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SOME PARAMETRIC OPTIMIZATION PROBLEMS IN MATHEMATICAL ECONOMICS

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SUMMARY

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Introduction

Mathematical economics is the application of mathematical methods to represent theories and analyze problems in economics. The language of mathematics allows one to address the latter with rigor, generality, and simplicity. Formal economic modeling began in the 19th century with the use of differential calculus to represent and explain economic behaviors, such as the utility maximization problem and the expenditure minimization problem, early applications of optimization in microeconomics. Economics became more mathematical as a discipline throughout the first half of the 20th century with the introduction of new and generalized techniques, including ones from calculus of variations and optimal control theory applied in dynamic analysis of economic growth models in macroeconomics.

Although consumption economics, production economics, and optimal economic growth have been studied intensively in many books (Takayama (1974), Intriligator (2002), Barro and Sala-i-Martin (2004), Chiang and Wainwright (2005), Acemoglu (2009), Nicholson and Snyder (2012), Rasmussen (2013), ...), and papers (Ramsey (1928), Harrod (1939), Domar (1946), Cass (1965), Koopmans (1965), Martinez-Legaz and Santos (1993), Crouzeix (1983, 2008), Penot (2013, 2014), Hadjisavvas and Penot (2015), ...), new results on qualitative properties of these models can be expected. They can lead to a deeper understanding of the classical models and to more effective uses of the latter. Fast progresses in optimization theory, set-valued and variational analysis, and optimal control theory allow us to hope that such new results are possible.

This dissertation focuses on qualitative properties (solution existence, optimality conditions, stability, and differential stability) of optimization problems arisen in consumption economics, production economics, and optimal economic growth models. Five chapters of the dissertation are divided into two parts.

Part I, which includes the first two chapters, studies the stability (the continuity property, the Lipschitz property, the Lipschitz-like property, and the Lipschitz-Hölder property) and the differential stability (the Fréchet/limitting coderivatives, the Fréchet/limitting subdifferentials of the infimal nuisance function, upper and lower estimates for the upper and the lower Dini directional derivatives of the indirect utility function) of the consumer problem named *maximizing utility subject to consumer budget constraint* with varying prices. Mathematically, this is a parametric optimization problem; and it is worthy to stress that the problem considered here also presents the producer problem named *maximizing profit subject to producer budget constraint* with varying input prices. Both problems are basic ones in microeconomics.

Part II of the dissertation includes the subsequent three chapters. In Chapters 3 and 4, a *maximum principle* for *finite horizon optimal control problems with state constraints* is analyzed via *parametric examples*. Each of those examples is an optimal control problem with five parameters. The difference among those are in the appearance of state constraints: The first one does not contain state constraints, the second one is a problem with unilateral state constraints, and the third one is a problem with bilateral state constraints. Since the maximum principle is only a necessary condition for local optimal processes, a large amount of additional investigations is needed to obtain a comprehensive synthesis of finitely many processes suspected for being local minimizers. The analysis in these chapters not only helps to understand advanced tools from optimal control theory (Filippov's existence theorems, the maximum principles) in depth, but also serves as a sample of applying them to meaningful prototypes of *economic optimal growth models* in macroeconomics. Chapter 5 establishes a series of theorems on solution existence for optimal economic growth problems in general forms as well as in some typical ones and synthesis of optimal processes for one of such typical problems. Some open questions and conjectures about the uniqueness and regularity of the global solutions of optimal economic growth problems are formulated in this chapter.

Last but not least, let us mention that, there are interpretations of the economic meanings for the majority of the mathematical concepts and obtained results in Chapter 1, 2, and 5, which form an indispensable part of the present dissertation. Needless to say that such economic interpretations of the new results are most desirable in a mathematical study related to economic topics.

Chapter 1

Stability of Parametric Consumer Problems

As in Penot (2013, 2014), we consider the consumer problem named *maximizing utility subject to consumer budget constraint* in the following infinite-dimensional setting. The set of *goods* is modeled by a nonempty, closed and convex cone X_+ in a reflexive Banach space X. The set of *prices* is

$$Y_+ := \{p \in X^* \ : \ \langle p, x\rangle \geq 0, \ \ \forall x \in X_+\},$$

where X^* is the topological dual space of X and $\langle p, x\rangle$ (or $p \cdot x$) is the value of p at x. We may normalize the prices and assume that the income of the consumer is 1. Then, the *budget map* is the set-valued map $B : Y_+ \rightrightarrows X_+$ associating to each price $p \in Y_+$ the *budget set*

$$B(p) := \Big\{x \in X_+ \ : \ \langle p, x\rangle \leq 1\Big\}. \tag{1.1}$$

We assume that the preferences of the consumer are presented by a function $u : X \to \overline{\mathbb{R}}$, called the *utility function*. This means that $u(x) \in \mathbb{R}$ for every $x \in X_+$, and a goods bundle $x \in X_+$ is preferred to another one $x' \in X_+$ if and only if $u(x) > u(x')$.

For a given price $p \in Y_+$, the problem is to maximize $u(x)$ subject to the constraint $x \in B(p)$. It is written formally as

$$\max\{u(x) \ : \ x \in B(p)\}. \tag{1.2}$$

The *indirect utility function* $v : Y_+ \to \overline{\mathbb{R}}$ of (1.2) is defined by setting

$$v(p) = \sup\{u(x) : x \in B(p)\}, \quad p \in Y_+.$$

The *demand map* of (1.2) is the set-valued map $D : Y_+ \rightrightarrows X_+$ defined by

$$D(p) = \{x \in B(p) \ : \ u(x) = v(p)\} , \quad p \in Y_+.$$

Mathematically, the problem (1.2) is an *parametric optimization problem*, where the prices p varying in Y_+ play as the role of *parameters*, the function $v(\cdot)$ is called the *optimal value function*, and the set-valued map $D(\cdot)$ is called the *solution map*.

Three illustrative examples of the consumer problem are presented in this section. The first one is the problem considered in finite dimension, while the second and the third are the ones in infinite-dimensional setting.

There are explanations why the consumer problem (1.2) consider in Chapters 1 and 2 has the same mathematical form to the producer problem named *maximizing profit subject to producer budget constraint with varying input prices* in the production theory, which is recalled in this section. Thus, all the results and proofs in these two chapters for the former problem are valid for the latter one.

In the dissertation, we have presented some concepts and results from set-valued analysis and variational inequalities in order to establish the stability properties of the function $v(\cdot)$ and the multifunctions $B(\cdot)$, $D(\cdot)$. The key concepts includes: the upper/lower semicontinuity of a set-valued map between topological spaces at a point/on a set and the Lipschitz-likenes of a set-valued map between Banach spaces at a point in its graph.

In the forthcoming statements, we consider X_+ (resp., Y_+) with the topologies *induced* from the topologies of X (resp., of Y). For example, an open set in the strong (resp., weak) topology X_+ is the intersection of X_+ and a subset of X, which is open in the strong (resp., weak) topology of X. Similarly, an open set in the strong (resp., weak, weak*) topology of Y_+ is the intersection of Y_+ and a subset of X^*, which is open in the strong (resp., weak, weak*) topology of X^*. By abuse of terminology, we shall speak about the weak and weak* topologies of X_+ (resp., of Y_+).

The lower semicontinuity property of the budget map can be stated as follows.

Proposition 1.1 *The set-valued map $B : Y_+ \rightrightarrows X_+$ is l.s.c. on Y_+ in the weak* topology of Y_+ and the strong topology of X_+. Hence, $B : Y_+ \rightrightarrows X_+$ is l.s.c. on Y_+ in the strong topologies of Y_+ and X_+.*

Unlike the preceding result on the l.s.c. property, the upper semicontinuity property of the budget map can be obtained only for internal points of the set of prices, and it requires a more stringent condition on topologies.

Proposition 1.2 *The set-valued map $B : Y_+ \rightrightarrows X_+$ is u.s.c. on $\operatorname{int} Y_+$ in the strong topology of Y_+ and the weak topology of X_+.*

From Propositions 1.1, 1.2, we obtain the next result on the continuity of the budget map.

Theorem 1.1 *The set-valued map $B : Y_+ \rightrightarrows X_+$ has nonempty weakly compact, convex values and is continuous on $\operatorname{int} Y_+$ in the strong topology of Y_+ and the weak topology of X_+. Specifically, if X is finite-dimensional, then $B(\cdot)$ has nonempty compact, convex values and is continuous on $\operatorname{int} Y_+$.*

Based on the above results, we are now in a position to present several continuity properties of the indirect utility function.

The forthcoming statement on the lower semicontinuity of $v(\cdot)$ is weaker than a lemma of Penot (2014), where it was only assumed that the utility function is lower radially l.s.c. on X_+. It is worthy to notice that our approach is new. Namely, we derive the desired result from the l.s.c. property of $B(\cdot)$, which is guaranteed by Proposition 1.1. In some sense, our proof arguments are simpler than those of Penot (2014).

Proposition 1.3 (cf. [Penot (2014), Lemma 3.1]) *If $u : X_+ \to \mathbb{R}$ is l.s.c. on X_+ in the strong topology of X_+, then $v : Y_+ \to \overline{\mathbb{R}}$ is l.s.c. on Y_+ in the weak* topology of Y_+.*

The next result on the upper semicontinuity of $v(\cdot)$ is due to Penot (2014). Here we give a new proof by using the u.s.c. property of $B(\cdot)$ provided by Proposition 1.2.

