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DO THAI DUONG

SOME PROBLEMS IN PLURIPOTENTIAL THEORY

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SUMMARY

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Supervisor: Prof. Dr.Sc. Pham Hoang Hiep
Institute of Mathematics
Vietnam Academy of Science and Technology

Prof. Dr.Sc. Dinh Tien Cuong
Department of Mathematics
National University of Singapore

Introduction

In the 19th century physics, two fundamental forces of nature known at the time, namely gravity and the electrostatic force, were believed to be derived from using functions called "potentials" which satisfied Laplace's equation. The term "potential theory" or "classical potential theory" arose to describe a linear theory associated to the Laplacian operator. This theory focused on harmonic functions, subharmonic functions, the Dirichlet problem, harmonic measure, Green's functions, potentials and capacity in several real variables.

The potential theory in two dimension, which always be considered as the potential theory in the complex plane, has attracted considerable interest since it is closely related to complex analysis. In particular, there is a close connect between Laplace's equation and analytic functions. While the real and imaginary parts of analytic functions of a complex variable satisfy the Laplace's equation in two dimensions, the solution to Laplace's equation is the real part of an analytic function. In general, some techniques of complex analysis, particularly conformal mapping, can be used to simplify proofs of some results in the potential theory while some theorems in potential theory have analogies and applications in complex analysis.

In the 20th century, pluripotential theory was developed as the several complex variables analogue of the classical potential theory in the complex plane. This theory is highly non-linear and associated to complex Monge-Ampère operators. The basic objects are plurisubharmonic functions of several complex variables which were defined in 1942 by Kiyoshi Oka and Pierre Lelong. This class is the natural counterpart of the class of subharmonic functions of one complex variable. The plurisubharmonic functions are also be considered as subharmonic functions on several real variables which are invariant with respect to biholomorphic mappings.

In this dissertation, with three chapters, we study some specific problems in pluripotential theory and potential theory.

In Chapter 1, we study some properties of subharmonic functions on several real variables. Motivated by the fact that two subharmonic functions which agree almost everywhere on a domain with respect to the Lebesgue measure must coincide everywhere on that domain, we are interested in the following problem.

Problem 1 *Whether we can conclude that two subharmonic functions which agree almost everywhere on a surface with respect to the surface measure must coincide everywhere on that surface?*

This chapter is devoted to answer Problem 1 completely. For this purpose, we prove two main theorems with similar assumptions. They concern restrictions of subharmonic functions in Ω to a Borel subset $K \subset \Omega$ which together with a measure μ is subject to some quite technical assumptions. These allow K to have co-dimension one (and a little more, but not two), with μ being more or less like a corresponding Hausdorff measure. The first main result is an extension of the mean value theorem, states that the mean value theorem, in an infinitesimal form, still holds when restricted to K, and with respect to μ. The second main result is a comparison theorem for subharmonic functions, states that a comparison between an upper semicontinuous function and a subharmonic function which holds almost everywhere (with respect to μ) on K actually holds at every point of K. By these theorems, we prove that Problem 1 has a positive answer in the case of hypersurfaces. We also provide a counterexample in the case of surfaces of higher co-dimension. In addition, we apply main theorems to Ahlfors-David regular sets to obtain some consequences and we prove other versions of the main results in terms of measure densities.

In Chapter 2, we study the Dirichlet problem for the complex Monge-Ampère equation. We are interested in the following problem.

Problem 2 *Find conditions for μ such that the solution u of the Dirichlet problem for the complex Monge-Ampère equation is continuous outside an analytic set but u may not be continuous in Ω.*

This problem arises from the fact that there are some plurisubharmonic functions which are not continuous in the whole domain even though they are continuous outside an analytic set. For example, $u(z) = -(-\log\|z\|)^{1/2}$ is not continuous in the whole unit ball $\mathbb{B}^{2n}$, but it is continuous in $\mathbb{B}^{2n}\backslash\{0\}$. In studying this problem, we prove a sufficient condition which relaxes assumptions of a well-known result of Kołodjiej (Theorem B in [26]) to some technical assumptions. These assumptions naturally lead to the following problem.

Problem 3 *Find conditions for α such that*

$$v = -(-\log(|f_1|^{\lambda_1} + ... + |f_m|^{\lambda_m}))^{\alpha}$$

belongs to domain of Monge-Ampère operator, where $f_1, ..., f_m$ are analytic functions.

Note that if $0 < \alpha < \frac{1}{n}$ then $v \in \mathcal{E}$ (see [7], [11]). Our result is a necessary and sufficient condition where we further assume some conditions of $f_1, ..., f_m$ and the non-singularity of their zero-sets.

