closed, then* $(\ker A)^\perp = \operatorname{rge}(A^*)$, *and there is* $c > 0$ *such that for every* $x^* \in \operatorname{rge}(A^*)$ *there exists* $y^* \in Y^*$ *with* $||y^*|| \le c||x^*||$ *and* $x^* = A^*y^*$.
(iii) *If, in addition,* $\operatorname{rge} A = Y$, *i.e.,* A *is onto, then* A^* *is one-to-one and there exists* $c > 0$ *such that* $||y^*|| \le c||A^*y^*||$, *for all* $y^* \in Y^*$.
(iv) $(\ker A^*)^\perp = \mathrm{cl}(\operatorname{rge} A)$.

We now recall some results from functional analysis related to Banach spaces, which can be found in [22, pp. 20–22].

For every $p \in [1, \infty)$, the symbol $L^p([0,1], \mathbb{R}^n)$ denotes the Banach space of Lebesgue measurable functions x from $[0,1]$ to $\mathbb{R}^n$ for which the integral $\displaystyle\int_0^1 \|x(t)\|^p dt$ is finite. The norm in $L^p([0,1], \mathbb{R}^n)$ is given by

$$||x||_p = \left(\int_0^1 \|x(t)\|^p dt \right)^{\frac{1}{p}}.$$

The dual space of $L^p([0,1], \mathbb{R}^n)$ is $L^q([0,1], \mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1$. In other words, for every continuous linear functional φ on the space $L^p([0,1], \mathbb{R}^n)$, there exists a unique element $x^* \in L^q([0,1], \mathbb{R}^n)$ such that

$$\varphi(x) = \langle \varphi, x \rangle = \int_0^1 x^*(t)x(t)dt \quad \forall x \in L^p([0,1], \mathbb{R}^n).$$

Moreover, one has $||\varphi|| = ||x^*||_q$.

The Sobolev space $W^{1,p}([0,1], \mathbb{R}^n)$ consisting of absolutely continuous functions $x : [0,1] \to \mathbb{R}^n$ such that $\dot{x} \in L^p([0,1], \mathbb{R}^n)$ is equipped with the norm

$$||x||_{1,p}^1 = \|x(0)\| + ||\dot{x}||_p,$$

or the norm $||x||_{1,p}^2 = ||x||_p + ||\dot{x}||_p$. The norms $||x||_{1,p}^1$ and $||x||_{1,p}^2$ are equivalent (see, e.g., [22, p. 21]). Every continuous linear functional φ on $W^{1,p}([0,1], \mathbb{R}^n)$ can be represented in the form

$$\langle \varphi, x \rangle = \langle a, x(0) \rangle + \int_0^1 \dot{x}(t)u(t)dt,$$

where the vector $a \in \mathbb{R}^n$ and the function $u \in L^q([0,1], \mathbb{R}^n)$ are uniquely defined. This means that, $\left(W^{1,p}([0,1], \mathbb{R}^n)\right)^* = \mathbb{R}^n \times L^q([0,1], \mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1$ (see, e.g., [22, p. 21]). In the case of $p = 2$, $W^{1,2}([0,1], \mathbb{R}^n)$ is a Hilbert space with the inner product being given by

$$\langle x, y \rangle = \langle x(0), y(0) \rangle + \int_0^1 \dot{x}(t)\dot{y}(t)dt,$$

for all $x, y \in W^{1,2}([0,1], \mathbb{R}^n)$.

Next, we recall two results on normal cones to convex sets. Suppose that $A_0, A_1, \ldots, A_n$ are convex subsets of a Hausdorff locally convex topological vector space X and $A = A_0 \cap A_1 \cap \cdots \cap A_n$. By $\operatorname{int} A_i$, for $i = 1, \ldots, n$, we denote the interior of A_i.

Proposition 1.4 (See [22, Proposition 1, p. 205]) *If one has*

$$A_0 \cap (\operatorname{int} A_1) \cap \cdots \cap (\operatorname{int} A_n) \neq \emptyset, \tag{1.12}$$

then

$$N(x; A) = N(x; A_0) + N(x; A_1) + \cdots + N(x; A_n)$$

for any point $x \in A$. In other words, if the regularity condition (1.12) *is satisfied, then the normal cone to the intersection of sets is equal to the sum of the normal cones to these sets.*

Proposition 1.5 (See [22, Proposition 3, p. 206]) *If one has* $\operatorname{int} A_i \neq \emptyset$ *for* $i = 1, 2, \ldots, n$ *then, for any $x_0 \in A$, the following statements are equivalent:*
(a) $A_0 \cap (\operatorname{int} A_1) \cap \cdots \cap (\operatorname{int} A_n) = \emptyset$*;*
(b) *There exist $x_i^* \in N(x_0; A_i)$ for $i = 0, 1, \ldots, n$, not all zero, such that*

$$x_0^* + x_1^* + \cdots + x_n^* = 0.$$

In the sequel, we will need the following fundamental calculus rule of convex analysis.

Theorem 1.1 (The Moreau-Rockafellar Theorem) (See [22, Theorem 0.3.3 on pp. 47–50, Theorem 1 on p. 200]) *Let $f_1, \ldots, f_m$ be proper convex functions on X. Then*

$$\partial(f_1 + \cdots + f_m)(x) \supset \partial f_1(x) + \cdots + \partial f_m(x)$$

for all $x \in X$. If, at a point $x^0 \in \operatorname{dom} f_1 \cap \cdots \cap \operatorname{dom} f_m$, all the functions $f_1, \ldots, f_m$, except, possibly, one are continuous, then

$$\partial(f_1 + \cdots + f_m)(x) = \partial f_1(x) + \cdots + \partial f_m(x)$$

for all $x \in X$.

The forthcoming theorem characterizes the continuity of extended-real-valued convex functions defined on Hausdorff locally convex topological vector spaces.

Theorem 1.2 (See [22, Theorem 1, p. 170]) *Let f be a proper convex function on a Hausdorff locally convex topological vector space X. Then the following assertions are equivalent:*

(i) *f is bounded from above on a neighborhood of a point $x \in X$;*
(ii) *f is continuous at a point $x \in X$;*
(iii) $\operatorname{int}(\operatorname{epi} f) \neq \emptyset$;
(iv) $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$ *and f is continuous on* $\operatorname{int}(\operatorname{dom} f)$. *Moreover,*

$$\operatorname{int}(\operatorname{epi} f) = \{(\alpha, x) \in \mathbb{R} \times X \mid x \in \operatorname{int}(\operatorname{dom} f),\ \alpha > f(x)\}.$$

1.6 Conclusions

This chapter presents several basic results from convex analysis, two types of general parametric optimization problems, and some facts from functional analysis which will be used repeatedly in the subsequent chapters. Moreover, Theorems 1, 2 and 7 in [29], which are the motivations for the research leading to our results in the next two chapters, are also briefly analyzed in this chapter.

Chapter 2

Differential Stability in Parametric Convex Programming Problems

Motivated by the work of Mordukhovich, Nam, and Yen [29] on the optimal value function in parametric programming under inclusion constraints, this chapter establishes some new results about differential stability of convex optimization problems under inclusion constraints and functional constraints. By using a version of the Moreau-Rockafellar Theorem, which has been recalled in Theorem 1.1, and appropriate regularity conditions, we obtain formulas for computing the subdifferential and the singular subdifferential of the optimal value function.

The chapter is written on the basis of [4].

2.1 Differential Stability of Convex Optimization Problems under Inclusion Constraints

The following theorem provides us with formulas for computing the subdifferential and the singular subdifferential of μ given in (1.6).

Theorem 2.1 *Suppose that $G : X \rightrightarrows Y$ is a convex set-valued mapping and $\varphi : X \times Y \to \overline{\mathbb{R}}$ is a proper convex function. If at least one of the following regularity conditions is satisfied:*

(a) $\operatorname{int}(\operatorname{gph} G) \cap \operatorname{dom} \varphi \neq \emptyset$,

(b) *φ is continuous at a point* $(x^0, y^0) \in \operatorname{gph} G$, *then for any* $\bar{x} \in \operatorname{dom}\mu$*, with* $\mu(\bar{x}) \neq -\infty$*, and for any* $\bar{y} \in M(\bar{x})$ *we have*

$$\partial\mu(\bar{x}) = \bigcup_{(x^*,y^*)\in\partial\varphi(\bar{x},\bar{y})} \{x^* + D^*G(\bar{x},\bar{y})(y^*)\} \tag{2.1}$$

and

$$\partial^\infty\mu(\bar{x}) = \bigcup_{(x^*,y^*)\in\partial^\infty\varphi(\bar{x},\bar{y})} \{x^* + D^*G(\bar{x},\bar{y})(y^*)\}. \tag{2.2}$$

Therefore, if $M(\bar{x})$ *is nonempty then* $\partial\mu(\bar{x})$ *and* $\partial^\infty\mu(\bar{x})$ *can be computed respectively by the formulas* (2.1) *and* (2.2)*, where the right-hand-sides do not depend on the choice of* $\bar{y} \in M(\bar{x})$*.*

Proof. Let $\bar{x} \in \operatorname{dom}\mu$ and $\bar{y} \in M(\bar{x})$. To prove the inclusion " $\subset$ " in (2.1), take an arbitrary element $\bar{x}^* \in \partial\mu(\bar{x})$. Since the optimal value function μ is convex, we have

$$\mu(x) - \mu(\bar{x}) \geq \langle \bar{x}^*, x - \bar{x}\rangle,\ \forall x \in X.$$

Now, taking an arbitrary $u \in X$ and selecting a $v \in G(u)$, from the above property we get

$$\begin{aligned}\varphi(u,v) - \varphi(\bar{x},\bar{y}) = \varphi(u,v) - \mu(\bar{x}) &\geq \mu(u) - \mu(\bar{x})\\ &\geq \langle \bar{x}^*, u - \bar{x}\rangle + \langle 0, v - \bar{y}\rangle.\end{aligned}$$

Therefore,

$$\varphi(u,v) - \varphi(\bar{x},\bar{y}) \geq \langle (\bar{x}^*,0), (u,v) - (\bar{x},\bar{y})\rangle,\ \forall (u,v) \in \operatorname{gph} G.$$

Hence

$$\begin{aligned}(\varphi + \delta(\cdot;\operatorname{gph} G))(u,v) \ &- (\varphi + \delta(\cdot;\operatorname{gph} G))(\bar{x},\bar{y})\\ &\geq \langle (\bar{x}^*,0), (u,v) - (\bar{x},\bar{y})\rangle,\ \forall (u,v) \in X \times Y,\end{aligned} \tag{2.3}$$

where $\delta((x,y);\operatorname{gph} G) = 0$ if $(x,y) \in \operatorname{gph}$ G, and $\delta((x,y);\operatorname{gph} G) = +\infty$ if $(x,y) \notin \operatorname{gph}$ G, is the indicator function of $\operatorname{gph} G$. From (2.3) we have

$$(\bar{x}^*,0) \in \partial(\varphi + \delta(\cdot;\operatorname{gph} G))(\bar{x},\bar{y}). \tag{2.4}$$

Since $\operatorname{gph} G$ is convex, $\delta(\cdot;\operatorname{gph} G) : X \times Y \to \overline{\mathbb{R}}$ is convex. Obviously, $\delta(\cdot;\operatorname{gph} G)$ is continuous at every point belonging to $\operatorname{int}(\operatorname{gph} G)$.

Consequently, if the regularity condition (a) is satisfied, then $\delta(\cdot;\operatorname{gph} G)$ is continuous at a point in $\operatorname{dom}\varphi$. By Theorem 1.1, from (2.4) we have

$$\begin{aligned}(\bar{x}^*,0) \in\ & \partial\varphi(\bar{x},\bar{y}) + \partial\delta(\cdot;\operatorname{gph} G)(\bar{x},\bar{y})\\ =\ & \partial\varphi(\bar{x},\bar{y}) + N((\bar{x},\bar{y});\operatorname{gph} G).\end{aligned} \tag{2.5}$$

Thus, there exists $(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})$ such that

$$(\bar{x}^*, 0) \in (x^*, y^*) + N((\bar{x}, \bar{y}); \operatorname{gph} G),$$

or

$$(\bar{x}^* - x^*, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{gph} G),$$

i.e.,

$$\bar{x}^* - x^* \in D^*G(\bar{x}, \bar{y})(y^*).$$

The last inclusion implies that

$$\bar{x}^* \in x^* + D^*G(\bar{x}, \bar{y})(y^*). \tag{2.6}$$

Consider the case where the regularity condition (b) is fulfilled. Since

$$\operatorname{dom} \delta(\cdot; \operatorname{gph} G) = \operatorname{gph} G,$$

from (b) it follows that φ is continuous at a point in $\operatorname{dom} \delta(\cdot; \operatorname{gph} G)$. Therefore, by Theorem 1.1, from (2.4) we also have (2.5). Thus, there exists $(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})$ such that (2.6) is satisfied.

In both cases, since $\bar{x}^* \in \partial\mu(\bar{x})$ can be taken arbitrarily, by (2.6) we can deduce that

$$\partial\mu(\bar{x}) \subset \bigcup_{(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\}.$$

To establish the opposite inclusion, we need to prove that for each element $(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})$ the following holds true:

$$x^* + D^*G(\bar{x}, \bar{y})(y^*) \subset \partial\mu(\bar{x}).$$

Taking an arbitrary vector $u^* \in x^* + D^*G(\bar{x}, \bar{y})(y^*)$, we have to show that $u^* \in \partial\mu(\bar{x})$. The inclusion $u^* \in x^* + D^*G(\bar{x}, \bar{y})(y^*)$ yields

$$u^* - x^* \in D^*G(\bar{x}, \bar{y})(y^*). \tag{2.7}$$

Clearly, condition (2.7) can be written equivalently as

$$\begin{aligned} &(u^* - x^*, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{gph} G) \\ \Leftrightarrow\ &(u^* - x^*, -y^*) \in \partial\delta((\bar{x}, \bar{y}); \operatorname{gph} G) \\ \Leftrightarrow\ &(u^*, 0) \in (x^*, y^*) + \partial\delta((\bar{x}, \bar{y}); \operatorname{gph} G). \end{aligned}$$

Therefore, we have

$$(u^*, 0) \in \partial\varphi(\bar{x}, \bar{y}) + \partial\delta((\bar{x}, \bar{y}); \operatorname{gph} G).$$

Without any regularity condition, the last inclusion implies that

$$(u^*, 0) \in \partial(\varphi + \delta(\cdot; \operatorname{gph} G))(\bar{x}, \bar{y}).$$

Hence

$$\varphi(x, y) - \varphi(\bar{x}, \bar{y}) \geq \langle u^*, x - \bar{x}\rangle + \langle 0, y - \bar{y}\rangle, \quad \forall (x, y) \in \operatorname{gph} G. \tag{2.8}$$

For each fixed element $x \in \operatorname{dom} G$, taking infimum on both sides of (2.8) on $y \in G(x)$ and remembering that $\mu(\bar{x}) = \varphi(\bar{x}, \bar{y})$, we obtain

$$\mu(x) - \mu(\bar{x}) \geq \langle u^*, x - \bar{x}\rangle.$$

Since $\mu(x) = +\infty$ for all $x \notin \operatorname{dom} G$, from the last property it follows that $u^* \in \partial\mu(\bar{x})$. Hence (2.1) is valid.

We are going to proof the equality (2.2). Observe that $x \in \operatorname{dom} \mu$ if and only if

$$\mu(x) = \inf\{\varphi(x, y) \mid y \in G(x)\} < \infty.$$

Since the last inequality holds if and only if there exists an $y \in G(x)$ with $(x, y) \in \operatorname{dom} \varphi$, we have

$$\delta(x; \operatorname{dom} \mu) = \inf\{\delta((x, y); \operatorname{dom} \varphi) \mid y \in G(x)\}. \tag{2.9}$$

The representation (2.9) for $\delta(x; \operatorname{dom} \mu)$ allows us to get (2.2) as a corollary of (2.1). Indeed, since $\operatorname{dom} \delta(\cdot; \operatorname{dom} \varphi) = \operatorname{dom} \varphi$, if the regularity requirement in (a) is satisfied then $\operatorname{int}(\operatorname{gph} G) \cap \operatorname{dom} \delta(\cdot; \operatorname{dom} \varphi) \neq \emptyset$. Next, if the condition (b) is fulfilled then $(x^0, y^0) \in \operatorname{int}(\operatorname{dom} \varphi)$; so $\delta(\cdot; \operatorname{dom} \varphi)$ is continuous at $(x^0, y^0) \in \operatorname{gph} G$. Now, consider the optimization problem (1.5) with $\varphi(x, y)$ replaced by $\delta((x, y); \operatorname{dom} \varphi)$. By (2.9), the corresponding optimal value function $\mu(x)$ coincides with $\delta(x; \operatorname{dom} \mu)$. Therefore, in accordance with (2.1), we have

$$\partial\delta(\cdot; \operatorname{dom} \mu)(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial\delta(\cdot; \operatorname{dom} \varphi)(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\}.$$

The latter yields (2.2) because

$$\partial\delta(\cdot; \operatorname{dom} \mu)(\bar{x}) = N(\bar{x}; \operatorname{dom} \mu) = \partial^\infty \mu(\bar{x})$$

and

$$\partial\delta(\cdot; \operatorname{dom} \varphi)(\bar{x}, \bar{y}) = N((\bar{x}, \bar{y}); \operatorname{dom} \varphi) = \partial^\infty \varphi(\bar{x}, \bar{y})$$

by Proposition 1.1. □

Here are two simple examples designed to illustrate Theorem 2.1.

Example 2.1 Let $X = Y = \mathbb{R}$ and $\bar{x} = 0$. Consider the optimal value function $\mu(x)$ in (1.6) with $\varphi(x,y) = |y|$ and $G(x) = \{y \mid y \geq \frac{1}{2}|x|\}$ for all $x \in \mathbb{R}$. Then we have $\mu(x) = \frac{1}{2}|x|$ for all $x \in \mathbb{R}$. So $\partial\mu(\bar{x}) = [-\frac{1}{2}, \frac{1}{2}]$, $\partial^\infty\mu(\bar{x}) = \{0\}$, and $M(\bar{x}) = \{0\}$. For $\bar{y} := 0 \in M(\bar{x})$, $\partial\varphi(\bar{x},\bar{y}) = \{0\} \times [-1,1]$ and $\partial^\infty\varphi(\bar{x},\bar{y}) = \{(0,0)\}$. Since G is a convex set-valued mapping, we have

$$\begin{aligned}
N((\bar{x},\bar{y});&\mathrm{gph\,G}) \\
&= \{(x^*,y^*) \in \mathbb{R}^2 \mid \langle (x^*,y^*),(x,y)-(0,0)\rangle \leq 0,\ \forall (x,y) \in \mathrm{gph\,G}\} \\
&= \{(x^*,y^*) \in \mathbb{R}^2 \mid y^* \leq -2|x^*|\}
\end{aligned}$$

and

$$D^*G(\bar{x},\bar{y})(y^*) = \begin{cases} [-\frac{1}{2}y^*, \frac{1}{2}y^*] & \text{if } y^* \geq 0, \\ \emptyset & \text{if } y^* < 0. \end{cases}$$

Thus the right-hand-sides of (2.1) and of (2.2) can be computed as follows:

$$\begin{aligned}
\bigcup_{(x^*,y^*)\in\partial\varphi(\bar{x},\bar{y})} \{x^* + D^*G(\bar{x},\bar{y})(y^*)\} &= \bigcup_{y^*\in[-1,1]} D^*G(\bar{x},\bar{y})(y^*) \\
&= \bigcup_{y^*\in[-1,1]} \Big[-\frac{1}{2}y^*, \frac{1}{2}y^*\Big] = \Big[-\frac{1}{2}, \frac{1}{2}\Big], \\
\bigcup_{(x^*,y^*)\in\partial^\infty\varphi(\bar{x},\bar{y})} \{x^* + D^*G(\bar{x},\bar{y})(y^*)\} &= D^*G(\bar{x},\bar{y})(0) = \{0\}.
\end{aligned}$$

As $\partial\mu(\bar{x}) = \big[-\frac{1}{2}, \frac{1}{2}\big]$ and $\partial^\infty\mu(\bar{x}) = \{0\}$, the equalities (2.1) and (2.2) hold.

Example 2.2 Choose $X = Y = \mathbb{R}$ and $\bar{x} = 0$. Let $\mu(x)$ be defined by (1.6), where $\varphi(x,y) = |x| + y$ for all $(x,y) \in \mathbb{R}^2$ and

$$G(x) = \begin{cases} \{y \mid y \geq -\sqrt{x}\} & \text{if } x \geq 0, \\ \emptyset & \text{if } x < 0. \end{cases}$$

We have $\mu(x) = |x| - \sqrt{x}$ for all $x \geq 0$, $\mu(x) = +\infty$ for all $x < 0$, and $M(\bar{x}) = \{0\}$. Hence $\partial\mu(\bar{x}) = \emptyset$ and $\partial^\infty\mu(\bar{x}) = (-\infty, 0]$. For $\bar{y} := 0 \in M(\bar{x})$, $\partial\varphi(\bar{x},\bar{y}) = [-1,1] \times \{1\}$ and $\partial^\infty\varphi(\bar{x},y) = \{(0,0)\}$. By the convexity of G we have

$$\begin{aligned}
N((\bar{x},\bar{y});&\mathrm{gph\,G}) \\
&= \{(x^*,y^*) \in \mathbb{R}^2 \mid \langle (x^*,y^*),(x,y)-(0,1)\rangle \leq 0,\ \forall (x,y) \in \mathrm{gph\,G}\} \\
&= (-\infty, 0] \times \{0\};
\end{aligned}$$

so $D^*G(\bar{x},\bar{y})(0) = (-\infty, 0]$ and $D^*G(\bar{x},\bar{y})(y^*) = \emptyset$ for every nonzero y^*. Then we can calculate the right-hand-sides of (2.1) and of (2.2) as follows:

$$\bigcup_{(x^*,y^*)\in\partial\varphi(\bar{x},\bar{y})} \{x^* + D^*G(\bar{x},\bar{y})(y^*)\} = \bigcup_{(x^*,y^*)\in[-1,1]\times\{1\}} \{x^* + D^*G(\bar{x},\bar{y})(y^*)\} = \emptyset,$$

$$\bigcup_{(x^*,y^*)\in\partial^\infty\varphi(\bar{x},\bar{y})} \{x^* + D^*G(\bar{x},\bar{y})(y^*)\} = D^*G(\bar{x},\bar{y})(0) = (-\infty, 0].$$

As $\partial\mu(\bar{x}) = \emptyset$ and $\partial^\infty\mu(\bar{x}) = (-\infty, 0]$, the equalities (2.1) and (2.2) are valid.

2.2 Convex Programming Problems under Functional Constraints

We now apply the above general results to convex optimization problems under geometrical and functional constraints. As in the preceding section, X and Y are Hausdorff locally convex topological vector spaces. Consider the problem

$$\min\{\varphi(x,y) \mid (x,y) \in C,\ \ g_i(x,y) \le 0,\ i \in I,\ \ h_j(x,y) = 0,\ j \in J\}, \quad (2.10)$$

in which $\varphi : X \times Y \to \overline{\mathbb{R}}$ is a convex function, $C \subset X \times Y$ is a convex set, $I = \{1, \dots, m\}$, $J = \{1, \dots, k\}$, $g_i : X \times Y \to \mathbb{R}$ ($i \in I$) are continuous convex functions, and $h_j : X \times Y \to \mathbb{R}$ ($j \in J$) are continuous affine functions. For each $x \in X$, we put

$$G(x) = \{y \in Y \,|\, (x,y) \in C,\ g_i(x,y) \le 0,\ i \in I,\ h_j(x,y) = 0,\ j \in J\}. \quad (2.11)$$

It clear that the set-valued map $G(\cdot)$ given by (2.11) is convex and

$$\operatorname{gph} G = C \cap \Big(\bigcap_{i\in I} \Omega_i\Big) \cap \Big(\bigcap_{j\in J} Q_j\Big), \quad (2.12)$$

where

$$\Omega_i := \{(x,y) \mid g_i(x,y) \le 0\} \ \ (i \in I)$$

and

$$Q_j := \{(x,y) \mid h_j(x,y) = 0\} \ \ (j \in J)$$

are convex sets.

The following infinite-dimensional version of the Farkas lemma [35, p. 200] has been obtained by Bartl [9].

Lemma 2.1 (See [9, Lemma 1]) *Let W be a vector space over $\mathbb{R}$. Let $A : W \to \mathbb{R}^m$ be a linear mapping and $\gamma : W \to \mathbb{R}$ a linear functional. Suppose that A is represented in the form $A = (\alpha_i)_i^m$, where each $\alpha_i : W \to \mathbb{R}$ is a linear functional (i.e., for each $x \in W$, $A(x)$ is a column vector whose i-th component is $\alpha_i(x)$, for $i = 1, \ldots, m$). Then, the inequality $\gamma(x) \leq 0$ is a consequence of the inequalities system*

$$\alpha_1(x) \leq 0,\ \alpha_2(x) \leq 0, \ldots,\ \alpha_m(x) \leq 0$$

if and only if there exist nonnegative real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ such that

$$\gamma = \lambda_1\alpha_1 + \cdots + \lambda_m\alpha_m.$$

The following lemma describes the normal cone of the intersection of finitely many affine hyperplanes.

Lemma 2.2 *Let X, Y be Hausdorff locally convex topological vector spaces. Let $(x_j^*, y_j^*) \in X^* \times Y^*$ and $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, m$, be given. Set*

$$Q_j = \{(x, y) \mid \langle (x_j^*, y_j^*), (x, y)\rangle = \alpha_j\}.$$

Then, for each $(\bar{x}, \bar{y}) \in \bigcap_{j=1}^{m} Q_j$, we have

$$N\left((\bar{x}, \bar{y}); \bigcap_{j=1}^{m} Q_j\right) = \operatorname{span}\left\{(x_j^*, y_j^*) \mid j = 1, \ldots, m\right\}, \tag{2.13}$$

where span$\left\{(x_j^*, y_j^*) \mid j = 1, \ldots, m\right\}$ *denotes the linear subspace generated by the vectors (x_j^*, y_j^*), $j = 1, \ldots, m$.*

Proof. Set $\mathcal{Q} = \bigcap_{j=1}^{m} Q_j$ and note that $(\bar{x}, \bar{y}) \in \mathcal{Q}$. Fixing an arbitrary element $(x^*, y^*) \in N((\bar{x}, \bar{y}); \mathcal{Q})$, one has

$$\langle (x^*, y^*), (x, y) - (\bar{x}, \bar{y})\rangle \leq 0,\ \ \forall (x, y) \in \mathcal{Q}. \tag{2.14}$$

The following property is valid:

$$(x, y) \in \mathcal{Q} \ \text{ iff } \ (u, v) := (x, y) - (\bar{x}, \bar{y}) \ \text{ belongs to } \bigcup_{j=1}^{m} \mathcal{Q}_j^0, \tag{2.15}$$

where

$$\mathcal{Q}_j^0 := \left\{(u, v) \mid \langle (x_j^*, y_j^*), (u, v)\rangle \leq 0,\ \langle -(x_j^*, y_j^*), (u, v)\rangle \leq 0\right\},\ j = 1, \ldots, m.$$

Indeed, the inclusion $(x, y) \in \mathcal{Q}$ implies

$$\begin{cases} \langle (x_j^*, y_j^*), (x, y) \rangle \le \alpha_j, \;\; j = 1, \dots, m \\ \langle -(x_j^*, y_j^*), (x, y) \rangle \le \alpha_j, \;\; j = 1, \dots, m. \end{cases} \tag{2.16}$$

Moreover, the condition $(\bar{x}, \bar{y}) \in \mathcal{Q}$ assures that

$$\langle (x_j^*, y_j^*), (\bar{x}, \bar{y}) \rangle = \alpha_j, \;\; j = 1, \dots, m. \tag{2.17}$$

Combining (2.16) and (2.17) yields

$$\begin{cases} \langle (x_j^*, y_j^*), (x, y) - (\bar{x}, \bar{y}) \rangle \le 0, \;\; j = 1, \dots, m \\ \langle -(x_j^*, y_j^*), (x, y) - (\bar{x}, \bar{y}) \rangle \le 0, \;\; j = 1, \dots, m. \end{cases}$$

This latter means that

$$\begin{cases} \langle (x_j^*, y_j^*), (u, v) \rangle \le 0, \;\; j = 1, \dots, m \\ \langle -(x_j^*, y_j^*), (u, v) \rangle \le 0, \;\; j = 1, \dots, m; \end{cases}$$

so one has $(u, v) \in \bigcup\limits_{j=1}^{m} \mathcal{Q}_j^0$. Conversely, suppose that $(x, y) \in X \times Y$ and $(x, y) - (\bar{x}, \bar{y})$ belongs to $\bigcup\limits_{j=1}^{m} \mathcal{Q}_j^0$. Then,

$$\begin{cases} \langle (x_j^*, y_j^*), (x, y) - (\bar{x}, \bar{y}) \rangle \le 0, \;\; j = 1, \dots, m \\ \langle -(x_j^*, y_j^*), (x, y) - (\bar{x}, \bar{y}) \rangle \le 0, \;\; j = 1, \dots, m. \end{cases} \tag{2.18}$$

Since $(\bar{x}, \bar{y}) \in \mathcal{Q}$, (2.17) holds. So, from (2.18) one has

$$\begin{cases} \langle (x_j^*, y_j^*), (x, y) \rangle \le \alpha_j, \;\; j = 1, \dots, m \\ \langle -(x_j^*, y_j^*), (x, y) \rangle \le \alpha_j, \;\; j = 1, \dots, m. \end{cases}$$

which obviously implies that $(x, y) \in \mathcal{Q}$.

