VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY

INSTITUTE OF MATHEMATICS

DO THAI DUONG

SOME PROBLEMS IN PLURIPOTENTIAL THEORY

DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY IN MATHEMATICS

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Supervisor: Prof. Dr.Sc. PHAM HOANG HIEP
Prof. Dr.Sc. DINH TIEN CUONG

HANOI - 2021

Confirmation

This dissertation was written on the basis of my research works carried out at Institute of Mathematics, Vietnam Academy of Science and Technology, under the supervision of Prof. Dr.Sc. Pham Hoang Hiep and Prof. Dr.Sc. Dinh Tien Cuong. All the presented results have never been published by others.

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The author

Do Thai Duong

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Finally, I would like to thank my family for their endless love and unconditional support.

Contents

Table of Notations

$\mathbb{N}$	the set of positive integers.
$\mathbb{R}$	the set of real numbers.
$\mathbb{C}$	the set of complex numbers.
$\mathbb{R}^n$	the real vector space of dimension n.
$\mathbb{C}^n$	the complex vector space of dimension n.
$\mathbb{B}^n$	the unit ball in $\mathbb{R}^n$.
$\mathbb{B}^{2n}$	the unit ball in $\mathbb{C}^n$.
$\partial\mathbb{B}^n$	the unit sphere in $\mathbb{R}^n$.
$\partial\mathbb{B}^{2n}$	the unit sphere in $\mathbb{C}^n$.
$\mathbb{B}(x,r)$	the open ball of center x and radius r in real vector space or complex vector space.
$\overline{\mathbb{B}}(x,r)$	the closed ball of center x and radius r in real vector space or complex vector space.
$\partial\mathbb{B}(x,r)$	the sphere of center x and radius r in real vector space or complex vector space.
V_n	the Lebesgue measure on $\mathbb{R}^n$.
V_{2n}	the Lebesgue measure on $\mathbb{C}^n$.
σ	the surface measure in any dimension on any surface.
$\emptyset$	the empty set.
$\|\|x\|\|$	the norm of a vector x.
$C(\Omega)$	the set of continuous functions on Ω.
$C^k(\Omega)$	the set of $k-$times differentiable functions with derivatives of order k are continuous on Ω.
$C_0^k(\Omega)$	the set of $k-$times differentiable functions with derivatives of order k are continuous and compact support on Ω.
$C^\infty(\Omega)$	the set of smooth functions on Ω.

$C_0^\infty(\Omega)$	the set of smooth functions with compact support on Ω.
$\mathcal{H}(\Omega)$	the set of harmonic functions on Ω.
$\mathcal{USC}(\Omega)$	the set of upper semicontinuous functions on Ω.
$L^\infty(\Omega)$	the set of bounded functions on Ω.
$L^\infty_{\rm loc}(\Omega)$	the set of locally bounded functions on Ω.
$L^p(\Omega)$	the set of p-th power integrable functions on Ω.
$L^p_{\rm loc}(\Omega)$	the set of locally p-th power integrable functions on Ω.
$\mathcal{SH}(\Omega)$	the set of subharmonic functions on Ω.
$\mathcal{PSH}(\Omega)$	the set of plurisubharmonic functions on Ω.
$\mathcal{PSH}^-(\Omega)$	the set of negative plurisubharmonic functions on Ω.
$\mathcal{MPSH}(\Omega)$	the set of maximal plurisubharmonic functions on Ω.
$\mathcal{O}_{X,z}$	the space of germs of holomorphic functions at a point $z \in X$.
$u * v$	the convolution of u and v.

Introduction

In the 19$^{\text{th}}$ century physics, two fundamental forces of nature known at the time, namely gravity and the electrostatic force, were believed to be derived from using functions called "potentials" which satisfied Laplace's equation. The term "potential theory" or "classical potential theory" arose to describe a linear theory associated to the Laplacian operator. This theory focused on harmonic functions, subharmonic functions, the Dirichlet problem, harmonic measure, Green's functions, potentials and capacity in several real variables.

The potential theory in two dimension, which always be considered as the potential theory in the complex plane, has attracted considerable interest since it is closely related to complex analysis. In particular, there is a close connect between Laplace's equation and analytic functions. While the real and imaginary parts of analytic functions of a complex variable satisfy the Laplace's equation in two dimensions, the solution to Laplaces equation is the real part of an analytic function. In general, some techniques of complex analysis, particularly conformal mapping, can be used to simplify proofs of some results in the potential theory while some theorems in potential theory have analogies and applications in complex analysis.

In the 20$^{\text{th}}$ century, pluripotential theory was developed as the several complex variables analogue of the classical potential theory in the complex plane. This theory is highly non-linear and associated to complex Monge-Ampère operators. The basic objects are plurisubharmonic functions of several complex variables which were defined in 1942 by Kiyoshi Oka and Pierre Lelong. This class is the natural counterpart of the class of subharmonic functions of one complex variable. The plurisubharmonic functions are also be considered as subharmonic functions on several real variables which are invariant with respect to biholomorphic mappings.

In this dissertation, with three chapters, we study some specific problems in pluripotential theory and potential theory.

In Chapter 1, we study some properties of subharmonic functions on several real variables. Motivated by the fact that two subharmonic functions which agree almost everywhere on a domain with respect to the Lebesgue measure must coincide everywhere on that domain, we are interested in the following problem.

Problem 1. *Whether we can conclude that two subharmonic functions which agree almost everywhere on a surface with respect to the surface measure must coincide everywhere on that surface?*

This chapter is devoted to answer Problem 1 completely. For this purpose, we prove two main theorems with similar assumptions. They concern restrictions of subharmonic functions in Ω to a Borel subset $K \subset \Omega$ which together with a measure μ is subject to some quite technical assumptions. These allow K to have co-dimension one (and a little more, but not two), with μ being more or less like a corresponding Hausdorff measure. The first main result is an extension of the mean value theorem, states that the mean value theorem, in an infinitesimal form, still holds when restricted to K, and with respect to μ. The second main result is a comparison theorem for subharmonic functions, states that a comparison between an upper semicontinuous function and a subharmonic function which holds almost everywhere (with respect to μ) on K actually holds at every point of K. By these theorems, we prove that Problem 1 has a positive answer in the case of hypersurfaces. We also provide a counterexample in the case of surfaces of higher co-dimension. In addition, we apply main theorems to Ahlfors-David regular sets to obtain some consequences and we prove other versions of the main results in terms of measure densities.

In Chapter 2, we study the Dirichlet problem for the complex Monge-Ampère equation. We are interested in the following problem.

Problem 2. *Find conditions for μ such that the solution u of the Dirichlet problem for the complex Monge-Ampère equation is continuous outside an analytic set but u may not be continuous in Ω.*

This problem arises from the fact that there are some plurisubharmonic functions which are not continuous in the whole domain even though they are continuous outside an analytic set. For example, $u(z) = -(-\log \|z\|)^{1/2}$ is not continuous in the whole unit ball $\mathbb{B}^{2n}$, but it is continuous in $\mathbb{B}^{2n} \setminus \{0\}$. In studying this problem, we prove a sufficient condition which relaxes assumptions of a well-known result of Kołodjiej (Theorem B in [26]) to some technical assumptions. These assumptions naturally lead to the following problem.

Problem 3. *Find conditions for* α *such that*

$$v = -(-\log(|f_1|^{\lambda_1} + ... + |f_m|^{\lambda_m}))^{\alpha}$$

belongs to domain of Monge-Ampère operator, where $f_1, ..., f_m$ *are analytic functions.*

Note that if $0 < \alpha < \frac{1}{n}$ then $v \in \mathcal{E}$ (see [7], [11]). Our result is a necessary and sufficient condition where we further assume some conditions of $f_1, ..., f_m$ and the non-singularity of their zero-sets.

Chapter 3 is devoted to study the behavior near boundary of the function from class $\mathcal{F}$ in strictly pseudoconvex domain Ω, in particular the estimate of the sublevel set of the plurisubharmonic function near the boundary of a domain. In fact, we are motivated by two sources: firstly, we are inspired from [8] where the author gave the characterization of the class $\mathcal{F}$ in terms of the capacity of the sublevel set and secondly from the fact that, for a bounded hyperconvex domain Ω, although, for every $u \in \mathcal{F}(\Omega)$,

$$\limsup_{z \to \partial\Omega} u(z) = 0,$$

being in $\mathcal{F}(\Omega)$ does not necessarily give that $\limsup_{z \to \partial\Omega} u(z) = 0$. We are interested in the following problem.

Problem 4. *Find necessary and sufficient conditions of the volume of the sublevel sets near certain boundary points of functions from the class* $\mathcal{F}$ *in strictly pseudoconvex domains.*

In studying this problem, we give an necessary condition for upper bound of their volume and its consequences. On special case when the domain is the unit ball, we prove a necessary condition and a sufficient condition for membership of the class $\mathcal{F}$. They both concern decay near boundary of volume of the sublevel sets.

The dissertation is written on the basis of the paper [19] published in Annales Polonici Mathematici, the paper [20] published in Acta Mathematica Vietnamica and the paper [21] published in Results in Mathematics.

The results of this dissertation were presented at

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- Analysis session of “The 9th Vietnam Mathematical Congress” (August 14-18, 2018, Telecommunications University, Nha Trang, Vietnam);
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Chapter 1

A comparison theorem for subharmonic functions

The present chapter, which is written on the basis of the paper "Thai Duong Do, *A comparison theorem for subharmonic functions,* Results Math. **74** (2019), Paper No. 176, 13pp", devote to study Problem 1 mentioned at the Introduction. This chapter is organized as follows.

- In Section 1.1 and 1.2, we recall some basic properties of subharmonic functions and some basic properties of Hausdorff measure which will be used in the sequel.
- In Section 1.3, we prove our first main result "An extension of the mean value theorem" and its corollaries.
- In Section 1.4, we prove our second main result "A comparison theorem for subharmonic functions" by using the first main result as main tool. We also establish its corollaries and construct a counterexample which answer the Problem 1 completely.
- In Section 1.5, we prove other versions of the main results in terms of measure densities.

Throughout this chapter, we always assume that Ω is a domain of $\mathbb{R}^n$ $(n \geq 2)$.

1.1 Some basic properties of subharmonic functions

In this section, we recall definition and basic properties of subharmonic functions. For more details, the reader is referred to [25, 32, 22].

Definition 1.1.1. *Let $u : \Omega \longrightarrow \mathbb{R}$ be a C^2-function. The function u is said to be harmonic in Ω if it satisfies the Laplace equation:*

$$\triangle u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} \equiv 0 \text{ in } \Omega.$$

We denote by $\mathcal{H}(\Omega)$ the family of all harmonic functions in Ω.

Definition 1.1.2. *Let $u : \Omega \longrightarrow [-\infty, +\infty)$ be an upper semicontinuous function which is not identically $-\infty$. The function u is said to be subharmonic if for every relatively compact open subset U of Ω and every function $\varphi \in \mathcal{H}(U) \cap \mathcal{C}(\bar{U})$, the following implication is true:*

$$u \leq \varphi \text{ on } \partial U \Longrightarrow u \leq \varphi \text{ on } U.$$

We denote by $\mathcal{SH}(\Omega)$ the family of all subharmonic functions in Ω.

The following elementary result, which is an immediate consequence of Definition 1.1.2, helps us to generate further examples of subharmonic functions.

Theorem 1.1.3. *Let $u, v \in \mathcal{SH}(\Omega)$. Then:*

1, $\max\{u, v\} \in \mathcal{SH}(\Omega)$*;*

2, $\alpha u + \beta v \in \mathcal{SH}(\Omega)$ *for all* $\alpha, \beta \geq 0$.

The following theorem gives us a characterization of subharmonicity in terms of integral means.

Theorem 1.1.4. *Let $u : \Omega \longrightarrow [-\infty, +\infty)$ be an upper semicontinuous function which is not identically $-\infty$. Then the following conditions are equivalent:*

1, $u \in \mathcal{SH}(\Omega)$*;*

2, if $\overline{\mathbb{B}}(a, R) \subset \Omega$*, then*

$$u(a) \leq \frac{1}{\sigma(\partial\mathbb{B}(0,1))R^{n-1}} \int\limits_{\partial\mathbb{B}(a,R)} u(x) d\sigma(x);$$

3, if $\overline{\mathbb{B}}(a, R) \subset \Omega$*, then*

$$u(a) \leq \frac{1}{V_n(\mathbb{B}(0,1))R^n} \int\limits_{\mathbb{B}(a,R)} u(x) dV_n(x).$$

The locally integrability of subharmonic functions is an immediate consequence of Theorem 1.1.4.

Corollary 1.1.5. *Let $u \in \mathcal{SH}(\Omega)$. Then $u \in L^1_{loc}(\Omega)$.*

Theorem 1.1.4 implies also that we can sometimes glue subharmonic functions together to give a new subharmonic function.

Corollary 1.1.6. *Let Ω be a domain in $\mathbb{R}^n$, ω be a non-empty proper open subset of Ω, and let $u \in \mathcal{SH}(\Omega)$, $v \in \mathcal{SH}(\omega)$. Suppose that*

$$\limsup_{x \to y} v(y) \leq u(y),$$

for each $y \in \partial\omega \cap \Omega$. Then the formula

$$w = \begin{cases} \max\{u, v\} & \text{in } \omega \\ u & \text{in } \Omega \backslash \omega \end{cases}$$

defines a subharmonic function in Ω.

The next theorem follows by the semi-continuity of subharmonic function and Theorem 1.1.4. It is the motivation of this chapter as mentioned in the beginning.

Theorem 1.1.7. *Let $u, v \in \mathcal{SH}(\Omega)$. If $u = v$ almost everywhere in Ω then $u = v$ in Ω.*

The following theorem, known as Riesz Decomposition theorem, will be used as a major technical tool in proving Extension of the mean value theorem.

Theorem 1.1.8. *Let $u \in \mathcal{SH}(\Omega)$ and U be a relatively compact open subset of Ω. Then we can decompose u as*

$$u(x) = \frac{-1}{\max\{1, n-2\}\sigma(\partial\mathbb{B}(0,1))} \int\limits_U g(|x - w|) d\nu(w) + \varphi(x),$$

on U, where $\nu = \Delta u|_U$, $\varphi \in \mathcal{H}(U)$ and the kernel $g : (0, +\infty) \to \mathbb{R}$ is defined by

$$g(r) = \begin{cases} -\log r & (n{=}2) \\ r^{2-n} & (n{>}2) \end{cases}. \tag{1.1}$$

Next, we recall the notation of convolution and its application on approximating a subharmonic function by a family of smooth subharmonic functions.

Definition 1.1.9. *Let $u, v \in L^1(\mathbb{R}^n)$, then the convolution $u * v$ is defined by the formula*

$$(u * v)(x) = \int\limits_{\mathbb{R}^n} u(x - y) v(y) dV_n(y).$$

Moreover, the convolution $u * v$ is also well-defined if $u \in L^1_{loc}(\mathbb{R}^n)$ and $v \in L^1(\mathbb{R}^n)$ has a compact support.

We define $\chi : \mathbb{R} \to \mathbb{R}$ by the formula

$$\chi(t) = \begin{cases} e^{-1/t} & (t > 0) \\ 0 & (t \leq 0). \end{cases}$$

And we define $\rho : \mathbb{R}^n \to \mathbb{R}$ as following

$$\rho(x) = K\chi(1 - \|x\|^2),$$

where

$$K = \Big(\int\limits_{B(0,1)} \chi(1 - \|x\|^2) d\lambda(x) \Big)^{-1}.$$

Obviously, $\rho \in C^\infty(\mathbb{R}^n)$, supp$\rho = \overline{\mathbb{B}}(0,1)$ and

$$\int\limits_{R^n} \rho(x) dV_n(x) = 1.$$

For $\epsilon > 0$, we define

$$\rho_\epsilon(x) = \frac{1}{\epsilon^n} \rho(\frac{x}{\epsilon}).$$

The next theorem is known as the main approximation theorem for subharmonic functions.

Theorem 1.1.10. *Let Ω be a domain in $\mathbb{R}^n$ and let $u \in \mathcal{SH}(\Omega)$. If $\epsilon > 0$ such that*

$$\Omega_\epsilon := \{x \in \Omega : dist(x, \partial\Omega) > \epsilon\} \neq \emptyset,$$

then $u * \rho_\epsilon \in C^\infty \cap \mathcal{SH}(\Omega_\epsilon)$. Moreover, $u * \rho_\epsilon$ monotomcally decreases with decreasing ϵ and

$$\lim_{\epsilon \to 0} u * \rho_\epsilon(x) = u(x)$$

for each $x \in \Omega$.

We end this section with the characterization of subharmonicity. The following theorem gives us a characterization of smooth subharmonic functions.

