VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY

**INSTITUTE OF MATHEMATICS**

**VU THI HUONG**

# SOME PARAMETRIC OPTIMIZATION PROBLEMS IN MATHEMATICAL ECONOMICS

**DISSERTATION**

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

**DOCTOR OF PHILOSOPHY IN MATHEMATICS**

**HANOI - 2020**

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**Speciality: Applied Mathematics**

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**Supervisor:** Prof. Dr.Sc. NGUYEN DONG YEN

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# Confirmation

This dissertation was written on the basis of my research works carried out at Institute of Mathematics, Vietnam Academy of Science and Technology, under the supervision of Prof. Dr.Sc. Nguyen Dong Yen. All the presented results have never been published by others.

February 26, 2020

The author

Vu Thi Huong

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# Contents

# Table of Notations

| | |
|---|---|
| $\mathbb{R}$ | the set of real numbers |
| $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ | the extended real line |
| $\emptyset$ | the empty set |
| $\|\|x\|\|$ | the norm of a vector $x$ |
| $\operatorname{int} A$ | the topological interior of $A$ |
| $\operatorname{cl} A$ (or $\overline{A}$) | the topological closure of a set $A$ |
| $\operatorname{cl}^* A$ | the closure of a set $A$ in the weak$^*$ topology |
| $\operatorname{cone} A$ | the cone generated by $A$ |
| $\operatorname{conv} A$ | the convex hull of $A$ |
| $\operatorname{dom} f$ | the effective domain of a function $f$ |
| $\operatorname{epi} f$ | the epigraph of $f$ |
| resp. | respectively |
| w.r.t. | with respect to |
| l.s.c. | lower semicontinuous |
| u.s.c. | upper semicontinuous |
| i.s.c. | inner semicontinuous |
| a.e. | almost everywhere |
| $\widehat{N}(x;\Omega)$ | the Fréchet normal cone to $\Omega$ at $x$ |
| $N(x;\Omega)$ | the limiting/Mordukhovich normal cone to $\Omega$ at $x$ |
| $\widehat{D}^*F(\bar{x},\bar{y})$ | the Fréchet coderivative of $F$ at $(\bar{x},\bar{y})$ |
| $D^*F(\bar{x},\bar{y})$ | the limiting/Mordukhovich coderivative of $F$ at $(\bar{x},\bar{y})$ |
| $\widehat{\partial}\varphi(\bar{x})$ | the Fréchet subdifferential of $\varphi$ at $\bar{x}$ |
| $\partial\varphi(\bar{x})$ | the limiting/Mordukhovich subdifferential of $\varphi$ at $\bar{x}$ |
| $\partial^\infty\varphi(\bar{x})$ | the singular subdifferential of $\varphi$ at $\bar{x}$ |
| $\widehat{\partial}^+\varphi(\bar{x})$ | the Fréchet upper subdifferential |

| | |
|---|---|
| | of $\varphi$ at $\bar{x}$ |
| $\partial^+\varphi(\bar{x})$ | the limiting/Mordukhovich upper subdifferential of $\varphi$ at $\bar{x}$ |
| $\partial^{\infty,+}\varphi(\bar{x})$ | the singular upper subdifferential of $\varphi$ at $\bar{x}$ |
| SNC | sequentially normally compact |
| $T_\Omega(\bar{x})$ | the Clarke tangent cone to $\Omega$ at $\bar{x}$ |
| $N_\Omega(\bar{x})$ | the Clarke normal cone to $\Omega$ at $\bar{x}$ |
| $\partial_C\varphi(\bar{x})$ | the Clarke subdifferential of $\varphi$ at $\bar{x}$ |
| $d^-v(\bar{p};q)$ | the lower Dini directional derivative of $v$ at $\bar{p}$ in direction $q$ |
| $d^+v(\bar{p};q)$ | the upper Dini directional derivative of $v$ at $\bar{p}$ in direction $q$ |
| $W^{1,1}([t_0,T],\mathbb{R}^n)$ | The Sobolev space of the absolutely continuous functions $x:[t_0,T]\to\mathbb{R}^n$ endowed with the norm $\|x\|_{W^{1,1}}=\|x(t_0)\|+\int_{t_0}^T\|\dot{x}(t)\|dt$ |
| $\mathcal{B}^m$ | The $\sigma$-algebra of the Borel sets in $\mathbb{R}^m$ |
| $\int_{[t_0,T]} x(t)dv(t)$ | the Riemann-Stieltjes integral of $x$ with respect to $v$ |
| $\partial_x^> h(t,x)$ | the partial hybrid subdifferential of $h$ at $(t,x)$ |
| $\mathcal{H}(t,x,p,u)$ | Hamiltonian |

# Introduction

*Mathematical economics* is the application of mathematical methods to represent theories and analyze problems in economics. The language of mathematics allows one to address the latter with rigor, generality, and simplicity. Formal economic modeling began in the 19th century with the use of differential calculus to represent and explain economic behaviors, such as the utility maximization problem and the expenditure minimization problem, early applications of optimization in microeconomics. Economics became more mathematical as a discipline throughout the first half of the 20th century with the introduction of new and generalized techniques, including ones from calculus of variations and optimal control theory applied in dynamic analysis of economic growth models in macroeconomics.

Although consumption economics, production economics, and optimal economic growth have been studied intensively (see the fundamental textbooks [19, 42, 61, 71, 79], the papers [44, 47, 55, 64, 65, 80] on consumption economics or production economics, the papers [4, 7, 51] on optimal economic growth, and the references therein), new results on qualitative properties of these models can be expected. They can lead to a deeper understanding of the classical models and to more effective uses of the latter. Fast progresses in optimization theory, set-valued and variational analysis, and optimal control theory allow us to hope that such new results are possible.

This dissertation focuses on qualitative properties (solution existence, optimality conditions, stability, and differential stability) of optimization problems arisen in consumption economics, production economics, and optimal economic growth models. Five chapters of the dissertation are divided into two parts.

Part I, which includes the first two chapters, studies the stability and the differential stability of the consumer problem named *maximizing utility sub-*

*ject to consumer budget constraint* with varying prices. Mathematically, this is a parametric optimization problem; and it is worthy to stress that the problem considered here also presents the producer problem named *maximizing profit subject to producer budget constraint* with varying input prices. Both problems are basic ones in microeconomics.

Part II of the dissertation includes the subsequent three chapters. We analyze a *maximum principle* for *finite horizon optimal control problems with state constraints* via *parametric examples* in Chapters 3 and 4. Our analysis serves as a sample of applying advanced tools from optimal control theory to meaningful prototypes of *economic optimal growth models* in macroeconomics. Chapter 5 is devoted to solution existence of optimal economic growth problems and synthesis of optimal processes for one typical problem.

We now briefly review some basic facts related to the consumer problem considered in the first two chapters of the dissertation.

In consumption economics, the following two classical problems are of common interest. The first one is *maximizing utility subject to consumer budget constraint* (see Intriligator [42, p. 149]); and the second one is *minimizing consumer's expenditure for the utility of a specified level* (see Nicholson and Snyder [61, p. 132]). In Chapters 1 and 2, we pay attention to the first one.

Qualitative properties of this consumer problem have been studied by Takayama [79, pp. 241–242, 253–255], Penot [64, 65], Hadjisavvas and Penot [32], and many other authors. Diewert [25], Crouzeix [22], Martínez-Legaz and Santos [54], and Penot [65] studied the duality between the utility function and the indirect utility function. Relationships between the differentiability properties of the utility function and of the indirect utility function have been discussed by Crouzeix [22, Sections 2 and 6], who gave sufficient conditions for the indirect utility function in finite dimensions to be differentiable. He also established [23] some relationships between the second-order derivatives of the direct and indirect utility functions. Subdifferentials of the indirect utility function in infinite-dimensional consumer problems have been computed by Penot [64].

Penot's recent papers [64, 65] on the first consumer problem stimulated our study and lead to the results presented in Chapters 1 and 2. In some sense, the aims of Chapter 1 (resp., Chapter 2) are similar to those of [65] (resp., [64]). We also adopt the general infinite-dimensional setting of the consumer problem which was used in [64, 65]. But our approach and results are quite different from the ones of Penot [64, 65].

Namely, various stability properties and a result on solution sensitivity of the consumer problem are presented in Chapter 1. Focusing on some nice features of the budget map, we are able to establish the continuity and the locally Lipschitz continuity of the indirect utility function, as well as the Lipschitz-Hölder continuity of the demand map under minimal assumptions. Our approach seems to be new. An implicit function theorem of Borwein [15] and a theorem of Yen [86] on solution sensitivity of parametric variational inequalities are the main tools in the subsequent proofs. To the best of our knowledge, the results on the Lipschitz-like property of the budget map, the Lipschitz property of the indirect utility function, and the Lipschitz-Hölder continuity of the demand map in the present chapter have no analogues in the literature.

In Chapter 2, by an intensive use of some theorems from Mordukhovich [58], we will obtain sufficient conditions for the budget map to be Lipschitz-like at a given point in its graph under weak assumptions. Formulas for computing the Fréchet coderivative and the limiting coderivative of the budget map can be also obtained by the results of [58] and some advanced calculus rules from [56]. The results of Mordukhovich *et al.* [60] and the just mentioned coderivative formulas allow us to get new results on differential stability of the consumer problem where the price is subject to change. To be more precise, we establish formulas for computing or estimating the Fréchet, limiting, and singular subdifferentials of the infimal nuisance function, which is obtained from the indirect utility function by changing its sign. Subdifferential estimates for the infimal nuisance function can lead to interesting economic interpretations. Namely, we will show that if the current price moves forward a direction then, under suitable conditions, the *instant rate of the change of the maximal satisfaction of the consumer* is bounded above and below by real numbers defined by subdifferentials of the infimal nuisance function.

The second part of this dissertation studies some optimal control problems, especially, ones with state constraints. It is well-known that optimal control problems with state constraints are models of importance, but one usually faces with a lot of difficulties in analyzing them. These models have been considered since the early days of the optimal control theory. For instance, the whole Chapter VI of the classical work [69, pp. 257–316] is devoted to problems with restricted phase coordinates. There are various forms of the maximum principle for optimal control problems with state constraints; see, e.g., [34], where the relations between several forms are shown and a series of numerical illustrative examples have been solved.

To deal with state constraints, one has to use functions of bounded variation, Borel measurable functions, Lebesgue-Stieltjes integral, nonnegative measures on the $\sigma-$algebra of the Borel sets, the Riesz Representation Theorem for the space of continuous functions, and so on.

By using the maximum principle presented in [43, pp. 233–254], Phu [66,67] has proposed an ingenious method called *the method of region analysis* to solve several classes of optimal control problems with one state variable and one control variable, which have both state and control constraints. Minimization problems of the Lagrange type were considered by the author and, among other things, it was assumed that integrand of the objective function is strictly convex with respect to the control variable. To be more precise, the author considered *regular problems*, i.e., the optimal control problems where the Pontryagin function is strictly convex with respect to the control variable.

In Chapters 3 and 4, the maximum principle for finite horizon state constrained problems from the book by Vinter [82, Theorem 9.3.1] is analyzed via parametric examples. The latter has origin in a recent paper by Basco, Cannarsa, and Frankowska [12, Example 1], and resembles the optimal economic growth problems in macroeconomics (see, e.g., [79, pp. 617–625]). The solution existence of these parametric examples, which are *irregular optimal control problems* in the sense of Phu [66, 67], is established by invoking Filippov's existence theorem for Mayer problems [18, Theorem 9.2.i and Section 9.4]. Since the maximum principle is only a necessary condition for local optimal processes, a large amount of additional investigations is needed to obtain a comprehensive synthesis of finitely many processes suspected for being local minimizers. *Our analysis not only helps to understand the principle in depth, but also serves as a sample of applying it to meaningful prototypes of economic optimal growth models. In the vast literature on optimal control, we have not found any synthesis of optimal processes of parametric problems* *like the ones presented herein.*

Just to have an idea about the importance of analyzing maximum principles via typical optimal control problems, observe that Section 22.1 of the book by Clarke [21] presents a maximum principle [21, Theorem 22.2] for an optimal control problem without state constraints denoted by $(OC)$. The whole Section 22.2 of [21] (see also [21, Exercise 26.1]) is devoted to solving a very special example of $(OC)$ having just one parameter. The analysis contains a series of additional propositions on the properties of the unique global solution.

Note that the *maximum principle* for finite horizon state constrained problems in [82, Chapter 9] covers several known ones for smooth problems and allows us to deal with nonsmooth problems by using the concepts of *limiting normal cone* and *limiting subdifferential* of Mordukhovich [56, 57, 59]. This principle is a necessary optimality condition which asserts the existence of a nontrivial multipliers set consisting of an absolutely continuous function, a function of bounded variation, a Borel measurable function, and a real number, such that the four conditions (i)–(iv) in Theorem 3.1 in Chapter 3 are satisfied. *The relationships between these conditions are worthy a detailed analysis.* Towards that aim, we will use the maximum principle to analyze in details three parametric examples of optimal control problems of the Lagrange type, which have five parameters: the first one appears in the description of the objective function, the second one appears in the differential equation, the third one is the initial value, the fourth one is the initial time, and the fifth one is the terminal time. Observe that, in Example 1 of [12], the terminal time is infinity, the initial value and the initial time are fixed.

Problems without state constraints, as well as problems with unilateral state constraints, are studied in Chapter 3. Problems with bilateral state constraints are considered in Chapter 4. To deal with bilateral state constraints, we have to prove a series of nontrivial auxiliary lemmas. Moreover, the synthesis of finitely many processes suspected for being local minimizers is rather sophisticated, and it requires a lot of refined arguments.

Models of economic growth have played an essential role in economics and mathematical studies since the 30s of the twentieth century. Based on different consumption behavior hypotheses, they allow ones to analyze, plan, and predict relations between global factors, which include *capital*, *labor force*, *production technology*, and *national product*, of a particular economy in a given planning interval of time. Principal models and their basic properties have been investigated by Ramsey [70], Harrod [33], Domar [26], Solow [77], Swan [78], and others. Details about the development of the economic growth theory can be found in the books by Barro and Sala-i-Martin [11] and Acemoglu [1].

Along with the analysis of the global economic factors, another major issue regarding an economy is the so-called *optimal economic growth problem*, which can be roughly stated as follows: Define the amount of consumption (and therefore, saving) at each time moment *to maximize a certain target of consumption satisfaction* while fulfilling given relations in the growth model of that economy. Economically, this is a basic problem in macroeconomics, while, in mathematical form, it is an optimal control problem. This optimal consumption/saving problem was first formulated and solved to a certain extent by Ramsey [70]. Later, significant extensions of the model in [70] were suggested by Cass [17] and Koopmans [50].

Characterizations of the solutions of optimal economic growth problems (necessary optimality conditions, sufficient optimality conditions, etc.) have been discussed in the books [79, Chapter 5], [68, Chapters 5, 7, 10, and 11], [19, Chapter 20], [1, Chapters 7 and 8], and some papers cited therein. However, results on the solution existence of these problems seem to be quite rare. For infinite horizon models, some solution existence results were given in [1, Example 7.4] and [24, Subsection 4.1]. For finite horizon models, our careful searching in the literature leads just to [24, Subsection 4.1 and Corollary 1] and [62, Theorem 1]. This observation motivates the investigations in the first part of Chapter 5.

The first part of Chapter 5 considers the solution existence of finite horizon optimal economic growth problems of an *aggregative economy*; see, e.g., [79, Sections C and D in Chapter 5]. It is worthy to stress that we do not assume any special saving behavior, such as *the constancy of the saving rate* as in growth models of Solow [77] and Swan [78] or the *classical saving behavior* as in [79, p. 439]. Our main tool is Filippov's Existence Theorem for optimal control problems with state constraints of the Bolza type from the monograph of Cesari [18]. Our new results on the solution existence are obtained under some mild conditions on the *utility function* and the *per capita production function*, which are two major inputs of the model in question. The results for general problems are also specified for typical ones with the production function and the utility function being either in the form of *AK functions* or *Cobb–Douglas ones* (see, e.g., [11] and [79]). Some interesting open questions and conjectures about the *regularity of the global solutions* of finite horizon optimal economic growth problems are formulated in the final part of the paper. Note that, since the saving policy on a compact segment of time would be implementable if it has an infinite number of discontinuities, our concept of regularity of the solutions of the optimal economic growth problem has a clear practical meaning.

The solution existence theorems in this Chapter 5 for finite horizon optimal economic growth problems cannot be derived from the above cited results in [24, Subsection 4.1 and Corollary 1] and [62, Theorem 1], because the assumptions of the latter are more stringent and more complicated than ours. For solution existence theorems in optimal control theory, apart from [18], the reader is referred to [52], [10], and the references therein.

Our focus point in the second part of Chapter 5 is to *solve* one of the four typical optimal economic growth problems mentioned in the first part of the same chapter. More precisely, our aim is to give a *complete synthesis* of the optimal processes for the parametric finite horizon optimal economic growth problem, where the production function and the utility function are both in the form of *AK functions* (see, e.g., [11]). By using a solution existence theorem in the first part of this chapter and the maximum principle for optimal control problems with state constraints in the book by Vinter [82, Theorem 9.3.1], we are able to prove that the problem has a unique local solution, which is also a global one, provided that the data triple satisfies a strict linear inequality. Our main theorem will be obtained via a series of nine lemmas and some involved technical arguments. Roughly speaking, we will combine an intensive treatment of the system of necessary optimality conditions given by the maximum principle with the specific properties of the given parametric optimal economic growth problem. The approach adopted herein has the origin in preceding Chapters 3 and 4. From the obtained results it follows that *if the total factor productivity $A$ is relatively small, then an expansion of the production facility does not lead to a higher total consumption satisfaction of the society.*

Last but not least, notice that there are interpretations of the economic meanings for the majority of the mathematical concepts and obtained results in Chapter 1, 2, and 5, which form an indispensable part of the present dissertation. Needless to say that such economic interpretations of new results are most desirable in a mathematical study related to economic topics.

So, as mentioned above, the dissertation has five chapters. It also has a list of the related papers of the author, a section of general conclusions, and a list of references. A brief description of the contents of each chapter is as follows.

In Chapter 1, we study the stability of a parametric consumer problem. The stability properties presented in this chapter include: the upper continuity, the lower continuity, and the continuity of the budget map, of the indirect utility function, and of the demand map; the Robinson stability and the Lipschitz-like property of the budget map; the Lipschitz property of the indirect utility function; the Lipschitz-Hölder property of the demand map.

Chapter 2 is devoted to differential stability of the parametric consumer problem considered in the preceding chapter. The differential stability here appears in the form of formulas for computing the Fréchet/limitting coderivatives of the budget map; the Fréchet/limitting subdifferentials of the infimal nuisance function (which is obtained from the indirect utility function by changing its sign), upper and lower estimates for the upper and the lower Dini directional derivatives of the indirect utility function. In addition, another result on the Lipschitz-like property of the budget map is also given in this chapter.

In Chapters 3 and 4, a maximum principle for finite horizon optimal control problems with state constraints is analyzed via parametric examples. The difference among those are in the appearance of state constraints: The first one does not contain state constraints, the second one is a problem with unilateral state constraints, and the third one is a problem with bilateral state constraints. The first two problems are studied in Chapter 3. The last one with bilateral state constraints is addressed in Chapter 4.

Chapter 5 establishes three theorems on solution existence for optimal economic growth problems in general forms as well as in some typical ones and a synthesis of optimal processes for one of such typical problems. Some open questions and conjectures about the uniqueness and regularity of the global solutions of optimal economic growth problems are formulated in this chapter.

The dissertation is written on the basis of the paper [35] published in *Journal of Optimization Theory and Applications*, the papers [36] and [37] published in *Journal of Nonlinear and Convex Analysis*, the paper [40] published in *Taiwanese Journal of Mathematics*, and two preprints [38,39], which were submitted for publication.

The results of this dissertation were presented at

- The weekly seminar of the Department of Numerical Analysis and Scientific Computing, Institute of Mathematics, Vietnam Academy of Science and Technology (08 talks);

- The $16^{\text{th}}$ and $17^{\text{th}}$ Workshops on "Optimization and Scientific Computing" (April 19–21, 2018 and April 18–20, 2019, Ba Vi, Vietnam) [contributed talks];

- International Conference "New trends in Optimization and Variational Analysis for Applications" (December 7–10, 2016, Quy Nhon, Vietnam) [a contributed talk];

- "Vietnam-Korea Workshop on Selected Topics in Mathematics" (February 20–24, 2017, Danang, Vietnam) [a contributed talk];

- "International Conference on Analysis and its Applications" (November 20–22, 2017, Aligarh Muslim University, Aligarh, India) [a contributed talk];

- International Conference "Variational Analysis and Optimization Theory" (December 19–21, 2017, Hanoi, Vietnam) [a contributed talk];

- "Taiwan-Vietnam Workshop on Mathematics" (May 9–11, 2018, Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan) [a contributed talk];

- International Workshop "Variational Analysis and Related Topics" (December 13–15, 2018, Hanoi Pedagogical University 2, Xuan Hoa, Phuc Yen, Vinh Phuc, Vietnam) [a contributed talk];

- "Vietnam-USA Joint Mathematical Meeting" (June 10–13, 2019, Quy Nhon, Vietnam) [a poster presentation, which has received an Excellent Poster Award].

# Chapter 1

# Stability of Parametric Consumer Problems

The present chapter, which is written on the basis of the paper [35], studies the stability of parametric consumer problems. Namely, we will establish sufficient conditions for

- the upper continuity, the lower continuity, and the continuity of the budget map, of the indirect utility function, and of the demand map;

- the Robinson stability and the Lipschitz-like property of the budget map;

- the Lipschitz property of the indirect utility function; the Lipschitz-Hölder property of the demand map.

Throughout this dissertation, we use the following notations. For a norm space $X$, the norm of a vector $x$ is denoted by $||x||$. The topological dual space of $X$ is denoted by $X^*$. The notations $\langle x^*, x\rangle$ or $x^* \cdot x$ are used for the value $x^*(x)$ of an element $x^* \in X^*$ at $x \in X$. The interior (resp., the closure) of a subset $\Omega \subset X$ in the norm topology is abbreviated to $\operatorname{int}\Omega$ (resp., $\overline{\Omega}$). The open (resp., closed) unit ball in $X$ is denoted by $B_X$ (resp., $\bar{B}_X$).

The set of real numbers (resp., nonnegative real numbers, nonpositive real numbers, extended real numbers, and positive integers) is denoted by $\mathbb{R}$ (resp., $\mathbb{R}_+$, $\mathbb{R}_-$, $\overline{\mathbb{R}}$, and $\mathbb{N}$).

## 1.1 Maximizing Utility Subject to Consumer Budget Constraint

Following [64, 65], we consider the consumer problem named *maximizing utility subject to consumer budget constraint* in the subsequent infinite-dimensional setting.

The set of *goods* is modeled by a nonempty, closed and convex cone $X_+$ in a *reflexive Banach space* $X$. The set of *prices* is

$$Y_+ := \{p \in X^* \ : \ \langle p, x\rangle \geq 0, \quad \forall x \in X_+\}. \tag{1.1}$$

It is well-known (see, e.g., [14, Proposition 2.40]) that $Y_+$ is a closed and convex cone in $X^*$, and

$$X_+ = \{x \in X \ : \ \langle p, x\rangle \geq 0, \quad \forall p \in Y_+\}.$$

We may normalize the prices and assume that the budget of the consumer is 1. Then, the *budget map* is the set-valued map $B : Y_+ \rightrightarrows X_+$ associating to each price $p \in Y_+$ the *budget set*

$$B(p) := \big\{x \in X_+ \ : \ \langle p, x\rangle \leq 1\big\}. \tag{1.2}$$

We assume that the preferences of the consumer are presented by a function $u : X \to \overline{\mathbb{R}}$, called the *utility function*. This means that $u(x) \in \mathbb{R}$ for every $x \in X_+$, and a goods bundle $x \in X_+$ is preferred to another one $x' \in X_+$ if and only if $u(x) > u(x')$. *For a given price $p \in Y_+$, the problem is to maximize $u(x)$ subject to the constraint $x \in B(p)$.* It is written formally as

$$\max\{u(x) \ : \ x \in B(p)\}. \tag{1.3}$$

The *indirect utility function* $v : Y_+ \to \overline{\mathbb{R}}$ of (1.3) is defined by

$$v(p) = \sup\{u(x) : x \in B(p)\}, \quad p \in Y_+. \tag{1.4}$$

The *demand map* of (1.3) is the set-valued map $D : Y_+ \rightrightarrows X_+$ defined by

$$D(p) = \{x \in B(p) \ : \ u(x) = v(p)\}, \quad p \in Y_+. \tag{1.5}$$

For convenience, we can put $B(p) = \emptyset$ and $D(p) = \emptyset$ for every $p \in X^* \setminus Y_+$. In this way, we have set-valued maps $B$ and $D$ defined on $X^*$ with values in $X$. As $B(p) = \emptyset$ and $\sup \emptyset = -\infty$ by an usual convention, one has $v(p) = -\infty$

for all $p \notin X^* \setminus Y_+$, meaning that $v$ is an extended real-valued function defined on $X^*$.

Mathematically, the problem (1.3) is an *parametric optimization problem*, where the prices $p$ varying in $Y_+$ play as the role of *parameters*, the function $v(\cdot)$ is called the *optimal value function*, and the set-valued map $D(\cdot)$ is called the *solution map*.

Let us present three illustrative examples of the consumer problem. The first one is the problem considered in finite dimension, while the second and the third are the ones in infinite-dimensional setting.