Chapter 3 is devoted to study the behavior near boundary of the function from class $\mathcal{F}$ in strictly pseudoconvex domain Ω, in particular the estimate of the sublevel set of the plurisubharmonic function near the boundary of a domain. In fact, we are motivated by two sources: firstly, we are inspired from [8] where the author gave the characterization of the class $\mathcal{F}$ in terms of the capacity of the sublevel set and secondly from the fact that, for a bounded hyperconvex domain Ω, although, for every $u \in \mathcal{F}(\Omega)$,

$$\limsup_{z\to\partial\Omega} u(z) = 0,$$

being in $\mathcal{F}(\Omega)$ does not necessarily give that $\limsup_{z\to\partial\Omega} u(z) = 0$. We are interested in the following problem.

Problem 4 *Find necessary and sufficient conditions of the volume of the sublevel sets near certain boundary points of functions from the class $\mathcal{F}$ in strictly pseudoconvex domains.*

In studying this problem, we give an necessary condition for upper bound of their volume and its consequences. On special case when the domain is the unit ball, we prove a necessary condition and a sufficient condition for membership of the class $\mathcal{F}$. They both concern decay near boundary of volume of the sublevel sets.

Chapter 1

A comparison theorem for subharmonic functions

The present chapter, which is written on the basis of the paper "Thai Duong Do, *A comparison theorem for subharmonic functions,* Results Math. **74** (2019), Paper No. 176, 13pp", devote to study Problem 1 mentioned at the Introduction.

- In Section 1.1 and 1.2, we recall some basic properties of subharmonic functions and some basic properties of Hausdorff measure which will be used in the sequel. For more details, the reader is referred to [17, 22, 25, 29, 30, 32, 36].
- In Section 1.3, we prove our first main result "An extension of the mean value theorem" and its corollaries.

Firstly, we introduce a family of admissible functions as follow.

$$H = \Big\{ h : (0,+\infty) \to (0,+\infty) : \exists\, M > 0,\, c > 4 \text{ such that } \int_0^{c\epsilon} \frac{h(r)}{r^{n-1}} dr \le M \frac{h(\epsilon)}{\epsilon^{n-2}}, \text{ for } \epsilon \text{ small enough} \Big\}.$$

Our extension of the mean value theorem is the following.

Theorem 1.1 (Extension of the mean value theorem) *Let u be a subharmonic function in a domain Ω, in $\mathbb{R}^n$ $(n \ge 2)$, K be a Borel subset of Ω, x_0 be a point of K and $h \in H$. Suppose that there exist a positive Borel measure μ, real numbers $A, B > 0$ and $\epsilon_0 > 0$ satisfying the following:*

1. *$\mu\big(K \cap \mathbb{B}(x_0, \epsilon)\big) \ge Ah(\epsilon)$ for all $\epsilon < \epsilon_0$,*
2. *$\mu\big(K \cap \mathbb{B}(x, \epsilon)\big) \le Bh(\epsilon)$ for all $x \in K$ and $\epsilon < \epsilon_0$.*

Then

$$\lim_{\epsilon \to 0} \frac{1}{\mu\big(K \cap \mathbb{B}(x_0, \epsilon)\big)} \int_{K \cap \mathbb{B}(x_0,\epsilon)} u(x) d\mu(x) = u(x_0).$$

The proof of Theorem 1.1 uses the Riesz Decomposition theorem and the Lebesgue's Dominated Convergence theorem as main tools.

Note that when K is appropriate and h is a gauge, we can apply Theorem 1.1 for μ as Generalized Hausdorff h-measures. In particular, by choosing $h(r) = r^k$, where $k > n-2$, we have a direct consequence as following.

Corollary 1.1 *Let u be a subharmonic function in a domain Ω, in $\mathbb{R}^n$ $(n \geq 2)$ and $U \subset \Omega$ be an Ahlfors-David regular set with dimension $k > n-2$. Then*

$$\lim_{\epsilon \to 0} \frac{1}{H^k(U \cap \mathbb{B}(x,\epsilon))} \int\limits_{U \cap \mathbb{B}(x,\epsilon)} u(y) dH^k(y) = u(x)$$

for all $x \in U$.

Moreover, if we consider K as a hypersurface, μ is the hypersurface measure and $h(r) = r^{n-1}$, we obtain the following immediate corollary:

Corollary 1.2 *Let u be a subharmonic function in a domain Ω, in $\mathbb{R}^n$ $(n \geq 2)$ and $\mathbb{H}$ be a hypersurface. Then*

$$\lim_{\epsilon \to 0} \frac{1}{\sigma\big(\mathbb{H} \cap \mathbb{B}(x_0,\epsilon)\big)} \int\limits_{\mathbb{H} \cap \mathbb{B}(x_0,\epsilon)} u(x) d\sigma(x) = u(x_0)$$

for all $x_0 \in \mathbb{H} \cap \Omega$, where σ is the surface measure on $\mathbb{H}$.