Proposition 1.4 (See [Penot (2014), Proposition 3.2]) *If $u : X_+ \to \mathbb{R}$ is u.s.c. on X_+ in the weak topology of X_+, then $v : Y_+ \to \overline{\mathbb{R}}$ is u.s.c. on $\operatorname{int} Y_+$ in the strong topology of Y_+.*

As a consequence of Propositions 1.3 and 1.4, we get the following result on the continuity of the indirect utility function.

Theorem 1.2 *If u is weakly u.s.c. and strongly l.s.c. on X_+, then v is strongly continuous on $\operatorname{int} Y_+$. Especially, if X is finite-dimensional and u is continuous on X_+, then v is continuous on $\operatorname{int} Y_+$.*

The following statement is an analogue of a proposition in Penot (2014). Here we do not use any assumption on the indirect utility function $v(\cdot)$.

Proposition 1.5 *If u is weakly u.s.c. and strongly l.s.c. on X_+, then the demand map $D : Y_+ \rightrightarrows X_+$ is u.s.c. on $\operatorname{int} Y_+$ in the strong topology of Y_+ and the weak topology of X_+.*

The next theorem is about a stability property of the budget map in the form of a uniform local error bound. The principal tool in the proofs of this theorem is Theorem 3.2 from the paper of J. M. Borwein [*Stability and regular points of inequality systems*, J. Optim. Theory Appl. **48** (1986), 9–52] on the Robinson stability property of a constraint system depending on parameters.

Theorem 1.3 *For any $p_0 \in \operatorname{int} Y_+$ and $x_0 \in B(p_0)$, there exists $\mu \geq 0$ along with a neighborhood U of p_0 and a neighborhood V of x_0 such that*

$$d(x, B(p)) \leq \mu[p \cdot x - 1]_+, \quad \forall p \in U \cap Y_+, \ \forall x \in V \cap X_+, \tag{1.3}$$

where $\alpha_+ := \max\{0, \alpha\}$.

It is shown in the proof of the next theorem, the Robinson stability property (1.3) of the constraint system $f(x, p) = p \cdot x - 1 \leq 0, x \in X_+$ depending on the parameter $p \in Y_+$, implies the Lipschitz-likeness of $B(\cdot)$ at (p_0, x_0).

Theorem 1.4 *For any $p_0 \in \operatorname{int} Y_+$ and $x_0 \in B(p_0)$, the map $B : Y_+ \rightrightarrows X_+$ is Lipschitz-like at (p_0, x_0) in the sense that there exist a neighborhood U of p_0, a neighborhood V of x_0, and a constant $\ell > 0$ satisfying*

$$B(p) \cap V \subset B(p') + \ell\|p - p'\|\overline{B}_X, \quad \forall p, p' \in U \cap Y_+.$$

.

The Lipschitz property of the indirect utility function is investigated by using the Lipschitz-like property of the budget map.

Theorem 1.5 *Suppose that X is finite-dimensional and $u : X_+ \to \mathrm{I\!R}$ is locally Lipschitz on X_+. Then the indirect utility function $v : Y_+ \to \overline{\mathrm{I\!R}}$ is locally Lipschitz on* $\operatorname{int} Y_+$.

We now describe the Lipschitz-Hölder property of the demand map by using the Lipschitz-like property of the budget map. Theorem 2.1 and Remark 2.3 from a paper by N. D. Yen [*Hölder continuity of solutions to a parametric variational inequality*, Applied Math. Optim. **31** (1995), 245–255] on solution sensitivity of a parametric variational inequality are the principal tools in our investigations.

Assume that X is a Hilbert space, M is a parameter set in a norm space, and

$$u : X_+ \times M \to \mathrm{I\!R}$$

is a utility function depending on the parameter $\mu \in M$. The appearance of μ signifies the fact that the utility function is subject to change, due to the changes of customs, the scale of values, time, etc. Consider the *parametric consumer problem*

$$\max\{u(x,\mu) \;:\; x \in B(p)\} \tag{1.4}$$

depending on a pair $(\mu, p) \in M \times Y_+$ where, as before, $B : Y_+ \rightrightarrows X_+$ is the budget map given by (1.1). It is clear that (1.4) is a generalization of (1.2).

In the sequel, it is assumed that there exists an open set Ω containing X_+ such that u is defined on $\Omega \times M$ and, for each $\mu \in M$, $u(\cdot, \mu)$ is Fréchet differentiable at every point of X_+. By $\nabla_x u(x, \mu)$ we denote the Fréchet derivative of $u(\cdot, \mu)$ at $x \in X_+$. Let x_0 be a solution of (1.4) at a given pair of parameters $(\mu_0, p_0) \in M \times Y_+$. Suppose that there exist a closed and convex neighborhood V of x_0, a neighborhood W of μ_0, and constants $\alpha > 0$, $\ell > 0$ satisfying

$$\|\nabla_x u(x', \mu') - \nabla_x u(x, \mu)\| \le \ell(\|x' - x\| + \|\mu' - \mu\|), \quad \forall x, x' \in V,\ \forall \mu, \mu' \in M \cap W \tag{1.5}$$

and

$$\langle \nabla_x(-u)(x', \mu) - \nabla_x(-u)(x, \mu), x' - x\rangle \ge \alpha\|x' - x\|^2, \quad \forall x, x' \in V,\ \forall \mu \in M \cap W. \tag{1.6}$$

Theorem 1.6 *Assume that, for every $\mu \in M$, the function $u(\cdot, \mu)$ is concave on X_+ and the operator $\nabla_x(-u)(\cdot, \mu) : X_+ \to X^*$ is continuous, where the dual space X^* is considered with the weak topology. Suppose that x_0 is a solution to the parametric consumer problem* (1.4) *with respect to a given pair of parameters $(\mu_0, p_0) \in M \times \operatorname{int} Y_+$ and conditions* (1.5), (1.6) *are satisfied. Then, there exist constants $\kappa_{\mu_0} > 0$, $\kappa_{p_0} > 0$, and neighborhoods W_1 of μ_0, U_1 of p_0 such that*

(a) *For every $(\mu, p) \in (M \cap W_1) \times (Y_+ \cap U_1)$,* (1.4) *has a unique solution, denoted by $x(\mu, p)$, and $x(\mu, p) \in \operatorname{int} V$;*

(b) *For all $(\mu', p'), (\mu, p) \in (M \cap W_1) \times (Y_+ \cap U_1)$,*

$$\|x(\mu', p') - x(\mu, p)\| \le \kappa_{\mu_0}\|\mu' - \mu\| + \kappa_{p_0}\|p' - p\|^{1/2}.$$

There are an illustrative example for Theorem 1.6 and an example in which the utility function fulfills the technical assumptions (1.5) and (1.6) in Theorem 1.6.

In the dissertation, we have given some economic interpretations for mathematical concepts involving directly to the consumer problem: the continuity, Lipschitz continuity, and Lipschitz-Hölder continuity.

Chapter 2

Differential Stability of Parametric Consumer Problems

In Section 2.1 "Auxiliary Concepts and Results" in the dissertation, we recall some concepts of generalized differentiation from the two-volume book of Mordukhovich (2006), as well as some tools from a paper of Mordukhovich (2004) and a paper of Mordukhovich, Nam, and Yen (2009).

The key concepts are the Fréchet normal cone $\widehat{N}(\bar{x};\Omega)$ and the limiting normal cone $N(\bar{x};\Omega)$ of a subset Ω in a Banach space X at $\bar{x} \in X$; the Fréchet coderivative $\widehat{D}^*F(\bar{x},\bar{y})$ and the limiting coderivative $D^*F(\bar{x},\bar{y})$ of a set-valued map $F: X \rightrightarrows Y$ between Banach spaces X, Y at $(\bar{x},\bar{y}) \in X \times Y$; the Fréchet subdifferential $\widehat{\partial}\varphi(\bar{x})$, the limiting subdifferential $\partial\varphi(\bar{x})$, and the singular subdifferential $\partial^\infty\varphi(\bar{x})$ of a function $\varphi: X \to \overline{\mathrm{IR}}$ at $\bar{x} \in X$; the Fréchet upper subdifferential $\widehat{\partial}^+\varphi(\bar{x})$, the limiting upper subdifferential $\partial^+\varphi(\bar{x})$, and the singular upper subdifferential $\partial^{\infty,+}\varphi(\bar{x})$ of a function $\varphi: X \to \overline{\mathrm{IR}}$ at $\bar{x} \in X$; and the sequentially normal compactness (SNC) of a subset of a Banach space at its point.

The main tools are two theorems from a paper of Mordukhovich (2004) on parametric generalized equations and three theorems from a paper of Mordukhovich, Nam, and Yen (2009) on parametric optimization problems. The first one is about formulas for estimating the limiting coderivative of the solution map of a given parametric generalized equation. The second one states a necessary and sufficient condition for Lipschitz-like property of that solution map. Three last theorems are about formulas for estimating Fréchet/ limiting/ singular subdifferentials of the optimal value function of parametric optimization problems via subdifferentials of the objective function and coderivatives of the constraint map.

In Chapter 1, a Lipschitz-Hölder property of the demand map $D(\cdot)$ was obtained by using the Lipschitz-like property of the budget map $B(\cdot)$ at point $(\bar{p},\bar{x}) \in \operatorname{gph} B$ with $\bar{p} \in Y_+$. Here, we show that if $\bar{x} \neq 0$ and X_+ is SNC at $\bar{x}$, then we can get the Lipschitz-like property of $B(\cdot)$ without imposing the condition $\bar{p} \in \operatorname{int} Y_+$. Hence, Theorem 1.6 in the previous chapter can be extended to the case where $\bar{p}$ may belong to the boundary of Y_+.