**Example 1.1** (See [42, pp. 143–148]) Suppose that there are $n$ types of available goods. The quantities of each of these goods purchased by the consumer are summarized by the good bundle $x = (x_1, \ldots, x_n)$, where $x_i$ is the quantity of $i^{th}$ good purchased by the consumer, $i = 1, \ldots, n$. Assume that each good is perfectly divisible so that any nonnegative quantity can be purchased. Good bundles are vectors in the commodity space $X := \mathbb{R}^n$. The set of all possible good bundles

$$X_+ := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_1 \geq 0, \ldots, x_n \geq 0\}$$

is the nonnegative orthant of $\mathbb{R}^n$. The set of prices is

$$Y_+ = \{p = (p_1, \ldots, p_n) \in \mathbb{R}^n \ : \ p_1 \geq 0, \ldots, p_n \geq 0\}.$$

For every $p = (p_1, \ldots, p_n) \in Y_+$, $p_i$ is the price of $i^{th}$ good, $i = 1, \ldots, n$. If the consumer's budget is 1 unit of money, then the budget constraint, that the total expenditure cannot exceed the budget, can be written as

$$B(p) = \Big\{x = (x_1, \ldots, x_n) \in X_+ \ : \ \sum_{i=1}^{n} p_i x_i \leq 1\Big\}, \quad p \in Y_+.$$

If the preferences of the consumer are presented by an utility function in the logarithmic type

$$u(x) := \sum_{i=1}^{n} \mu_i \log(x_i + \varepsilon_i), \quad x \in X_+$$

with $\mu_i > 0$, $\varepsilon_i > 0$ for all $i = 1, \ldots, n$, being given numbers, then the consumer problem (1.3) is to choose a "most preferred" good bundle in the budget set $B(p)$.

**Example 1.2** (See [79, p. 59]) Consider a consumer who wants to maximize the sum of the utility stream $U(x(t))$ attained by the consumption stream $x(t)$ over the lifetime $[0, T]$. Suppose that at any time $t \in [0, T]$, the consumer knows the budget $y(t)$, and the price of goods $P(t)$. Let $\rho$ and $r$ respectively denote the subjective *discount rate* and the market *rate of interest*, both of which are assumed to be positive constants. Assume that the choice of $x(t)$ does not affect the price $P(t)$ and rate $r$ that prevail in the market. Then the problem can be formulated as follows: Maximize

$$u(x(\cdot)) := \int_0^T U(x(t))e^{-\rho t}dt$$

subject to

$$p(x(\cdot)) := \int_0^T P(t)x(t)e^{-rt}dt \leq M, \ \ x(\cdot) \in X_+$$

with $M := \int_0^T y(t)e^{-rt}dt$ being the total budget and $X_+$ being a closed and convex cone in a suitable space of functions, say, $L^p([0\ T], \mathbb{R})$, $p \in (1, \infty)$. This is a problem in the form of (1.3), where the budget set is

$$B(p(\cdot)) = \left\{x(\cdot) \in X_+ \,:\, \frac{1}{M}p(x(\cdot)) \leq 1\right\}.$$

**Example 1.3** A goods bundle usually contains a finite number of nonzero components representing the quantities of different goods (rice, bread, milk, vegetable oil, cloths, electronic appliances, books,...) purchased by the consumer. Since there are thousands different goods available in the market and since the need of the consumer changes from time to time, it is not always reasonable to assume that the set of goods belongs to an Euclidean space of fixed dimension. To deal with that situation, one can embed goods bundles into the subspace of the Banach space $X = \ell^p$ with $p \in (1, +\infty)$, denoted by $X_0$, which is formed by sequences of real numbers having finitely many nonzero components. As $\overline{X}_0 = X$, every continuous linear functional $p_0 : X_0 \to \mathbb{R}$ has a unique continuous linear extension $p : X \to \mathbb{R}$ with $\langle p, x\rangle = \langle p_0, x\rangle$ for all $x \in X_0$. In particular, given a nonempty closed convex cone $X_{0,+} \subset X_0$, one sees that any continuous linear functional $p_0$ on $X_0$ satisfying $\langle p, x\rangle \geq 0$ for all $x \in X_{0,+}$ (a price defined on $X_{0,+}$) has a unique continuous linear extension $p$ on $X$ satisfying $\langle p, x\rangle \geq 0$ for all $x \in X_+$, where $X_+$ is the topological closure of $X_{0,+}$ in $X$. Naturally, $X_+$ can be interpreted as a set of goods in $X$ and $p$ belongs to $Y_+$, where $Y_+$ is defined by (1.1). So, $p$ is a price defined on $X_+$. Any function $u : X \to \overline{\mathbb{R}}$ with $u(x) \in \mathbb{R}$ for every $x \in X_+$ defines a utility function on $X$, which can be considered as an extension of the utility function $u_0$ on $X_0$, where $u_0(x) := u(x)$ for $x \in X_0$. In this sense, the consumer problem in (1.3) is an extension of the consumer problem $\max\{u_0(x) \,:\, x \in B_0(p)\}$ with $B_0(p) := \{x \in X_{0,+} \,:\, p_0 \cdot x \leq 1\}$.

It worthy to stress that the consumer problem (1.3) considered in Chapters 1 and 2 has the same mathematical form to the producer problem named *maximizing profit subject to producer budget constraint* with varying input prices in the production theory, which is recalled bellow. Thus, *all the results and proofs in these two chapters for the former problem are valid for the latter one.*

Assume that a firm produces a single product under the circumstances of *pure competition.* The price of both inputs and output must be taken as *exogenous.* Keeping the same mathematical setting of problem (1.3), let each $x \in X_+$ be a collection of inputs which costs a corresponding price $p \in Y_+$. The utility function $u(\cdot)$ is replaced by $Q(\cdot)$, the *production function*, whose values represent the output quantities. Denote by $\bar{p}$ the price of the output. The manufacturer's aim is to maximize the *profit*

$$\Pi := \bar{p}Q(x) - \langle p, x\rangle,$$

where $TR := \bar{p}Q(x)$ is the *total revenue*, $TC := \langle p, x\rangle$ is the *total cost.* If the manufacturer takes a given amount of total cost, say, 1 unit of money, for implementing the production process, then the task of maximizing the profit leads to a maximization of the total revenue. As the output price $\bar{p}$ is exogenous, this amounts to maximize the quantity $Q(x)$. The problem of *maximizing profit subject to producer budget constraint* (see, e.g., [71, p. 38]) is the following:

$$\max\{Q(x) \,:\, x \in B(p)\}, \tag{1.6}$$

where $B(p) := \{x \in X_+ \,:\, \langle p, x\rangle \leq 1\}$ is the budget constraint for the producer at a price $p \in Y_+$ of inputs. It is not hard to see that (1.6) has the same structure as that of (1.3).

## 1.2 Auxiliary Concepts and Results

In order to establish the stability properties of the function $v(\cdot)$ and the multifunctions $B(\cdot)$, $D(\cdot)$, we need some concepts and results from set-valued analysis and variational inequalities.

Let $T : E \rightrightarrows F$ be a set-valued map between two topological spaces. The *graph* of $T$ is defined by $\operatorname{gph} T := \{(a, b) \in E \times F \, : \, b \in T(a)\}$. If $\operatorname{gph} T$ is closed in the product topology of $E \times F$, then $T$ is said to be *closed*. The map $T$ is said to be *upper semicontinuous* (u.s.c.) at $a \in E$ if, for each open subset $V \subset F$ with $T(a) \subset V$, there exists a neighborhood $U$ of $a$ satisfying $T(a') \subset V$ for all $a' \in U$. One says that $T$ is *lower semicontinuous* (l.s.c.) at $a$ if, for each open subset $V \subset F$ with $T(a) \cap V \neq \emptyset$, there exists a neighborhood $U$ of $a$ such that $T(a') \cap V \neq \emptyset$ for every $a' \in U$. If $T$ is u.s.c. (resp., l.s.c.) at every point $a$ in a subset $M \subset E$, then $T$ is said to be u.s.c. (resp., l.s.c.) on $M$.

If $T$ is both l.s.c. and u.s.c. at $a$, we say that it is *continuous* at $a$. If $T$ is continuous at every point $a$ in a subset $M \subset E$, then $T$ is said to be continuous on $M$. Thus, the verification of the continuity of the set-valued map $T$ consists of the verifications of the lower semicontinuity and of the upper semicontinuity of $T$.

One says that $T$ is *inner semicontinuous* (i.s.c.) at $(a, b) \in \operatorname{gph} T$ if, for each open subset $V \subset F$ with $b \in V$, there exists a neighborhood $U$ of $a$ such that $T(a') \cap V \neq \emptyset$ for every $a' \in U$. (In [56, p. 42], the terminology "inner semicontinuous map" has a little bit different meaning.) Clearly, $T$ is l.s.c. at $a$ if and only if it is i.s.c. at any point $(a, b) \in \operatorname{gph} T$.

If $E$ and $F$ are some norm spaces, one says that $T$ is *Lipschitz-like* or $T$ has *the Aubin property*, at a point $(a_0, b_0) \in \operatorname{gph} T$, if there exists a constant $l > 0$ along with neighborhoods $U$ of $a_0$ and $V$ of $b_0$, such that

$$T(a) \cap V \subset T(a') + l \parallel a - a' \| \bar{B}_F, \quad \forall a, a' \in U.$$

This fundamental concept was suggested by Aubin [8]. As it has been noted in [87, Proposition 3.1] (see also the related proof), if $T$ is Lipschitz-like $(a_0, b_0) \in \operatorname{gph} T$ and $l > 0$, $U$, $V$ are as above, then the map $\widetilde{T} : U \rightrightarrows F$, $\widetilde{T}(a) := T(a) \cap V$ for all $a \in U$, is lower semicontinuous on $U$. In particular, both $\widetilde{T}$ and $T$ are i.s.c. at $(a_0, b_0)$.

Let $A$ be a closed subset of a Banach space $X$, $x_0 \in A$. The *Clarke tangent* *cone* to $A$ at $x_0$ is

$$T_A(x_0) := \{v \in X \ : \ \forall (t_k \downarrow 0, x_k \to x_0, x_k \in A) \\ \exists x'_k \to x_0,\ x'_k \in A,\ t_k^{-1}(x'_k - x_k) \to v\};$$

see [20, p. 51 and Theorem 2.4.5], [15, pp. 16–17], and [9, p. 127]. This tangent cone is closed and convex. Clearly, if $x_0 \in \operatorname{int} A$, then $T_A(x_0) = X$. By [9, Lemma 4.2.5], if $A$ is a closed and convex cone of $X$, then $T_A(x_0) = \overline{A + \mathbb{R}x_0}$. The *Clarke normal cone* (see [20, p. 51]) to $A$ at $x_0$ is

$$N_A(x_0) := \{x^* \in X^* \ : \ \langle x^*, x\rangle \le 0 \ \ \forall x \in T_A(x_0)\}.$$

The notation $N_A^\times(x_0)$ will be used to indicate the set $N_A(x_0) \setminus \{0\}$.

Given a function $f : X \times P \to \mathbb{R}$, where $X$ is a Banach space and $P$ is a metric space, as in [15, p. 14], we say that $f$ is *locally equi-Lipschitz* in $x$ at $(x_0, p_0)$ if there exists $\gamma > 0$ such that

$$|f(x,p) - f(x',p)| \le \gamma \|x - x'\|$$

for all $x, x'$ in a neighborhood of $x_0$, all $p$ in a neighborhood of $p_0$. Slightly modifying the terminology of Borwein [15], we call the number

$$d_x^0 f(x_0, p_0; d) := \limsup_{x \to x_0, p \to p_0, t \downarrow 0} [f(x + td, p) - f(x,p)]/t$$

the *partial generalized derivative* of $f$ at $(x_0, p_0)$ in a direction $d \in X$, and the set

$$\partial_x f(x_0, p_0) := \{x^* \in X^* \ : \ d_x^0 f(x_0, p_0; d) \ge x^*.d \ \ \forall d \in X\}$$

the *partial subdifferential* of $f$ with respect to $x$ at $(x_0, p_0)$.

Let $B$ and $C$ be nonempty closed subsets of $\mathbb{R}$ and $X$, respectively. As in [15], we consider the set-valued map $\Omega : X \rightrightarrows P$,

$$\Omega(x) := \begin{cases} \{p \in P \ : \ f(x,p) \in B\}, & x \in C, \\ \emptyset, & x \notin C, \end{cases} \tag{1.7}$$

where $f$ is given above. The inverse of $\Omega$ is the *implicit set-valued map* $\Omega^{-1} : P \rightrightarrows X$ defined by

$$\Omega^{-1}(p) := \{x \in C \ : \ f(x,p) \in B\} \quad (p \in P). \tag{1.8}$$

One says that $\Omega$ is *metrically regular* at $(x_0, p_0) \in \operatorname{gph} \Omega$ if there exist $\mu \ge 0$, and neighborhoods $V$ of $x_0$ and $U$ of $p_0$ such that

$$d(x, \Omega^{-1}(p)) \le \mu d(f(x,p), B) \quad \forall x \in V \cap C,\ \forall p \in U. \tag{1.9}$$

Here, $d(\cdot, K)$ stands for the *distance function* to a nonempty closed subset $K$ in a Banach space, i.e., $d(x, K) = \inf \{\|x - u\| \ : \ u \in K\}$.

**Remark 1.1** In the terminology of Gfrerer and Mordukhovich [30, Definition 1.1], the property in (1.9) is the *Robinson stability* of the constraint system

$$f(x, p) \in B \ \text{ with } \ x \in C \ \text{ and } \ p \in P$$

at $(x_0, p_0)$ with modulus $\mu \geq 0$.

The next statement is a special case of [15, Theorem 3.2].

**Theorem 1.1** (See [15, p. 20]) *Let $X$ be a Banach space and $P$ be a metric space. Suppose that $f : X \times P \to \mathbb{R}$ is continuous and locally equi-Lipschitz in $x$ at $(x_0, p_0)$ and that*

$$0 \notin N_B^{\times}(f(x_0, p_0))\partial_x f(x_0, p_0) + N_C(x_0). \tag{1.10}$$

*Then, the set-valued map $\Omega$ given by (1.7) is metrically regular at $(x_0, p_0)$.*

Finally, let us recall a result of [86] on solution sensitivity of a parametric variational inequality. Suppose that $X$ is a Hilbert space, $M$ and $\Lambda$ are subsets of some norm spaces. Given a function $f : X \times M \to X$ and a set-valued map $K : \Lambda \rightrightarrows X$ with nonempty, closed and convex values, we consider the following *parametric variational inequality* depending on a pair of parameters $(\mu, \lambda) \in M \times \Lambda$:

$$\text{Find } x \in K(\lambda) \text{ such that } \langle f(x, \mu), y - x \rangle \geq 0 \text{ for all } y \in K(\lambda), \tag{1.11}$$

where $\langle ., . \rangle$ stands for the scalar product in $X$.

Let $\bar{x}$ be a solution to problem (1.11) at $(\bar{\mu}, \bar{\lambda}) \in M \times \Lambda$. As in [86], we make two assumptions on $f$ around the point $(\bar{x}, \bar{\mu})$. Namely, suppose that there exist a closed and convex neighborhood $V$ of $\bar{x}$, a neighborhood $W$ of $\bar{\mu}$ and constants $\alpha > 0$, $\ell > 0$ satisfying

$$\|f(x', \mu') - f(x, \mu)\| \leq \ell(\|x' - x\| + \|\mu' - \mu\|) \quad \forall x, x' \in V, \ \forall \mu, \mu' \in M \cap W \tag{1.12}$$

and

$$\langle f(x', \mu) - f(x, \mu), x' - x \rangle \geq \alpha \|x' - x\|^2 \quad \forall x, x' \in V, \ \forall \mu \in M \cap W. \tag{1.13}$$

**Remark 1.2** Condition (1.12) states that $f$ is *locally Lipschitz* at $(\bar{x}, \bar{\mu})$, while (1.13) is the requirement that $f(\cdot, \mu)$ is *locally strongly monotone* at $\bar{x}$ with a coefficient independent of $\mu \in M \cap W$.

**Theorem 1.2** (See Theorem 2.1 and Remark 2.3 in [86]) *Assume that $\bar{x}$ is a solution to problem* (1.11) *with respect to the given parameters $(\bar{\mu}, \bar{\lambda}) \in M \times \Lambda$, condition* (1.12) *and* (1.13) *hold, and the set-valued map $K : \Lambda \rightrightarrows X$ is Lipschitz-like at $(\bar{\lambda}, \bar{x})$. Then, there exist constants $\kappa_{\bar{\mu}} > 0$, $\kappa_{\bar{\lambda}} > 0$, and neighborhoods $W_1$ of $\bar{\mu}$, $U_1$ of $\bar{\lambda}$ such that:*

(i) *For every $(\mu, \lambda) \in (M \cap W_1) \times (\Lambda \cap U_1)$, a unique solution to* (1.11), *denoted by $x(\mu, \lambda)$, exists in $V$.*

(ii) *For all $(\mu', \lambda'), (\mu, \lambda) \in (M \cap W_1) \times (\Lambda \cap U_1)$,*

$$\|x(\mu', \lambda') - x(\mu, \lambda)\| \leq \kappa_{\bar{\mu}} \|\mu' - \mu\| + \kappa_{\bar{\lambda}} \|\lambda' - \lambda\|^{1/2}.$$

## 1.3 Continuity Properties

Adopting the above notations and assumptions, we shall study continuity properties of the budget map, the indirect utility function, and the demand map.

The following property of the values of the budget map $B : Y_+ \rightrightarrows X_+$ is a known result (see [65, Proof of Proposition 4.1]). The proof below is given to make our presentation shelf-contained.

**Lemma 1.1** *For every $p \in \operatorname{int} Y_+$, $B(p)$ is a nonempty, closed, convex and bounded set in $X$. Hence, it is a weakly compact set.*

**Proof.** For any $p \in \operatorname{int} Y_+$, by the assumptions made on $X_+$ and the inclusion $p \in X^*$, we can assert that $B(p)$ is closed and convex. In addition, $B(p) \neq \emptyset$ because $0 \in B(p)$. We shall prove that $B(p)$ is bounded. Since $p \in \operatorname{int} Y_+$, there exists $\alpha_p > 0$ satisfying $p + \alpha_p \bar{B}_{X^*} \subset Y_+$. Taking any $y \in X^* \setminus \{0\}$, we have $\|y\|^{-1} y \in \bar{B}_{X^*}$. It follows that $p - \alpha_p \|y\|^{-1} y \in Y_+$ and $p + \alpha_p \|y\|^{-1} y \in Y_+$. Hence, for each $x \in B(p)$, $(p - \alpha_p \|y\|^{-1} y) \cdot x \geq 0$ and $(p + \alpha_p \|y\|^{-1} y) \cdot x \geq 0$. Therefore, one has

$$y \cdot x \leq \alpha_p^{-1} \|y\| p \cdot x = \alpha_p^{-1} \|y\|,$$

and

$$-y \cdot x \leq \alpha_p^{-1}\|y\| p \cdot x = \alpha_p^{-1}\|y\|,$$

which yield $|y \cdot x| \leq \alpha_p^{-1}\|y\|$. This inequality is also valid for $y = 0$; so we have $\|x\| \leq \alpha_p^{-1}$. It follows that $B(p)$ is bounded by $\alpha_p^{-1}$. Since $X$ is reflexive, by the Banach-Alaoglu Theorem and the Mazur Lemma [76, Theorems 3.15 and 3.12], the last property implies that the closed, convex set $B(p)$ is weakly compact. □

In the forthcoming statements, we consider $X_+$ (resp., $Y_+$) with the topologies *induced* from the topologies of $X$ (resp., of $Y$). For example, an open set in the strong (resp., weak) topology $X_+$ is the intersection of $X_+$ and a subset of $X$, which is open in the strong (resp., weak) topology of $X$. Similarly, an open set in the strong (resp., weak, weak*) topology of $Y_+$ is the intersection of $Y_+$ and a subset of $X^*$, which is open in the strong (resp., weak, weak*) topology of $X^*$. By abuse of terminology, we shall speak about the weak and weak* topologies of $X_+$ (resp., of $Y_+$). Note that if $X$ is finite-dimensional, then the weak topology of $X_+$ (resp., the weak$^*$ topology of $Y_+$) coincides with its norm topology.

The lower semicontinuity property of the budget map can be stated as follows.

**Proposition 1.1** *The set-valued map $B : Y_+ \rightrightarrows X_+$ is l.s.c. on $Y_+$ in the weak* topology of $Y_+$ and the strong topology of $X_+$. Hence, $B : Y_+ \rightrightarrows X_+$ is l.s.c. on $Y_+$ in the strong topologies of $Y_+$ and $X_+$.*

**Proof.** Let $p_0 \in Y_+$, and let $V$ be an open set in the strong topology of $X_+$ such that $B(p_0) \cap V \neq \emptyset$. Take any $x \in B(p_0) \cap V$. For every $t \in (0, 1)$, $x_t := (1 - t)x$ belongs to $X_+$. Since $x_t \to x$ when $t \to 0$, and $V$ is a neighborhood of $x$, one can find $t_0 \in (0, 1)$ such that $x_{t_0} \in V$. As

$$p_0 \cdot x_{t_0} = \langle p_0, (1 - t_0)x\rangle = (1 - t_0)p_0 \cdot x < 1,$$

$U := \{p \in Y_+ : \ p \cdot x_{t_0} < 1\}$ is an open neighborhood of $p_0$ in the weak* topology of $Y_+$. For every $p \in U$, since $p \cdot x_{t_0} < 1$ and $x_{t_0} \in X_+$, one has $x_{t_0} \in B(p)$. It follows that $V \cap B(p) \neq \emptyset$ for all $p \in U$. The proof of the first claim is complete. The second claim is immediate from the first one. □

Unlike the preceding result on the l.s.c. property, the upper semicontinuity property of the budget map can be obtained only for internal points of the set of prices, and it requires a more stringent condition on topologies.

**Proposition 1.2** *The set-valued map $B : Y_+ \rightrightarrows X_+$ is u.s.c. on* $\operatorname{int} Y_+$ *in the strong topology of $Y_+$ and the weak topology of $X_+$.*

**Proof.** Let $p_0 \in \operatorname{int} Y_+$. For a given number $r \in (0,1)$, we set $q_0 = rp_0$. It is easily seen that $q_0 \in \operatorname{int} Y_+$ and $(1-r)p_0 \in \operatorname{int} Y_+$. Since

$$p_0 = q_0 + (1-r)p_0 \in q_0 + \operatorname{int} Y_+ \subset q_0 + Y_+,$$

the set $q_0 + Y_+$ is a strong neighborhood of $p_0$. Taking any $p \in q_0 + Y_+$, one can find $y \in Y_+$ such that $p = q_0 + y$. Hence, for every $x \in B(p)$, one has

$$1 \geq p \cdot x = q_0 \cdot x + y \cdot x \geq q_0 \cdot x$$

which implies that $x \in B(q_0)$. Thus, $B(p) \subset B(q_0)$ for all $p \in q_0 + Y_+$.

To prove that $B(\cdot)$ is u.s.c. at $p_0$ in the strong topology of $Y_+$ and the weak topology of $X_+$, we assume the contrary that there exist a weakly open set $V$ containing $B(p_0)$ and a sequence $\{p_k\}_{k=1}^{\infty}$ in $q_0 + Y_+$, which converges in norm to $p_0$ such that $B(p_k) \setminus V \neq \emptyset$ for all $k \in I\!N$. For each $k \in I\!N$, select a point $x_k \in B(p_k) \setminus V$. Due to the choice of $q_0$ and the above arguments, we have $B(p_k) \subset B(q_0)$ for $k \in I\!N$. Since $B(q_0)$ is a weakly compact set in $X$ by Lemma 1.1 and $V$ is weakly open, the whole sequence $\{x_k\}_{k=1}^{\infty}$ lies the the weakly compact set $B(q_0) \setminus V$. By taking a subsequence if necessary, we may assume that $\{x_k\}_{k=1}^{\infty}$ converges weakly to a point $\bar{x} \in B(q_0) \setminus V$. Since $B(p_0) \subset V$, we must have $\bar{x} \notin B(p_0)$. As $\{x_k\}_{k=1}^{\infty}$ converges weakly to $\bar{x}$ and $\{p_k\}_{k=1}^{\infty}$ converges strongly to $p_0$, using a well-known result [16, Proposition 3.5] we can assert that $\{p_k \cdot x_k\}_{k=1}^{\infty}$ converges to $p_0 \cdot \bar{x}$. Hence, passing the inequality $p_k \cdot x_k \leq 1$ to the limit when $k \to \infty$, we get $p_0 \cdot \bar{x} \leq 1$. It follows that $\bar{x} \in B(p_0)$. We have thus arrived at a contradiction.

The proof is complete. □

From Lemma 1.1 and Propositions 1.1, 1.2, we obtain the next result on the continuity of the budget map.

**Theorem 1.3** *The set-valued map $B : Y_+ \rightrightarrows X_+$ has nonempty weakly compact, convex values and is continuous on* $\operatorname{int} Y_+$ *in the strong topology of $Y_+$ and the weak topology of $X_+$. Specifically, if $X$ is finite-dimensional, then $B(\cdot)$ has nonempty compact, convex values and is continuous on* $\operatorname{int} Y_+$.

Based on the above results, we are now in a position to present several continuity properties of the indirect utility function.

The forthcoming statement on the lower semicontinuity of $v(\cdot)$ is weaker than Lemma 3.1 from [65], where it was only assumed that the utility function is lower radially l.s.c. on $X_+$. It is worthy to notice that our approach is new. Namely, we derive the desired result from the l.s.c. property of $B(\cdot)$, which is guaranteed by Proposition 1.1. In some sense, our proof arguments are simpler than those of [65].