- In Section 1.4, we prove our second main result "A comparison theorem for subharmonic functions". We also establish its corollaries and construct a counterexample which answer the Problem 1 completely.

First, we introduce our comparison theorem for subharmonic functions.

Theorem 1.2 (Comparison theorem for subharmonic functions) *Let u be an upper semicontinuous function, v be a subharmonic function in a domain Ω, in $\mathbb{R}^n$ $(n \geq 2)$ and let K be a Borel subset of Ω, $h \in H$. Suppose that there exist a positive Borel measure μ, real numbers $A, B > 0$, $\epsilon_0 > 0$ and $N \subset K$ satisfying the following:*

1. $\mu(N) = 0$,
2. $Ah(\epsilon) \leq \mu\big(K \cap \mathbb{B}(x,\epsilon)\big) \leq Bh(\epsilon)$ *for all $x \in K$ and $\epsilon < \epsilon_0$,*
3. $u \geq v$ *on* $K \setminus N$.

Then $u \geq v$ on K.

For the proof of Theorem 1.2, we use Theorem 1.1 and the upper semicontiniuty of u as main tools.

Next, by applying Theorem 1.2 for $k-$dimensional Hausdorff measure $(k > n-2)$ and hypersurface measure, we obtain some direct consequences as followings.

Corollary 1.3 *Let Ω be a domain in $\mathbb{R}^n$ $(n \geq 2)$ and $E \subset \Omega$ be an Ahlfors-David regular set with dimension $k > n-2$. Let u be an upper semicontinuous function and v be a subharmonic function in Ω. Suppose that $u \geq v$ almost everywhere on E with respect to k-dimensional Hausdorff measure. Then $u \geq v$ on E.*

Corollary 1.4 *Let Ω be a domain in $\mathbb{R}^n$ $(n \geq 2)$ and let $\mathbb{H}$ be a hypersurface such that $\mathbb{H} \cap \Omega \neq \emptyset$. Let u be an upper semicontinuous function and v be a subharmonic function in Ω. Suppose that $u \geq v$ almost everywhere on $\mathbb{H} \cap \Omega$ with respect to the surface measure on $\mathbb{H}$. Then $u \geq v$ on $\mathbb{H} \cap \Omega$.*

As a direct consequence of Corollary 1.4, we can conclude that two subharmonic functions which agree almost everywhere on a hypersurface with respect to the surface measure must coincide everywhere on that hypersurface. In other words, Question 1 has a positive answer in the case of hypersurfaces.

We end this section by constructing a counterexample to show that Question 1 has a negative answer in the case of surfaces of higher co-dimension.

Example 1.1 (Counterexample) *In $\mathbb{R}^n$ $(n \geq 3)$, we denote by $\mathbb{B}_{n-2}(0, R)$ the open ball in $\mathbb{R}^{n-2}$, with center at 0 and radius $R > 0$. For $i \geq 2$, let μ_i be the measure defined on $\left(\mathbb{B}_{n-2}(0, i) \setminus \mathbb{B}_{n-2}(0, \frac{1}{i})\right) \times \{0\} \times \{0\}$ generated by the $(n-2)$-dimensional Lebesgue measure on $\left(\mathbb{B}_{n-2}(0, i) \setminus \mathbb{B}_{n-2}(0, \frac{1}{i})\right)$. We define potentials $p_{\mu_i} : \mathbb{R}^n \longrightarrow [-\infty, \infty)$ by*

$$p_{\mu_i}(x) = \int_{\mathbb{R}^n} \frac{-1}{|x - w|^{n-2}} d\mu_i(w).$$

Then we obtain the sequence $\{p_{\mu_i}\}_{i \geq 2} \subset \mathcal{SH}(\mathbb{R}^n)$ which satisfies these properties:

$$\begin{cases} p_{\mu_i} \leq 0 \ \text{on } \mathbb{R}^n, \\ p_{\mu_i} = -\infty \ \text{on } \left(\mathbb{B}_{n-2}(0, i) \setminus \mathbb{B}_{n-2}(0, \frac{1}{i})\right) \times \{0\} \times \{0\}, \\ -\infty < p_{\mu_i}(0) < 0, \end{cases}$$