Theorem 2.1 *If $\bar{p} \in Y_+$, $\bar{x} \in B(\bar{p}) \setminus \{0\}$, and X_+ is SNC at $\bar{x}$, then the budget map $B(\cdot)$ is Lipschitz-like at $(\bar{p},\bar{x})$.*

Under some mild conditions, we can have exact formulas for both Fréchet and limiting coderivatives of the budget map.

Theorem 2.2 *Suppose that* $\bar{p} \in \operatorname{int} Y_+$, $\bar{x} \in B(\bar{p}) \setminus \{0\}$, *and* X_+ *is SNC at* $\bar{x}$. *Then the budget map* $B : Y_+ \rightrightarrows X_+$ *is graphically regular at* $(\bar{p}, \bar{x})$. *Moreover, for every* $x^* \in X^*$, *one has*

$$\widehat{D}^*B(\bar{p},\bar{x})(x^*) = D^*B(\bar{p},\bar{x})(x^*) = \begin{cases} \{\lambda\bar{x} \ : \ \lambda \geq 0, \ x^* + \lambda\bar{p} \in -N(\bar{x}; X_+)\} & \text{if } \langle \bar{p}, \bar{x}\rangle = 1 \\ \{0\} \quad \text{if } \langle \bar{p}, \bar{x}\rangle < 1, \ x^* \in -N(\bar{x}; X_+) \\ \emptyset \quad \text{if } \langle \bar{p}, \bar{x}\rangle < 1, \ x^* \notin -N(\bar{x}; X_+). \end{cases}$$

Technically, we will transform the consumer problem into an equivalent minimization one and then apply the results in the paper of Mordukhovich, Nam, and Yen (2009) on estimating subdifferentials of the optimal value function. By that way, we will get

$$-v(p) = \inf\{-u(x) : x \in B(p)\}, \quad p \in Y_+;$$

hence, we can consider a counterpart of $v(\cdot)$, the *infimal nuisance function* $-v(\cdot)$ obtained from the former by changing its sign, as the role of the optimal value function of the corresponding minimization problem.

Results on estimating the Fréchet subdifferential of the function $-v$ are presented in this section, while those on estimating the limiting one will be addressed in the next section.

Theorem 2.3 *Let* $\bar{p} \in \operatorname{int} Y_+$ *and* $\bar{x} \in D(\bar{p}) \setminus \{0\}$ *be such that* $D(\bar{p}) \neq \emptyset$, X_+ *is SNC at* $\bar{x}$, *and* $\widehat{\partial}u(\bar{x}) \neq \emptyset$. *The following assertions hold:*

(i) *If* $\langle \bar{p}, \bar{x}\rangle = 1$, *then*

$$\widehat{\partial}(-v)(\bar{p}) \subset \bigcap_{x^* \in -\widehat{\partial}u(\bar{x})} \{\lambda\bar{x} \ : \ \lambda \geq 0, \ x^* + \lambda\bar{p} \in -N(\bar{x}; X_+)\};$$

(ii) *If* $\langle \bar{p}, \bar{x}\rangle < 1$, *then* $\widehat{\partial}(-v)(\bar{p}) \subset \{0\}$;

(iii) *If* $\langle \bar{p}, \bar{x}\rangle < 1$ *and* $\widehat{\partial}u(\bar{x}) \setminus N(\bar{x}; X_+) \neq \emptyset$, *then* $\widehat{\partial}(-v)(\bar{p}) = \emptyset$;

(iv) *If* u *is Fréchet differentiable at* $\bar{x}$, *and the map* $D : \operatorname{dom} B \rightrightarrows X_+$ *admits a local upper Lipschitzian selection at* $(\bar{p}, \bar{x})$, *then*

$$\widehat{\partial}(-v)(\bar{p}) = \begin{cases} \{\langle \nabla u(\bar{x}), \bar{x}\rangle \bar{x}\} & \text{if } \langle \bar{p}, \bar{x}\rangle = 1 \\ \{0\} & \text{if } \langle \bar{p}, \bar{x}\rangle < 1. \end{cases}$$

Two corollaries of Theorem 2.3 and an example illustrated for one of those corollaries are provided in the dissertation.

Our results on limiting and singular subdifferentials of $-v$ are stated in the next theorem.

Theorem 2.4 *Let* $\bar{p} \in \operatorname{int} Y_+$ *and* $\bar{x} \in D(\bar{p}) \setminus \{0\}$ *be such that* $D(\bar{p}) \neq \emptyset$, X_+ *is SNC at* $\bar{x}$, u *is upper semicontinuous at* $\bar{x}$, *and* D *is* v*-inner semicontinuous at* $(\bar{p}, \bar{x})$. *Assume that either* hypo u *is SNC at* $(\bar{x}, \varphi(\bar{x}))$ *or* X *is finite-dimensional, and the qualification condition*

$$\partial^{\infty,+}u(\bar{x}) \cap N(\bar{x}; X_+) = \{0\} \tag{2.1}$$

is satisfied. Then, the following assertions hold:

(i) *If* $\langle \bar{p}, \bar{x} \rangle = 1$, *then*

$$\partial(-v)(\bar{p}) \quad \subset \bigcup_{x^* \in \partial^+ u(\bar{x})} \{\lambda \bar{x} \,:\, \lambda \geq 0,\ x^* - \lambda \bar{p} \in N(\bar{x}; X_+)\},$$
$$\partial^\infty(-v)(\bar{p}) \quad \subset \bigcup\nolimits_{x^* \in \partial^{\infty,+} u(\bar{x})} \{\lambda \bar{x} \,:\, \lambda \geq 0,\ x^* - \lambda \bar{p} \in N(\bar{x}; X_+)\};$$

(ii) *If* $\langle \bar{p}, \bar{x} \rangle < 1$, *then* $\partial(-v)(\bar{p}) \subset \{0\}$ *and* $\partial^\infty(-v)(\bar{p}) = \{0\}$;

(iii) *If* $\langle \bar{p}, \bar{x} \rangle < 1$ *and* $\partial^+ u(\bar{x}) \cap N(\bar{x}; X_+) = \emptyset$, *then* $\partial(-v)(\bar{p}) = \emptyset$;

(iv) *If* u *is strictly differentiable at* $\bar{x}$, *and the map* $D : \operatorname{dom} B \rightrightarrows X_+$ *admits a local upper Lipschitzian selection at* $(\bar{p}, \bar{x})$, *then* $(-v)$ *is lower regular at* $\bar{x}$ *and*

$$\partial(-v)(\bar{p}) = \begin{cases} \{\langle \nabla u(\bar{x}), \bar{x} \rangle \bar{x}\} & \text{if } \langle \bar{p}, \bar{x} \rangle = 1 \\ \{0\} & \text{if } \langle \bar{p}, \bar{x} \rangle < 1. \end{cases}$$

Let us present a counterpart of Theorem 2.4, where the assumption on the v-inner semicontinuity of D at $(\bar{p}, \bar{x})$ is removed. In fact, here one has the v-inner semicompactness of D at $\bar{p}$, which is guaranteed by the assumptions saying that X is finite-dimensional and $\bar{p} \in \operatorname{int} Y_+$.

Theorem 2.5 *Suppose that* X *is a finite-dimensional Banach space, the non satiety condition is satisfied, and* u *is upper semicontinuous on* X_+. *For any* $\bar{p} \in \operatorname{int} Y_+$, *if the qualification condition* (2.1) *is satisfied for every* $\bar{x} \in D(\bar{p})$, *then one has*

$$\partial(-v)(\bar{p}) \quad \subset \bigcup_{\bar{x} \in D(\bar{p})} \ \bigcup_{x^* \in \partial^+ u(\bar{x})} \{\lambda \bar{x} \,:\, \lambda \geq 0,\ x^* - \lambda \bar{p} \in N(\bar{x}; X_+)\},$$
$$\partial^\infty(-v)(\bar{p}) \quad \subset \bigcup\nolimits_{\bar{x} \in D(\bar{p})} \bigcup\nolimits_{x^* \in \partial^{\infty,+} u(\bar{x})} \{\lambda \bar{x} \,:\, \lambda \geq 0,\ x^* - \lambda \bar{p} \in N(\bar{x}; X_+)\}.$$

An illustrative example for Theorem 2.4 and a corollary of Theorem 2.5 are provided in the dissertation. We also give some economic interpretations for the obtained results on estimating subdifferentials. To do so, we have clarified the relationships between the concepts of subdifferential and derivative.

Chapter 3

Parametric Optimal Control Problems with Unilateral State Constraints

The Sobolev space $W^{1,1}([t_0, T], \mathbb{R}^n)$ is the linear space of the absolutely continuous functions $x : [t_0, T] \to \mathbb{R}^n$ endowed with the norm

$$\|x\|_{W^{1,1}} = \|x(t_0)\| + \int_{t_0}^{T} \|\dot{x}(t)\| dt.$$

The normal cone $N(\bar{x};\Omega)$ of a subset $\Omega \subset \mathbb{R}^n$ (resp., the subdifferential $\partial\varphi(\bar{x})$ of an extended real-valued function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$) at a point $\bar{x}$ is understood in the sense of the Mordukhovich normal cone (resp., the Mordukhovich subdifferential).