**Proposition 1.3** (cf. [65, Lemma 3.1]) *If $u : X_+ \to \mathbb{R}$ is l.s.c. on $X_+$ in the strong topology of $X_+$, then $v : Y_+ \to \overline{\mathbb{R}}$ is l.s.c. on $Y_+$ in the weak* topology of $Y_+$.*

**Proof.** Let $p_0 \in Y_+$ and $\varepsilon > 0$ be given arbitrarily. On one hand, since $B(p_0)$ is nonempty and $v(p_0) = \sup\{u(x) : x \in B(p_0)\}$, there is $x_0 \in B(p_0)$ with $u(x_0) > v(p_0) - 2^{-1}\varepsilon$. On the other hand, as $u$ is l.s.c. at $x_0$ in the strong topology of $X_+$, we can choose a strong neighborhood $V$ of $x_0$ such that $u(x) > u(x_0) - 2^{-1}\varepsilon$ for all $x \in V$. Therefore, $u(x) > v(p_0) - \varepsilon$ for all $x \in V$. Besides, the lower semicontinuity of $B(\cdot)$ at $p_0$ in Proposition 1.1 implies that there is a weak* neighborhood $U$ of $p_0$ satisfying $V \cap B(p) \neq \emptyset$ for all $p \in U$. For every $p \in U$, by taking an element $x_p \in V \cap B(p)$, we have

$$v(p) \geq u(x_p) > v(p_0) - \varepsilon.$$

This shows that $v$ is l.s.c. at $p_0$ in the weak* topology of $Y_+$ and completes the proof. $\square$

The next result on the upper semicontinuity of $v(\cdot)$ is due to Penot [65]. Here we give a new proof by using the u.s.c. property of $B(\cdot)$ provided by Proposition 1.2.

**Proposition 1.4** (See [65, Proposition 3.2]) *If $u : X_+ \to \mathbb{R}$ is u.s.c. on $X_+$ in the weak topology of $X_+$, then $v : Y_+ \to \overline{\mathbb{R}}$ is u.s.c. on $\operatorname{int} Y_+$ in the strong topology of $Y_+$.*

**Proof.** Fix any point $p_0 \in \operatorname{int} Y_+$ and let $\varepsilon > 0$. For every $x \in B(p_0)$, since $u$ is u.s.c. at $x$ in the weak topology of $X_+$, there exists a weakly open neighborhood $V_x$ of $x$ satisfying $u(z) < u(x) + \varepsilon$ for all $z \in V_x$. By Lemma 1.1, $B(p_0)$ is a weakly compact set. Hence, there is a finite covering $\{V_{x_i}\}_{i \in I}$ of $B(p_0)$. For every $z \in V := \bigcup_{i \in I} V_{x_i}$, one can find an index $i(z) \in I$ such that

$z \in V_{x_{i(z)}}$; so $u(z) < u(x_{i(z)}) + \varepsilon$. Thus

$$u(z) < \max\{u(x_i)\ :\ i \in I\} + \varepsilon \le \sup\{u(x)\ :\ x \in B(p_0)\} + \varepsilon = v(p_0) + \varepsilon \tag{1.14}$$

for every $z \in V$. According to Proposition 1.2, $B(\cdot)$ is u.s.c. at $p_0$ in the strong topology of $Y_+$ and the weak topology of $X_+$. So, having the weakly open set $V$ that contains $B(p_0)$, we can find a strong neighborhood $U$ of $p_0$ such that $B(p) \subset V$ for all $p \in U$. Consequently, for every $p \in U$, by invoking (1.14) we have

$$v(p) = \sup\{u(x)\ :\ x \in B(p)\} \le \sup\{u(z)\ :\ z \in V\} < v(p_0) + \varepsilon.$$

It follows that $v$ is strongly u.s.c. at $p_0$. □

As a consequence of Propositions 1.3 and 1.4, we get the following result on the continuity of the indirect utility function.

**Theorem 1.4** *If $u$ is weakly u.s.c. and strongly l.s.c. on $X_+$, then $v$ is strongly continuous on $\operatorname{int} Y_+$. Especially, if $X$ is finite-dimensional and $u$ is continuous on $X_+$, then $v$ is continuous on $\operatorname{int} Y_+$.*

We conclude this section by considering some properties of the demand map. The first assertion of the next proposition follows from Lemma 1.1 and the Weierstrass theorem. By the same theorem and a delicate argument, one can get the second assertion. Let $\partial Y_+ := Y_+ \setminus \operatorname{int} Y_+$ be the boundary of $Y_+$.

**Proposition 1.5** (See [65, Proposition 4.1]) *Suppose that $u$ is weakly u.s.c. on $X_+$. Then, for every $p \in \operatorname{int} Y_+$, the demand set $D(p)$ is nonempty and weakly compact. If $p \in \partial Y_+$ and if one has*

$$v(p) > \limsup_{x \in B(p),\, \|x\| \to \infty} u(x), \tag{1.15}$$

*then $D(p)$ is nonempty.*

Concerning this proposition, we first remark that for every $p \in \partial Y_+$, the budget set $B(p)$ is unbounded. To verify this claim, we can apply the separation theorem [76, Theorem 3.4] for the disjoint convex nonempty subsets $\{p\}$ and $\operatorname{int} Y_+$ of $X^*$, that is equipped with the norm topology, to find a nonzero vector $x \in X = X^{**}$ such that

$$p \cdot x \le q \cdot x \quad (\forall q \in \operatorname{int} Y_+). \tag{1.16}$$

From (1.16) it follows that $p \cdot x \le 0$ and $q \cdot x \ge 0$ for all $q \in Y_+$. Hence $x \in X_+$ and, moreover, $tx \in B(p)$ for every $t > 0$. Since $x \neq 0$, the last property shows that $B(p)$ is unbounded. Next, let us give a simple illustrative example where $u$ is quasiconcave and continuous on $X_+$, and the special assumption (1.15) is fulfilled. Let $X = \mathbb{R}$, $X_+ = \mathbb{R}_+$, and

$$u(x) = \begin{cases} 1, & \text{for } x \in [0,1], \\ -\frac{1}{2}x + \frac{3}{2}, & \text{for } x \in (1,2], \\ \frac{1}{2}, & \text{for } x \in (2,+\infty). \end{cases}$$

Since $Y_+ = \mathbb{R}_+$, $p := 0$ belongs to $\partial Y_+$ and $B(p) = \mathbb{R}_+$. As $v(p) = 1$ and $\limsup_{x \in B(p),\, \|x\| \to \infty} u(x) = 1/2$, condition (1.15) is satisfied. Clearly, $D(p) = [0,1]$.

An example given by Penot [65, p. 1082] shows that the demand map $D(\cdot)$ may fail to be i.s.c. at a point $(p_0, x_0) \in \operatorname{gph} D$, where $p_0 \in \operatorname{int} Y_+$, even if $u$ is continuous and concave on $X_+$, with $X$ being a finite-dimensional Euclidean space. This means that the lower semicontinuity of the demand map requires rather strong assumptions. Later, we will give a set of conditions guaranteeing the continuity of $D(\cdot)$; see Theorem 1.8 in Section 4.

The following statement is an analogue of [65, Proposition 6.5]. Here we do not use any assumption on the indirect utility function $v(\cdot)$.

**Proposition 1.6** *If $u$ is weakly u.s.c. and strongly l.s.c. on $X_+$, then the demand map $D : Y_+ \rightrightarrows X_+$ is u.s.c. on $\operatorname{int} Y_+$ in the strong topology of $Y_+$ and the weak topology of $X_+$.*

**Proof.** First, we shall prove that the set-valued map $\Delta : Y_+ \rightrightarrows X_+$ given by

$$\Delta(p) = \{x \in X_+ \,:\, u(x) \ge v(p)\} \quad (p \in Y_+)$$

has closed graph w.r.t. the weak* topology of $Y_+$ and the weak topology of $X_+$. Suppose that $(\bar{p}, \bar{x}) \in Y_+ \times X_+$, $\bar{x} \notin \Delta(\bar{p})$. Then $u(\bar{x}) < v(\bar{p})$, so we can choose an $\alpha > 0$ satisfying $u(\bar{x}) < \alpha < v(\bar{p})$. Since $u$ is weakly u.s.c. at $\bar{x}$, there exists a neighborhood $V$ of $\bar{x}$ in the weak topology of $X_+$ such that $u(x) < \alpha$ for all $x \in V$. According to Proposition 1.3, the lower semicontinuity of $u(\cdot)$ in the strong topology of $X_+$ implies that $v$ is l.s.c. at $\bar{p}$ in the weak* topology of $Y_+$. Hence, there is a weak* neighborhood $U$ of $\bar{p}$ satisfying $\alpha < v(p)$ for all $p \in U$. Taking any $(p, x) \in U \times V$, we have

$u(x) < v(p)$. This implies that $\operatorname{gph}\Delta \cap (U \times V) = \emptyset$ and proves the closedness of $\operatorname{gph}\Delta$.

Now, in order to show that $D$ is u.s.c. on $\operatorname{int} Y_+$, fix a point $p_0 \in \operatorname{int} Y_+$. For $p \in Y_+$, since $D(p) = \{x \in B(p) : u(x) = v(p)\}$, we have $D(p) = B(p) \cap \Delta(p)$. Let $W$ be a weakly open set with $D(p_0) \subset W$.

If $B(p_0) \subset W$, then the upper semicontinuity of $B(\cdot)$ in Proposition 1.2 implies the existence of a neighborhood $U$ of $p_0$ in the strong topology of $Y_+$ with $B(p) \subset W$ for every $p \in U$. Then we have $D(p) \subset W$ for every $p \in U$.

Consider the case where $B(p_0) \not\subset W$. Since $B(p_0)$ is a nonempty and weakly compact set by Lemma 1.1, $K := B(p_0) \setminus W$ is nonempty and weakly compact. For every $x \in K$, as $(p_0, x) \notin \operatorname{gph}\Delta$ and $\Delta(\cdot)$ has closed graph w.r.t. the weak* topology of $Y_+$ and the weak topology of $X_+$, there exist a weak* neighborhood $U_x$ of $p_0$ and a weakly open neighborhood $V_x$ of $x$ satisfying $\operatorname{gph}\Delta \cap (U_x \times V_x) = \emptyset$. Then we have $\Delta(p) \cap V_x = \emptyset$ for all $p \in U_x$. Therefore, by the weak compactness of $K$, we can find a finite family of points $\{x_i\}_{i \in I}$ of $K$ such that $K \subset \bigcup\limits_{i \in I} V_{x_i}$. Setting $V = \bigcup\limits_{i \in I} V_{x_i}$, $V \cup W$ is a weakly open set in $X_+$ satisfying $B(p_0) \subset V \cup W$. Hence, by using again the upper semicontinuity of $B(\cdot)$ in Proposition 1.2, we can find a neighborhood $U'$ of $p_0$ in the strong topology of $Y_+$ with $B(p) \subset V \cup W$ for all $p \in U'$. The set $U := (\bigcap\limits_{i \in I} U_{x_i}) \bigcap U'$ is a neighborhood of $p_0$ in the strong topology of $Y_+$. For every $p \in U$, by the construction of $V$ and $U$, we have $\Delta(p) \cap V = \emptyset$ and $B(p) \subset V \cup W$, which implies that $D(p) = \Delta(p) \cap B(p) \subset W$.

Thus we have proved the existence of a neighborhood $U$ of $p_0$ in the strong topology of $Y_+$ with the property $D(p) \subset W$ for every $p \in U$ and, therefore, we get the desired upper semicontinuity of $D(\cdot)$ at $p_0$. □

## 1.4 Lipschitz-like and Lipschitz Properties

We shall investigate the Lipschitz property of the indirect utility function by using the Lipschitz-like property of the budget map. The results from [15], which have been recalled in Section 1.2, will be the principal tools in our proofs.

Let us start with a stability property of the budget map in the form of a uniform local error bound.

**Theorem 1.5** *For any* $p_0 \in \operatorname{int} Y_+$ *and* $x_0 \in B(p_0)$*, there exists* $\mu \geq 0$ *along with a neighborhood* $U$ *of* $p_0$ *and a neighborhood* $V$ *of* $x_0$ *such that*

$$d(x, B(p)) \leq \mu[p \cdot x - 1]_+, \quad \forall p \in U \cap Y_+, \ \forall x \in V \cap X_+, \tag{1.17}$$

*where* $\alpha_+ := \max\{0, \alpha\}$.

**Proof.** We now apply Theorem 1.1 with $X$ being the reflexive Banach space containing the closed and convex cone $X_+$ that appeared in formula (1.2) for $B(p)$, $P = Y_+$ being the positive dual cone of $X_+$ with the metric induced by the norm of $X^*$. For the function $f : X \times P \mapsto \mathbb{R}$ with $f(x,p) := p \cdot x - 1$, $B := \mathbb{R}_-$, and $C := X_+$, formula (1.8) gives us

$$\Omega^{-1}(p) = \{x \in X_+ : \ f(x,p) \in \mathbb{R}_-\} = B(p), \quad \forall p \in Y_+.$$

The fact that $f(x,p) = p \cdot x - 1$ is continuous on $X \times P$ is obvious.

Let $p_0 \in \operatorname{int} Y_+$ and $x_0 \in B(p_0)$. Then $(x_0, p_0) \in \operatorname{gph} \Omega$. To prove that $f$ is locally equi-Lipschitz in $x$ at $(x_0, p_0)$, select an $r > 0$ as small as $\bar{B}(p_0, r) \subset Y_+$. For all $p \in \bar{B}(p_0, r)$ and $x, x' \in X$, we have

$$|f(x,p) - f(x',p)| = |p \cdot (x - x')| \leq \|p\| \|x - x'\| \leq (r + \|p_0\|)\|x - x'\|.$$

This shows that $f$ is locally equi-Lipschitz in $x$ at $(x_0, p_0)$.

Next, let us show that condition (1.10) is fulfilled. Since $x_0 \in B(p_0)$, we have $f(x_0, p_0) \in \mathbb{R}_-$. If $f(x_0, p_0) < 0$, then $T_{\mathbb{R}_-}(f(x_0, p_0)) = \mathbb{R}$. Therefore, $N_{\mathbb{R}_-}(f(x_0, p_0)) = 0$ and one has $N^{\times}_{\mathbb{R}_-}(f(x_0, p_0)) = \emptyset$. Hence, (1.10) is fulfilled. If $f(x_0, p_0) = 0$, i.e., $p_0 \cdot x_0 = 1$, then

$$T_{\mathbb{R}_-}(f(x_0, p_0)) = \overline{\mathbb{R}_- + \mathbb{R}.0} = \mathbb{R}_-.$$

Hence, $N_{\mathbb{R}_-}(f(x_0, p_0)) = \mathbb{R}_+$ and $N^{\times}_{\mathbb{R}_-}(f(x_0, p_0)) = \{t \in \mathbb{R} : \ t > 0\}$. It holds that $\partial_x f(x_0, p_0) = \{p_0\}$. Indeed, for every $d \in X$,

$$\begin{aligned} d^0_x f(x_0, p_0; d) &= \limsup_{x \to x_0, p \to p_0, t \downarrow 0} [f(x + td, p) - f(x,p)]/t \\ &= \limsup_{x \to x_0, p \to p_0, t \downarrow 0} [(p \cdot (x + td) - 1) - (p \cdot x - 1)]/t \\ &= \limsup_{x \to x_0, p \to p_0, t \downarrow 0} p \cdot d = p_0 \cdot d. \end{aligned}$$

Consequently, the partial subdifferential of $f$ w.r.t. $x$ at $(x_0, p_0)$ is

$$\begin{aligned} \partial_x f(x_0, p_0) &= \{p \in X^* : \ d^0_x f(x_0, p_0; d) \geq p \cdot d \ \forall d \in X\} \\ &= \{p \in X^* : \ p_0 \cdot d \geq p \cdot d \ \forall d \in X\} = \{p_0\}. \end{aligned}$$

Then, we have

$$N^{\times}_{\mathbb{R}_-}(f(x_0,p_0))\partial_x f(x_0,p_0)+N_{X_+}(x_0)=\{tp_0\;:\;\;t>0\}+N_{X_+}(x_0).$$

If (1.10) is invalid, then $0\in\{tp_0\;:\;\;t>0\}+N_{X_+}(x_0)$. So, one can choose $t_0<0$ such that $t_0p_0\in N_{X_+}(x_0)$. Since $X_+$ is a closed and convex cone of $X$, we get $T_{X_+}(x_0)=\overline{X_++\mathbb{R}x_0}$ (see Section 1.2). Clearly,

$$-x_0=x_0-2x_0\in\overline{X_++\mathbb{R}x_0}.$$

It follows that $(t_0p_0)\cdot(-x_0)\le 0$. Hence, the equality $p_0\cdot x_0=1$ forces $t_0\ge 0$, which contradicts the choice of $t_0$. And thus, we also have (1.10) in the case $f(x_0,p_0)=0$.

In summary, we have proved that all the assumptions of Theorem 1.1 are satisfied. Therefore, the set-valued map $\Omega$ given by (1.7) is metrically regular at $(x_0,p_0)$. Hence, there exist $\mu>0$ and neighborhoods $V$ of $x_0$ and $U_0$ of $p_0$ in $P$ such that

$$d(x,\Omega^{-1}(p))\le\mu d(f(x,p),\mathbb{R}_-)=\mu[f(x,p)]_+,\quad\forall x\in V\cap X_+,\;\forall p\in U_0.$$

Choosing a neighborhood $U$ of $p_0$ in the norm topology of $X^*$ such that $U_0=U\cap Y_+$ and substituting $f(x,p)=p\cdot x-1$, $\Omega^{-1}(p)=B(p)$, into the last expression, we obtain (1.17). The proof is complete. □

We now show that the Robinson stability property (1.17) of the constraint system

$$f(x,p)=p\cdot x-1\le 0,\quad x\in X_+$$

depending on the parameter $p\in Y_+$, implies the Lipschitz-likeness of $B(\cdot)$ at $(p_0,x_0)$. It is worthy to remark that the Robinson stability property and the Lipschitz-likeness of an implicit set-valued map are not equivalent (see [45]). However, under some additional assumptions, the first property implies the second one; see [30, 41].

**Theorem 1.6** *For any $p_0\in\operatorname{int}Y_+$ and $x_0\in B(p_0)$, the map $B:\;Y_+\rightrightarrows X_+$ is Lipschitz-like at $(p_0,x_0)$.*

**Proof.** Fix any $p_0\in\operatorname{int}Y_+$, $x_0\in B(p_0)$. For a given $r\in(0,1)$, as shown in the proof of Proposition 1.2, the vector $q_0:=rp_0$ belongs to $\operatorname{int}Y_+$, the set $q_0+\operatorname{int}Y_+$ is a strong neighborhood of $p_0$ in $Y_+$, and $B(p)\subset B(q_0)$ for every $p\in q_0+\operatorname{int}Y_+$. According to Lemma 1.1, we can find $\alpha_{q_0}>0$ such that $\|x\|\le\alpha_{q_0}^{-1}$ for all $x\in B(q_0)$. By Theorem 1.5, there exists $\mu>0$ along with a neighborhood $U$ of $p_0$ and a neighborhood $V$ of $x_0$ satisfying (1.17). Therefore, for any $p, p'$ from the neighborhood $U_0 := U \cap (q_0 + \operatorname{int} Y_+)$ of $p_0$, and for any $x \in V \cap B(p) \subset V \cap B(q_0)$, we have

$$d(x, B(p')) \le \mu[p' \cdot x - 1]_+ \le \mu|p' \cdot x - 1 - a|,$$

where $a := p \cdot x - 1 \le 0$. Hence,

$$\begin{aligned} d(x, B(p')) &\le \mu(|p' \cdot x - 1 - a| - |p \cdot x - 1 - a|) \\ &\le \mu|(p' \cdot x - 1 - a) - (p \cdot x - 1 - a)| \\ &= \mu|p' \cdot x - p \cdot x| \le \mu\alpha_{q_0}^{-1}\|p - p'\|. \end{aligned}$$

Moreover, since the function $u \mapsto \|x - u\|$ is convex and continuous (hence it is weakly lower semicontinuous by the Mazur Lemma [76, Theorems 3.15 and 3.12]) and the set $B(p')$ is weakly compact by Lemma 1.1, there exists $x' \in B(p')$ such that $\|x - x'\| \le \mu\alpha_{q_0}^{-1}\|p - p'\|$. Thus, we have

$$B(p) \cap V \subset B(p') + \mu\alpha_{q_0}^{-1}\|p - p'\|\bar{B}_X, \quad \forall p, p' \in U_0.$$

The Lipschitz-like property of $B(\cdot)$ at $(p_0, x_0)$ has been established. □

From Theorem 1.6 it follows that $B(\cdot)$ is i.s.c. at every $(p_0, x_0) \in \operatorname{gph} B$, where $p_0 \in \operatorname{int} Y_+$. Hence $B(\cdot)$ is l.s.c. at every $p_0 \in \operatorname{int} Y_+$ (see Section 2), provided that $Y_+$ and $X_+$ are considered with the strong topologies. This fact has been obtained in Proposition 1.1 for any $p_0 \in Y_+$.

**Theorem 1.7** *Suppose that $X$ is finite-dimensional and $u : X_+ \to \mathbb{R}$ is locally Lipschitz on $X_+$. Then the indirect utility function $v : Y_+ \to \overline{\mathbb{R}}$ is locally Lipschitz on* $\operatorname{int} Y_+$.

**Proof.** (Some arguments from [88, pp. 217–219] will be used in this proof.) Given a point $p_0 \in \operatorname{int} Y_+$, we have to prove that there exist $\gamma > 0$ and a neighborhood $U_0$ of $p_0$ such that

$$|v(p) - v(p')| \le \gamma\|p - p'\| \quad \forall p,\ p' \in U_0. \tag{1.18}$$

As it has been shown in the proof of Proposition 1.2, for any $r \in (0, 1)$ we have $q_0 := rp_0 \in \operatorname{int} Y_+$, $U := q_0 + \operatorname{int} Y_+$ is an open neighborhood of $p_0$, and $B(p) \subset B(q_0)$ for every $p \in U$. Hence, $B(q_0)$ is a compact set by Theorem 1.3 and $D(p)$ is nonempty for all $p \in U$ by Proposition 1.5. In addition, we have

$$\emptyset \ne D(p) \subset B(q_0), \quad \forall p \in U.$$

Since $u : X_+ \to \mathbb{R}$ is locally Lipschitz on $X_+$, for each $x \in B(q_0)$, there exist a neighborhood $V_x$ of $x$, and $l_x > 0$ satisfying

$$|u(x'') - u(x')| \leq l_x \|x'' - x'\|, \quad \forall x', x'' \in V_x. \tag{1.19}$$

The compactness of $B(q_0)$ implies that $B(q_0) \subset \bigcup_{i \in I} V_{x_i}$, with $I$ being a finite set. Taking any $x, x' \in B(q_0)$, we remark that the whole segment $[x, x']$ lies in $B(q_0)$ as the latter is convex. Consequently, there exists a sequence of points $a_0 := x, a_1, a_2, \ldots, a_s := x'$ of the segment $[x, x']$ such that for each index $j \in \{0, 1, \ldots, s-1\}$, there exists $i \in I$ satisfying $[a_j, a_{j+1}] \subset V_{x_i}$. Hence, for $l := \max\{l_{x_i} \ : \ i \in I\}$, by using (1.19) we have

$$\begin{aligned} |u(x) - u(x')| &\leq |u(a_0) - u(a_1)| + |u(a_1) - u(a_2)| + \cdots + |u(a_{s-1}) - u(a_s)| \\ &\leq l\|a_0 - a_1\| + l\|a_1 - a_2\| + \cdots + l\|a_{s-1} - a_s\| = l\|x - x'\|. \end{aligned}$$

For every $z \in B(p_0)$, since $B(\cdot) : Y_+ \rightrightarrows X_+$ is Lipschitz-like at $(p_0, z)$, there exist a neighborhood $U_z$ of $p_0$, a neighborhood $W_z$ of $z$, and $k_z > 0$ such that

$$B(p) \cap W_z \subset B(p') + k_z\|p - p'\| \quad \forall p, p' \in U_z. \tag{1.20}$$

Besides, as $D(p_0) \subset B(p_0)$ and $D(p_0)$ is nonempty and compact by Proposition 1.5, one can find a finite covering $\{W_{z_j}\}_{j \in J}$ of $D(p_0)$. Now, by the upper semicontinuity of $D(\cdot) : Y_+ \rightrightarrows X_+$ at $p_0$ (see Proposition 1.6) and by the inclusion $D(p_0) \subset \bigcup_{i \in J} W_{z_j}$, we can find a neighborhood $U_1$ of $p_0$ such that $D(p) \subset \bigcup_{i \in J} W_{z_j}$ for all $p \in U_1$. Set $U_0 = U \cap U_1 \cap \Big( \bigcap_{j \in J} U_{z_j} \Big)$. For every $p, p' \in U_0$, since $D(p) \neq \emptyset$, we can select an $x \in D(p)$. Let $j_0 \in J$ be such that $x \in W_{z_{j_0}}$. Applying (1.20) for $z = z_{j_0}$, we have

$$x \in D(p) \cap W_{z_{j_0}} \subset B(p) \cap W_{z_{j_0}} \subset B(p') + k_{z_{j_0}}\|p - p'\|\bar{B}_X.$$

Hence, there exists $x' \in B(p')$ satisfying $\|x - x'\| \leq k_{z_{j_0}}\|p - p'\|$. Moreover, since $p, p' \in U$, one has $B(p) \subset B(q_0)$ and $B(p') \subset B(q_0)$; so $x, x' \in B(q_0)$. Therefore, $|u(x) - u(x')| \leq l\|x - x'\|$. It follows that

$$u(x) - u(x') \leq |u(x) - u(x')| \leq l\|x - x'\| \leq l k_{z_{j_0}}\|p - p'\|. \tag{1.21}$$

As the inclusions $x \in D(p)$, $x' \in B(p')$ yield $u(x) = v(p)$ and $u(x') \leq v(p')$, from (1.21) we can deduce that $v(p) \leq v(p') + l k_{z_{j_0}}\|p - p'\|$. Similarly, we can show that $v(p') \leq v(p) + l k_{z_{j_0}}\|p - p'\|$. So, setting $\gamma = l k_{z_{j_0}}$, we get (1.18).