Next, (2.14) and (2.15) show that $(x^*, y^*) \in N((\bar{x}, \bar{y}); \mathcal{Q})$ if and only if the inequality $\langle (x^*, y^*), (u, v) \rangle \le 0$ is a consequence of the inequalities system

$$\begin{cases} \langle (x_j^*, y_j^*), (u, v) \rangle \le 0, \;\; j = 1, \dots, m \\ \langle -(x_j^*, y_j^*), (u, v) \rangle \le 0, \;\; j = 1, \dots, m. \end{cases}$$

According to Lemma 2.1, this property is available if and only if there exist $\lambda_j \ge 0, \; \tilde{\lambda}_j \ge 0, \; j = 1, \dots, m$ such that

$$(x^*, y^*) = \sum_{j=1}^{m} \lambda_j (x_j^*, y_j^*) + \sum_{j=1}^{m} \tilde{\lambda}_j (-x_j^*, y_j^*).$$

The latter means that $(x^*, y^*) \in \operatorname{span}\{(x_j^*, y_j^*),\ j = 1, \dots, m\}$. Formula (2.13) has been proved. □

The next lemma from [22], which has a very brief proof, describes the normal cone of the sublevel set of a convex function. Due to the importance of this result, here we will give a detailed proof.

Lemma 2.3 (See [22, p. 206]) *Let f be a proper convex function on X, which is continuous at a point $x_0 \in X$. Assume that $f(x_1) < f(x_0) = \alpha_0$ for some $x_1 \in X$. Then,*

$$N(x_0; \mathcal{L}_{\alpha_0} f) = K_{\partial f(x_0)},$$

where $\mathcal{L}_{\alpha_0} f := \{x \mid f(x) \le \alpha_0\}$ is a sublevel set of f and

$$K_{\partial f(x_0)} := \{u^* \in X^* \mid u^* = \lambda x^*,\ \lambda \ge 0,\ x^* \in \partial f(x_0)\}$$

is the cone generated by the subdifferential of f at x_0.

Proof. Put $A = \mathcal{L}_{\alpha_0} f$. Since f is convex, A is a convex set. It is clear that $x_0 \in A$. We need to prove that $N(x_0; A) = K_{\partial f(x_0)}$.

First, let us prove that $K_{\partial f(x_0)} \subset N(x_0; A)$. Take an arbitrary element $u^* \in K_{\partial f(x_0)}$. Then $u^* = \lambda x^*$, with $x^* \in \partial f(x_0)$ and $\lambda \ge 0$. As $x^* \in \partial f(x_0)$,

$$\langle x^*, x - x_0 \rangle \le f(x) - f(x_0), \quad \forall x \in X.$$

Therefore, for every x in A, since $f(x) \le f(x_0)$, $\langle x^*, x - x_0 \rangle \le 0$. This shows that $x^* \in N(x_0; A)$. Hence $u^* = \lambda x^* \in N(x_0; A)$. Thus $K_{\partial f(x_0)} \subset N(x_0; A)$.

Next, we will prove that $N(x_0; A) \subset K_{\partial f(x_0)}$. Take an arbitrary vector $x^* \in N(x_0; A)$. If $x^* = 0$, then the inclusion $x^* \in K_{\partial f(x_0)}$ is obvious. Consider the case where $x^* \ne 0$. Note that

$$\mathcal{H} := \{(\alpha, x) \in \mathbb{R} \times X \mid \alpha = f(x_0),\ \langle x^*, x - x_0 \rangle = 0\}$$

is an affine set. As f is convex, $\operatorname{epi} f$ is a convex set. By the assumption that f is continuous at x_0, f bounded on a neighborhood of x_0. Invoking Theorem 1.2 we have $\operatorname{int}(\operatorname{epi} f) \ne \emptyset$. Besides, by the same theorem, we also have $\operatorname{int}(\operatorname{dom} f) \ne \emptyset$, f is continuous on $\operatorname{int}(\operatorname{dom} f)$, and we can determine the set $\operatorname{int}(\operatorname{epi} f)$ by the formula given in property (iv) of Theorem 1.2. We will show that $\mathcal{H} \cap \operatorname{int}(\operatorname{epi} f) = \emptyset$. Suppose on the contrary that there exists $(\bar\alpha, \bar x) \in \mathcal{H} \cap \operatorname{int}(\operatorname{epi} f)$. The last property means that

$$\bar\alpha = f(x_0), \quad \langle x^*, \bar x - x_0 \rangle = 0$$

and

$$\bar{\alpha} > f(\bar{x}), \quad \bar{x} \in \operatorname{int}(\operatorname{dom} f).$$

Since $\alpha_0 = f(x_0) = \bar{\alpha} > f(\bar{x})$ and f is continuous at $\bar{x}$, there exists a neighborhood $U \in \mathcal{N}(0)$ such that

$$\alpha_0 > f(\bar{x} + u), \quad \forall u \in U.$$

It follows that

$$\bar{x} + U \subset \mathcal{L}_{\alpha_0} f = A.$$

As $x^* \in N(x_0; A)$, $\langle x^*, (\bar{x} + u) - x_0 \rangle \leq 0$ for all $u \in U$. Since $\langle x^*, \bar{x} - x_0 \rangle = 0$, we have

$$\langle x^*, u \rangle \leq 0, \quad \forall u \in U.$$

Thus $\langle x^*, v \rangle = 0$ for all $v \in U \cap (-U)$; hence $x^* = 0$ (a contradiction). In conclusion, $\mathcal{H} \cap \operatorname{int}(\operatorname{epi} f) = \emptyset$. By the separation theorem for convex sets [40, Theorem 3.4(a)], there exists $(\alpha^*, y^*) \in (\mathbb{R} \times X^*) \setminus \{(0,0)\}$ satisfying

$$\langle (\alpha^*, y^*), (\alpha, x) \rangle \leq \langle (\alpha^*, y^*), (\alpha', x') \rangle, \ \forall (\alpha, x) \in \mathcal{H}, \ \forall (\alpha', x') \in \operatorname{epi} f. \quad (2.19)$$

If $\alpha^* < 0$ then, by substituting $(\alpha, x) = (\alpha_0, x_0)$ to the left-hand-side and

$$(\alpha', x') = (\alpha_0 + \mu, x_0) = (f(x_0) + \mu, x_0),$$

with $\mu \geq 0$, to the right-hand-side of (2.19), and letting $\mu \to +\infty$, we get a contradiction. So we can assume that $\alpha^* \geq 0$. If $\alpha^* = 0$ then (2.19) implies

$$\langle y^*, x \rangle \leq \langle y^*, x' \rangle, \ \forall (\alpha, x) \in \mathcal{H}, \ \forall (\alpha', x') \text{ with } \alpha' \geq f(x').$$

By substituting $(\alpha, x) = (\alpha_0, x_0)$ (observe that $(\alpha_0, x_0) \in \mathcal{H}$) and choosing $(\alpha', x') = (f(x), x) \in \operatorname{epi} f$ with $x \in \operatorname{dom} f$, from this property we have

$$\langle y^*, x_0 \rangle \leq \langle y^*, x \rangle, \quad \forall x \in \operatorname{dom} f.$$

Since $x_0 \in \operatorname{int}(\operatorname{dom} f)$, there exists $U_1 \in \mathcal{N}(0)$ such that $U_1 \subset \operatorname{dom} f$. Therefore,

$$\langle y^*, x_0 \rangle \leq \langle y^*, x_0 + u \rangle, \quad \forall u \in U_1,$$

i.e.,

$$\langle y^*, u \rangle \geq 0, \quad \forall u \in U_1.$$

Hence $y^* = 0$ (this is a contradiction, because $(\alpha^*, y^*) \neq (0,0)$). Thus the case $\alpha^* = 0$ cannot happen.

Consider the case where $\alpha^* > 0$. If we choose $(\alpha', x') = (\alpha_0, x_0) \in \operatorname{epi} f$, then by formula (2.19) we have

$$\langle (\alpha^*, y^*), (\alpha, x) \rangle \leq \langle (\alpha^*, y^*), (\alpha_0, x_0) \rangle, \quad \forall (\alpha, x) \in \mathcal{H}.$$

Hence,

$$\langle (\alpha^*, y^*), (\alpha, x) - (\alpha_0, x_0) \rangle \leq 0, \quad \forall (\alpha, x) \in \mathcal{H}. \tag{2.20}$$

Since $\mathcal{H}$ is an affine set and since $(\alpha_0, x_0) \in \mathcal{H}$, $M := \mathcal{H} - (\alpha_0, x_0)$ is a subspace parallel to $\mathcal{H}$. According to (2.20), we have

$$\langle (\alpha^*, y^*), (\beta, u) \rangle \leq 0, \quad \forall (\beta, u) \in M,$$

hence

$$\langle (\alpha^*, y^*), (\beta, u) \rangle = 0, \quad \forall (\beta, u) \in M.$$

Thus, from (2.20) we can deduce that

$$\langle (\alpha^*, y^*), (\alpha, x) \rangle = \langle (\alpha^*, y^*), (\alpha_0, x_0) \rangle, \quad \forall (\alpha, x) \in \mathcal{H}.$$

Set $\beta_0 = \langle (\alpha^*, y^*), (\alpha_0, x_0) \rangle$ and rewrite the last property as follows:

$$\langle (\alpha^*, y^*), (\alpha, x) \rangle = \beta_0, \quad \forall (\alpha, x) \in \mathcal{H}. \tag{2.21}$$

Consider the hyperplane $\Pi := \{(\alpha, x) \in \mathbb{R} \times X \mid \langle (\alpha^*, y^*), (\alpha, x) \rangle = \beta_0\}$ of the product space $\mathbb{R} \times X$. Due to (2.21), $\mathcal{H} \subset \Pi$. For each $(\alpha, x) \in \Pi$, we have

$$\begin{aligned} &\langle (\alpha^*, y^*), (\alpha, x) \rangle = \alpha^* \alpha_0 + \langle y^*, x_0 \rangle \\ &\Leftrightarrow \ \alpha^* \alpha + \langle y^*, x \rangle = \alpha^* f(x_0) + \langle y^*, x_0 \rangle \\ &\Leftrightarrow \ \alpha + \langle (\alpha^*)^{-1} y^*, x - x_0 \rangle = f(x_0), \end{aligned}$$

and this is equivalent to

$$\alpha = \langle -(\alpha^*)^{-1} y^*, x - x_0 \rangle + f(x_0). \tag{2.22}$$

Setting $\widetilde{y}^* = -(\alpha^*)^{-1} y^*$, from (2.22) we get

$$\alpha = \langle \widetilde{y}^*, x - x_0 \rangle + f(x_0), \quad \forall (\alpha, x) \in \Pi. \tag{2.23}$$

The inclusion $\mathcal{H} \subset \Pi$ yields $\widetilde{y}^* = \gamma x^*$, with $\gamma \in \mathbb{R}$. Indeed, as $\mathcal{H} \subset \Pi$, by (2.23) we have

$$\alpha = \langle \widetilde{y}^*, x - x_0 \rangle + f(x_0), \quad \forall (\alpha, x) \in \mathcal{H}.$$

Since $\alpha = f(x_0)$ for each $(\alpha, x) \in \mathcal{H}$, this means that $0 = \langle \widetilde{y}^*, x - x_0 \rangle$ for every x satisfying the condition $\langle x^*, x - x_0 \rangle = 0$. Thus from $\langle x^*, x - x_0 \rangle = 0$ it follows that $\langle \widetilde{y}^*, x - x_0 \rangle = 0$, or $\langle x^*, u \rangle = 0$ implies $\langle \widetilde{y}^*, u \rangle = 0$. So the inequality $\langle \widetilde{y}^*, u \rangle \leq 0$ is a consequence of the inequalities system

$$\langle x^*, u \rangle \leq 0, \quad \langle -x^*, u \rangle \leq 0.$$

By Lemma 2.1, there exist $\alpha_1 \geq 0,\ \alpha_2 \geq 0$ satisfying

$$\widetilde{y}^* = \alpha_1 x^* + \alpha_2(-x^*) = (\alpha_1 - \alpha_2)x^* = \gamma x^*,$$

with $\gamma := \alpha_1 - \alpha_2 \in \mathbb{R}$. We will show that $\gamma < 0$. Indeed, since $\alpha^* > 0$, we can replace the pair (α^*, y^*) in (2.19) by $(1, y^*/\alpha^*)$, i.e., we can suppose that $\alpha^* = 1$. By the continuity of f at x_0 and by (2.19), there exists $U_2 \in \mathcal{N}(0)$ such that

$$\langle (1, \widetilde{y}^*), (\alpha_0, x_0)\rangle \leq \langle (1, \widetilde{y}^*), (f(x), x)\rangle, \ \ \forall x \in x_0 + U_2.$$

This is equivalent to

$$\alpha_0 \leq f(x) + \langle \widetilde{y}^*, x - x_0\rangle, \ \ \forall x \in x_0 + U_2. \tag{2.24}$$

Substituting $x = (1-t)x_0 + tx_1$, with $t \in (0,1)$ being chosen as small as $x \in x_0 + U_2$, into the last inequality, we obtain

$$\begin{aligned}\alpha_0 &\leq f((1-t)x_0 + tx_1) + t\langle \widetilde{y}^*, x_1 - x_0\rangle \\ &\leq (1-t)f(x_0) + tf(x_1) + t\langle \widetilde{y}^*, x_1 - x_0\rangle.\end{aligned}$$

Since $\alpha_0 = f(x_0)$, from this we deduce that

$$f(x_0) \leq f(x_1) + \langle \widetilde{y}^*, x_1 - x_0\rangle. \tag{2.25}$$

If $\gamma = 0$ then $\widetilde{y}^* = 0$. Hence (2.25) forces $f(x_0) \leq f(x_1)$, contradicting the assumption $f(x_1) < f(x_0)$. If $\gamma > 0$ then $\widetilde{y}^* = \gamma x^* \in N(x_0; A)$. Thus, on one hand, for all $x \in A = \mathcal{L}_{\alpha_0} f$ we have $\langle \widetilde{y}^*, x - x_0\rangle \leq 0$. In particular,

$$\langle \widetilde{y}^*, x_1 - x_0\rangle \leq 0.$$

One the other hand, by (2.25) we obtain

$$\langle \widetilde{y}^*, x_1 - x_0\rangle \geq f(x_0) - f(x_1) > 0.$$

We have arrived at a contradiction. Thus $\gamma < 0$.

From formula (2.24) we get

$$f(x_0) \leq f(x) + \langle \widetilde{y}^*, x - x_0\rangle, \quad \forall x \in x_0 + U_2.$$

This shows that the convex function $x \mapsto f(x) + \langle \widetilde{y}^*, x - x_0\rangle$ reaches a local minimum at x_0. Then, by the convexity of f we have

$$\langle -\widetilde{y}^*, x - x_0\rangle \leq f(x) - f(x_0), \quad \forall x \in X,$$

i.e., $-\widetilde{y}^* \in \partial f(x_0)$, or $-\gamma x^* \in \partial f(x_0)$. As $\gamma < 0$, it follows that

$$x^* \in \frac{1}{-\gamma}\partial f(x_0) \in K_{\partial f(x_0)}.$$

This is exactly what we have to prove. □

Let us go back to considering the parametric convex programming problem (2.10). Our first result in this section can be formulated as follows.

Theorem 2.2 *Suppose that the equality constraints* $h_j(x,y) = 0$ $(j \in J)$ *are absent in* (2.10). *If at least one of the following regularity conditions*

(a1) *There exists a point* $(u^0, v^0) \in \operatorname{dom}\varphi$ *such that* $(u^0, v^0) \in \operatorname{int} C$ *and* $g_i(u^0, v^0) < 0$ *for all* $i \in I$,

(b1) φ *is continuous at a point* $(x^0, y^0) \in \operatorname{int} C$ *where* $g_i(x^0, y^0) < 0$ *for all* $i \in I$,

is satisfied, then for any $\bar{x} \in \operatorname{dom}\mu$, *with* $\mu(\bar{x}) \neq -\infty$, *and for any* $\bar{y} \in M(\bar{x})$ *we have*

$$\partial\mu(\bar{x}) = \bigcup_{(x^*,y^*)\in\partial\varphi(\bar{x},\bar{y})} \{x^* + Q_0^*\} \tag{2.26}$$

and

$$\partial^\infty\mu(\bar{x}) = \bigcup_{(x^*,y^*)\in\partial^\infty\varphi(\bar{x},\bar{y})} \{x^* + Q_0^*\}, \tag{2.27}$$

where

$$Q_0^* := \left\{ u^* \in X^* \mid (u^*, -y^*) \in N((\bar{x},\bar{y}); C) + \sum_{i\in I(\bar{x},\bar{y})} \operatorname{cone}\partial g_i(\bar{x},\bar{y}) \right\} \tag{2.28}$$

with $I(\bar{x},\bar{y}) := \{i \mid g_i(\bar{x},\bar{y}) = 0\}$ *and* $\operatorname{cone} M := \{tz \,|\, t \geq 0, z \in M\}$ *denoting the cone generated by* M.

Proof. Recall that $G : X \rightrightarrows Y$, with $G(x)$ being defined by (2.11), is a convex set-valued mapping, and the objective function $\varphi(x,y)$ of (2.10) is convex.

If (a1) is satisfied, then it is clear that $(u^0, v^0) \in \operatorname{int}(\operatorname{gph} G)$; hence the condition (a) in Theorem 2.1 is fulfilled. If (b1) is satisfied, then φ is continuous at the point (x^0, y^0) which belongs to $\operatorname{gph} G$; so the condition (b) in Theorem 2.1 is satisfied. Therefore, our assumptions guarantee that (2.1) and (2.2) hold.

By the definition of coderivative,

$$D^*G(\bar{x},\bar{y})(y^*) = \{u^* \in X^* \mid (u^*, -y^*) \in N((\bar{x},\bar{y}); \operatorname{gph} G)\}. \tag{2.29}$$

Since the constraints $h_j(x,y) = 0$ $(j \in J)$ are absent in (2.10), formula (2.12) becomes

$$\operatorname{gph} G = C \cap \left(\bigcap_{i\in I} \Omega_i \right). \tag{2.30}$$

If (a1) is satisfied, then $(u^0, v^0) \in (\operatorname{int} C) \cap \left(\bigcap_{i\in I} \operatorname{int} \Omega_i\right)$. If (b1) is valid, then $(x^0, y^0) \in C \cap \left(\bigcap_{i\in I} \operatorname{int} \Omega_i\right)$. So, in both cases we can use Proposition 1.4 and formula (2.30) to compute the normal cone to $\operatorname{gph} G$ at $(\bar{x}, \bar{y})$ as follows

$$N((\bar{x}, \bar{y}); \operatorname{gph} G) = N((\bar{x}, \bar{y}); C) + \sum_{i\in I} N((\bar{x}, \bar{y}); \Omega_i).$$

Since $N((\bar{x}, \bar{y}); \Omega_i) = \{(0, 0)\}$ for every $i \notin I(\bar{x}, \bar{y})$, this formula can be written in the equivalent form

$$N((\bar{x}, \bar{y}); \operatorname{gph} G) = N((\bar{x}, \bar{y}); C) + \sum_{i\in I(\bar{x},\bar{y})} N((\bar{x}, \bar{y}); \Omega_i). \tag{2.31}$$

By Lemma 2.3, for every $i \in I(\bar{x}, \bar{y})$ we have

$$N((\bar{x}, \bar{y}); \Omega_i) = K_{\partial g_i(\bar{x},\bar{y})} = \operatorname{cone} \partial g_i(\bar{x}, \bar{y}).$$

Combining this with (2.28), (2.29), (2.31), we get (2.26) from (2.1) and (2.27) from (2.2). The proof is complete. □

Let us consider the following illustrative example.

Example 2.3 Let $X = Y = \mathbb{R}$, $C = X \times Y$, $\varphi(x, y) = |x + y|$, $m = 1$, $k = 0$ (no equality functional constraint), $g_1(x, y) = y$ for all $(x, y) \in X \times Y$. Choosing $\bar{x} = 0$, we note that $M(\bar{x}) = \{\bar{y}\}$, with $\bar{y} = 0$. Since

$$\varphi(x, y) = |x + y| = \max\{x + y, -x - y\},$$

by applying a well known formula for computing the subdifferential of the maximum function [22, Theorem 3, pp. 201–202] we get

$$\partial\varphi(\bar{x}, \bar{y}) = \operatorname{co}\left\{(1, 1)^T, (-1, -1)^T\right\},$$

where $\operatorname{co}\Omega$ denotes the *convex hull* of Ω. On one hand,

$$\mu(x) = \inf\{|x + y| \mid y \le 0\} = \begin{cases} 0, & \text{if } x \ge 0, \\ -x, & \text{if } x < 0. \end{cases}$$

So we find $\partial\mu(\bar{x}) = [-1, 0]$. On the other hand,

$$\begin{aligned} Q_0^* &= \left\{u^* \in X^* \mid (u^*, -y^*) \in N((\bar{x},\bar{y}); C) + \sum_{i \in I(\bar{x},\bar{y})} \operatorname{cone} \partial g_i(\bar{x},\bar{y})\right\} \\ &= \left\{u^* \in X^* \mid (u^*, -y^*) \in N((\bar{x},\bar{y}); X \times Y) + \operatorname{cone} \partial g_1(\bar{x},\bar{y})\right\} \\ &= \left\{u^* \in X^* \mid (u^*, -y^*) \in \{(0,0)\} + \{0\} \times [0, +\infty)\right\} \\ &= \begin{cases} \{0\} & \text{if } y^* \le 0 \\ \emptyset & \text{if } y^* > 0. \end{cases} \end{aligned}$$

Thus,

$$\bigcup_{(x^*,y^*) \in \partial\varphi(\bar{x},\bar{y})} \{x^* + Q_0^*\} = \bigcup_{0 \le \alpha \le \frac{1}{2}} \{2\alpha - 1\} = [-1, 0].$$

Moreover, since the function φ is Lipschitz continuous around $(\bar{x},\bar{y})$, we have $\partial^\infty\varphi(\bar{x},\bar{y}) = \{(0,0)\}$. It is easy to show that $\partial^\infty\mu(\bar{x}) = \{0\}$. Therefore, (2.26) and (2.27) are valid.

We now consider the case where the affine constraints $h_j(x,y) = 0$ $(j \in J)$ are available in (2.10). The second result of this section reads as follows.

Theorem 2.3 *For every $j \in J$, suppose that*

$$h_j(x,y) = \langle (x_j^*, y_j^*), (x,y) \rangle - \alpha_j, \quad \alpha_j \in \mathbb{R}.$$

If φ is continuous at a point (x^0, y^0) with $(x^0, y^0) \in \operatorname{int} C$, $g_i(x^0, y^0) < 0$, for all $i \in I$ and $h_j(x^0, y^0) = 0$, for all $j \in J$, then for any $\bar{x} \in \operatorname{dom}\mu$, with $\mu(\bar{x}) \neq -\infty$, and for any $\bar{y} \in M(\bar{x})$ we have

$$\partial\mu(\bar{x}) = \bigcup_{(x^*,y^*) \in \partial\varphi(\bar{x},\bar{y})} \{x^* + \tilde{Q}^*\} \tag{2.32}$$

and

$$\partial^\infty\mu(\bar{x}) = \bigcup_{(x^*,y^*) \in \partial^\infty\varphi(\bar{x},\bar{y})} \{x^* + \tilde{Q}^*\}, \tag{2.33}$$

where

$$\tilde{Q}^* := \left\{u^* \in X^* \mid (u^*, -y^*) \in A + N((\bar{x},\bar{y}); C)\right\} \tag{2.34}$$

with

$$A := \sum_{i \in I(\bar{x},\bar{y})} \operatorname{cone} \partial g_i(\bar{x},\bar{y}) + \operatorname{span}\{(x_j^*, y_j^*),\ j \in J\}. \tag{2.35}$$

Proof. (This proof follows the same scheme as the proof of Theorem 2.2.) For the set-valued map $G(\cdot)$ defined by (2.11), we have $(x^0, y^0) \in \operatorname{gph} G$. Hence the condition (b) in Theorem 2.1 is satisfied, and we know that (2.1) and (2.2) hold. By our assumptions,

$$(u^0, v^0) \in (\operatorname{int} C) \cap \left(\bigcap_{i \in I} \operatorname{int} \Omega_i \right) \cap \left(\bigcap_{j \in J} Q_j \right).$$

Therefore, according to Proposition 1.4 and formula (2.12), we have

$$N((\bar{x}, \bar{y}); \operatorname{gph} G) = N((\bar{x}, \bar{y}); C) + \sum_{i \in I} N((\bar{x}, \bar{y}); \Omega_i) + N\left((\bar{x}, \bar{y}); \bigcap_{j \in J} Q_j \right).$$

Since $N\left((\bar{x}, \bar{y}); \bigcap_{j \in J} Q_j \right) = \operatorname{span}\left\{ (x_j^*, y_j^*) \mid j \in J \right\}$ by Lemma 2.2,

$$N((\bar{x}, \bar{y}); \Omega_i) = \operatorname{cone} \partial g_i(\bar{x}, \bar{y})$$

for every $i \in I(\bar{x}, \bar{y})$ by Lemma 2.3, and since $N((\bar{x}, \bar{y}); \Omega_i) = \{(0,0)\}$ for every $i \notin I(\bar{x}, \bar{y})$, this formula can be written in the form

$$\begin{aligned} &N((\bar{x}, \bar{y}); \operatorname{gph} G) \\ &= N((\bar{x}, \bar{y}); C) + \sum_{i \in I(\bar{x}, \bar{y})} N((\bar{x}, \bar{y}); \Omega_i) + \operatorname{span}\left\{ (x_j^*, y_j^*) \mid j \in J \right\}. \end{aligned} \tag{2.36}$$

Using (2.29), (2.36), and the definition of the set Q^*, we easily obtain (2.32) from (2.1), (2.33) from (2.2). □

2.3 Conclusions

In this chapter, we obtained formulas for computing the subdifferential and the singular subdifferential of the optimal value function for parametric convex optimization problems. Theorems 2.1, 2.2 and 2.3 are our main results here. By using the convexity of the problem under consideration and the Moreau-Rockafellar Theorem, we do not need the assumption $\widehat{\partial}^+ \varphi(\bar{x}, \bar{y}) \neq \emptyset$ in [29, Theorem 1], the requirement that the solution map $M : \operatorname{dom} G \rightrightarrows Y$ has a local upper Lipschitzian selection at $(\bar{x}, \bar{y})$ in [29, Theorem 2], the sequentially normally compact property of φ, and the μ-inner semicontinuity or the μ-inner semicompactness conditions on the solution map $M(\cdot)$ in [29, Theorem 7].

Chapter 3

Stability Analysis using Aubin's Regularity Condition

This chapter develops some aspects of Chapter 2. Namely, motivated by the idea of Aubin [6, Problem 35 - Subdifferentials of Marginal Functions, p. 335], we obtain formulas for computing the subdifferentials of the optimal value function for parametric convex programs under three assumptions:

(a_1) The objective function is closed;

(a_2) The constraint multifunction has closed graph;

(a_3) An interior regularity condition (we will call it *Aubin's regularity condition*) is satisfied.

Our exposition is based on [2].

3.1 Differential Stability under Aubin's Regularity Condition

Let X be a Banach space, $f : X \to \overline{\mathbb{R}}$ is a function having values in the extended real line. If $\operatorname{epi} f$ is a closed subset of $X \times \mathbb{R}$, f is said to be a *closed* function.

Denoting the set of all the neighborhoods of x by $\mathcal{N}(x)$, one says that f is *lower semicontinuous* (l.s.c.) at $x \in X$ if for every $\varepsilon > 0$ there exists $U \in \mathcal{N}(x)$ such that $f(x') \geq f(x) - \varepsilon$ for any $x' \in U$. If f is l.s.c. at every $x \in X$, f is said to be l.s.c. on X. It is easy to show that: *f is l.s.c. on X if and only if f is closed and* $\operatorname{dom} f$ *is closed too.* The simple example with $f(x) = x^{-1}$ for $x > 0$ and $f(x) = +\infty$ for all nonpositive real numbers x shows that the closedness of $\operatorname{epi} f$ alone does not imply that f is l.s.c. on X.

By a different approach, Aubin [6, Problem 35 - Subdifferentials of Marginal Functions, p. 335] has studied a problem similar to that one considered in the preceding chapters. Namely, in our notation, Aubin has studied the parametric problem:

$$\min\{\varphi_0(y) \mid y \in G(x)\}, \tag{3.1}$$

where X, Y are Hilbert spaces, $\varphi_0 : Y \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, *lower semicontinuous* function, $G : X \rightrightarrows Y$ is convex, *of closed graph.* The optimal value function of that problem is given by

$$\mu(x) = \inf\{\varphi_0(y) \mid y \in G(x)\}. \tag{3.2}$$

Using the notion of conjugate function, the above Fenchel-Moreau theorem, and some auxiliary results related to continuous linear mappings, convex functions, and convex sets on Hilbert spaces, Aubin has proved the following theorem.

Theorem 3.1 (See [6, p. 335]) *Assume that*

$$0 \in \operatorname{int}(\operatorname{dom}\varphi_0 - \operatorname{dom} G^{-1}), \tag{3.3}$$

and that $\bar{y} \in G(\bar{x})$ is a solution of problem (3.1). *Then, $x^* \in \partial\mu(\bar{x})$ if and only if there exists $y^* \in \partial\varphi_0(\bar{y})$ such that*

$$(-y^*, x^*) \in N((\bar{y}, \bar{x}); \operatorname{gph} G^{-1}),$$

or

$$(x^*, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{gph} G).$$

Hence,

$$\partial\mu(\bar{x}) = D^*G(\bar{x}, \bar{y})(\partial\varphi_0(\bar{y})).$$

We now recall a sum rule for subdifferentials based on a regularity condition of Aubin's type. Note that Aubin [6, Theorem 4.4, p. 67] has proved the next theorem in a Hilbert space setting, but he has also observed that the result is valid in a reflexive Banach space setting.