Let $a > \lambda > 0$, $T > t_0 \geq 0$, and $x_0 \in \mathbb{R}$ be given as five parameters. In this chapter, we consider two finite horizon optimal control problems of the Lagrange type denoted by (FP_1) and (FP_2). The first problem (FP_1) is the following

$$\text{Minimize} \;\; J(x,u) = \int_{t_0}^{T} \big[- e^{-\lambda t}(x(t) + u(t))\big]\, dt$$

over $x \in W^{1,1}([t_0,T],\mathbb{R})$ and measurable functions $u : [t_0,T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{x}(t) = -au(t), & \text{a.e. } t \in [t_0,T] \\ x(t_0) = x_0 \\ u(t) \in [-1,1], & \text{a.e. } t \in [t_0,T]. \end{cases} \tag{3.1}$$

The second problem (FP_2) is formed from (FP_1) by adding the requirement $x(t) \leq 1$ for all $t \in [t_0,T]$ to the constraint system (3.1).

We will treat (FP_1) and (FP_2) as problems of the Mayer type by setting

$$x(t) \;=\; (x_1(t), x_2(t)),$$

where $x_1(t)$ plays the role of the variable $x(t)$ in (FP_1) and (FP_2), and

$$x_2(t) := \int_{t_0}^{t} \big[- e^{-\lambda \tau}(x_1(\tau) + u(\tau))\big]\, d\tau$$

for all $t \in [0,T]$. Then, (FP_1) is equivalent to the problem

$$\text{Minimize} \;\; x_2(T)$$

over $x = (x_1,x_2) \in W^{1,1}([t_0,T],\mathbb{R}^2)$ and measurable functions $u : [t_0,T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{x}_1(t) = -au(t), & \text{a.e. } t \in [t_0,T] \\ \dot{x}_2(t) = -e^{-\lambda t}(x_1(t) + u(t)), & \text{a.e. } t \in [t_0,T] \\ (x(t_0), x(T)) \in \{(x_0,0)\} \times \mathbb{R}^2 \\ u(t) \in [-1,1], & \text{a.e. } t \in [t_0,T], \end{cases}$$

which is abbreviated to (FP_{1a}). Similarly, (FP_2) is equivalent to the problem (FP_{2a}), which is formed from (FP_{1a}) by adding the requirement $x_1(t) \leq 1, \forall t \in [t_0,T]$ to the constraint system of the latter.

As in the book by R. Vinter [*Optimal Control*, Birkhäuser, Boston, 2000; p. 321], we consider the following *finite horizon optimal control problem of the Mayer type*, denoted by $\mathcal{M}$,

$$\text{Minimize} \;\; g(x(t_0), x(T)),$$

over $x \in W^{1,1}([t_0,T],\mathbb{R}^n)$ and measurable functions $u : [t_0,T] \to \mathbb{R}^m$ satisfying

$$\begin{cases} \dot{x}(t) = f(t,x(t),u(t)), & \text{a.e. } t \in [t_0,T] \\ (x(t_0), x(T)) \in C \\ u(t) \in U(t), & \text{a.e. } t \in [t_0,T] \\ h(t,x(t)) \leq 0, & \forall t \in [t_0,T], \end{cases} \tag{3.2}$$

where $[t_0, T]$ is a given interval, $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $f : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $h : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}$ are given functions, $C \subset \mathbb{R}^n \times \mathbb{R}^n$ is a closed set, and $U : [t_0, T] \rightrightarrows \mathbb{R}^m$ is a set-valued map.

A measurable function $u : [t_0, T] \to \mathbb{R}^m$ satisfying $u(t) \in U(t)$ for almost every $t \in [t_0, T]$ is called a *control function.* A *process* (x, u) consists of a control function u and an arc $x \in W^{1,1}([t_0, T]; \mathbb{R}^n)$ that is a solution to the differential equation in (3.2). A *state trajectory* x is the first component of some process (x, u). A process (x, u) is called *feasible* if the state trajectory satisfies the *endpoint constraint* $(x(t_0), x(T)) \in C$ and the *state constraint* $h(t, x(t)) \le 0$ for all $t \in [t_0, T]$. A feasible process $(\bar{x}, \bar{u})$ is called a $W^{1,1}$ *local minimizer* for $\mathcal{M}$ if there exists $\delta > 0$ such that $g(\bar{x}(t_0), \bar{x}(T)) \le g(x(t_0), x(T))$ for any feasible process (x, u) satisfying $\|\bar{x} - x\|_{W^{1,1}} \le \delta$. A feasible process $(\bar{x}, \bar{u})$ is called a $W^{1,1}$ *global minimizer* for $\mathcal{M}$ if, for any feasible process (x, u), one has $g(\bar{x}(t_0), \bar{x}(T)) \le g(x(t_0), x(T))$.

The *Hamiltonian* $\mathcal{H} : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ of (3.2) is defined by

$$\mathcal{H}(t, x, p, u) := \langle p, f(t, x, u)\rangle = \sum_{i=1}^{n} p_i f_i(t, x, u).$$

The *partial hybrid subdifferential* $\partial_x^> h(t, x)$ of $h(t, x)$ w.r.t. x is given by

$$\partial_x^> h(t, x) := \operatorname{co}\big\{\xi \,:\, \text{there exists } (t_i, x_i) \xrightarrow{h} (t, x) \text{ such that } h(t_k, x_k) > 0 \text{ for all } k \text{ and } \nabla_x h(t_k, x_k) \to \xi\big\},$$

where $(t_k, x_k) \xrightarrow{h} (t, x)$ means that $(t_k, x_k) \to (t, x)$ and $h(t_k, x_k) \to h(t, x)$ when $k \to \infty$.

Due to the appearance of the state constraint $h(t, x(t)) \le 0$ in $\mathcal{M}$, one has to introduce a multiplier that is an element in the topological dual $C^*([t_0, T]; \mathbb{R})$ of the space of continuous functions $C([t_0, T]; \mathbb{R})$ with the supremum norm. By the Riesz Representation Theorem, any bounded linear functional f on $C([t_0, T]; \mathbb{R})$ can be uniquely represented in the form $f(x) = \int_{[t_0,T]} x(t)dv(t)$, where v is *a function of bounded variation* on $[t_0, T]$ which vanishes at t_0 and which are continuous from the right at every point $\tau \in (t_0, T)$, and $\displaystyle\int_{[t_0,T]} x(t)dv(t)$ is the Riemann-Stieltjes integral of x with respect to v. The set of the elements of $C^*([t_0, T]; \mathbb{R})$ which are given by nondecreasing functions v is denoted by $C^\oplus(t_0, T)$. Every $v \in C^*([t_0, T]; \mathbb{R})$ corresponds to *a finite regular measure*, denoted by μ_v, on the σ-algebra $\mathcal{B}$ of the Borel subsets of $[t_0, T]$ by the formula $\mu_v(A) := \int_{[t_0,T]} \chi_A(t)dv(t)$, where $\chi_A(t) = 1$ for $t \in A$ and $\chi_A(t) = 0$ if $t \notin A$. Due to the correspondence $v \mapsto \mu_v$, we call every element $v \in C^*([t_0, T]; \mathbb{R})$ a "measure" and identify v with μ_v. Clearly, the measure corresponding to each $v \in C^\oplus(t_0, T)$ is nonnegative. The integrals $\displaystyle\int_{[t_0,t)} \nu(s)d\mu(s)$ and $\displaystyle\int_{[t_0,T]} \nu(s)d\mu(s)$ of a Borel measurable function ν in the next theorem are understood in the sense of the Lebesgue-Stieltjes integration.

Theorem 3.1 (Theorem 9.3.1 in the cited book of Vinter (2000)) *Let $(\bar{x}, \bar{u})$ be a $W^{1,1}$ local minimizer for $\mathcal{M}$. Assume that for some $\delta > 0$, the following hypotheses are satisfied:*

(H1) *$f(\cdot, x, \cdot)$ is $\mathcal{L} \times \mathcal{B}^m$ measurable, for fixed x. There exists a Borel measurable function $k(\cdot, \cdot) : [t_0, T] \times \mathbb{R}^m \to \mathbb{R}$ such that $t \mapsto k(t, \bar{u}(t))$ is integrable and*

$$\|f(t, x, u) - f(t, x', u)\| \le k(t, u)\|x - x'\|, \forall x, x' \in \bar{x}(t) + \delta\bar{B}, \forall u \in U(t), \text{ a.e.;}$$

(H2) $\operatorname{gph} U$ *is a Borel set in* $[t_0, T] \times \mathbb{R}^m$*;*

(H3) g *is Lipschitz continuous on the ball* $(\bar{x}(t_0), \bar{x}(T)) + \delta\bar{B}$*;*

(H4) h *is upper semicontinuous and there exists* $K > 0$ *such that*

$$\|h(t,x) - h(t,x')\| \leq K\|x - x'\|, \quad \forall x, x' \in \bar{x}(t) + \delta\bar{B},\ \forall t \in [t_0, T].$$

Then there exist $p \in W^{1,1}([t_0, T]; \mathbb{R}^n)$*,* $\gamma \geq 0$*,* $\mu \in C^{\oplus}(t_0, T)$*, and a Borel measurable function* $\nu : [t_0, T] \to \mathbb{R}^n$ *such that* $(p, \mu, \gamma) \neq (0, 0, 0)$*, and for* $q(t) := p(t) + \eta(t)$ *with*

$$\eta(t) := \int_{[t_0,t)} \nu(s)d\mu(s), \quad \text{if } t \in [t_0, T)$$

and $\eta(T) := \displaystyle\int_{[t_0,T]} \nu(s)d\mu(s)$*, the following holds true:*

(i) $\nu(t) \in \partial_x^{>} h(t, \bar{x}(t))$ *μ-a.e.;*

(ii) $-\dot{p}(t) \in \operatorname{co} \partial_x \mathcal{H}(t, \bar{x}(t), q(t), \bar{u}(t))$ *a.e.;*

(iii) $(p(t_0), -q(T)) \in \gamma\partial g(\bar{x}(t_0), \bar{x}(T)) + N((\bar{x}(t_0), \bar{x}(T)); C)$*;*

(iv) $\mathcal{H}(t, \bar{x}(t), q(t), \bar{u}(t)) = \max_{u \in U(t)} \mathcal{H}(t, \bar{x}(t), q(t), u)$ *a.e.*

Using *Filippov's Existence Theorem for Mayer problems* in the book by L. Cesari [*Optimization Theory and Applications*, Springer-Verlag, New York, 1983; Theorem 9.2.i and Section 9.4], we have proved that (FP_{1a}) (resp., (FP_{2a})) has a $W^{1,1}$ global minimizer. Therefore, (FP_1) (resp., (FP_2)) has a $W^{1,1}$ global minimizer by the equivalence of (FP_{1a}) and (FP_1) (resp., by the equivalence of (FP_{2a}) and (FP_2)).