The proof is complete. □

## 1.5 Lipschitz-Hölder Property

We shall study the Lipschitz-Hölder property of the demand map by using the Lipschitz-like property of the budget map. The results from [86], which have been recalled in Section 1.2, will be the principal tools in our proofs.

Now, assume that $X$ is a Hilbert space, $M$ is a parameter set in a norm space, and $u : X_+ \times M \to \mathbb{R}$ is a utility function depending on the parameter $\mu \in M$. The appearance of $\mu$ signifies the fact that the utility function is subject to change, due to the changes of customs, the scale of values, time, etc. Consider the *parametric consumer problem*

$$\max \{u(x, \mu) \ : \ x \in B(p)\} \tag{1.22}$$

depending on a pair $(\mu, p) \in M \times Y_+$ where, as before, $B : Y_+ \rightrightarrows X_+$ is the budget map given by (1.2). It is clear that (1.22) is a generalization of (1.3). Indeed, if $M$ reduces to a singleton, then (1.22) coincides with (1.3).

In the sequel, it is assumed that there exists an open set $\Omega$ containing $X_+$ such that $u$ is defined on $\Omega \times M$ and, for each $\mu \in M$, $u(\cdot, \mu)$ is Fréchet differentiable at every point of $X_+$. By $\nabla_x u(x, \mu)$ we denote the Fréchet derivative of $u(\cdot, \mu)$ at $x \in X_+$. Let $x_0$ be a solution of (1.22) at a given pair of parameters $(\mu_0, p_0) \in M \times Y_+$. Suppose that there exist a closed and convex neighborhood $V$ of $x_0$, a neighborhood $W$ of $\mu_0$, and constants $\alpha > 0$, $\ell > 0$ satisfying

$$\|\nabla_x u(x', \mu') - \nabla_x u(x, \mu)\| \leq \ell(\|x' - x\| + \|\mu' - \mu\|), \ \forall x, x' \in V, \ \forall \mu, \mu' \in M \cap W \tag{1.23}$$

and

$$\langle \nabla_x(-u)(x', \mu) - \nabla_x(-u)(x, \mu), x' - x\rangle \geq \alpha\|x' - x\|^2, \ \forall x, x' \in V, \ \forall \mu \in M \cap W. \tag{1.24}$$

Condition (1.23) states that the map $\nabla_x u(\cdot) : X_+ \times M \to X$ is *locally Lipschitz* at $(x_0, \mu_0)$, while (1.24) requires that $\nabla_x(-u)(\cdot, \mu)$ is *locally strongly monotone* on $V$ uniformly w.r.t. $\mu \in M \cap W$. The latter is equivalent [81] to the requirement that $(-u)(\cdot, \mu)$ is *locally strongly convex* on $V$ uniformly w.r.t. $\mu \in M \cap W$, i.e., there exists $\alpha > 0$ such that

$$(-u)((1-t)x + tx', \mu) \leq (1-t)(-u)(x, \mu) + t(-u)(x', \mu) - \frac{1}{2}\alpha t(1-t)\|x' - x\|^2 \tag{1.25}$$

for all $x, x' \in V$, $\mu \in M \cap W$. According [81, Lemma 1, p. 184], (1.25) is valid if and only if, for all $\mu \in M \cap W$, the function $x \mapsto (-u)(x,\mu) - \frac{\alpha}{2}\|x\|^2$ is convex on $V$.

**Theorem 1.8** *Assume that, for every $\mu \in M$, the function $u(\cdot,\mu)$ is concave on $X_+$ and the operator $\nabla_x(-u)(\cdot,\mu) : X_+ \to X^*$ is continuous, where the dual space $X^*$ is considered with the weak topology. Suppose that $x_0$ is a solution to the parametric consumer problem* (1.22) *with respect to a given pair of parameters $(\mu_0, p_0) \in M \times \operatorname{int} Y_+$ and conditions* (1.23), (1.24) *are satisfied. Then, there exist constants $\kappa_{\mu_0} > 0$, $\kappa_{p_0} > 0$, and neighborhoods $W_1$ of $\mu_0$, $U_1$ of $p_0$ such that*

(a) *For every $(\mu, p) \in (M \cap W_1) \times (Y_+ \cap U_1)$,* (1.22) *has a unique solution, denoted by $x(\mu,p)$, and $x(\mu,p) \in \operatorname{int} V$;*

(b) *For all $(\mu', p'), (\mu, p) \in (M \cap W_1) \times (Y_+ \cap U_1)$,*

$$\|x(\mu',p') - x(\mu,p)\| \le \kappa_{\mu_0}\|\mu' - \mu\| + \kappa_{p_0}\|p' - p\|^{1/2}. \tag{1.26}$$

**Proof.** First, let us show that the parametric consumer problem (1.22) is equivalent to a parametric variational inequality of the form (1.11). To do so, choose $\Lambda = Y_+$ and write $p$ for $\lambda$. Put $f(x,\mu) = \nabla_x(-u)(x,\mu)$, $K(p) = B(p)$ for every $(x,\mu) \in X_+ \times M$ and $p \in Y_+$. Fix any $(\mu,p) \in M \times \operatorname{int} Y_+$. Let $\bar{x}$ be a solution to (1.22). For every $x \in B(p)$ and $t \in (0,1)$, set $x_t = (1-t)\bar{x} + tx$. Since $B(p)$ is convex, $x_t \in B(p)$ for all $t \in (0,1)$. Hence,

$$u(\bar{x},\mu) \ge u(x_t,\mu) = u((1-t)\bar{x} + tx, \mu) = u(\bar{x} + t(x - \bar{x}), \mu), \quad \forall t \in (0,1).$$

It follows that

$$\frac{(-u)(\bar{x} + t(x-\bar{x}),\mu) - (-u)(\bar{x},\mu)}{t} \ge 0, \quad \forall t \in (0,1).$$

Letting $t \to 0^+$ and using the Fréchet differentiability of $u(\cdot,\mu)$ at $\bar{x}$, we obtain

$$\langle \nabla_x(-u)(\bar{x},\mu), x - \bar{x}\rangle \ge 0, \quad \forall x \in B(p);$$

so $\bar{x}$ is a solution of (1.11) (with the above-defined $f$, $K$, $\Lambda$, and $\lambda$ being replaced by $p$). Conversely, let $\bar{x}$ be a solution of the latter problem (1.11). By the convexity of $B(p)$ and the concavity of $u(\cdot,\mu)$,

$$u((1-t)\bar{x} + tx, \mu) \ge (1-t)u(\bar{x},\mu) + tu(x,\mu), \quad \forall x \in B(p), \forall t \in (0,1).$$

Equivalently,

$$u(\bar{x},\mu) - u(x,\mu) \geq \frac{(-u)(\bar{x}+t(x-\bar{x}),\mu) - (-u)(\bar{x},\mu)}{t}$$

for all $x \in B(p)$ and $t \in (0,1)$. Hence, by letting $t \to 0^+$, we obtain

$$u(\bar{x},\mu) - u(x,\mu) \geq \langle \nabla_x(-u)(\bar{x},\mu), x - \bar{x}\rangle, \quad \forall x \in B(p).$$

Since $\bar{x}$ is a solution of the problem (1.11) (with the above-defined $f$, $K$, $\Lambda$, and $\lambda$ being replaced by $p$), the latter shows that $\bar{x}$ is a solution of (1.22).

Now, let $x_0$ be a solution to (1.22) for $(\mu,p) = (\mu_0,p_0) \in M \times \operatorname{int} Y_+$. Since $p_0 \in \operatorname{int} Y_+$ and $x_0 \in B(p_0)$, the budget map $B : Y_+ \rightrightarrows X_+$ is Lipschitz-like at $(p_0,x_0)$ by Theorem 1.6. Besides, the assumptions (1.23) and (1.24) make the requirements (1.12) and (1.13) on $f(x,\mu) = \nabla_x(-u)(x,\mu)$ be fulfilled with $\bar{x} := x_0, \bar{\mu} := \mu_0, \bar{\lambda} := p_0$. Hence, according to Theorem 1.2, there exist constants $\kappa_{\mu_0} > 0$, $\kappa_{p_0} > 0$, and neighborhoods $W_1$ of $\mu_0$, $U_1$ of $p_0$ such that

(a') *For every* $(\mu,p) \in (M \cap W_1) \times (Y_+ \cap U_1)$, (1.11) *has a unique solution, denoted by* $x(\mu,p)$, *in* $V$

and (1.26) holds. It remains to prove that (a') implies (a). The property $x(\mu,p) \in \operatorname{int} V$ for all $(\mu,p) \in (M \cap W_1) \times (Y_+ \cap U_1)$ has been established in [86, formula (2.17)]. By the equivalence between (1.11) and (1.22), this vector $x(\mu,p)$ is a solution of (1.22). To show that it is the unique solution of (1.22), suppose on the contrary that the problem has another solution $\widetilde{x}(\mu,p)$, which also is a solution of (1.11). By (a'), $\widetilde{x}(\mu,p) \notin V$. As the function $u(\cdot,\mu)$ is concave on $X_+$, the operator $\nabla_x(-u)(\cdot,\mu) : X_+ \to X^*$ is monotone (see the characterization of convexity of a function in [81] and note the proof is valid for a Hilbert space setting). By virtue of this monotonicity and the assumed continuity of $\nabla_x(-u)(\cdot,\mu)$, we can apply the Minty Lemma [48, Lemma 1.5 in Chap. III] to assert that the solution of (1.11) coincides with the solution set of the following *Minty variational inequality*:

Find $x \in B(p)$ such that $\langle \nabla_x(-u)(y,\mu), y - x\rangle \geq 0$ for all $y \in B(p)$.

Hence, the solution set of (1.11) is the intersection of the closed and convex sets

$$\{x \in B(p) \ : \ \langle \nabla_x(-u)(y,\mu), y - x\rangle \geq 0\}, \quad y \in B(p).$$

It follows that the solution set of (1.11) is closed and convex; so the solution set of (1.22) is closed and convex. In particular, the whole line segment

$[x(\mu,p),\widetilde{x}(\mu,p)]$ lies in that solution set. Since $x(\mu,p)\in \operatorname{int} V$, (1.22) must have some solutions in $V$ which are different from $x(\mu,p)$. We have arrived at a contradiction. The solution uniqueness of the problem (1.22) for all $(\mu,p)\in (M\cap W_1)\times(Y_+\cap U_1)$ together with the implication "(a') ⇒ (a)" has been proved. □

Clearly, (1.26) implies that $x(\mu,p)$ is continuous at $(\mu_0,p_0)$.

**Remark 1.3** In the above proof, the fact saying that if $\bar{x}$ is a solution to the optimization problem (1.22) then it is a solution to a parametric variational inequality of the form (1.11) is not new. In fact, it is the content of the celebrated *generalized Fermat rule* (see [48, Proposition 5.1 in Chap. I] and note that the proof is effective for a Banach space setting). The converse statement, which holds under the convexity (or pseudo-convexity) of $(-u)(\cdot,\mu)$ is also well-known.

We now give an illustrative example for Theorem 1.8.

**Example 1.4** Let $X=\mathbb{R}^n$, $X_+=\mathbb{R}^n_+$, where $\mathbb{R}^n_+$ denotes the nonnegative orthant of $\mathbb{R}^n$, and let

$$M=\{\mu=(\mu_1,\dots,\mu_n)\in\mathbb{R}^n\ :\ \mu_i\in(0,1),\ i=1,\dots,n\}.$$

One has $Y_+=\mathbb{R}^n_+$. Given positive constants $\varepsilon_i>0$, $i=1,\dots,n$, we consider the parametric utility function

$$u(x,\mu):=\sum_{i=1}^{n}\mu_i\ln(x_i+\varepsilon_i).$$

(Since $\exp u(x,\mu)=(x_1+\varepsilon_1)^{\mu_1}\dots(x_n+\varepsilon_n)^{\mu_n}$ is a *modified Cobb-Douglas function*, this utility function $u(x,\mu)$ is the image of the latter after the logarithmic transformation.) Setting $\Omega=\{x\in\mathbb{R}^n\ :\ x_i>-\varepsilon_i,\ i=1,\dots,n\}$, we see that $\Omega$ contains $X_+$, $u$ is defined on $\Omega\times M$ and, for each $\mu\in M$, $u(\cdot,\mu)$ is Fréchet differentiable at every point of $X_+$. For a given pair of parameters $(\mu_0,p_0)\in M\times Y_+$, denote by $x_0$ the unique solution of (1.22), where $(\mu_0,p_0)$ plays the role of $(\mu,p)$. The existence of a closed and convex neighborhood $V$ of $x_0$, a neighborhood $W$ of $\mu_0$, and constants $\alpha>0$, $\ell>0$ satisfying (1.23) and (1.24) follows from the fact that $u$ is twice continuously differentiable on $\Omega\times M$ and [81, Theorem 4, p. 185]. Thus, all the assumptions of Theorem 1.8 are satisfied. It follows that there are constants $\kappa_{\mu_0}>0$, $\kappa_{p_0}>0$, and neighborhoods $W_1$ of $\mu_0$, $U_1$ of $p_0$ such that the claims (a) and (b) are valid.

Note that the technical assumptions (1.23) and (1.24) play a decisive role for Theorem 1.8. We now provide an example of a general utility function which fulfills both assumptions.

**Example 1.5** In practice, it may happen that, for a good bundle $x$, its coordinates represent amounts of distinct goods having different levels of importance for the consumer. The utilities of different groups of goods are not the same for the latter. So, different utilities functions are applied to different groups of goods, and the combined utility is the sum of the sectional utilities. These ideas lead to the following model of consumption. Let $X_1, X_2$ be Hilbert spaces, $(X_1)_+ \subset X_1$ and $(X_2)_+ \subset X_2$ be nonempty, closed and convex cones. Let $\Omega_1 \subset X_1$ and $\Omega_2 \subset X_2$ be open sets containing $(X_1)_+$ and $(X_2)_+$, respectively. Clearly, $X := X_1 \times X_2$ is a Hilbert space with the inner product

$$\langle x, y\rangle := \langle x_1, y_1\rangle + \langle x_2, y_2\rangle, \quad \forall x = (x_1, x_2), y = (y_1, y_2) \in X,$$

where the right-hand side is the sum of the inner products in $X_1$ and $X_2$. Put $X_+ = (X_1)_+ \times (X_2)_+$ and $\Omega = \Omega_1 \times \Omega_2$. The positive dual cone of $X_+$ is given by $Y_+ := (Y_1)_+ \times (Y_2)_+$, where $(Y_i)_+$ is the positive dual cone of $(X_i)_+$, $i = 1, 2$. Let $M$ be a parameter set in some norm space. Suppose that $u_i : \Omega_i \times M \to \mathbb{R}$, $i = 1, 2$, are such that for each $\mu \in M$, $u_i(\cdot, \mu)$ is concave and Fréchet differentiable on $(X_i)_+$, $i = 1, 2$. Consider the function $u : \Omega \times M \to \mathbb{R}$ mapping each $x = (x_1, x_2) \in \Omega$ to $u_1(x_1, \mu) + u_2(x_2, \mu)$. Then, for any $\mu \in M$, $u(\cdot, \mu)$ is concave and Fréchet differentiable on $X_+$. Besides,

$$\nabla_x u(\bar{x}, \bar{\mu}) = (\nabla_{x_1} u_1(\bar{x}_1, \bar{\mu}), \nabla_{x_2} u_2(\bar{x}_2, \bar{\mu})), \quad \forall \bar{x} = (\bar{x}_1, \bar{x}_2) \in X_+, \bar{\mu} \in M.$$

Take a pair of parameters $(\bar{\mu}, \bar{p}) \in M \times Y_+$ and suppose that $\bar{x}$ is a unique solution of (1.22), where $(\bar{\mu}, \bar{p})$ plays the role of $(\mu, p)$. In addition, suppose that both functions $u_1, u_2$ satisfy the conditions (1.23) and (1.24) (see Example 1.4 for some functions of this type). It is easy to show that those conditions are also valid for $u$.

**Remark 1.4** In Theorem 1.8, one can replace the assumption "for every $\mu \in M$, the function $u(\cdot, \mu)$ is concave on $X_+$ and $\nabla_x(-u)(\cdot, \mu) : X_+ \to X^*$ is continuous, where $X^*$ is considered with the weak topology" with the following one: "for every $\mu \in M$, the function $(-u)(\cdot, \mu)$ is pseudo-convex on $X_+$ and the operator $\nabla_x(-u)(\cdot, \mu) : X_+ \to X^*$ is hemicontinuous, where $X^*$ is considered with the weak topology". The proof of the theorem remains the same. The interested reader is referred to [31, 83–85] for the definitions of hemicontinuity of function, pseudo-convex function, pseudo-monotone operator, relationships between the pseudo-convexity of functions and the pseudo-monotonicity of their gradient mappings, and the generalized Minty lemma, which works for variational inequalities with pseudo-monotone and hemicontinuous operators on reflexive Banach spaces.

**Remark 1.5** Note that there are some results on the local Lipschitz property of the demand map provided that it is single-valued. For instance, such a result is given by Beggs in [13, Lemma 3 and the subsequent comments] under special assumptions including the following: $X = \mathbb{R}^n$, $X_+$ is given by finitely linear constraints, and $u$ is twice continuously differentiable, strictly concave in the neighborhood of $x_0$.

## 1.6 Some Economic Interpretations

Going back to the consumer problem (1.3), we see that the budget map $B(\cdot)$ is l.s.c. at $\bar{p} \in Y_+$ if and only if for any $\bar{x} \in B(\bar{p})$ and for any open set $V \subset X$ satisfying $\bar{x} \in V$, there exists a neighborhood $U$ of $\bar{p}$ such that $B(p) \cap V \neq \emptyset$ for every $p \in U \cap Y_+$. This means that although the *good bundle* $\bar{x}$ which the consumer can buy when the price is $\bar{p}$ may not be purchased if the price of goods is shifted to the value $p \in U \cap Y_+$ (this happens when $\bar{x} \notin B(p)$) but, in the given neighborhood $V \cap X_+$ of $\bar{x}$, there still exist some goods (namely the elements of $B(p) \cap V$) that the consumer can afford to buy, using his fixed budget. Similar interpretations are valid for the l.s.c. property of the demand map $D(\cdot)$ at $\bar{p} \in Y_+$, where $D(\bar{p})$ (resp., $D(p)$) is the collection of *the optimal good bundles* for the consumer when the price is $\bar{p}$ (resp., when the price is $p$). Thus, the l.s.c. property of $B(\cdot)$ (resp., of $D(\cdot)$) at $\bar{p}$ signifies the fact that every element $\bar{x} \in B(\bar{p})$ (resp., $\bar{x} \in D(\bar{p})$) is *persistent* to the changes of the price $\bar{p}$. In contrast to the l.s.c. property, the budget map $B(\cdot)$ is u.s.c. at $\bar{p} \in Y_+$ if and only if for any open set $V \subset X$ with $B(\bar{p}) \subset V$, one can find a neighborhood $U$ of $\bar{p}$ such that $B(p) \subset V$ for every $p \in U \cap Y_+$. The meaning of the latter is the following: If the reference price is perturbed slightly (in the sense that $\bar{p}$ is shifted to another price $p \in U \cap Y_+$), then the whole set of good bundles that the consumer can purchase must lie within the given open set $V$. Thus, for any price $p \in U \cap Y_+$, no good bundles outside $V$ can be bought by the consumer. Similar interpretations are valid for the u.s.c. property of the demand map $D(\cdot)$ at $\bar{p} \in Y_+$.

The continuity of the function $v(\cdot) : Y_+ \to \overline{\mathbb{R}}$ at $\bar{p} \in Y_+$ in the strong topology means that if the norm $\|p - \bar{p}\|$, $p \in Y_+$, tends to 0, then the absolute value $|v(p) - v(\bar{p})|$ also tends to 0. Hence, if the function $v(\cdot)$ is *discontinuous* at the price $\bar{p}$, then there must exists $\varepsilon > 0$ and a sequence of prices $\{p_k\} \subset Y_+$, $p_k \to \bar{p}$ as $k \to \infty$, such that $|v(p_k) - v(\bar{p})| \geq \varepsilon$ for all $k$. Thus, in any neighborhood of $\bar{p}$ there exist prices at which the indirect utility differs greatly from $v(\bar{p})$. Such a price $\bar{p}$ is *abnormal* as it should be treated by the model analyst with a care. If $v(\cdot)$ is not only continuous, but also *locally Lipschitz* at $\bar{p}$ (see Theorem 1.7 and formula (1.7) in Section 4), then the magnitude $|v(p) - v(p')|$ of the change of the indirect utilities is bounded by the magnitude $\|p - p'\|$ of the change in the prices multiplied by a fixed constant, i.e., the ratio $\dfrac{|v(p) - v(p')|}{\|p - p'\|}$ is bounded above by a constant for all $p, p' \in Y_+$ from a neighborhood of $\bar{p}$. In other words, *the rate of the change of the indirect utility w.r.t. the change of prices around $\bar{p}$ can be controlled.* Similar interpretations can be made for the Lipschitz-like property of the map $B(\cdot)$ (resp., the map $D(\cdot)$) at a point $(\bar{p}, \bar{x})$ in the graph of $B(\cdot)$ (resp., of $D(\cdot)$).

The demand map $D : M \times Y_+ \rightrightarrows X_+$ with its values

$$D(\mu, p) := \{x \in B(p) \ : \ u(x, \mu) = v(\mu, p)\}$$

where, for each $\mu \in M$, $v(\mu, \cdot)$ is the indirect utility function of the problem (1.22) with $p \in Y_+$ being subject to change, is aslo called the *Walrasian or Marshallian demand correspondence* (see, e.g., [32]), and plays a crucial role in microeconomics. Economists are always interested in the sensitivity of the vector $x(\mu, p)$ demanded to price $p$ at the perturbation $\mu$ of the utility function, provided that $D(\mu, p)$ is a singleton for each pair $(\mu, p) \in M \times Y_+$. For some goods, a small change in price results in a significant change in quantities demanded. Meanwhile, for other goods, a big change in price results in a little bit change in quantities demanded. To estimate such sensitivity, they use a measure called the *price elasticity of demand* (see, e.g., [3, p. 92]). It is defined to be the *percentage change in the quantities demanded resulting from a 1 percent change in price* and denoted by $\varepsilon$. More precisely, if $D(\mu, \cdot)$ is single-valued around a price $p_0$, and Fréchet differentiable at that point, then one has (see, e.g., [3, p .92])

$$\varepsilon(\mu, p_0) = \frac{p_0}{x(\mu, p_0)}.x'_p(\mu, p_0),$$

where $x'_p(\mu, p_0)$ denotes the partial derivative of $x(\mu, p)$ w.r.t. $p$ at $(\mu, p_0)$. However, if $D(\mu, \cdot)$ is not Fréchet differentiable at $p_0$ or we do not know explicitly about the demand function $D(\cdot)$, then (1.26) gives us a type of sensitivity estimate, which can replace the price elasticity of demand.

## 1.7 Conclusions

We have studied various stability properties and a result on solution sensitivity of a consumer problem in this chapter. Especially, by focusing on some nice features of the budget map, we have been able to establish the continuity and the Lipschitz continuity of the indirect utility function, as well as the Lipschitz-Hölder continuity of the demand map under a minimal set of assumptions. We have also presented some economic interpretations for mathematical concepts and property involving directly to the consumer problem: the continuity, Lipschitz continuity, and Lipschitz-Hölder continuity.

# Chapter 2

# Differential Stability of Parametric Consumer Problems

Written on the basis of the paper [36], the present chapter is devoted to differential stability of the parametric consumer problem considered in the preceding chapter. The differential stability here appears in the form of formulas for computing the Fréchet/limitting coderivatives of the budget map; the Fréchet/limitting subdifferentials of the infimal nuisance function (which is obtained from the indirect utility function by changing its sign), upper and lower estimates for the upper and the lower Dini directional derivatives of the indirect utility function. In addition, a new result on the Lipschitz-like property of the budget map is also given in this chapter.

## 2.1 Auxiliary Concepts and Results

In this section, it is assumed that $X, Y$ are Banach spaces.

Let $F : X \rightrightarrows Y$ be a set-valued map. The *graph* of $F$ is the set

$$\operatorname{gph} F := \{(x, y) \in X \times Y \ : \ y \in F(x)\}.$$

If $\operatorname{gph} F$ is closed (resp., convex) in $X \times Y$, then $F$ is said to be *closed* (resp., *convex*). Here, the norm in the product space $X \times Y$ is given by $\|(x, y)\| = \|x\| + \|y\|$ with the first norm in the right-hand side denoting the norm in $X$ and the second one standing for the norm in $Y$. For an extended real-valued function $\varphi : X \to \overline{\mathbb{R}}$, the *epigraph* and *hypograph* of $\varphi$ are defined respectively by

$$\operatorname{epi}\varphi = \{(x,\mu) \in X \times \mathbb{R} \,:\, \mu \geq \varphi(x)\}$$

and

$$\operatorname{hypo}\varphi = \{(x,\mu) \in X \times \mathbb{R} \,:\, \mu \leq \varphi(x)\}.$$

For a set-valued map $F : X \rightrightarrows X^*$, the notation

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* \,:\, \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{\omega^*} x^* \\ \text{with } x_k^* \in F(x_k) \text{ for all } k \in \mathbb{N}\Big\}$$

signifies the *sequential Painlevé-Kuratowski upper/outer limit* w.r.t. the norm topology of $X$ and the weak* topology (that denoted by $\omega^*$) of $X^*$. For a function $\varphi : X \to \overline{\mathbb{R}}$ and a set $\Omega \subset X$, the notations $x \xrightarrow{\varphi} \bar{x}$ and $x \xrightarrow{\Omega} \bar{x}$, respectively, mean $x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$ and $x \to \bar{x}$ with $x \in \Omega$.