Theorem 3.2 (See Bonnans and Shapiro [11, Theorem 2.168]) *Let X and Y be Banach spaces. Let $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ be proper, closed, convex functions, $A : X \to Y$ a continuous linear operator, and $F(x) := f(x) + g(Ax)$. Suppose that*

$$0 \in \operatorname{int}(A(\operatorname{dom} f) - \operatorname{dom} g) \tag{3.4}$$

holds. Then, for any $x \in \operatorname{dom} F$, we have

$$\partial F(x) = \partial f(x) + A^*(\partial g(Ax)), \tag{3.5}$$

where $A^* : Y^* \to X^*$ is the conjugate operator of A.

The proof of Theorem 3.2 given in [11] is long and complicated, as it requires various facts from the abstract duality theory for optimization problems and from functional analysis.

Let us consider an example clarifying the necessity of the regularity condition (3.4) for the assertion (3.5).

Example 3.1 Let $X = Y = \mathbb{R}$, $A \equiv I$, where I denotes the indentity mapping, and f be defined by $f(x) = 0$ if $x = 0$ and $f(x) = +\infty$ if $x \neq 0$. Let g be given by $g(y) = -\sqrt{y}$ if $y \geq 0$ and $g(y) = +\infty$ if $y < 0$. Then

$$g(Ax) = g(x) = \begin{cases} -\sqrt{x} & \text{if } x \geq 0, \\ +\infty & \text{if } x < 0. \end{cases}$$

We have $A(\operatorname{dom} f) = \operatorname{dom} f = \{0\},\ \operatorname{dom} g = [0, +\infty)$. Hence

$$0 \notin \operatorname{int}(A(\operatorname{dom} f) - \operatorname{dom} g).$$

In addition,

$$F(x) = f(x) + g(Ax) = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{if } x \neq 0. \end{cases}$$

For $\bar{x} := 0 \in \operatorname{dom} F$, one has $\partial F(\bar{x}) = \mathbb{R}$ whereas $\partial f(\bar{x}) + A^*(\partial g(A\bar{x})) = \emptyset$.

Taking $X = Y$, $A = I$, from Theorem 3.2 we obtain the following result, which is a geometrical form of the traditional Moreau–Rockafellar Theorem which has been recalled in Theorem 1.1.

Theorem 3.3 (See [11, Remark 2.169]) *If $f, g : X \to \overline{\mathbb{R}}$ are proper, closed, convex functions and the regularity condition*

$$0 \in \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g) \tag{3.6}$$

holds, then for any $x \in (\operatorname{dom} f) \cap (\operatorname{dom} g)$ we have

$$\partial(f + g)(x) = \partial f(x) + \partial g(x). \tag{3.7}$$

We now show that the assumption on the closedness of f and g cannot be dropped from Theorem 3.3.

Example 3.2 (See [11, Remark 2.169]) Let X be an infinite-dimensional Banach space. Then one can construct a linear discontinuous function $f : X \to \mathbb{R}$. For $g := -f$, we have $\operatorname{dom} f = \operatorname{dom} g = X$, so the regularity condition (3.6) is satisfied. Since f and g are discontinuous linear functionals, none of them can be closed. On one hand, $\partial f(x) = \partial g(x) = \emptyset$ for any $x \in X$. On the other hand, since $f(x) + g(x) \equiv 0$, one has $\partial(f+g)(x) = \{0\}$. Therefore, (3.7) fails to hold. We have seen that the regularity condition (3.6) alone cannot assure the fulfillment of (3.7), unless f and g are closed.

In the sequel, if Z is Banach space then $\mathbb{B}_Z$ stands for the open unit ball of Z.

Note that the convex programming problem considered in [6, Problem 3.35] is a particular case of the convex version of problem (1.5), which has been studied in Chapter 2, with $\varphi_0(y) := \varphi(x, y)$ for every $(x, y) \in X \times Y$, and the regularity condition (3.3) being rewritten as

$$(0, 0) \in \operatorname{int}(\operatorname{dom} \varphi - \operatorname{gph} G). \tag{3.8}$$

To verify the last claim, note that by (3.3) there exists $\gamma > 0$ such that every $v \in \gamma\mathbb{B}_Y$ can be written as $v = y_1 - y_2$, with $y_1 \in \operatorname{dom} \varphi_0$, $y_2 \in \operatorname{dom} G^{-1}$. The inclusion $y_2 \in \operatorname{dom} G^{-1}$ implies that there exists $x_2 \in G^{-1}(y_2)$, hence $y_2 \in G(x_2)$. Take any $u \in \gamma\mathbb{B}_X$ and put $x_1 = u + x_2$. Then, one has

$$(u, v) = (x_1 - x_2, y_1 - y_2) = (x_1, y_1) - (x_2, y_2) \in \operatorname{dom} \varphi - \operatorname{gph} G.$$

Therefore (3.8) holds. The fact that (3.8) implies (3.3) can be proved easily.

From now on, let $G : X \rightrightarrows Y$ be a convex multifunction between *Banach spaces*, whose graph is *closed*. Let $\varphi : X \times Y \to \overline{\mathbb{R}}$ be a proper, *closed*, convex function. Consider the *parametric optimization problem under an inclusion constraint*

$$\min \{\varphi(x, y) \mid y \in G(x)\}. \tag{3.9}$$

Using the regularity condition (3.8), we will derive formulas for computing the subdifferential and the singular subdifferential of the *optimal value function* $\mu : X \to \overline{\mathbb{R}}$ of (3.9), which is given by

$$\mu(x) = \inf\{\varphi(x, y) \mid y \in G(x)\}. \tag{3.10}$$

The first main result of this chapter reads as follows.

Theorem 3.4 *If the regularity condition* (3.8) *is satisfied, then for every* $\bar{x} \in \operatorname{dom} \mu$ *with* $\mu(\bar{x}) \neq -\infty$*, and for every* $\bar{y} \in M(\bar{x})$*, we have*

$$\partial\mu(\bar{x}) = \bigcup_{(x^*,y^*)\in\partial\varphi(\bar{x},\bar{y})} \{x^* + D^*G(\bar{x},\bar{y})(y^*)\}. \tag{3.11}$$

Proof. (This proof follows the scheme used for proving Theorem 2.1.) Fix any $\bar{x}^* \in \partial\mu(\bar{x})$. As μ is convex, one has $\mu(x) - \mu(\bar{x}) \geq \langle \bar{x}^*, x - \bar{x}\rangle$ for all $x \in X$. Taking any $u \in X$ and selecting a vector $v \in G(u)$, from the last property we get

$$\begin{aligned}\varphi(u,v) - \varphi(\bar{x},\bar{y}) = \varphi(u,v) - \mu(\bar{x}) &\geq \mu(u) - \mu(\bar{x}) \\ &\geq \langle \bar{x}^*, u - \bar{x}\rangle + \langle 0, v - \bar{y}\rangle.\end{aligned}$$

Therefore, $\varphi(u,v) - \varphi(\bar{x},\bar{y}) \geq \langle (\bar{x}^*,0), (u,v) - (\bar{x},\bar{y})\rangle$ for all $(u,v) \in \operatorname{gph} G$. Hence

$$\begin{aligned}(\varphi + \delta(\cdot;\operatorname{gph} G))(u,v) &- (\varphi + \delta(\cdot;\operatorname{gph} G))(\bar{x},\bar{y}) \\ &\geq \langle (\bar{x}^*,0), (u,v) - (\bar{x},\bar{y})\rangle, \quad \forall (u,v) \in X \times Y.\end{aligned} \tag{3.12}$$

By (3.12),

$$(\bar{x}^*, 0) \in \partial(\varphi + \delta(\cdot;\operatorname{gph} G))(\bar{x},\bar{y}). \tag{3.13}$$

Since $\operatorname{gph} G$ is a convex and closed set, the indicator function $\delta(\cdot;\operatorname{gph} G) : X \times Y \to \overline{\mathbb{R}}$ is convex and closed. As $\operatorname{dom} \delta(\cdot;\operatorname{gph} G) = \operatorname{gph} G$, by (3.8) we have

$$(0,0) \in \operatorname{int}(\operatorname{dom}\varphi - \operatorname{dom}\delta(\cdot;\operatorname{gph} G)).$$

Therefore, using Theorem 3.3, one deduces from (3.13) that

$$\begin{aligned}(\bar{x}^*, 0) &\in \partial(\varphi + \delta(\cdot;\operatorname{gph} G))(\bar{x},\bar{y}) \\ &= \partial\varphi(\bar{x},\bar{y}) + \partial\delta(\cdot;\operatorname{gph} G)(\bar{x},\bar{y}) \\ &= \partial\varphi(\bar{x},\bar{y}) + N((\bar{x},\bar{y});\operatorname{gph} G).\end{aligned}$$

So one can find a vector $(x^*, y^*) \in \partial\varphi(\bar{x},\bar{y})$ such that

$$(\bar{x}^*, 0) \in (x^*, y^*) + N((\bar{x},\bar{y});\operatorname{gph} G).$$

The latter means that $(\bar{x}^* - x^*, -y^*) \in N((\bar{x},\bar{y});\operatorname{gph} G)$. This is equivalent to writing $\bar{x}^* - x^* \in D^*G(\bar{x},\bar{y})(y^*)$. The last inclusion yields

$$\bar{x}^* \in x^* + D^*G(\bar{x},\bar{y})(y^*). \tag{3.14}$$

Since the vector $\bar{x}^* \in \partial\mu(\bar{x})$ can be taken arbitrarily, (3.14) gives

$$\partial\mu(\bar{x}) \subset \bigcup_{(x^*,y^*)\in\partial\varphi(\bar{x},\bar{y})} \{x^* + D^*G(\bar{x},\bar{y})(y^*)\}.$$

To obtain the opposite inclusion, given any $(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})$, we need to show that

$$x^* + D^*G(\bar{x}, \bar{y})(y^*) \subset \partial\mu(\bar{x}). \tag{3.15}$$

Taking an arbitrary vector $u^* \in x^* + D^*G(\bar{x}, \bar{y})(y^*)$. This inclusion clearly yields

$$u^* - x^* \in D^*G(\bar{x}, \bar{y})(y^*). \tag{3.16}$$

Condition (3.16) can be equivalently transformed as follows:

$$\begin{aligned} &(u^* - x^*, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{gph} G) \\ &\Leftrightarrow \ (u^* - x^*, -y^*) \in \partial\delta((\bar{x}, \bar{y}); \operatorname{gph} G) \\ &\Leftrightarrow \ (u^*, 0) \in (x^*, y^*) + \partial\delta((\bar{x}, \bar{y}); \operatorname{gph} G). \end{aligned}$$

Therefore, we have $(u^*, 0) \in \partial\varphi(\bar{x}, \bar{y}) + \partial\delta((\bar{x}, \bar{y}); \operatorname{gph} G)$. Without any regularity condition, the last inclusion implies that $(u^*, 0) \in \partial(\varphi + \delta(\cdot; \operatorname{gph} G))(\bar{x}, \bar{y})$. Hence

$$\varphi(x, y) - \varphi(\bar{x}, \bar{y}) \geq \langle u^*, x - \bar{x}\rangle + \langle 0, y - \bar{y}\rangle, \quad \forall (x, y) \in \operatorname{gph} G. \tag{3.17}$$

For each $x \in \operatorname{dom} G$, taking infimum on both sides of (3.17) on $y \in G(x)$ and using the equality $\varphi(\bar{x}, \bar{y}) = \mu(\bar{x})$, we obtain $\mu(x) - \mu(\bar{x}) \geq \langle u^*, x - \bar{x}\rangle$. Since $\mu(x) = +\infty$ for all $x \notin \operatorname{dom} G$, the validity of the last property for all $x \in \operatorname{dom}$ G implies that $u^* \in \partial\mu(\bar{x})$; so (3.15) is valid.

We have completed the proof of (3.11). □

The next theorem gives us a formula for computing the singular subdifferential of the optimal value function of (3.9).

Theorem 3.5 *In addition to the assumption of Theorem 3.4, suppose that the set* $\operatorname{dom} \varphi$ *is closed. Then*

$$\partial^\infty \mu(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial^\infty \varphi(\bar{x}, \bar{y})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*)\}. \tag{3.18}$$

Proof. (This proof is based on the scheme used for proving Theorem 2.1.) We have $x \in \operatorname{dom} \mu$ if and only if

$$\mu(x) = \inf\{\varphi(x, y) \mid y \in G(x)\} < \infty. \tag{3.19}$$

Since the inequality in (3.19) holds if and only if there exists an $y \in G(x)$ with $(x, y) \in \operatorname{dom} \varphi$, we have

$$\delta(x; \operatorname{dom} \mu) = \inf\{\delta((x, y); \operatorname{dom} \varphi) \mid y \in G(x)\}. \tag{3.20}$$

We will apply Theorem 3.4 for problem (3.9) with $\varphi(x,y)$ being replaced by $\delta((x,y);\operatorname{dom}\varphi)$. Note that $\operatorname{dom}\varphi$ is nonempty, convex, and closed by our assumptions. Therefore $\delta(\cdot;\operatorname{dom}\varphi)$ is a proper, closed, convex function. It is clear that $\operatorname{dom}\delta(\cdot;\operatorname{dom}\varphi)=\operatorname{dom}\varphi$.

On one hand, (3.20) shows that $\delta(\cdot;\operatorname{dom}\mu)$ is the optimal value function of the new problem. On the other hand, the regularity condition (3.8) yields

$$(0,0)\in\operatorname{int}\big(\operatorname{dom}\delta(\cdot;\operatorname{dom}\varphi)-\operatorname{gph}\mathrm{G}\big).$$

So, by Theorem 3.4,

$$\partial\delta(\cdot;\operatorname{dom}\mu)(\bar{x})=\bigcup_{(x^*,y^*)\in\partial\delta(\cdot;\operatorname{dom}\varphi)(\bar{x},\bar{y})}\{x^*+D^*G(\bar{x},\bar{y})(y^*)\}. \tag{3.21}$$

Invoking Proposition 1.1, we have

$$\partial\delta(\cdot;\operatorname{dom}\mu)(\bar{x})=N(\bar{x};\operatorname{dom}\mu)=\partial^\infty\mu(\bar{x})$$

and

$$\partial\delta(\cdot;\operatorname{dom}\varphi)(\bar{x},\bar{y})=N((\bar{x},\bar{y});\operatorname{dom}\varphi)=\partial^\infty\varphi(\bar{x},\bar{y}).$$

Thus (3.18) follows from (3.21). □

The next example of an optimization problem in infinite-dimensional Banach spaces is designed to illustrate the results in Theorems 3.4 and 3.5.

Example 3.3 Let $X=Y=C[a,b]$ where $C[a,b]$ is the Banach space of real-valued continuous functions defined on the interval $[a,b]\subset\mathbb{R}$, $a<b$, with the supremum norm. By the Riesz representation theorem (see [24, p. 374] and [25, pp. 113–115]), the dual space of $C[a,b]$ is the normalized space $NBV[a,b]$ of functions of bounded variation on $[a,b]$, which vanish at a and which are continuous from the left on (a,b). For any $x^*\in NBV[a,b]$ and $x\in C[a,b]$, one has $\langle x^*,x\rangle=\int_a^b x(t)dx^*(t)$, where the left-hand-side is the Riemann-Stieltjes integral of x with respect to x^* on $[a,b]$. Define a function $\varphi: X\times Y\to\mathbb{R}$ and a multifunction $G: X\rightrightarrows Y$ by setting $\varphi(x,y)=\big(x(\frac{b+a}{2})\big)^2+(y(b))^2$, $G(x)=\{0\}$ for $x=0$ and $G(x)=\emptyset$ for every $x\neq 0$. Then $\operatorname{gph}G=\{(0,0)\}$ and $\operatorname{dom}\varphi=C[a,b]\times C[a,b]$. Hence the regularity condition (3.8) is satisfied. Also, we have $\mu(x)=0$ if $x=0$, $\mu(x)=+\infty$ if $x\neq 0$. Choose $\bar{x}=\bar{y}=0$. On one hand,

$$\begin{aligned}\partial\mu(\bar{x})&=\partial^\infty\mu(\bar{x})\\&=\{x^*\in NBV[a,b]\mid\langle x^*,x-\bar{x}\rangle\le\mu(x)-\mu(\bar{x}),\ \forall x\in C[a,b]\}\\&=\{x^*\in NBV[a,b]\mid\langle x^*,x\rangle\le\mu(x),\ \forall x\in C[a,b]\}\\&=NBV[a,b].\end{aligned}$$

On the other hand,

$$\begin{aligned}
&N((\bar{x},\bar{y}); \operatorname{gph} G) \\
&= \{(x^*,y^*) \in NBV[a,b] \times NBV[a,b] \mid \langle (x^*,y^*),(x,y)\rangle \le 0,\ \forall (x,y) \in \operatorname{gph} G\} \\
&= NBV[a,b] \times NBV[a,b].
\end{aligned}$$

So $D^*G(\bar{x},\bar{y})(0) = NBV[a,b]$. Moreover, $\partial\varphi(\bar{x},\bar{y}) = \partial^\infty\varphi(\bar{x},\bar{y}) = \{(0,0)\}$. Therefore, the equalities (3.11) and (3.18) are valid.

3.2 An Analysis of the Regularity Conditions

First, we consider an example satisfying Aubin's regularity condition (3.8), but both regularity conditions (a) and (b) in Theorem 2.1 are not fulfilled, whereas the conclusion of the Theorem 3.4 holds true.

Example 3.4 Let $X = Y = \mathbb{R}^2$ and $(\bar{x},\bar{y}) = (0,0)$. Consider the optimal value function $\mu(x)$ defined by (3.10) with

$$\varphi_0(y) = \begin{cases} 0 & \text{if } y_1 = 0, \\ +\infty & \text{if } y_1 \neq 0, \end{cases}$$

for every $y = (y_1,y_2) \in Y$, and

$$G(x) = \begin{cases} \mathbb{R} \times \{0\} & \text{if } x = 0, \\ \emptyset & \text{if } x \neq 0, \end{cases}$$

for every $x = (x_1,x_2) \in X$. Clearly, φ_0 is a proper, closed, convex function with $\operatorname{dom}\varphi_0$ being closed. In addition, G is a convex multifunction of closed graph. Setting $\varphi(x,y) = \varphi_0(y)$ for all $(x,y) \in X \times Y$, we have $\operatorname{gph} G = \{0_{\mathbb{R}^2}\} \times \mathbb{R} \times \{0\}$ and $\operatorname{dom}\varphi = \mathbb{R}^2 \times \{0\} \times \mathbb{R}$. Since $\operatorname{int}(\operatorname{gph} G) = \emptyset$, the regularity condition $\operatorname{int}(\operatorname{gph} G) \cap \operatorname{dom}\varphi \neq \emptyset$ fails to hold. Obviously, φ is discontinuous at any point $(x^0,y^0) \in \operatorname{gph} G$. Meanwhile, $\operatorname{dom}\varphi - \operatorname{gph} G = X \times Y$, so (3.8) is satisfied. It is easy to see that

$$\mu(x) = \inf\{\varphi_0(y) \mid y \in G(x)\} = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{if } x \neq 0. \end{cases}$$

A simple calculation shows that $\partial\mu(\bar{x}) = \mathbb{R}^2$ and $\partial\varphi(\bar{x},\bar{y}) = \{0_{\mathbb{R}^2}\} \times \mathbb{R} \times \ \{0\}$.

For any $y^* = (y_1^*, 0) \in \mathbb{R} \times \{0\}$, we have

$$\begin{aligned} D^*G(\bar{x}, \bar{y})(y^*) &= \{x^* \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{gph} G)\} \\ &= \begin{cases} \mathbb{R}^2 & \text{if } y_1^* = 0, \\ \emptyset & \text{if } y_1^* \neq 0. \end{cases} \end{aligned}$$

Hence equality (3.11) is valid. So, the condition (3.8) is really weaker than both regularity assumptions (a) and (b) in Theorem 2.1

The next example shows the necessity of the regularity condition (3.8) for Theorem 3.4.

Example 3.5 For $X = Y = \mathbb{R}$ and $\bar{x} = 0$, consider the optimal value function $\mu(x)$ in (3.2) with

$$\varphi(x, y) = \begin{cases} +\infty & \text{if } y < 0, \\ -\sqrt{y} & \text{if } y \geq 0, \end{cases}$$

and

$$G(x) = \begin{cases} \{0\} & \text{if } x = 0, \\ \emptyset & \text{if } x \neq 0. \end{cases}$$

Since $\operatorname{dom} \varphi = \mathbb{R} \times [0, +\infty)$ and $\operatorname{gph} G = \{(0, 0)\}$, (3.8) fails to hold. Meanwhile,

$$\mu(x) = \inf \{\varphi(x, y) \mid y \in G(x)\} = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{if } x \neq 0. \end{cases}$$

Hence $\partial\mu(\bar{x}) = \mathbb{R}$. As $M(\bar{x}) = \{0\}$ and $\partial\varphi(0, 0) = \emptyset$, the set on the right-hand-side of (3.11) is empty. Hence equality (3.11) does not hold.

Next, we will show that, under a mild assumption, condition (a) in Theorem 2.1 is equivalent to (3.8).

Proposition 3.1 *If the assumption*

$$\operatorname{int}(\operatorname{gph} G) \neq \emptyset \tag{3.22}$$

is fulfilled, then the regularity condition (a) *in Theorem 2.1 is equivalent to Aubin's regularity condition* (3.8).

Proof. If the condition $\mathrm{int}(\mathrm{gph}\, G) \cap \mathrm{dom}\, \varphi \neq \emptyset$ is satisfied, then there exist $(x_0, y_0) \in \mathrm{gph}\, G$ with $(x_0, y_0) \in \mathrm{dom}\, \varphi$ and open sets $U \in \mathcal{N}(0_X)$, $V \in \mathcal{N}(0_Y)$ such that

$$(x_0 + U) \times (y_0 + V) \subset \mathrm{gph}\, G.$$

It follows that

$$\begin{aligned} (0,0) \in (-U) \times (-V) &= (x_0, y_0) - [(x_0 + U) \times (y_0 + V)] \\ &\subset \mathrm{dom}\, \varphi - \mathrm{gph}\, G. \end{aligned} \tag{3.23}$$

Since $(-U) \times (-V)$ is an open subset of $X \times Y$, the first inclusion in (3.23) implies that $(-U) \times (-V) \in \mathcal{N}((0,0))$. So (3.23) assures that $(0,0) \in \mathrm{int}(\mathrm{dom}\, \varphi - \mathrm{gph}\, G)$; thus (3.8) is valid.

Conversely, suppose that (3.8) is satisfied. If condition (a) in Theorem 2.1 is violated, then $\mathrm{int}(\mathrm{gph}\, G) \cap \mathrm{dom}\, \varphi = \emptyset$. As $\mathrm{int}(\mathrm{gph}\, G) \neq \emptyset$, by the separation theorem [40, Theorem 3.4(a)], we can find a functional $(x^*, y^*) \in X^* \times Y^* \setminus \{(0,0)\}$ separating the convex sets $\mathrm{gph}\, G$ and $\mathrm{dom}\, \varphi$, i.e.,

$$\langle (x^*, y^*), (u, v) \rangle \geq \langle (x^*, y^*), (x, y) \rangle, \ \forall (u,v) \in \mathrm{dom}\, \varphi, \ \forall (x,y) \in \mathrm{gph}\, G. \tag{3.24}$$

On one hand, the assumption $(0,0) \in \mathrm{int}(\mathrm{dom}\, \varphi - \mathrm{gph}\, G)$ implies that there exists a neighborhood $U \times V$ of $(0,0)$ in $X \times Y$ such that $U \times V \subset \mathrm{dom}\, \varphi - \mathrm{gph}\, G$. On the other hand, by (3.24) we have

$$\langle (x^*, y^*), z_1 - z_2 \rangle \geq 0, \quad \forall z_1 \in \mathrm{dom}\, \varphi, \ \forall z_2 \in \mathrm{gph}\, G.$$

Therefore $\langle (x^*, y^*), (\tilde{u}, \tilde{v}) \rangle \geq 0$ for every $(\tilde{u}, \tilde{v}) \in U \times V$. The latter yields $(x^*, y^*) = (0,0)$. We have arrived a contradiction.

The proof is complete. □

In connection with Proposition 3.1, a natural question arises: *Is there any condition guaranteeing the equivalence between condition* (b) *in Theorem 2.1 and condition* (3.8)*, or not?*

By invoking a basic result on the continuity of proper, closed, convex functions on the Banach spaces, we are going to give an answer to the above question.

Proposition 3.2 *If the assumption*

$$\mathrm{int}(\mathrm{dom}\, \varphi) \neq \emptyset \tag{3.25}$$

is satisfied, then the regularity condition (b) *in Theorem 2.1 and the condition* (3.8) *are equivalent.*

Proof. First, let us prove that condition (b) in Theorem 2.1 implies (3.8) without using (3.25). If φ is continuous at a point $(x_0, y_0) \in \operatorname{gph} G$ then, for any $\varepsilon > 0$, there exist $U \in \mathcal{N}(0_X)$ and $V \in \mathcal{N}(0_Y)$ such that

$$|\varphi(x_0 + u, y_0 + v) - \varphi(x_0, y_0)| < \varepsilon, \quad \forall u \in U, \ \forall v \in V.$$

In particular, $(x_0 + u, y_0 + v) \in \operatorname{dom} \varphi$ for all $(u, v) \in U \times V$. Then we get

$$\begin{aligned}(0, 0) \in U \times V &= \{(x_0 + u, y_0 + v) - (x_0, y_0) \mid (u, v) \in U \times V\} \\ &\subset \operatorname{dom} \varphi - \operatorname{gph} G,\end{aligned}$$

which establishes (3.8).

Now, suppose that (3.8) and (3.25) are satisfied. We need to show that φ is continuous at a point in $\operatorname{gph} G$. Since φ is proper, closed, convex function on the Banach space $X \times Y$, by [11, Proposition 2.111], we see that φ is continuous on $\operatorname{int}(\operatorname{dom} \varphi)$. Therefore, to order to prove that φ is continuous at a point in $\operatorname{gph} G$, it suffices to show that

$$\operatorname{int}(\operatorname{dom} \varphi) \cap \operatorname{gph} G \neq \emptyset. \tag{3.26}$$

If (3.26) was false, then by the separation theorem [40, Theorem 3.4(a)] we would find $(x^*, y^*) \in X^* \times Y^* \setminus \{(0, 0)\}$ such that

$$\langle (x^*, y^*), (u, v) \rangle \geq \langle (x^*, y^*), (x, y) \rangle, \ \forall (u, v) \in \operatorname{dom} \varphi, \ \forall (x, y) \in \operatorname{gph} G. \tag{3.27}$$

Due to (3.8), there exists an open neighborhood $U \times V$ of $(0, 0) \in X \times Y$ such that $U \times V \subset \operatorname{dom} \varphi - \operatorname{gph} G$. Then, from (3.27) we can deduce that $\langle (x^*, y^*), (u', v') \rangle \geq 0$, for any $(u', v') \in U \times V$. This gives $(x^*, y^*) = (0, 0)$, a contradiction. Thus (3.26) is valid.

The proof is complete. □

We conclude this section with two sufficient conditions for having (3.22).

Proposition 3.3 *Let $K \subset Y$ be a convex cone with nonempty interior, and $g : X \to Y$ a continuous function at a point $x_0 \in X$. Then* (3.22) *holds for the multifunction $G : X \rightrightarrows Y$ defined by $G(x) = g(x) + K$.*

Proof. Take any $x^0 \in X$ and $v^0 \in \operatorname{int} K$. If we can prove that

$$(x^0, g(x^0) + v^0) \in \operatorname{int}(\operatorname{gph} G),$$

then (3.22) is valid. Since $v^0 \in \operatorname{int} K$, there exists an open set $V_0 \in \mathcal{N}(0_Y)$ such that $v^0 + V_0 \subset K$. Define $h : X \times Y \to Y$ by setting

$$h(x, v) = g(x^0) - g(x) + v.$$

By the continuity of g at x^0, h is continuous at $(x^0, 0)$. As $h(x^0, 0_Y) = 0_Y \in V_0$, we can find open sets $U \in \mathcal{N}(x^0)$ and $V_1 \in \mathcal{N}(0_Y)$ such that $h(U \times V_1) \subset V_0$. Then, for every $(x, v) \in U \times V_1$, it holds that $g(x^0) - g(x) + v \in V_0$. The latter yields

$$v^0 + g(x^0) - g(x) + v \in v^0 + V_0 \subset K;$$

hence

$$v^0 + g(x^0) + v \in g(x) + K = G(x).$$

It follows that

$$(x, g(x^0) + v^0 + v) \in \operatorname{gph} G, \ \ \forall x \in U, \forall v \in V_1.$$

So, setting $W = U \times (g(x^0) + v^0 + V_1)$, we see that $W \subset \operatorname{gph} G$, and W is a neighborhood of $(x^0, g(x^0) + v^0)$. This completes the proof. □

As usual, if K is a convex cone in a vector space Z, then one writes $a \leq_K b$ if and only if $a - b \in -K$.

Proposition 3.4 *Let X, Y, Z be Banach spaces and K a convex cone in Z with nonempty interior. Suppose $\psi : X \times Y \to Z$ is a K-convex function, i.e.,*

$$\psi((1-t)(x, y) + t(\tilde{x}, \tilde{y})) \leq_K (1-t)\psi(x, y) + t\psi(\tilde{x}, \tilde{y}),$$

for all $(x, y), (\tilde{x}, \tilde{y}) \in X \times Y$ and for all $t \in [0, 1]$. Let $G : X \rightrightarrows Y$ be defined by

$$G(x) = \{y \in Y \mid \psi(x, y) \leq_K 0\}.$$

If there exists $(u^0, v^0) \in X \times Y$ such that $\psi(u^0, v^0) \in -\mathrm{int} K$ and ψ is continuous at (u^0, v^0), then $\mathrm{int}(\operatorname{gph} G) \neq \emptyset$.