Applying Theorem 3.1 to unconstrained optimal control problems, one gets Theorem 6.2.1 in the book of Vinter (2000). By the latter and a relatively simple additional analysis, we have obtained the following result.

Theorem 3.2 *Given any* a, λ *with* $a > \lambda > 0$*, define* $\rho = \dfrac{1}{\lambda} \ln \dfrac{a}{a - \lambda} > 0$ *and* $\bar{t} = T - \rho$*. Then, problem* (FP_1) *has a unique local solution* $(\bar{x}, \bar{u})$*, which is a global solution, where* $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ *for almost every* $t \in [t_0, T]$ *and* $\bar{x}(t)$ *can be described as follows:*

(a) *If* $t_0 \geq \bar{t}$ *(i.e.,* $T - t_0 \leq \rho$*), then*

$$\bar{x}(t) = x_0 - a(t - t_0), \quad t \in [t_0, T].$$

(b) *If* $t_0 < \bar{t}$ *(i.e.,* $T - t_0 > \rho$*), then*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}] \\ x_0 - a(t + t_0 - 2\bar{t}), & t \in (\bar{t}, T]. \end{cases}$$

For the problem (FP_2), based on Theorem 3.1 and a series of three propositions, we have obtained in the next theorem.

Theorem 3.3 *Given any a, λ with $a > \lambda > 0$, define $\rho = \dfrac{1}{\lambda} \ln \dfrac{a}{a-\lambda} > 0$, $\bar{t} = T - \rho$, $\bar{x}_0 = 1 - a(\bar{t} - t_0)$, and $\alpha_1 = t_0 + a^{-1}(1 - x_0)$. Then, the problem (FP_2) has a unique local solution $(\bar{x}, \bar{u})$, which is a global solution, where $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ for almost every $t \in [t_0, T]$ and $\bar{x}(t)$ is described as follows:*

(a) *If $t_0 \geq \bar{t}$ (i.e, $T - t_0 \leq \rho$), then*

$$\bar{x}(t) = x_0 - a(t - t_0), \quad t \in [t_0, T].$$

(b) *If $t_0 < \bar{t}$ and $x_0 < \bar{x}_0$ (i.e, $\rho < T - t_0 < \rho + a^{-1}(1 - x_0)$), then*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}] \\ x_0 - a(t + t_0 - 2\bar{t}), & t \in (\bar{t}, T]. \end{cases}$$

(c) *If $t_0 < \bar{t}$ and $x_0 = \bar{x}_0$ (i.e, $0 < T - t_0 = \rho + a^{-1}(1 - x_0)$), then*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}) \\ 1 - a(t - \bar{t}), & t \in [\bar{t}, T]. \end{cases}$$

(d) *If $t_0 < \bar{t}$ and $x_0 > \bar{x}_0$ (i.e, $T - t_0 > \rho + a^{-1}(1 - x_0)$, then*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \alpha_1) \\ 1, & t \in [\alpha_1, \bar{t}) \\ 1 - a(t - \bar{t}), & t \in [\bar{t}, T]. \end{cases}$$

Geometrically, depending on the parameters tube $(\lambda, a, x_0, t_0, T)$, by Theorem 3.3 we know that the unique optimal trajectory $\{(t, \bar{x}(t)) : t \in [t_0, T]\} \subset \mathbb{R}^2$ of (FP_{2a}) must be of one the following four types:

(a) "*interior straight trajectory*" – a line segment, which does not touch the boundary line $x = 1$;

(b) "*interior triangular trajectory*" – the union of two line segments, which does not touch the boundary line $x = 1$ and which has a turning point at the time moment $t = \bar{t}$;

(c) "*boundary triangular trajectory*" – the union of two line segments, which touches the boundary line $x = 1$ and has a turning point at the time moment $t = \bar{t}$;

(d) "*boundary trapezoidal trajectory*" – the union of three line segments, which moves on the boundary $x = 1$ from the time moment $t = \alpha_1$ to the time moment $t = \bar{t}$ and has two turning points at the moments α_1 and $\bar{t}$.

Correspondingly, the optimal control function $\bar{u}(\cdot)$ may have no switching point (in situation (a)), one switching point (in the situations (b) and (c)), or two switching points (in situation (d)).

Chapter 4

Parametric Optimal Control Problems with Bilateral State Constraints

By (FP_3) we denote the finite horizon optimal control problem of the Lagrange type

$$\text{Minimize } \; J(x,u) = \int_{t_0}^{T} \big[-e^{-\lambda t}(x(t)+u(t)) \big]\, dt \tag{4.1}$$

over $x \in W^{1,1}([t_0,T],\mathbb{R})$ and measurable functions $u : [t_0,T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{x}(t) = -au(t), & \text{a.e. } t \in [t_0,T] \\ x(t_0) = x_0 \\ u(t) \in [-1,1], & \text{a.e. } t \in [t_0,T] \\ -1 \le x(t) \le 1, & \forall t \in [t_0,T] \end{cases} \tag{4.2}$$

with $a > \lambda > 0$, $T > t_0 \ge 0$, and $-1 \le x_0 \le 1$ being given as five parameters. Thus, it is instantly seen that the only difference among (FP_3) and the problems (FP_1) and (FP_2) is the appearance of the bilateral state constraints $-1 \le x(t) \le 1$, for all $t \in [t_0,T]$.

Similarly as it was done in the case of the problems (FP_1) and (FP_2), we now treat (FP_3) in (4.1)–(4.2) as a problem of the Mayer type by setting $x(t) = (x_1(t), x_2(t))$, where $x_1(t)$ plays the role of $x(t)$ in (FP_3) and

$$x_2(t) := \int_{t_0}^{t} \big[-e^{-\lambda \tau}(x_1(\tau)+u(\tau)) \big]\, d\tau, \quad \forall t \in [0,T].$$

Thus, (FP_3) is equivalent to the problem

$$\text{Minimize } \; x_2(T) \tag{4.3}$$

over $x = (x_1,x_2) \in W^{1,1}([t_0,T],\mathbb{R}^2)$ and measurable functions $u : [t_0,T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{x}_1(t) = -au(t), & \text{a.e. } t \in [t_0,T] \\ \dot{x}_2(t) = -e^{-\lambda t}(x_1(t)+u(t)), & \text{a.e. } t \in [t_0,T] \\ (x(t_0), x(T)) \in \{(x_0,0)\} \times \mathbb{R}^2 \\ u(t) \in [-1,1], & \text{a.e. } t \in [t_0,T] \\ -1 \le x_1(t) \le 1, & \forall t \in [t_0,T]. \end{cases} \tag{4.4}$$

The problem (4.3)–(4.4) is abbreviated to (FP_{3a}).

Using once again Filippov's Existence Theorem for Mayer problems in the book by L. Cesari (1983) [Theorem 9.2.i and Section 9.4], we can show that (FP_{3a}) has a $W^{1,1}$ global minimizer.

Thus, by the equivalence of (FP_3) and (FP_{3a}), we can assert that (FP_3) has a $W^{1,1}$ global minimizer.

To solve problem (FP_3) by applying Theorem 3.1, we have to go through seven basic lemmas. Using the given constants a, λ with $a > \lambda > 0$, we define $\rho = \dfrac{1}{\lambda} \ln \dfrac{a}{a-\lambda} > 0$. This number ρ is a characteristic constant of (FP_3). We can obtain a complete synthesis of optimal processes. Due to the complexity of the possible trajectories, we present our results in six separate theorems. The first one treats the situation where $\rho \geq 2a^{-1}$, while the other five deal with the situation where $\rho < 2a^{-1}$.

Theorem 4.1 *If $\rho \geq 2a^{-1}$, then problem (FP_3) has a unique local solution $(\bar{x}, \bar{u})$, which is a unique global solution, where $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ for almost every $t \in [t_0, T]$ and $\bar{x}(t)$ can be described as follows:*

(a) *If $T - t_0 \leq a^{-1}(1 + x_0)$, then*

$$\bar{x}(t) = x_0 - a(t - t_0), \quad t \in [t_0, T].$$

(b) *If $a^{-1}(1 + x_0) < T - t_0 < a^{-1}(3 - x_0)$, then*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_\zeta] \\ -1 - a(t - T), & t \in (t_\zeta, T] \end{cases}$$

with $t_\zeta := 2^{-1}[T + t_0 - a^{-1}(1 + x_0)]$.