We now recall several basic concepts of generalized differentiation. For more details, the reader is referred to [56, 57].

Let $\Omega$ be a subset of $X$. Given $x \in \Omega$ and $\varepsilon \geq 0$, define the set of *$\varepsilon$-normals* to $\Omega$ at $x$ by

$$\widehat{N}_\varepsilon(x;\Omega) := \Big\{x^* \in X^* \,:\, \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \leq \varepsilon\Big\}. \tag{2.1}$$

When $\varepsilon = 0$, elements of (2.1) are called *Fréchet normals* and their collection, denoted by $\widehat{N}(x;\Omega)$, is the *prenormal cone* or *Fréchet normal cone* to $\Omega$ at $x$. If $x \notin \Omega$, we put $\widehat{N}_\varepsilon(x;\Omega) = \emptyset$ for all $\varepsilon \geq 0$. Consider a vector $\bar{x} \in \Omega$. One say that $x^* \in X^*$ is a *limiting/Mordukhovich normal* to $\Omega$ at $\bar{x}$ if there are sequences $\varepsilon_k \downarrow 0, x_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* \xrightarrow{\omega^*} x^*$ such that $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k;\Omega)$ for all $k \in \mathbb{N}$. The collection of such normals

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{x\to\bar{x}\ \ \varepsilon\downarrow 0} \widehat{N}_\varepsilon(x;\Omega)$$

is the *limiting/Mordukhovich normal cone* to $\Omega$ at $x$. Put $N(\bar{x};\Omega) = \emptyset$ for $\bar{x} \notin \Omega$.

Let $F : X \rightrightarrows Y$ be a set-valued map. Given $(\bar{x},\bar{y}) \in X \times Y$ and $\varepsilon \geq 0$, the *$\varepsilon$-coderivative* of $F$ at $(\bar{x},\bar{y})$ is the set-valued map $\widehat{D}^*_\varepsilon F(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$ with the values

$$\widehat{D}^*_\varepsilon F(\bar{x},\bar{y})(y^*) := \Big\{x^* \in X^* \,:\, (x^*, -y^*) \in \widehat{N}_\varepsilon((\bar{x},\bar{y}); \operatorname{gph} F)\Big\}. \tag{2.2}$$

When $\varepsilon = 0$ in (2.2), this construction is called the *precoderivative* or *Fréchet coderivative* of $F$ at $(\bar{x}, \bar{y})$ and is denoted by $\widehat{D}^*F(\bar{x}, \bar{y})$. The set-valued map $D^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$, which is defined by

$$D^*F(\bar{x}, \bar{y})(y^*) = \{x^* \in X^* \ : \ (x^*, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{gph} F)\},$$

is said to be the *limiting/Mordukhovich coderivative* of $F$ at $(\bar{x}, \bar{y})$. It follows from the definition that $\widehat{D}^*_\varepsilon F(\bar{x}, \bar{y})(y^*) = D^*F(\bar{x}, \bar{y})(y^*) = \emptyset$ for all $\varepsilon \geq 0$ and $y^* \in Y^*$ if $(\bar{x}, \bar{y}) \notin \operatorname{gph} F$. We shall omit $\bar{y}$ in the coderivative notation if $F(\bar{x}) = \{\bar{y}\}$. It follows from above definitions that

$$\widehat{D}^*F(\bar{x}, \bar{y})(y^*) \subset D^*F(\bar{x}, \bar{y})(y^*) \tag{2.3}$$

for any $y^* \in Y^*$. There are examples showing that inclusions in (2.3) is strict (see [56, p. 43]). When (2.3) holds as equality, $F$ is said to be *graphically regular* at $(\bar{x}, \bar{y})$. As shown in the following propositions, the class of graphically regular mappings includes convex set-valued maps and strictly differentiable functions. The reader is referred to [56, Def. 1.13] for the definition of strictly differentiable function.

**Proposition 2.1** (See [56, Prop. 1.37]) *Let $F : X \rightrightarrows Y$ be a convex set-valued map. Then $F$ is graphically regular at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and one has the codervative representations*

$$\begin{aligned}\widehat{D}^*F(\bar{x}, \bar{y})(y^*) &= D^*F(\bar{x}, \bar{y})(y^*) \\ &= \Big\{x^* \in X^* \ : \ \langle x^*, \bar{x}\rangle - \langle y^*, \bar{y}\rangle = \max_{(x,y)\in \operatorname{gph} F}[\langle x^*, x\rangle - \langle y^*, y\rangle]\Big\}.\end{aligned}$$

**Proposition 2.2** (See [56, Prop. 1.38]) *Let $f : X \to Y$ be Fréchet differentiable at $\bar{x}$. Then*

$$\widehat{D}^*f(\bar{x})(y^*) = \{\nabla f(\bar{x})^* y^*\}, \quad \forall y^* \in Y^*,$$

*where $\nabla f(\bar{x})^* : Y^* \to X^*$ denotes the adjoint operator of the Fréchet derivative $\nabla f(\bar{x}) : X \to Y$. If, moreover, $f$ is strictly differentiable at $\bar{x}$, then*

$$D^*f(\bar{x})(y^*) = \widehat{D}^*f(\bar{x})(y^*) = \{\nabla f(\bar{x})^* y^*\}, \quad \forall y^* \in Y^*,$$

*and thus $f$ is graphically regular at $\bar{x}$.*

A set $\Omega \subset X$ is said to be *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if for any sequence $(\varepsilon_k, x_k, x_k^*) \in [0, \infty) \times \Omega \times X^*$ satisfying

$$\varepsilon_k \downarrow 0, \ \ x_k \to \bar{x}, \ \ x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega), \ \text{ and } x_k^* \xrightarrow{\omega^*} 0$$

one has $\|x_k^*\| \to 0$ as $k \to \infty$. A map $F : X \rightrightarrows Y$ is *sequentially normally compact* (SNC) at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if its graph is SNC at $(\bar{x}, \bar{y})$. Let $\Omega_1$ and $\Omega_2$ be respectively subsets of Banach spaces $X_1$ and $X_2$ and $(x_1, x_2) \in \Omega_1 \times \Omega_2$. It follows from definition of SNC of sets and properties of the normal cone to a Cartesian product (see [56, Proposition 1.2]) that $\Omega_1 \times \Omega_2$ is SNC at $(x_1, x_2)$ if $\Omega_1$ is SNC at $x_1$ and $\Omega_2$ is SNC at $x_2$.

One says that $\Omega \subset X$ is *locally closed* around $\bar{x} \in \Omega$ if there is a neighborhood $U$ of $\bar{x}$ for which $\Omega \cap U$ is closed. If the set $\operatorname{gph} F$ of some set-valued map $F$ is locally closed around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, then $F$ is said to be *locally closed* around $(\bar{x}, \bar{y})$. Clearly, if $\operatorname{gph} F$ is a closed set, then $F$ is locally closed around any point belonging to its graph.

Let $\varphi : X \to \overline{\mathrm{IR}}$ be an extended real-valued function. The *Fréchet subdifferential*, *limiting/Mordukhovich subdifferential*, and the *singular subdifferential* of $\varphi$ at $\bar{x} \in X$ with $|\varphi(\bar{x})| < \infty$ are defined, respectively, by

$$\widehat{\partial}\varphi(\bar{x}) := \left\{x^* \in X^* \ : \ (x^*, -1) \in \widehat{N}\big((\bar{x}, \varphi(\bar{x})); \operatorname{epi}\varphi\big)\right\},$$

$$\partial\varphi(\bar{x}) := \left\{x^* \in X^* \ : \ (x^*, -1) \in N\big((\bar{x}, \varphi(\bar{x})); \operatorname{epi}\varphi\big)\right\},$$

and

$$\partial^\infty\varphi(\bar{x}) := \left\{x^* \in X^* \ : \ (x^*, 0) \in N\big((\bar{x}, \varphi(\bar{x})); \operatorname{epi}\varphi\big)\right\}.$$

If $|\varphi(x)| = \infty$, then one puts $\widehat{\partial}\varphi(\bar{x}) = \partial\varphi(\bar{x}) = \partial^\infty\varphi(\bar{x}) = \emptyset$. It follows from above definitions that for all $\bar{x} \in X$, one has $\widehat{\partial}\varphi(\bar{x}) \subset \partial\varphi(\bar{x})$. When this holds as equality, one says that $\varphi$ is *lower regular* at $\bar{x}$. The *Fréchet upper subdifferential*, *limiting/Mordukhovich upper subdifferential*, and *singular upper subdifferential* of $\varphi$ at $\bar{x}$ are respectively defined by

$$\widehat{\partial}^+\varphi(\bar{x}) := -\widehat{\partial}(-\varphi)(\bar{x}), \quad \partial^+\varphi(\bar{x}) := -\partial(-\varphi)(\bar{x}),$$

and

$$\partial^{\infty,+}\varphi(\bar{x}) := -\partial^\infty(-\varphi)(\bar{x}).$$

When $\varphi(\bar{x})$ is finite, it follows from definition of the singular subdifferential and the singular upper subdifferential that $\partial^\infty\varphi(\bar{x})$ and $\partial^{\infty,+}\varphi(\bar{x})$ always contain zero, while (see [56, Corollary 1.81]) $\partial^\infty\varphi(\bar{x}) = \{0\}$ (and therefore, $\partial^{\infty,+}\varphi(\bar{x}) = \{0\}$) if $\varphi$ is *locally Lipschitz* around $\bar{x}$, i.e., there is a neighborhood $U$ of $\bar{x}$ and a constant $\ell \geq 0$ such that

$$|\varphi(x) - \varphi(x')| \leq \ell\|x - x'\|, \quad \forall x,\ x' \in U.$$

We will need two theorems from [58] on parametric generalized equations, which are recalled now.

Let $f : X \times Y \to Z$ be a single-valued map, and $Q : X \times Y \rightrightarrows Z$ be a set-valued map between Banach spaces $X, Y, Z$. Consider a *parametric generalized equation*

$$0 \in f(x, y) + Q(x, y) \tag{2.4}$$

with the decision variable $y$ and the parameter $x$. The *solution map* to (2.4) is the set-valued map given by

$$S(x) := \{y \in Y \ : \ 0 \in f(x, y) + Q(x, y)\}, \quad x \in X. \tag{2.5}$$

The limiting coderivative of the solution map (2.5) can be estimated or computed in term of the initial data of (2.4) by using the following result.

**Theorem 2.1** (See [58, Theorem 4.1]) *Suppose that $X, Y, Z$ are Asplund spaces, $(\bar{x}, \bar{y})$ satisfy (2.4), $f$ is continuous around $(\bar{x}, \bar{y})$, and $Q$ is locally closed around $(\bar{x}, \bar{y}, \bar{z})$ with $\bar{z} := -f(\bar{x}, \bar{y})$. If $Q$ is SNC at $(\bar{x}, \bar{y}, \bar{z})$, and*

$$[(x^*, y^*) \in D^*f(\bar{x}, \bar{y})(z^*) \cap (-D^*Q(\bar{x}, \bar{y}, \bar{z})(z^*))] \Longrightarrow (x^*, y^*, z^*) = (0, 0, 0), \tag{2.6}$$

*then the inclusion*

$$D^*S(\bar{x}, \bar{y})(y^*) \subset \big\{x^* \in X^* \ : \exists z^* \in Z^* \text{ with } (x^*, -y^*) \in D^*f(\bar{x}, \bar{y})(z^*) + D^*Q(\bar{x}, \bar{y}, \bar{z})(z^*)\big\} \tag{2.7}$$

*holds for every $y^* \in Y^*$. If, in addition, $f$ is strictly differentiable at $(\bar{x}, \bar{y})$ and $Q$ is graphically regular at $(\bar{x}, \bar{y}, \bar{z})$, then $S$ is graphically regular at $(\bar{x}, \bar{y})$ and (2.7) holds as equality.*

The next theorem states a necessary and sufficient condition for Lipschitz-like property of the solution map (2.5).

**Theorem 2.2** (See [58, Theorem 4.2]) *Let $X, Y, Z$ be Asplund spaces and let $(\bar{x}, \bar{y})$ satisfy (2.4). Suppose that $f$ is strictly differentiable at $(\bar{x}, \bar{y})$ and that $Q$ is locally closed around $(\bar{x}, \bar{y}, \bar{z})$ with $\bar{z} := -f(\bar{x}, \bar{y})$, graphically regular and SNC at this point. If*

$$\big[(0, 0) \in \nabla f(\bar{x}, \bar{y})^* z^* + D^*Q(\bar{x}, \bar{y}, \bar{z})(z^*)\big] \Longrightarrow z^* = 0, \tag{2.8}$$

*then $S$ is Lipschitz-like at $(\bar{x}, \bar{y})$ if and only if*

$$\big[(x^*, 0) \in \nabla f(\bar{x}, \bar{y})^* z^* + D^*Q(\bar{x}, \bar{y}, \bar{z})(z^*)\big] \Longrightarrow [x^* = 0,\ z^* = 0]. \tag{2.9}$$

Following [60], we consider the *parametric optimization problem*

$$\min\{\varphi(x,y) \ : \ y \in G(x)\}, \tag{2.10}$$

where $\varphi : X \times Y \to \overline{\mathrm{IR}}$ is a *cost* function, $G : X \rightrightarrows Y$ is a *constraint* set-valued map between Banach spaces. The *marginal function* or the *optimal value function* $\mu(\cdot) : X \to \overline{\mathrm{IR}}$ and the *solution map* $M(\cdot) : X \rightrightarrows Y$ of this problem are defined, respectively, by

$$\mu(x) := \inf\{\varphi(x,y) \ : \ y \in G(x)\},$$

and

$$M(x) := \{y \in G(x) \ : \ \mu(x) = \varphi(x,y)\}\,.$$

The forthcoming theorem gives an upper estimate for the Fréchet subdifferential of the general marginal function $\mu(\cdot)$ at a given point $\bar{x}$ via the Fréchet coderivative of the constraint mapping $G$ and the Fréchet upper subdifferential of the value function $\varphi$.

**Theorem 2.3** (See [60, Theorem 1]) *Let $\bar{x}$ be such that $M(\bar{x}) \neq \emptyset$ and $|\mu(\bar{x})| \neq \infty$, and let $\bar{y} \in M(\bar{x})$ be such that $\widehat{\partial}^{+}\varphi(\bar{x},\bar{y}) \neq \emptyset$. Then*

$$\widehat{\partial}\mu(\bar{x}) \subset \bigcap_{(x^*,y^*)\in\widehat{\partial}^{+}\varphi(\bar{x},\bar{y})} \big[x^* + \widehat{D}^*G(\bar{x},\bar{y})(y^*)\big]. \tag{2.11}$$

To recall the sufficient conditions of [60] for the inclusion in (2.11) to hold as equality, we need the following definitions. Let $X, Y$ be Banach spaces, $D \subset X$. A map $h : D \to Y$ is said to be *locally upper Lipschitzian* at $\bar{x} \in D$ if there are a neighborhood $U$ of $\bar{x}$ and a constant $\ell > 0$ such that

$$\|h(x) - h(\bar{x})\| \leq \ell\|x - \bar{x}\|, \quad \forall x \in D \cap U.$$

We say that a set-valued map $F : D \rightrightarrows Y$ admits *a local upper Lipschitzian selection* at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if there is a map $h : D \to Y$, such that $h$ is locally upper Lipschitzian at $\bar{x}$, $h(\bar{x}) = \bar{y}$, and $h(x) \in F(x)$ for all $x \in D$ in a neighborhood of $\bar{x}$.

**Theorem 2.4** (See [60, Theorem 2]) *In addition to the assumptions of Theorem 2.3, suppose that $\varphi$ is Fréchet differentiable at $(\bar{x},\bar{y})$ and the map $M : \operatorname{dom} G \rightrightarrows Y$ admits a local upper Lipschitzian selection at $(\bar{x},\bar{y})$. Then*

$$\widehat{\partial}\mu(\bar{x}) = x^* + \widehat{D}^*G(\bar{x},\bar{y})(y^*),$$

*where $(x^*,y^*) := \nabla\varphi(\bar{x},\bar{y})$.*

Next, we will use Theorem 7 from [60] to estimate the limiting and singular subdifferentials of the function $-v$. The formulation of that theorem is based on some definitions related to the solution map $M(\cdot)$ of problem (2.10). Let $\bar{x} \in \operatorname{dom} M$ and $\bar{y} \in M(\bar{x})$. One says that $M(\cdot)$ is *$\mu$-inner semicontinuous* at $(\bar{x}, \bar{y})$ if for every sequence $x_k \xrightarrow{\mu} \bar{x}$ there is a sequence $y_k \in M(x_k)$ that contains a subsequence converging to $\bar{y}$. The map $M(\cdot)$ is said to be *$\mu$-inner semicompact* at $\bar{x}$ if for every sequence $x_k \xrightarrow{\mu} \bar{x}$ there is a sequence $y_k \in M(x_k)$ that contains a convergent subsequence.

**Theorem 2.5** (See [60, Theorem 7]) *Let $X$ and $Y$ be two Asplund spaces, $M(\cdot) : X \rightrightarrows Y$ be the solution map of the parametric problem* (2.10), *and let $(\bar{x}, \bar{y}) \in \operatorname{gph} M$ be such that $\varphi$ is lower semicontinuous at $(\bar{x}, \bar{y})$ and $G$ is locally closed around this point. The following statements hold:*

(i) *Assume that $M(\cdot)$ is $\mu$-inner semicontinuous at $(\bar{x}, \bar{y})$, that either* $\operatorname{epi} \varphi$ *is SNC at $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$ or $G$ is SNC at $(\bar{x}, \bar{y})$, and that the qualification condition*

$$\partial^\infty \varphi(\bar{x}, \bar{y}) \cap (-N((\bar{x}, \bar{y}); \operatorname{gph} G)) = \{(0,0)\} \tag{2.12}$$

*is satisfied; the above assumptions are automatic if $\varphi$ is locally Lipschitz around $(\bar{x}, \bar{y})$. Then one has the inclusions*

$$\partial \mu(\bar{x}) \subset \bigcup \{x^* + D^*G(\bar{x}, \bar{y})(y^*) : (x^*, y^*) \in \partial \varphi(\bar{x}, \bar{y})\},$$
$$\partial^\infty \mu(\bar{x}) \subset \bigcup \{x^* + D^*G(\bar{x}, \bar{y})(y^*) : (x^*, y^*) \in \partial^\infty \varphi(\bar{x}, \bar{y})\}.$$

(ii) *Assume that $M(\cdot)$ is $\mu$-inner semicompact at $\bar{x}$ and that the other assumption of* (i) *are satisfied at any $(\bar{x}, \bar{y}) \in \operatorname{gph} M$. Then one has the inclusions*

$$\partial \mu(\bar{x}) \subset \bigcup_{\bar{y} \in M(\bar{x})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*) : (x^*, y^*) \in \partial \varphi(\bar{x}, \bar{y})\},$$
$$\partial^\infty \mu(\bar{x}) \subset \bigcup_{\bar{y} \in M(\bar{x})} \{x^* + D^*G(\bar{x}, \bar{y})(y^*) : (x^*, y^*) \in \partial^\infty \varphi(\bar{x}, \bar{y})\}.$$

(iii) *In addition to* (i), *assume that $\varphi$ is strictly differentiable at $(\bar{x}, \bar{y})$, the map $M : \operatorname{dom} G \rightrightarrows Y$ admits a local upper Lipschitzian selection at $(\bar{x}, \bar{y})$, and $G$ is normally regular at $(\bar{x}, \bar{y})$. Then $\mu$ is lower regular at $\bar{x}$ and*

$$\partial \mu(\bar{x}) = \nabla_x \varphi(\bar{x}, \bar{y}) + D^*G(\bar{x}, \bar{y})(\nabla_y \varphi(\bar{x}, \bar{y})).$$

## 2.2 Coderivatives of the Budget Map

To study the budget map $B : Y_+ \rightrightarrows X_+$ of the consumer problem (1.3) in Section 1.1 with the values $B(p) = \{x \in X_+ \ : \ \langle p, x\rangle \le 1\}$, $p \in Y_+$ given in (1.2), we define a single-valued map $f : X^* \times X \to \mathbb{R} \times X$ and a set-valued map $Q : X^* \times X \rightrightarrows \mathbb{R} \times X$ by

$$f(x^*, x) = (\langle x^*, x\rangle - 1, -x), \quad Q(x^*, x) = \mathbb{R}_+ \times X_+ \tag{2.13}$$

for all $(x^*, x) \in X^* \times X$. From the definition for $B(p)$, we have

$$B(p) = \{x \in X \ : \ 0 \in f(p, x) + Q(p, x)\}, \quad \forall p \in Y_+.$$

Thus, the budget map $B : Y_+ \rightrightarrows X_+$ is the restriction on $Y_+$ of the solution map $\widetilde{B} : X^* \rightrightarrows X$,

$$\widetilde{B}(x^*) = \{x \in X \ : \ 0 \in f(x^*, x) + Q(x^*, x)\}, \quad x^* \in X^*, \tag{2.14}$$

of the parametric generalized equation $0 \in f(x^*, x) + Q(x^*, x)$ with $x^* \in X^*$ playing as parameters.

**Lemma 2.1** *The single-valued map $f : X^* \times X \to \mathbb{R} \times X$ is strictly differentiable on $X^* \times X$. Besides, for any $(\bar{x}^*, \bar{x}) \in X^* \times X$, the coderivative $D^*f(\bar{x}^*, \bar{x}) : \mathbb{R} \times X^* \to (X^* \times X)^*$ is the adjoint operator of $\nabla f(\bar{x}^*, \bar{x})$ mapping each $(\lambda, y^*) \in \mathbb{R} \times X^*$ to an element of $(X^* \times X)^*$ which is determined by*

$$\begin{aligned}\big\langle D^*f(\bar{x}^*, \bar{x})(\lambda, y^*), (u^*, u)\big\rangle &= \big\langle \nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*), (u^*, u)\big\rangle \\ &= \lambda\langle \bar{x}^*, u\rangle + \lambda\langle u^*, \bar{x}\rangle - \langle y^*, u\rangle\end{aligned} \tag{2.15}$$

*for all $(u^*, u) \in X^* \times X$. Moreover, if $\bar{x}$ is a nonzero vector, then the linear operator $D^*f(\bar{x}^*, \bar{x})$ is injective.*

**Proof.** Fix any $(\bar{x}^*, \bar{x}) \in X^* \times X$. Consider $T(\bar{x}^*, \bar{x}) : X^* \times X \to \mathbb{R} \times X$ defined by

$$T(\bar{x}^*, \bar{x})(u^*, u) = \big(\langle \bar{x}^*, u\rangle + \langle u^*, \bar{x}\rangle, -u\big) \ \ ((u^*, u) \in X^* \times X). \tag{2.16}$$

Clearly, $T(\bar{x}^*, \bar{x})$ is a linear operator. Putting $\gamma = \max\{\|\bar{x}^*\| + 1, \|\bar{x}\|\}$, for every $(u^*, u) \in X^* \times X$, we have

$$\begin{aligned}\|T(\bar{x}^*, \bar{x})(u^*, u)\| &= \big|\langle \bar{x}^*, u\rangle + \langle u^*, \bar{x}\rangle\big| + \|u\| \\ &\le (\|\bar{x}^*\| + 1)\|u\| + \|u^*\|\|\bar{x}\| \\ &\le \gamma(\|u^*\| + \|u\|) = \gamma\|(u^*, u)\|.\end{aligned}$$

Thus, $T(\bar{x}^*, \bar{x})$ is a bounded linear operator. Moreover,

$$\begin{aligned}
&\lim_{(u^*,u)\to(\bar{x}^*,\bar{x})} \frac{f(u^*,u) - f(\bar{x}^*,\bar{x}) - T(\bar{x}^*,\bar{x})((u^*,u)-(\bar{x}^*,\bar{x}))}{\|(u^*,u)-(\bar{x}^*,\bar{x})\|} \\
= &\lim_{(u^*,u)\to(\bar{x}^*,\bar{x})} \Big[\frac{\big(\langle u^*,u\rangle - 1, -u\big) - \big(\langle \bar{x}^*,\bar{x}\rangle - 1, -\bar{x}\big)}{\|(u^*-\bar{x}^*, u-\bar{x})\|} \\
&\qquad - \frac{\langle \bar{x}^*, u-\bar{x}\rangle + \langle u^*-\bar{x}^*, \bar{x}\rangle, -(u-\bar{x})}{\|(u^*-\bar{x}^*, u-\bar{x})\|}\Big] \\
= &\lim_{(u^*,u)\to(\bar{x}^*,\bar{x})} \left(\frac{\langle u^*-\bar{x}^*, u-\bar{x}\rangle}{\|(u^*-\bar{x}^*, u-\bar{x})\|},\ 0\right).
\end{aligned}$$

Since

$$\frac{|\langle u^*-\bar{x}^*, u-\bar{x}\rangle|}{\|(u^*-\bar{x}^*, u-\bar{x})\|} \le \frac{\|u^*-\bar{x}^*\|\|u-\bar{x}\|}{\|u^*-\bar{x}^*\| + \|u-\bar{x}\|} \le \frac{\|u^*-\bar{x}^*\|\|u-\bar{x}\|}{\|u^*-\bar{x}^*\|} = \|u-\bar{x}\|$$

and $\|u-\bar{x}\|$ converges to 0 when $u$ tends to $\bar{x}$, this implies that $f$ is Fréchet differentiable at $(\bar{x}^*, \bar{x})$, and we have $\nabla f(\bar{x}^*, \bar{x}) = T(\bar{x}^*, \bar{x})$. From (2.16) it follows that the operator $(\bar{x}^*, \bar{x}) \mapsto T(\bar{x}^*, \bar{x})$ from $X^* \times X$ to the space of bounded linear operators $L(X^*\times X, \mathbb{R}\times X)$ is continuous on $X^*\times X$. Hence, $f$ is strictly differentiable at $(\bar{x}^*, \bar{x})$. Therefore, for each $(\lambda, y^*) \in \mathbb{R}\times X^*$, by Proposition 2.2 we have

$$\widehat{D}^* f(\bar{x}^*, \bar{x})(\lambda, y^*) = D^* f(\bar{x}^*, \bar{x})(\lambda, y^*) = \{\nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*)\},$$

where $\nabla f(\bar{x}^*, \bar{x})^*$ denotes the adjoint operator of $\nabla f(\bar{x}^*, \bar{x})$. In addition, by (2.16) and the equality $\nabla f(\bar{x}^*, \bar{x}) = T(\bar{x}^*, \bar{x})$ one has

$$\begin{aligned}
\big\langle \nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*), (u^*, u)\big\rangle &= \big\langle (\lambda, y^*), \nabla f(\bar{x}^*, \bar{x})(u^*, u)\big\rangle \\
&= \big\langle (\lambda, y^*), \big(\langle \bar{x}^*, u\rangle + \langle u^*, \bar{x}\rangle, -u\big)\big\rangle \\
&= \lambda\langle \bar{x}^*, u\rangle + \lambda\langle u^*, \bar{x}\rangle - \langle y^*, u\rangle
\end{aligned}$$

for every $(u^*, u) \in X^* \times X$. This establishes formula (2.15).