for all $i \geq 2$. By setting

$$u_i(x) = -\frac{p_{\mu_i}(x)}{p_{\mu_i}(0)},$$

the sequence $\{u_i\}_{i \geq 2} \subset \mathcal{SH}(\mathbb{R}^n)$ has these properties:

$$\begin{cases} u_i \leq 0 \ \text{on } \mathbb{R}^n, \\ u_i = -\infty \ \text{on } \left(\mathbb{B}_{n-2}(0, i) \setminus \mathbb{B}_{n-2}(0, \frac{1}{i})\right) \times \{0\} \times \{0\}, \\ u_i(0) = -1, \end{cases}$$

for all $i \geq 2$. Now we define

$$u(x) = \sum_{i=2}^{\infty} \frac{1}{2^{i-1}} u_i(x),$$

hence

$$\begin{cases} u \in \mathcal{SH}(\mathbb{R}^n), \\ u = -\infty \ \text{on } \left(\mathbb{R}^{n-2} \setminus \{0\}\right) \times \{0\} \times \{0\}, \\ u(0) = -1. \end{cases}$$

Therefore, the function $\widetilde{u} = \max(u, -2)$ satisfies:

$$\begin{cases} \widetilde{u} \in \mathcal{SH}(\mathbb{R}^n), \\ \widetilde{u} = -2 \ \text{on } \left(\mathbb{R}^{n-2} \setminus \{0\}\right) \times \{0\} \times \{0\}, \\ \widetilde{u}(0) = -1. \end{cases}$$

Finally, by setting $\widetilde{v} \equiv -2$*, we conclude that* $\widetilde{v} \geq \widetilde{u}$ *almost everywhere on* $\mathbb{R}^{n-2} \times \{0\} \times \{0\}$ *with respect to* $(n-2)$*-dimensional the Lebesgue measure on* $\mathbb{R}^{n-2} \times \{0\} \times \{0\}$*, but not everywhere as* $\widetilde{v}(0) < \widetilde{u}(0)$*.*

- In Section 1.5, we prove other versions of the main results in terms of measure densities.

First, for a Borel measure η on $\mathbb{R}^n$ and a Borel set K, we define:

$$\eta_K(E) = \eta(K \cap E).$$

It is clear that η_K is also a Borel measure on $\mathbb{R}^n$.

Next, we note that the main step of the proof of Theorem 1.1 is to bound the functions f_ϵ by integrable functions with respect to ν. This step can be modified by using upper and lower densities of a measure. The following theorem, therefore, can be considered as an another version of Theorem 1.1.

Theorem 1.3 *Let* Ω *be a domain in* $\mathbb{R}^n$ $(n \geq 2)$ *and* $u \in \mathcal{SH}(\Omega)$*. Let* K *be a Borel subset of* Ω *and* x_0 *be a point of* K*. Suppose that there exist a positive Borel measure* μ*, a relatively compact open subset* U *of* Ω *that contains* x_0 *and a positive number* $s > n-2$ *satisfying the following:*

$$\frac{1}{\Theta^s_*(\mu_K, x_0)} \int_U \frac{\Theta^{*s}(\mu_K, w)}{|w|^{n-2}} d\nu(w) < +\infty$$

where $\nu = \Delta u|_U$*. Then*

$$\lim_{\epsilon \to 0} \frac{1}{\mu\big(K \cap \mathbb{B}(x_0, \epsilon)\big)} \int_{K \cap \mathbb{B}(x_0,\epsilon)} u(x) d\mu(x) = u(x_0).$$

The next result, as a consequence of Theorem 1.3, is another version of Theorem 1.2. Their proofs are the same.

Theorem 1.4 *Let* Ω *be a domain in* $\mathbb{R}^n$ $(n \geq 2)$ *and* K *be a Borel subset of* Ω*. Let* u *be an upper semicontinuous function and* v *be a subharmonic function in* Ω*. Suppose that there exist a positive Borel measure* μ *and a positive number* $s > n-2$ *such that for all* $x \in K$*, there exists a relatively compact open subset* U_x *of* Ω *that contains* x *satisfying:*

$$\frac{1}{\Theta^s_*(\mu_K, x)} \int_{U_x} \frac{\Theta^{*s}(\mu_K, w)}{|w|^{n-2}} d\nu(w) < +\infty$$

where $\nu = \Delta v|_{U_x}$*. If* $u \geq v$ *almost everywhere on* K *with respect to* μ *then* $u \geq v$ *on* K*.*

Chapter 2

Complex Monge-Ampère equation in strictly pseudoconvex domains

Written on the basis of the paper "Hoang Son Do, Thai Duong Do and Hoang Hiep Pham, *Complex Monge-Ampère equation in strictly pseudoconvex domains,* Acta Math. Vietnam. **45** (2020), 93–-101", the present chapter is devoted to study the Dirichlet problem for the complex Monge-Ampère equation $(dd^c u)^n = \mu$ in a strictly pseudoconvex domain Ω with the boundary condition $u = \varphi$, where $\varphi \in C(\partial\Omega)$.