(c) *If $T - t_0 \geq a^{-1}(3 - x_0)$, then*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_0 + a^{-1}(1 - x_0)] \\ 1, & t \in (t_0 + a^{-1}(1 - x_0), T - 2a^{-1}] \\ -1 - a(t - T), & t \in (T - 2a^{-1}, T]. \end{cases}$$

If $\rho < 2a^{-1}$, then the locally optimal processes of (FP_3) *depend greatly on the relative position of* x_0 in the segment $[-1, 1]$. We distinguish five alternatives (one instance must occur, and any instance excludes others):

(i) $x_0 = -1$;

(ii) $x_0 > -1$ and $\rho < a^{-1}(1 + x_0) < \rho + a^{-1}(1 - x_0)$;

(iii) $x_0 > -1$ and $a^{-1}(1 + x_0) = \rho + a^{-1}(1 - x_0)$;

(iv) $x_0 > -1$ and $a^{-1}(1 + x_0) > \rho + a^{-1}(1 - x_0)$;

(v) $x_0 > -1$ and $a^{-1}(1 + x_0) \leq \rho$.

It is worthy to stress that to describe the possibilities (i)–(v) we have used just the parameters a, λ, and x_0. In each one of the situations (i)–(v), the synthesis of the trajectories suspected for local minimizers of (FP_3) is obtained by considering *the position of the number* $T - t_0 > 0$ *on the half-line* $[0, +\infty)$, which is divided into sections by the values ρ, 2ρ, $\rho + 2a^{-1}$, $4a^{-1}$, and other constants appeared in (i)–(v).

Among the remaining 5 theorems of this chapter, due to the lack of space, we present only the following one.

Theorem 4.2 *If $\rho < 2a^{-1}$ and $x_0 = -1$, then any local solution of problem (FP_3) must have the form $(\bar{x}, \bar{u})$, where $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ for a.e. $t \in [t_0, T]$ and $\bar{x}(t)$ is described as follows:*

(a) *If $T - t_0 \le 2\rho$, then*

$$\bar{x}(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, 2^{-1}(t_0 + T)] \\ -1 - a(t - T), & t \in (2^{-1}(t_0 + T), T]. \end{cases} \tag{4.5}$$

In this situation, $(\bar{x}, \bar{u})$ is a unique local solution of (FP_3), which is also a unique global solution of the problem.

(b) *If $2\rho < T - t_0 < \rho + 2a^{-1}$, then $\bar{x}(t)$ is given by either*

$$\bar{x}(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, \bar{t}] \\ -1 - a(t + t_0 - 2\bar{t}), & t \in (\bar{t}, T] \end{cases} \tag{4.6}$$

or (4.5).

(c) *If $T - t_0 = \rho + 2a^{-1}$, then $\bar{x}(t)$ is given by either*

$$\bar{x}(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T] \end{cases} \tag{4.7}$$

or (4.5).

(d) *If $\rho + 2a^{-1} < T - t_0 \le 4a^{-1}$, then $\bar{x}(t)$ is given by either*

$$\bar{x}(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, t_0 + 2a^{-1}] \\ 1, & t \in (t_0 + 2a^{-1}, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T] \end{cases} \tag{4.8}$$

or (4.5).

(e) *If $T - t_0 > 4a^{-1}$, then $\bar{x}(t)$ is given by either* (4.8) *or*

$$\bar{x}(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, t_0 + 2a^{-1}] \\ 1, & t \in (t_0 + 2a^{-1}, T - 2a^{-1}] \\ -1 - a(t - T), & t \in (T - 2a^{-1}, T]. \end{cases}$$

In situations (b)–(e)*, the unique global solution of the problem (FP_3) is given correspondingly by* (4.6), (4.7), (4.8), *and* (4.8)*, where the last switching time of the optimal control function $\bar{u}(\cdot)$ is $\bar{t}$.*

Chapter 5

Finite Horizon Optimal Economic Growth Problems

Following Takayama (1974) [Sections C and D in Chapter 5], we consider the problem of *optimal growth of an aggregative economy*. Suppose that the economy can be characterized by one sector, which produces the *national product* $Y(t)$ at time t. Suppose that $Y(t)$ depends on two factors, the *labor* $L(t)$ and the *capital* $K(t)$, and the dependence is described by a *production function* F. Namely, one has

$$Y(t) = F(K(t), L(t)), \quad \forall t \geq 0.$$

It is assumed that $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a function defined on the nonnegative orthant $\mathbb{R}^2_+$ of $\mathbb{R}^2$ having nonnegative real values, and that it exhibits *constant returns to scale*, i.e.,

$$F(\alpha K, \alpha L) = \alpha F(K, L) \tag{5.1}$$

for any $(K, L) \in \mathbb{R}^2_+$ and $\alpha > 0$.

For every $t \geq 0$, by $C(t)$ and $I(t)$, respectively, we denote the *consumption amount* and the *investment amount* of the economy. The *equilibrium relation* in the output market is depicted by

$$Y(t) = C(t) + I(t), \quad \forall t \geq 0. \tag{5.2}$$

The relationship between the capital $K(t)$ and the investment amount $I(t)$ is given by the differential equation

$$\dot{K}(t) = I(t), \quad \forall t \geq 0. \tag{5.3}$$

If the investment function $I(\cdot)$ is continuous, then one can compute the capital stock $K(t)$ at time t by the formula

$$K(t) = K(0) + \int_0^t I(\tau)d\tau,$$

where the integral is Riemannian and $K(0)$ signifies the initial capital stock. In particular, the rate of increase of the capital stock $\dot{K}(t)$ at every time moment t exists and it is finite. If the initial labor amount is $L_0 > 0$ and the *rate of labor force* is a constant $\sigma > 0$ (i.e., $\dot{L}(t) = \sigma L(t)$ for all $t \geq 0$), then the labor amount at time moment t is

$$L(t) = L_0 e^{\sigma t}, \quad \forall t \geq 0. \tag{5.4}$$

For any $t \geq 0$, as $L(t) > 0$, from (5.1) we have

$$\frac{Y(t)}{L(t)} = F\Big(\frac{K(t)}{L(t)}, 1\Big), \quad \forall t \geq 0.$$

By introducing the *capital-to-labor ratio* $k(t) := \dfrac{K(t)}{L(t)}$ and the function $\phi(k) := F(k, 1)$ for $k \geq 0$, from the last equality we have

$$\phi(k(t)) = \frac{Y(t)}{L(t)}, \quad \forall t \geq 0. \tag{5.5}$$

Due to (5.5), one calls $\phi(k(t))$ the *output per capita* at time t and $\phi(\cdot)$ the *per capita production function.* Since F has nonnegative values, so does ϕ. Combining the continuous differentiability of $K(\cdot)$ and $L(\cdot)$, which is guaranteed by (5.3) and (5.4), with the equality defining the capital-to-labor ratio, one can asserts that $k(\cdot)$ is continuously differentiable. Thus, from the relation $K(t) = k(t)L(t)$ one obtains

$$\dot{K}(t) = \dot{k}(t)L(t) + k(t)\dot{L}(t), \quad \forall t \geq 0.$$

Dividing both sides of the above equality by $L(t)$ and invoking $\dot{L}(t) = \sigma L(t)$, we get

$$\frac{\dot{K}(t)}{L(t)} = \dot{k}(t) + \sigma k(t), \quad \forall t \geq 0. \tag{5.6}$$

Similarly, dividing both sides of the equality in (5.3) by $L(t)$ and using (5.2), we have

$$\frac{\dot{K}(t)}{L(t)} = \frac{Y(t)}{L(t)} - \frac{C(t)}{L(t)}, \quad \forall t \geq 0.$$

So, by considering the *per capita consumption* $c(t) := \dfrac{C(t)}{L(t)}$ of the economy at time t and invoking (5.5), one obtains

$$\frac{\dot{K}(t)}{L(t)} = \phi(k(t)) - c(t), \quad \forall t \geq 0.$$

Combining this with (5.6) yields

$$\dot{k}(t) = \phi(k(t)) - \sigma k(t) - c(t), \quad \forall t \geq 0. \tag{5.7}$$

The amount of consumption at time t is

$$C(t) = (1 - s(t))Y(t), \quad \forall t \geq 0, \tag{5.8}$$

with $s(t) \in [0, 1]$ being the *propensity to save* at time t (thus, $1 - s(t)$ is the *propensity to consume* at time t). Then, by dividing both sides of (5.8) by $L(t)$ and referring to (5.5), one gets

$$c(t) = (1 - s(t))\phi(k(t)), \quad \forall t \geq 0. \tag{5.9}$$

Thanks to (5.9), one can rewrite (5.7) equivalently as

$$\dot{k}(t) = s(t)\phi(k(t)) - \sigma k(t), \quad \forall t \geq 0. \tag{5.10}$$

In the special case where $s(\cdot)$ is a constant function, i.e., $s(t) = s > 0$ for all $t \geq 0$, relation (5.10) is the fundamental equation of the *neo-classical aggregate growth model* of Solow (1956).