Theorem 1.1.11. *Let $u \in C^2(\Omega)$. Then $u \in \mathcal{SH}(\Omega)$ if and only if $\triangle u \geq 0$ in Ω.*

Since subharmonic functions are locally integrable, their Laplacians can be evaluated in the sense of distributions. The next theorem shows that we can apply the characterization of subharmonicity described in Theorem 1.1.11 in a much wider context.

Theorem 1.1.12. *Let $u \in \mathcal{SH}(\Omega)$. Then $\triangle u \geq 0$ in the sense of distributions, i.e.*

$$\int_{\Omega} u(x)\triangle\varphi(x)dV_n(x) \geq 0$$

for any non-negative test function $\varphi \in C_0^\infty(\Omega)$. Conversely, if $v \in L^1_{loc}(\Omega)$ is such that $\triangle v \geq 0$ in Ω in the sense of distributions, then the function $u = \lim_{\epsilon\to 0}(v * \rho_\epsilon)$ is well-defined, subharmonic in Ω, and equal to v almost everywhere in Ω.

1.2 Some basic properties of Hausdorff measure

In this section, we recall definitions and basic properties of Hausdorff measure, the Ahlfors-David regular set, the upper and lower densities of a Radon measure. For more details, the reader is referred to [17, 29, 30, 36]. First, we recall the Carathédory's construction.

Definition 1.2.1. *Let X be a metric space, F be a family of subsets of X and φ be a non-negative function on F. Assume that*

1, for every $\delta > 0$, there is $\{E_i\}_{i=1,2,...} \subset F$ such that $X = \cup_{i=1}^\infty E_i$ and $d(E_i) \leq \delta$, where $d(E)$ is the diameter of E:

$$d(E) = \sup_{x,y\in E} |x-y|;$$

2, for every $\delta > 0$, there is $E \in F$ such that $\varphi(E) \leq \delta$ and $d(E) \leq \delta$.

For $0 < \delta \leq \infty$ and $U \subset X$, we define

$$\psi_\delta(U) = \inf\{\sum_i \varphi(E_i) : \ U \subset \bigcup_i E_i, \ d(E_i) \leq \delta, \ E_i \in F\},$$

$$\psi(U) = \lim_{\delta\downarrow 0} \psi_\delta(U).$$

Theorem 1.2.2. *1, ψ is a Borel measure.*

2, If the members of F are Borel sets then ψ is Borel regular.

Now let X be separable and $0 \leq k < \infty$, we choose $F = \{U : U \subset X\}$ and $\varphi(U) = d(U)^k$ with the interpretations $0^0 = 1$ and $d(\emptyset)^k = 0$.The resulting measure ψ is called the $k-$dimensional Hausdorff measure.

Definition 1.2.3. *Let $U \subset \mathbb{R}^n$, we define:*

$$H^k_\delta(U) = \inf\{\sum_i d(E_i)^k : U \subset \bigcup_i E_i, d(E_i) \leq \delta\}.$$

The k-dimensional Hausdorff measure of U, denoted by $H^k(U)$, is defined by

$$H^k(U) = \lim_{\delta \downarrow 0} H^k_\delta(U).$$

Example 1.2.4.

- *For $k = 0$, H^0 is the counting measure.*
- *For $k = m$, where m is an integer and $1 \le m < n$, let U be a $m-$dimensional surface in $\mathbb{R}^n$, then the restriction $H^m|_U$ gives a constant multiple of the surface measure on U.*
- *For $k = n$,*

  $$H^n = \frac{2^n}{V_n(\mathbb{B}(0,1))} V_n.$$

  In particular, $H^n(\mathbb{B}(a,R)) = (2R)^n$ for $a \in \mathbb{R}^n$ and $0 < R < \infty$.
- *For $k > n$, since $H^k(\mathbb{R}^n) = 0$, H^k in $\mathbb{R}^n$ is uninteresting.*

The Hausdorff measures play a special role. Notice that Hausdorff measures is a Borel measure, but usually it is not a Radon measure since it need not be locally finite. For example, if $k < n$ then every non-empty open set in $\mathbb{R}^n$ has non-σ-finite H^k measure.

The next elementary result, as a direct consequence of Definition 1.2.3, show that Hausdorff measures behave nicely under translations and dilations in $\mathbb{R}^n$.

Theorem 1.2.5. *For $U \subset \mathbb{R}^n$, $a \in \mathbb{R}^n$ and $0 < \alpha < \infty$, we have*

1, $H^k(U + a) = H^k(U)$ where $U + a = \{x + a : \ x \in U\}$,

2, $H^k(\alpha U) = \alpha^k H^k(U)$ where $\alpha U = \{\alpha x : \ x \in U\}$.

The next result helps us compare measures H^k with each other.

Theorem 1.2.6. *For $0 \le k < l < \infty$ and $U \subset \mathbb{R}^n$, we have:*

1, if $H^k(U) < \infty$ then $H^l(U) = 0$;

2, if $H^l(U) > 0$ then $H^k(U) = \infty$.

According to Theorem 1.2.6, we may define the Hausdorff dimension as following.

Definition 1.2.7. *Let $U \in \mathbb{R}^n$. Then the Hausdorff dimension of U is*

$$\begin{aligned} \dim U &= \sup\{k : H^k(U) > 0\} \\ &= \sup\{k : H^k(U) = \infty\} \\ &= \inf\{k : H^k(U) < \infty\} \\ &= \inf\{k : H^k(U) = 0\}. \end{aligned}$$

In other words, $\dim U$ is the unique number (maybe ∞) for which

1, if $k < \dim U$ then $H^t(U) = \infty$;

2, if $k > \dim U$ then $H^t(U) = 0$.

Example 1.2.8. • $\dim \mathbb{R}^n = n$.

• $\dim \mathcal{C} = \log 2/\log 3$ *where* $\mathcal{C}$ *is the Cantor ternary set.*

Notice that, at the borderline case when $k = \dim U$, all three cases $H^k(U) = 0$, $0 < H^k(U) < \infty$, $H^k(U) = \infty$ are possible and we cannot have any general nontrivial information about the value $H^k(U)$. In fact, for a given subset U of $\mathbb{R}^n$, there may be no value k for which U has positive and finite H^k measure. Therefore, the values of the Hausdorff measures often do not give much extra information. But, by replacing $\varphi(U) = d(U)^k$ by some other function of the diameter, we can construct the Generalized Hausdorff measures which measure the given set in a more delicate manner.

Definition 1.2.9. *Let* $h : [0, +\infty) \to [0, +\infty)$ *is a gague, i.e.* h *is non-decreasing, right-continuous and equal to* 0 *only at* 0. $U \in \mathbb{R}^n$, *we define the Generalized Hausdorff* h*-measure as following:*

$$\Lambda_h(U) = \lim_{\delta \downarrow 0} \inf\{\sum_i h(d(E_i)) : U \subset \bigcup_i E_i, d(E_i) \leq \delta\}.$$

The next definitions is about Ahlfors-David regular and densities of Radon measures.

Definition 1.2.10. *A subset* U *of* $\mathbb{R}^n$ *is said to be Ahlfors-David regular with dimension* k *if it is closed and if there is a constant* $C_0 > 0$ *such that*

$$C_0^{-1} R^k \leq H^k(U \cap \mathbb{B}(x, R)) \leq C_0 R^k,$$

for all $x \in U$ *and* $0 < R < d(U)$.

Definition 1.2.11. *Let* $0 \leq k < \infty$ *and let* η *be a Radon measure on* $\mathbb{R}^n$. *The upper and lower* k*-densities of* η *at* $x \in \mathbb{R}^n$ *are defined by*

$$\Theta^{*k}(\eta, x) = \limsup_{r \downarrow 0} \frac{\eta(\mathbb{B}(x, r))}{(2r)^k},$$

$$\Theta^k_*(\eta, x) = \liminf_{r \downarrow 0} \frac{\eta(\mathbb{B}(x, r))}{(2r)^k}.$$

If they agree, their common value

$$\Theta^k(\eta, x) = \Theta^{*k}(\eta, x) = \Theta^k_*(\eta, x)$$

is called the $k-$density of η at x.

Information on upper $k-$densities can be used to compare η with H^k.

Theorem 1.2.12. *Let η be a Radon measure on $\mathbb{R}^n$, $U \subset \mathbb{R}^n$, and $0 < \alpha < \infty$.*

1, If $\Theta^{*k}(\eta, x) \leq \alpha$ for $x \in U$, then $\eta(U) \leq 2^k \alpha H^k(U)$.

2, If $\Theta^{*k}(\eta, x) \geq \alpha$ for $x \in U$, then $\eta(U) \geq \alpha H^k(U)$.

1.3 An extension of the mean value theorem

In this section, we consider an extension of the mean value theorem and its corollaries. Firstly, we introduce a family of admissible functions as follow.

$$H = \left\{ h : (0, +\infty) \to (0, +\infty) : \exists\, M > 0,\ c > 4 \text{ such that } \int_0^{c\epsilon} \frac{h(r)}{r^{n-1}} dr \leq M \frac{h(\epsilon)}{\epsilon^{n-2}}, \text{ for } \epsilon \text{ small enough} \right\}.$$

Remark 1.3.1. *The family H contains many functions such as r^k, $r^k|\log r|$ where $k > n-2$. If h_1, $h_2 \in H$ then $h_1 + h_2 \in H$ and $a \cdot h_1 \in H$ for every positive number a. Here is one simple way of interpreting the admissible functions h. For a function $F : (0, +\infty) \to (0, +\infty)$, we could consider the following property:*

$$\exists c > 4 \text{ such that } \limsup_{\epsilon \to 0^+} \frac{\frac{1}{c\epsilon} \int_0^{c\epsilon} F(r) dr}{F(\epsilon)} < \infty. \tag{$*$}$$

This can be viewed as a very weak asymptotic one-dimensional mean value property as we are comparing integral means with the value of the integrated function inside the interval of integration. A function h belongs to H if and only if $F(r) = h(r)/r^{n-1}$ satisfies $(*)$.

The following classical result will be used in the sequel.

Theorem 1.3.2 (see Theorem 1.15 in [29]). *Let μ be a Borel measure and f be a non-negative Borel function on a separable metric space X. Then*

$$\int_X f d\mu = \int_0^{+\infty} \mu(\{x \in X : f(x) \geq t\})dt.$$

Next, we introduce our extension of the mean value theorem and its proof.

Theorem 1.3.3 (Extension of the mean value theorem). *Let u be a subharmonic function in a domain Ω, in $\mathbb{R}^n$ $(n \geq 2)$, K be a Borel subset of Ω, x_0 be a point of K and $h \in H$. Suppose that there exist a positive Borel measure μ, real numbers $A, B > 0$ and $\epsilon_0 > 0$ satisfying the following:*

1. $\mu\big(K \cap \mathbb{B}(x_0, \epsilon)\big) \geq Ah(\epsilon)$ *for all* $\epsilon < \epsilon_0$,
2. $\mu\big(K \cap \mathbb{B}(x, \epsilon)\big) \leq Bh(\epsilon)$ *for all* $x \in K$ *and* $\epsilon < \epsilon_0$.

Then

$$\lim_{\epsilon \to 0} \frac{1}{\mu\big(K \cap \mathbb{B}(x_0, \epsilon)\big)} \int_{K \cap \mathbb{B}(x_0,\epsilon)} u(x) d\mu(x) = u(x_0).$$

Proof. It is sufficient to prove the theorem when $x_0 = 0$. We choose $\epsilon_1 < \frac{\epsilon_0}{c}$ such that

$$\int_0^{c\epsilon} \frac{h(r)}{r^{n-1}} dr \leq M \frac{h(\epsilon)}{\epsilon^{n-2}},$$

for $\epsilon \leq \epsilon_1$ and we choose $\gamma > 1$, $p > 1$ such that $c = 2\gamma(1 + p)$. For convenience, we set

$$K_\epsilon = K \cap \mathbb{B}(0, \epsilon).$$

Our goal is to establish:

$$\lim_{\epsilon \to 0} \frac{1}{\mu(K_\epsilon)} \int_{K_\epsilon} u(x) d\mu(x) = u(0). \tag{1.2}$$

If $u(0) = -\infty$, then we have

$$\limsup_{\epsilon \to 0} \frac{1}{\mu(K_\epsilon)} \int_{K_\epsilon} u(x) d\mu(x) \leq \limsup_{\epsilon \to 0} \sup_{y \in K_\epsilon} u(y) \leq u(0) = -\infty.$$

Hence,

$$\lim_{\epsilon \to 0} \frac{1}{\mu(K_\epsilon)} \int_{K_\epsilon} u(x) d\mu(x) = u(0).$$

We now turn to the case where $u(0) > -\infty$. Since the problem is local, we can assume that $\mathbb{B}(0,1) \Subset \Omega$. By the Riesz Decomposition theorem, we can write

$$u(x) = \frac{-1}{\max\{1, n-2\}\sigma\big(\partial\mathbb{B}(0,1)\big)} \int\limits_{\mathbb{B}(0,1)} g(|x-w|)d\nu(w) + \varphi(x), \tag{1.3}$$

on $\mathbb{B}(0,1)$, where $\nu = \Delta u|_{\mathbb{B}(0,1)}$ is a Radon nonnegative measure, $\varphi \in \mathcal{H}\big(\mathbb{B}(0,1)\big)$ and the kernel g is defined by (1.1). It thus follows from equality (1.3), Fubini's theorem and the continuity of φ that

$$\begin{aligned} &\lim_{\epsilon\to 0} \frac{1}{\mu(K_\epsilon)} \int\limits_{K_\epsilon} u(x)d\mu(x) = \\ &\lim_{\epsilon\to 0} \frac{-1}{\max\{1, n-2\}\sigma\big(\partial\mathbb{B}(0,1)\big)} \int\limits_{\mathbb{B}(0,1)} f_\epsilon(w)d\nu(w) + \varphi(0), \end{aligned} \tag{1.4}$$

where

$$f_\epsilon(w) = \frac{1}{\mu(K_\epsilon)} \int\limits_{K_\epsilon} g(|x-w|)d\mu(x).$$

By replacing x by 0 in equality (1.3), we have

$$u(0) = \frac{-1}{\max\{1, n-2\}\sigma\big(\partial\mathbb{B}(0,1)\big)} \int\limits_{\mathbb{B}(0,1)} g(|w|)d\nu(w) + \varphi(0). \tag{1.5}$$

By (1.4) and (1.5), the equality (1.2) is equivalent to

$$\lim_{\epsilon\to 0} \int\limits_{\mathbb{B}(0,1)} f_\epsilon(w)d\nu(w) = \int\limits_{\mathbb{B}(0,1)} g(|w|)d\nu(w). \tag{1.6}$$

We prove (1.6) by three steps:

Step 1: We claim that

$$f_\epsilon(w) \to g(|w|),\ \epsilon \to 0,$$

almost everywhere on $\mathbb{B}(0,1)$ with respect to ν. Indeed, consider (1.5), since the measure ν is nonnegative and the function g is positive and decreasing on $(0,1)$, we have

$$\begin{aligned} u(0) &\le \frac{-1}{\max\{1, n-2\}\sigma\big(\partial\mathbb{B}(0,1)\big)} \int\limits_{\mathbb{B}(0,\delta)} g(|w|)d\nu(z) + \varphi(0) \\ &\le \frac{-1}{\max\{1, n-2\}\sigma\big(\partial\mathbb{B}(0,1)\big)} g(\delta)\nu\big(\mathbb{B}(0,\delta)\big) + \varphi(0), \end{aligned}$$

for all $0 < \delta < 1$. This implies that

$$\nu(\mathbb{B}(0,\delta)) \le \max\{1, n-2\}\sigma(\partial\mathbb{B}(0,1)) \frac{-u(0) + \varphi(0)}{g(\delta)},$$

for all $0 < \delta < 1$. By letting $\delta \to 0$, we conclude that $\nu(\mathbb{B}(0,\delta)) \searrow 0$ when $\delta \searrow 0$. Hence $\nu(\{0\}) = 0$. Combining this with the fact that

$$f_\epsilon(w) \to g(|w|),$$

pointwise for all $w \in \mathbb{B}(0,1)\setminus\{0\}$, we see that the claim holds.

Step 2: We will show that there exist constants $C_1, C_2 > 0$ such that

$$f_\epsilon(w) \le C_1 g(|w|) + C_2,$$

for all $w \in \mathbb{B}(0,1)\setminus\{0\}$ and $\epsilon < \epsilon_1$. We split the proof into two cases: Case I, where $|w| > p\epsilon$, and Case II, where $|w| \le p\epsilon$.