If $\bar{x} \neq 0$, then $\nabla f(\bar{x}^*, \bar{x}) : X^* \times X \to \mathbb{R} \times X$ is surjective. Indeed, let us show that for any $(\mu, v) \in \mathbb{R} \times X$ there exits $(u^*, u) \in X^* \times X$ with $\nabla f(\bar{x}^*, \bar{x})(u^*, u) = (\mu, v)$. Since $\bar{x} \neq 0$, by the Hahn-Banach Theorem we can find $u_1^* \in X^*$ such that $\langle u_1^*, \bar{x}\rangle = 1$. Setting $u^* = (\mu + \langle \bar{x}^*, v\rangle)u_1^*$ and $u = -v$, we have

$$\begin{aligned}
&u = -v, \quad \langle u^*, \bar{x}\rangle = \mu + \langle \bar{x}^*, v\rangle \\
\Leftrightarrow\ &[-u = v, \quad \langle \bar{x}^*, u\rangle + \langle u^*, \bar{x}\rangle = \mu] \\
\Leftrightarrow\ &\big(\langle \bar{x}^*, u\rangle + \langle u^*, \bar{x}\rangle, -u\big) = (\mu, v) \\
\Leftrightarrow\ &T(\bar{x}^*, \bar{x})(u^*, u) = (\mu, v).
\end{aligned}$$

Hence, $\nabla f(\bar{x}^*, \bar{x})(u^*, u) = (\mu, v)$. Now, since $D^*f(\bar{x}^*, \bar{x}) = \nabla f(\bar{x}^*, \bar{x})^*$ and $\nabla f(\bar{x}^*, \bar{x})$ is surjective, we obtain the desired injectivity of $D^*f(\bar{x}^*, \bar{x})$ from [76, Theorem 4.15] and complete the proof. □

**Lemma 2.2** *The map $Q : X^* \times X \rightrightarrows \mathbb{R} \times X$ given in* (2.13) *is locally closed around and graphically regular at every point of* $\operatorname{gph} Q = X^* \times X \times \mathbb{R}_+ \times X_+$. *For any $(\bar{x}^*, \bar{x}, \bar{\mu}, \bar{y}) \in \operatorname{gph} Q$, we have*

$$\widehat{D}^*Q(\bar{x}^*, \bar{x}, \bar{\mu}, \bar{y})(\lambda, y^*) = D^*Q(\bar{x}^*, \bar{x}, \bar{\mu}, \bar{y})(\lambda, y^*) = \begin{cases} \{(0,0)\} & \textit{if} -\lambda \in N(\bar{\mu}; \mathbb{R}_+) \textit{ and } -y^* \in N(\bar{y}; X_+) \\ \emptyset & \textit{otherwise} \end{cases} \tag{2.17}$$

*with $(\lambda, y^*) \in \mathbb{R} \times X^*$. Moreover, $Q$ is SNC at $(\bar{x}^*, \bar{x}, \bar{\mu}, \bar{y}) \in \operatorname{gph} Q$ whenever $X_+$ is SNC at $\bar{y}$. Especially, if* $\operatorname{int} X_+ \neq \emptyset$, *then $Q$ is SNC at every point of its graph.*

**Proof.** Since $X_+$ is closed and convex, $\operatorname{gph} Q = X^* \times X \times \mathbb{R}_+ \times X_+$ is a closed and convex subset of $X^* \times X \times \mathbb{R} \times X$. Take any $(\bar{x}^*, \bar{x}, \bar{\mu}, \bar{y}) \in \operatorname{gph} Q$. The closeness of $\operatorname{gph} Q$ implies that $Q$ is locally closed around $(\bar{x}^*, \bar{x}, \bar{\mu}, \bar{y})$, while the the convexity of $\operatorname{gph} Q$ and Proposition 2.1 yields that $Q$ is graphically regular at this point, and

$$\widehat{D}^*Q(\bar{x}^*, \bar{x}, \bar{\mu}, \bar{y}) = D^*Q(\bar{x}^*, \bar{x}, \bar{\mu}, \bar{y}).$$

Take any $(\lambda, y^*) \in \mathbb{R} \times X^*$. By the definition of limiting coderivative and properties of the normal cone to a Cartesian product (see [56, Proposition 1.2]), we have

$$\begin{aligned} &D^*Q(\bar{x}^*, \bar{x}, \bar{\mu}, \bar{y})(\lambda, y^*) \\ &= \{(x, x^*) \,:\, (x, x^*, -\lambda, -y^*) \in N((\bar{x}^*, \bar{x}, \bar{\mu}, \bar{y}); \operatorname{gph} Q)\} \\ &= \{(x, x^*) : (x, x^*, -\lambda, -y^*) \in N(\bar{x}^*; X^*) \times N(\bar{x}; X) \times N(\bar{\mu}; \mathbb{R}_+) \times N(\bar{y}; X_+)\} \\ &= \begin{cases} \{(0,0)\} & \text{if } -\lambda \in N(\bar{\mu}; \mathbb{R}_+) \text{ and } -y^* \in N(\bar{y}; X_+) \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, formula (2.17) is valid.

Next, suppose that $(\bar{x}^*, \bar{x}, \bar{\mu}, \bar{y}) \in \operatorname{gph} Q$. By [56, Theorem 1.21], any convex set with nonempty interior is SNC at every point belonging to it. Hence, $X^*$, $X$, $\mathbb{R}_+$ are SNC at $\bar{x}^*, \bar{x}, \bar{\mu}$, respectively. If, in addition, $X_+$ is SNC at $\bar{y}$ (which is automatically satisfied if $\operatorname{int} X_+ \neq \emptyset$), then $\operatorname{gph} Q = X^* \times X \times \mathbb{R}_+ \times X_+$ is SNC at $(\bar{x}^*, \bar{x}, \bar{\mu}, \bar{y})$. So, $Q$ is SNC at $(\bar{x}^*, \bar{x}, \bar{\mu}, \bar{y})$ whenever $X_+$ is SNC at $\bar{y}$.

The proof is complete. $\square$

**Lemma 2.3** *Let $\bar{x}^* \in X^*$, $\bar{x} \in \widetilde{B}(\bar{x}^*)$, where $\widetilde{B}$ is the map in* (2.14), *be such that $\bar{x} \neq 0$ and $X_+$ is SNC at $\bar{x}$. Then $\widetilde{B} : X^* \rightrightarrows X$ is graphically regular at $(\bar{x}^*, \bar{x})$. Moreover, for every $x^* \in X^*$,*

$$\begin{aligned}&\widehat{D}^*\widetilde{B}(\bar{x}^*, \bar{x})(x^*) = D^*\widetilde{B}(\bar{x}^*, \bar{x})(x^*)\\ &= \begin{cases} \{\lambda\bar{x} \ : \ \lambda \geq 0, \ x^* + \lambda\bar{x}^* \in -N(\bar{x}; X_+)\} & \textit{if } \langle \bar{x}^*, \bar{x}\rangle = 1 \\ \{0\} \quad \textit{if } \langle \bar{x}^*, \bar{x}\rangle < 1, \ x^* \in -N(\bar{x}; X_+) \\ \emptyset \quad \textit{if } \langle \bar{x}^*, \bar{x}\rangle < 1, \ x^* \notin -N(\bar{x}; X_+). \end{cases}\end{aligned} \tag{2.18}$$

**Proof.** Suppose that $\bar{x}^* \in X^*$, $\bar{x} \in \widetilde{B}(\bar{x}^*) \setminus \{0\}$, and $X_+$ is SNC at $\bar{x}$. Since $X$ is reflexive, so are $X^*$, $X^* \times X$, and $\mathbb{R} \times X$. Hence, these spaces are Asplund and $(X^*)^* = X$, $(X^* \times X)^* = X \times X^*$, $(\mathbb{R} \times X)^* = \mathbb{R} \times X^*$. The proof is divided into two steps. In the first step, we will apply Theorem 2.1 to show that, for every $x^* \in X^*$,

$$\begin{aligned}D^*\widetilde{B}(\bar{x}^*, \bar{x})(x^*) = \big\{x \in X \ : \ &\exists \lambda \in -N(1 - \langle \bar{x}^*, \bar{x}\rangle; \mathbb{R}_+), \ \exists y^* \in -N(\bar{x}; X_+) \\ &\text{such that } \ (x, -x^*) = \nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*)\big\}.\end{aligned} \tag{2.19}$$

In the second step, we will prove that the set $A(x^*)$ in the right-hand side of (2.19) can be computed by the formula

$$A(x^*) = \begin{cases} \{\lambda\bar{x} : \lambda \geq 0, \ x^* + \lambda\bar{x}^* \in -N(\bar{x}; X_+)\}, & \text{if } \langle \bar{x}^*, \bar{x}\rangle = 1 \\ \{0\}, \quad \text{if } \langle \bar{x}^*, \bar{x}\rangle < 1, x^* \in -N(\bar{x}; X_+) \\ \emptyset, \quad \text{if } \langle \bar{x}^*, \bar{x}\rangle < 1, x^* \notin -N(\bar{x}; X_+). \end{cases} \tag{2.20}$$

Step 1. From Lemma 2.1, $f$ is strictly differentiable at $(\bar{x}^*, \bar{x})$ and its coderivative is given by formula (2.15). Besides, since $\bar{x} \in \widetilde{B}(\bar{x}^*)$, $\bar{x} \in X_+$ and $1 - \langle \bar{x}^*, \bar{x}\rangle \in \mathbb{R}_+$. Thus, $(\bar{x}^*, \bar{x}, 1 - \langle \bar{x}^*, \bar{x}\rangle, \bar{x}) \in \operatorname{gph} Q$. Lemma 2.2 implies that $Q$ is locally closed around $(\bar{x}^*, \bar{x}, 1 - \langle \bar{x}^*, \bar{x}\rangle, \bar{x})$, graphically regular and SNC at this point if $X_+$ is SNC at $\bar{x}$. So, if condition (2.6) is satisfied, then $\widetilde{B}$ is graphically regular at $(\bar{x}^*, \bar{x})$ and we can estimate the coderivatives of $\widetilde{B}$ at $(\bar{x}^*, \bar{x})$ by formula (2.7) when it holds as equality. To check (2.6), we fix any $(x, x^*, \lambda, y^*) \in X \times X^* \times \mathbb{R} \times X^*$ satisfying the inclusion

$$(x, x^*) \in D^*f(\bar{x}^*, \bar{x})(\lambda, y^*) \bigcap \Big( - D^*Q(\bar{x}^*, \bar{x}, 1 - \langle \bar{x}^*, \bar{x}\rangle, \bar{x})(\lambda, y^*)\Big). \tag{2.21}$$

By Lemma 2.2, one has $D^*Q(\bar{x}^*, \bar{x}, 1 - \langle \bar{x}^*, \bar{x}\rangle, \bar{x})(\lambda, y^*) \subset \{(0,0)\} \subset X \times X^*$. Combining this with (2.21) and noting that $D^*f(\bar{x}^*, \bar{x})(\lambda, y^*)$ is a singleton, we obtain $(x, x^*) = (0,0)$ and $D^*f(\bar{x}^*, \bar{x})(\lambda, y^*) = (0,0)$. In addition, since $\bar{x} \neq 0$, it follows from Lemma 2.1 that $D^*f(\bar{x}^*, \bar{x})$ is injective. This implies that $(\lambda, y^*) = (0,0) \in \mathbb{R} \times X^*$. Thus, if $(x, x^*, \lambda, y^*) \in X \times X^* \times \mathbb{R} \times X^*$ satisfies (2.21), then $(x, x^*, \mu, y^*) = (0,0,0,0)$. Thus, condition (2.6) is fulfilled.

By the second assertion of Theorem 2.1, the set-valued map $\widetilde{B} : X^* \rightrightarrows X$ is graphically regular at $(\bar{x}^*, \bar{x})$, and the coderivative $D^*\widetilde{B}(\bar{x}^*, \bar{x}) : X^* \rightrightarrows X$ maps each $x^* \in X^*$ to the set

$$\begin{aligned} &D^*\widetilde{B}(\bar{x}^*, \bar{x})(x^*) \\ &= \big\{x \in X \ : \ \exists(\lambda, y^*) \in \mathbb{R} \times X^* \text{ such that} \\ &\qquad (x, -x^*) \in D^*f(\bar{x}^*, \bar{x})(\lambda, y^*) + D^*Q(\bar{x}^*, \bar{x}, 1 - \langle \bar{x}^*, \bar{x}\rangle, \bar{x})(\lambda, y^*)\big\}. \end{aligned}$$

In addition, by (2.17),

$$\begin{aligned} &D^*Q(\bar{x}^*, \bar{x}, 1 - \langle \bar{x}^*, \bar{x}\rangle, \bar{x})(\lambda, y^*) \\ &= \begin{cases} \{(0,0)\} & \text{if } -\lambda \in N(1 - \langle \bar{x}^*, \bar{x}\rangle; \mathbb{R}_+) \text{ and } -y^* \in N(\bar{x}; X_+) \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, we obtain formula (2.19).

Step 2. For any $x^* \in X^*$, $x \in X$, $\lambda \in \mathbb{R}$, $y^* \in X^*$, the equality

$$(x, -x^*) = \nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*) \tag{2.22}$$

holds if and only if

$$\langle (x, -x^*), (u^*, u)\rangle = \langle \nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*), (u^*, u)\rangle, \quad \forall u^* \in X^*, \forall u \in X.$$

So, by (2.15), (2.22) means that

$$\langle (x - \lambda\bar{x}, y^* - x^* - \lambda\bar{x}^*), (u^*, u)\rangle = 0, \quad \forall u^* \in X^*, \forall u \in X.$$

Clearly, the latter is equivalent to the following system

$$x = \lambda\bar{x}, \quad y^* = x^* + \lambda\bar{x}^*. \tag{2.23}$$

Now, fix an $x^* \in X^*$. If $\langle \bar{x}^*, \bar{x}\rangle < 1$, then $-N(1 - \langle \bar{x}^*, \bar{x}\rangle; \mathbb{R}_+) = \{0\}$. So, $x \in A(x^*)$ if and only if the equality (2.22) holds for $\lambda = 0$ and for some $y^* \in -N(\bar{x}; X_+)$. From this property and (2.23) it follows that $A(x^*) = \{0\}$ when $x^* \in -N(\bar{x}; X_+)$ and $A(x^*) = \emptyset$ when $x^* \notin -N(\bar{x}; X_+)$. If $\langle \bar{x}^*, \bar{x}\rangle = 1$, then $-N(1 - \langle \bar{x}^*, \bar{x}\rangle; \mathbb{R}_+) = \mathbb{R}_+$. Therefore, $x \in A(x^*)$ if and only if (2.22) holds for some $\lambda \geq 0$ and $y^* \in -N(\bar{x}; X_+)$. So, $x \in A(x^*)$ if and only if $x$ fulfills (2.23) with $\lambda \in \mathbb{R}_+$ satisfying the condition $x^* + \lambda\bar{x}^* \in -N(\bar{x}; X_+)$. Formula (2.20) has been obtained.

The proof is complete. $\square$

In Chapter 1, a Lipschitz-Hölder property of the demand map $D(\cdot)$ was obtained by using the Lipschitz-like property of the budget map $B(\cdot)$ at point $(\bar{p}, \bar{x}) \in \operatorname{gph} B$ with $\bar{p}$ being an interior point of the cone of prices $Y_+$. Now, we will show that if $\bar{x} \neq 0$ and $X_+$ is SNC at $\bar{x}$, then we can get the Lipschitz-like property of $B(\cdot)$ without imposing the condition $\bar{p} \in \operatorname{int} Y_+$. Hence, Theorem 1.8 in the previous chapter can be extended to the case where $\bar{p}$ may belong to the boundary of $Y_+$.

**Theorem 2.6** *Assume that $\bar{x}^* \in X^*$, $\bar{x} \in \widetilde{B}(\bar{x}^*) \setminus \{0\}$, and $X_+$ is SNC at $\bar{x}$. Then, the solution map $\widetilde{B}(\cdot)$ is Lipschitz-like at $(\bar{x}^*, \bar{x}) \in \operatorname{gph} \widetilde{B}$. Consequently, if $\bar{p} \in Y_+$, $\bar{x} \in B(\bar{p}) \setminus \{0\}$, and $X_+$ is SNC at $\bar{x}$, then the budget map $B(\cdot)$ is Lipschitz-like at $(\bar{p}, \bar{x})$ in the sense that there exist a neighborhood $U$ of $\bar{p}$, a neighborhood $V$ of $\bar{x}$, and a constant $\ell > 0$ satisfying*

$$B(p) \cap V \subset B(p') + \ell\|p - p'\|\bar{B}_X, \quad \forall p, p' \in U \cap Y_+. \tag{2.24}$$

**Proof.** Suppose that $\bar{x}^* \in X^*$, $\bar{x} \in \widetilde{B}(\bar{x}^*) \setminus \{0\}$, and $X_+$ is SNC at $\bar{x}$. We apply Theorem 2.2 with $f$, $Q$ being given by (2.13), $(\bar{x}^*, \bar{x})$ and $\widetilde{B}(\cdot)$ playing the roles of $(\bar{x}, \bar{y})$ and $S(\cdot)$, respectively. Thus, we are dealing with the generalized equation $0 \in f(x^*, x) + Q(x^*, x)$ appeared in (2.14).

Since $X$ is reflexive, so are $X^*$ and $\mathbb{R} \times X$. Hence, $X^*$, $X$, and $\mathbb{R} \times X$ are Asplund spaces. Besides, it follows from Lemma 2.1 that $f$ is strictly differentiable at $(\bar{x}^*, \bar{x})$. Moreover, as $\bar{x} \in \widetilde{B}(\bar{x}^*)$, one has $\bar{x} \in X_+$ and $1 - \langle \bar{x}^*, \bar{x} \rangle \in \mathbb{R}_+$. Thus, for $\bar{z} := (1 - \langle \bar{x}^*, \bar{x} \rangle, \bar{x})$, one has

$$(\bar{x}^*, \bar{x}, \bar{z}) \in \operatorname{gph} Q = X^* \times X \times \mathbb{R}_+ \times X_+.$$

Since $X_+$ is SNC at $\bar{x}$, Lemma 2.2 assures that $Q$ is locally closed around $(\bar{x}^*, \bar{x}, \bar{z})$, graphically regular and SNC at this point. Therefore, it remains to check (2.8) and (2.9).

Condition (2.8) requires

$$\left[(0,0) \in \nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*) + D^*Q(\bar{x}^*, \bar{x}, \bar{z})(\lambda, y^*)\right] \Longrightarrow (\lambda, y^*) = (0, 0).$$

Fix any $(\lambda, y^*) \in \mathbb{R} \times X^*$ with the property

$$(0,0) \in \nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*) + D^*Q(\bar{x}^*, \bar{x}, \bar{z})(\lambda, y^*). \tag{2.25}$$

By (2.17), the inclusion $D^*Q(\bar{x}^*, \bar{x}, \bar{z})(\lambda, y^*) \subset \{(0,0)\}$ is valid. Hence, (2.25) implies that $\nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*) = (0,0)$. On one hand, combining this with (2.15), we get $D^*f(\bar{x}^*, \bar{x})(\lambda, y^*) = \nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*) = (0,0)$. On the other hand, since $\bar{x} \neq 0$, $D^*f(\bar{x}^*, \bar{x})$ is injective by Lemma 2.1. It follows that $(\lambda, y^*) = (0,0)$. Thus, condition (2.8) is satisfied.

Condition (2.9) is the following

$$\big[(x,0) \in \nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*) + D^*Q(\bar{x}^*, \bar{x}, \bar{z})(\lambda, y^*)\big] \implies (x, \lambda, y^*) = (0,0,0).$$

Let $(x, \lambda, y^*) \times \mathbb{R} \times X^*$ be such that

$$(x,0) \in \nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*) + D^*Q(\bar{x}^*, \bar{x}, \bar{z})(\lambda, y^*).$$

In combination with (2.17), this inclusion implies that $\lambda \in -N(1-\langle \bar{x}^*, \bar{x}\rangle; \mathbb{R}_+)$, $y^* \in -N(\bar{x}; X_+)$, and $(x,0) = \nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*)$. Hence, using formula (2.19) for $x^* = 0$, we obtain $x \in D^*\widetilde{B}(\bar{x}^*, \bar{x})(0)$. Let us show that

$$D^*\widetilde{B}(\bar{x}^*, \bar{x})(0) = \{0\}. \tag{2.26}$$

Since $X_+$ is a convex cone, $N(\bar{x}; X_+)$ is the normal cone in the sense of convex analysis. Clearly, $0 \in -N(\bar{x}; X_+)$. As $\bar{x} \in \widetilde{B}(\bar{x}^*)$, one has $\langle \bar{x}^*, \bar{x}\rangle \le 1$. In the case $\langle \bar{x}^*, \bar{x}\rangle < 1$, by formula (2.18), one gets (2.26). In the case $\langle \bar{x}^*, \bar{x}\rangle = 1$, (2.18) implies

$$D^*\widetilde{B}(\bar{x}^*, \bar{x})(0) = \{t\bar{x} \,:\, t \ge 0,\ t\bar{x}^* \in -N(\bar{x}; X_+)\}. \tag{2.27}$$

By (2.27), if $D^*\widetilde{B}(\bar{x}^*, \bar{x})(0)$ contains a nonzero vector, then the latter must have the form $t\bar{x}$ with $t > 0$ and $-t\bar{x}^* \in N(\bar{x}; X_+)$. For $\tilde{x} := 0 \in X_+$, we have

$$\langle -t\bar{x}^*, \tilde{x} - \bar{x}\rangle = t\langle \bar{x}^*, \bar{x}\rangle = t > 0,$$

which contradicts the inclusion $-t\bar{x}^* \in N(\bar{x}; X_+)$. Thus, (2.26) is valid. Since $x$ belongs to $D^*\widetilde{B}(\bar{x}^*, \bar{x})(0)$, it follows from (2.26) that $x = 0$. So, as $\nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*) = (x,0)$, we get $\nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*) = (0,0)$. Hence, by (2.15),

$$D^*f(\bar{x}^*, \bar{x})(\lambda, y^*) = \nabla f(\bar{x}^*, \bar{x})^*(\lambda, y^*) = (0,0).$$

Since $D^*f(\bar{x}^*, \bar{x})$ is injective by Lemma 2.1, this yields $(\lambda, y^*) = (0,0)$. We have shown that $(x, \lambda, y^*) = (0,0,0)$; thus (2.9) is fulfilled.

The above analysis allows us to invoke Theorem 2.2 to assert that the map $\widetilde{B} : X^* \rightrightarrows X$ is Lipschitz-like at $(\bar{x}^*, \bar{x})$. Now, take any $\bar{p} \in Y_+$ and $\bar{x} \in B(\bar{p}) \setminus \{0\}$. Since $\bar{x} \in B(\bar{p}) = \widetilde{B}(\bar{p})$, one has $\bar{x} \in \widetilde{B}(\bar{p})$. Thus, if $X_+$ is

SNC at $\bar{x}$, then $\widetilde{B}$ is Lipschitz-like at $(\bar{p}, \bar{x})$. Hence, there exist a neighborhood $U$ of $\bar{p}$, a neighborhood $V$ of $\bar{x}$, and a constant $\ell > 0$ satisfying

$$\widetilde{B}(p) \cap V \subset \widetilde{B}(p') + \ell\|p - p'\|\bar{B}_X, \quad \forall p, p' \in U. \tag{2.28}$$

Remembering that $\widetilde{B}(p) = B(p)$ for all $p \in U \cap Y_+$, we obtain (2.24) from (2.28) and complete the proof. □

Under some mild conditions, we can have exact formulas for both Fréchet and limiting coderivatives of the budget map.