Proof. By our assumptions, there exist open sets U and V with $u_0 \in U$, $v_0 \in V$ such that $\psi(U \times V) \subset -\mathrm{int}\, K \subset -K$. Then $U \times V \subset \mathrm{int}(\operatorname{gph} G)$. □

3.3 Conclusions

We have obtained formulas for computing the subdifferential and the singular subdifferential of the optimal value function of parametric convex programs under three assumptions: the objective function is closed, the constraint multifunction is of closed graph, and Aubin's regularity condition is satisfied. Relationships between various regularity conditions have been analyzed.

Theorems 3.4 and 3.5, Example 3.4, Propositions 3.1 and 3.2 are the main results of this chapter. On one hand, these results complement our results in Chapter 2. On the other hand, they led us to get a better insight into differential stability of parametric convex optimization problems. In combination with Chapter 2, this chapter gives our stability of differential properties of the optimal value function a complete form.

Chapter 4

Subdifferential Formulas Based on Multiplier Sets

This chapter discusses the connection between the subdifferentials of the optimal value function of parametric convex mathematical programming problems under geometrical and/or functional constraints and certain multiplier sets. Optimality conditions for convex optimization problems under inclusion constraints and functional constraints are formulated too.

This chapter is written on the basis of [5].

4.1 Optimality Conditions for Convex Optimization

Optimality conditions for convex optimization problems, which can be derived from the calculus rules of convex analysis, have been presented in many books and research papers. To make our presentation self-contained and easy for reading, we are going to present systematically some optimality conditions for convex programs under inclusion constraints and for convex optimization problems under geometrical and functional constraints. These conditions lead to certain Lagrange multiplier sets which are used in our subsequent differential stability analysis of parametric convex programs. Observe that Theorems 4.1 - 4.3 below are consequences of [22, Proposition 1, p. 81] and the Moreau-Rockafellar Theorem (see Theorem 1.1).

Let X and Y be Hausdorff locally convex topological vector spaces. Given a convex function $\varphi : X \times Y \to \overline{\mathbb{R}}$, we denote by $\partial_x \varphi(\bar{x}, \bar{y})$ (resp., $\partial_y \varphi(\bar{x}, \bar{y})$) its *partial subdifferential* in the first variable (resp., in the second variable) at $(\bar{x}, \bar{y})$. Thus, $\partial_x \varphi(\bar{x}, \bar{y}) = \partial \varphi(., \bar{y})(\bar{x})$ and $\partial_y \varphi(\bar{x}, \bar{y}) = \partial \varphi(\bar{x}, .)(\bar{y})$, provided that the expressions on the right-hand-sides are well defined. One has

$$\partial\varphi(\bar{x},\bar{y}) \subset \partial_x\varphi(\bar{x},\bar{y}) \times \partial_y\varphi(\bar{x},\bar{y}). \tag{4.1}$$

Indeed, for any $(x^*, y^*) \in \partial\varphi(\bar{x},\bar{y})$, the inequality

$$\langle (x^*, y^*), (x, y) - (\bar{x},\bar{y})\rangle \le \varphi(x,y) - \varphi(\bar{x},\bar{y})$$

is valid for all $(x, y) \in X \times Y$. Hence, taking $y = \bar{y}$ one obtains the inequality $\langle x^*, x - \bar{x}\rangle \le \varphi(x,\bar{y}) - \varphi(\bar{x},\bar{y})$ for every $x \in X$. So, $x^* \in \partial_x\varphi(\bar{x},\bar{y})$. Similarly, one has $y^* \in \partial_y\varphi(\bar{x},\bar{y})$. Thus, property (4.1) has been proved.

Let us show that the inclusion in (4.1) can be strict.

Example 4.1 Let $X = Y = \mathbb{R}$, $\varphi(x, y) = |x + y|$, and $\bar{x} = \bar{y} = 0$. As it has been shown in Example 2.3,

$$\partial\varphi(\bar{x},\bar{y}) = \mathrm{co}\left\{(1,1)^T,\ (-1,-1)^T\right\}.$$

By a simple computation, we have

$$\partial_x\varphi(\bar{x},\bar{y}) = \partial_y\varphi(\bar{x},\bar{y}) = [-1, 1].$$

So, $\partial_x\varphi(\bar{x},\bar{y}) \times \partial_y\varphi(\bar{x},\bar{y}) \not\subset \partial\varphi(\bar{x},\bar{y})$.

4.1.1 Problems under Inclusion Constraints

Let $\varphi : X \times Y \to \overline{\mathbb{R}}$ be a proper convex function, $G : X \rightrightarrows Y$ a convex multifunction between Hausdorff locally convex topological vector spaces. Consider the *parametric optimization problem under an inclusion constraint*

$$(P_x) \qquad\qquad \min\{\varphi(x,y) \mid y \in G(x)\}$$

depending on the parameter x. The *optimal value function* $\mu : X \to \overline{\mathbb{R}}$ of problem (P_x) is

$$\mu(x) := \inf\{\varphi(x,y) \mid y \in G(x)\}.$$

The usual convention $\inf\emptyset = +\infty$ forces $\mu(x) = +\infty$ for every $x \notin \mathrm{dom}\, G$. The *solution map* $M : \mathrm{dom}\, G \rightrightarrows Y$ of that problem is defined by

$$M(x) := \{y \in G(x) \mid \mu(x) = \varphi(x,y)\}.$$

The next theorems describe some necessary and sufficient optimality conditions for (P_x) at a given parameter $\bar{x} \in X$.

Theorem 4.1 *Let $\bar{x} \in X$. Suppose that at least one of the following regularity conditions is satisfied:*

(a) $\operatorname{int} G(\bar{x}) \cap \operatorname{dom} \varphi(\bar{x}, .) \neq \emptyset$,

(b) $\varphi(\bar{x}, .)$ *is continuous at a point belonging to* $G(\bar{x})$.

Then, one has $\bar{y} \in M(\bar{x})$ if and only if

$$0 \in \partial_y \varphi(\bar{x}, \bar{y}) + N(\bar{y}; G(\bar{x})). \tag{4.2}$$

Proof. Consider the function $\mathcal{L}(y) = \varphi(\bar{x}, y) + \delta(y; G(\bar{x}))$, where $\delta(\cdot; G(\bar{x}))$ is the indicator function of the convex set $G(\bar{x})$. The latter means that $\delta(y; G(\bar{x})) = 0$ for $y \in G(\bar{x})$ and $\delta(y; G(\bar{x})) = +\infty$ for $y \notin G(\bar{x})$. It is clear that $\bar{y} \in M(\bar{x})$ if and only if the function $\mathcal{L}$ attains its minimum at $\bar{y}$. Hence, by [22, Proposition 1, p. 81], $\bar{y} \in M(\bar{x})$ if and only if

$$0 \in \partial \mathcal{L}(\bar{y}) = \partial \Big(\varphi(\bar{x}, .) + \delta(.; G(\bar{x})) \Big)(\bar{y}). \tag{4.3}$$

Since $G(\bar{x})$ is convex, $\delta(\cdot; G(\bar{x}))$ is convex. Clearly, $\delta(\cdot; G(\bar{x}))$ is continuous at every point belonging to $\operatorname{int} G(\bar{x})$. Thus, if the condition (a) is fulfilled, then $\delta(\cdot; G(\bar{x}))$ is continuous at a point in $\operatorname{dom} \varphi(\bar{x}, .)$. By Theorem 1.1, from (4.3) one has

$$\begin{aligned} 0 \in \partial \Big(\varphi(\bar{x}, .) + \delta(.; G(\bar{x})) \Big)(\bar{y}) &= \partial_y \varphi(\bar{x}, \bar{y}) + \partial \delta(\bar{y}; G(\bar{x})) \\ &= \partial_y \varphi(\bar{x}, \bar{y}) + N(\bar{y}; G(\bar{x})). \end{aligned}$$

Consider the case where (b) holds. Since $\operatorname{dom} \delta(\cdot; G(\bar{x})) = G(\bar{x})$, $\varphi(\bar{x}, .)$ is continuous at a point in $\operatorname{dom} \delta(\cdot; G(\bar{x}))$. Then, by Theorem 1.1 one can obtain (4.2) from (4.3). □

The sum rule in Theorem 3.3 allows us to get the following result.

Theorem 4.2 *Let X, Y be Banach spaces, $\varphi : X \times Y \to \overline{\mathbb{R}}$ a proper, closed, convex function. Suppose that $G : X \rightrightarrows Y$ is a convex multifunction, whose graph is closed. Let $\bar{x} \in X$ be such that the regularity condition*

$$0 \in \operatorname{int} \big(\operatorname{dom} \varphi(\bar{x}, .) - G(\bar{x}) \big) \tag{4.4}$$

is satisfied. Then, $\bar{y} \in M(\bar{x})$ if and only if

$$0 \in \partial_y \varphi(\bar{x}, \bar{y}) + N(\bar{y}; G(\bar{x})).$$

Proof. The proof is similar to that of Theorem 4.1. Namely, if condition (4.4) is fulfilled, then we can apply Theorem 3.3 instead of Theorem 1.1 to the case where $X \times Y$, $\varphi(\bar{x}, .)$, and $\delta(\cdot; G(\bar{x}))$, respectively, play the roles of X, f, and g. □

4.1.2 Problems under Geometrical and Functional Constraints

We now study optimality conditions for *convex optimization problems under geometrical and functional constraints.* Consider the program

$$(\widetilde{P}_x) \qquad \min\{\varphi(x,y) \mid (x,y) \in C,\ g_i(x,y) \le 0,\ i \in I,\ h_j(x,y) = 0,\ j \in J\}$$

depending on the parameter x, where $C \subset X \times Y$ is a convex set, $g_i : X \times Y \to \mathbb{R}$ $(i \in I)$, with $I := \{1, \ldots, m\}$, are continuous convex functions, $h_j : X \times Y \to \mathbb{R}$ $(j \in J)$, with $J := \{1, \ldots, k\}$, are continuous affine functions. For each $x \in X$, we put

$$G(x) = \{y \in Y \mid (x,y) \in C,\ g(x,y) \le 0,\ h(x,y) = 0\}, \tag{4.5}$$

where

$$g(x,y) := (g_1(x,y), \ldots, g_m(x,y))^T, \quad h(x,y) := (h_1(x,y), \ldots, h_k(x,y))^T,$$

with ^{T} denoting matrix transposition, and the inequality $z \le w$ between two vectors in $\mathbb{R}^m$ means that every coordinate of z is less than or equal to the corresponding coordinate of w. It is easy to show that the multifunction $G(\cdot)$ given by (4.5) is convex. Fix a point $\bar{x} \in X$ and put

$$C_{\bar{x}} := \{y \in Y \mid (\bar{x}, y) \in C\}. \tag{4.6}$$

Optimality conditions for convex optimization problems under geometrical and functional constraints can be formulated as follows.

Theorem 4.3 *If $\varphi(\bar{x}, .)$ is continuous at a point $y^0 \in \operatorname{int} C_{\bar{x}}$, $g_i(\bar{x}, y^0) < 0$ for all $i \in I$ and $h_j(\bar{x}, y^0) = 0$ for all $j \in J$, then for a point $\bar{y} \in G(\bar{x})$ to be a solution of $(\widetilde{P}_{\bar{x}})$, it is necessary and sufficient that there exist $\lambda_i \ge 0$, $i \in I$, and $\mu_j \in \mathbb{R}$, $j \in J$, such that*
(a) $0 \in \partial_y \varphi(\bar{x}, \bar{y}) + \sum_{i \in I} \lambda_i \partial_y g_i(\bar{x}, \bar{y}) + \sum_{j \in J} \mu_j \partial_y h_j(\bar{x}, \bar{y}) + N(\bar{y}; C_{\bar{x}})$;
(b) $\lambda_i g_i(\bar{x}, \bar{y}) = 0,\ i \in I$.

Proof. For any $\bar{x} \in X$, let $\bar{y} \in G(\bar{x})$ be given arbitrarily. Note that $(\widetilde{P}_{\bar{x}})$ can be written in the form

$$\min\{\varphi(\bar{x}, y) \mid y \in G(\bar{x})\}.$$

If $\varphi(\bar{x}, .)$ is continuous at a point y^0 with $y^0 \in \operatorname{int} C_{\bar{x}}$, $g_i(\bar{x}, y^0) < 0$ for all $i \in I$, and $h_j(\bar{x}, y^0) = 0$ for all $j \in J$, then the regularity condition (b) in Theorem 4.1 is satisfied. Consequently, $\bar{y} \in M(\bar{x})$ if and only if

$$0 \in \partial_y \varphi(\bar{x}, \bar{y}) + N(\bar{y}; G(\bar{x})).$$

We now show that

$$N(\bar{y}; G(\bar{x})) = \left\{ \sum_{i\in I(\bar{x},\bar{y})} \lambda_i \partial_y g_i(\bar{x},\bar{y}) + \sum_{j\in J} \mu_j \partial_y h_j(\bar{x},\bar{y}) + N(\bar{y}; C_{\bar{x}}) \right\}, \tag{4.7}$$

with $I(\bar{x},\bar{y}) := \{i \mid g_i(\bar{x},\bar{y}) = 0,\ i \in I\}$, $\lambda_i \geq 0$, $i \in I$, $\mu_j \in \mathbb{R}$, $j \in J$. First, observe that

$$G(\bar{x}) = \left(\bigcap_{i\in I} \Omega_i(\bar{x}) \right) \cap \left(\bigcap_{j\in J} \mathcal{Q}_j(\bar{x}) \right) \cap C, \tag{4.8}$$

where $\Omega_i(\bar{x}) = \{y \mid g_i(\bar{x}, y) \leq 0\}(i \in I)$ and $\mathcal{Q}_j(\bar{x}) = \{y \mid h_j(\bar{x}, y) = 0\}(j \in J)$ are convex sets. By our assumptions, we have

$$y^0 \in \left(\bigcap_{i\in I} \operatorname{int} \Omega_i(\bar{x}) \right) \cap \left(\bigcap_{j\in J} \mathcal{Q}_j(\bar{x}) \right) \cap (\operatorname{int} C).$$

Therefore, according to Proposition 1.4 and formula (4.8), one has

$$N(\bar{y}; G(\bar{x})) = \sum_{i\in I} N(\bar{y}; \Omega_i(\bar{x})) + N\left(\bar{y}; \bigcap_{j\in J} \mathcal{Q}_j(\bar{x}) \right) + N(\bar{y}; C_{\bar{x}}). \tag{4.9}$$

On one hand, by Lemma 2.3, for every $i \in I(\bar{x}, \bar{y})$ we have

$$N(\bar{y}; \Omega_i(\bar{x})) = K_{\partial_y g_i(\bar{x},\bar{y})} = \{\lambda_i y^* \mid \lambda_i \geq 0,\ y^* \in \partial_y g_i(\bar{x}, \bar{y})\}. \tag{4.10}$$

On the other hand, according to Lemma 2.2 and the fact that

$$h_j(x, y) = \langle x_j^*, x\rangle + \langle y_j^*, y\rangle - \alpha_j \quad \big((x_j^*, y_j^*) \in X^* \times Y^*,\ \alpha_j \in \mathbb{R}\big),$$

we can assert that

$$N\left(\bar{y}; \bigcap_{j\in J} \mathcal{Q}_j(\bar{x}) \right) = \operatorname{span}\{y_j^* \mid j \in J\} = \operatorname{span}\{\partial_y h_j(\bar{x}, \bar{y}) \mid j \in J\}, \tag{4.11}$$

Combining (4.9), (4.10), and (4.11), we obtain (4.7). So, the assertion of the theorem is valid. □

4.2 Subdifferential Estimates via Multiplier Sets

Our aim in this section is to derive formulas for computing or estimating the subdifferential of the optimal value function of $(\tilde{P}_x)$ through suitable multiplier sets.

The *Lagrangian function* corresponding to the parametric problem $(\widetilde{P}_x)$ is

$$L(x, y, \lambda, \mu) := \varphi(x, y) + \lambda^T g(x, y) + \mu^T h(x, y) + \delta((x, y); C), \tag{4.12}$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbb{R}^m$ and $\mu = (\mu_1, \mu_2, ..., \mu_k) \in \mathbb{R}^k$. For each pair $(x, y) \in X \times Y$, by $\Lambda_0(x, y)$ we denote the set of all the multipliers $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^k$ with $\lambda_i \geq 0$ for all $i \in I$ and $\lambda_i = 0$ for every $i \in I \setminus I(x, y)$, where $I(x, y) = \{i \in I \mid g_i(x, y) = 0\}$.

For a parameter $\bar{x}$, the *Lagrangian function* corresponding to the unperturbed problem $(\widetilde{P}_{\bar{x}})$ is

$$L(\bar{x}, y, \lambda, \mu) = \varphi(\bar{x}, y) + \lambda^T g(\bar{x}, y) + \mu^T h(\bar{x}, y) + \delta((\bar{x}, y); C). \tag{4.13}$$

Denote by $\Lambda(\bar{x}, \bar{y})$ the *Lagrange multiplier set* corresponding to an optimal solution $\bar{y}$ of problem $(\widetilde{P}_{\bar{x}})$. Thus, $\Lambda(\bar{x}, \bar{y})$ consists of the pairs $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^k$ satisfying

$$\begin{cases} 0 \in \partial_y L(\bar{x}, \bar{y}, \lambda, \mu), \\ \lambda_i g_i(\bar{x}, \bar{y}) = 0, \ i = 1, \ldots, m, \\ \lambda_i \geq 0, \ i = 1, \ldots, m, \end{cases}$$

where $\partial_y L(\bar{x}, \bar{y}, \lambda, \mu)$ is the subdifferential of the function $L(\bar{x}, ., \lambda, \mu)$ defined by (4.13) at $\bar{y}$. It is clear that $\delta((\bar{x}, y); C) = \delta(y; C_{\bar{x}})$, where $C_{\bar{x}}$ has been defined by (4.6).

Based on the multiplier set $\Lambda_0(x, y)$, the next theorem provides us with a formula for computing the subdifferential of the optimal value function $\mu(x)$.

Theorem 4.4 *Suppose that $h_j(x, y) = \langle (x_j^*, y_j^*), (x, y) \rangle - \alpha_j, \ \alpha_j \in \mathbb{R}, \ j \in J$, and $M(\bar{x})$ is nonempty for some $\bar{x} \in \operatorname{dom} \mu$. If φ is continuous at a point $(x^0, y^0) \in \operatorname{int} C$, $g_i(x^0, y^0) < 0$ for all $i \in I$ and $h_j(x^0, y^0) = 0$ for all $j \in J$ then, for any $\bar{y} \in M(\bar{x})$, one has*

$$\partial \mu(\bar{x}) = \left\{ \bigcup_{(\lambda, \mu) \in \Lambda_0(\bar{x}, \bar{y})} \operatorname{pr}_{X^*} \Big(\partial L(\bar{x}, \bar{y}, \lambda, \mu) \cap \big(X^* \times \{0\} \big) \Big) \right\}, \tag{4.14}$$

where $\partial L(\bar{x}, \bar{y}, \lambda, \mu)$ is the subdifferential of the function $L(., ., \lambda, \mu)$ at $(\bar{x}, \bar{y})$ and, for any $(x^*, y^*) \in X^* \times Y^*$, $\operatorname{pr}_{X^*}(x^*, y^*) := x^*$.

Proof. To prove the inclusion "$\subset$" in (4.14), take any $\bar{x}^* \in \partial \mu(\bar{x})$. By Theorem 2.3, there are $(x^*, y^*) \in \partial \varphi(\bar{x}, \bar{y})$ and $u^* \in \tilde{Q}^*$ with $\bar{x}^* = x^* + u^*$. According to (2.34), the condition $u^* \in \tilde{Q}^*$ means that

$$(u^*, -y^*) \in N((\bar{x}, \bar{y}); C) + A, \tag{4.15}$$

where A is given by (2.35). Adding the inclusion $(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})$ and that one in (4.15) yields

$$(x^* + u^*, 0) \in (x^*, y^*) + A + N((\bar{x}, \bar{y}); C).$$

Hence,

$$(\bar{x}^*, 0) \in \partial\varphi(\bar{x}, \bar{y}) + A + N((\bar{x}, \bar{y}); C). \tag{4.16}$$

For every $(\lambda, \mu) \in \Lambda_0(\bar{x}, \bar{y})$, the assumptions made on the functions φ, g_i, h_j, and the set C allow us to apply the Moreau–Rockafellar Theorem (see Theorem 1.1) to the Lagrangian function $L(x, y, \lambda, \mu)$ defined by (4.12) to get

$$\begin{aligned} \partial L(\bar{x}, \bar{y}, \lambda, \mu) = \partial\varphi(\bar{x}, \bar{y}) + \sum_{i \in I(\bar{x}, \bar{y})} \lambda_i \partial g_i(\bar{x}, \bar{y}) + \sum_{j \in J} \mu_j \partial h_j(\bar{x}, \bar{y}) \\ + N((\bar{x}, \bar{y}); C). \end{aligned} \tag{4.17}$$

Since $\partial h_j(\bar{x}, \bar{y}) = \{(x_j^*, y_j^*)\}$, from (4.17) it follows that

$$\partial\varphi(\bar{x}, \bar{y}) + A + N((\bar{x}, \bar{y}); C) = \bigcup_{(\lambda, \mu) \in \Lambda_0(\bar{x}, \bar{y})} \partial L(\bar{x}, \bar{y}, \lambda, \mu). \tag{4.18}$$

So, (4.16) means that

$$\bar{x}^* \in \bigcup_{(\lambda, \mu) \in \Lambda_0(\bar{x}, \bar{y})} \mathrm{pr}_{X^*}\Big(\partial L(\bar{x}, \bar{y}, \lambda, \mu) \cap \big(X^* \times \{0\}\big)\Big). \tag{4.19}$$

Thus, the inclusion "$\subset$" in (4.14) is valid. To obtain the reverse inclusion, fixing any $\bar{x}^*$ satisfying (4.19) we have to show that $\bar{x}^* \in \partial\mu(\bar{x})$. As it has been noted before, (4.19) is equivalent to (4.16). Select a pair $(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})$ satisfying

$$(\bar{x}^*, 0) \in (x^*, y^*) + A + N((\bar{x}, \bar{y}); C).$$

Then, for $u^* := \bar{x}^* - x^*$, one has

$$(x^* + u^*, 0) \in (x^*, y^*) + A + N((\bar{x}, \bar{y}); C).$$

Therefore, the inclusion (4.15) holds. Hence, thanks to (2.34) and (2.32), the vector $\bar{x}^* = x^* + u^*$ belongs to $\partial\mu(\bar{x})$. The proof is complete. $\square$

As an illustration for Theorem 4.4, let us review Example 2.3 of Chapter 2.

Example 4.2 Let $X = Y = \mathbb{R}$, $C = X \times Y$, $\varphi(x, y) = |x + y|$, $m = 1$, $k = 0$ (no equality functional constraint), $g_1(x, y) = y$ for all $(x, y) \in X \times Y$.

Choosing $\bar{x} = 0$, we note that $M(\bar{x}) = \{\bar{y}\}$, with $\bar{y} = 0$. We have $\Lambda_0(\bar{x}, \bar{y}) = [0, \infty)$ and $L(x, y, \lambda) = \varphi(x, y) + \lambda y$. As it has been shown in Example 2.3,

$$\partial\varphi(\bar{x}, \bar{y}) = \text{co}\left\{(1,1)^T,\, (-1,-1)^T\right\}.$$

Since $\partial L(\bar{x}, \bar{y}, \lambda) = \partial\varphi(\bar{x}, \bar{y}) + \{(0, \lambda)\}$, by (4.14) we can compute

$$\begin{aligned}
\partial\mu(\bar{x}) &= \left\{ \bigcup_{\lambda\in\Lambda_0(\bar{x},\bar{y})} \text{pr}_{X^*}\Big(\partial L(\bar{x}, \bar{y}, \lambda) \cap \big(X^* \times \{0\}\big)\Big) \right\} \\
&= \text{pr}_{X^*}\left[\Big(\bigcup_{\lambda\in\Lambda_0(\bar{x},\bar{y})} \partial L(\bar{x}, \bar{y}, \lambda) \Big) \cap \big(X^* \times \{0\}\big) \right] \\
&= \text{pr}_{X^*}\Big\{ \Big[\text{co}\left\{(1,1)^T,\, (-1,-1)^T\right\} + \Big(\{0\} \times \mathbb{R}_+\Big)\Big] \cap \big(X^* \times \{0\}\big) \Big\} \\
&= [-1, 0].
\end{aligned}$$

Thus, we also obtain the equality $\partial\mu(\bar{x}) = [-1, 0]$, which justifies the validity of (4.14) for the problem under consideration.

We have seen that Theorem 4.4 is an effective tool to compute the subdifferential $\partial\mu(\bar{x})$ via the multiplier set $\Lambda_0(\bar{x}, \bar{y})$. It is worthy to stress that the formulas for computing the subdifferential of μ in Theorems 2.2 and 2.3 are based on the subdifferential of the objective function and the coderivative of the constraint set mapping.

We are now in a position to establish an upper estimate for the subdifferential $\mu(.)$ at $\bar{x}$ by using the Lagrange multiplier set $\Lambda(\bar{x}, \bar{y})$ corresponding to a solution $\bar{y}$ of $(\widetilde{P}_{\bar{x}})$.

Theorem 4.5 *Under the assumptions of Theorem 4.4, one has*

$$\partial\mu(\bar{x}) \subset \bigcup_{(\lambda,\mu)\in\Lambda(\bar{x},\bar{y})} \partial_x L(\bar{x}, \bar{y}, \lambda, \mu), \tag{4.20}$$

where $\partial_x L(\bar{x}, \bar{y}, \lambda, \mu)$ *stands for the subdifferential of* $L(., \bar{y}, \lambda, \mu)$ *at* $\bar{x}$.

Proof. Fix an arbitrary vector $\bar{x}^* \in \partial\,\mu(\bar{x})$. The arguments in the first part of the proof of Theorem 4.4 show that (4.16) and (4.18) are valid. Hence, we can find a vector $(\lambda, \mu) \in \Lambda_0(\bar{x}, \bar{y})$ such that

$$(\bar{x}^*, 0) \in \partial L(\bar{x}, \bar{y}, \lambda, \mu). \tag{4.21}$$

Using the definition of the subdifferential, from (4.21) we can deduce that

$$\langle \bar{x}^*, x - \bar{x}\rangle \le L(x, \bar{y}, \lambda, \mu) - L(\bar{x}, \bar{y}, \lambda, \mu), \quad \forall x \in X$$

and

$$\langle 0, y - \bar{y} \rangle \leq L(\bar{x}, y, \lambda, \mu) - L(\bar{x}, \bar{y}, \lambda, \mu), \quad \forall y \in Y.$$

Hence,

$$\bar{x}^* \in \partial_x L(\bar{x}, \bar{y}, \lambda, \mu), \;\; 0 \in \partial_y L(\bar{x}, \bar{y}, \lambda, \mu). \tag{4.22}$$

Since $(\lambda, \mu) \in \Lambda_0(\bar{x}, \bar{y})$, one has $\lambda_i g_i(\bar{x}, \bar{y}) = 0$ and $\lambda_i \geq 0$ for every $i \in I$. Therefore, the second inclusion in (4.22) implies that $(\lambda, \mu) \in \Lambda(\bar{x}, \bar{y})$. Then, formula (4.20) follows from the first inclusion in (4.22). $\square$

The next example shows that the inclusion in Theorem 4.5 can be strict.

Example 4.3 Let $X = Y = \mathbb{R}$, $C = X \times Y$, $\varphi(x, y) = |x + y|$, $m = 1$, $k = 0$ (no equality functional constraint), $g_1(x, y) = y$ for all $(x, y) \in X \times Y$. Choosing $\bar{x} = 0$, we note that $M(\bar{x}) = \{\bar{y}\}$, with $\bar{y} = 0$. We have $L(x, y, \lambda) = \varphi(x, y) + \lambda y$ and

$$\begin{aligned} \Lambda(\bar{x}, \bar{y}) &= \{\lambda \geq 0 \mid 0 \in \partial_y L(\bar{x}, \bar{y}, \lambda)\} \\ &= \{\lambda \geq 0 \mid 0 \in [-1, 1] + \lambda\} \\ &= [0, 1]. \end{aligned}$$

As in Example 4.2, one has $\partial \mu(\bar{x}) = [-1, 0]$. We now compute the right-hand-side of (4.20). By simple computation, we obtain $\partial_x L(\bar{x}, \bar{y}, \lambda) = [-1, 1]$ for all $\lambda \in \Lambda(\bar{x}, \bar{y})$. Then $\bigcup\limits_{\lambda \in \Lambda(\bar{x}, \bar{y})} \partial_x L(\bar{x}, \bar{y}, \lambda) = [-1, 1]$. Therefore, in this example, inclusion (4.20) is strict.

4.3 Computation of the Singular Subdifferential

First, we observe that $x \in \operatorname{dom} \mu$ if and only if

$$\mu(x) = \inf\{\varphi(x, y) \mid y \in G(x)\} < \infty,$$

with $G(x)$ being given by (4.5). Since the strict inequality holds if and only if there exists $y \in G(x)$ with $(x, y) \in \operatorname{dom} \varphi$, we have

$$\delta(x; \operatorname{dom} \mu) = \inf\{\delta((x, y); \operatorname{dom} \varphi) \mid y \in G(x)\}. \tag{4.23}$$

To compute the singular subdifferential of $\mu(.)$, let us consider the minimization problem

$$(\widehat{P}_x) \quad \begin{cases} \delta((x, y); \operatorname{dom} \varphi) \to \inf \\ \text{subject to } \; (x, y) \in C, \; g_i(x, y) \leq 0, \; i \in I, \; h_j(x, y) = 0, \; j \in J. \end{cases}$$

The Lagrangian function corresponding to $(\widehat{P}_x)$ is

$$\widehat{L}(x,y,\lambda,\mu)=\delta((x,y);\operatorname{dom}\varphi)+\lambda^T g(x,y)+\mu^T h(x,y)+\delta((x,y);C), \quad (4.24)$$

where $\lambda=(\lambda_1,\lambda_2,...,\lambda_m)\in\mathbb{R}^m$, $\mu=(\mu_1,\mu_2,...,\mu_k)\in\mathbb{R}^k$.