We first consider **Case I**, the case where $|w| > p\epsilon$. We observe that, for $x \in K_\epsilon$,

$$|x - w| \ge |w| - |x| \ge |w| - \epsilon > |w| - \frac{|w|}{p} = \frac{p-1}{p}|w|.$$

Therefore, by the definition of f_ϵ and the decrease of the function g on $(0,1)$,

$$\begin{aligned} f_\epsilon(w) &\le \frac{1}{\mu(K_\epsilon)} \int\limits_{K_\epsilon} g(|w| - |x|) d\mu(x) \\ &\le \frac{1}{\mu(K_\epsilon)} \int\limits_{K_\epsilon} g\Big(\frac{p-1}{p}|w|\Big) d\mu(x) \\ &= g\Big(\frac{p-1}{p}|w|\Big). \end{aligned}$$

The proof of Case I is finished by noticing that

$$g\Big(\frac{p-1}{p}|w|\Big) = \begin{cases} g(|w|) + g\Big(\frac{p-1}{p}\Big) & \text{(n=2)} \\ \Big(\frac{p-1}{p}\Big)^{2-n} g(|w|) & \text{(n>2)} \end{cases}.$$

Next, we study **Case II**, the case where $|w| \le p\epsilon$. By Theorem 1.3.2 and the definition of f_ϵ, we have

$$\begin{aligned} f_\epsilon(w) &= \frac{1}{\mu(K_\epsilon)} \int\limits_0^{+\infty} \mu\Big(\{x \in K_\epsilon : g(|x - w|) \ge t\}\Big) dt \\ &= \frac{1}{\mu(K_\epsilon)} \int\limits_0^{+\infty} \mu\Big(K_\epsilon \cap \overline{\mathbb{B}}(w, g^{-1}(t))\Big) dt \end{aligned}$$

Therefore, by splitting the integral at $\alpha(\epsilon)$ where $\alpha(\epsilon) = g(|w| + \epsilon)$,

$$f_\epsilon(w) = \frac{1}{\mu(K_\epsilon)} \int\limits_0^{\alpha(\epsilon)} \mu\Big(K_\epsilon \cap \overline{\mathbb{B}}(w, g^{-1}(t))\Big) dt + \frac{1}{\mu(K_\epsilon)} \int\limits_{\alpha(\epsilon)}^{+\infty} \mu\Big(K_\epsilon \cap \overline{\mathbb{B}}(w, g^{-1}(t))\Big) dt.$$

For the first term of the right-hand side, it is clear that

$$\mathbb{B}(0,\epsilon) \subset \mathbb{B}\big(w, g^{-1}(t)\big)$$

and hence

$$\mu\Big(K_\epsilon \cap \overline{\mathbb{B}}\big(w, g^{-1}(t)\big)\Big) = \mu(K_\epsilon), \text{ for all } 0 \le t \le \alpha(\epsilon). \tag{1.7}$$

For the second term of the right-hand side, if $K \cap \mathbb{B}\big(w, \gamma g^{-1}(t)\big) = \emptyset$ then

$$\mu\Big(K_\epsilon \cap \overline{\mathbb{B}}\big(w, g^{-1}(t)\big)\Big) = 0.$$

Otherwise,

$$K_\epsilon \cap \overline{\mathbb{B}}\big(w, g^{-1}(t)\big) \subset K \cap \mathbb{B}\big(w, \gamma g^{-1}(t)\big) \subset K \cap \mathbb{B}\big(w_0, 2\gamma g^{-1}(t)\big),$$

where $w_0 \in K \cap \mathbb{B}\big(w, \gamma g^{-1}(t)\big)$. Therefore, by the assumption of the theorem and the fact that

$$2\gamma g^{-1}(t) < \epsilon_0 \text{ for all } t \ge \alpha(\epsilon) \text{ and } \epsilon < \epsilon_1,$$

we obtain

$$\mu\Big(K_\epsilon \cap \overline{\mathbb{B}}\big(w, g^{-1}(t)\big)\Big) \le Bh\big(2\gamma g^{-1}(t)\big), \text{ for all } t \ge \alpha(\epsilon). \tag{1.8}$$

Thus, combining (1.7), (1.8) with the expression of f_ϵ, we have, for any $\epsilon < \epsilon_1$,

$$\begin{aligned}
f_\epsilon(w) &\le \frac{1}{\mu(K_\epsilon)} \int_0^{\alpha(\epsilon)} \mu(K_\epsilon) dt + \frac{B}{\mu(K_\epsilon)} \int_{\alpha(\epsilon)}^{+\infty} h(2\gamma g^{-1}(t)) dt \\
&= \alpha(\epsilon) + \frac{(2\gamma)^{n-2} \max\{1, n-2\} B}{\mu(K_\epsilon)} \int_0^{2\gamma(|w|+\epsilon)} \frac{h(r)}{r^{n-1}} dr \\
&\le g\big(|w| + \epsilon\big) + \frac{(2\gamma)^{n-2} \max\{1, n-2\} B}{Ah(\epsilon)} \int_0^{2\gamma(p+1)\epsilon} \frac{h(r)}{r^{n-1}} dr \\
&\le g(|w|) + \frac{(2\gamma)^{n-2} \max\{1, n-2\} BM}{A\epsilon^{n-2}},
\end{aligned}$$

the last inequality is deduced from the assumption of h. Furthermore, by the assumption of Case II, we get

$$f_\epsilon(w) \le g(|w|) + \frac{(2\gamma p)^{n-2} \max\{1, n-2\} BM}{A|w|^{n-2}}, \tag{1.9}$$

for all $\epsilon < \epsilon_1$. It is clear that the second term in the right-hand side of (1.9) is a constant when $n = 2$ and is equal to

$$\frac{(2\gamma p)^{n-2}(n-2)BM}{A} g(|w|)$$

when $n > 2$. The proof of Case II and, therefore, of Step 2 is complete.

Step 3: By conclusions of Step 1, Step 2 and the Lebesgue's Dominated Convergence theorem, we derive (1.6), which completes the proof. □

Note that when K is appropriate and h is a gauge, we can apply Theorem 1.3.3 for μ as Generalized Hausdorff h-measures. In particular, by choosing $h(r) = r^k$, where $k > n - 2$, we have a direct consequence as following.

Corollary 1.3.4. *Let u be a subharmonic function in a domain Ω, in $\mathbb{R}^n$ $(n \geq 2)$ and $U \subset \Omega$ be an Ahlfors-David regular set with dimension $k > n - 2$. Then*

$$\lim_{\epsilon \to 0} \frac{1}{H^k(U \cap \mathbb{B}(x, \epsilon))} \int\limits_{U \cap \mathbb{B}(x,\epsilon)} u(y) dH^k(y) = u(x)$$

for all $x \in U$.

Moreover, if we consider K as a hypersurface, μ is the hypersurface measure and $h(r) = r^{n-1}$, we obtain the following immediate corollary:

Corollary 1.3.5. *Let u be a subharmonic function in a domain Ω, in $\mathbb{R}^n$ $(n \geq 2)$ and $\mathbb{H}$ be a hypersurface. Then*

$$\lim_{\epsilon \to 0} \frac{1}{\sigma\big(\mathbb{H} \cap \mathbb{B}(x_0, \epsilon)\big)} \int\limits_{\mathbb{H} \cap \mathbb{B}(x_0,\epsilon)} u(x) d\sigma(x) = u(x_0)$$

for all $x_0 \in \mathbb{H} \cap \Omega$, where σ is the surface measure on $\mathbb{H}$.

1.4 A comparison theorem for subharmonic functions

In this section, we prove a comparison theorem for subharmonic functions and its corollaries. By these results and a counterexample, we also give a complete answer to Question 1. First, we introduce our comparison theorem for subharmonic functions and its proof.

Theorem 1.4.1 (Comparison theorem for subharmonic functions)**.** *Let u be an upper semicontinuous function, v be a subharmonic function in a domain Ω, in $\mathbb{R}^n$ $(n \geq 2)$ and let K be a Borel subset of Ω, $h \in H$. Suppose that there exist a positive Borel measure μ, real numbers $A, B > 0$, $\epsilon_0 > 0$ and $N \subset K$ satisfying the following:*

1. *$\mu(N) = 0$,*

2. $Ah(\epsilon) \le \mu\big(K \cap \mathbb{B}(x,\epsilon)\big) \le Bh(\epsilon)$ *for all* $x \in K$ *and* $\epsilon < \epsilon_0$*,*

3. $u \ge v$ *on* $K \diagdown N$*.*

Then $u \ge v$ *on* K*.*

Proof. Let $x_0 \in K$, it is sufficient to show that $u(x_0) \ge v(x_0)$. By the upper semicontiniuty of u, we have

$$u(x_0) \ge \lim_{\epsilon \to 0} \frac{1}{\mu\big(K \cap \mathbb{B}(x_0,\epsilon)\big)} \int\limits_{K \cap \mathbb{B}(x_0,\epsilon)} u(x) d\mu(x).$$

Following Theorem 1.3.3, we get

$$v(x_0) = \lim_{\epsilon \to 0} \frac{1}{\mu\big(K \cap \mathbb{B}(x_0,\epsilon)\big)} \int\limits_{K \cap \mathbb{B}(x_0,\epsilon)} v(x) d\mu(x).$$

Since $u \ge v$ almost everywhere on K with respect to μ, we infer that for every $\epsilon > 0$,

$$\int\limits_{K \cap \mathbb{B}(x_0,\epsilon)} u(x) d\mu(x) \ge \int\limits_{K \cap \mathbb{B}(x_0,\epsilon)} v(x) d\mu(x).$$

Combining the above inequalities gives $u(x_0) \ge v(x_0)$, as desired. □

Next, by applying Theorem 1.4.1 for $k-$dimensional Hausdorff measure ($k > n-2$) and hypersurface measure, we obtain some direct consequences as followings.

Corollary 1.4.2. *Let* Ω *be a domain in* $\mathbb{R}^n$ $(n \ge 2)$ *and* $E \subset \Omega$ *be an Ahlfors-David regular set with dimension* $k > n-2$*. Let* u *be an upper semicontinuous function and* v *be a subharmonic function in* Ω*. Suppose that* $u \ge v$ *almost everywhere on* E *with respect to* k*-dimensional Hausdorff measure. Then* $u \ge v$ *on* E*.*

Corollary 1.4.3. *Let* Ω *be a domain in* $\mathbb{R}^n$ $(n \ge 2)$ *and let* $\mathbb{H}$ *be a hypersurface such that* $\mathbb{H} \cap \Omega \ne \emptyset$*. Let* u *be an upper semicontinuous function and* v *be a subharmonic function in* Ω*. Suppose that* $u \ge v$ *almost everywhere on* $\mathbb{H} \cap \Omega$ *with respect to the surface measure on* $\mathbb{H}$*. Then* $u \ge v$ *on* $\mathbb{H} \cap \Omega$*.*

As a direct consequence of Corollary 1.4.3, we can conclude that two subharmonic functions which agree almost everywhere on a hypersurface with respect to the surface measure must coincide everywhere on that hypersurface. In other words, Question 1 has a positive answer in the case of hypersurfaces.

We end this section by constructing a counterexample to show that Question 1 has a negative answer in the case of surfaces of higher co-dimension.

Example 1.4.4 (Counterexample). *In $\mathbb{R}^n$ $(n \geq 3)$, we denote by $\mathbb{B}_{n-2}(0,R)$ the open ball in $\mathbb{R}^{n-2}$, with center at 0 and radius $R > 0$. For $i \geq 2$, let μ_i be the measure defined on $\big(\mathbb{B}_{n-2}(0,i)\diagdown\mathbb{B}_{n-2}(0,\frac{1}{i})\big) \times \{0\} \times \{0\}$ generated by the $(n-2)$-dimensional Lebesgue measure on $\big(\mathbb{B}_{n-2}(0,i)\diagdown\mathbb{B}_{n-2}(0,\frac{1}{i})\big)$. We define potentials $p_{\mu_i} : \mathbb{R}^n \longrightarrow [-\infty,\infty)$ by*

$$p_{\mu_i}(x) = \int\limits_{\mathbb{R}^n} \frac{-1}{|x-w|^{n-2}} d\mu_i(w).$$

Then we obtain the sequence $\{p_{\mu_i}\}_{i\geq 2} \subset \mathcal{SH}(\mathbb{R}^n)$ which satisfies these properties:

$$\begin{cases} p_{\mu_i} \leq 0 \text{ on } \mathbb{R}^n, \\ p_{\mu_i} = -\infty \text{ on } \big(\mathbb{B}_{n-2}(0,i)\diagdown\mathbb{B}_{n-2}(0,\frac{1}{i})\big) \times \{0\} \times \{0\}, \\ -\infty < p_{\mu_i}(0) < 0, \end{cases}$$

for all $i \geq 2$. By setting

$$u_i(x) = -\frac{p_{\mu_i}(x)}{p_{\mu_i}(0)},$$

the sequence $\{u_i\}_{i\geq 2} \subset \mathcal{SH}(\mathbb{R}^n)$ has these properties:

$$\begin{cases} u_i \leq 0 \text{ on } \mathbb{R}^n, \\ u_i = -\infty \text{ on } \big(\mathbb{B}_{n-2}(0,i)\diagdown\mathbb{B}_{n-2}(0,\frac{1}{i})\big) \times \{0\} \times \{0\}, \\ u_i(0) = -1, \end{cases}$$

for all $i \geq 2$. Now we define

$$u(x) = \sum_{i=2}^{\infty} \frac{1}{2^{i-1}} u_i(x),$$

hence

$$\begin{cases} u \in \mathcal{SH}(\mathbb{R}^n), \\ u = -\infty \text{ on } \big(\mathbb{R}^{n-2}\diagdown\{0\}\big) \times \{0\} \times \{0\}, \\ u(0) = -1. \end{cases}$$

Therefore, the function $\widetilde{u} = \max(u,-2)$ satisfies:

$$\begin{cases} \widetilde{u} \in \mathcal{SH}(\mathbb{R}^n), \\ \widetilde{u} = -2 \text{ on } \big(\mathbb{R}^{n-2}\diagdown\{0\}\big) \times \{0\} \times \{0\}, \\ \widetilde{u}(0) = -1. \end{cases}$$

Finally, by setting $\widetilde{v} \equiv -2$, we conclude that $\widetilde{v} \geq \widetilde{u}$ almost everywhere on $\mathbb{R}^{n-2} \times \{0\} \times \{0\}$ with respect to $(n-2)$-dimensional the Lebesgue measure on $\mathbb{R}^{n-2} \times \{0\} \times \{0\}$, but not everywhere as $\widetilde{v}(0) < \widetilde{u}(0)$.

1.5 Other versions of main results

In this section, we give other versions of the main results in terms of measure densities. First, we present a notation which will be used in the sequel. For a Borel measure η on $\mathbb{R}^n$ and a Borel set K, we define:

$$\eta_K(E) = \eta(K \cap E).$$

It is clear that η_K is also a Borel measure on $\mathbb{R}^n$.

Next, we note that the main idea of the proof of Theorem 1.3.3 is that the functions f_ϵ is bounded by integrable functions with respect to ν. By upper and lower densities of a measure, this theorem below is another version of Theorem 1.3.3.

Theorem 1.5.1. *Let Ω be a domain in $\mathbb{R}^n$ $(n \geq 2)$ and $u \in \mathcal{SH}(\Omega)$. Let K be a Borel subset of Ω and x_0 be a point of K. Suppose that there exist a positive Borel measure μ, a relatively compact open subset U of Ω that contains x_0 and a positive number $s > n-2$ satisfying the following:*

$$\frac{1}{\Theta_*^s(\mu_K, x_0)} \int\limits_U \frac{\Theta^{*s}(\mu_K, w)}{|w|^{n-2}} d\nu(w) < +\infty$$

where $\nu = \Delta u|_U$. Then

$$\lim_{\epsilon \to 0} \frac{1}{\mu(K \cap \mathbb{B}(x_0, \epsilon))} \int\limits_{K \cap \mathbb{B}(x_0, \epsilon)} u(x) d\mu(x) = u(x_0).$$

Proof. We can assume that $x_0 = 0$ and $U = \mathbb{B}(0,1)$. Adapting to the technique used in the proof of Theorem 1.3.3, it remains to show that f_ϵ is bounded from above by integrable function with respect to ν. Under the assumption of densities, it is sufficient to show that there exist constants C_1, C_2, C_3 such that

$$f_\epsilon(w) \leq C_1 g(|w|) + \frac{C_2}{\Theta_*^s(\mu_K, 0)} . \frac{\Theta^{*s}(\mu_K, w)}{|w|^{n-2}} + C_3,$$

for all $w \in \mathbb{B}(0,1) \diagdown \{0\}$ and $\epsilon > 0$ small enough. For $p > 1$, we also consider two cases as before. The proof differs only in the case II where $|w| \leq p\epsilon$. In this case, we have

$$\begin{aligned}
f_\epsilon(w) &\leq \frac{1}{\mu(K_\epsilon)}\int\limits_0^{\alpha(\epsilon)} \mu(K_\epsilon)dt + \frac{1}{\mu(K_\epsilon)}\int\limits_{\alpha(\epsilon)}^{+\infty} \mu(K\cap \mathbb{B}(w, 2g^{-1}(t)))dt \\
&\leq \alpha(\epsilon) + \frac{1}{(2\epsilon)^s\Theta_*^s(\mu_K,0)}\int\limits_{\alpha(\epsilon)}^{+\infty}(4g^{-1}(t))^s\Theta^{*s}(\mu_K,w)dt \\
&\leq g(|w|) + \frac{2^s\max\{1,n-2\}}{s-(n-2)}.\frac{\Theta^{*s}(\mu_K,w)}{\Theta_*^s(\mu_K,0)}.\Big(\frac{|w|+\epsilon}{\epsilon}\Big)^s.\frac{1}{(|w|+\epsilon)^{n-2}} \\
&\leq g(|w|) + \frac{2^s.(p+1)^s\max\{1,n-2\}}{s-(n-2)}.\frac{\Theta^{*s}(\mu_K,w)}{\Theta_*^s(\mu_K,0)}.\frac{1}{|w|^{n-2}},
\end{aligned}$$

as desired. □

The next result, as a consequence of Theorem 1.5.1, is another version of Theorem 1.4.1. Their proofs are the same.