**Theorem 2.7** *Suppose that $\bar{p} \in \operatorname{int} Y_+$, $\bar{x} \in B(\bar{p}) \setminus \{0\}$, and $X_+$ is SNC at $\bar{x}$. Then the budget map $B : Y_+ \rightrightarrows X_+$ is graphically regular at $(\bar{p}, \bar{x})$. Moreover, for every $x^* \in X^*$, one has*

$$\begin{aligned}
&\widehat{D}^*B(\bar{p}, \bar{x})(x^*) = D^*B(\bar{p}, \bar{x})(x^*) \\
&\qquad = \begin{cases} \{\lambda\bar{x} \ : \ \lambda \geq 0, \ x^* + \lambda\bar{p} \in -N(\bar{x}; X_+)\} & \text{if } \langle \bar{p}, \bar{x}\rangle = 1 \\ \{0\} \quad \text{if } \langle \bar{p}, \bar{x}\rangle < 1, \ x^* \in -N(\bar{x}; X_+) \\ \emptyset \quad \text{if } \langle \bar{p}, \bar{x}\rangle < 1, \ x^* \notin -N(\bar{x}; X_+). \end{cases}
\end{aligned} \tag{2.29}$$

**Proof.** Since $\bar{p} \in \operatorname{int} Y_+$, there exists an open set $U$ in the norm topology of $X^*$ such that $\bar{p} \in U \subset Y_+$. Then $B(p) = \widetilde{B}(p)$ for all $p \in U$. It follows that

$$(\operatorname{gph} B) \cap (U \times X) = (\operatorname{gph} \widetilde{B}) \cap (U \times X). \tag{2.30}$$

By definitions of Fréchet and limiting normal cones, $\widehat{N}((\bar{p}, \bar{x}); \operatorname{gph} B)$ and $N((\bar{p}, \bar{x}); \operatorname{gph} B)$ (resp., $\widehat{N}((\bar{p}, \bar{x}); \operatorname{gph} \widetilde{B})$ and $N((\bar{p}, \bar{x}); \operatorname{gph} \widetilde{B})$) are defined just by vectors of $\operatorname{gph} B$ (resp., of $\operatorname{gph} \widetilde{B}$) in a neighborhood $W$ of $(\bar{p}, \bar{x})$. Choosing $W = U \times X$, from (2.30) one has $\widehat{N}((\bar{p}, \bar{x}); \operatorname{gph} B) = \widehat{N}((\bar{p}, \bar{x}); \operatorname{gph} \widetilde{B})$ and $N((\bar{p}, \bar{x}); \operatorname{gph} B) = N((\bar{p}, \bar{x}); \operatorname{gph} \widetilde{B})$. Consequently, for any $x^* \in X^*$, $\widehat{D}^*B(\bar{p}, \bar{x})(x^*) = \widehat{D}^*\widetilde{B}(\bar{p}, \bar{x})(x^*)$ and $D^*B(\bar{p}, \bar{x})(x^*) = D^*\widetilde{B}(\bar{p}, \bar{x})(x^*)$. Hence, letting $\bar{p}$ play the role of $\bar{x}^*$ in (2.18), we obtain formula (2.29) from the latter.

The proof is complete. □

Theorem 2.7 requires that $X_+$ is SNC at $\bar{x}$ and $\operatorname{int} Y_+ \neq \emptyset$. As any convex set with nonempty interior is SNC at every point belonging to it (see the proof of Lemma 2.2), the cones of goods and cones of prices constructed in the following example satisfy these requirements.

**Example 2.1** Let $X$ be a reflexive Banach space. Take any $a \in X \setminus \{0\}$ and fix a value $\varepsilon \in (0, \frac{1}{2}\|a\|)$. As the set of goods, we choose $X_+ = \overline{\text{cone}\,(\bar{B}(a,\varepsilon))}$, where

$$\text{cone}\,\Omega := \{tv \,:\, v \in \Omega,\ t > 0\}$$

denotes the cone generated by $\Omega$. Clearly, $X_+$ is a closed convex cone in $X$ with nonempty interior. Select a unit vector $\bar{x}^* \in X^*$ such that $\langle \bar{x}^*, a\rangle > \frac{3}{4}\|a\|$. It is a simple matter to show that $\bar{B}(\bar{x}^*, \frac{1}{8})$ is contained in the cone of prices $Y_+$ corresponding to $X_+$. Thus, $Y_+ \subset X^*$ is a closed convex cone with nonempty interior.

If $\langle \bar{p}, \bar{x}\rangle = 1$ then, for any $x^* \in X^*$, using (2.29) one can compute the coderivative values $\hat{D}^*B(\bar{p},\bar{x})(x^*)$ and $D^*B(\bar{p},\bar{x})(x^*)$ via the set of real numbers $\{\lambda \geq 0 \,:\, x^* + \lambda\bar{p} \in -N(\bar{x}; X_+)\}$. The forthcoming lemma, which will be used intensively in Sections 2.3 and 2.4, describes explicitly latter set in a situation where $\bar{x} \in D(\bar{p})$ and $x^* = -\nabla u(\bar{x})$.

**Lemma 2.4** *Suppose that* $(\bar{p},\bar{x}) \in \text{gph}\, D$ *and* $u$ *is Fréchet differentiable at* $\bar{x}$. *Then*

$$\{\lambda \geq 0 \,:\, \lambda\bar{p} \in \nabla u(\bar{x}) - N(\bar{x}; X_+)\} = \{\langle \nabla u(\bar{x}), \bar{x}\rangle\} \tag{2.31}$$

*when* $\langle \bar{p}, \bar{x}\rangle = 1$, *and* $\nabla u(\bar{x}) \in N(\bar{x}; X_+)$ *when* $\langle \bar{p}, \bar{x}\rangle < 1$.

**Proof.** Let $\bar{p}, \bar{x}, u$ satisfy our assumptions.

Suppose that $\langle \bar{p}, \bar{x}\rangle = 1$. We first establish the inclusion "$\subset$" in (2.31), and then show that the set on the left-hand side, which will be denoted by $\Lambda$, is nonempty.

Given any $\lambda \in \Lambda$, we have $\lambda\bar{p} = \nabla u(\bar{x}) - z^*$ for some $z^* \in N(\bar{x}; X_+)$. From the last equality and the condition $\langle \bar{p}, \bar{x}\rangle = 1$, we have $\lambda = \langle \nabla u(\bar{x}), \bar{x}\rangle - \langle z^*, \bar{x}\rangle$. Besides, since $z^* \in N(\bar{x}; X_+)$ and $X_+$ is convex, $\langle z^*, x - \bar{x}\rangle \leq 0$ for all $x \in X_+$. In addition, as $X_+$ is a nonempty closed cone, $x_1 := 0$ and $x_2 := 2\bar{x}$ belong to $X_+$. Therefore, $-\langle z^*, \bar{x}\rangle = \langle z^*, x_1 - \bar{x}\rangle \leq 0$ and $\langle z^*, \bar{x}\rangle = \langle z^*, x_2 - \bar{x}\rangle \leq 0$. Hence, we must have $\langle z^*, \bar{x}\rangle = 0$. Thus, $\lambda = \langle \nabla u(\bar{x}), \bar{x}\rangle$ and we obtain the inclusion "$\subset$" in (2.31).

Next, fix an arbitrary $x \in B(\bar{p})$. As $B(\bar{p})$ is convex, $x_t := tx + (1-t)\bar{x}$ belongs to $B(\bar{p})$ for all $t \in (0,1)$. Since $\bar{x} \in D(\bar{p})$, $u(\bar{x}) \geq u(x_t)$ for all $t \in (0,1)$. Thus, $[u(\bar{x}+t(x-\bar{x})) - u(\bar{x})]/t \leq 0$ for all $t \in (0,1)$. Letting $t \to 0^+$ and using the Fréchet differentiability of $u$ at $\bar{x}$, we obtain $\langle \nabla u(\bar{x}), x - \bar{x}\rangle \leq 0$.

Since the latter holds for any $x \in B(\bar{p})$, by the convexity of $B(\bar{p})$ we have $\nabla u(\bar{x}) \in N(\bar{x}; B(\bar{p}))$. Now, as $B(\bar{p}) = X_+ \cap \Omega$ with $\Omega := \{x \in X : \langle \bar{p}, x \rangle \leq 1\}$ and $0 \in X_+ \cap \operatorname{int} \Omega$, applying the fundamental intersection rule of convex analysis [43, Proposition 1, p. 205], we have

$$N(\bar{x}; B(\bar{p})) = N(\bar{x}; X_+) + N(\bar{x}; \Omega). \tag{2.32}$$

Clearly, $N(\bar{x}; \Omega) = \mathbb{R}_+ \bar{p}$ when $\langle \bar{p}, \bar{x} \rangle = 1$. So, the inclusion $\nabla u(\bar{x}) \in N(\bar{x}; B(\bar{p}))$ and (2.32) imply the existence of $\lambda \geq 0$ satisfying $\nabla u(\bar{x}) \in N(\bar{x}; X_+) + \lambda \bar{p}$. This shows that $\Lambda \neq \emptyset$.

If $\langle \bar{p}, \bar{x} \rangle < 1$, then $N(\bar{x}; \Omega) = \{0\}$. Thus, the inclusion $\nabla u(\bar{x}) \in N(\bar{x}; B(\bar{p}))$ and (2.32) yield $\nabla u(\bar{x}) \in N(\bar{x}; X_+)$.

The proof is complete. □

## 2.3 Fréchet Subdifferential of the Function $-v$

As discussed in Section 1.1, the consumer problem (1.3) is a maximization problem depending on parameters $p$ when the prices $p$ vary in the cone $Y_+$. Thus, we can consider the optimal value function and the solution map of the problem, which are respectively called the indirect utility function and the demand map in the economic terminology and their values are respectively defined by (1.4) and (1.5). Concerning a parametric optimization problem, whether a maximization or a minimization one, people always pay attention to study the stability (the continuity, the Lipschitz/Lipschitz-like/Lipschitz-Höder property) and the differential stability (in some sense) of its optimal value function and solution map. To continue the investigations presented in Chapter 1, we will study the Fréchet and limiting subdifferentials of the optimal value function of (1.3). A natural question arises: *Why are Fréchet and limiting subdifferentials instead of simpler ones, for examples, one in the sense of convex analysis?* To answer, we raise another question: *Is this optimal value function convex or concave so that we can consider the subdifferential in the latter sense?* As will be shown in Example 2.3, this optimal value function $v(\cdot)$ is generally neither convex nor concave.

Technically, we will transform this maximization problem into an equivalent minimization one which has available advanced results on estimating subdifferentials of the optimal value function. By that way, we will get

$$-v(p) = \inf\{-u(x) : x \in B(p)\}, \quad p \in Y_+;$$

hence, we can consider a counterpart of $v(\cdot)$, the *infimal nuisance function* $-v(\cdot)$ obtained from the former by changing its sign, as the role of the optimal value function of the corresponding minimization problem.

Results on estimating Fréchet subdifferential of the function $-v$ are presented in this section, while those on estimating limiting one will be addressed in the next section.

**Theorem 2.8** *Let $\bar{p} \in \operatorname{int} Y_+$ and $\bar{x} \in D(\bar{p}) \setminus \{0\}$ be such that $D(\bar{p}) \neq \emptyset$, $X_+$ is SNC at $\bar{x}$, and $\widehat{\partial} u(\bar{x}) \neq \emptyset$. The following assertions hold:*

(i) *If $\langle \bar{p}, \bar{x} \rangle = 1$, then*

$$\widehat{\partial}(-v)(\bar{p}) \subset \bigcap_{x^* \in -\widehat{\partial} u(\bar{x})} \{\lambda \bar{x} \ : \ \lambda \geq 0, \ x^* + \lambda \bar{p} \in -N(\bar{x}; X_+)\}; \tag{2.33}$$

(ii) *If $\langle \bar{p}, \bar{x} \rangle < 1$, then*

$$\widehat{\partial}(-v)(\bar{p}) \subset \{0\}; \tag{2.34}$$

(iii) *If $\langle \bar{p}, \bar{x} \rangle < 1$ and $\widehat{\partial} u(\bar{x}) \setminus N(\bar{x}, X_+) \neq \emptyset$, then*

$$\widehat{\partial}(-v)(\bar{p}) = \emptyset; \tag{2.35}$$

(iv) *If $u$ is Fréchet differentiable at $\bar{x}$, and the map $D : \operatorname{dom} B \rightrightarrows X_+$ admits a local upper Lipschitzian selection at $(\bar{p}, \bar{x})$, then*

$$\widehat{\partial}(-v)(\bar{p}) = \begin{cases} \{\langle \nabla u(\bar{x}), \bar{x} \rangle \bar{x}\} & \text{if } \langle \bar{p}, \bar{x} \rangle = 1 \\ \{0\} & \text{if } \langle \bar{p}, \bar{x} \rangle < 1. \end{cases} \tag{2.36}$$

**Proof.** To transform the problem (1.3) to the minimization problem (2.10), we let $X^*$ (resp., $X$) play the role of $X$ (resp., $Y$). Put $\varphi(x^*, x) = -u(x)$ for $(x^*, x) \in X^* \times X$. Besides, let $G(x^*) = \{x \in X_+ : \langle x^*, x \rangle \leq 1\}$ for $x^* \in Y_+$, $G(x^*) = \emptyset$ otherwise. From (1.2), (1.4), (1.5), and the conventions made, one deduces that $G(x^*) = B(x^*)$, $\mu(x^*) = -v(x^*)$, and $M(x^*) = D(x^*)$ for all $x^* \in X^*$.

Let $\bar{p} \in \operatorname{int} Y_+$ and $\bar{x} \in D(\bar{p}) \setminus \{0\}$ satisfy the assumptions of the theorem. Using the definitions of Fréchet subdifferential and Fréchet upperdifferential, one can show that

$$\widehat{\partial}^+ \varphi(x^*, x) = \{0\} \times \big(-\widehat{\partial} u(x)\big), \quad \forall (x^*, x) \in X^* \times X.$$

Hence, the assumption $\widehat{\partial} u(\bar{x}) \neq \emptyset$ implies $\widehat{\partial}^+ \varphi(\bar{p}, \bar{x}) \neq \emptyset$. Applying Theorem 2.3 for $\mu(\cdot) = (-v)(\cdot)$, $M(\cdot) = D(\cdot)$, and $(\bar{x}, \bar{y}) := (\bar{p}, \bar{x})$, we have

$$\widehat{\partial}(-v)(\bar{p}) \subset \bigcap_{(x,x^*) \in \widehat{\partial}^+ \varphi(\bar{p},\bar{x})} \left[x + \widehat{D}^* B(\bar{p}, \bar{x})(x^*)\right] = \bigcap_{x^* \in -\widehat{\partial}(u)(\bar{x})} \widehat{D}^* B(\bar{p}, \bar{x})(x^*). \tag{2.37}$$

(i) If $\langle \bar{p}, \bar{x} \rangle = 1$, then (2.37) and (2.29) imply (2.33).

(ii) If $\langle \bar{p}, \bar{x} \rangle < 1$, then by (2.29) one has $\widehat{D}^* B(\bar{p}, \bar{x})(x^*) \subset \{0\}$ for every $x^* \in X^*$. It follows that

$$\bigcap_{x^* \in -\widehat{\partial}(u)(\bar{x})} \widehat{D}^* B(\bar{p}, \bar{x})(x^*) \subset \{0\}.$$

Hence, (2.37) yields (2.34).

(iii) If $\langle \bar{p}, \bar{x} \rangle < 1$ and $\widehat{\partial} u(\bar{x}) \setminus N(\bar{x}; X_+) \neq \emptyset$, then there exist $\bar{x}^* \in -\widehat{\partial} u(\bar{x})$ such that $\bar{x}^* \notin -N(\bar{x}; X_+)$. By (2.37) and (2.29), we have

$$\widehat{\partial}(-v)(\bar{p}) \subset \bigcap_{x^* \in -\widehat{\partial}(u)(\bar{x})} \widehat{D}^* B(\bar{p}, \bar{x})(x^*) \subset \widehat{D}^* B(\bar{p}, \bar{x})(\bar{x}^*) = \emptyset.$$

Hence, (2.35) is valid.

(iv) Now, suppose that the function $u(\cdot)$ is Fréchet differentiable at $\bar{x}$, and the map $D : \mathrm{dom} B \rightrightarrows X_+$ admits a local upper Lipschitzian selection at $(\bar{p}, \bar{x})$. Since $u$ is Fréchet differentiable at $\bar{x}$, $\nabla \varphi(\bar{p}, \bar{x}) = (0, -\nabla u(\bar{x}))$. By Theorem 2.4, one gets

$$\widehat{\partial}(-v)(\bar{p}) = \widehat{D}^* B(\bar{p}, \bar{x})(-\nabla u(\bar{x})). \tag{2.38}$$

Since $\widehat{D}^* B(\bar{p}, \bar{x})(-\nabla u(\bar{x}))$ can be computed by (2.29) with $x^* := -\nabla u(\bar{x})$ and by Lemma 2.4, formula (2.36) follows from (2.38). □

The upper estimates for the subdifferential $\widehat{\partial}(-v)(\bar{p})$ provided by Theorem 2.8 are sharp. Moreover, under some mild conditions, the set on the right-hand side of (2.33) is a singleton. In addition, if the indirect utility function $v$ is Fréchet differentiable at $\bar{p}$, then its derivative at $\bar{p}$ can be easily computed by using (2.33) and (2.34). The following statement justifies our observations.

**Corollary 2.1** *Let $\bar{p} \in \mathrm{int}\, Y_+$ and $\bar{x} \in D(\bar{p}) \setminus \{0\}$ be such that $X_+$ is SNC at $\bar{x}$, and $u$ is Fréchet differentiable at $\bar{x}$. Then*

$$\widehat{\partial}(-v)(\bar{p}) \subset \begin{cases} \{\langle \nabla u(\bar{x}), \bar{x} \rangle \bar{x}\} & \text{if } \langle \bar{p}, \bar{x} \rangle = 1 \\ \{0\} & \text{if } \langle \bar{p}, \bar{x} \rangle < 1. \end{cases} \tag{2.39}$$

*Consequently, if the indirect utility function $v$ is Fréchet differentiable at $\bar{p}$, then*

$$\nabla v(\bar{p}) = \begin{cases} -\langle \nabla u(\bar{x}), \bar{x}\rangle \bar{x} & \text{if } \langle \bar{p}, \bar{x}\rangle = 1 \\ 0 & \text{if } \langle \bar{p}, \bar{x}\rangle < 1. \end{cases} \tag{2.40}$$

**Proof.** Let $\bar{p}$, $\bar{x}$, and $u$ satisfy our assumptions. Then, $\widehat{\partial} u(\bar{x}) = \{\nabla u(\bar{x})\}$. Hence, (2.39) follows from Theorem 2.8 and Lemma 2.4.

If $v$ is Fréchet differentiable at $\bar{p}$, then $\widehat{\partial} v(\bar{p}) = -\widehat{\partial}(-v)(\bar{p}) = \{\nabla v(\bar{p})\}$ by [56, Proposition 1.87]. So, (2.40) follows from (2.39).

The proof is complete. □

To obtain another corollary from Theorem 2.8, we now recall a well-known concept in mathematical economics. One says (see, e.g., [65, p. 1076]) that the consumer problem (1.3) satisfies the *non satiety condition* (NSC) if

$$[(x,p) \in X_+ \times Y_+,\ \langle p, x\rangle < 1] \Longrightarrow [\exists x' \in X_+,\ u(x') > u(x)]\,.$$

As it has been noted in [65], NSC is equivalent to the following condition:

$$[(x,p) \in X_+ \times Y_+,\ \langle p, x\rangle < 1] \Longrightarrow u(x) < v(p).$$

Moreover, one can easily prove the next lemma, which characterizes NSC via the demand map.

**Lemma 2.5** (See [65, Lemma 4.2]) *The NSC is satisfied if and only if for all $p \in Y_+$, one has $D(p) \subset \{x \in X \,:\, \langle p, x\rangle = 1\}$.*

Consumer problems with the Cobb-Douglas utility functions satisfy the non satiety condition.

**Example 2.2** Suppose that there are $n$ types of available goods. The quantities of goods purchased by the consumer form the good bundle $x = (x_1, \ldots, x_n)$, where $x_i$ is the purchased quantity of the $i$-th good, $i = 1, \ldots, n$. Assume that each good is perfectly divisible so that any nonnegative quantity can be purchased. Good bundles are vectors in the commodity space $X := \mathbb{R}^n$. The set of all possible good bundles

$$X_+ := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, x_1 \geq 0, \ldots, x_n \geq 0\}$$

is the nonnegative orthant of $\mathbb{R}^n$. The set of prices is

$$Y_+ = \{p = (p_1, \ldots, p_n) \in \mathbb{R}^n \,:\, p_1 \geq 0, \ldots, p_n \geq 0\}.$$

For every $p = (p_1, \dots, p_n) \in Y_+$, $p_i$ is the price of the $i$-th good, $i = 1, \dots, n$. Given some numbers $A > 0$, $\alpha_1, \alpha_2, \dots, \alpha_n \in (0, 1)$, consider the utility function $u : X \to \mathrm{I\!R}_+$ defined by $u(x) = Ax_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ for any $x \in \operatorname{int} X_+$. (Recall [89, p. 96] that $0^\alpha = 0$ for any $\alpha > 0$.) Clearly, $u$ is strictly increasing in each variable on $\operatorname{int} X_+$. Take $p = (p_1, \dots, p_n) \in Y_+$ and $x = (x_1, \dots, x_n) \in X_+$ satisfying $\langle p, x\rangle < 1$. If $x \notin \operatorname{int} X_+$, then we choose $x' = (x_1', x_2', \dots, x_n')$ such that $x_i' \in (0, 1)$ and $\sum_1^n p_i x_i' \leq 1$. Then $\langle p, x'\rangle \leq 1$ and $x' \in \operatorname{int} X_+$, and therefore $u(x') > 0 = u(x)$. Consider the case where $x \in \operatorname{int} X_+$. If $p = 0$, then by choosing $x' = (x_1', x_2, \dots, x_n)$ with $x_1' > x_1$, one gets $\langle p, x'\rangle < 1$ and $x' \in \operatorname{int} X_+$. Hence, $u(x') > u(x)$ as $u$ is strictly increasing on $\operatorname{int} \mathrm{I\!R}_+$ w.r.t. the first variable. If $p \neq 0$, then there exists $i_0$ such that $p_{i_0} > 0$. We choose $x' = (x_1', x_2', \dots, x_n')$ with $x_{i_0}' = (1 - \sum_{i \neq i_0} p_i x_i)/p_{i_0}$ and $x_i' = x_i$ for all $i \neq i_0$. It follows that $\langle p, x'\rangle = 1$, $x' \in \operatorname{int} X_+$, and $x_{i_0}' > x_{i_0}$. As $u$ is strictly increasing on $\operatorname{int} \mathrm{I\!R}_+$ w.r.t. the $i_0$-th variable, one gets $u(x') > u(x)$. We have shown that, for any pair $(p, x) \in Y_+ \times X_+$ with $\langle p, x\rangle < 1$, there exists $x' \in X_+$ such that $\langle p, x'\rangle \leq 1$ and $u(x') > u(x)$. Thus, the NSC is satisfied.

In the following corollary, it is not assumed a priori that the demand set $D(\bar p)$ is nonempty.

**Corollary 2.2** *Suppose that $X_+$ has nonempty interior and the NSC is satisfied. If $u(\cdot)$ is concave and upper semicontinuous on $X_+$, and Fréchet differentiable on $D(\bar p)$ then, for any $\bar p \in \operatorname{int} Y_+$, one has*

$$\widehat{\partial}(-v)(\bar p) \subset \{\langle \nabla u(\bar x), \bar x\rangle \bar x \ : \ \bar x \in D(\bar p)\}. \tag{2.41}$$

**Proof.** Let $\bar p \in \operatorname{int} Y_+$. Since $u(\cdot)$ is concave and upper semicontinuous on $X_+$, it is weakly upper semicontinuous on $X_+$. Hence, by [65, Proposition 4.1] we have $D(\bar p) \neq \emptyset$. Take any $\bar x \in D(\bar p)$. Since $X_+$ is a convex set with nonempty interior, it is SNC at $\bar x$. Besides, as the NSC is satisfied, Lemma 2.5 implies $\langle \bar p, \bar x\rangle = 1$. Due to this and the Fréchet differentiability of $u(\cdot)$ on $D(\bar p)$, formula (2.33) and Lemma 2.4 give $\widehat{\partial}(-v)(\bar p) \subset \{\langle \nabla u(\bar x), \bar x\rangle \bar x\}$. Since $\bar x \in D(\bar p)$ is arbitrarily chosen, from the last inclusion we obtain (2.41). $\square$

## 2.4 Limiting Subdifferential of the Function $-v$

Our results on limiting and singular subdifferentials of $-v$ are stated in the following theorem.