Interpreting $(\widehat{P}_x)$ as a problem of the form $(\widetilde{P}_x)$, where $\delta((x,y);\operatorname{dom}\varphi)$ plays the role of $\varphi(x,y)$, we can apply Theorem 4.4 (resp., Theorem 4.5) to compute (resp., estimate) the singular subdifferential of $\mu(.)$ as follows.

Theorem 4.6 *Under the hypotheses of Theorem 4.4, for any $\bar{y}\in M(\bar{x})$, one has*

$$\partial^\infty\mu(\bar{x})=\left\{\bigcup_{(\lambda,\mu)\in\Lambda_0(\bar{x},\bar{y})}\operatorname{pr}_{X^*}\Big(\partial\widehat{L}(\bar{x},\bar{y},\lambda,\mu)\cap\big(X^*\times\{0\}\big)\Big)\right\}, \quad (4.25)$$

where

$$\begin{aligned}\partial\widehat{L}(\bar{x},\bar{y},\lambda,\mu)=\partial^\infty\varphi(\bar{x},\bar{y})+\sum_{i\in I(\bar{x},\bar{y})}\lambda_i\partial g_i(\bar{x},\bar{y})+\sum_{j\in J}\mu_j\partial h_j(\bar{x},\bar{y})\\ +N((\bar{x},\bar{y});C)\end{aligned} \quad (4.26)$$

is the subdifferential of the function $\widehat{L}(.,.,\lambda,\mu)$ at $(\bar{x},\bar{y})$, provided that a pair $(\lambda,\mu)\in\Lambda_0(\bar{x},\bar{y})$ has been chosen.

Proof. The inclusion $\bar{y}\in M(\bar{x})$ implies that $(\bar{x},\bar{y})\in\operatorname{dom}\varphi$ and $\bar{y}\in G(\bar{x})$. So, $\delta((\bar{x},\bar{y});\operatorname{dom}\varphi)=0$ and $\bar{y}$ is a feasible point of problem $(\widehat{P}_{\bar{x}})$. As $\delta((\bar{x},y);\operatorname{dom}\varphi)\geq 0$ for all $y\in G(\bar{x})$, we can assert that $\bar{y}$ is a solution of $(\widehat{P}_{\bar{x}})$. The corresponding optimal value is $\delta(\bar{x};\operatorname{dom}\mu)=0$ (see (4.23)). Hence, by Theorem 4.4 and formula (4.23), we have

$$\partial\delta(\bar{x};\operatorname{dom}\mu)=\left\{\bigcup_{(\lambda,\mu)\in\Lambda_0(\bar{x},\bar{y})}\operatorname{pr}_{X^*}\Big(\partial\widehat{L}(\bar{x},\bar{y},\lambda,\mu)\cap\big(X^*\times\{0\}\big)\Big)\right\}.$$

Since $\partial\delta(\bar{x};\operatorname{dom}\mu)=\partial^\infty\mu(\bar{x})$, the last equality implies (4.25).

For every $(\lambda,\mu)\in\Lambda_0(\bar{x},\bar{y})$, remembering that h_j, $j\in J$, are affine functions, φ is continuous at a point (x^0,y^0) with $(x^0,y^0)\in\operatorname{int}C$, $g_i(x^0,y^0)<0$ for all $i\in I$ and $h_j(x^0,y^0)=0$ for all $j\in J$, we can apply Theorem 1.1 to the Lagrangian function $\widehat{L}(x,y,\lambda,\mu)$ defined by (4.24) to obtain

$$\begin{aligned}&\partial\widehat{L}(\bar{x},\bar{y},\lambda,\mu)\\ &=\partial\delta((\bar{x},\bar{y});\operatorname{dom}\varphi)+\sum_{i\in I(\bar{x},\bar{y})}\lambda_i\partial g_i(\bar{x},\bar{y})+\sum_{j\in J}\mu_j\partial h_j(\bar{x},\bar{y})+N((\bar{x},\bar{y});C).\end{aligned}$$

Combining this with the equality $\partial\delta((\bar{x},\bar{y});\operatorname{dom}\varphi)=\partial^\infty\varphi(\bar{x},\bar{y})$ yields (4.26). □

Remark 4.1 The result in Theorem 4.6 can be derived from formula (2.33) by a proof analogous to that of Theorem 4.4.

Next, denote by $\Lambda^\infty(\bar{x}, \bar{y})$ the *singular Lagrange multiplier set* corresponding to an optimal solution $\bar{y}$ of problem $(\widehat{P}_{\bar{x}})$, which consists of the pairs $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^k$ satisfying

$$\begin{cases} 0 \in \partial_y \widehat{L}(\bar{x}, \bar{y}, \lambda, \mu), \\ \lambda_i g_i(\bar{x}, \bar{y}) = 0, \ i = 1, \ldots, m, \\ \lambda_i \geq 0, \ i = 1, \ldots, m. \end{cases}$$

Here $\partial_y \widehat{L}(\bar{x}, \bar{y}, \lambda, \mu)$ is the subdifferential of the function $\widehat{L}(\bar{x}, ., \lambda, \mu)$, with $\widehat{L}(x, y, \lambda, \mu)$ being given by (4.24), at $\bar{y}$.

Theorem 4.7 *Under the assumptions of Theorem 4.4, for any $\bar{y} \in M(\bar{x})$, one has*

$$\partial^\infty \mu(\bar{x}) \subset \bigcup_{(\lambda,\mu) \in \Lambda^\infty(\bar{x},\bar{y})} \partial_x \widehat{L}(\bar{x}, \bar{y}, \lambda, \mu), \tag{4.27}$$

where $\partial_x \widehat{L}(\bar{x}, \bar{y}, \lambda, \mu)$ stands for the subdifferential of $\widehat{L}(., \bar{y}, \lambda, \mu)$ at $\bar{x}$.

Proof. To get (4.27), it suffices to apply Theorem 4.5 to the parametric problem $(\widehat{P}_{\bar{x}})$, keeping in mind that $\bar{y}$ is a solution of $(\widehat{P}_{\bar{x}})$. Indeed, taking account of Theorem 4.5 and (4.20), one has

$$\partial \delta(\bar{x}; \operatorname{dom} \mu) \subset \bigcup_{(\lambda,\mu) \in \Lambda^\infty(\bar{x},\bar{y})} \partial_x \widehat{L}(\bar{x}, \bar{y}, \lambda, \mu).$$

As $\partial \delta(\bar{x}; \operatorname{dom} \mu) = \partial^\infty \mu(\bar{x})$, this inclusion is equivalent to (4.27). □

4.4 Conclusions

In this chapter, we focus on differential stability of convex programming problems under functional constraints. The obtained results are different from those in Chapter 2. Namely, coderivative of the constraint multifunction, subdifferential, and singular subdifferential of the objective function are the main ingredients in the formulas in Chapter 2. Meanwhile, Theorems 4.4 and 4.6 present formulas for computing the subdifferentials of the optimal value function via suitable multiplier sets. Theorems 4.5 and 4.7 give upper estimates for the subdifferentials via the Lagrange multiplier sets.

Chapter 5

Stability Analysis of Convex Discrete Optimal Control Problems

Following the recent works of Chieu and Yao [15], Toan and Yao [46], in this chapter we present some new results on differential stability of convex discrete optimal control problems. Our assumptions are weaker than those in [15] and [46] applied to the convex case. In addition, instead of the finite-dimensional spaces setting in those papers, here we can use a Banach space setting. The main tools of our analysis are the formulas for computing subdifferentials of the optimal value function from Chapter 2.

Our presentation is based on [1].

5.1 Control Problem

Let X_k, U_k, W_k, for $k = 0, 1, \ldots, N-1$, and X_N be Banach spaces, where N is a positive natural number. Let there be given

- convex sets $\Omega_0 \subset U_0, \ldots, \Omega_{N-1} \subset U_{N-1}$, and $C \subset X_0$;

- continuous linear operators $A_k : X_k \to X_{k+1},\ B_k : U_k \to X_{k+1},\ T_k : W_k \to X_{k+1}$, for $k = 0, 1, \ldots, N-1$;

- functions $h_k : X_k \times U_k \times W_k \to \mathbb{R}$, for $k = 0, 1, \ldots, N-1$, and $h_N : X_N \to \mathbb{R}$, which are convex.

We are going to describe a control system where the state variable (resp., the control variable) at time k is x_k (resp., u_k), and the objective function is the sum of the functions h_k, for $k = 0, 1, \ldots, N$. We interpret X_k as the space of state variables at stage k, and U_k (resp., W_k) the space of control variables (resp., space of random parameters) at stage k.

Put $W = W_0 \times W_1 \times \cdots \times W_{N-1}$. For every vector $w = (w_0, w_1, \ldots, w_{N-1}) \in W$, consider the following *convex discrete optimal control problem*: Find a pair (x, u) where $x = (x_0, x_1, \ldots, x_N) \in X_0 \times X_1 \times \cdots \times X_N$ is a trajectory and $u = (u_0, u_1, \ldots, u_{N-1}) \in U_0 \times U_1 \times \cdots \times U_{N-1}$ is a control sequence, which minimizes the *objective function*

$$\sum_{k=0}^{N-1} h_k(x_k, u_k, w_k) + h_N(x_N) \tag{5.1}$$

and satisfies the *linear state equations*

$$x_{k+1} = A_k x_k + B_k u_k + T_k w_k, \ k = 0, 1, \ldots, N-1, \tag{5.2}$$

the *initial condition*

$$x_0 \in C, \tag{5.3}$$

and the *control constraints*

$$u_k \in \Omega_k \subset U_k, \quad k = 0, 1, \ldots, N-1. \tag{5.4}$$

A well-known classical example for problem (5.1)–(5.4) is the *inventory control problem* in economics, where x_k plays stock available at the beginning of the kth period, u_k plays stock ordered (and immediately delivered) at the beginning of the kth period, w_k is the demand during the kth period (in practice, $w_0, \ldots, w_{N-1}$ are independent random variables with a given probability distribution), and the objective function has the form

$$h(x_N) + \sum_{k=0}^{N-1} h(x_k, u_k, w_k)$$

together with the state equation $x_{k+1} = x_k + u_k - w_k$ (see [10, pp. 2–6, 13–14, 162–168] for details).

Setting $X = X_0 \times X_1 \times \cdots \times X_N$, $U = U_0 \times U_1 \times \cdots \times U_{N-1}$. For every parameter $w = (w_0, w_1, \ldots, w_{N-1}) \in W$, denote by $V(w)$ the optimal value of problem (5.1)–(5.4), and by $S(w)$ the solution set of that problem. The extended-real-valued function $V : W \to \overline{\mathbb{R}}$ is called *the optimal value function* of problem (5.1)–(5.4). It is assumed that V is finite at a certain parameter $\bar{w} = (\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{N-1}) \in W$ and $(\bar{x}, \bar{u})$ is a solution of (5.1)–(5.4), that is $(\bar{x}, \bar{u}) \in S(\bar{w})$ where $\bar{x} = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_N), \bar{u} = (\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1})$.

For each $w = (w_0, w_1, \dots, w_{N-1}) \in W$, let

$$f(x, u, w) = \sum_{k=0}^{N-1} h_k(x_k, u_k, w_k) + h_N(x_N).$$

Then, setting $\Omega = \Omega_0 \times \Omega_1 \times \cdots \times \Omega_{N-1}$, $\widetilde{X} = X_1 \times X_2 \times \cdots \times X_N$, and

$$G(w) = \{(x, u) \in X \times U \,|\, x_{k+1} = A_k x_k + B_k u_k + T_k w_k,\ k = 0, 1, \dots, N-1\},$$

we have

$$V(w) = \inf_{(x,u) \in G(w) \cap (C \times \widetilde{X} \times \Omega)} f(x, u, w).$$

5.2 Differential Stability of the Parametric Mathematical Programming Problem

By using some results from Chapter 2 on differential stability of parametric convex optimization problems under inclusion constraints, we will establish a theorem, which is the main tool for our subsequent investigations on the discrete optimal control problem.

First, we specify Theorem 2.1 in Chapter 2 for a case where $\operatorname{gph} G$ is a linear subspace of a product space. Suppose that X, W and Z are Banach spaces with the dual spaces X^*, W^* and Z^*, respectively. Assume that $M : Z \to X$ and $T : W \to X$ are continuous linear operators. Let $M^* : X^* \to Z^*$ and $T^* : X^* \to W^*$ be the adjoint operators of M and T, respectively. Let $f : W \times Z \to \overline{\mathbb{R}}$ be a convex function and Ω a convex subset of Z with nonempty interior. For each $w \in W$, put $H(w) = \big\{z \in Z \,|\, Mz = Tw\big\}$ and consider the optimization problem

$$\min\{f(z, w) \ | \ z \in H(w) \cap \Omega\}. \tag{5.5}$$

We want to compute the subdifferential and the singular subdifferential of the optimal value function

$$h(w) := \inf_{z \in H(w) \cap \Omega} f(z, w) \tag{5.6}$$

of the parametric problem (5.5). Denote by $\widehat{S}(w)$ the solution set of (5.5).

Define the linear operator $\Phi : W \times Z \to X$ by setting $\Phi(w, z) = -Tw + Mz$ for all $(w, z) \in W \times Z$.

Lemma 5.1 *For each* $(\bar{w}, \bar{z}) \in \operatorname{gph} H$, *one has*

$$N\big((\bar{w}, \bar{z}); \operatorname{gph} H\big) = \operatorname{cl}^*\big\{(-T^*x^*, M^*x^*) \,|\, x^* \in X^*\big\}. \tag{5.7}$$

Moreover, if Φ *has closed range, then*

$$N\big((\bar{w}, \bar{z}); \operatorname{gph} H\big) = \big\{(-T^*x^*, M^*x^*) \,|\, x^* \in X^*\big\}. \tag{5.8}$$

In particular, if Φ *is surjective, then* (5.8) *is valid.*

Proof. First, note that Φ is continuous by the continuity of T and M. Second, observe that

$$\operatorname{gph} H = \{(w, z) \mid \Phi(w, z) = 0\} = \Phi^{-1}(0) = \ker \Phi.$$

On one hand, we have

$$\Phi^*(x^*) = (-T^*x^*, M^*x^*) \ \ \forall x^* \in X^*, \tag{5.9}$$

because

$$\begin{aligned}\langle \Phi^*(x^*), (w, z)\rangle &= \langle x^*, \Phi(w, z)\rangle \\ &= \langle x^*, -Tw\rangle + \langle x^*, Mz\rangle \\ &= \langle -T^*x^*, w\rangle + \langle M^*x^*, z\rangle \\ &= \langle (-T^*x^*, M^*x^*), (w, z)\rangle\end{aligned}$$

for every $(w, z) \in W \times Z$. On the other hand, since $\operatorname{gph} H$ is a linear subspace of $W \times Z$,

$$N((\bar{w}, \bar{z}); \operatorname{gph} H) = (\operatorname{gph} H)^\perp = (\ker \Phi)^\perp, \tag{5.10}$$

where

$$(\ker \Phi)^\perp = \{(w^*, z^*) \in W^* \times Z^* \mid \langle (w^*, z^*), (w, z)\rangle = 0 \ \ \forall (w, z) \in \ker \Phi\}.$$

Hence, by the first assertion of Proposition 1.3, (5.7) follows from (5.9) and (5.10). If Φ has closed range, then the weak* closure sign in (5.7) can be removed due to the second assertion of Proposition 1.3. Thus, (5.8) is valid. If Φ is a surjective, then it has closed range; so (5.8) holds true. □

Lemma 5.2 *If* Φ *has closed range and* $\ker T^* \subset \ker M^*$, *then one has for each* $(\bar{w}, \bar{z}) \in \operatorname{gph} H$ *the equality*

$$N\big((\bar{w}, \bar{z}); (W \times \Omega) \cap \operatorname{gph} H\big) = \{0\} \times N(\bar{z}; \Omega) + N\big((\bar{w}, \bar{z}); \operatorname{gph} H\big). \tag{5.11}$$

Proof. First, let us show that

$$N((\bar{w},\bar{z}); W\times\Omega)\cap[-N((\bar{w},\bar{z}); \operatorname{gph} H)] = \{(0,0)\}. \tag{5.12}$$

To obtain this property, take any

$$(w^*,z^*)\in N((\bar{w},\bar{z}); W\times\Omega)\cap[-N((\bar{w},\bar{z}); \operatorname{gph} H)].$$

Since $N((\bar{w},\bar{z}); W\times\Omega)=\{0\}\times N(\bar{z};\Omega)$, we must have $w^*=0$, $z^*\in N(\bar{z};\Omega)$. As Φ has closed range, (5.8) is valid by Lemma 5.1. Therefore, the inclusion $(w^*,z^*)\in -N((\bar{w},\bar{z}); \operatorname{gph} H)$ implies the existence of $x^*\in X^*$ with $0=T^*x^*$ and $z^*=-M^*x^*$. Combining this with the inclusion $\ker T^*\subset \ker M^*$, we obtain $z^*=0$. Property (5.12) has been proved.

Next, since $\operatorname{int}\Omega\neq\emptyset$, we see that $W\times\Omega$ is a convex set with nonempty interior. Let $A_0:=\operatorname{gph} H$ and $A_1:=W\times\Omega$. Due to (5.12), one cannot find any $(w_0^*,z_0^*)\in N((\bar{w},\bar{z}); A_0)$ and $(w_1^*,z_1^*)\in N((\bar{w},\bar{z}); A_1)$, not all zero, such that $(w_0^*,z_0^*)+(w_1^*,z_1^*)=0$. Hence, applying Proposition 1.5 to the sets A_0 and A_1 and the point $(\bar{w},\bar{z})\in A_0\cap A_1$, we can assert that $A_0\cap \operatorname{int} A_1\neq\emptyset$. Therefore, by Proposition 1.4 we have

$$N((\bar{w},\bar{z}); A_0\cap A_1) = N((\bar{w},\bar{z}); A_0)+N((\bar{w},\bar{z}); A_1). \tag{5.13}$$

Since $N((\bar{w},\bar{z}); A_0)=N\big((\bar{w},\bar{z}); \operatorname{gph} H\big)$ and $N((\bar{w},\bar{z}); A_1)=\{0\}\times N(\bar{z};\Omega)$, equality (5.11) follows from (5.13). □

Theorem 5.1 *Suppose that Φ has closed range and $\ker T^*\subset\ker M^*$. If the optimal value function h in (5.6) is finite at $\bar{w}\in\operatorname{dom}\widehat{S}$ and f is continuous at $(\bar{w},\bar{z})\in(W\times\Omega)\cap\operatorname{gph} H$, then*

$$\partial h(\bar{w}) = \bigcup_{(w^*,z^*)\in\partial f(\bar{z},\bar{w})}\ \bigcup_{v^*\in N(\bar{z};\Omega)} \big[w^*+T^*\big((M^*)^{-1}(z^*+v^*)\big)\big] \tag{5.14}$$

and

$$\partial^\infty h(\bar{w}) = \bigcup_{(w^*,z^*)\in\partial^\infty f(\bar{z},\bar{w})}\ \bigcup_{v^*\in N(\bar{z};\Omega)} \big[w^*+T^*\big((M^*)^{-1}(z^*+v^*)\big)\big], \tag{5.15}$$

where $\big(M^*\big)^{-1}(z^*+v^*)=\{x^*\in X^*\mid M^*x^*=z^*+v^*\}$.

Proof. (This proof is based on Theorem 2.1, Lemma 5.1, and Lemma 5.2.) We apply Theorem 2.1 to the case where w, z, $f(z,w)$, $H(w)\cap\Omega$, and $h(w)$ respectively play the roles of x, y, $\varphi(x,y)$, $G(x)$, and $\mu(x)$. By the assumptions of the theorem, f is continuous at $(\bar{w},\bar{z})\in(W\times\Omega)\cap\operatorname{gph} H$. Hence, the regularity condition (b) of Theorem 2.1 is satisfied. Therefore,

$$\partial h(\bar{w}) = \bigcup_{(w^*,z^*)\in\partial f(\bar{z},\bar{w})} \big\{w^*+D^*\widetilde{G}(\bar{w},\bar{z})(z^*)\big\} \tag{5.16}$$

and

$$\partial^\infty h(\bar{w}) = \bigcup_{(w^*,z^*)\in\partial^\infty f(\bar{z},\bar{w})} \{w^* + D^*\widetilde{G}(\bar{w},\bar{z})(z^*)\}, \tag{5.17}$$

where $\widetilde{G}(w) := H(w)\cap\Omega$ for all $w\in W$. Clearly, $\operatorname{gph}\widetilde{G} = (W\times\Omega)\cap\operatorname{gph}H$. Let us show that

$$D^*\widetilde{G}(\bar{w},\bar{z})(z^*) = \bigcup_{v^*\in N(\bar{z};\Omega)} \{T^*[(M^*)^{-1}(z^*+v^*)]\}. \tag{5.18}$$

By the definition of coderivative,

$$\begin{aligned} D^*\widetilde{G}(\bar{w},\bar{z})(z^*) &= \{\widetilde{w}^*\in W^* \mid (\widetilde{w}^*,-z^*)\in N((\bar{w},\bar{z});\operatorname{gph}\widetilde{G})\} \\ &= \{\widetilde{w}^*\in W^* \mid (\widetilde{w}^*,-z^*)\in N\big((\bar{w},\bar{z});(W\times\Omega)\cap\operatorname{gph}H\big)\}. \end{aligned}$$

So, the assumptions made allow us to use formula (5.11) in Lemma 5.2 to have

$$\begin{aligned} D^*\widetilde{G}(\bar{w},\bar{z})(z^*) &= \{\widetilde{w}^*\in W^* \mid (\widetilde{w}^*,-z^*)\in\{0\}\times N(\bar{z},\Omega)+N((\bar{w},\bar{z});\operatorname{gph}H)\} \\ &= \bigcup_{v^*\in N(\bar{z};\Omega)} \{\widetilde{w}^*\in W^* \mid (\widetilde{w}^*,-z^*)-(0,v^*)\in N((\bar{w},\bar{z});\operatorname{gph}H)\} \\ &= \bigcup_{v^*\in N(\bar{z};\Omega)} \{\widetilde{w}^*\in W^* \mid (\widetilde{w}^*,-z^*-v^*)\in N((\bar{w},\bar{z});\operatorname{gph}H)\}. \end{aligned}$$

Furthermore, as Φ has closed range, (5.8) is valid. Hence, $\widetilde{w}^*\in D^*\widetilde{G}(\bar{w},\bar{z})(z^*)$ if and only if there exist $v^*\in N(\bar{z};\Omega)$ and $x^*\in X^*$ such that

$$(\widetilde{w}^*,-z^*-v^*) = (-T^*x^*, M^*x^*).$$

It follows that $x^*\in(M^*)^{-1}(-z^*-v^*)$ and $\widetilde{w}^* = -T^*x^*$. Therefore, one has $\widetilde{w}^*\in D^*\widetilde{G}(\bar{w},\bar{z})(z^*)$ if and only if $\widetilde{w}^*\in T^*[(M^*)^{-1}(z^*+v^*)]$ for some $z^*\in N(\bar{z};\Omega)$. Thus, (5.18) has been proved. Combining (5.16) with (5.18), we obtain equality (5.14). Finally, from (5.17) and (5.18) we can easily get equality (5.15). □

When the objective function f is Fréchet differentiable at $(\bar{z},\bar{w})$, it holds that $\partial f(\bar{z},\bar{w}) = \{\nabla f(\bar{z},\bar{w})\}$. Hence (5.14) has a simpler form. Namely, the following statement is valid.

Theorem 5.2 *Under the assumptions of Theorem 5.1, suppose additionally that the function f is Fréchet differentiable at $(\bar{z},\bar{w})$. Then*

$$\partial h(\bar{w}) = \bigcup_{v^*\in N(\bar{z};\Omega)} \big[\nabla_w f(\bar{z},\bar{w}) + T^*\big((M^*)^{-1}(\nabla_z f(\bar{z},\bar{w})+v^*)\big)\big],$$

where $\nabla_z f(\bar{z},\bar{w})$ and $\nabla_w f(\bar{z},\bar{w})$, respectively, stand for the Fréchet derivatives of $f(\cdot,\bar{w})$ at $\bar{z}$ and of $f(\bar{z},\cdot)$ at $\bar{w}$.

5.3 Differential Stability of the Control Problem

Based on Theorem 5.1, we can obtain formulas for computing or estimating the subdifferential and singular subdifferential of the optimal value function $V(w)$ of the parametric control problem (5.1)–(5.4).

In the notation of Section 5.1, put $Z = X \times U$ and $K = C \times \widetilde{X} \times \Omega$ and note that $V(w)$ can be expressed as

$$V(w) = \inf_{z \in G(w) \cap K} f(z, w), \tag{5.19}$$

where

$$G(w) = \big\{ z = (x, u) \in Z \,|\, Mz = Tw \big\}$$

with $M : Z \to \widetilde{X}$ and $T : W \to \widetilde{X}$ being defined, respectively, by

$$Mz = \begin{pmatrix} -A_0 & I & 0 & 0 & \dots & 0 & 0 & -B_0 & 0 & 0 & \dots & 0 \\ 0 & -A_1 & I & 0 & \dots & 0 & 0 & 0 & -B_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -A_{N-1} & I & 0 & 0 & 0 & \dots & -B_{N-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \\ u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix},$$

$$Tw = \begin{pmatrix} T_0 w_0 \\ T_1 w_1 \\ \vdots \\ T_{N-1} w_{N-1} \end{pmatrix}.$$

Then problem (5.1)–(5.4) reduces to the mathematical programming problem (5.5). For every $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, ..., \tilde{x}_N^*) \in \widetilde{X}^*$, one has

$$T^* \tilde{x}^* = \big(T_0^* \tilde{x}_1^*, T_1^* \tilde{x}_2^*, \cdots, T_{N-1}^* \tilde{x}_N^* \big) \in W^* = W_0^* \times W_1^* \times \cdots \times W_{N-1}^* \tag{5.20}$$

and

$$M^*\tilde{x}^* = \begin{pmatrix} -A_0^* & 0 & 0 & \dots & 0 \\ I & -A_1^* & 0 & \dots & 0 \\ 0 & I & & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -A_{N-1}^* \\ 0 & 0 & 0 & \dots & I \\ -B_0^* & 0 & 0 & \dots & 0 \\ 0 & -B_1^* & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -B_{N-1}^* \end{pmatrix} \begin{pmatrix} \tilde{x}_1^* \\ \tilde{x}_2^* \\ \vdots \\ \tilde{x}_N^* \end{pmatrix}, \tag{5.21}$$

where T^*, M^*, A_i^*, and B_i^* are the adjoint operators of T, M, A_i, and B_i, respectively.

The next theorem gives us an upper estimate for the subdifferential of the optimal value function $V(\cdot)$ in (5.19), where the objective function may be nondifferentiable.