Theorem 1.5.2. *Let Ω be a domain in $\mathbb{R}^n$ $(n\geq 2)$ and K be a Borel subset of Ω. Let u be an upper semicontinuous function and v be a subharmonic function in Ω. Suppose that there exist a positive Borel measure μ and a positive number $s>n-2$ such that for all $x\in K$, there exists a relatively compact open subset U_x of Ω that contains x satisfying:*

$$\frac{1}{\Theta_*^s(\mu_K,x)}\int\limits_{U_x}\frac{\Theta^{*s}(\mu_K,w)}{|w|^{n-2}}d\nu(w) < +\infty$$

where $\nu = \Delta v|_{U_x}$. If $u\geq v$ almost everywhere on K with respect to μ then $u\geq v$ on K.

Chapter 2

Complex Monge-Ampère equation in strictly pseudoconvex domains

Written on the basis of the paper "Hoang Son Do, Thai Duong Do, Hoang Hiep Pham, *Complex Monge-Ampère equation in strictly pseudoconvex domains,* Acta Math. Vietnam. **45** (2020), 93–101", the present chapter is devoted to study the Dirichlet problem for the complex Monge-Ampère equation $(dd^c u)^n = \mu$ in a strictly pseudoconvex domain Ω with the boundary condition $u = \varphi$, where $\varphi \in C(\partial\Omega)$. This chapter is organized as follows.

- In Section 2.1, Section 2.2 and Section 2.3, we recall some basic properties of plurisubharmonic functions, some basic properties of relative capacity, domain of Monge-Ampère operator and notions of Cegrell classes which will be used in the sequel.

- In Section 2.4, we recall comparison principles and some sufficient conditions for Dirichlet problem. After that, we prove our main result which uses these results as main tools. This main result is also our attempt to study Problem 2 mentioned in the Introduction.

- In Section 2.5, we prove our results on studying Problem 3.

Throughout Chapter 2 and Chapter 3, we always assume that Ω is a domain of $\mathbb{C}^n$.

2.1 Some properties of plurisubharmonic functions

In this section, we recall definition and basic properties of plurisubharmonic functions, definition of hyperconvex domains and strictly pseudoconvex domains. For more details, the reader is referred to [25, 32, 22].

Definition 2.1.1. *Let $u : \Omega \to [-\infty, \infty)$ be an upper semicontinuous function which is not identically $-\infty$. We say that u is plurisubharmonic if for each complex line*

$$\{z_1 + \lambda z_2 \in \Omega : \lambda \in \mathbb{C}\},$$

the function $\lambda \mapsto u(z_1 + \lambda z_2)$ is subharmonic or identically $-\infty$ on the set $\{\lambda \in \mathbb{C} : \ z_1 + \lambda z_2 \in \Omega\}$. We denote by $\mathcal{PSH}(\Omega)$ and $\mathcal{PSH}^-(\Omega)$ respectively the family of all plurisubharmonic functions and the family of all negative plurisubharmonic functions in Ω.

Similarly as in the case of subharmonic functions, we shall recall the main approximation theorem for plurisubharmonic functions. The notation we use here is the same as in Section 1.1.

Theorem 2.1.2. *Let Ω be a domain in $\mathbb{C}^n$ and let $u \in \mathcal{PSH}(\Omega)$. If $\epsilon > 0$ such that $\Omega_\epsilon \neq \emptyset$, then $u * \rho_\epsilon \in C^\infty \cap \mathcal{PSH}(\Omega_\epsilon)$. Moreover, $u * \rho_\epsilon$ monotonically decreases with decreasing ϵ and*

$$\lim_{\epsilon \to 0} u * \rho_\epsilon(z) = u(z)$$

for each $z \in \Omega$.

The following properties of families of plurisubharmonic functions is an immediate consequence of its definition.

Theorem 2.1.3. *Let Ω be a domain in $\mathbb{C}^n$.*

1, *The family $\mathcal{PSH}(\Omega)$ is a convex cone, i.e. if $u, v \in \mathcal{PSH}(\Omega)$ then $\alpha u + \beta v \in \mathcal{PSH}(\Omega)$ for all $\alpha, \beta \geq 0$.*

2, *If $\{u_j\}_{i \in \mathbb{N}} \subset \mathcal{PSH}(\Omega)$ is a decreasing sequence then $u = \lim_{j \to \infty} u_j \in \mathcal{PSH}(\Omega)$ or $u \equiv -\infty$.*

3, *Let $\{u_\alpha\}_{\alpha \in A} \subset \mathcal{PSH}(\Omega)$ be such that its upper envelope $u = \sup_{\alpha \in A} u_\alpha$ is locally bounded above. Then the upper semicontinuous regularization u^* is plurisubharmonic in Ω and $u = u^*$ almost everywhere. Moreover, if $\epsilon > 0$ such that $\Omega_\epsilon \neq \emptyset$, then $u * \rho_\epsilon \in C^\infty \cap \mathcal{PSH}(\Omega_\epsilon)$, $u * \rho_\epsilon$ monotonically decreases with decreasing ϵ and $\lim_{\epsilon \to 0} u * \rho_\epsilon(z) = u^*(z)$ for each $z \in \Omega$.*

Plurisubharmonicity can also be characterized in terms of distributional derivatives.

Theorem 2.1.4. *Let Ω be a domain in $\mathbb{C}^n$ and $u \in \mathcal{PSH}(\Omega)$. Then for each $\zeta = (\zeta_1, \zeta_2, ..., \zeta_n) \in \mathbb{C}^n$, we have*

$$\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} \zeta_j \overline{\zeta}_k \geq 0$$

in Ω, in the sense of distributions, i.e.

$$\int_{\Omega} u(z) \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \zeta_j \overline{\zeta}_k dV_{2n}(z) \geq 0,$$

for any non-negative test functions $\varphi \in C_0^\infty(\Omega)$. Conversely, if $v \in L^1_{loc}(\Omega)$ is such that for each $\zeta = (\zeta_1, \zeta_2, ..., \zeta_n) \in \mathbb{C}^n$, we have

$$\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} \zeta_j \overline{\zeta}_k \geq 0$$

in Ω, in the sense of distributions, then the function $u = \lim_{\epsilon \to 0} (v * \rho_\epsilon)$ is well-defined, plurisubharmonic in Ω and equal to v almost everywhere in Ω.

By the main approximation theorem and characterizations of subharmonicity, plurisubharmonicity, we have the following inclusion.

Corollary 2.1.5. *If Ω is a domain in $\mathbb{C}^n$, then $\mathcal{PSH}(\Omega) \subset \mathcal{SH}(\Omega)$.*

Similarly as in the case of subharmonic functions, we can sometimes glue plurisubharmonic functions together to give a new plurisubharmonic functions.

Proposition 2.1.6. *Let Ω be a domain in $\mathbb{C}^n$, ω be a non-empty proper open subset of Ω, and let $u \in \mathcal{PSH}(\Omega)$, $v \in \mathcal{PSH}(\omega)$. Suppose that*

$$\limsup_{x \to y} v(y) \leq u(y),$$

for each $y \in \partial\omega \cap \Omega$. Then the formula

$$w = \begin{cases} \max\{u, v\} & \text{in } \omega \\ u & \text{in } \Omega \backslash \omega \end{cases}$$

defines a plurisubharmonic function in Ω.

Plurisubharmonicity is preserved by holomorphic substitutions.

Proposition 2.1.7. *Let Ω and Ω' be domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. If $u \in \mathcal{PSH}(\Omega)$ and $f : \Omega \to \Omega'$ is a holomorphic mapping, then $u \circ f \in \mathcal{PSH}(\Omega')$.*

We end this section with the definition of strictly plurisubharmonic functions, maximal plurisubharmonic functions and pluripolar sets which will be used in the next section.

Definition 2.1.8. *Let Ω be a domain in $\mathbb{C}^n$. A real-valued function $u \in C^2(\Omega)$ is said to be strictly plurisubharmonic if for every $z \in \Omega$ and $0 \neq \zeta \in \mathbb{C}^n$, we have*

$$\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z)\zeta_j \overline{\zeta}_j > 0.$$

Definition 2.1.9. *Let Ω be a domain in $\mathbb{C}^n$ and let $u : \Omega \to \mathbb{R}$ be a plurisubharmonic function. We say that u is maximal if for every relatively compact open subset U of Ω and every function $\varphi \in \mathcal{PSH}(U) \cap \mathcal{USC}(\bar{U})$, the following implication is true:*

$$\varphi \leq u \text{ on } \partial U \Longrightarrow \varphi \leq u \text{ on } U.$$

We denote by $\mathcal{MPSH}(\Omega)$ the family of all maximal plurisubharmonic functions in Ω.

Definition 2.1.10. *A set $E \subset \mathbb{C}^n$ is called pluripolar if for each $z \in E$, there is an open set $U \ni z$ and $u \in \mathcal{PSH}(\Omega)$ such that $E \cap U \subset \{u = -\infty\}$.*

2.2 Domain of Monge-Ampère operator and notions of Cegrell classes

In pluripotential theory, the Monge-Ampère operator plays a crucial role in establishing many important properties of plurisubharmonic functions. In this section, we recall the definition and some important properties of Monge-Ampère operator. We also recall the definition of Cegrell's classes and their generalizations which help us study the most general definition of the Monge-Ampère operator and Dirichlet problem. For more details, the reader is referred to [1, 5, 9, 10, 12, 13, 14, 18, 24, 28, 34].

On $\mathbb{C}^n$, we write $d = \partial + \overline{\partial}$ and $d^c = i(\overline{\partial} - \partial)$, so that $dd^c u = 2i\partial\overline{\partial}u$. We denote by

$$dV = \left(\frac{i}{2}\right)^n \prod_{j=1}^{n} dz_j \wedge d\overline{z}_j$$

the usual volume form. In the classical sense, if u is a smooth plurisubharmonic function on a domain of $\mathbb{C}^n$ then its Monge-Ampère operator is a regular Radon measure given by

$$(dd^c u)^n = c_n \det\Big(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_j}\Big) dV,$$

where $c_n > 0$ depends only on n and the power on the left is taken with respect to the wedge product. Shiffman and Taylor have found an example showing that we can not define $(dd^c u)^n$ as a regular Radon measure for arbitrary plurisubharmonic function u if $n \geq 2$. Kiselman has simplified this example as following.

Example 2.2.1. *In $\mathbb{C}^n$, we define*

$$u(z) = (-\log|z_1|)^{1/n}(|z_2|^2 + ... + |z_n|^n - 1).$$

Then u is plurisubharmonic near the origin, smooth away from the hyperplane $z_1 = 0$, but $(dd^c u)^n$ is unbounded near $z_1 = 0$.

On the other hand, Bedford and Taylor have defined $(dd^c u)^n$ as a closed positive current in the case where u is a continuous plurisubharmonic function and then in the case where u is a locally bounded plurisubharmonic function. In general, the operator

$$(u_1, ..., u_n) \mapsto dd^c u_1 \wedge ... \wedge dd^c u_n,$$

which is also referred to as the Monge-Ampère operator, is well-defined if $u_1, ..., u_n$ are locally bounded plurisubharmonic functions on a domain of $\mathbb{C}^n$. We recall some important properties of the Monge-Ampère operator acting on locally bounded plurisubharmonic functions. The first result is known as Chern-Levine-Nirenberg inequality.

Theorem 2.2.2. *Let Ω be a domain and $\Omega' \Subset \Omega$. Then there exists a constant $C > 0$ which depends only on Ω and Ω' such that*

$$\int_{\Omega'} dd^c u_1 \wedge ... \wedge dd^c u_n \leq C\|u_1\|_\Omega ... \|u_n\|_\Omega,$$

for any set $\{u_j\}_{j=1}^n \subset \mathcal{PSH}(\Omega) \cap L^\infty_{loc}(\Omega)$, where $\|\cdot\|$ denotes the sup norm of a function.

The second result, known as convergence theorem, shows that monotone convergence of locally bounded plurisubharmonic functions implies the convergence of the corresponding currents.

Theorem 2.2.3. *Let Ω be a domain in $\mathbb{C}^n$ and $u_1, u_2, ..., u_n \in \mathcal{PSH}(\Omega) \cap L^\infty_{loc}(\Omega)$. Let $\{u_k^j\}_{j\in\mathbb{N}} \subset \mathcal{PSH}(\Omega)$ be an increasing (or a decreasing) sequence for $k = 1, 2, ..., n$ such that $\lim_{j\to\infty} u_k^j = u_k$ almost everywhere, for every k. Then*

$$dd^c u_1^j \wedge ... \wedge dd^c u_n^j \to dd^c u_1 \wedge ... \wedge dd^c u_n$$

in the weak topology of currents.

After that, Demailly has proved that the domain of Monge-Amère operator can be extended to the set of all plurisubharmonic functions which are bounded near the boundary of the domain. He also proved an improved version of the Chern-Levine-Nirenberg inequality and a result on the continuity of the Monge-Ampère operator. In fact, since a plurisubharmonic function can be approximated by a decreasing sequence of smooth plurisubharmonic functions (by Theorem 2.1.2), the continiuty of the Monge-Ampère operator under decreasing limits is a matter of interest. The choice of decreasing sequences for considering continuity of the Monge-Ampère operator is also motivated by the following example where Cegrell has constructed two sequences of smooth plurisubharmonic functions which both decrease to a plurisubharmonic function but the limits of their Monge-Ampère operators are not the same.

Example 2.2.4. *In $\mathbb{C}^n$, let*

$$\begin{aligned} u(z) &= 2\log|z_1...z_n|, \\ u_j(z) &= \log(|z_1...z_n|^{1/2} + 1/j), \\ v_j(z) &= \log(|z_1|^{1/2} + 1/j) + ... + \log(|z_n|^{1/2} + 1/j). \end{aligned}$$

We have

$$\begin{cases} \{u_j\}_{j\in\mathbb{N}}, \{v_j\}_{j\in\mathbb{N}} \subset \mathcal{PSH} \cap C^\infty(\mathbb{C}^n), \\ u_j \searrow u,\ v_j \searrow u, \end{cases}$$

but $(dd^c u_j)^n$ and $(dd^c v_j)^n$ tend respectively weakly to 0 and $n!4^n\delta_0$, where δ_0 is the Dirac measure at the origin.

Therefore, for an open subset $\Omega \in \mathbb{C}^n$, it is natural to define the subclass $\mathcal{D}(\Omega) \subset \mathcal{PSH}(\Omega)$, for which the Monge-Ampère operator can be well-defined, as follows: a plurisubharmonic function u belongs to $\mathcal{D}(\Omega)$ if there exists a non-negative Radon measure μ on Ω such that if $\Omega' \subset \Omega$ is open and a sequence $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{PSH} \cap C^\infty(\Omega')$ decreases to u in Ω' then $(dd^c u_j)^n$ tends weakly to μ in Ω'. Błocki has given a precise characterization of $\mathcal{D}(\Omega)$ and has shown that the Monge-Ampère operator is continuous for decreasing sequences in $\mathcal{D}(\Omega)$.

On the other hand, with the development of notions of convexity, the Monge-Ampère operator has been studied in hyperconvex domains and strictly pseudoconvex domains which give more information than usual domains.

Definition 2.2.5. *Let Ω be a domain in $\mathbb{C}^n$. Then Ω is said to be a hyperconvex domain if it admits an exhaustion function which is non-positive and plurisubharmonic, i.e. there exists $\varphi \in \mathcal{PSH}^-(\Omega)$ such that $\{z \in \Omega : \varphi(z) < -c\} \Subset \Omega$ for all $c > 0$.*

Definition 2.2.6. *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2-smooth boundary. Then Ω is said to be strictly pseudoconvex if there exists a C^2-smooth defining function for Ω which is strictly plurisubharmonic on a neighbourhood of $\overline{\Omega}$.*

Example 2.2.7. *The set $\{(z, w) \in \mathbb{C}^2 : \ \|z\|^2 + \|w\|^4 < 1\}$ is a hyperconex domain, but not a strictly pseudoconvex domain.*

For a bounded hyperconvex domain, Cegrell introduced finite energy classes of plurisubharmonic functions which are now known as Cegrell's classes.