One major concern of the planners is to choose a pair of functions (k, c) (or (k, s)) defined on a planning interval $[t_0, T] \subset [0, +\infty]$, that satisfies (5.7) (or (5.10)) and the initial condition $k(t_0) = k_0$, to maximize a certain target of consumption. Here $k_0 > 0$ is a given value. As the target function one may choose is $\int_{t_0}^{T} c(t)dt$, which is the total amount of per capita consumption on the time period $[t_0, T]$. A more general kind of the target function is $\int_{t_0}^{T} \omega(c(t))e^{-\lambda t}dt$, where $\omega : \mathbb{R}_+ \to \mathbb{R}$ is a *utility function* associated with the representative individual consumption $c(t)$ in the society and $e^{-\lambda t}$ is the time discount factor. The number $\lambda \geq 0$ is called the *real interest rate.* Clearly, the former target function is a particular case of the latter one with $\omega(c) = c$ being a linear utility function and the real interest rate $\lambda = 0$. The just mentioned planning task is an *optimal control problem.* Interpreting $k(t)$ as the state trajectory and $s(t)$ as the control function, we can formulate the problem as follows.

Let there be given a production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ satisfying (5.1) for any (K, L) from $\mathbb{R}^2_+$ and $\alpha > 0$. Define the function $\phi(k)$ on $\mathbb{R}_+$ by setting $\phi(k) = F(k, 1)$. Assume that a finite time interval $[t_0, T]$ with $T > t_0 \geq 0$, a utility function $\omega : \mathbb{R}_+ \to \mathbb{R}$, and a time discount rate $\lambda \geq 0$ are given. Since $c(t) = (1 - s(t))\phi(k(t))$ by (5.9), the target function can be expressed via $k(t)$ and $s(t)$ as

$$\int_{t_0}^{T} \omega(c(t))e^{-\lambda t}dt = \int_{t_0}^{T} \omega[(1 - s(t))\phi(k(t))]e^{-\lambda t}dt.$$

So, the problem of finding an optimal growth process for an aggregative economy is the following one:

$$\text{Maximize } \ I(k, s) := \int_{t_0}^{T} \omega[(1 - s(t))\phi(k(t))]e^{-\lambda t}dt \tag{5.11}$$

over $k \in W^{1,1}([t_0, T], \mathbb{R})$ and measurable functions $s : [t_0, T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{k}(t) = s(t)\phi(k(t)) - \sigma k(t), & \text{a.e. } t \in [t_0, T] \\ k(t_0) = k_0 & \\ s(t) \in [0, 1], & \text{a.e. } t \in [t_0, T] \\ k(t) \geq 0, & \forall t \in [t_0, T]. \end{cases} \tag{5.12}$$

This problem has five parameters: t_0, T, $\lambda \geq 0$, $\sigma > 0$, and $k_0 > 0$. The optimal control problem in (5.11)–(5.12) will be denoted by (GP).

To make (GP) competent with the given modeling presentation, one has to explain why the state trajectory can be sought in $W^{1,1}([t_0, T], \mathbb{R})$ and the control function is just required to be measurable. This explanation is available in the dissertation.

To obtain solution existence theorems for (GP), we have used the notations, concepts, and results given in the cited monograph of Cesari (1983) [Sections 9.2, 9.3, and 9.5]. Note that *Filippov's Existence Theorem for Bolza problems* in Cesari (1983) [Theorem 9.3.i, p. 317, and Section 9.5] is an analogue of Filippov's Existence Theorem addressing the solution existence for Mayer problems, which has been applied in the preceding two chapters.

Our first result on the solution existence of the finite horizon optimal economic growth problem (GP) in (5.11)–(5.12) is stated as follows.

Theorem 5.1 *For the problem (GP), suppose that $\omega(\cdot)$ and $\phi(\cdot)$ are continuous on $\mathbb{R}_+$. If, in addition, $\omega(\cdot)$ is concave on $\mathbb{R}_+$ and the function $\phi(\cdot)$ satisfies the condition*

(c_1) *There exists* $c \geq 0$ *such that* $\phi(k) \leq (c - \sigma)k + c$ *for all* $k \in \mathbb{R}_+$,

then (GP) *has a global solution.*

In Theorem 5.1, it is not required that $\phi(\cdot)$ is concave on $\mathbb{R}_+$. It turns out that if the concavity of $\phi(\cdot)$ is available, then there is no need to check (c_1). Since the assumption saying that the per capita production function $\phi(k) := F(k, 1)$ is concave on $\mathbb{R}_+$ is reasonable in practice, the next theorem seems to be interesting.

Theorem 5.2 *If both functions* $\omega(\cdot)$ *and* $\phi(\cdot)$ *are continuous and concave on* $\mathbb{R}_+$*, then* (GP) *has a global solution.*

The next proposition reveals the nature of condition (c_1), which is essential for the validity of Theorem 5.1.

Proposition 5.1 *Condition* (c_1) *and the conditions* (c') *and* (c'_0)*, which were formulated in the proof of Theorem 5.1, are equivalent. Moreover, each of these conditions is equivalent to the condition*

$$\limsup_{k \to +\infty} \frac{\phi(k)}{k} < +\infty$$

on the asymptotic behavior of ϕ.

As observed by Takayama (1974) [p. 450], the production function given by

$$F(K, L) = \frac{1}{a} K, \quad \forall (K, L) \in \mathbb{R}^2_+, \tag{5.13}$$

where $a > 0$ is a constant representing the *capital-to-output ratio*, is of a great importance. This function is in the form of the *AK function* with *the diminishing returns to capital being absent*, which is a key property of endogenous growth models. The function in (5.13) is also referred to in connection with the Harrod-Domar model of which a main assumption is that the labor factor is not explicitly involved in the production function. By (5.13) one has

$$\phi(k) = \frac{1}{a} k, \quad \forall k \geq 0.$$

So, the differential equation in (5.12) becomes

$$\dot{k}(t) = \frac{1}{a} s(t)k(t) - \sigma k(t), \quad \text{a.e. } t \in [t_0, T].$$

If the production function F is the *Cobb-Douglas function*, i.e.,

$$F(K, L) = AK^{\alpha} L^{1-\alpha}, \quad \forall (K, L) \in \mathbb{R}^2_+, \tag{5.14}$$

where $A > 0$ and a constant $\alpha \in (0, 1)$ are given, then F *exhibits diminishing returns to capital and labor*. The latter means that the *marginal products* of both capital and labor are diminishing. The presence of diminishing returns to capital, which plays a very important role in many results of the basic growth model, distinguishes the production given by (5.14) with the one in (5.13). The per capita production function corresponding to (5.14) is

$$\phi(k) = Ak^{\alpha}, \quad \forall k \geq 0. \tag{5.15}$$

Therefore, (5.12) collapses to

$$\dot{k}(t) = As(t)k^{\alpha}(t) - \sigma k(t), \quad \text{a.e. } t \in [t_0, T]. \tag{5.16}$$

Since (5.13) can be written in the form of (5.14) with $\alpha := 1$ and $A := 1/a$, one can combine the above two types of production functions in a general one by considering (5.14) with $A > 0$ and $\alpha \in (0, 1]$. This means that one has deal with the model (5.15)–(5.16), where $A > 0$ and $\alpha \in (0, 1]$ are given constants. In the same manner, concerning the utility function $\omega(\cdot)$, the formula

$$\omega(c) = c^{\beta}, \quad \forall c \geq 0 \tag{5.17}$$

with $\beta \in (0, 1]$ can be considered. For $\beta = 1$, $\omega(\cdot)$ is a linear function. For $\beta \in (0, 1)$, it is a Cobb-Douglas function.

For the problem (GP), we now assume that $\phi(\cdot)$ and $\omega(\cdot)$ are given respectively by (5.15) and (5.17). Then, the target function of (GP) is

$$I(k, s) = \int_{t_0}^{T} [1 - s(t)]^{\beta} \phi^{\beta}(k(t)) e^{-\lambda t} dt = A \int_{t_0}^{T} [1 - s(t)]^{\beta} k^{\alpha\beta}(t) e^{-\lambda t} dt.$$

Thus, we have to solve the following optimal control problem:

$$\text{Maximize} \quad \int_{t_0}^{T} [1 - s(t)]^{\beta} k^{\alpha\beta}(t) e^{-\lambda t} dt \tag{5.18}$$

over $k \in W^{1,1}([t_0, T], \mathbb{R})$ and measurable functions $s : [t_0, T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{k}(t) = Ak^{\alpha}(t)s(t) - \sigma k(t), & \text{a.e. } t \in [t_0, T] \\ k(t_0) = k_0 & \\ s(t) \in [0, 1], & \text{a.e. } t \in [t_0, T] \\ k(t) \in [0, +\infty), & \forall\, t \in [t_0, T] \end{cases} \tag{5.19}$$

with $T > t_0 \geq 0$, $\lambda \geq 0$, $A > 0$, $\sigma > 0$, and $k_0 > 0$ being given parameters. The forthcoming result is a consequence of Theorem 5.2.

Theorem 5.3 *For any constants $\alpha \in (0, 1]$ and $\beta \in (0, 1]$, the optimal economic growth problem in* (5.18)–(5.19) *possesses a global solution.*

Depending on the displacement of α and β on $(0, 1]$, we have four types of the model (5.18)–(5.19):

(T1) "Linear-linear": $\phi(k) = Ak$ and $\omega(c) = c$ (both the per capita production function and the utility function are linear);

(T2) "Linear-nonlinear": $\phi(k) = Ak$ and $\omega(c) = c^{\beta}$ with $\beta \in (0, 1)$ (the per capita production function is linear, but the utility function is nonlinear);

(T3) "Nonlinear-linear": $\phi(k) = Ak^{\alpha}$ and $\omega(c) = c$ with $\alpha \in (0, 1)$ (the per capita production function is nonlinear, but the utility function is linear);

(T4) "Nonlinear-nonlinear": $\phi(k) = Ak^{\alpha}$ and $\omega(c) = c^{\beta}$ with $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ (both the per capita production function and the utility function are nonlinear).