Theorem 5.3 *Suppose that $h_k,\ k = 0, 1, \ldots, N$, are continuous and the interiors of Ω_k, for $k = 0, 1, \ldots, N-1$, are nonempty. Suppose in addition that the following conditions are satisfied:*

(i) $\ker T^* \subset \ker M^*$;

(ii) *The operator $\Phi : W \times Z \to \widetilde{X}$ defined by $\Phi(w, z) = -Tw + Mz$ has closed range.*

If $\tilde{w}^* = (\tilde{w}_0^*, \tilde{w}_1^*, \ldots, \tilde{w}_{N-1}^*) \in \partial V(\bar{w})$, then there exist vectors $x_0^* \in N(\bar{x}_0; C)$, $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, \ldots, \tilde{x}_N^*) \in \widetilde{X}^*$, and $u^* = (u_0^*, u_1^*, \ldots, u_{N-1}^*) \in N(\bar{u}; \Omega)$, such that

$$\begin{cases} \tilde{x}_N^* \in \partial h_N(\bar{x}_N), \\ \tilde{x}_k^* \in \partial_{x_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) + A_k^* \tilde{x}_{k+1}^*,\ \ k = 1, 2, ..., N-1, \\ x_0^* \in -\partial_{x_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) - A_0^* \tilde{x}_1^*, \\ u_k^* \in -\partial_{u_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) - B_k^* \tilde{x}_{k+1}^*,\ \ k = 0, 1, ..., N-1, \\ \tilde{w}_k^* \in \partial_{w_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) + T_k^* \tilde{x}_{k+1}^*,\ \ k = 0, 1, ..., N-1. \end{cases} \tag{5.22}$$

Proof. Since the functions $h_k,\ k = 0, 1, \ldots, N$, are continuous, the objective function f of (5.5) is continuous. Note that $V(w)$ coincides with the optimal value function $h(w)$ in (5.6) of (5.5). As the operator Φ has closed range and $\ker T^* \subset \ker M^*$, applying Theorem 5.1 to (5.5) yields

$$\partial V(\bar{w}) = \bigcup_{(z^*, w^*) \in \partial f(\bar{z}, \bar{w})} \ \bigcup_{v^* \in N(\bar{z}; K)} \big[w^* + T^*\big((M^*)^{-1}(z^* + v^*)\big)\big]. \tag{5.23}$$

Consider the function $\tilde{h}_k : X \times U \times W \to \mathbb{R}$, $k = 0, 1, ..., N-1$, given by $\tilde{h}_k(x, u, w) = h_k(x_k, u_k, w_k)$. Let $\tilde{h}_N : X \times U \times W \to \mathbb{R}$ be defined by $\tilde{h}_N(x, u, w) = h_N(x_N)$. From (5.23) one has $\tilde{w}^* \in \partial V(\bar{w})$ if and only if there exist $(z_1^*, w_1^*) \in \partial f(\bar{z}, \bar{w})$ and $v_1^* = (x^*, u^*) \in N(\bar{z}; K)$ such that

$$\tilde{w}^* \in w_1^* + T^*((M^*)^{-1})(z_1^* + v_1^*).$$

The last inclusion means that there exists $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, ..., \tilde{x}_N^*) \in \widetilde{X}^*$ such that

$$\begin{cases} M^* \tilde{x}^* = z_1^* + v_1^*, \\ \tilde{w}^* \in w_1^* + T^* \tilde{x}^*. \end{cases} \tag{5.24}$$

Denote by $\partial_z f(\bar{z}, \bar{w}), \partial_w f(\bar{z}, \bar{w})$ the subdifferentials of $f(., \bar{w})$ at $\bar{z}$ and $f(\bar{z}, .)$ at $\bar{w}$, respectively. According to (4.1), $\partial f(\bar{z}, \bar{w}) \subset \partial_z f(\bar{z}, \bar{w}) \times \partial_w f(\bar{z}, \bar{w})$. Therefore, since $(z_1^*, w_1^*) \in \partial f(\bar{z}, \bar{w})$, (5.24) implies

$$\begin{cases} M^* \tilde{x}^* \in \partial_z f(\bar{z}, \bar{w}) + v_1^*, \\ \tilde{w}^* \in \partial_w f(\bar{z}, \bar{w}) + T^* \tilde{x}^*. \end{cases} \tag{5.25}$$

By the continuity of $\tilde{h}_k(.)$, $k = 0, 1, \ldots, N$, applying the Moreau–Rockafellar Theorem (see Theorem 1.1), we have

$$\begin{aligned} \partial_z f(\bar{z}, \bar{w}) &= \partial_z \left(\sum_{k=0}^{N} \tilde{h}_k \right) (\bar{z}, \bar{w}) = \sum_{k=0}^{N} \partial_z \tilde{h}_k(\bar{z}, \bar{w}) \\ &\subset \sum_{k=0}^{N} \partial_{x_k} \tilde{h}_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) \times \partial_{u_k} \tilde{h}_k(\bar{x}_k, \bar{u}_k, \bar{w}_k). \end{aligned}$$

It is easy to see that

$$\begin{aligned} &\partial_{x_0} \tilde{h}_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) = \partial_{x_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) \times \{0\} \times ... \times \{0\}, \\ &\partial_{x_1} \tilde{h}_1(\bar{x}_1, \bar{u}_1, \bar{w}_1) = \{0\} \times \partial_{x_1} h_1(\bar{x}_1, \bar{u}_1, \bar{w}_1) \times ... \times \{0\}, \\ &... \\ &\partial_{x_{N-1}} \tilde{h}_{N-1}(\bar{x}_{N-1}, \bar{u}_{N-1}, \bar{w}_{N-1}) = \{0\} \times ... \times \partial_{x_{N-1}} h_{N-1}(\bar{x}_{N-1}, \bar{u}_{N-1}, \bar{w}_{N-1}) \times \{0\}, \\ &\partial_{x_N} \tilde{h}_N(\bar{x}_N) = \partial h_N(\bar{x}_N). \end{aligned}$$

Similarly,

$$\begin{aligned} &\partial_{u_0} \tilde{h}_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) = \partial_{u_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) \times \{0\} \times ... \times \{0\}, \\ &\partial_{u_1} \tilde{h}_1(\bar{x}_1, \bar{u}_1, \bar{w}_1) = \{0\} \times \partial_{u_1} h_1(\bar{x}_1, \bar{u}_1, \bar{w}_1) \times ... \times \{0\}, \\ &... \\ &\partial_{u_{N-1}} \tilde{h}_{N-1}(\bar{x}_{N-1}, \bar{u}_{N-1}, \bar{w}_{N-1}) = \{0\} \times ... \times \partial_{u_{N-1}} h_{N-1}(\bar{x}_{N-1}, \bar{u}_{N-1}, \bar{w}_{N-1}) \times \{0\}. \end{aligned}$$

Hence

$$\begin{aligned}&\partial_z f(\bar{z},\bar{w})\\&\quad\subset \partial_{x_0} h_0(\bar{x}_0,\bar{u}_0,\bar{w}_0)\times ...\times \partial_{x_{N-1}} h_{N-1}(\bar{x}_{N-1},\bar{u}_{N-1},\bar{w}_{N-1})\times \partial h_N(\bar{x}_N)\\&\qquad\times \partial_{u_0} h_0(\bar{x}_0,\bar{u}_0,\bar{w}_0)\times ...\times \partial_{u_{N-1}} h_{N-1}(\bar{x}_{N-1},\bar{u}_{N-1},\bar{w}_{N-1}).\end{aligned}\tag{5.26}$$

In the same manner, we obtain

$$\begin{aligned}\partial_w f(\bar{z},\bar{w}) &\subset \sum_{k=0}^{N-1}\partial_{w_k}\tilde{h}_k(\bar{x}_k,\bar{u}_k,\bar{w}_k)\\&\subset \partial_{w_0} h_0(\bar{x}_0,\bar{u}_0,\bar{w}_0)\times ...\times \partial_{w_{N-1}} h_{N-1}(\bar{x}_{N-1},\bar{u}_{N-1},\bar{w}_{N-1}).\end{aligned}\tag{5.27}$$

Since

$$\begin{aligned}v_1^*\in N(\bar{z};K) &= N(\bar{x}_0;C)\times\{0_{\tilde{X}^*}\}\times N(\bar{u};\Omega)\\&= N(\bar{x}_0;C)\times\{0_{\tilde{X}^*}\}\times N(\bar{u}_0;\Omega_0)\times ...\times N(\bar{u}_{N-1};\Omega_{N-1}),\end{aligned}$$

there exist $x_0^*\in N(\bar{x}_0;C)$ and $u^*=(u_0^*,u_1^*,\ldots,u_{N-1}^*)$ with $u_k^*\in N(\bar{u}_k;\Omega_k)$ $(k=0,1,\ldots,N-1)$ such that $v_1^*=(x_0^*,0,u^*)$. Therefore, from the first inclusion in (5.25) and from (5.21), (5.26), we get

$$\begin{aligned}&\begin{pmatrix}-A_0^* & 0 & 0 & \ldots & 0\\ I & -A_1^* & 0 & \ldots & 0\\ 0 & I & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & -A_{N-1}^*\\ 0 & 0 & 0 & \ldots & I\end{pmatrix}\begin{pmatrix}\tilde{x}_1^*\\ \tilde{x}_2^*\\ \vdots\\ \tilde{x}_N^*\end{pmatrix}\\&\in\big(\partial_{x_0} h_0(\bar{x}_0,\bar{u}_0,\bar{w}_0)+x_0^*\big)\times\partial_{x_1}h_1(\bar{x}_1,\bar{u}_1,\bar{w}_1)\times ...\times\\&\qquad\partial_{x_{N-1}} h_{N-1}(\bar{x}_{N-1},\bar{u}_{N-1},\bar{w}_{N-1})\times\partial h_N(\bar{x}_N)\end{aligned}$$

and

$$\begin{aligned}&\begin{pmatrix}-B_0^* & 0 & 0 & \ldots & 0\\ 0 & -B_1^* & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & -B_{N-1}^*\end{pmatrix}\begin{pmatrix}\tilde{x}_1^*\\ \tilde{x}_2^*\\ \vdots\\ \tilde{x}_N^*\end{pmatrix}\\&\in\big(\partial_{u_0} h_0(\bar{x}_0,\bar{u}_0,\bar{w}_0)+u_0^*\big)\times ...\times\big(\partial_{u_{N-1}} h_{N-1}(\bar{x}_{N-1},\bar{u}_{N-1},\bar{w}_{N-1})+u_{N-1}^*\big).\end{aligned}$$

This implies that

$$\begin{cases} -x_0^* \in A_0^*\tilde{x}_1^* + \partial_{x_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0), \\ \tilde{x}_1^* \in A_1^*\tilde{x}_2^* + \partial_{x_1} h_1(\bar{x}_1, \bar{u}_1, \bar{w}_1), \\ \ldots \\ \tilde{x}_{N-1}^* \in A_{N-1}^*\tilde{x}_N^* + \partial_{x_{N-1}} h_{N-1}(\bar{x}_{N-1}, \bar{u}_{N-1}, \bar{w}_{N-1}), \\ \tilde{x}_N^* \in \partial h_N(\bar{x}_N) \end{cases} \tag{5.28}$$

and

$$-B_k^*\tilde{x}_{k+1}^* \in \partial_{u_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) + u_k^*, \;\; k = 0, 1, ..., N-1. \tag{5.29}$$

Now we can derive from the second inclusion in (5.25) and from (5.20), (5.27), the following

$$\tilde{w}_k^* \in \partial_{w_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) + T_k^*\tilde{x}_{k+1}^*, \;\; k = 0, 1, ..., N-1. \tag{5.30}$$

Combining (5.28)–(5.30), we obtain (5.22). The proof is complete. □

If the objective function is differentiable, the obtained upper estimate in Theorem 5.3 becomes an equality. The second main result of this section reads as follows.

Theorem 5.4 *Under the assumptions of Theorem 5.3, suppose additionally that the functions h_k, for $k = 0, 1, \ldots, N$, are Fréchet differentiable. Then, a vector $\tilde{w}^* = (\tilde{w}_0^*, \tilde{w}_1^*, \ldots, \tilde{w}_{N-1}^*) \in W^*$ belongs to $\partial V(\bar{w})$ if and only if there exist $x_0^* \in N(\bar{x}_0; C)$, $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, \ldots, \tilde{x}_N^*) \in \widetilde{X}^*$, and $u^* = (u_0^*, u_1^*, \ldots, u_{N-1}^*) \in N(\bar{u}; \Omega)$ such that*

$$\begin{cases} \tilde{x}_N^* = \nabla h_N(\bar{x}_N), \\ \tilde{x}_k^* = \nabla_{x_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) + A_k^*\tilde{x}_{k+1}^*, \;\; k = 1, 2, \ldots, N-1, \\ x_0^* = -\nabla_{x_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) - A_0^*\tilde{x}_1^*, \\ u_k^* = -\nabla_{u_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) - B_k^*\tilde{x}_{k+1}^*, \;\; k = 0, 1, \ldots, N-1, \\ \tilde{w}_k^* = \nabla_{w_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) + T_k^*\tilde{x}_{k+1}^*, \;\; k = 0, 1, \ldots, N-1, \end{cases} \tag{5.31}$$

where $\nabla h_N(\bar{x}_N), \nabla_{x_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k)$, $\nabla_{u_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k)$ and $\nabla_{w_k} h_k(\bar{x}_k, \bar{u}_k, \bar{w}_k)$, respectively, are the Fréchet derivatives of $h_N(.)$, $h_k(., \bar{u}_k, \bar{w}_k)$, $h_k(\bar{x}_k, ., \bar{w}_k)$ and $h_k(\bar{x}_k, \bar{u}_k, .)$ at $\bar{x}_N, \bar{x}_k$, $\bar{u}_k$, and $\bar{w}_k$.

Proof. It is well known that if $\varphi : Y \to \overline{\mathbb{R}}$ is a convex function defined on a normed space Y and φ is Fréchet differentiable at $\bar{y} \in Y$, then $\partial\varphi(\bar{y}) = \{\nabla\varphi(\bar{y})\}$ (see, e.g., [22, pp. 197–198]). Hence, since h_k, $k = 0, 1, ..., N$, are

Fréchet differentiable by our assumptions, the inclusions in (5.25)–(5.27) become equalities. Namely, we have

$$\begin{cases} M^*\tilde{x}^* = \nabla_z f(\bar{z}, \bar{w}) + v_1^* \\ \tilde{w}^* = \nabla_w f(\bar{z}, \bar{w}) + T^*\tilde{x}^*, \end{cases}$$

$$\nabla_z f(\bar{z}, \bar{w}) = \sum_{k=0}^{N} \big(\nabla_{x_k}\tilde{h}_k(\bar{x}_k, \bar{u}_k, \bar{w}_k), \nabla_{u_k}\tilde{h}_k(\bar{x}_k, \bar{u}_k, \bar{w}_k)\big),$$

$$\nabla_z f(\bar{z}, \bar{w}) = \big(\nabla_{x_0}h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0), ..., \nabla_{x_{N-1}}h_{N-1}(\bar{x}_{N-1}, \bar{u}_{N-1}, \bar{w}_{N-1}), \nabla h_N(\bar{x}_N),$$
$$\nabla_{u_0}h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0), ..., \nabla_{u_{N-1}}h_{N-1}(\bar{x}_{N-1}, \bar{u}_{N-1}, \bar{w}_{N-1})\big),$$

and

$$\nabla_w f(\bar{z}, \bar{w}) = \big(\nabla_{w_0}h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0), ..., \nabla_{w_{N-1}}h_{N-1}(\bar{x}_{N-1}, \bar{u}_{N-1}, \bar{w}_{N-1})\big).$$

Consequently, by the proof of Theorem 5.3 we can conclude that a vector $\tilde{w}^* = (\tilde{w}_0^*, \tilde{w}_1^*, \ldots, \tilde{w}_{N-1}^*) \in W^*$ belongs to $\partial V(\bar{w})$ if and only if there exist $x_0^* \in N(\bar{x}_0; C)$, $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, \ldots, \tilde{x}_N^*) \in \widetilde{X}^*$ and $u^* = (u_0^*, u_1^*, \ldots, u_{N-1}^*) \in N(\bar{u}; \Omega)$ such that (5.31) is satisfied. □

The next theorem shows that the singular subdifferential of $V(\cdot)$ always consists of the origin of the dual space.

Theorem 5.5 *Under the assumptions of Theorem 5.3, we have*

$$\partial^\infty V(\bar{w}) = \{0_{W^*}\}.$$

Proof. Similarly as in the proof of Theorem 5.3, having in mind that $V(w)$ coincides with the optimal value function $h(w)$ in (5.6) of (5.5), we can apply Theorem 5.1 to (5.5) to get

$$\partial^\infty V(\bar{w}) = \bigcup_{(z^*,w^*)\in\partial^\infty f(\bar{z},\bar{w})} \ \bigcup_{v^*\in N(\bar{z};K)} \big[w^* + T^*\big((M^*)^{-1}(z^* + v^*)\big)\big]. \quad (5.32)$$

Since $\operatorname{dom} f = Z \times W$ and $\partial^\infty f(\bar{z}, \bar{w}) = N((\bar{z}, \bar{w}); \operatorname{dom} f)$, we have

$$\partial^\infty f(\bar{z}, \bar{w}) = \{(0_{Z^*}, 0_{W^*})\}.$$

Therefore, from (5.32) it follows that

$$\partial^\infty V(\bar{w}) = \bigcup_{v^*\in N(\bar{z};K)} \big[T^*\big((M^*)^{-1}(v^*)\big)\big].$$

Thus, $\tilde{w}^*$ belongs to $\partial^\infty V(\bar{w})$ if and only if there exist $v^* \in N(\bar{z}; K)$ and $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, \ldots, \tilde{x}_N^*) \in \widetilde{X}^*$ such that

$$M^*\tilde{x}^* = v^*, \ \ \tilde{w}^* = T^*\tilde{x}^*.$$

As $v^* \in N(\bar{z}; K) = N(\bar{x}_0; C) \times \{0_{\tilde{X}^*}\} \times N(\bar{u}_0; \Omega_0) \times ... \times N(\bar{u}_{N-1}; \Omega_{N-1})$, we can find $x_0^* \in N(\bar{x}_0; C)$ and $u^* = (u_0^*, u_1^*, \ldots, u_{N-1}^*)$ with $u_k^* \in N(\bar{u}_k; \Omega_k)$ for $k = 0, 1, \ldots, N-1$. So $v^* = (x_0^*, 0_{\tilde{X}^*}, u_0^*, ..., u_{N-1}^*)$. By (5.20) and (5.21), we see that $\tilde{w}^* \in \partial^\infty V(\bar{w})$ if and only if there exist $x_0^* \in N(\bar{x}_0; C)$ and $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, \ldots, \tilde{x}_N^*) \in \widetilde{X}^*$, and $u^* = (u_0^*, u_1^*, ..., u_{N-1}^*)$ with $u_k^* \in N(\bar{u}_k; \Omega_k)$, $k = 0, 1, ..., N-1$, such that

$$\begin{cases} \tilde{x}_N^* = 0, \\ \tilde{x}_k^* = A_k^* \tilde{x}_{k+1}^*, \ k = 1, 2, \ldots, N-1, \\ x_0^* = -A_0^* \tilde{x}_1^*, \\ u_k^* = -B_k^* \tilde{x}_{k+1}^*, \ k = 0, 1, \ldots, N-1, \\ w_k^* = T_k^* \tilde{x}_{k+1}^*, \ k = 0, 1, \ldots, N-1. \end{cases} \tag{5.33}$$

From (5.33) we can easily deduce that $\partial^\infty V(\bar{w}) = \{0_{W^*}\}$. □

Let us describe a typical situation where the assumptions of Theorem 5.3 are automatically satisfied.

Remark 5.1 If $T_0, T_1, ..., T_{N-1}$ are surjective, the operator $T : W \to \widetilde{X}$ is surjective too. Hence $\ker T^* = \{0\}$, and, therefore, condition (i) in Theorem 5.3 is satisfied. Moreover, condition (ii) of that theorem is also fulfilled, because $\mathrm{rge}\, \Phi = \widetilde{\mathrm{X}}$.

5.4 Applications

In this section we apply the just obtained results to some examples. First, we give an auxiliary result related to a convex optimization problem under linear constraints.

Let X be Banach space with the dual denoted by X^*. Consider the problem

$$(P) \qquad \min \{\varphi(x) \mid \langle a_i, x\rangle \le \alpha_i, \ \langle b_j, x\rangle = \beta_j, \ i = 1, \ldots, m, \ j = 1, \ldots, k\},$$

where $\varphi : X \to \mathbb{R}$ is a continuous convex function, $a_i, b_j \in X^*$, $i = 1, \ldots, m$, $j = 1, \ldots, k$.

Denote by Ω and $\mathrm{Sol}\,(P)$, respectively, the constraint set and the solution set of (P).

Based on Lemma 2.1 and a standard Fermat rule for convex programs, one can obtain the next proposition on necessary and sufficient optimality conditions for (P), which is very useful for dealing with problem (5.1)–(5.4) when C and Ω_i, $i = 0, 1, ..., N-1$, are polyhedral convex sets. For clarity, we provide here a detailed proof of this result.

Proposition 5.1 *For a point $\bar{x} \in \Omega$ to be a solution of (P), it is necessary and sufficient, that there exist $\lambda_i \geq 0$, $i = 1, \ldots, m$ and $\mu_j \in \mathbb{R}$, $j = 1, \ldots, k$, such that*
(a) $0 \in \partial\varphi(\bar{x}) + \sum\limits_{i=1}^{m} \lambda_i a_i + \sum\limits_{j=1}^{k} \mu_j b_j$;
(b) $\lambda_i(\langle a_i, \bar{x}\rangle - \alpha_i) = 0, \ i = 1, \ldots, m.$

Proof. Fix any $\bar{x} \in \Omega$. Rewriting (P) in the form $\min\{\varphi(x) \mid x \in \Omega\}$ and applying [22, Proposition 2, p. 81], we see that $\bar{x} \in \mathrm{Sol}\,(P)$ if and only if

$$0 \in \partial\varphi(\bar{x}) + N(\bar{x}; \Omega). \tag{5.34}$$

We now show that

$$N(\bar{x}; \Omega) = \left\{ \sum_{i \in I(\bar{x})} \lambda_i a_i + \sum_{j=1}^{k} \mu_j b_j \mid \lambda_i \geq 0,\, i = 1, ..., m,\ \mu_j \in \mathbb{R},\, j = 1, ..., k \right\}, \tag{5.35}$$

where $I(\bar{x}) = \{i \mid \langle a_i, \bar{x}\rangle = \alpha_i,\, i = 1, \ldots, m\}$. Take any $x^* \in N(\bar{x}; \Omega)$. Let $v \in X$ be such that $\langle a_i, v\rangle \leq 0$ for $i \in I(\bar{x})$, $\langle b_j, v\rangle \leq 0$ and $\langle -b_j, v\rangle \leq 0$ for $j = 1, \ldots, k$. If $t > 0$ is chosen small enough, then

$$\langle a_i, \bar{x} + tv\rangle = \langle a_i, \bar{x}\rangle + t\langle a_i, v\rangle \leq \alpha_i, \quad i = 1, \ldots, m,$$

and

$$\langle b_j, \bar{x} + tv\rangle = \beta_j, \quad j = 1, \ldots, k.$$

Thus $\bar{x} + tv \in \Omega$; so we have

$$0 \geq \langle x^*, (\bar{x} + tv) - \bar{x}\rangle = t\langle x^*, v\rangle.$$

It follows that $\langle x^*, v\rangle \leq 0$. Hence, the inequality $\langle x^*, v\rangle \leq 0$ is a consequence of the inequalities system

$$\begin{cases} \langle a_i, v\rangle \leq 0,\ i \in I(\bar{x}), \\ \langle b_j, v\rangle \leq 0,\ j = 1, \ldots, k, \\ \langle -b_j, v\rangle \leq 0,\ j = 1, \ldots, k. \end{cases}$$

By Lemma 2.1, there exist $\lambda_i \geq 0$, $i \in I(\bar{x})$, $\mu_j^1 \geq 0$ and $\mu_j^2 \geq 0$, $j = 1, \ldots, k$, such that

$$x^* = \sum_{i \in I(\bar{x})} \lambda_i a_i + \sum_{j=1}^{k} \mu_j^1 b_j + \sum_{j=1}^{k} \mu_j^2 (-b_j).$$

Setting $\mu_j := \mu_j^1 - \mu_j^2$, from the last equality we can deduce that x^* belongs to the right-hand-side of (5.35).

Now, suppose that $x^* = \sum\limits_{i \in I(\bar{x})} \lambda_i a_i + \sum\limits_{j=1}^{k} \mu_j b_j$, where $\lambda_i \geq 0$ for every $i \in I(\bar{x})$, $\mu_j \in \mathbb{R}$ for $j = 1, \ldots, k$. Given any $x \in \Omega$, we have

$$\langle a_i, x - \bar{x} \rangle = \langle a_i, x \rangle - \langle a_i, \bar{x} \rangle \leq \alpha_i - \alpha_i = 0, \quad i \in I(\bar{x}).$$

Therefore,

$$\begin{aligned}
\langle x^*, x - \bar{x} \rangle &= \left\langle \sum_{i \in I(\bar{x})} \lambda_i a_i + \sum_{j=1}^{k} \mu_j b_j, x - \bar{x} \right\rangle \\
&= \sum_{i \in I(\bar{x})} \lambda_i \langle a_i, x - \bar{x} \rangle \\
&\leq 0.
\end{aligned}$$

It follows that $x^* \in N(\bar{x}; \Omega)$. Combining (5.34) and (5.35), we obtain the assertion of the proposition. □

The next example is designed to show how Theorem 5.4 can work for parametric optimal control problems with differentiable objective functions.

Example 5.1 Let $N = 1$, $X_0 = \mathbb{R}$, $X_1 = \mathbb{R}$, $U_0 = \mathbb{R}$, $W_0 = \mathbb{R}$, $\Omega_0 = [-1, +\infty)$ and $C = (-\infty, 2]$. Let $A_0 : X_0 \to X_1$, $B_0 : U_0 \to X_1$, $T_0 : W_0 \to X_1$ be defined by $A_0 x_0 = x_0$, $B_0 u_0 = -u_0$, and $T_0 w_0 = 2w_0$. Furthermore, let $h_0 : X_0 \times U_0 \times W_0 \to \mathbb{R}$ and $h_1 : X_1 \to \mathbb{R}$ be given, respectively, by

$$\begin{aligned}
h_0(x_0, u_0, w_0) &= x_0^2 + x_0 u_0 + u_0^2 + \frac{1}{2} w_0, \\
h_1(x_1) &= (x_1 + 1)^2.
\end{aligned}$$

Consider the control problem (5.1)–(5.4) and choose $\bar{w} = 0$ belonging to $W = W_0 = \mathbb{R}$. Then, problem (5.1)–(5.4) becomes

$$\begin{cases}
f(x, u, \bar{w}) = x_0^2 + x_0 u_0 + u_0^2 + (x_1 + 1)^2 \to \inf, \\
x_1 = x_0 - u_0, \\
x_0 \leq 2, \\
u_0 \geq -1.
\end{cases} \tag{P_1}$$

Using Proposition 5.1, it is easy to show that $(\bar{x}_0, \bar{x}_1, \bar{u}_0) = \left(-\frac{2}{5}, -\frac{4}{5}, \frac{2}{5} \right)$ is the unique solution of (P_1). Hence $\bar{x} = (\bar{x}_0, \bar{x}_1) = \left(-\frac{2}{5}, -\frac{4}{5} \right)$ and $\bar{u} = \bar{u}_0 = \frac{2}{5}$, we have $(\bar{x}, \bar{u}) \in S(\bar{w})$. Clearly, the mapping $\Phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$\Phi(w, z) = Mz - Tw = -x_0 + x_1 + u_0 - 2w_0$$

has closed range and $\ker T^* = \ker M^* = \{0\}$. Hence, by Theorem 5.4, a vector $w_0^* \in \partial V(\bar{w})$ if and only if there exist $x_0^* \in N(\bar{x}_0; C)$, $\tilde{x}_1^* \in \mathbb{R}$, and $u_0^* \in N(\bar{u}_0; \Omega_0)$ such that

$$\begin{cases} \tilde{x}_1^* = \nabla h_1(\bar{x}_1), \\ x_0^* = -\nabla_{x_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) - A_0^* \tilde{x}_1^*, \\ u_0^* = -\nabla_{u_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) - B_0^* \tilde{x}_1^*, \\ w_0^* = \nabla_{w_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) + T_0^* \tilde{x}_1^*. \end{cases} \tag{5.36}$$

We have

$$\nabla h_1(\bar{x}_1) = \nabla h_1\left(-\frac{4}{5}\right) = \frac{2}{5},$$
$$\nabla_{x_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) = \nabla_{x_0} h_0\left(-\frac{2}{5}, \frac{2}{5}, 0\right) = -\frac{2}{5},$$
$$\nabla_{u_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) = \nabla_{u_0} h_0\left(-\frac{2}{5}, \frac{2}{5}, 0\right) = \frac{2}{5},$$
$$\nabla_{w_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) = \nabla_{w_0} h_0\left(-\frac{2}{5}, \frac{2}{5}, 0\right) = \frac{1}{2},$$

$N(\bar{x}_0; C) = N\left(-\frac{2}{5}; (-\infty, 2]\right) = \{0\}$, $N(\bar{u}_0; \Omega_0) = N\left(\frac{2}{5}; [-1, +\infty)\right) = \{0\}$, $A_0^* = 1$, $B_0^* = -1$, and $T_0^* = 2$. Thus, from (5.36) we have $\tilde{x}_1^* = \frac{2}{5}$, $x_0^* = 0$, $u_0^* = 0$, $w_0^* = \frac{13}{10}$. Hence $\partial V(\bar{w}) = \left\{\frac{13}{10}\right\}$.

We now give an illustrative example for the result in Theorem 5.3, where $h_0, ..., h_N$ are not required to be differentiable.

Example 5.2 Choose $N = 2$, $X_0 = X_1 = X_2 = \mathbb{R}$, $U_0 = U_1 = \mathbb{R}$, $W_0 = W_1 = \mathbb{R}$, $C = (-\infty, 1]$, and $\Omega_0 = \Omega_1 = \mathbb{R}$. Let $A_0 : X_0 \to X_1$, $B_0 : U_0 \to X_1$, $T_0 : W_0 \to X_1$, $A_1 : X_1 \to X_2$, $B_1 : U_1 \to X_2$, and $T_1 : W_1 \to X_2$ be given by $A_0 x_0 = -x_0$, $B_0 u_0 = 0$, $T_0 w_0 = -w_0$, $A_1 x_1 = x_1$, $B_1 u_1 = -u_1$, and $T_1 w_1 = w_1$. Furthermore, define $h_0 : X_0 \times U_0 \times W_0 \to \mathbb{R}$, $h_1 : X_1 \times U_1 \times W_1 \to \mathbb{R}$, and $h_2 : X_2 \to \mathbb{R}$ by

$$h_0(x_0, u_0, w_0) = (x_0 + u_0)^2 + \frac{1}{2} w_0^2,$$
$$h_1(x_1, u_1, w_1) = |x_1 - 1| + |w_1|,$$
$$h_2(x_2) = |x_2|.$$

Then, at the parameter $\bar{w} = (\bar{w}_0, \bar{w}_1) := (0, 0)$, problem (5.1)–(5.4) becomes

$$\begin{cases} f(x, u, \bar{w}) = (x_0 + u_0)^2 + |x_1 - 1| + |x_2| \to \inf, \\ x_1 = -x_0, \\ x_2 = x_1 - u_1, \\ x_0 \leq 1. \end{cases} \tag{P_2}$$

Using Proposition 5.1, it is not difficult to show that $S(\bar{w}) = \{\bar{z}\}$, where $\bar{z} = (\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{u}_0, \bar{u}_1) = (-1, 1, 0, 1, 1)$. Moreover, the linear operator $\Phi : \mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^2$,

$$\Phi(w, z) = Mz - Tw = \begin{pmatrix} x_0 + x_1 + w_0 \\ -x_1 + x_2 + u_1 - w_1 \end{pmatrix},$$

has closed range. The assumption $\ker T^* \subset \ker M^*$ is satisfied, because $\ker T^* = \ker M^* = \{(0, 0)\}$. Hence, by Theorem 5.3, if $w^* = (w_0^*, w_1^*)$ belongs to $\partial V(\bar{w})$ then we can find vectors $x_0^* \in N(\bar{x}_0; C)$, $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*)$, and $u^* = (u_0^*, u_1^*) \in N(\bar{u}; \Omega)$ such that (5.22) is satisfied. It is clear that

$$\begin{aligned} &\partial h_2(\bar{x}_2) = \partial h_2(0) = [-1, 1], \\ &\partial_{x_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) = \partial_{x_0} h_0(-1, 1, 0) = \{0\}, \\ &\partial_{x_1} h_1(\bar{x}_1, \bar{u}_1, \bar{w}_1) = \partial_{x_1} h_1(1, 1, 0) = [-1, 1], \\ &\partial_{u_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) = \partial_{u_0} h_0(-1, 1, 0) = \{0\}, \\ &\partial_{u_1} h_1(\bar{x}_1, \bar{u}_1, \bar{w}_1) = \partial_{u_1} h_1(1, 1, 0) = \{0\}, \\ &\partial_{w_0} h_0(\bar{x}_0, \bar{u}_0, \bar{w}_0) = \partial_{w_0} h_0(-1, 1, 0) = \{0\}, \\ &\partial_{w_1} h_1(\bar{x}_1, \bar{u}_1, \bar{w}_1) = \partial_{w_1} h_1(1, 1, 0) = [-1, 1]. \end{aligned}$$

In addition, we have $A_0^* = -1$, $A_1^* = 1$, $B_0^* = 0$, $B_1^* = -1$, $T_0^* = -1$, and $T_1^* = 1$. Therefore, (5.22) yields $\tilde{x}_2^* \in [-1, 1]$, $\tilde{x}_1^* \in [-2, 2]$, $x_0^* \in [-2, 2]$, $u_0^* = 0$, $u_1^* \in [-1, 1]$. Combining this with the conditions $x_0^* \in N(\bar{x}_0; C) = \{0\}$, $u^* = (u_0^*, u_1^*) \in N(\bar{u}; \Omega) = \{(0, 0)\}$, we obtain $\tilde{x}_2^* \in [-1, 1]$, $\tilde{x}_1^* \in [-2, 2]$, $x_0^* = 0$, $u_0^* = u_1^* = 0$. Thus, the last N inclusions of (5.22) imply

$$w_0^* \in [-2, 2], \quad w_1^* \in [-2, 2].$$

We have shown that $\partial V(\bar{w}) \subset [-2, 2] \times [-2, 2]$.