Definition 2.2.8. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Define*

$$\begin{aligned}
\mathcal{E}_0(\Omega) &= \{u \in \mathcal{PSH}^-(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} u(z) = 0, \int_\Omega (dd^c u)^n < \infty\},\\
\mathcal{F}(\Omega) &= \{u \in \mathcal{PSH}^-(\Omega) : \exists \{u_j\} \subset \mathcal{E}_0(\Omega),\ u_j \searrow u,\ \sup_j \int_\Omega (dd^c u_j)^n < \infty\},\\
\mathcal{E}(\Omega) &= \{u \in \mathcal{PSH}^-(\Omega) : \forall K \Subset \Omega, \exists u_K \in \mathcal{F}(\Omega) \text{ such that } u_K = u \text{ on } K\},\\
\mathcal{N}(\Omega) &= \{u \in \mathcal{E}(\Omega) : \text{ the smallest maximal plurisubharmonic majorant} = 0\}.
\end{aligned}$$

It is clearly that $\mathcal{E}_0 \subset \mathcal{F} \subset \mathcal{N} \subset \mathcal{E}$. In the case where Ω is bounded hyperconvex domain, Cegrell has shown that the class $\mathcal{E}(\Omega)$ is the largest subclass of $\mathcal{PSH}^-(\Omega)$ on which the Monge-Ampère operator is well-defined, in other words,

$$\mathcal{E}(\Omega) = \mathcal{D}(\Omega) \cap \mathcal{PSH}^-(\Omega).$$

We end this section with the following generalizations of the classes $\mathcal{E}_0, \mathcal{F}, \mathcal{N}$ which has been used to study the Dirichlet problem with smooth boundary data.

Definition 2.2.9. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$.*
i) Let H be a maximal plurisubharmonic function in Ω. For $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}, \mathcal{N}\}$, we denote

$$\mathcal{K}(H) = \mathcal{K}(H, \Omega) = \{u \in \mathcal{PSH}(\Omega) : \exists \phi \in \mathcal{K}, H \geq u \geq \phi + H\}.$$

ii) Let $f \in C(\partial\Omega)$. For $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}, \mathcal{N}\}$, we denote

$$\mathcal{K}(f) = K(f,\Omega) = \{u \in \mathcal{PSH}(\Omega) : \exists \phi \in \mathcal{K}, U(0,f) \geq u \geq \phi + U(0,f)\},$$

where $U(0,f)$ is the unique solution of

$$\begin{cases} U(0,f) \in \mathcal{PSH}(\Omega) \cap L^\infty(\overline{\Omega}), \\ (dd^c U(0,f))^n = 0 \text{ in } \Omega, \\ U(0,f) = f \text{ in } \partial\Omega. \end{cases} \tag{2.1}$$

Note that, if Ω is strictly pseudoconvex then $U(0,f)$ is always continuous.

2.3 Some basic properties of Relative capacity

Relative capacity is very useful in the studies of plurisubharmonic functions and the Monge-Ampère equation. In this section, we recall the definition and some important properties of the relative capacity, especially in strictly pseudoconvex domain. For more details, the reader is referred to [3, 6, 25, 27, 35].

Definition 2.3.1. *Let Ω be an open subset of $\mathbb{C}^n$ and K be a compact subset of Ω. The relative capacity of K in Ω is defined by*

$$Cap(K,\Omega) = \sup\Big\{ \int_K (dd^c v)^n | v \in \mathcal{PSH}(\Omega, [-1,0]) \Big\}.$$

According to the Chern-Levine-Nirenberg inequality (Theorem 2.2.2), we have, for every compact subset $K \subset \Omega$,

$$Cap(K,\Omega) < \infty.$$

Definition 2.3.2. *If $E \subset \Omega$ then the relative capacity of E in Ω is defined by*

$$Cap(E,\Omega) = \sup\{Cap(K,\Omega) | K \text{ is a compact subset of } E\}.$$

We recall some properties of relative capacity.

Proposition 2.3.3. *Let Ω, Ω' be domains in $\mathbb{C}^n$.*

1, If $E \subset \Omega$ is a Borel set then

$$Cap(E,\Omega) = \sup\Big\{ \int_E (dd^c v)^n | v \in \mathcal{PSH}(\Omega, [-1,0]) \Big\}.$$

2, If $E' \subset E \subset \Omega \subset \Omega'$ then

$$Cap(E,\Omega) \geq Cap(E',\Omega) \text{ and } Cap(E,\Omega) \geq Cap(E,\Omega').$$

3, If $\{E_j\}_{j\in\mathbb{N}} \subset \Omega$ *then*

$$Cap\Big(\bigcup_{j=1}^{\infty} E_j, \Omega\Big) \leq \sum_{j=1}^{\infty} Cap(E_j, \Omega).$$

4, If $E_1 \subset E_2 \subset \ldots$ *are Borel subsets of* Ω*, then*

$$Cap\Big(\bigcup_{j=1}^{\infty} E_j, \Omega\Big) = \lim_{j\to\infty} Cap(E_j, \Omega).$$

5, If E *is a pluripolar set then* $Cap(E, \Omega) = 0$.

Next, we recall some results of relative capacity when we require the assumption of strictly pseudoconvexity.

Theorem 2.3.4. *Let* $\Omega'', \Omega', \Omega$ *be strictly pseudoconvex domains satisfying* $\Omega'' \Subset \Omega' \Subset \Omega$. *Then, there exists a constant* $A > 0$ *such that*

$$Cap(K, \Omega) \leq Cap(K, \Omega') \leq ACap(K, \Omega),$$

for every compact subset $K \subset \Omega''$.

Theorem 2.3.5. *Let* Ω *be a strictly pseudoconvex domain and* E *be a open subset of* Ω. *Then there exists a constant* $A > 0$ *which depends only on* E *and* Ω *such that*

$$V_{2n}(E') \leq ACap(E', \Omega),$$

for every $E' \subset E$.

Relative capacity is used to describe the behaviour of plurisubharmonic functions as followings.

Theorem 2.3.6. *Let* Ω *be a strictly pseudoconvex domain and* $u \in \mathcal{PSH}(\Omega)$. *Then, for each* $\epsilon > 0$, *there exists an open subset* $U \subset \Omega$ *with* $Cap(U, \Omega) < \epsilon$ *such that* $u|_{\Omega\setminus U}$ *is continuous.*

By this result, we say that plurisubharmonic functions are quasicontinuous.

Theorem 2.3.7. *Let* Ω *be a strictly pseudoconvex domain and* $u \in \mathcal{PSH}(\Omega)$. *If* $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{PSH}(\Omega)$ *is a decreasing sequence, convergent to* u *in* Ω *and* $u_j = u$ *in a neighbourhood of* $\partial\Omega$ *then we have*

$$\lim_{j\to\infty} Cap(\{u_j > u + \epsilon\}, \Omega) = 0,$$

for any $\epsilon > 0$.

We end this section by the continuity of the Monge-Ampère operator on sequences converging with respect to capacity.

Theorem 2.3.8. *Let Ω be a strictly pseudoconvex domain and $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{PSH}(\Omega)$ be a uniformly bounded sequence converging with respect to capacity to $u \in \mathcal{PSH}(\Omega)$, i.e. for any $\epsilon > 0$ and $K \Subset \Omega$,*

$$\lim_{j\to\infty} Cap(K \cap \{|u - u_j| > \epsilon\}, \Omega) = 0.$$

Then we have

$$(dd^c u_j)^n \to (dd^c u)^n,$$

in the sense of currents.

2.4 Dirichlet problem for the Monge-Ampère equation is strictly pseudoconvex

In this section, we establish a sufficient condition for the continuity of the solution outside an analytic set. To prove this result, we need some auxiliary results including comparison principles and some sufficient conditions for Dirichlet problem. For more details, the reader is referred to [1, 2, 6, 14, 26, 31].

First, we recall the Bedford-Taylor comparison principle which reflects an "elliptic" nature of the Monge-Ampère operator. The idea of this comparison principle is to use the comparison between the Monge-Ampère operators of two plurisubharmonic functions u, v to compare u and v.

Theorem 2.4.1. *Let Ω be a bounded open set in $\mathbb{C}^n$. Let $u, v \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$ such that*

$$\liminf_{\Omega \ni z \to \partial\Omega} (u(z) - v(z)) \geq 0.$$

Then

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

Theorem 2.4.1 has been generalized in several directions. One of improved versions is the following.

Theorem 2.4.2. *Let Ω be a hyperconvex domain and $u, v \in \mathcal{E}(\Omega)$. Assume that one of the following conditions holds* *(i)* $\liminf_{\Omega\ni z\to\partial\Omega}(u(z)-v(z))\geq 0$.

(ii) $u\in\mathcal{N}(H,\Omega)$ *for some maximal plurisubharmonic function* $H\leq 0$, *and* $v\leq H$.

Then,

$$\frac{1}{n!}\int_{\{u<v\}}(v-u)^n dd^c w_1\wedge...\wedge dd^c w_n+\int_{\{u<v\}}-w_1(dd^c v)^n\leq \int_{\{u<v\}\cup\{u=v=-\infty\}}-w_1(dd^c u)^n,$$

for any $w_1,...,w_n\in\mathcal{PSH}(\Omega,[-1,0])$.

The following corollary of Theorem 2.4.2 will be used to prove the main theorem.

Theorem 2.4.3. *Let* Ω *be a hyperconvex domain. Let* $u,v\in\mathcal{E}(\Omega)$ *such that* $(dd^c u)^n$ *vanishes on all pluripolar sets and* $(dd^c u)^n\leq(dd^c v)^n$. *Assume that one of the following conditions holds*

(i) $\liminf_{\Omega\ni z\to\partial\Omega}(u(z)-v(z))\geq 0$.

(ii) $u\in\mathcal{N}(H,\Omega)$ *for some maximal plurisubharmonic function* $H\leq 0$, *and* $v\leq H$.

Then $u\geq v$ *in* Ω.

Next, we recall a well-known sufficient conditions for Dirichlet problem of Kołodjiej where the author gives possibly detailed description of the nonnegative Borel measures which rises to plurisubharmonic solutions satisfying continuity and boundedness.

Theorem 2.4.4. *Let* Ω *be a strictly pseudoconvex,* μ *be a Borel probability measure in* Ω *and* $\varphi\in C(\partial\Omega)$. *Consider an increasing function* $h:\mathbb{R}\to(1,\infty)$ *satisfying*

$$\int_1^\infty (yh^{1/n}(y))^{-1}dy<\infty. \tag{2.2}$$

If μ *satisfies the inequality*

$$\mu(K)\leq ACap(K,\Omega)h^{-1}((Cap(K,\Omega))^{-1/n}), \tag{2.3}$$

for any $K \subset \Omega$ compact and regular then there exists a unique $u \in \mathcal{PSH}(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} (dd^c u)^n = \mu \text{ in } \Omega, \\ u = \varphi \text{ in } \partial\Omega. \end{cases} \tag{2.4}$$

Moreover, $\|u\|_{L^\infty}$ is bounded by a constant $B = B(h, A, \varphi, \Omega)$ which does not depend on μ.

Next, we recall some sufficient conditions which lead to solutions in some larger classes of plurisubharmonic functions.

Theorem 2.4.5. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and μ be a positive measure on Ω. If $\mu(\Omega) < +\infty$ and μ vanishes on all pluripolar sets then there exists a unique solution $u \in \mathcal{F}(\Omega)$ of the equation $(dd^c u)^n = \mu$.*

Theorem 2.4.6. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and μ be a non-negative Radon measure on Ω. If there exists a function $v \in \mathcal{E}(\Omega)$ such that $(dd^c v)^n \geq \mu$ then for every maximal plurisubharmonic function $H \in \mathcal{E}(\Omega)$, there exists a function $u \in \mathcal{E}(\Omega)$ such that $v + H \leq u \leq H$ and $(dd^c u)^n = \mu$.*

Theorem 2.4.7. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$, μ be a positive measure on Ω and $f : \partial\Omega \to \mathbb{R}$ be a continuous function. If μ vanishes on all pluripolar sets and $U(0, f) \in C(\overline{\Omega})$ then there exists a unique solution $u \in \mathcal{F}(f, \Omega)$ of the equation $(dd^c u)^n = \mu$.*

We are ready to establish our result which can be considered as a sufficient condition for the continuity of the solution outside an analytic set.

Theorem 2.4.8. *Suppose that Ω is strictly pseudoconvex, μ be a Borel probability measure in Ω, $\varphi \in C(\partial\Omega)$ and $v \in \mathcal{E}(\Omega)$. Assume that there exists a sequence $\{M_j\}_{j=1}^{\infty}$ of postive real numbers with $\lim\limits_{j\to\infty} M_j = \infty$ such that*

(i) For any $j \in \mathbb{Z}^+$, $\chi_{U_j}\mu \leq \dfrac{1}{2^j}\chi_{U_j}(dd^c v)^n$, where $U_j = \{z \in \Omega | v(z) < -M_j\}$.

(ii) For any $j \in \mathbb{Z}^+$, there exist $h = h_j, A = A_j$ satisfying (2.2) and (2.3) for every compact set $K \subset V_j := \Omega \setminus U_j$.

Then, there exists a unique function u satisfying

$$\begin{cases} u \in \mathcal{F}(\varphi, \Omega), \\ (dd^c u)^n = \mu, \\ u \in C(V_j), \forall j \in \mathbb{Z}^+. \end{cases} \tag{2.5}$$

Moreover, for each $j \in \mathbb{Z}^+$, for any $z \in \partial\Omega \cap \overline{V}_j$,

$$\lim_{V_j \ni \xi \to z} u(\xi) = \varphi(z).$$

Proof. First, we show that μ vanishes on all pluripolar sets. Let $F \subset \Omega$ be an arbitrary pluripolar set. For every compact set $K \subset \Omega$ and for every $j \in \mathbb{Z}^+$, we have

$$\mu(F \cap K \cap V_j) \le A_j Cap(F \cap K \cap V_j, \Omega) h_j^{-1}((Cap(F \cap K \cap V_j, \Omega))^{-1/n}) = 0,$$

and

$$\mu(F \cap K \cap U_j) \le \frac{1}{2^j} \int\limits_{F \cap K \cap U_j} (dd^c v)^n \le \frac{1}{2^j} \int\limits_K (dd^c v)^n.$$

Hence,

$$\mu(F \cap K) = \mu(F \cap K \cap U_j) + \mu(F \cap K \cap V_j) \le \frac{1}{2^j} \int\limits_K (dd^c v)^n.$$

Letting $j \to \infty$, we get

$$\mu(F \cap K) = 0.$$

Since K is arbitrary, we have $\mu(F) = 0$. Then, μ vanishes on all pluripolar sets.

Now, by using Theorem 2.4.7, there exists a unique function u satisfying

$$\begin{cases} u \in \mathcal{F}(\varphi, \Omega), \\ (dd^c u)^n = \mu. \end{cases}$$

It remains to show that $u \in C(V_j \cup \partial\Omega)$ if we define $u = \varphi$ in $\partial\Omega$.

By Theorem 2.4.4, for any $j \in \mathbb{Z}^+$, there exists a unique solution u_j of the equation

$$\begin{cases} (dd^c u_j)^n = \chi_{V_j} \mu \text{ in } \Omega, \\ u = \varphi \text{ in } \partial\Omega. \end{cases} \tag{2.6}$$

It is easy to check that

$$(dd^c u_j)^n \le (dd^c u)^n \le (dd^c (u_j + \frac{v}{2^{j/n}}))^n,$$

for every $j \in \mathbb{Z}^+$.

By Theorem 2.4.3, we have

$$u_j + \frac{v}{2^{j/n}} \leq u \leq u_j, \tag{2.7}$$

for every $j \in \mathbb{Z}^+$.

Let j_0 be an arbitrary positive integer. For any $\epsilon > 0$, there exists $j \gg 1$ such that

$$\frac{M_{j_0}}{2^{j/n}} < \frac{\epsilon}{2}. \tag{2.8}$$

By (2.7), (2.8) and by $u|_{\partial\Omega} = \varphi$, we have

$$u_j - \frac{\epsilon}{2} \leq u \leq u_j, \tag{2.9}$$

in $V_{j_0} \cup \partial\Omega$.

By the continuity of u_j, there exists $\delta > 0$ such that,

$$|u_j(z) - u_j(w)| < \frac{\epsilon}{2}, \tag{2.10}$$

for all $z, w \in \overline{\Omega}, |z - w| < \delta$.