To apply Theorem 3.1 for finding optimal processes for (GP_1), we have to interpret (GP_1) in the form of the Mayer problem $\mathcal{M}$ in Chapter 3. For doing so, we set $x(t) = (x_1(t), x_2(t))$, where $x_1(t)$ plays the role of $k(t)$ in (5.18)–(5.19) and and

$$x_2(t) := -\int_{t_0}^{t} [1 - s(\tau)]^{\beta} x_1^{\alpha\beta}(\tau) e^{-\lambda\tau} d\tau$$

for all $t \in [0, T]$. Thus, (GP_1) is equivalent to the following problem:

$$\text{Minimize } \ x_2(T) \tag{5.20}$$

over $x = (x_1, x_2) \in W^{1,1}([t_0, T], \mathbb{R}^2)$ and measurable functions $s : [t_0, T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{x}_1(t) = A x_1^{\alpha}(t) s(t) - \sigma x_1(t), & \text{a.e. } t \in [t_0, T] \\ \dot{x}_2(t) = -[1 - s(t)]^{\beta} x_1^{\alpha\beta}(t) e^{-\lambda t}, & \text{a.e. } t \in [t_0, T] \\ (x(t_0), x(T)) \in \{(k_0, 0)\} \times \mathbb{R}^2 & \\ s(t) \in [0, 1], & \text{a.e. } t \in [t_0, T] \\ x_1(t) \in [0, +\infty), & \forall t \in [t_0, T]. \end{cases} \tag{5.21}$$

The optimal control problem in (5.20)–(5.21) is denoted by (GP_{1a}).

Let $(\bar{x}, \bar{s})$ be a $W^{1,1}$ local minimizer for (GP_{1a}). To satisfy the assumption (H1) in Theorem 3.1, for any $s \in [0, 1]$, the function $f(t, \cdot, s)$ must be locally Lipschitz around $\bar{x}(t)$ for almost every $t \in [t_0, T]$. This requirement cannot be satisfied if $\alpha \in (0, 1)$ and the set of $t \in [t_0, T]$ when the curve $\bar{x}_1(t)$ hits the lower bound $x_1 = 0$ of the state constraint $x_1(t) \in [0, +\infty)$ has a positive measure. To overcome this situation, we may use one of the following two additional assumptions:

(A1) $\alpha = 1$;

(A2) $\alpha \in (0, 1)$ *and the set* $\{t \in [t_0, T] : \bar{x}_1(t) = 0\}$ *has the Lebesgue measure 0*, i.e., $\bar{x}_1(t) > 0$ for almost every $t \in [t_0, T]$.

Regarding the exponent $\beta \in (0, 1]$ in the formula of $\omega(\cdot)$, we distinguish two cases:

(B1) $\beta = 1$;

(B2) $\beta \in (0, 1)$.

Theorem 5.4 *Suppose that the assumptions* (A1) *and* (B1) *are satisfied. If*

$$A < \sigma + \lambda,$$

then (GP_{1a}) *has a unique* $W^{1,1}$ *local minimizer* $(\bar{x}, \bar{s})$, *which is a global minimizer, where* $\bar{s}(t) = 0$ *for a.e.* $t \in [t_0, T]$ *and* $\bar{x}_1(t) = k_0 e^{-\sigma(t - t_0)}$ *for all* $t \in [t_0, T]$. *This means that the problem* (GP_1) *has a unique solution* $(\bar{k}, \bar{s})$, *where* $\bar{s}(t) = 0$ *for a.e.* $t \in [t_0, T]$ *and* $\bar{k}(t) = k_0 e^{-\sigma(t - t_0)}$ *for all* $t \in [t_0, T]$.

The coefficient A in the expression $\phi(k) = Ak$ of the per capita production function $\phi(\cdot)$ expresses the *total factor productivity*. Recall that σ is the *rate of labor force* (closely related to the population growth rate) and λ is the *real interest rate* (which indicates the rate of the decrease along time of the satisfaction level of the society w.r.t. the same amount of consumption). Theorem 5.4 says that if the total factor productivity A is smaller than the sum of the rate of labor force σ and the real interest rate λ, then optimal strategy is to keep the saving equal to 0. In other words, *if the total factor productivity A is relatively small, then an expansion of the production facility does not lead to a higher total consumption satisfaction of the society.*

The barrier $A = \sigma+\lambda$ for the total factor productivity appears for the first time in the paper by V. T. Huong, J.-C. Yao, and N. D. Yen (Taiwanes J. Math., 2020). Due to Theorem 5.4, the notions of *weak economy* (with $A < \sigma + \lambda$) and *strong economy* (with $A > \sigma + \lambda$) can have exact meanings. Moreover, the behaviors of a weak economy and of a strong economy might be very different.

By Theorem 5.4 we have solved the problem (GP_1) in the situation where $A < \sigma + \lambda$. A natural question arises: *What happens if $A > \sigma + \lambda$?* The latter condition means that if the total factor productivity A is relatively large. In this situation, it is likely that the optimal strategy requires to make the maximum saving until a special time $\bar{t} \in (t_0, T)$, which depends on the data tube (A, σ, λ), then switch the saving to minimum. Further investigations in this direction are going on.

General Conclusions

In this dissertation, we have applied different tools from set-valued analysis, variational analysis, optimization theory, and optimal control theory to study qualitative properties (solution existence, optimality conditions, stability, and sensitivity) of some optimization problems arisen in consumption economics, production economics, optimal economic growths and their prototypes in the form of parametric optimal control problems.

The main results of the dissertation include:

1. Sufficient conditions for: the upper continuity, the lower continuity, and the continuity of the budget map, the indirect utility function, and the demand map; the Robinson stability and the Lipschitz-like property of the budget map; the Lipschitz property of the indirect utility function; the Lipschitz-Hölder property of the demand map.

2. Formulas for computing the Fréchet/ limitting coderivatives of the budget map; the Fréchet/limitting subdifferentials of the infimal nuisance function, upper and lower estimates for the upper and the lower Dini directional derivatives of the indirect utility function.

3. The syntheses of finitely many processes suspected for being local minimizers for parametric optimal control problems without/with state constraints.

4. Three theorems on solution existence for optimal economic growth problems in general forms as well as in some typical ones, and the synthesis of optimal processes for one of such typical problems.

5. Interpretations of the economic meanings for most of the obtained results.

List of Author's Related Papers

1. Vu Thi Huong, Jen-Chih Yao, Nguyen Dong Yen, *On the stability and solution sensitivity of a consumer problem*, Journal of Optimization Theory and Applications, **175** (2017), 567–589. (SCI)

2. Vu Thi Huong, Jen-Chih Yao, Nguyen Dong Yen, *Differentiability properties of a parametric consumer problem*, Journal of Nonlinear and Convex Analysis, **19** (2018), 1217–1245. (SCI-E)

3. Vu Thi Huong, Jen-Chih Yao, Nguyen Dong Yen, *Analyzing a maximum principle for finite horizon state constrained problems via parametric examples. Part 1: Unilateral state constraints*, Journal of Nonlinear and Convex Analysis, **21** (2020), 157–182. (SCI-E)

4. Vu Thi Huong, Jen-Chih Yao, Nguyen Dong Yen, *Analyzing a maximum principle for finite horizon state constrained problems via parametric examples. Part 2: Bilateral state constraints*, preprint, 2019. (Submitted)

5. Vu Thi Huong, *Solution existence theorems for finite horizon optimal economic growth problems*, preprint, 2019. (https://arxiv.org/abs/2001.03298; submitted)

6. Vu Thi Huong, Jen-Chih Yao, Nguyen Dong Yen, *Optimal processes in a parametric optimal economic growth model*, Taiwanese Journal of Mathematics, https://doi.org/10.11650/tjm/200203 (2020). (SCI)

The results of this dissertation have been presented at

- The weekly seminar of the Department of Numerical Analysis and Scientific Computing, Institute of Mathematics, Vietnam Academy of Science and Technology (08 talks);

- The 16$^{\text{th}}$ and 17$^{\text{th}}$ Workshops on "Optimization and Scientific Computing" (April 19–21, 2018 and April 18–20, 2019, Ba Vi, Vietnam) [contributed talks];

- International Conference "New trends in Optimization and Variational Analysis for Applications" (December 7–10, 2016, Quy Nhon, Vietnam) [a contributed talk];

- "Vietnam-Korea Workshop on Selected Topics in Mathematics" (February 20–24, 2017, Danang, Vietnam) [a contributed talk];

- "International Conference on Analysis and its Applications" (November 20–22, 2017, Aligarh Muslim University, Aligarh, India) [a contributed talk];

- International Conference "Variational Analysis and Optimization Theory" (December 19–21, 2017, Hanoi, Vietnam) [a contributed talk];

- "Taiwan-Vietnam Workshop on Mathematics" (May 9–11, 2018, Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan) [a contributed talk];

- International Workshop "Variational Analysis and Related Topics" (December 13–15, 2018, Hanoi Pedagogical University 2, Xuan Hoa, Phuc Yen, Vinh Phuc, Vietnam) [a contributed talk];

- "Vietnam-USA Joint Mathematical Meeting" (June 10–13, 2019, Quy Nhon, Vietnam) [a poster presentation, which has received an Excellent Poster Award].