5.5 Conclusions

Differential stability of convex discrete optimal control problems in Banach spaces is studied in this chapter. By using several results of Chapter 2 on differential stability of parametric convex optimization problems under inclusion constraints, we obtain an upper estimate for the subdifferential of the optimal value function of a parametric convex discrete optimal control problem, where the objective function may be nondifferentiable (Theorem 5.3). If the objective function is differentiable, then the obtained upper estimate becomes an equality (Theorem 5.4). It is shown that the singular subdifferential of the just mentioned optimal value function always consists of the origin of the dual space (Theorem 5.5).

Chapter 6

Stability Analysis of Convex Continuous Optimal Control Problems

Very recently, Thuy and Toan [43] have obtained a formula for computing the subdifferential of the optimal value function to a parametric unconstrained convex optimal control problem with a convex objective function and linear state equations. In this chapter, we develop the approach of [43] to deal with *constrained control problems*. Namely, based on the result of Chapter 5 about differential stability of parametric convex mathematical programming problems, we will get new formulas for computing the subdifferential and the singular subdifferential of the optimal value function. The computation procedures and several illustrative examples are presented.

Our exposition is based on [3].

6.1 Problem Setting and Auxiliary Results

Let $W^{1,p}([0,1],\mathbb{R}^n)$, $1 \le p < \infty$, be the Sobolev space consisting of absolutely continuous functions $x : [0,1] \to \mathbb{R}^n$ such that $\dot{x} \in L^p([0,1],\mathbb{R}^n)$. Let there be given

- matrix-valued functions $A(t) = (a_{ij}(t))_{n\times n}$, $B(t) = (b_{ij}(t))_{n\times m}$, and $C(t) = (c_{ij}(t))_{n\times k}$;

- real-valued functions $g : \mathbb{R}^n \to \mathbb{R}$ and $L : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$;

- a convex set $\mathcal{U} \subset L^p([0,1],\mathbb{R}^m)$;

- a pair of parameters $(\alpha,\theta) \in \mathbb{R}^n \times L^p([0,1],\mathbb{R}^k)$.

Put

$$X = W^{1,p}([0,1],\mathbb{R}^n),\ U = L^p([0,1],\mathbb{R}^m),\ Z = X \times U,$$
$$\Theta = L^p([0,1],\mathbb{R}^k),\ W = \mathbb{R}^n \times \Theta.$$

Consider the constrained fixed time optimal control problem which depends on a pair of parameters (α, θ): *Find a pair (x, u), where $x \in W^{1,p}([0,1],\mathbb{R}^n)$ is a trajectory and $u \in L^p([0,1],\mathbb{R}^m)$ is a control function, which minimizes the objective function*

$$g(x(1)) + \int_0^1 L(t, x(t), u(t), \theta(t))dt \tag{6.1}$$

and satisfies the linear ordinary differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)\theta(t) \ \ a.e.\ t \in [0,1], \tag{6.2}$$

the initial value

$$x(0) = \alpha, \tag{6.3}$$

and the control constraint

$$u \in \mathcal{U}. \tag{6.4}$$

It is well known that X, U, Z, and Θ are Banach spaces. For every element $w = (\alpha, \theta) \in W$, denote by $V(w)$ and $S(w)$, respectively, the optimal value and the solution set of problem (6.1)–(6.4). We call $V : W \to \overline{\mathbb{R}}$ the *optimal value function* of the problem in question. If for each $w = (\alpha, \theta) \in W$ we put

$$J(x, u, w) = g(x(1)) + \int_0^1 L(t, x(t), u(t), \theta(t))dt, \tag{6.5}$$

$$G(w) = \big\{z = (x, u) \in X \times U \ \mid (6.2) \text{ and } (6.3) \text{ are satisfied}\big\}, \tag{6.6}$$

and

$$K = X \times \mathcal{U}, \tag{6.7}$$

then (6.1)–(6.4) can be written formally as $\min\{J(z,w) \mid z \in G(w) \cap K\}$, and

$$V(w) = \inf\{J(z,w) \mid z = (x,u) \in G(w) \cap K\}. \tag{6.8}$$

It is assumed that V is finite at $\bar{w} = (\bar{\alpha}, \bar{\theta}) \in W$ and $(\bar{x}, \bar{u})$ is a solution of the corresponding problem, that is $(\bar{x}, \bar{u}) \in S(\bar{w})$.

Similarly as in [24, p. 310], a vector-valued function $g : [0,1] \to Y$, where Y is a normed space, is called *essentially bounded* if there exists a constant $\gamma > 0$ such that the set $T := \{t \in [0,1] \mid ||g(t)|| \leq \gamma\}$ is of full Lebesgue measure. The latter means $\mu([0,1] \setminus T) = 0$, with μ denoting the Lebesgue measure on $[0,1]$.

Consider the following assumptions:

(A1) The matrix-valued functions $A : [0,1] \to M_{n,n}(\mathbb{R})$, $B : [0,1] \to M_{n,m}(\mathbb{R})$, and $C : [0,1] \to M_{n,k}(\mathbb{R})$, are measurable and essentially bounded.

(A2) The functions $g : \mathbb{R}^n \to \mathbb{R}$ and $L : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ are such that $g(\cdot)$ is convex and continuously differentiable on $\mathbb{R}^n$, $L(\cdot, x, u, v)$ is measurable for all $(x, u, v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$, $L(t, \cdot, \cdot, \cdot)$ is convex and continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$ for almost every $t \in [0,1]$, and there exist constants $c_1 > 0$, $c_2 > 0$, $r \geq 0$, $p \geq p_1 \geq 0$, $p - 1 \geq p_2 \geq 0$, and a nonnegative function $w_1 \in L^p([0,1], \mathbb{R})$, such that

$$|L(t,x,u,v)| \leq c_1\big(w_1(t) + ||x||^{p_1} + ||u||^{p_1} + ||v||^{p_1}\big),$$

$$\begin{aligned}\max\big\{|L_x(t,x,u,v)|, |L_u(t,x,u,v)|, |L_v(t,x,u,v)|\big\}\\ \leq c_2\big(||x||^{p_2} + ||u||^{p_2} + ||v||^{p_2}\big) + r,\end{aligned}$$

for all $(t,x,u,v) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$.

6.2 Differential Stability of the Control Problem

Keeping the notation of Section 6.1, we consider the linear mappings $\mathcal{A} : X \to X$, $\mathcal{B} : U \to X$, $\mathcal{M} : X \times U \to X$, and $\mathcal{T} : W \to X$ which are defined by setting

$$\mathcal{A}x := x - \int_0^{(.)} A(\tau)x(\tau)d\tau, \tag{6.9}$$

$$\mathcal{B}u := -\int_0^{(.)} B(\tau)u(\tau)d\tau, \tag{6.10}$$

$$\mathcal{M}(x,u) := \mathcal{A}x + \mathcal{B}u, \tag{6.11}$$

and

$$\mathcal{T}(\alpha, \theta) := \alpha + \int_0^{(.)} C(\tau)\theta(\tau)d\tau, \tag{6.12}$$

where the writing $\int_0^{(.)} g(\tau)d\tau$ for a function $g \in L^p([0,1],\mathbb{R}^n)$ abbreviates the function $t \mapsto \int_0^t g(\tau)d\tau$, which belongs to $X = W^{1,p}([0,1],\mathbb{R}^n)$.

Under assumption (**A1**), we can write the linear ordinary differential equation (6.2) in the integral form

$$x = x(0) + \int_0^{(.)} A(\tau)x(\tau)d\tau + \int_0^{(.)} B(\tau)u(\tau)d\tau + \int_0^{(.)} C(\tau)\theta(\tau)d\tau.$$

Combining this with the initial value in (6.3), one gets

$$x = \alpha + \int_0^{(.)} A(\tau)x(\tau)d\tau + \int_0^{(.)} B(\tau)u(\tau)d\tau + \int_0^{(.)} C(\tau)\theta(\tau)d\tau.$$

Thus, in accordance with (6.9)–(6.12), (6.6) can be written as

$$\begin{aligned} G(w) &= \left\{(x,u) \in X \times U \mid x = \alpha + \int_0^{(\cdot)} Axd\tau + \int_0^{(\cdot)} Bud\tau + \int_0^{(\cdot)} C\theta d\tau\right\} \\ &= \left\{(x,u) \in X \times U \mid x - \int_0^{(.)} Axd\tau - \int_0^{(.)} Bud\tau = \alpha + \int_0^{(.)} C\theta d\tau\right\} \\ &= \{(x,u) \in X \times U \mid \mathcal{M}(x,u) = \mathcal{T}(w)\}. \end{aligned}$$

Hence, the control problem (6.1)–(6.4) reduces to the mathematical programming problem (5.5), where the function $J(\cdot)$, the multifunction $G(\cdot)$, and the set K defined by (6.5)–(6.7), play the roles of $f(\cdot)$, $H(\cdot)$, and Ω.

We shall need several lemmas.

Lemma 6.1 (See [45, Lemma 2.3]) *Under assumption* (**A1**), *the following are valid:*
(i) *The linear operators* $\mathcal{M}$ *in* (6.11) *and* $\mathcal{T}$ *in* (6.12) *are continuous;*
(ii) $\mathcal{T}^*(a,u) = (a, C^T u)$ *for every* $(a,u) \in \mathbb{R}^n \times L^q([0,1],\mathbb{R}^n)$;
(iii) $\mathcal{M}^*(a,u) = (\mathcal{A}^*(a,u), \mathcal{B}^*(a,u))$, *where*

$$\begin{aligned} &\mathcal{A}^*(a,u) \\ &\quad = \left(a - \int_0^1 A^T(t)u(t)dt,\ u + \int_0^{(.)} A^T(\tau)u(\tau)d\tau - \int_0^1 A^T(t)u(t)dt\right), \end{aligned} \tag{6.13}$$

and $\mathcal{B}^*(a,u) = -B^T u$ *for every* $(a,u) \in \mathbb{R}^n \times L^q([0,1],\mathbb{R}^n)$.

Lemma 6.2 (See [45, Lemma 3.1 (b)]) *If* (**A1**) *and* (**A2**) *are satisfied, then the functional* $J : X \times U \times W \to \mathbb{R}$ *is Fréchet differentiable at* $(\bar{z},\bar{w})$ *and* $\nabla J(\bar{z},\bar{w})$ *is given by*

$$\nabla_w J(\bar{z},\bar{w}) = \big(0_{\mathbb{R}^n}, L_\theta(\cdot,\bar{x}(\cdot),\bar{u}(\cdot),\bar{\theta}(\cdot))\big),$$

$$\nabla_z J(\bar{z}, \bar{w}) = \big(J_x(\bar{x}, \bar{u}, \bar{\theta}), J_u(\bar{x}, \bar{u}, \bar{\theta})\big),$$

with

$$J_x(\bar{x}, \bar{u}, \bar{\theta}) = \bigg(g'(\bar{x}(1)) + \int_0^1 L_x(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t))dt,$$
$$g'(\bar{x}(1)) + \int_{(.)}^1 L_x(\tau, \bar{x}(\tau), \bar{u}(\tau), \bar{\theta}(\tau))d\tau\bigg),$$

and $J_u(\bar{x}, \bar{u}, \bar{\theta}) = L_u(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot))$.

Let

$$\varPsi_A : L^q([0,1], \mathbb{R}^n) \to \mathbb{R}, \quad \varPsi_B : L^q([0,1], \mathbb{R}^n) \to L^q([0,1], \mathbb{R}^m),$$
$$\varPsi_C : L^q([0,1], \mathbb{R}^n) \to L^q([0,1], \mathbb{R}^k), \quad \varPsi : L^q([0,1], \mathbb{R}^n) \to L^q([0,1], \mathbb{R}^n)$$

be defined by

$$\varPsi_A(v) = \int_0^1 A^T(t)v(t)dt, \quad \varPsi_B(v)(t) = -B^T(t)v(t) \text{ a.e. } t \in [0,1],$$
$$\varPsi_C(v)(t) = C^T(t)v(t) \text{ a.e. } t \in [0,1], \quad \varPsi(v) = -\int_0^{(.)} A^T(\tau)v(\tau)d\tau.$$

We will employ the following two assumptions.

(A3) Suppose that

$$\ker \varPsi_C \subset \big(\ker \varPsi_A \cap \ker \varPsi_B \cap \operatorname{Fix}\varPsi\big), \tag{6.14}$$

where $\operatorname{Fix}\varPsi := \{x \in L^q([0,1], \mathbb{R}^n) \mid \varPsi(x) = x\}$ is the set of the fixed points of $\varPsi$, and $\ker \varPsi_A$ (resp., $\ker \varPsi_B$, $\ker \varPsi_C$) denotes the kernel of $\varPsi_A$ (resp., $\varPsi_B$, $\varPsi_C$).

(A4) The operator $\varPhi : W \times Z \to X$, which is given by

$$\varPhi(w, z) = x - \int_0^{(.)} A(\tau)x(\tau)d\tau - \int_0^{(.)} B(\tau)v(\tau)d\tau - \alpha - \int_0^{(.)} C(\tau)\theta(\tau)d\tau$$

for every $w = (\alpha, \theta) \in W$ and $z = (x, v) \in Z$, has closed range.

Lemma 6.3 *If* (**A3**) *is satisfied, then* $\ker \mathcal{T}^* \subset \ker \mathcal{M}^*$.

Proof. For any $(a, v) \in \ker \mathcal{T}^*$, it holds that $(a, v) \in \mathbb{R}^n \times L^q([0,1], \mathbb{R}^n)$ and $\mathcal{T}^*(a, v) = 0$. Then $(a, C^T v) = 0$. So $a = 0$ and $C^T v = 0$. The latter means that $C^T(t)v(t) = 0$ a.e. on $[0,1]$. Hence $v \in \ker \varPsi_C$. By (6.14),

$$v \in \ker \varPsi_A \cap \ker \varPsi_B \cap \operatorname{Fix}\varPsi.$$

The condition $\Psi_B(v) = 0$ implies that $-B^T(t)v(t) = 0$ a.e. on $[0, 1]$. As $\mathcal{B}^*(a, v) = -B^T v$ by Lemma 6.1, this yields

$$\mathcal{B}^*(a, v) = 0. \tag{6.15}$$

According to the condition $v \in \ker \Psi_A$, we have $\displaystyle\int_0^1 A^T(t)v(t)dt = 0$. Lastly, the condition $v \in \text{Fix}\, \Psi$ implies

$$v = -\int_0^{(.)} A^T(\tau)v(\tau)d\tau,$$

hence $v + \displaystyle\int_0^{(.)} A^T(\tau)v(\tau)d\tau = 0$. Consequently, using formula (6.13), we can assert that

$$\mathcal{A}^*(a, v) = 0. \tag{6.16}$$

Since $\mathcal{M}^*(a, v) = (\mathcal{A}^*(a, v), \mathcal{B}^*(a, v))$ by Lemma 6.1, from (6.15) and (6.16) it follows that $\mathcal{M}^*(a, v) = 0$. Thus, we have shown that $\ker \mathcal{T}^* \subset \ker \mathcal{M}^*$. □

The assumption (H_3) in [43] can be stated as follows

(A5) There exists a constant $c_3 > 0$ such that, for every $v \in \mathbb{R}^n$,

$$||C^T(t)v|| \geq c_3||v|| \ \text{ a.e. } t \in [0, 1].$$

Proposition 6.1 *If* (**A5**) *is satisfied, then* (**A3**) *and* (**A4**) *are fulfilled.*

Proof. By (**A5**) and the definition of Ψ_C, for any $v \in L^q([0, 1], \mathbb{R}^n)$, one has

$$||\Psi_C(v)(t)|| \geq c_3||v(t)|| \ \text{ a.e. } t \in [0, 1].$$

So, if $\Psi_C(v) = 0$, i.e., $\Psi_C(v)(t) = 0$ a.e. $t \in [0, 1]$, then $v(t) = 0$ a.e. $t \in [0, 1]$. This means that $\ker \Psi_C = \{0\}$. Therefore, condition (6.14) in (**A3**) is satisfied. By Lemma 6.1, we have $\mathcal{T}^*(a, v) = (a, C^T v)$ for every $(a, v) \in \mathbb{R}^n \times L^q([0, 1], \mathbb{R}^n)$. It follows that

$$||\mathcal{T}^*(a, v)|| = ||a|| + ||C^T v||_q.$$

Using (**A5**), we get

$$||\mathcal{T}^*(a, v)|| = ||a|| + ||C^T v||_q \geq c_1(||a|| + ||v||_q),$$

where $c_1 = \min\{1, c_3\}$. This means $||\mathcal{T}^* x^*|| \geq c_1||x^*||$ for every $x^* \in X^*$. By [40, Theorem 4.13, p. 100], $\mathcal{T} : W \to X$ is surjective. Since $\Phi(w, z)$ can be rewritten as $\Phi(w,z) = -\mathcal{T}w + \mathcal{M}z$, the surjectivity of $\mathcal{T}$ implies that $\{\Phi(w,0) \mid w \in W\} = X$. Hence Φ has closed range. $\square$

We are now in a position to formulate our main results on differential stability of problem (6.1)–(6.4). The following theorems not only completely describe the subdifferential and the singular subdifferential of the optimal value function, but also explain in detail the process of finding vectors belonging to the subdifferentials. In particular, from the results it follows that each subdifferential is either a singleton or an empty set.

Theorem 6.1 *Suppose that the optimal value function V in (6.8) is finite at $\bar{w} = (\bar{\alpha}, \bar{\theta})$, $\mathrm{int}\,\mathcal{U} \neq \emptyset$, and $(\mathbf{A1}) - (\mathbf{A4})$ are fulfilled. In addition, suppose that problem (6.1)–(6.4), with $\bar{w} = (\bar{\alpha}, \bar{\theta})$ playing the role of $w = (\alpha, \theta)$, has a solution $(\bar{x}, \bar{u})$. Then, a vector $(\alpha^*, \theta^*) \in \mathbb{R}^n \times L^q([0,1], \mathbb{R}^k)$ belongs to $\partial V(\bar{\alpha}, \bar{\theta})$ if and only if*

$$\alpha^* = g'(\bar{x}(1)) + \int_0^1 L_x(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t))dt - \int_0^1 A^T(t)y(t)dt,$$

$$\theta^*(t) = -C^T(t)y(t) + L_\theta(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)) \ \ a.e.\ t \in [0,1],$$

where $y \in W^{1,q}([0,1], \mathbb{R}^n)$ is the unique solution of the system

$$\begin{cases} \dot{y}(t) + A^T(t)y(t) = L_x(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)) \ \ a.e.\ \mathrm{t} \in [0,1], \\ y(1) = -g'(\bar{x}(1)), \end{cases}$$

such that the function $u^* \in L^q([0,1], \mathbb{R}^m)$ defined by

$$u^*(t) = B^T(t)y(t) - L_u(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)) \ \ a.e.\ t \in [0,1]$$

satisfies the condition $u^* \in N(\bar{u}; \mathcal{U})$.

Proof. We apply Theorem 5.2 in the case where $J(z,w)$, K and $V(w)$, respectively, play the roles of $f(z,w)$, Ω and $h(w)$. By Lemmas 6.2 and 6.3, conditions $(\mathbf{A1}) - (\mathbf{A4})$ guarantee that all the assumptions of Theorem 5.2 are satisfied. So, we have

$$\partial V(\bar{w}) = \bigcup_{v^* \in N(\bar{z}; K)} \left[\nabla_w J(\bar{z}, \bar{w}) + \mathcal{T}^*\big((\mathcal{M}^*)^{-1}(\nabla_z J(\bar{z}, \bar{w}) + v^*)\big)\right]. \quad (6.17)$$

From (6.17), $(\alpha^*, \theta^*) \in \partial V(\bar{w})$ if and only if

$$(\alpha^*, \theta^*) - \nabla_w J(\bar{z}, \bar{w}) \in \mathcal{T}^*\big((\mathcal{M}^*)^{-1}(\nabla_z J(\bar{z}, \bar{w}) + v^*)\big) \quad (6.18)$$

for some $v^* \in N(\bar{z}; K)$. Note that $\nabla_w J(\bar{z}, \bar{w}) = (0_{\mathbb{R}^n}, J_\theta(\bar{z}, \bar{w}))$ and $v^* = (0_{X^*}, u^*)$ for some $u^* \in N(\bar{u}; \mathcal{U})$. Hence, from (6.18) we get

$$(\alpha^*, \theta^* - J_\theta(\bar{z}, \bar{w})) \in \mathcal{T}^*\big((\mathcal{M}^*)^{-1}(\nabla_z J(\bar{z}, \bar{w}) + v^*)\big).$$

Thus, there exists $(a, v) \in \mathbb{R}^n \times L^q([0,1], \mathbb{R}^n)$ such that

$$(\alpha^*, \theta^* - J_\theta(\bar{z}, \bar{w})) \in \mathcal{T}^*(a, v) \ \text{ and } \ \nabla_z J(\bar{z}, \bar{w}) + v^* = \mathcal{M}^*(a, v). \tag{6.19}$$

By virtue of Lemma 6.1, we see that (6.19) is equivalent to the following

$$\begin{cases} \alpha^* = a, \ \theta^* - J_\theta(\bar{z}, \bar{w}) = C^T(\cdot)v(\cdot), \\ \big(J_x(\bar{x}, \bar{u}, \bar{w}), J_u(\bar{x}, \bar{u}, \bar{w}) + u^*\big) = (\mathcal{A}^*(a, v), \mathcal{B}^*(a, v)). \end{cases}$$

Invoking Lemma 6.2, we can rewrite this system as

$$\begin{cases} \alpha^* = a, \ \theta^* = L_\theta(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)) + C^T(\cdot)v(\cdot), \\ g'(\bar{x}(1)) + \displaystyle\int_0^1 L_x(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t))dt = a - \int_0^1 A^T(t)v(t)dt, \\ g'(\bar{x}(1)) + \displaystyle\int_{(\cdot)}^1 L_x(\tau, \bar{x}(\tau), \bar{u}(\tau), \bar{\theta}(\tau))d\tau = v(\cdot) + \int_0^{(\cdot)} A^T(\tau)v(\tau)d\tau \\ \qquad\qquad\qquad - \displaystyle\int_0^1 A^T(t)v(t)dt, \\ L_u(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)) + u^* = -B^T(\cdot)v(\cdot), \end{cases}$$

Clearly, the latter is equivalent to

$$\begin{cases} \alpha^* = a, \ \theta^* = L_\theta(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)) + C^T(\cdot)v(\cdot), \\ g'(\bar{x}(1)) + \displaystyle\int_0^1 L_x(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t))dt = a - \int_0^1 A^T(t)v(t)dt, \\ g'(\bar{x}(1)) - \displaystyle\int_1^{(\cdot)} L_x(\tau, \bar{x}(\tau), \bar{u}(\tau), \bar{\theta}(\tau))d\tau = v(\cdot) + \int_1^{(\cdot)} A^T(\tau)v(\tau)d\tau, \\ L_u(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)) + u^* = -B^T(\cdot)v(\cdot). \end{cases} \tag{6.20}$$

The third equality in (6.20) and the condition $v(\cdot) \in L^q([0,1], \mathbb{R}^n)$ imply that $v(\cdot)$ is absolutely differentiable on $[0,1]$ and, moreover, $\dot{v}(\cdot) \in L^q([0,1], \mathbb{R}^n)$. Hence $v(\cdot) \in W^{1,q}([0,1], \mathbb{R}^n)$. In addition, the third equality in (6.20) implies that $v(1) = g'(\bar{x}(1))$. Moreover, by differentiating, we get

$$-\dot{v}(\cdot) - A^T(\cdot)v(\cdot) = L_x(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)).$$

Therefore, (6.20) can be written as the following

$$\begin{cases} \alpha^* = a, \, v \in W^{1,q}([0,1], \mathbb{R}^n), \\ \alpha^* = g'(\bar{x}(1)) + \displaystyle\int_0^1 L_x(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t))dt + \int_0^1 A^T(t)v(t)dt, \\ \theta^* = L_\theta(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)) + C^T(\cdot)v(\cdot), \\ v(1) = g'(\bar{x}(1)), \\ -\dot{v}(\cdot) - A^T(\cdot)v(\cdot) = L_x(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)), \\ -B^T(\cdot)v(\cdot) = L_u(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)) + u^*. \end{cases} \tag{6.21}$$

Defining $y := -v$ and omitting the vector $\alpha^* = a \in \mathbb{R}^n$, we can put (6.21) in the form

$$\begin{cases} y \in W^{1,q}([0,1], \mathbb{R}^n), \\ \alpha^* = g'(\bar{x}(1)) + \displaystyle\int_0^1 L_x(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t))dt - \int_0^1 A^T(t)y(t)dt, \\ \theta^*(t) = L_\theta(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)) - C^T(t)y(t) \ \text{ a.e. } t \in [0,1], \\ \dot{y}(t) + A^T(t)y(t) = L_x(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)) \ \text{ a.e. } t \in [0,1], \\ y(1) = -g'(\bar{x}(1)), \\ B^T(t)y(t) - u^*(t) = L_u(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)) \ \text{ a.e. } t \in [0,1]. \end{cases}$$

The assertion of the theorem follows easily from this system. □

Next, let us show that how the singular subdifferential of $V(\cdot)$ can be computed.

Theorem 6.2 *Suppose that all the assumptions of Theorem 6.1 are satisfied. Then, a vector* $(\alpha^*, \theta^*) \in \mathbb{R}^n \times L^q([0,1], \mathbb{R}^k)$ *belongs to* $\partial^\infty V(\bar{w})$ *if and only if*

$$\alpha^* = \int_0^1 A^T(t)v(t)dt,$$

$$\theta^*(t) = C^T(t)v(t) \ \textit{ a.e. } t \in [0,1],$$

where $v \in W^{1,q}([0,1], \mathbb{R}^n)$ *is the unique solution of the system*

$$\begin{cases} \dot{v}(t) = -A^T(t)v(t) \ \textit{ a.e. } t \in [0,1], \\ v(0) = \alpha^*, \end{cases}$$

such that the function $u^* \in L^q([0,1], \mathbb{R}^m)$ *given by*

$$u^*(t) = -B^T(t)v(t) \ \textit{ a.e. } t \in [0,1]$$

belongs to $N(\bar{u}, \mathcal{U})$.