Combining (2.9) and (2.10), we get

$$u(z) - u(w) \leq u_j(z) - (u_j(w) - \frac{\epsilon}{2}) < \epsilon,$$

for all $z, w \in V_{j_0} \cap \partial\Omega, |z - w| < \delta$.

Hence, $u \in C(V_{j_0} \cup \partial\Omega)$. □

We end this section by a direct consequence of Theorem 2.4.8.

Corollary 2.4.9. *Assume that the assumption of Theorem 2.4.8 is satisfied. If there exist $\alpha \in (0,1)$, $\lambda_1, ..., \lambda_m > 0$ and analytic functions $f_1, ..., f_m \in \mathcal{A}(\mathbb{C}^n)$ such that $v = -(-\log(|f_1|^{\lambda_1} + ... + |f_m|^{\lambda_m}))^\alpha$ in Ω then $u \in C(\Omega \setminus F)$, where $F = \{f_1 = f_2 = ... = f_m = 0\}$. Moreover, for any $z \in \partial\Omega \setminus F$,*

$$\lim_{\Omega\setminus F \ni \xi \to z} u(\xi) = \varphi(z).$$

2.5 A remark on the class $\mathcal{E}$

We discuss the condition "$v = -(-\log(|f_1|^{\lambda_1} + ... + |f_m|^{\lambda_m}))^\alpha \in \mathcal{E}(\Omega)$" in Corollary 2.4.9. If $0 < \alpha < \frac{1}{n}$ then $v \in \mathcal{E}(\Omega)$ (see [7, 11]). In the case where F is non-singular, the set $\{\alpha \in (0,1) | v \in \mathcal{E}(\Omega)\}$ can be clearly described as the following.

Proposition 2.5.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and $\lambda_1, ..., \lambda_m > 0$. Let $f_1, ..., f_m \in \mathcal{A}(\Omega)$ such that $|f_1|^{\lambda_1} + ... + |f_m|^{\lambda_m} < 1$ in Ω. Assume that $F = \{f_1 = ... = f_m = 0\}$ is non-singular and $n > dim_{\mathbb{C}} F = n - k > 0$. Then $v = -(-\log(|f_1|^{\lambda_1} + ... + |f_m|^{\lambda_m}))^\alpha \in \mathcal{D}(\Omega)$ iff $\alpha \in (0, \frac{k}{n})$.*

Note that if $dim_{\mathbb{C}} F = 0$ (i.e. F is a finite set) then $\log(|f_1|^{\lambda_1} + ... + |f_m|^{\lambda_m}) \in \mathcal{D}(\Omega)$. As a consequence, $v \in \mathcal{D}(\Omega)$ for any $\alpha \in (0, 1)$.

In order to prove Proposition 2.5.1, we need the following lemma.

Lemma 2.5.2. *Let $0 < k < n$. In the ball $B = \{z \in \mathbb{C}^n : |z| < 1/2\}$, consider the plurisubharmonic functions*

$$u_{\alpha\epsilon} = -(-\log(|z_1|^2 + ... + |z_k|^2 + \epsilon))^\alpha,$$

where $\epsilon \in (0, 1/2), \alpha \in (0, 1)$. Then,

$$\limsup_{\epsilon \to 0} \int_B |u_{\alpha\epsilon}|^{n-p-2} du_{\alpha\epsilon} \wedge d^c u_{\alpha\epsilon} \wedge (dd^c u_{\alpha\epsilon})^p \wedge \omega^{n-p-1} < \infty,$$

for any $p = 0, 1, ..., n-2$ iff $\alpha < \frac{k}{n}$. Here $\omega = dd^c|z|^2$.

Proof. For $\epsilon \in (0, 1/2)$, we denote

$$u_\epsilon(z) = \log(|z_1|^2 + ... + |z_k|^2 + \epsilon).$$

Then, for any $\alpha \in (0, 1)$,

$u_{\alpha\epsilon} = -(-u_\epsilon)^\alpha,$

$du_{\alpha\epsilon} = \alpha(-u_\epsilon)^{\alpha-1} du_\epsilon,$

$du_{\alpha\epsilon} \wedge d^c u_{\alpha\epsilon} = \alpha^2(-u_\epsilon)^{2(\alpha-1)} du_\epsilon \wedge d^c u_\epsilon,$

$dd^c u_{\alpha\epsilon} = \alpha(-u_\epsilon)^{\alpha-1} dd^c u_\epsilon + \alpha(1-\alpha)(-u_\epsilon)^{\alpha-2} du_\epsilon \wedge d^c u_\epsilon.$

Since $u_{\alpha\epsilon}$ depends only on k variables, $du_{\alpha\epsilon} \wedge d^c u_{\alpha\epsilon} \wedge (dd^c u_{\alpha\epsilon})^p = 0$ for any $p \geq k$. For any $p = 0, 1, ..., k-1$, we have

$$du_{\alpha\epsilon} \wedge d^c u_{\alpha\epsilon} \wedge (dd^c u_{\alpha\epsilon})^p = \alpha^{p+1} |u_\epsilon|^{\alpha n - p - 2} du_\epsilon \wedge d^c u_\epsilon \wedge (dd^c u_\epsilon)^p. \qquad (2.11)$$

By calculating, we have

$$\left(\frac{\partial u_\epsilon}{\partial z_j}\frac{\partial u_\epsilon}{\partial \bar{z}_l}\right)_{j,l=\overline{1,k}} = e^{-2u_\epsilon}\left(\bar{z}_j z_l\right)_{j,l=\overline{1,k}} =: A \text{ and } \left(\frac{\partial^2 u_\epsilon}{\partial z_j \partial \bar{z}_l}\right)_{j,l=\overline{1,k}} = e^{-u_\epsilon} Id_k - A.$$

Since $rank A \in \{0,1\}$, there exists a $k\times k$ unita matrix U such that

$$A = U^* diag(e^{-2u_\epsilon}(|z_1|^2 + ... + |z_k|^2), 0, ..., 0)U,$$

and then

$$e^{-u_\epsilon} Id_k - A = e^{-u_\epsilon} U^* diag(\epsilon e^{-u_\epsilon}, 1, ..., 1)U.$$

Then, for any $p = 0, 1, ..., k-1$, we have

$$du_\epsilon \wedge d^c u_\epsilon \wedge (dd^c u_\epsilon)^p \wedge \omega^{n-p-1} = C(n,p) e^{-(p+2)u_\epsilon}(|z_1|^2 + |z_2|^2 + ... + |z_k|^2) dV_{2n}, \tag{2.12}$$

where $C(n,p) > 0$ depends only on n and p.

By combining (2.11), (2.12) and by Fubini's theorem, we have

$$\begin{aligned}
&\int_B du_{\alpha\epsilon} \wedge d^c u_{\alpha\epsilon} \wedge (dd^c u_{\alpha\epsilon})^p \wedge \omega^{n-p-1} \\
&\sim \int_{\{|z_1|^2+...+|z_k|^2<1/4\}} \frac{|u_\epsilon|^{n\alpha-p-2}(|z_1|^2 + |z_2|^2 + ... + |z_k|^2) dV_{2k}}{e^{(p+2)u_\epsilon}} \\
&\sim \int_0^{1/2} \frac{(-\log(t^2+\epsilon))^{n\alpha-p-2} t^2 t^{2k-1} dt}{(t^2+\epsilon)^{p+2}} = \int_0^{1/2} \frac{(-\log(t^2+\epsilon))^{n\alpha-p-2} t^{2k+1} dt}{(t^2+\epsilon)^{p+2}},
\end{aligned}$$

for $p = 0, ..., k-1$. Here, $A \sim B$ means that $c_1 A \le B \le c_2 A$, where $c_1, c_2 > 0$ are independent on ϵ, α.

For any $p \le k-2$, we have

$$\int_0^{1/2} \frac{(-\log(t^2+\epsilon))^{n\alpha-p-2} t^{2k+1} dt}{(t^2+\epsilon)^{(p+2)}} \le \int_0^{1/2} (-\log(t^2))^{n-2} t dt < \infty,$$

for every $\alpha \in (0,1), \epsilon \in (0, 1/2)$.

If $p = k-1$ then

$$\int_0^{1/2} \frac{(-\log(t^2+\epsilon))^{n\alpha-p-2} t^{2k+1} dt}{(t^2+\epsilon)^{(p+2)}} = \int_0^{1/2} \frac{(-\log(t^2+\epsilon))^{n\alpha-k-1} t^{2k+1} dt}{(t^2+\epsilon)^{(k+1)}}.$$

If $0 < \alpha < \dfrac{k}{n}$ then, for any $0 < \epsilon < 1/3$,

$$\begin{aligned}\int_0^{1/2}\frac{(-\log(t^2+\epsilon))^{n\alpha-k-1}t^{2k+1}dt}{(t^2+\epsilon)^{(k+1)}} &\le \int_0^{1/2}\frac{(-\log(t^2+\epsilon))^{n\alpha-k-1}dt}{(t^2+\epsilon)^{1/2}}\\ &\le 2^{n\alpha-k-1/2}\int_0^{1/2}\frac{(-\log(t+\epsilon))^{n\alpha-k-1}dt}{(t+\epsilon)}\\ &\le 2^{n\alpha-k-1/2}\int_0^{5/6}\frac{(-\log t)^{n\alpha-k-1}dt}{t}\\ &< \infty.\end{aligned}$$

If $\alpha \geq \frac{k}{n}$ then, by Fatou's lemma,

$$\liminf_{\epsilon\to 0}\int_0^{1/2}\frac{(-\log(t^2+\epsilon))^{n\alpha-k-1}t^{2k+1}dt}{(t^2+\epsilon)^{(k+1)}} \geq \int_0^{1/2}\frac{(-\log(t^2))^{n\alpha-k-1}dt}{t} = \infty.$$

This completes the proof. □

By [10] and Lemma 2.5.2, we have the following.

Corollary 2.5.3. *Let $0 < k < n$. In the ball $B = \{z \in \mathbb{C}^n : |z| < 1/2\}$, consider the plurisubharmonic functions*

$$u_\alpha = -(-\log(|z_1|^2 + ... + |z_k|^2))^\alpha,$$

where $\alpha \in (0,1)$. Then, $u_\alpha \in \mathcal{D}(B)$ iff $0 < \alpha < \frac{k}{n}$.

Now, we prove the proposition 2.5.1.

Proof of Proposition 2.5.1. Let $a \in \Omega$. If $a \notin F$ then there exists an open neighbourhood U of a such that v is bounded in U (and then $v \in \mathcal{D}(U)$).

If $a \in F$ then there exist an open neighbourhood U of a and a biholomorphic function $\phi : U \to B = \{z \in \mathbb{C}^n : |z| < 1/2\}$ such that $\phi(U \cap F) = \{z \in B | z_1 = ... = z_k = 0\}$. By Hilbert's Nullstellensatz theorem (see, for example, [23, p.19]), there exist $M, N > 0$ such that

$$(f_1 \circ \phi^{-1})^M, ..., (f_m \circ \phi^{-1})^M \in \langle z_1, ..., z_k\rangle \subset \mathcal{O}_{\mathbb{C}^n,\phi(a)},$$

and

$$z_1^N, ..., z_k^N \in \langle f_1 \circ \phi^{-1}, ..., f_m \circ \phi^{-1}\rangle \subset \mathcal{O}_{\mathbb{C}^n,\phi(a)}.$$

Then, there exists an open set $V \subset U$ such that, in $\phi(V)$,

$$(|f_1 \circ \phi^{-1}|^2 + ... + |f_m \circ \phi^{-1}|^2)^M \leq C_1(|z_1|^2 + ... + |z_k|^2),$$

and

$$(|z_1|^2 + ... + |z_k|^2)^N \leq C_2(|f_1 \circ \phi^{-1}|^2 + ...|f_m \circ \phi^{-1}|^2),$$

where $C_1, C_2 > 0$.

Hence, there exist $C_3, C_4 > 0$ and an open neighbourhood $W \subset V$ of a such that

$$-C_3(-\log(|z_1|^2 + ... + |z_k|^2))^\alpha \leq v \circ \phi^{-1} \leq -C_4(-\log(|z_1|^2 + ... + |z_k|^2))^\alpha,$$

in $\phi(W)$.

It follows from [10] that if $v_1 \in \mathcal{D}, v_2 \in PSH$ and $v_1 \leq v_2$ then $v_2 \in \mathcal{D}$. By Corollary 2.5.3, we conclude that $v \circ \phi^{-1} \in \mathcal{D}(\phi(W))$ iff $0 < \alpha < \frac{k}{n}$. Hence $v \in \mathcal{D}(W)$ iff $0 < \alpha < \frac{k}{n}$.

Moreover, it follows from [10] that belonging in $\mathcal{D}$ is a local property. Thus $v \in \mathcal{D}(\Omega)$ iff $0 < \alpha < \frac{k}{n}$. □

Chapter 3

Decay near boundary of volume of sublevel sets of plurisubharmonic functions

Written on the basis of the paper "Hoang Son Do, Thai Duong Do, *Some remarks on Cegrell's class* $\mathcal{F}$, Ann. Polon. Math. **125** (2020), 13–24", the present chapter is devoted to study the behavior near boundary of the function from class $\mathcal{F}$ in strictly pseudoconvex domain Ω, in particular the estimate of the sublevel set of the plurisubharmonic function near the boundary of a domain. This chapter is organized as follows.

- In Section 3.1, we recall some properties of the class $\mathcal{F}$ which will be used in the sequel.

- In Section 3.2, we prove our result "An integral theorem for the class $\mathcal{F}$" which might be of independent interest. This result is also used to prove our result in the next Section.

- In Section 3.3, we establish our results about necessary conditions for membership of the class $\mathcal{F}$.

- In Section 3.4, we prove a sufficient condition for membership of the class $\mathcal{F}$ in case of unit ball.

3.1 Some properties of the class $\mathcal{F}$

In this section, we recall some properties of the class $\mathcal{F}$. The reader can find more details in [1, 2, 8, 14, 16, 31].

Proposition 3.1.1. *Let Ω be a hyperconvex domain. If $u \in \mathcal{F}(\Omega)$ and $\varphi \in \mathcal{PSH}^-(\Omega)$ then* $\max(u,\varphi) \in \mathcal{F}(\Omega)$.

Theorem 3.1.2. *Let Ω be a hyperconvex domain. Assume that $u^p \in \mathcal{F}(\Omega)$, $1 \le p \le n$ and $\varphi \in \mathcal{PSH}^-(\Omega)$. If $g_j^p \in \mathcal{E}_0(\Omega)$ decreases to u^p as $j \to \infty$, then*

$$\lim_{j\to+\infty} \int \varphi dd^c g_j^1 \wedge ... \wedge dd^c g_j^n = \int \varphi dd^c u^1 \wedge ... \wedge dd^c u^n.$$

Corollary 3.1.3. *Let Ω be a hyperconvex domain. Suppose $u \in \mathcal{F}(\Omega)$. If $u_j \in \mathcal{E}_0(\Omega)$ decreases to u as $j \to \infty$ then*

$$\lim_{j\to\infty} \int_\Omega (dd^c u_j)^n = \int_\Omega (dd^c u)^n.$$

By the definition of the class $\mathcal{F}$ and Corollary 3.1.3, we can see that

$$\int_\Omega (dd^c u)^n < \infty,$$

for every $u \in \mathcal{F}$.

As a consequence of [31, Theorem 3.7], we have:

Proposition 3.1.4. *Let Ω be a hyperconvex domain. Let $u_j \in \mathcal{F}(\Omega)$, $j \in \mathbb{N}$, such that u_j converges almost everywhere to a function $u \in \mathcal{PSH}^-(\Omega)$ as $j \to \infty$. If* $\sup_{j>0} \int_\Omega (dd^c u_j)^n < \infty$ *then $u \in \mathcal{F}(\Omega)$.*

Note that, in the above proposition, $\int_\Omega (dd^c u)^n$ may be different from $\limsup_{j\to\infty} \int_\Omega (dd^c u_j)^n$ unless we further assume that u_j decreases to u.

Proposition 3.1.5. *Let Ω be a hyperconvex domain. Suppose $u_1, ..., u_n \in \mathcal{F}(\Omega)$. Then*

$$\int_\Omega dd^c u_1 \wedge ... \wedge dd^c u_n \le \left(\int_\Omega (dd^c u_1)^n\right)^{1/n} ... \left(\int_\Omega (dd^c u_n)^n\right)^{1/n}.$$

Proposition 3.1.5 implies that $\mathcal{F}$ is a convex cone. Moreover, if $u, v \in \mathcal{F}$ with $\int_\Omega (dd^c u)^n = a^n$, $\int_\Omega (dd^c v)^n = b^n$ then

$$\int_\Omega (dd^c(tu + sv))^n \le (ta + sb)^n, \text{ for all } t, s \ge 0.$$

The next result is sometimes called the strongly comparison principle in the class $\mathcal{F}$.