- In Section 2.1, Section 2.2 and Section 2.3, we recall some basic properties of plurisubharmonic functions, some basic properties of relative capacity, domain of Monge-Ampère operator and notions of Cegrell classes which will be used in the sequel. For convenience, we recall some important definitions here.

Definition 2.1 *Let Ω be a open set in $\mathbb{C}^n$. We define the subclass $\mathcal{D}(\Omega) \subset \mathcal{PSH}(\Omega)$, for which the Monge-Ampère operator can be well-defined, as follows: a plurisubharmonic function u belongs to $\mathcal{D}(\Omega)$ if there exists a nonnegative Radon measure μ on Ω such that if $\Omega' \subset \Omega$ is open and a sequence $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{PSH} \cap C^\infty(\Omega')$ decreases to u in Ω' then $(dd^c u_j)^n$ tends weakly to μ in Ω'.*

Definition 2.2 *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Define*

$$\mathcal{E}_0(\Omega) = \{u \in \mathcal{PSH}^-(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} u(z) = 0, \int_\Omega (dd^c u)^n < \infty\},$$

$$\mathcal{F}(\Omega) = \{u \in \mathcal{PSH}^-(\Omega) : \exists \{u_j\} \subset \mathcal{E}_0(\Omega),\ u_j \searrow u, \sup_j \int_\Omega (dd^c u_j)^n < \infty\},$$

$$\mathcal{E}(\Omega) = \{u \in \mathcal{PSH}^-(\Omega) : \forall K \Subset \Omega, \exists u_K \in \mathcal{F}(\Omega) \text{ such that } u_K = u \text{ on } K\},$$

$$\mathcal{N}(\Omega) = \{u \in \mathcal{E}(\Omega) : \text{ the smallest maximal plurisubharmonic majorant} = 0\}.$$

For more details, the reader is referred to [1, 3, 5, 6, 9, 10, 12, 13, 14, 18, 22, 24, 25, 27, 28, 32, 34, 35].

- In Section 2.4, we recall comparison principles and some sufficient conditions for Dirichlet problem. For more details, the reader is referred to [1, 2, 6, 14, 26, 31]. After that, we prove our main result which uses these results as main tools. This main result, which can be considered as a sufficient condition for the continuity of the solution outside an analytic set, is also our attempt to study Problem 2 mentioned in the Introduction.

Theorem 2.1 *Suppose that Ω is strictly pseudoconvex, μ be a Borel probability measure in Ω, $\varphi \in C(\partial\Omega)$ and $v \in \mathcal{E}(\Omega)$. Assume that there exists a sequence $\{M_j\}_{j=1}^\infty$ of* *postive real numbers with* $\lim_{j\to\infty} M_j = \infty$ *such that*

(i) For any $j \in \mathbb{Z}^+$, $\chi_{U_j}\mu \leq \frac{1}{2^j}\chi_{U_j}(dd^c v)^n$, *where* $U_j = \{z \in \Omega | v(z) < -M_j\}$.

(ii) For any $j \in \mathbb{Z}^+$, *there exist an increasing function* $h_j : \mathbb{R} \to (1, \infty)$ *and nonnegative constant* A_j *satisfying*

$$\int_1^\infty (y h_j^{1/n}(y))^{-1} dy < \infty, \tag{2.1}$$

and

$$\mu(K) \leq A_j Cap(K, \Omega) h_j^{-1}((Cap(K, \Omega))^{-1/n}), \tag{2.2}$$

for every compact set $K \subset V_j := \Omega \setminus U_j$.

Then, there exists a unique function u *satisfying*

$$\begin{cases} u \in \mathcal{F}(\varphi, \Omega), \\ (dd^c u)^n = \mu, \\ u \in C(V_j), \forall j \in \mathbb{Z}^+. \end{cases} \tag{2.3}$$

Moreover, for each $j \in \mathbb{Z}^+$, *for any* $z \in \partial\Omega \cap \overline{V}_j$,

$$\lim_{V_j \ni \xi \to z} u(\xi) = \varphi(z).$$

The proof proceeds roughly as follows. First, by the second assumption, we imply that μ vanishes on all pluripolar sets. Then the existence of solution is deduced from the recalled result of Åhag. For the continiuty of the solution, we use the first assumption, well-known result of Kołodjiej and comparison principle.