Proof. We apply Theorem 5.1 in the case where $J(z,w)$, K and $V(w)$, respectively, play the roles of $f(z,w)$, Ω and $h(w)$. By Lemmas 6.2 and 6.3, conditions (**A1**) – (**A4**) guarantee that all the assumptions of Theorem 5.1 are satisfied. Hence, by (5.15) we have

$$\partial^\infty V(\bar w) = \bigcup_{(w^*,z^*)\in\partial^\infty J(\bar z,\bar w)} \ \bigcup_{v^*\in N(\bar z;K)} \big[w^* + \mathcal{T}^*\big((\mathcal{M}^*)^{-1}(z^*+v^*)\big)\big]. \quad (6.22)$$

Since $\operatorname{dom} J = Z \times W$ and $\partial^\infty J(\bar z,\bar w) = N((\bar z,\bar w); \operatorname{dom} J)$, it holds that $\partial^\infty J(\bar z,\bar w) = \{(0_{Z^*}, 0_{W^*})\}$. Therefore, from (6.22) one gets

$$\partial^\infty V(\bar w) = \bigcup_{v^*\in N(\bar z;K)} \big[\mathcal{T}^*\big((\mathcal{M}^*)^{-1}(v^*)\big)\big].$$

Thus, a vector $(\alpha^*,\theta^*) \in \mathbb{R}^n \times L^q([0,1],\mathbb{R}^k)$ belongs to $\partial^\infty V(\bar w)$ if and only if one can find $v^* \in N(\bar z;K)$ and $(a,v) \in \mathbb{R}^n \times L^q([0,1],\mathbb{R}^n)$ such that

$$\mathcal{M}^*(a,v) = v^* \ \text{ and } \ \mathcal{T}^*(a,v) = (\alpha^*,\theta^*). \quad (6.23)$$

Since $N(\bar z;K) = \{0_{\mathbb{R}^n}\} \times N(\bar u;\mathcal{U})$, we must have $v^* = (0_{\mathbb{R}^n}, u^*)$ for some $u^* \in N(\bar u;\mathcal{U})$. By Lemma 6.1, we can rewrite (6.23) equivalently as

$$\begin{cases} a - \displaystyle\int_0^1 A^T(t)v(t)dt = 0, \\ v(\cdot) + \displaystyle\int_0^{(.)} A^T(\tau)v(\tau)d\tau - \int_0^1 A^T(t)v(t)dt = 0, \\ -B^T(\cdot)v(\cdot) = u^*(\cdot), \\ a = \alpha^*, \\ C^T(\cdot)v(\cdot) = \theta^*(\cdot). \end{cases}$$

Omitting the vector $a \in \mathbb{R}^n$, we transform this system to the form

$$\begin{cases} \alpha^* = \displaystyle\int_0^1 A^T(t)v(t)dt, \\ v(\cdot) = -\displaystyle\int_0^{(.)} A^T(\tau)v(\tau)d\tau + \int_0^1 A^T(t)v(t)dt, \\ u^*(\cdot) = -B^T(\cdot)v(\cdot), \\ \theta^*(\cdot) = C^T(\cdot)v(\cdot). \end{cases}$$

The second equality of the last system implies that $v \in W^{1,q}([0,1],\mathbb{R}^n)$ (see the detailed explanation in the proof of Theorem 6.1). Hence, that system is equivalent to the following

$$\begin{cases} v \in W^{1,q}([0,1], \mathbb{R}^n), \\ \alpha^* = \displaystyle\int_0^1 A^T(t)v(t)dt, \\ \dot{v}(t) = -A^T(t)v(t) \ \text{ a.e. } t \in [0,1], \\ v(0) = \displaystyle\int_0^1 A^T(t)v(t)dt, \\ u^*(t) = -B^T(t)v(t) \ \text{ a.e. } t \in [0,1], \\ \theta^*(t) = C^T(t)v(t) \ \text{ a.e. } t \in [0,1]. \end{cases}$$

These properties and the inclusion $u^* \in N(\bar{u}; \mathcal{U})$ show that the conclusion of the theorem is valid. □

6.3 Illustrative Examples

We shall apply the results obtained in Theorems 6.1 and 6.2 to an optimal control problem which has a clear mechanical interpretation.

Following Pontryagin *et al.* [34, Example 1, p. 23], we consider a vehicle of mass 1 moving without friction on a straight road, marked by an origin, under the impact of a force $u(t) \in \mathbb{R}$ depending on time $t \in [0,1]$. Denoting the coordinate of the vehicle at t by $x_1(t)$ and its velocity by $x_2(t)$. According to Newton's Second Law, we have $u(t) = 1 \times \ddot{x}_1(t)$; hence

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = u(t). \end{cases} \tag{6.24}$$

Suppose that the vehicle's initial coordinate and velocity are, respectively, $x_1(0) = \bar{\alpha}_1$ and $x_2(0) = \bar{\alpha}_2$. The problem is to minimize both the distance of the vehicle to the origin and its velocity at terminal time $t = 1$. Formally, it is required that the sum of squares $[x_1(1)]^2 + [x_2(1)]^2$ must be minimum when the measurable control $u(\cdot)$ satisfies the constraint $\displaystyle\int_0^1 |u(t)|^2 dt \leq 1$ (*an energy-type control constraint*).

It is worthy to stress that the above problem is different from the one considered in [34, Example 1, p. 23], where *the pointwise control constraint* $u(t) \in [-1, 1]$ was considered and the authors' objective is to minimize the terminal time moment $T \in [0, \infty)$ at which $x_1(T) = 0$ and $x_2(T) = 0$. The latter conditions mean that the vehicle arrives at the origin with the velocity 0. As far as we know, the classical Maximum Principle [34, Theorem 1, p. 19] cannot be applied to our problem.

We will analyze model (6.24) with the control constraint $\int_0^1 |u(t)|^2 dt \le 1$ by using the results of the preceding section. Let $X = W^{1,2}([0,1],\mathbb{R}^2)$, $U = L^2([0,1],\mathbb{R})$, $\Theta = L^2([0,1],\mathbb{R}^2)$. Choose $A(t) = A$, $B(t) = B$, $C(t) = C$ for all $t \in [0,1]$, where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Put $g(x) = \|x\|^2$ for $x \in \mathbb{R}^2$ and $L(t,x,u,\theta) = 0$ for $(t,x,u,\theta) \in [0,1] \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$. Let $\mathcal{U} = \{u \in L^2([0,1],\mathbb{R}) \mid ||u||_2 \le 1\}$. With the above described data set, the optimal control problem (6.1)–(6.4) becomes

$$\begin{cases} J(x,u,w) = x_1^2(1) + x_2^2(1) \to \inf \\ \dot{x}_1(t) = x_2(t) + \theta_1(t), \ \dot{x}_2(t) = u(t) + \theta_2(t), \\ x_1(0) = \alpha_1, \ x_2(0) = \alpha_2, \ u \in \mathcal{U}. \end{cases} \tag{6.25}$$

The perturbation $\theta_1(t)$ may represent a noise in the velocity, that is caused by a small wind. Similarly, the perturbation $\theta_2(t)$ may indicate a noise in the force, that is caused by the inefficiency and/or improperness of the reaction of the vehicle's engine in response to a human control decision. We define the function $\bar{\theta} \in \Theta$ by setting $\bar{\theta}(t) = (0,0)$ for all $t \in [0,1]$. The vector $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2) \in \mathbb{R}^2$ will be chosen in several ways.

In next examples, optimal solutions of (6.25) is sought for $\theta = \bar{\theta}$ and $\alpha = \bar{\alpha}$, where $\bar{\alpha}$ is taken from certain subsets of $\mathbb{R}^2$. These optimal solutions are used in the subsequent two examples, where we compute the subdifferential and the singular subdifferential of the optimal value function $V(w)$, $w = (\alpha, \theta) \in \mathbb{R}^2 \times \Theta$, of (6.25) at $\bar{w} = (\bar{\alpha}, \bar{\theta})$ by applying Theorems 6.1 and 6.2.

Example 6.1 Consider problem (6.25) at the parameter $w = \bar{w}$:

$$\begin{cases} J(x,u,\bar{w}) = x_1^2(1) + x_2^2(1) \to \inf \\ \dot{x}_1(t) = x_2(t), \ \dot{x}_2(t) = u(t), \\ x_1(0) = \bar{\alpha}_1, \ x_2(0) = \bar{\alpha}_2, \ u \in \mathcal{U}. \end{cases} \tag{6.26}$$

In the notation of Section 6.2, we interpret (6.26) as the parametric optimization problem

$$\begin{cases} J(x,u,\bar{w}) = x_1^2(1) + x_2^2(1) \to \inf \\ (x,u) \in G(\bar{w}) \cap K, \end{cases}$$

where $G(\bar{w}) = \{(x,u) \in X \times U \mid \mathcal{M}(x,u) = \mathcal{T}(\bar{w})\}$ and $K = X \times \mathcal{U}$. Then, in accordance with [22, Proposition 2, p. 81], $(\bar{x},\bar{u})$ is a solution of (6.26) if and only if

$$(0_{X^*}, 0_{U^*}) \in \partial_{x,u} J(\bar{x},\bar{u},\bar{w}) + N((\bar{x},\bar{u}); G(\bar{w}) \cap K). \tag{6.27}$$

Step 1 (computing the cone $N((\bar{x},\bar{u}); G(\bar{w}))$). We have

$$N((\bar{x},\bar{u}); G(\bar{w})) = \mathrm{rge}(\mathcal{M}^*) := \{\mathcal{M}^* x^* \mid x^* \in X^*\}. \tag{6.28}$$

Indeed, since $G(\bar{w}) = \{(x,u) \in X \times U \mid \mathcal{M}(x,u) = \mathcal{T}(\bar{w})\}$ is an affine manifold,

$$N((\bar{x},\bar{u}); G(\bar{w})) = (\ker \mathcal{M})^{\perp} \tag{6.29}$$

For any $z(\cdot) = (z_1(\cdot), z_2(\cdot)) \in X$, if we choose $x_2(t) = z_2(0)$ and $x_1(t) = z_1(t) + z_2(0)t$ for all $t \in [0,1]$, and $u(t) = -\dot{z}_2(t)$ for a.e. $t \in [0,1]$, then $(x,u) \in X \times U$ and $\mathcal{M}(x,u) = z$. This shows that the continuous linear operator $\mathcal{M} : X \times U \to X$ is surjective. In particular, $\mathcal{M}$ has closed range. Therefore, by Proposition 1.3, from (6.29) we get

$$N((\bar{x},\bar{u}); G(\bar{w})) = (\ker \mathcal{M})^{\perp} = \mathrm{rge}(\mathcal{M}^*) = \{\mathcal{M}^* x^* \mid x^* \in X^*\};$$

so (6.28) is valid.

Step 2 (decomposing the cone $N((\bar{x},\bar{u}); G(\bar{w}) \cap K)$). To prove that

$$N((\bar{x},\bar{u}); G(\bar{w}) \cap K) = \{0_{X^*}\} \times N(\bar{u}; \mathcal{U}) + N((\bar{x},\bar{u}); G(\bar{w})), \tag{6.30}$$

we first notice that

$$N((\bar{x},\bar{u}); K) = \{0_{X^*}\} \times N(\bar{u}; \mathcal{U}). \tag{6.31}$$

Next, let us verify the normal qualification condition

$$N((\bar{x},\bar{u}); K) \cap [-N((\bar{x},\bar{u}); G(\bar{w}))] = \{(0,0)\} \tag{6.32}$$

for the convex sets K and $\mathrm{gph}\, G$. Take any

$$(x_1^*, u_1^*) \in N((\bar{x},\bar{u}); K) \cap [-N((\bar{x},\bar{u}); G(\bar{w}))].$$

On one hand, by (6.31) we have $x_1^* = 0$ and $u_1^* \in N(\bar{u}; \mathcal{U})$. On the other hand, by (6.28) and the third assertion of Lemma 6.1, we can find an element

$$x^* = (a, v) \in X^* = \mathbb{R}^2 \times L^2([0,1], \mathbb{R}^2)$$

such that $x_1^* = -\mathcal{A}^*(a, v)$ and $u_1^* = -\mathcal{B}^*(a, v)$. Then

$$0 = \mathcal{A}^*(a, v), \;\; u_1^* = -\mathcal{B}^*(a, v). \tag{6.33}$$

Write $a = (a_1, a_2)$, $v = (v_1, v_2)$ with $a_i \in \mathbb{R}$ and $v_i \in L^2([0, 1], \mathbb{R})$, $i = 1, 2$. According to Lemma 6.1, (6.33) is equivalent to the following system

$$\begin{cases} a_1 = 0, \; a_2 - \displaystyle\int_0^1 v_1(t)dt = 0, \\ v_1 = 0, \\ v_2 + \displaystyle\int_0^{(.)} v_1(\tau)d\tau - \int_0^1 v_1(t)dt = 0, \\ u_1^* = v_2. \end{cases} \tag{6.34}$$

From (6.34) it follows that $(a_1, a_2) = (0, 0)$, $(v_1, v_2) = (0, 0)$ and $u_1^* = 0$. Thus $(x_1^*, u_1^*) = (0, 0)$. Hence, (6.32) is fulfilled.

Furthermore, since $\mathcal{U} = \{u \in L^2([0, 1], \mathbb{R}) \mid ||u||_2 \leq 1\}$, we have $\mathrm{int}\,\mathcal{U} \neq \emptyset$; so K is a convex set with nonempty interior. Due to (6.32), one cannot find any $(x_0^*, u_0^*) \in N((\bar{x}, \bar{u}); K)$ and $(x_1^*, u_1^*) \in N((\bar{x}, \bar{u}); G(\bar{w}))$, not all zero, with $(x_0^*, u_0^*) + (x_1^*, u_1^*) = 0$. Hence, by Proposition 1.5, $G(\bar{w}) \cap \mathrm{int}\, K \neq \emptyset$. Moreover, according to Proposition 1.4, we have

$$N((\bar{x}, \bar{u}); G(\bar{w}) \cap K) = N((\bar{x}, \bar{u}); K) + N((\bar{x}, \bar{u}); G(\bar{w})).$$

Hence, combining the last equation with (6.31) yields (6.30).

Step 3 (computing the partial subdifferentials of $J(\cdot, \cdot, \bar{w})$ at $(\bar{x}, \bar{u})$). We first note that $J(x, u, \bar{w})$ is a convex function. Clearly, the assumptions (**A1**) and (**A2**) are satisfied. Hence, by Lemma 6.2, the function $J(x, u, \bar{w}) = g(x(1)) = x_1^2(1) + x_2^2(1)$ is Fréchet differentiable at $(\bar{x}, \bar{u})$, $J_u(\bar{x}, \bar{u}, \bar{w}) = 0_{U^*}$, and

$$J_x(\bar{x}, \bar{u}, \bar{w}) = \big(g'(\bar{x}(1)), g'(\bar{x}(1))\big) = \big((2\bar{x}_1(1), 2\bar{x}_2(1)), (2\bar{x}_1(1), 2\bar{x}_2(1))\big), \tag{6.35}$$

where the first symbol $(2\bar{x}_1(1), 2\bar{x}_2(1))$ is a vector in $\mathbb{R}^2$, while the second symbol $(2\bar{x}_1(1), 2\bar{x}_2(1))$ signifies the constant function $t \mapsto (2\bar{x}_1(1), 2\bar{x}_2(1))$ from $[0, 1]$ to $\mathbb{R}^2$. Therefore, one has

$$\partial J_{x,u}(\bar{x}, \bar{u}, \bar{w}) = \big\{\big(J_x(\bar{x}, \bar{u}, \bar{w}), 0_{U^*}\big)\big\} \tag{6.36}$$

with $J_x(\bar{x}, \bar{u}, \bar{w})$ being given by (6.35).

Step 4 (solving the optimality condition) By (6.28), (6.30), and (6.36), we can assert that (6.27) is fulfilled if and only if there exist $u^* \in N(\bar{u}; \mathcal{U})$

and $x^* = (a, v) \in \mathbb{R}^2 \times L^2([0,1], \mathbb{R}^2)$ with $a = (a_1, a_2) \in \mathbb{R}^2$, $v = (v_1, v_2) \in L^2([0,1], \mathbb{R}^2)$, such that

$$\big(((-2\bar{x}_1(1), -2\bar{x}_2(1)), (-2\bar{x}_1(1), -2\bar{x}_2(1))), -u^*\big) = \mathcal{M}^*(a, v). \tag{6.37}$$

According to Lemma 6.3, we have $\mathcal{M}^*(a, v) = (\mathcal{A}^*(a, v), \mathcal{B}^*(a, v))$, where

$$\mathcal{A}^*(a, v) = \Big(a - \int_0^1 A^T(t)v(t)dt,\ v + \int_0^{(.)} A^T(\tau)v(\tau)d\tau - \int_0^1 A^T(t)v(t)dt\Big),$$

and $\mathcal{B}^*(a, v) = -B^T v$. Combining this with (6.37) gives

$$\begin{cases} -2\bar{x}_1(1) = a_1,\ \ -2\bar{x}_2(1) = a_2 - \displaystyle\int_0^1 v_1(t)dt, \\ -2\bar{x}_1(1) = v_1,\ -2\bar{x}_2(1) = v_2 + \displaystyle\int_0^{(.)} v_1(\tau)d\tau - \int_0^1 v_1(t)dt, \\ u^* = v_2. \end{cases} \tag{6.38}$$

If we can choose $a = 0$ and $v = 0$ for (6.38), then $u^* = 0$; so $u^* \in N(\bar{u}; \mathcal{U})$. Moreover, (6.38) reduces to

$$\bar{x}_1(1) = 0,\ \ \bar{x}_2(1) = 0. \tag{6.39}$$

Besides, we observe that $(\bar{x}, \bar{u}) \in G(\bar{w})$ if and only if

$$\begin{cases} \dot{\bar{x}}_1(t) = \bar{x}_2(t),\ \ \dot{\bar{x}}_2(t) = \bar{u}(t), \\ \bar{x}_1(0) = \bar{\alpha}_1,\ \bar{x}_2(0) = \bar{\alpha}_2,\ \bar{u} \in \mathcal{U}. \end{cases} \tag{6.40}$$

Combining (6.39) with (6.40) yields

$$\begin{cases} \dot{\bar{x}}_1(1) = 0,\ \ \dot{\bar{x}}_1(0) = \bar{\alpha}_2, \\ \bar{x}_1(0) = \bar{\alpha}_1,\ \bar{x}_1(1) = 0, \\ \dot{\bar{x}}_1(t) = \bar{x}_2(t),\ \ \dot{\bar{x}}_2(t) = \bar{u}(t), \\ \bar{u} \in \mathcal{U}. \end{cases} \tag{6.41}$$

We shall find $\bar{x}_1(t)$ in the form $\bar{x}_1(t) = at^3 + bt^2 + ct + d$. Substituting this $\bar{x}_1(t)$ into the first four equalities in (6.41), we get

$$\begin{cases} 3a + 2b + c = 0,\ \ c = \bar{\alpha}_2, \\ d = \bar{\alpha}_1,\ \ a + b + c + d = 0. \end{cases}$$

Solving this system, we have $a = 2\bar{\alpha}_1 + \bar{\alpha}_2$, $b = -3\bar{\alpha}_1 - 2\bar{\alpha}_2$, $c = \bar{\alpha}_2$, $d = \bar{\alpha}_1$. Then $\bar{x}_1(t) = (2\bar{\alpha}_1 + \bar{\alpha}_2)t^3 - (3\bar{\alpha}_1 + 2\bar{\alpha}_2)t^2 + \bar{\alpha}_2 t + \bar{\alpha}_1$. So, from the fifth and the sixth equalities in (6.41) it follows that

$$\begin{cases} \bar{x}_2(t) = \dot{\bar{x}}_1(t) = 3(2\bar{\alpha}_1 + \bar{\alpha}_2)t^2 - 2(3\bar{\alpha}_1 + 2\bar{\alpha}_2)t + \bar{\alpha}_2, \\ \bar{u}(t) = \dot{\bar{x}}_2(t) = (12\bar{\alpha}_1 + 6\bar{\alpha}_2)t - (6\bar{\alpha}_1 + 4\bar{\alpha}_2). \end{cases}$$

Now, condition $\bar{u} \in \mathcal{U}$ in (6.41) means that

$$1 \geq \int_0^1 |\bar{u}(t)|^2 dt = \int_0^1 \left[(12\bar{\alpha}_1 + 6\bar{\alpha}_2)t - (6\bar{\alpha}_1 + 4\bar{\alpha}_2)\right]^2 dt. \tag{6.42}$$

By simple computation, we see that (6.42) is equivalent to

$$12\bar{\alpha}_1^2 + 12\bar{\alpha}_1\bar{\alpha}_2 + 4\bar{\alpha}_2^2 - 1 \leq 0. \tag{6.43}$$

Clearly, the set Ω of all the points $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2) \in \mathbb{R}^2$ satisfying (6.43) is an ellipse. We have shown that for every $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2)$ from Ω, problem (6.26) has an optimal solution $(\bar{x}, \bar{u})$, where

$$\begin{cases} \bar{x}_1(t) = (2\bar{\alpha}_1 + \bar{\alpha}_2)t^3 - (3\bar{\alpha}_1 + 2\bar{\alpha}_2)t^2 + \bar{\alpha}_2 t + \bar{\alpha}_1, \\ \bar{x}_2(t) = 3(2\bar{\alpha}_1 + \bar{\alpha}_2)t^2 - 2(3\bar{\alpha}_1 + 2\bar{\alpha}_2)t + \bar{\alpha}_2, \\ \bar{u}(t) = (12\bar{\alpha}_1 + 6\bar{\alpha}_2)t - (6\bar{\alpha}_1 + 4\bar{\alpha}_2). \end{cases} \tag{6.44}$$

In this case, the optimal value is $J(\bar{x}, \bar{u}'\bar{w}) = 0$.

In the forthcoming two examples, we will use Theorems 6.1 and 6.2 to compute the subdifferential and the singular subdifferential of the optimal value function $V(w)$ of (6.25) at $\bar{w} = (\bar{\alpha}, \bar{\theta})$, where $\bar{\alpha}$ satisfies condition (6.43). Recall that the set of all the points $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2) \in \mathbb{R}^2$ satisfying (6.43) is an ellipse, which has been denoted by Ω.

Example 6.2 *(Optimal trajectory is implemented by an internal optimal control)* For $\alpha = \bar{\alpha} := \left(\frac{1}{5}, 0\right)$, that belongs to $\operatorname{int}\Omega$, and $\theta = \bar{\theta}$ with $\bar{\theta}(t) = (0, 0)$ for all $t \in [0, 1]$, the control problem (6.25) becomes

$$\begin{cases} J(x, u) = ||x(1)||^2 \to \inf \\ \dot{x}_1(t) = x_2(t),\ \dot{x}_2(t) = u(t), \\ x_1(0) = \frac{1}{5},\ x_2(0) = 0,\ u \in \mathcal{U}. \end{cases} \tag{6.45}$$

For the parametric problem (6.25), it is clear that the assumptions (**A1**) and (**A2**) are satisfied. As $C(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for $t \in [0, 1]$, one has for every $v \in \mathbb{R}^2$ the following

$$||C^T(t)v|| = ||v|| \text{ a.e. } t \in [0, 1].$$

Hence, the assumption (**A5**) is also satisfied. Then, by Proposition 6.1, the assumptions (**A3**) and (**A4**) are fulfilled. According to (6.44) and the analysis given in Example 6.1, the pair $(\bar{x}, \bar{u}) \in X \times U$, where $\bar{x}(t) = \left(\frac{2}{5}t^3 - \frac{3}{5}t^2 + \frac{1}{5}, \frac{6}{5}t^2 - \frac{6}{5}t\right)$ and $\bar{u}(t) = \frac{12}{5}t - \frac{6}{5}$ for $t \in [0,1]$, is a solution of (6.45). In this case, $\bar{u}(t)$ is an interior point of $\mathcal{U}$ since $\int_0^1 |\bar{u}(t)|^2 dt = \frac{12}{25} < 1$. Thus, by Theorem 6.1, a vector $(\alpha^*, \theta^*) \in \mathbb{R}^2 \times L^2([0,1], \mathbb{R}^2)$ belongs to $\partial V(\bar{\alpha}, \bar{\theta})$ if and only if

$$\alpha^* = g'(\bar{x}(1)) - \int_0^1 A^T(t) y(t) dt \tag{6.46}$$

and

$$\theta^*(t) = -C^T(t) y(t) \ \text{ a.e. } t \in [0,1], \tag{6.47}$$

where $y \in W^{1,2}([0,1], \mathbb{R}^2)$ is the unique solution of the system

$$\begin{cases} \dot{y}(t) = -A^T(t) y(t) \ \text{ a.e. } t \in [0,1], \\ y(1) = -g'(\bar{x}(1)), \end{cases} \tag{6.48}$$

such that the function $u^* \in L^2([0,1], \mathbb{R})$ defined by

$$u^*(t) = B^T(t) y(t) \ \text{ a.e. } t \in [0,1] \tag{6.49}$$

satisfies the condition $u^* \in N(\bar{u}; \mathcal{U})$.

Since $\bar{x}(1) = (0,0)$, we have $g'(\bar{x}(1)) = (0,0)$. So, (6.48) can be rewritten as

$$\begin{cases} \dot{y}_1(t) = 0, \ \dot{y}_2(t) = -y_1(t), \\ y_1(1) = 0, \ y_2(1) = 0. \end{cases}$$

Clearly, $y(t) = (0,0)$ is the unique solution of this terminal value problem. Combining this with (6.46), (6.47) and (6.49), we obtain $\alpha^* = (0,0)$ and $\theta^*(t) = \theta^* = (0,0)$ a.e. $t \in [0,1]$, and $u^*(t) = 0$ a.e. $t \in [0,1]$. Since $u^*(t) = 0$ satisfies the condition $u^* \in N(\bar{u}; \mathcal{U})$, we have $\partial V(\bar{w}) = \{(\alpha^*, \theta^*)\}$, where $\alpha^* = (0,0)$ and $\theta^* = (0,0)$.

We now compute $\partial V^\infty(\bar{\alpha}, \bar{\theta})$. By Theorem 6.2, $(\tilde{\alpha}^*, \tilde{\theta}^*) \in \mathbb{R}^2 \times L^2([0,1], \mathbb{R}^2)$ belongs to $\partial^\infty V(\bar{w})$ if and only if

$$\tilde{\alpha}^* = \int_0^1 A^T(t) v(t) dt, \tag{6.50}$$

$$\tilde{\theta}^*(t) = C^T(t) v(t) \ \text{ a.e. } t \in [0,1], \tag{6.51}$$

where $v \in W^{1,2}([0,1], \mathbb{R}^2)$ is the unique solution of the system

$$\begin{cases} \dot{v}(t) = -A^T(t)v(t) \ \text{ a.e. } t \in [0,1], \\ v(0) = \tilde{\alpha}^*, \end{cases} \tag{6.52}$$

such that the function $\tilde{u}^* \in L^2([0,1], \mathbb{R})$ given by

$$\tilde{u}^*(t) = -B^T(t)v(t) \ \text{ a.e. } t \in [0,1] \tag{6.53}$$

belongs to $N(\bar{u}; \mathcal{U})$. Thanks to (6.50), we can rewrite (6.52) as

$$\begin{cases} \dot{v}_1(t) = 0, \ \dot{v}_2(t) = -v_1(t), \\ v_1(0) = 0, \ v_2(0) = \displaystyle\int_0^1 v_1(t)dt. \end{cases}$$

It is easy to show that $v(t) = (0,0)$ is the unique solution of this system. Hence, (6.50), (6.51) and (6.53) imply that $\tilde{\alpha}^* = (0,0)$, $\tilde{\theta}^* = (0,0)$ and $\tilde{u}^* = 0$. Since $\tilde{u}^* \in N(\bar{u}; \mathcal{U})$, we have $\partial^\infty V(\bar{w}) = \{(\tilde{\alpha}^*, \tilde{\theta}^*)\}$, where $\tilde{\alpha}^* = (0,0)$ and $\tilde{\theta}^* = (0,0)$.

Example 6.3 *(Optimal trajectory is implemented by a boundary optimal control)* For $\alpha = \bar{\alpha} := \left(0, \frac{1}{2}\right)$, that belongs to $\partial\Omega$, and $\theta = \bar{\theta}$ with $\bar{\theta}(t) = (0,0)$ for all $t \in [0,1]$, problem (6.25) becomes

$$\begin{cases} J(x,u) = ||x(1)||^2 \to \inf \\ \dot{x}_1(t) = x_2(t), \ \dot{x}_2(t) = u(t), \\ x_1(0) = 0, \ x_2(0) = \frac{1}{2}, \ u \in \mathcal{U}. \end{cases} \tag{6.54}$$

As it has been shown in Example 6.1,

$$(\bar{x}, \bar{u}) = \left(\frac{1}{2}t^3 - t^2 + \frac{1}{2}t, \frac{3}{2}t^2 - 2t + \frac{1}{2}, 3t - 2\right)$$

is a solution of (6.54). In this case, we have $\int_0^1 |\bar{u}(t)|^2 dt = \int_0^1 (3t-2)^2 dt = 1$. This means that $\bar{u}(t)$ is a boundary point of $\mathcal{U}$. So, $N(\bar{u}; \mathcal{U}) = \{\lambda \bar{u} \mid \lambda \geq 0\}$. Since $\bar{x}(1) = (0,0)$, arguing in the same manner as in Example 6.2, we obtain $\partial V(\bar{w}) = \{(\alpha^*, \theta^*)\}$ and $\partial^\infty V(\bar{w}) = \left\{\left(\tilde{\alpha}^*, \tilde{\theta}^*\right)\right\}$, where $\alpha^* = \tilde{\alpha}^* = (0,0)$ and $\theta^* = \tilde{\theta}^* = (0,0)$.

6.4 Conclusions

We have obtained some formulas for computing the subdifferentials of the optimal value function of parametric constrained optimal control problems with a convex objective function and linear state equations. In combination with Chapter 5, this chapter shows that the results in Chapter 2 can be apply to both convex discrete optimal control problem and convex continuous optimal control problem.

General Conclusions

The main results of this dissertation include:

1) Formulas for computing or estimating the subdifferential and the singular subdifferential of the optimal value function of parametric convex mathematical programming problems under inclusion constraints;

2) Formulas showing the connection between the subdifferentials of the optimal value function of parametric convex mathematical programming problems under geometrical and/or functional constraints and certain multiplier sets;

3) Formulas for computing the subdifferential and the singular subdifferential of the optimal value function of convex optimal control problems under linear constraints via the problem data.

List of Author's Related Papers

1. D.T.V. An and N.D. Yen, *Differential stability of convex optimization problems under inclusion constraints*, Applicable Analysis **94** (2015), 108–128. (SCIE)
2. D.T.V. An and J.-C. Yao, *Further results on differential stability of convex optimization problems*, Journal of Optimization Theory and Applications **170** (2016), 28–42. (SCI)
3. D.T.V. An and N.T. Toan, *Differential stability of convex discrete optimal control problems,* Acta Mathematica Vietnamica **43** (2018), 201–217. (Scopus, ESCI)
4. D.T.V. An, J.-C. Yao, and N.D. Yen, *Differential stability of a class of convex optimal control problems,* Applied Mathematics and Optimization (2017), DOI 10.1007/s00245-017-9475-4. (SCI)
5. D.T.V. An and N.D. Yen, *Subdifferential stability analysis for convex optimization problems via multiplier sets*, Vietnam Journal of Mathematics **46** (2018), 365–379. (Scopus, ESCI)

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