Theorem 3.1.6. *Let Ω be a hyperconvex domain. Let $u, v \in \mathcal{F}(\Omega)$ such that $u \leq v$ on Ω. Then, for $1 \leq k \leq n$,*

$$\frac{1}{k!}\int\limits_{\Omega}(v-u)^k dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\Omega}(r-w_1)(dd^c v)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n \leq \int\limits_{\Omega}(r-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n,$$

for all $w_j \in \mathcal{PSH}(\Omega),\ 0 \leq w_j \leq 1,\ j = 1, ..., k,\ w_{k+1}, ..., w_n \in \mathcal{F}$ and $r \geq 1$.

The following result can be considered as a consequence of the strongly comparison principle in the class $\mathcal{F}$.

Corollary 3.1.7. *Let Ω be a hyperconvex domain. Let $u, v \in \mathcal{F}(\Omega)$ such that $u \leq v$ in Ω. Then*

$$\int\limits_{\Omega}(dd^c v)^n \leq \int\limits_{\Omega}(dd^c u)^n.$$

The following theorem is usually referred to as the comparison principle in the class $\mathcal{F}$.

Theorem 3.1.8. *Let Ω be a hyperconvex domain. Let $u, v \in \mathcal{F}(\Omega)$ such that $(dd^c u)^n$ vanishes on every pluripolar set. If $(dd^c v)^n \geq (dd^c u)^n$ then $v \leq u$.*

For every $u \in \mathcal{PSH}^-(\Omega)$, there exists a non-positive maximal plurisubharmonic function $\bar{u} \in \mathcal{MPSH}(\Omega) \cap \mathcal{PSH}^-(\Omega)$ such that $u \leq \bar{u} \leq h$ for every $u \leq h \in \mathcal{MPSH}(\Omega)$. Then, by Corollary 3.1.8 we have $\bar{u} = 0$ for every $u \in \mathcal{F}(\Omega)$. It implies that

$$\limsup_{z \to \partial\Omega} u(z) = 0 \text{ for all } u \in \mathcal{F}(\Omega).$$

The following theorem shows that there exists $u \in \mathcal{F}(\Omega)$ with

$$\liminf_{z \to \partial\Omega} u(z) = -\infty.$$

Theorem 3.1.9. *Let Ω be a hyperconvex domain. If $E \subset \Omega$ is a pluripolar set then there exists $h \in \mathcal{F}(\Omega)$ such that $E \subset \{h = -\infty\}$.*

Note that, for a bounded hyperconvex domain Ω, if we choose E to be a countable set such that every point on $\partial\Omega$ is a limit point to E, then by Theorem 3.1.9, there exists $u \in \mathcal{F}(\Omega)$ such that $u = -\infty$ on E, and therefore

$$\liminf_{z \to \partial\Omega} u(z) = -\infty.$$

For every $H \in \mathcal{E} \cap \mathcal{MPSH}(\Omega)$, we denote

$$F(H) = \{v \in \mathcal{E} : f + H \le v \le H \text{ for some } f \in \mathcal{F}\}.$$

The following result was shown in [16].

Theorem 3.1.10. *Let Ω be a hyperconvex domain. Suppose $u \in \mathcal{E}(\Omega)$ and $\int_\Omega (dd^c u)^n < \infty$. Then, $u \in \mathcal{F}(\bar{u})$.*

By Corollary 3.1.8, Theorem 3.1.10 and Corollary 3.1.3, we can see that

$$\mathcal{F} = \{u \in \mathcal{N} : \int\limits_\Omega (dd^c u)^n < \infty\}. \tag{3.1}$$

The class $\mathcal{F}$ can be used to characterize the boundary behavior in the Dirichlet problem for the Monge-Ampère equation in the class $\mathcal{E}$. Formally, the problem can be stated as follows: given a positive Borel measures μ on Ω and a maximal plurisubharmonic function $H \in \mathcal{E} \cap \mathcal{MPSH}(\Omega)$, find (if it exists) $u \in \mathcal{E}$ such that $(dd^c u)^n = \mu$ and $\bar{u} = H$. By Theorem 3.1.10, if μ has finite mass then the solutions of the Dirichlet problem belong to $\mathcal{F}(H)$. It is inconvenient to recall here all results about the existence of solution. In order to give the readers some insight into the problem, we combine some results in [1, 2, 14] to state the following theorem:

Theorem 3.1.11. *Let μ_1, μ_2 be positive Borel measures on Ω such that*

- *$\mu = \mu_1 + \mu_2$ has finite mass;*
- *μ_1 vanishes on every pluripolar set;*
- *$(dd^c f)^n \ge \mu_2$ for some $f \in \mathcal{F}$.*

Then, for every $H \in \mathcal{E} \cap \mathcal{MPSH}(\Omega)$, there exists $u \in \mathcal{F}(H)$ such that $(dd^c u)^n = \mu$. Moreover, if $\mu_2 = 0$ then u is unique.

3.2 An integral theorem for the class $\mathcal{F}$

First, we introduce an integral theorem for plurisubharmonic functions which is a direct consequence of Theorem 2.6.5 in [25] and the definition of plurisubharmonic functions.

Lemma 3.2.1. *Let $\Omega \subset \mathbb{C}^n$ be a connected open set and (X, μ) be a $\sigma-$finite measure space. Suppose that $u : \Omega \times X \to [-\infty, 0)$ is a measurable function such that* *(i) For every $a \in X$, $u(\cdot, a) \in \mathcal{PSH}(\Omega)$.*

(ii) There exists $g \in L^1(\mu)$ such that for every $z \in \Omega,\ a \in X$, we have

$$u(z, a) \leq g(a).$$

Then the function

$$\tilde{u}(z) = \int\limits_X u(z, a) d\mu(a)$$

is either plurisubharmonic in Ω or identically $-\infty$.

Next, we establish an integral theorem for the class $\mathcal{F}$ which will be used to prove a necessary condition for membership of the class $\mathcal{F}$ in case of unit ball.

Lemma 3.2.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and (X, μ) be a totally bounded metric probability space. Let $u : \Omega \times X \to [-\infty, 0)$ is a measurable function such that*

(i) For every $a \in X$, $u(\cdot, a) \in \mathcal{F}(\Omega)$ and

$$\int\limits_\Omega (dd^c u(z, a))^n < M,$$

where $M > 0$ is a constant.

(ii) For every $z \in \Omega$, the function $u(z, \cdot)$ is upper semicontinuous in X.

Then $\tilde{u}(z) = \int\limits_X u(z, a) d\mu(a) \in \mathcal{F}(\Omega)$. Moreover

$$\int\limits_\Omega (dd^c \tilde{u})^n \leq M.$$

Proof. By Lemma 3.2.1, we have that either $\tilde{u} \in \mathcal{PSH}^-(\Omega)$ or $\tilde{u} \equiv -\infty$. We need to find a sequence of functions $\tilde{u}_j \in \mathcal{F}(\Omega)$ such that $\tilde{u}_j$ is decreasing to $\tilde{u}$ as $j \to \infty$ and $\sup_j \int_\Omega (dd^c \tilde{u}_j)^n \leq M$.

Since X is totally bounded, there exists a finite cover $\{X_k\}_{k=1}^{m_1}$ of X such that the diameter of each X_k is at most $1/2$. Denote

$$U_{1,1} = X_1, U_{1,2} = X_2 \setminus X_1, ..., U_{1,m_1} = X_{m_1} \setminus (\cup_{l=1}^{m_1 - 1} X_l).$$

Then $\{U_{1,k}\}_{k=1}^{m_1}$ is a finite cover of X such that its elements are pairwise disjoint and of diameter at most $1/2$. Moreover, $U_{1,k}$ is totally bounded for every k. By using induction, for every $j \in \mathbb{Z}^+$, we can divide X into a finite pairwise disjoint collection $\{U_{j,k}\}_{k=1}^{m_j}$ of sets of diameter at most 2^{-j} satisfying: for every $1 \leq k \leq m_{j+1}$, there exists $1 \leq l \leq m_j$ such that $U_{j+1,k} \subset U_{j,l}$.

For every $j \in \mathbb{Z}^+$, we define

$$u_j(z) = \sum_{k=1}^{m_j} \mu(U_{j,k}) \sup_{a \in U_{j,k}} u(z,a) \qquad \text{and} \qquad \tilde{u}_j = (u_j)^*.$$

Then $\tilde{u}_j \in \mathcal{F}(\Omega)$. Moreover, by using Corollary 3.1.7 for $\tilde{u}_j$ and $\sum_{k=1}^{m_j} \mu(U_{j,k})u(z,a_k)$ (with $a_k \in U_{j,k}$) and by applying Proposition 3.1.5, we have

$$\begin{aligned}
&\int_\Omega (dd^c \tilde{u}_j)^n \leq \int_\Omega (dd^c(\sum_{k=1}^{m_j} \mu(U_{j,k})u(z,a_k)))^n \\
&= \sum_{k_1+...+k_{m_j}=n} \frac{n!}{k_1!...k_{m_j}!} \prod_{l=1}^{m_j} \mu(U_{j,l})^{k_l} \int_\Omega (dd^c u(z,a_1))^{k_1} \wedge ... \wedge (dd^c u(z,a_{m_j}))^{k_{m_j}} \\
&\leq \sum_{k_1+...+k_{m_j}=n} \frac{n!}{k_1!...k_{m_j}!} \prod_{l=1}^{m_j} \mu(U_{j,l})^{k_l} \prod_{l=1}^{m_j} (\int_\Omega (dd^c u(z,a_l))^n)^{k_l/n} \\
&\leq M \sum_{k_1+...+k_{m_j}=n} \frac{n!}{k_1!...k_{m_j}!} \prod_{l=1}^{m_j} \mu(U_{j,l})^{k_l} \\
&= M(\mu(U_{j,1}) + ... + \mu(U_{j,k_{m_j}}))^n \\
&= M,
\end{aligned}$$

for all $j \in \mathbb{Z}^+$.

We will show that $\tilde{u}_j$ is decreasing to $\tilde{u}$ and use Proposition 3.1.4 to prove that $\tilde{u} \in \mathcal{F}(\Omega)$.

For every $z \in \Omega, a \in X$ and $j \in \mathbb{Z}^+$, we define

$$\phi_j(z,a) = \sum_{k=1}^{m_j} I_{U_{j,k}}(a) \sup_{a \in U_{j,k}} u(z,a) = \sup_{\xi \in U_{j,k(j,a)}} u(z,\xi),$$

where $I_{U_{j,k}}$ is the characteristic function of $U_{j,k}$ and $k(j,a)$ is given by $a \in U_{j,k(j,a)}$. Then, we have

$$u_j(z) = \int_X \phi_j(z,a) d\mu(a) \geq \int_X u(z,a) d\mu(a) = \tilde{u}(z), \tag{3.2}$$

for every $z \in \Omega$ and $j \in \mathbb{Z}^+$.

Note that

$$U_{j+1,k(j+1,a)} \cap U_{j,k(j,a)} \neq \emptyset.$$

Then, it follows from the construction of the sets $U_{j,k}$ that $U_{j+1,k(j+1,a)} \subset U_{j,k(j,a)}$. Hence

$$u_j(z) = \int\limits_X \phi_j(z,a)d\mu(a) \geq \int\limits_X \phi_{j+1}(z,a)d\mu(a) = u_{j+1}(z), \tag{3.3}$$

for every $z \in$ and $j \in \mathbb{Z}^+$.

By the semicontinuity of $u(z,\cdot)$, we have,

$$u(z,a) \geq \lim_{j\to\infty}(\sup\{u(z,\xi) : |\xi - a| \leq 2^{-j}\}) \geq \lim_{j\to\infty} \phi_j(z,a), \tag{3.4}$$

for every $z \in \Omega$ and $a \in X$. Integrating both sides of (3.4) with respect to a and using Fatou's lemma, we get

$$\tilde{u}(z) \geq \lim_{j\to\infty} u_j(z), \tag{3.5}$$

for every $z \in \Omega$.

Combining (3.2), (3.3) and (3.5), we get that u_j is decreasing to $\tilde{u}$ as $j \to \infty$. Note that, by Theorem 2.1.3, $u_j = \tilde{u}_j$ almost everywhere and then $\lim\limits_{j\to\infty} \tilde{u}_j = \tilde{u}$ almost everywhere. Since $\lim\limits_{j\to\infty} \tilde{u}_j$ is either plurisubharmonic or identically $-\infty$, we have $\lim\limits_{j\to\infty} \tilde{u}_j = \tilde{u}$ everywhere. Therefore, $\tilde{u}_j$ is decreasing to $\tilde{u}$ as $j \to \infty$.

By Proposition 3.1.4, $\max\{\tilde{u}, -k\} \in \mathcal{F}(\Omega)$ for $k > 0$ and it implies that $\tilde{u}$ is not identically $-\infty$. Then, by using Proposition 3.1.4 for $\tilde{u}$, we get that $\tilde{u} \in \mathcal{F}(\Omega)$. Moreover, since the sequence $\tilde{u}_j$ is decreasing, we have

$$\int\limits_\Omega (dd^c\tilde{u})^n \leq \liminf_{j\to\infty} \int\limits_\Omega (dd^c\tilde{u}_j)^n \leq M.$$

□

3.3 Some necessary conditions for membership of the class $\mathcal{F}$

In this section, we estimate the size of sublevel sets of the class $\mathcal{F}$. For convenience, we denote

$$W_d = \{z \in \Omega | d(z, \partial\Omega) < d\}.$$

Theorem 3.3.1. *Assume that Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{F}(\Omega)$. Then, there exists $C > 0$ depending only on Ω, n and u such that*

$$V_{2n}(\{z \in W_d, u(z) < -\epsilon\}) \leq \frac{C.d^{n+1-na}}{(1-a)a^n\epsilon^n}, \tag{3.6}$$

for any $0 < \epsilon, a < 1$ *and* $d > 0$.

Proof. Since Ω is bounded strictly pseudoconvex, there exists $\rho \in C^2(\bar{\Omega}, [-1, 0])$ such that $\Omega = \{z : \rho(z) < 0\}$ and

$$|D\rho| > C_1 \text{ in } \bar{\Omega}, \tag{3.7}$$

and

$$dd^c\rho \geq C_2 dd^c|z|^2 = C_2\omega, \tag{3.8}$$

where $C_1, C_2 > 0$ are constants.

By (3.7), there exist $C_3, C_4 > 0$ depending only on Ω and ρ such that

$$C_3 d(z, \partial\Omega) \leq -\rho(z) \leq C_4 d(z, \partial\Omega), \tag{3.9}$$

for every $z \in \Omega$.

For every $a \in (0, 1)$ and $z \in \Omega$, we have

$$dd^c\rho_a(z) := dd^c(-(-\rho(z))^a) = a(1-a)(-\rho)^{a-2}d\rho \wedge d^c\rho + a(-\rho)^{a-1}dd^c\rho.$$

Then

$$(dd^c\rho_a)^n \geq a^n(1-a)(-\rho)^{na-n-1}d\rho \wedge d^c\rho \wedge (dd^c\rho)^{n-1}. \tag{3.10}$$

Hence, by (3.7), (3.8) and (3.9), there exists $1 \gg d_0 > 0$ depending only on Ω and ρ such that, for every $0 < d < d_0$ and $z \in W_d$,

$$(dd^c\rho_a)^n \geq C_5(1-a)a^n d^{na-n-1}\omega^n, \tag{3.11}$$

where $C_5 > 0$ depends only on n and ρ.

Since $u \in \mathcal{F}(\Omega)$, there exists $\{u_j\}_{j=1}^{\infty} \subset \mathcal{E}_0(\Omega)$ such that $u_j \searrow u$ and

$$\int\limits_{\Omega} (dd^c u_j)^n < C_6, \tag{3.12}$$

for every $j \in \mathbb{Z}^+$, where $C_6 > 0$ depends only on u. By using (3.11), (3.12) and the Bedford-Taylor comparison principle [5, 6] (see also [25]), we have, for every $j \in \mathbb{Z}^+$, $\epsilon, d > 0$ and $a \in (0, 1)$,

$$\begin{aligned} C_6 > \int\limits_{\{u_j < \epsilon\rho_a\}} (dd^c u_j)^n &\geq \int\limits_{\{u_j < \epsilon\rho_a\}} (dd^c \epsilon\rho_a)^n \\ &\geq \frac{C_5(1-a)a^n\epsilon^n}{d^{n+1-na}} \int\limits_{\{u_j < \epsilon\rho_a\} \cap W_d} \omega^n. \end{aligned}$$

Hence, for every $0 < d < d_0$,

$$V_{2n}(\{z \in W_d | u_j(z) < -\epsilon\}) \leq \frac{C_7.d^{n+1-na}}{(1-a)a^n\epsilon^n},$$

where $C_7 > 0$ depends only on Ω, ρ, n and u. Letting $j \to \infty$, we get

$$V_{2n}(\{z \in W_d | u(z) < -\epsilon\}) \leq \frac{C_7.d^{n+1-na}}{(1-a)a^n\epsilon^n},$$

for every $0 < d < d_0$. By setting

$$C = \max\{C_7, \frac{V_{2n}(\Omega)}{d_0^{n+1}}\},$$

we have

$$V_{2n}(\{z \in W_d | u(z) < -\epsilon\}) \leq \frac{C.d^{n+1-na}}{(1-a)a^n\epsilon^n},$$

for every $d > 0$. This completes the proof of Theorem 3.3.1. $\square$

By Theorem 3.3.1, we have

$$\lim_{d\to 0} \frac{V_{2n}(\{z \in W_d | u(z) < -\epsilon\})}{d^t} = 0,$$

for every $0 < t < n+1$. It helps us to estimate the “density” of the the set $\{u < -\epsilon\}$ near the boundary. Moreover, by using Theorem 3.3.1 for $\epsilon = d^\alpha$ and $0 < a < 1-\alpha$, we have the following result.