The following result is a direct consequence of Theorem 2.1.

Corollary 2.1 *Assume that the assumption of Theorem 2.1 is satisfied. If there exist* $\alpha \in (0, 1)$, $\lambda_1, ..., \lambda_m > 0$ *and analytic functions* $f_1, ..., f_m \in \mathcal{A}(\mathbb{C}^n)$ *such that* $v = -(-\log(|f_1|^{\lambda_1} + ... + |f_m|^{\lambda_m}))^\alpha$ *in* Ω *then* $u \in C(\Omega \setminus F)$, *where* $F = \{f_1 = f_2 = ... = f_m = 0\}$. *Moreover, for any* $z \in \partial\Omega \setminus F$,

$$\lim_{\Omega \setminus F \ni \xi \to z} u(\xi) = \varphi(z).$$

• In Section 2.5, we prove our results on studying Problem 3. We discuss the condition *"*$v = -(-\log(|f_1|^{\lambda_1} + ... + |f_m|^{\lambda_m}))^\alpha \in \mathcal{E}(\Omega)$*"* in Corollary 2.1.

First, we prove the following lemma which will be used to prove the main result.

Lemma 2.1 *Let* $0 < k < n$. *In the ball* $B = \{z \in \mathbb{C}^n : |z| < 1/2\}$, *consider the plurisubharmonic functions*

$$u_{\alpha\epsilon} = -(-\log(|z_1|^2 + ... + |z_k|^2 + \epsilon))^\alpha,$$

where $\epsilon \in (0, 1/2), \alpha \in (0, 1)$. *Then,*

$$\limsup_{\epsilon \to 0} \int_B |u_{\alpha\epsilon}|^{n-p-2} du_{\alpha\epsilon} \wedge d^c u_{\alpha\epsilon} \wedge (dd^c u_{\alpha\epsilon})^p \wedge \omega^{n-p-1} < \infty,$$

for any $p = 0, 1, ..., n-2$ *iff* $\alpha < \frac{k}{n}$. *Here* $\omega = dd^c|z|^2$.

For the proof of Lemma 2.1, we calculate directly the integral and use Fubini's theorem, Fatou's lemma to estimate.

By [10] and Lemma 2.1, we have the following.

Corollary 2.2 *Let* $0 < k < n$. *In the ball* $B = \{z \in \mathbb{C}^n : |z| < 1/2\}$, *consider the plurisubharmonic functions*

$$u_\alpha = -(-\log(|z_1|^2 + ... + |z_k|^2))^\alpha,$$

where $\alpha \in (0, 1)$. *Then,* $u_\alpha \in \mathcal{D}(B)$ *iff* $0 < \alpha < \frac{k}{n}$.

Note that if $0 < \alpha < \frac{1}{n}$ then $v = -(-\log(|f_1|^{\lambda_1} + ... + |f_m|^{\lambda_m}))^\alpha \in \mathcal{E}(\Omega)$ (see [7], [11]). Our main result describes the set $\{\alpha \in (0, 1) | v \in \mathcal{E}(\Omega)\}$ in the case where F is non-singular.

Proposition 2.1 *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded domain and* $\lambda_1, ..., \lambda_m > 0$. *Let* $f_1, ..., f_m \in \mathcal{A}(\Omega)$ *such that* $|f_1|^{\lambda_1} + ... + |f_m|^{\lambda_m} < 1$ *in* Ω. *Assume that* $F = \{f_1 = ... = f_m = 0\}$ *is non-singular and* $n > dim_{\mathbb{C}} F = n - k > 0$. *Then* $v = -(-\log(|f_1|^{\lambda_1} + ... + |f_m|^{\lambda_m}))^\alpha \in \mathcal{D}(\Omega)$ *iff* $\alpha \in (0, \frac{k}{n})$.

The proof of Proposition 2.1 uses Corollary 2.2, Hilbert's Nullstellensatz theorem and local property of belonging in $\mathcal{D}$ as main tools.

Chapter 3

Decay near boundary of volume of sublevel sets of plurisubharmonic functions

Written on the basis of the paper "Hoang Son Do, Thai Duong Do, *Some remarks on Cegrell's class* $\mathcal{F}$, Ann. Polon. Math. **125** (2020), 13–24", the present chapter is devoted to study the behavior near boundary of the function from class $\mathcal{F}$ in strictly pseudoconvex domain Ω, in particular the estimate of the sublevel set of the plurisubharmonic function near the boundary of a domain.

- In Section 3.1, we recall some properties of the class $\mathcal{F}$ which will be used in the sequel. The reader can find more details in [1, 2, 8, 14, 16, 31].
- In Section 3.2, we prove our result "An integral theorem for the class $\mathcal{F}$" which might be of independent interest. This result is also used to prove our result in the next Section.

Lemma 3.1 *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and (X, μ) be a totally bounded metric probability space. Let $u : \Omega \times X \to [-\infty, 0)$ is a measurable function such that*

(i) For every $a \in X$, $u(\cdot, a) \in \mathcal{F}(\Omega)$ and

$$\int_\Omega (dd^c u(z,a))^n < M,$$

where $M > 0$ is a constant.

(ii) For every $z \in \Omega$, the function $u(z, \cdot)$ is upper semicontinuous in X.

Then $\tilde{u}(z) = \int_X u(z,a)d\mu(a) \in \mathcal{F}(\Omega)$. Moreover

$$\int_\Omega (dd^c \tilde{u})^n \leq M.$$

For the proof of Lemma 3.1, by dividing X into finite pairwise disjoint collections and taking upper envelopes, we construct a decreasing sequence of functions of the class $\mathcal{F}$ which tends to $\tilde{u}$.

- In Section 3.3, we establish our results about necessary conditions for membership of the class $\mathcal{F}$. For convenience, we denote

$$W_d = \{z \in \Omega | d(z, \partial\Omega) < d\}.$$

Theorem 3.1 *Assume that Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{F}(\Omega)$. Then, there exists $C > 0$ depending only on Ω, n and u such that*

$$V_{2n}(\{z \in W_d, u(z) < -\epsilon\}) \leq \frac{C.d^{n+1-na}}{(1-a)a^n\epsilon^n}, \tag{3.1}$$

for any $0 < \epsilon, a < 1$ and $d > 0$.

The proof of Theorem 3.1 uses the Bedford-Taylor comparison principle and stronger convexity assumptions on Ω as main tools.

By Theorem 3.1, we have

$$\lim_{d \to 0} \frac{V_{2n}(\{z \in W_d | u(z) < -\epsilon\})}{d^t} = 0,$$

for every $0 < t < n+1$. It helps us to estimate the "density" of the the set $\{u < -\epsilon\}$ near the boundary. Moreover, by using Theorem 3.1 for $\epsilon = d^\alpha$ and $0 < a < 1 - \alpha$, we have the following result.

Corollary 3.1 *Assume that Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{F}(\Omega)$. Then, for every $0 < \alpha < 1$,*

$$\lim_{d \to 0} \frac{V_{2n}(\{z \in W_d | u(z) < -d^\alpha\})}{d} = 0.$$

We try to improve this result when Ω is the unit ball. First we have the following lemma which gives a necessary and sufficient condition for a radial plurisubharmonic function to be in the class $\mathcal{F}$. Note that if u is a radial plurisubharmonic function then $u(z) = \phi(\log|z|)$ for some convex, increasing function ϕ.

Lemma 3.2 *Let $u = \phi(\log|z|)$ be a radial plurisubharmonic function in $\mathbb{B}^{2n}$. Then, $u \in \mathcal{F}(\mathbb{B}^{2n})$ iff the following conditions hold*

(i) $\lim_{t\to 0^-} \phi(t) = 0$;

(ii) $\lim_{t\to 0^-} \frac{\phi(t)}{t} < \infty$.

For the proof of Lemma 3.2, we note that, by Theorem 3.1, the condition (i) is a necessary condition for $u \in \mathcal{F}(\mathbb{B}^{2n})$. By comparison theorem, we derive that when (i) is satisfied, the condition $u \in \mathcal{F}(\mathbb{B}^{2n})$ is equivalent to (ii), therefore complete the proof.

The following result is a necessary condition for membership of the class $\mathcal{F}$ when the domain is the unit ball.

Theorem 3.2 *If $u \in \mathcal{F}(\mathbb{B}^{2n})$ then*

$$\lim_{r\to 1^-} \frac{\int_{\{|z|=r\}} |u(z)| d\sigma(z)}{1-r} < \infty.$$

In particular, there exists $C > 0$ such that

$$\limsup_{d\to 0^+} \frac{V_{2n}(\{z \in \mathbb{B}^{2n} : \|z\| > 1-d, u(z) < -Ad\})}{d} < \frac{C}{A},$$

for every $A > 0$.

The proof proceeds roughly as follows. First, we use a symmetrization trick to obtain a radial plurisubharmonic function which actually belongs to class $\mathcal{F}$ by Lemma 3.1. Then, by Lemma 3.2, we estimate the volume of the sublevel sets near the boundary.