Corollary 3.3.2. *Assume that Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{F}(\Omega)$. Then, for every $0 < \alpha < 1$,*

$$\lim_{d\to 0} \frac{V_{2n}(\{z \in W_d | u(z) < -d^\alpha\})}{d} = 0.$$

When Ω is the unit ball, this result can be improved as following.

Theorem 3.3.3. *If $u \in \mathcal{F}(\mathbb{B}^{2n})$ then*

$$\lim_{r\to 1^-} \frac{\int_{\{|z|=r\}} |u(z)| d\sigma(z)}{1-r} < \infty.$$

In particular, there exists $C > 0$ such that

$$\limsup_{d\to 0^+} \frac{V_{2n}(\{z \in \mathbb{B}^{2n} : \|z\| > 1-d, u(z) < -Ad\})}{d} < \frac{C}{A},$$

for every $A > 0$.

The proof of this theorem needs the following lemma which gives a necessary and sufficient condition for a radial plurisubharmonic function to be in the class $\mathcal{F}$. Note that if u is a radial plurisubharmonic function then

$$u(z) = \phi(\log|z|)$$

for some convex, increasing function ϕ.

Lemma 3.3.4. *Let $u = \phi(\log|z|)$ be a radial plurisubharmonic function in $\mathbb{B}^{2n}$. Then, $u \in \mathcal{F}(\mathbb{B}^{2n})$ iff the following conditions hold*

(i) $\lim_{t\to 0^-} \phi(t) = 0$;

(ii) $\lim_{t\to 0^-} \frac{\phi(t)}{t} < \infty$.

Proof. By Theorem 3.3.1, the condition (i) is a necessary condition for $u \in \mathcal{F}(\mathbb{B}^{2n})$. We need to show that, when (i) is satisfied, the condition $u \in \mathcal{F}(\mathbb{B}^{2n})$ is equivalent to (ii).

If (ii) is satisfied then there exists $k_0 \gg 1$ such that $k_0 t < \phi(t)$. Hence $u(z) > k_0 \log|z| \in \mathcal{F}(\mathbb{B}^{2n})$. Thus, $u \in \mathcal{F}(\mathbb{B}^{2n})$.

Conversely, if (ii) is not satisfied, we consider the functions $u_k = \max\{u, k\log|z|\}$. Then, for every k, $u_k > u$ near $\partial\mathbb{B}^{2n}$. Hence

$$\int_\Omega (dd^c u)^n \geq \int_\Omega (dd^c u_k)^n = k^n \int_\Omega (dd^c \log|z|)^n \overset{k\to\infty}{\longrightarrow} \infty.$$

Thus $u \notin \mathcal{F}(\mathbb{B}^{2n})$.

The proof is completed. □

Proof of Theorem 3.3.3. Denote by μ the unique invariant probability measure on the unitary group $U(n)$. For every $z \in \mathbb{B}^{2n}$, we define

$$\tilde{u}(z) = \int_{U(n)} u(\phi(z))d\mu(\phi) = \frac{1}{c_{2n-1}|z|^{2n-1}} \int_{\{|w|=|z|\}} u(w)d\sigma(w),$$

where c_{2n-1} is the $(2n-1)$-dimensional volume of $\partial\mathbb{B}^{2n}$. By Lemma 3.2.2, we have $\tilde{u} \in \mathcal{F}(\mathbb{B}^{2n})$. Since $\tilde{u}$ is radial, we have, by Lemma 3.3.4,

$$\lim_{|z|\to 1^-} \frac{\tilde{u}(z)}{|z|-1} = \lim_{|z|\to 1^-} \frac{\tilde{u}(z)}{\log|z|} < \infty.$$

Hence

$$\lim_{r\to 1^-}\frac{\int_{\{|z|=r\}}|u(z)|d\sigma(z)}{1-r}=M<\infty.$$

Consequently, we have, for $0<d\ll 1$,

$$\sigma(\{z\in\mathbb{B}^{2n}:\|z\|=1-d,u(z)<-Ad\})\le\frac{M+1}{A},\tag{3.13}$$

for all $A>0$. Note that

$$V_{2n}(\{z\in\mathbb{B}^{2n}:\|z\|>1-d,u(z)<-Ad\})=\int\limits_0^d\sigma(\{z\in\mathbb{B}^{2n}:\|z\|=1-t,u(z)<-Ad\})dt.$$

Hence, by (3.13), we have, for $0<d\ll 1$,

$$V_{2n}(\{z\in\mathbb{B}^{2n}:\|z\|>1-d,u(z)<-Ad\})\le\int\limits_0^d\frac{(M+1)}{Ad/t}dt=\frac{(M+1)d}{2A}.$$

Thus we get the last assertion of Theorem 3.3.3.

The proof is completed. □

3.4 A sufficient condition for membership of the class $\mathcal{F}$

Our second purpose is to find a sharp sufficient condition for u to belong to $\mathcal{F}(\Omega)$ based on the near-boundary behavior of u. We are interested in the following question:

Question 1. *Let Ω be a bounded strictly pseudoconvex domain. Assume that u is a negative plurisubharmonic function in Ω satisfying*

$$\lim_{d\to 0^+}\frac{V_{2n}(\{z\in W_d:u(z)<-Ad\})}{d}=0,$$

for some $A>0$. Then, do we have $u\in\mathcal{F}(\Omega)$?

In this section, we answer this question for the case where Ω is the unit ball.

Theorem 3.4.1. *Let $u\in PSH^-(\mathbb{B}^{2n})$. Assume that there exists $A>0$ such that*

$$\lim_{d\to 0^+}\frac{V_{2n}(\{z\in\mathbb{B}^{2n}:\|z\|>1-d,u(z)<-Ad\})}{d}=0.\tag{3.14}$$

Then $u\in\mathcal{F}(\mathbb{B}^{2n})$.

Proof. We will find a sequence of functions $u_j \in \mathcal{F}(\mathbb{B}^{2n})$ such that

$$\sup_{j\geq 0} \int_{\Omega} (dd^c u_j)^n < \infty$$

and u_j converges almost everywhere to u as $j \to \infty$. Then, by using Proposition 3.1.4, we will obtain $u \in \mathcal{F}(\mathbb{B}^{2n})$.

For every $0 < a < 1$, we denote $S_a = \{\phi \in U(n) : \|\phi - Id\| < a\}$.

For every $0 < \epsilon, a < 1$ and $z \in \mathbb{B}^{2n}_{1-\epsilon} := \{w \in \mathbb{C}^n : \|w\| < 1 - \epsilon\}$, we define

$$u_{a,\epsilon}(z) = (\sup\{u((1 + r)\phi(z)) : \phi \in S_a, 0 \leq r \leq \epsilon\})^*.$$

Then $u_{a,\epsilon}$ is plurisubharmonic in $\mathbb{B}^{2n}_{1-\epsilon}$ (see [25, Corollary 2.9.5] and [25, Theorem 2.9.14]) and, by the semicontinuity of u, we have

$$\lim_{\max\{a,\epsilon\}\to 0^+} u_{a,\epsilon}(z) = u(z), \tag{3.15}$$

for every $z \in \mathbb{B}^{2n}$. Moreover, for $z \neq 0$,

$$u_{a,\epsilon}(z) = (\sup\{u(\xi) : \xi \in B_{a,\epsilon,z}\})^*, \tag{3.16}$$

where

$$\begin{aligned} B_{a,\epsilon,z} &= \{\xi \in \mathbb{C}^n : \|\frac{z}{\|z\|} - \frac{\xi}{\|\xi\|}\| < a, \|z\| \leq \|\xi\| \leq (1 + \epsilon)\|z\|\} \\ &= \{t\xi : t \in [\|z\|, (1 + \epsilon)\|z\|], \xi \in \partial\mathbb{B}^{2n}, \|\xi - \frac{z}{\|z\|}\| < a\}. \end{aligned}$$

Denote

$$S_{z/\|z\|,a} = \{\xi \in \mathbb{C}^n : \|\xi\| = 1, \|\xi - \frac{z}{\|z\|}\| < a\}.$$

We have

$$\begin{aligned} V_{2n}(B_{a,\epsilon,z}) = \int_{S_{z/\|z\|,a}} \int_{\|z\|}^{(1+\epsilon)\|z\|} t dt dS(\xi) &= \frac{(2\epsilon + \epsilon^2)\|z\|^2}{2} \int_{S_{z/\|z\|,a}} dS(\xi) \\ &= \frac{(2\epsilon + \epsilon^2)\|z\|^2}{2} \int_{S_{(0,\ldots,0,1),a}} dS(\xi), \end{aligned}$$

the last equality holds since the volume of hypersurfaces are preserved under rotations.

We will show that, for all $1/2 > a > 0$, there exists $a > \epsilon_a > 0$ such that for every $\epsilon_a \geq 3\epsilon \geq 1 - \|z\| \geq \epsilon > 0$,

$$u_{a,\epsilon}(z) \geq -3A\epsilon, \tag{3.17}$$

where $A > 0$ is the constant in the condition (3.14). Consider the parameterization

$$p : \mathbb{B}^{2n-1} \to \partial\mathbb{B}^{2n} \cap \{z \in \mathbb{C}^n = \mathbb{R}^{2n} : y_n > 0\}$$
$$s = (s_1, ..., s_{2n-1}) \longmapsto p(s) = (s, \sqrt{1 - \|s\|^2}).$$

For each $s \in \mathbb{B}^{2n-1}$, we consider the angle α between the vectors $e_{2n} = (0, ..., 0, 1)$ and $p(s)$. We have

$$\sin(\frac{\alpha}{2}) = \frac{\|e_{2n} - p(s)\|}{2} \qquad \text{and} \qquad \sin(\alpha) = \|s\|.$$

Hence,

$$\|s\| = \|e_{2n} - p(s)\|\sqrt{1 - \frac{\|e_{2n} - p(s)\|^2}{4}}.$$

Then $p(\mathbb{B}^{2n-1}_{a\sqrt{1-a^2/4}}) = S_{e_{2n},a}$ and we have

$$\begin{aligned}
V_{2n}(B_{a,\epsilon,z}) &= \frac{(2\epsilon + \epsilon^2)\|z\|^2}{2} \int_{S_{e_{2n},a}} dS(\xi) \\
&= \frac{(2\epsilon + \epsilon^2)\|z\|^2}{2} \int_{\mathbb{B}^{2n-1}_{a\sqrt{1-a^2/4}}} \sqrt{1 + \|\nabla\sqrt{1 - \|\xi\|^2}\|^2} d\xi \\
&= \frac{(2\epsilon + \epsilon^2)\|z\|^2}{2} \int_{\mathbb{B}^{2n-1}_{a\sqrt{1-a^2/4}}} \frac{d\xi}{\sqrt{1 - \|\xi\|^2}}.
\end{aligned}$$

Therefore, there exist $C_1, C_2 > 0$ such that

$$C_1 a^{2n-1}\epsilon < V_{2n}(B_{a,\epsilon,z}) < C_2 a^{2n-1}\epsilon, \tag{3.18}$$

for every $0 < \epsilon, a < 1/2$ and $1/2 < \|z\| \le 1 - \epsilon$.

By (3.14), for every $1/2 > a > 0$, there exists $a > \epsilon_a > 0$ such that, for every $\epsilon_a \ge 3\epsilon > 0$,

$$V_{2n}\{\xi \in \mathbb{B}^{2n} : \|\xi\| > 1 - 3\epsilon, u(\xi) < -3A\epsilon\} < C_1 a^{2n-1}\epsilon,$$

and therefore, by (3.18), for every $3\epsilon \ge 1 - \|z\| \ge \epsilon$,

$$B_{a,\epsilon,z} \not\subseteq \{\xi \in \mathbb{B}^{2n} : \|\xi\| > 1 - 3\epsilon, u(\xi) < -3A\epsilon\}.$$

Then, by (3.16), for every $\epsilon_a \ge 3\epsilon \ge 1 - \|z\| \ge \epsilon > 0$, we have

$$u_{a,\epsilon}(z) \ge -3A\epsilon. \tag{3.19}$$

For each $1/2 > a > 0$ and $\epsilon_a \ge 3\epsilon > 0$, we consider the following function

$$\tilde{u}_{a,\epsilon}(z) = \begin{cases} 3A(-1+\|z\|^2) & \text{if} \quad 1-\epsilon \leq \|z\| \leq 1, \\ \max\{3A(-1+\|z\|^2), u_{a,\epsilon}(z) - 6A\epsilon\} & \text{if} \quad 1-3\epsilon \leq \|z\| \leq 1-\epsilon, \\ u_{a,\epsilon}(z) - 6A\epsilon & \text{if} \quad \|z\| \leq 1-3\epsilon. \end{cases}$$

By using the gluing theorem (see, for example, [25, Corollary 2.9.15]), we have $\tilde{u}_{a,\epsilon} \in \mathcal{PSH}(\mathbb{B}^{2n})$. For $m > 0$, we set $\tilde{u}^m_{a,\epsilon} = \max\{\tilde{u}_{a,\epsilon}, -m\}$. Then, we have $\tilde{u}^m_{a,\epsilon} \searrow \tilde{u}_{a,\epsilon}$, when $m \to \infty$. Moreover, since $\tilde{u}^m_{a,\epsilon} = 3A(-1+\|z\|^2)$ near $\partial\mathbb{B}^{2n}$, we have,

$$\int_{\mathbb{B}^{2n}} (dd^c \tilde{u}^m_{a,\epsilon})^n = \int_{\mathbb{B}^{2n}} (dd^c 3A(-1+\|z\|^2))^n < \infty,$$

for every $m > 0$. Then $\tilde{u}^m_{a,\epsilon} \in \mathcal{E}_0(\mathbb{B}^{2n})$. Therefore, $\tilde{u}_{a,\epsilon} \in \mathcal{F}(\mathbb{B}^{2n})$. Moreover, by Theorem 3.1.2, we have

$$\int_{\mathbb{B}^{2n}} (dd^c \tilde{u}_{a,\epsilon})^n = \lim_{m\to\infty} \int_{\mathbb{B}^{2n}} (dd^c \tilde{u}^m_{a,\epsilon})^n < \infty, \tag{3.20}$$

for every $1/2 > a > 0$ and $\epsilon_a \geq 3\epsilon > 0$.

For every $j \in \mathbb{N}$, we denote $u_j = \tilde{u}_{2^{-j}, 3^{-1}\epsilon_{2^{-j}}}$. By (3.15), we have u_j converges pointwise to u as j tends to ∞. By (3.20), we have $\sup_j \int_{\mathbb{B}^{2n}} (dd^c u_j)^n < \infty$. Then, by using Proposition 3.1.4, we have $u \in \mathcal{F}(\mathbb{B}^{2n})$.

The proof is completed. □

By (3.1) and Theorem 3.4.1, we get the following as a direct consequence.

Corollary 3.4.2. *Let $u \in \mathcal{N}(\mathbb{B}^{2n})$ such that $\int_{\mathbb{B}^{2n}} (dd^c u)^n = \infty$. Then, for every $A > 0$,*

$$\limsup_{d\to 0^+} \frac{V_{2n}(\{z \in \mathbb{B}^{2n} : \|z\| > 1-d, u(z) < -Ad\})}{d} > 0.$$

List of Author's Related Papers

1. Thai Duong Do, *A comparison theorem for subharmonic functions*, Results Math. **74** (2019), Paper No. 176, 13pp. (SCI-E)

2. Hoang Son Do, Thai Duong Do, *Some remarks on Cegrell's class $\mathcal{F}$*, Ann. Polon. Math. **125** (2020), 13–24. (SCI-E)

3. Hoang Son Do, Thai Duong Do, Hoang Hiep Pham, *Complex Monge-Ampère equation in strictly pseudoconvex domains*, Acta Math. Vietnam. **45** (2020), 93-101. (VAST1)

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