- In Section 3.4, we prove a sufficient condition for membership of the class $\mathcal{F}$ in case of unit ball. Our purpose is to find a sharp sufficient condition for u to belong to $\mathcal{F}(\Omega)$ based on the near-boundary behavior of u. We are interested in the following question:

Question 1 *Let Ω be a bounded strictly pseudoconvex domain. Assume that u is a negative plurisubharmonic function in Ω satisfying*

$$\lim_{d\to 0^+} \frac{V_{2n}(\{z \in W_d : u(z) < -Ad\})}{d} = 0,$$

for some $A > 0$. Then, do we have $u \in \mathcal{F}(\Omega)$?

In this section, we answer this question for the case where Ω is the unit ball.

Theorem 3.3 *Let $u \in PSH^-(\mathbb{B}^{2n})$. Assume that there exists $A > 0$ such that*

$$\lim_{d\to 0^+} \frac{V_{2n}(\{z \in \mathbb{B}^{2n} : \|z\| > 1-d, u(z) < -Ad\})}{d} = 0. \tag{3.2}$$

Then $u \in \mathcal{F}(\mathbb{B}^{2n})$.

The main idea of the proof of this theorem is to construct a sequence of functions which belong to class $\mathcal{F}$, converge almost everywhere to u and have bounded total mass. To do it, we take upper envelopes of a family generated by u and a sequence of rotations to construct a collection of plurisubharmonic $u_{a,\epsilon}$ which lives on slightly smaller balls. Next, we will exploit the assumption on the volume decay of the set $\{u < -Ad\}$ near the boundary to get a lower estimate of $u_{a,\epsilon}$ in terms of some defining function for $\mathbb{B}^n$. Then we will glue these data together to obtain the desired sequence.

By the fact that

$$\mathcal{F} = \{u \in \mathcal{N} : \int\limits_{\Omega} (dd^c u)^n < \infty\}$$

and Theorem 3.3, we get the following as a direct consequence.

Corollary 3.2 *Let $u \in \mathcal{N}(\mathbb{B}^{2n})$ such that $\int\limits_{\mathbb{B}^{2n}} (dd^c u)^n = \infty$. Then, for every $A > 0$,*

$$\limsup_{d \to 0^+} \frac{V_{2n}(\{z \in \mathbb{B}^{2n} : \|z\| > 1 - d, u(z) < -Ad\})}{d} > 0.$$

General Conclusions

In this dissertation, we study some specific problems in pluripotential theory and potential theory.

The main results of the dissertation include:

1. An extension of the mean value theorem, a comparison theorem for subharmonic functions and their other versions in terms of measure densities, a counterexample in the case of surfaces of co-dimension 2.
2. A sufficient condition which relaxes assumptions of a well-known result of Kołodjiej to some technical assumptions.
3. A necessary and sufficient condition for α such that

$$v = -(-\log(|f_1|^{\lambda_1} + ... + |f_m|^{\lambda_m}))^{\alpha}$$

   belongs to domain of Monge-Ampère operator.
4. An integral theorem for the class $\mathcal{F}$.
5. An necessary condition for upper bound of the volume of the sublevel sets near certain boundary points of functions from the class $\mathcal{F}$ in strictly pseudoconvex domains, a necessary condition and a sufficient condition for membership of the class $\mathcal{F}$ when the domain is the unit ball.

List of Author's Related Papers

1. Thai Duong Do, *A comparison theorem for subharmonic functions*, Results Math. **74** (2019), Paper No. 176, 13pp. (SCI-E)
2. Hoang Son Do, Thai Duong Do, *Some remarks on Cegrell's class* $\mathcal{F}$, Ann. Polon. Math. **125** (2020), 13–24. (SCI-E)
3. Hoang Son Do, Thai Duong Do and Hoang Hiep Pham, *Complex Monge-Ampère equation in strictly pseudoconvex domains*, Acta Math. Vietnam. **45** (2020), 93–-101. (VAST1)

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- The weekly seminar of the Department of Mathematical Analysis, Institute of Mathematics, Vietnam Academy of Science and Technology;
- Analysis session of the 9th Vietnam Mathematical Congress (August 14-18, 2018, Telecommunications University, Nha Trang, Vietnam);
- International Conference "Differential Equations and Dynamical Systems" (September 5-6, 2019, Hanoi Pedagogical University 2, Xuan Hoa, Phuc Yen, Vinh Phuc, Vietnam);
- "Vietnam-USA Joint Mathematical Meeting" (June 10-13, 2019, Quy Nhon, Vietnam) (a poster presentation).

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