**Theorem 2.9** *Let $\bar{p} \in \operatorname{int} Y_+$ and $\bar{x} \in D(\bar{p})\setminus\{0\}$ be such that $D(\bar{p}) \neq \emptyset$, $X_+$ is SNC at $\bar{x}$, $u$ is upper semicontinuous at $\bar{x}$, and $D$ is $v$-inner semicontinuous at $(\bar{p}, \bar{x})$. Assume that either* hypo $u$ *is SNC at $(\bar{x}, \varphi(\bar{x}))$ or $X$ is finite-dimensional, and the qualification condition*

$$\partial^{\infty,+}u(\bar{x}) \cap N(\bar{x}; X_+) = \{0\} \tag{2.42}$$

*is satisfied. Then, the following assertions hold:*

(i) *If $\langle \bar{p}, \bar{x}\rangle = 1$, then*

$$\partial(-v)(\bar{p}) \subset \bigcup_{x^* \in \partial^+ u(\bar{x})} \{\lambda \bar{x} \ : \ \lambda \geq 0, \ x^* - \lambda \bar{p} \in N(\bar{x}; X_+)\}, \tag{2.43}$$

$$\partial^\infty(-v)(\bar{p}) \subset \bigcup_{x^* \in \partial^{\infty,+} u(\bar{x})} \{\lambda \bar{x} \ : \ \lambda \geq 0, \ x^* - \lambda \bar{p} \in N(\bar{x}; X_+)\}; \tag{2.44}$$

(ii) *If $\langle \bar{p}, \bar{x}\rangle < 1$, then*

$$\partial(-v)(\bar{p}) \subset \{0\}, \tag{2.45}$$

$$\partial^\infty(-v)(\bar{p}) = \{0\}; \tag{2.46}$$

(iii) *If $\langle \bar{p}, \bar{x}\rangle < 1$ and $\partial^+ u(\bar{x}) \cap N(\bar{x}; X_+) = \emptyset$, then*

$$\partial(-v)(\bar{p}) = \emptyset; \tag{2.47}$$

(iv) *If $u$ is strictly differentiable at $\bar{x}$, and the map $D : \operatorname{dom} B \rightrightarrows X_+$ admits a local upper Lipschitzian selection at $(\bar{p}, \bar{x})$, then $(-v)$ is lower regular at $\bar{x}$ and*

$$\partial(-v)(\bar{p}) = \begin{cases} \{\langle \nabla u(\bar{x}), \bar{x}\rangle \bar{x}\} & \text{if } \langle \bar{p}, \bar{x}\rangle = 1 \\ \{0\} & \text{if } \langle \bar{p}, \bar{x}\rangle < 1. \end{cases} \tag{2.48}$$

**Proof.** Let $\bar{p} \in \operatorname{int} Y_+$ and $\bar{x} \in D(\bar{p}) \setminus \{0\}$ satisfy the assumptions of the theorem. At the beginning of the proof of Theorem 2.8, we have transformed the consumer problem in (1.3) to the minimization problem (2.10), where $\varphi(x^*, x) = -u(x)$ for $(x^*, x) \in X^* \times X$, $G(x^*) = B(x^*)$, $\mu(x^*) = -v(x^*)$, and $M(x^*) = D(x^*)$ for all $x^* \in X^*$.

Now, we will show that the limiting and singular sudifferential of $-v$ at $\bar{p}$ can be estimated by assertion (i) of Theorem 2.5. Since $X$ is a reflexive Banach space, so is $X^*$. Hence, $X$ and $X^*$ are Asplund spaces. Moreover, $B$ is locally closed around $(\bar{p}, \bar{x})$ because $\operatorname{gph} B$ is closed and $(\bar{p}, \bar{x}) \in \operatorname{gph} B$. Since $u$ is upper semicontinuous at $\bar{x}$, $\varphi$ is lower semicontinuous at $(\bar{p}, \bar{x})$.

If $\operatorname{hypo} u$ is SNC at $(\bar{x}, \varphi(\bar{x}))$, then $\operatorname{epi} \varphi$ is SNC at $(\bar{p}, \bar{x}, \varphi(\bar{p}, \bar{x}))$. If $X$ is finite-dimensional, $B$ is SNC at $(\bar{p}, \bar{x})$.

Letting $(\bar{p}, \bar{x})$ play the role of $(\bar{x}, \bar{y})$, we now show that (2.42) implies (2.12). The latter means that

$$\partial^\infty \varphi(\bar{p}, \bar{x}) \cap (-N((\bar{p}, \bar{x}); \operatorname{gph} B)) = \{(0, 0)\}. \tag{2.49}$$

The inclusion "$\supset$" is trivial. Take any $(x, x^*) \in X \times X^*$ belonging to the left-hand side of (2.49). On one hand, since $\partial^\infty \varphi(\bar{p}, \bar{x}) = \{0\} \times \partial^\infty(-u)(\bar{x})$, $x = 0$ and $x^* \in \partial^\infty(-u)(\bar{x})$. This implies that $-x^* \in \partial^{\infty,+}(-u)(\bar{x})$. On the other hand, because $(-x, -x^*) \in N((\bar{p}, \bar{x}); \operatorname{gph} B)$ and $x = 0$, it follows that $(0, -x^*) \in N((\bar{p}, \bar{x}); \operatorname{gph} B)$. Hence, $0 \in D^*B(\bar{p}, \bar{x})(x^*)$. Combining this with (2.29) implies $-x^* \in N(\bar{x}; X_+)$. Thus, one has

$$-x^* \in \partial^{\infty,+}(-u)(\bar{x}) \cap N(\bar{x}; X_+).$$

So, by (2.42) one obtains $x^* = 0$. We have just shown that $(x, x^*) = (0, 0)$; hence the inclusion "$\subset$" in (2.49) is true.

Since all the assumptions for the validity of assertion (i) of Theorem 2.5 are satisfied, we have

$$\partial(-v)(\bar{p}) \subset \bigcup \left\{x + D^*B(\bar{p}, \bar{x})(x^*) : (x, x^*) \in \partial\varphi(\bar{p}, \bar{x})\right\}, \tag{2.50}$$

$$\partial^\infty(-v)(\bar{p}) \subset \bigcup \left\{x + D^*B(\bar{p}, \bar{x})(x^*) : (x, x^*) \in \partial^\infty\varphi(\bar{p}, \bar{x})\right\}. \tag{2.51}$$

Since $\partial\varphi(\bar{p}, \bar{x}) = \{0\} \times \partial(-u)(\bar{x})$, and $\partial^\infty\varphi(\bar{p}, \bar{x}) = \{0\} \times \partial^\infty(-u)(\bar{x})$, the inclusions (2.50) and (2.51) respectively imply

$$\partial(-v)(\bar{p}) \subset \bigcup_{x^* \in \partial(-u)(\bar{x})} D^*B(\bar{p}, \bar{x})(x^*), \tag{2.52}$$

$$\partial^\infty(-v)(\bar{p}) \subset \bigcup_{x^* \in \partial^\infty(-u)(\bar{x})} D^*B(\bar{p}, \bar{x})(x^*). \tag{2.53}$$

(i) If $\langle \bar{p}, \bar{x} \rangle = 1$, then (2.52), (2.53), and (2.29) imply

$$\partial(-v)(\bar{p}) \subset \bigcup_{x^* \in \partial(-u)(\bar{x})} \{\lambda\bar{x} \ : \ \lambda \geq 0,\ x^* + \lambda\bar{p} \in -N(\bar{x}; X_+)\},$$

$$\partial^\infty(-v)(\bar{p}) \subset \bigcup_{x^* \in \partial^\infty(-u)(\bar{x})} \{\lambda\bar{x} \ : \ \lambda \geq 0,\ x^* + \lambda\bar{p} \in -N(\bar{x}; X_+)\}.$$

As $-\partial(-u)(\bar{x}) = \partial^+ u(\bar{x})$ and $-\partial^\infty(-u)(\bar{x}) = \partial^{\infty,+}(-u)(\bar{x})$, these inclusions yield (2.43) and (2.44), respectively.

(ii) If $\langle \bar{p}, \bar{x} \rangle < 1$, then by (2.29) one has $D^*B(\bar{p}, \bar{x})(x^*) \subset \{0\}$ for every $x^* \in X^*$. It follows that

$$\bigcup_{x^* \in \partial(-u)(\bar{x})} D^*B(\bar{p}, \bar{x})(x^*) \subset \{0\}, \qquad \bigcup_{x^* \in \partial^\infty(-u)(\bar{x})} D^*B(\bar{p}, \bar{x})(x^*) \subset \{0\}.$$

So, (2.52) implies (2.45), and (2.53) yields $\partial^\infty(-v)(\bar{p}) \subset \{0\}$. Remembering that $\partial^\infty(-v)(\bar{p})$ always contains the origin, one obtains (2.46).

(iii) If $\langle \bar{p}, \bar{x} \rangle < 1$ and $\partial^+ u(\bar{x}) \cap N(\bar{x}; X_+) = \emptyset$, then for any $x^* \in \partial(-u)(\bar{x})$ one has $x^* \notin -N(\bar{x}; X_+)$. Therefore, by (2.29), $D^*B(\bar{p}, \bar{x})(x^*) = \emptyset$ for all $x^* \in \partial(-u)(\bar{x})$. Combining this with (2.52) implies (2.47).

(iv) Since $\bar{p} \in \operatorname{int} Y_+$ and $\bar{x} \in B(\bar{p}) \setminus \{0\}$, $B$ is graphically regular at $(\bar{p}, \bar{x})$ by Theorem 2.7. Besides, as $u$ is strictly differentiable at $\bar{x}$, so is $\varphi$ at $(\bar{p}, \bar{x})$ and one has $\nabla\varphi(\bar{p}, \bar{x}) = (0, -\nabla u(\bar{x}))$. Applying assertion (iii) of Theorem 2.5, we have

$$\partial(-v)(\bar{p}) = D^*B(\bar{p}, \bar{x})(-\nabla u(\bar{x})). \tag{2.54}$$

Since $D^*B(\bar{p}, \bar{x})(-\nabla u(\bar{x}))$ can be computed via (2.29) with $x^* := -\nabla u(\bar{x})$ and Lemma 2.4, formula (2.48) follows from (2.54).

The proof is complete. $\square$

We now give an illustrative example for Theorem 2.9.

**Example 2.3** Choose $X = \mathbb{R}, X_+ = \mathbb{R}_+$, and define the concave utility function $u : X \to \mathbb{R}$ by

$$u(x) = \begin{cases} x & \text{if} \quad x \leq 1 \\ 1 & \text{if} \quad x > 1. \end{cases}$$

One has $Y_+ = \mathbb{R}_+$ and

$$B(p) = \begin{cases} [0, +\infty) & \text{if} \quad p = 0 \\ [0, 1/p] & \text{if} \quad p > 0. \end{cases}$$

It is easy to show that

$$v(p) = \begin{cases} 1 & \text{if} \quad 0 \leq p < 1 \\ 1/p & \text{if} \quad p \geq 1 \end{cases}$$

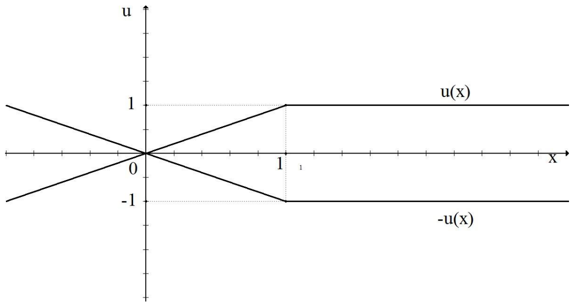


Figure 2.1: Utility function

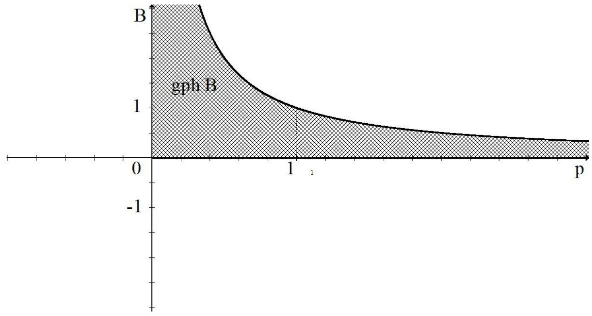


Figure 2.2: Budget map

and

$$D(p) = \begin{cases} [1, +\infty) & \text{if} \quad p = 0 \\ [1, 1/p] & \text{if} \quad 0 < p < 1 \\ \{1/p\} & \text{if} \quad p \geq 1. \end{cases}$$

We see that both functions $v$ and $-v$ are neither convex nor concave on

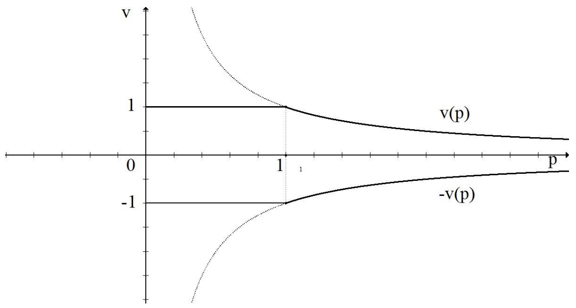


Figure 2.3: Indirect utility function

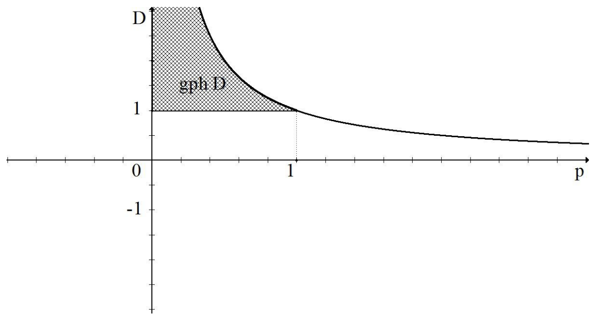


Figure 2.4: Demand map

$X_+$. Consider the pair $(\bar{p}, \bar{x}) = (1, 1)$ belonging to $\operatorname{gph} D$. It is obvious that $\bar{p} \in \operatorname{int} Y_+, \bar{x} \neq 0$, $X_+$ is SNC at $\bar{x}$, and $\langle \bar{p}, \bar{x} \rangle = 1$. Besides, as $u$ is locally Lipschitz on $X$, $u$ is upper semicontinuous at $\bar{x}$ and $\partial^{\infty,+} u(\bar{x}) = \{0\}$. Thus, the qualification (2.42) is satisfied. Moreover, the formulas of $v$ and $D$ imply that $D$ is $v$-inner semicontinuous at $(\bar{p}, \bar{x})$.

It follows from definitions that

$$\partial^+ u(\bar{x}) = [0, 1], \quad \partial(-v)(\bar{p}) = [0, 1], \quad \text{and} \quad \partial^\infty(-v)(\bar{p}) = \{0\}$$

while, by direct computation, we have

$$\bigcup_{x^*\in\partial^+u(\bar{x})} \{\lambda\bar{x} \ : \ \lambda \geq 0, \ x^* - \lambda\bar{p} \in N(\bar{x}; X_+)\} = [0,1],$$

$$\bigcup_{x^*\in\partial^{\infty,+}u(\bar{x})} \{\lambda\bar{x} \ : \ \lambda \geq 0, \ x^* - \lambda\bar{p} \in N(\bar{x}; X_+)\} = \{0\}.$$

Thus, the inclusions (2.43) and (2.44) hold as equalities.

Now, we present a counterpart of Theorem 2.9, where the assumption on the $v$-inner semicontinuity of $D$ at $(\bar{p}, \bar{x})$ is removed. In fact, here one has the $v$-inner semicompactness of $D$ at $\bar{p}$, which is guaranteed by the assumptions saying that $X$ is finite-dimensional and $\bar{p} \in \operatorname{int} Y_+$.

**Theorem 2.10** *Suppose that $X$ is a finite-dimensional Banach space, the non satiety condition is satisfied, and $u$ is upper semicontinuous on $X_+$. For any $\bar{p} \in \operatorname{int} Y_+$, if the qualification condition* (2.42) *is satisfied for every $\bar{x} \in D(\bar{p})$, then one has*

$$\partial(-v)(\bar{p}) \subset \bigcup_{\bar{x}\in D(\bar{p})} \ \bigcup_{x^*\in\partial^+u(\bar{x})} \{\lambda\bar{x} \ : \ \lambda \geq 0, \ x^* - \lambda\bar{p} \in N(\bar{x}; X_+)\}, \tag{2.55}$$

$$\partial^\infty(-v)(\bar{p}) \subset \bigcup_{\bar{x}\in D(\bar{p})} \ \bigcup_{x^*\in\partial^{\infty,+}u(\bar{x})} \{\lambda\bar{x} \ : \ \lambda \geq 0, \ x^* - \lambda\bar{p} \in N(\bar{x}; X_+)\}. \tag{2.56}$$

**Proof.** Fix a vector $\bar{p} \in \operatorname{int} Y_+$ and suppose that all the assumptions of the theorem are satisfied. We transform (1.3) to (2.10) in same manner as in the proofs of Theorem 2.8 and Theorem 2.9. Let us show that the limiting and singular sudifferential of $-v$ at $\bar{p}$ can be estimated by assertion (ii) of Theorem 2.5. To do so, we have to prove that the map $D$ is $v$-inner semicompact at $\bar{p}$.

On one hand, for a given $r \in (0,1)$, as shown in the proof of Proposition 3.2 in [35], the vector $\bar{q} := r\bar{p}$ belongs to $\operatorname{int} Y_+$, the set $U_1 := \bar{q} + \operatorname{int} Y_+$ is an open neighborhood of $\bar{p}$, and $B(p) \subset B(\bar{q})$ for every $p \in U_1$. On the other hand, since $X$ is a finite-dimensional Banach space, the weak topology of $X$ coincides with its norm topology. Hence, by the upper semicontinuity of $u$ on $X_+$ and [65, Proposition 4.1], $D(p)$ is nonempty for every $p \in \operatorname{int} Y_+$. So, the set $U := U_1 \cap \operatorname{int} Y_+$ is an open neighborhood of $\bar{p}$ and one has $\emptyset \neq D(p) \subset B(\bar{q})$ for every $p \in U$. Take any sequence $p_k \xrightarrow{v} \bar{p}$. Without loss of generality, one may assume that $p_k \in U$ for all $k$. Thus, for each $k$, one can select a vector $x_k \in D(p_k)$. Since $\{x_k\} \subset B(\bar{q})$, the compactness of $B(\bar{q})$ (see [65, Proof of Proposition 4.1]) implies that $\{x_k\}_{k\in\mathbb{N}}$ contains a convergent subsequence. We have thus proved that $D$ is $v$-inner semicompact at $\bar{p}$.

By assertion (ii) of Theorem 2.5, where we let $\varphi(x^*,x) = -u(x)$ for every $(x^*,x) \in X^* \times X$, $G(x^*) = B(x^*)$, $\mu(x^*) = -v(x^*)$, $M(x^*) = D(x^*)$ for every $x^* \in X^*$, and $\bar{x} = \bar{p}$, we have

$$\partial(-v)(\bar{p}) \subset \bigcup_{\bar{x}\in D(\bar{p})} \{x + D^*B(\bar{p},\bar{x})(x^*) : (x,x^*) \in \partial\varphi(\bar{p},\bar{x})\},$$
$$\partial^\infty(-v)(\bar{p}) \subset \bigcup_{\bar{x}\in D(\bar{p})} \{x + D^*B(\bar{p},\bar{x})(x^*) : (x,x^*) \in \partial^\infty\varphi(\bar{p},\bar{x})\}.$$

Since $\partial\varphi(\bar{p},\bar{x}) = \{0\} \times \partial(-u)(\bar{x})$, and $\partial^\infty\varphi(\bar{p},\bar{x}) = \{0\} \times \partial^\infty(-u)(\bar{x})$, these inclusions imply

$$\partial(-v)(\bar{p}) \subset \bigcup_{\bar{x}\in D(\bar{p})} \bigcup_{x^*\in\partial(-u)(\bar{x})} D^*B(\bar{p},\bar{x})(x^*), \tag{2.57}$$
$$\partial^\infty(-v)(\bar{p}) \subset \bigcup_{\bar{x}\in D(\bar{p})} \bigcup_{x^*\in\partial^\infty(-u)(\bar{x})} D^*B(\bar{p},\bar{x})(x^*). \tag{2.58}$$

As the NSC is satisfied, we have $\bar{x} \neq 0$ and $\langle\bar{p},\bar{x}\rangle = 1$ for any $\bar{x} \in D(\bar{p})$. Thus, it follows from (2.57), (2.58), and (2.29) that

$$\partial(-v)(\bar{p}) \subset \bigcup_{\bar{x}\in D(\bar{p})} \bigcup_{x^*\in\partial(-u)(\bar{x})} \{\lambda\bar{x} \ : \ \lambda \geq 0, \ x^* + \lambda\bar{p} \in -N(\bar{x};X_+)\}$$
$$\partial^\infty(-v)(\bar{p}) \subset \bigcup_{\bar{x}\in D(\bar{p})} \bigcup_{x^*\in\partial^\infty(-u)(\bar{x})} \{\lambda\bar{x} \ : \ \lambda \geq 0, \ x^* + \lambda\bar{p} \in -N(\bar{x};X_+)\}.$$

Remembering that $-\partial(-u)(\bar{x}) = \partial^+u(\bar{x})$ and $-\partial^\infty(-u)(\bar{x}) = \partial^{\infty,+}(-u)(\bar{x})$, one obtains (2.55) and (2.56) from these inclusions and thus completes the proof. □

**Corollary 2.3** *Let $\bar{p} \in \operatorname{int} Y_+$. If $X$ is finite-dimensional, the NSC is satisfied, and $u$ is strictly differentiable on $D(\bar{p})$, then*

$$\partial(-v)(\bar{p}) \subset \{\langle\nabla u(\bar{x}),\bar{x}\rangle\bar{x} \ : \ \bar{x} \in D(\bar{p})\}, \tag{2.59}$$
$$\partial^\infty(-v)(\bar{p}) = \{0\}. \tag{2.60}$$

**Proof.** Under the assumptions of the corollary, take any $\bar{x} \in D(\bar{p})$. By the strict differentiability of $u$ on $D(\bar{p})$, one has $\partial^+u(\bar{x}) = \{\nabla u(\bar{x})\}$ (see [56, Corollary 1.82]), and $u$ is locally Lipschitz around $\bar{x}$ (see [56, p. 19]). Thus,

$\partial^{\infty,+}u(\bar{x}) = \{0\}$; hence the qualification condition (2.42) is satisfied at $\bar{x}$. It follows from Theorem 2.10 that

$$\partial(-v)(\bar{p}) \subset \bigcup_{\bar{x}\in D(\bar{p})} \{\lambda\bar{x} \ : \ \lambda \geq 0,\ \lambda\bar{p} \in \nabla u(\bar{x}) - N(\bar{x}; X_+)\}, \tag{2.61}$$

$$\partial^{\infty}(-v)(\bar{p}) \subset \bigcup_{\bar{x}\in D(\bar{p})} \{\lambda\bar{x} \ : \ \lambda \geq 0,\ \lambda\bar{p} \in -N(\bar{x}; X_+)\}. \tag{2.62}$$

For any $\bar{x} \in D(\bar{p})$, by Lemma 2.4 one has

$$\{\lambda \geq 0 \ : \ \lambda\bar{p} \in \nabla u(\bar{x}) - N(\bar{x}; X_+)\} = \{\langle \nabla u(\bar{x}), \bar{x}\rangle\},$$

while $\{\lambda \geq 0 \ : \ \lambda\bar{p} \in -N(\bar{x}; X_+)\} = \{0\}$. Thus, (2.59) follows from (2.61), and (2.60) follows from (2.62), because $0 \in \partial^{\infty}(-v)(\bar{p})$.

The proof is complete. □

## 2.5 Some Economic Interpretations

In order to give some economic interpretations for the obtained results on estimating subdifferentials, we need to clarify the relationships between the concepts of subdifferential and derivative.

Suppose that $\bar{p} \in \operatorname{int} Y_+$ and $\bar{\xi} \in \widehat{\partial}(-v)(\bar{p})$. By [56, Theorem 1.88], there exists a function $s : X^* \to \overline{\mathbb{R}}$ that is finite around $\bar{p}$, Fréchet differentiable at $\bar{p}$, such that

$$s(\bar{p}) = -v(\bar{p}), \quad \nabla s(\bar{p}) = \bar{\xi}, \quad \text{and} \quad s(x^*) \leq -v(x^*) \quad \text{for all } x^* \in X^*. \tag{2.63}$$

Fix a vector $q \in X^*$. If $t > 0$ is small enough, then the Fréchet differentiability of $s$ at $\bar{p}$ implies

$$s(\bar{p} + tq) = s(\bar{p}) + t\langle \nabla s(\bar{p}), q\rangle + o(t)$$

with $\lim\limits_{t\to 0^+} \dfrac{o(t)}{t} = 0$. Combining this with (2.63) gives

$$-\langle \bar{\xi}, q\rangle \geq \frac{v(\bar{p} + tq) - v(\bar{p})}{t} + \frac{o(t)}{t},$$

for $t > 0$ small enough. Hence, we get

$$-\langle \bar{\xi}, q\rangle \geq \limsup_{t\to 0^+} \frac{v(\bar{p} + tq) - v(\bar{p})}{t} =: d^+v(\bar{p}; q),$$

where $d^+v(\bar{p};q)$ stands for the *upper Dini directional derivative* of $v$ at $\bar{p}$ in direction $q$. Thus, if $\widehat{\partial}(-v)(\bar{p})$ is nonempty, then

$$d^+v(\bar{p};q) \leq \inf_{\xi\in\widehat{\partial}(-v)(\bar{p})} [-\langle \xi, q\rangle]. \tag{2.64}$$

If a formula for exact computation of $\widehat{\partial}(-v)(\bar{p})$ is available, then (2.64) provides us with a sharp upper estimate for the value $d^+v(\bar{p};q)$.

Since $\dfrac{v(\bar{p}+tq)-v(\bar{p})}{t}$ is the average rate of the change of the maximal satisfaction, represented by the indirect utility function $v$, of the consumer when the price moves slightly forward direction $q$ from the current price $\bar{p}$, *the upper Dini directional derivative $d^+v(\bar{p};q)$ can be interpreted as an upper bound for the instant rate of the change of the maximal satisfaction of the consumer.* Therefore, the estimate given by (2.64) reads as follows: *If the current price is $\bar{p}$ and the price moves forward a direction $q \in X^*$, then the instant rate of the change of the maximal satisfaction of the consumer is bounded above by the real number $\inf_{\xi\in\widehat{\partial}(-v)(\bar{p})} [-\langle \xi, q\rangle]$.*

If $\widehat{\partial}(-v)(\bar{p}) = \emptyset$, then the estimate in (2.64) is trivial because $\inf\emptyset = +\infty$. If $u$ is weakly upper semicontinuous and strongly lower semicontinuous on $X_+$, then $v$ is strongly continuous on $\operatorname{int} Y_+$ by [35, Theorem 3.2]. In particular, if $X$ is finite-dimensional and $u$ is continuous on $X_+$, then $v$ is continuous on $\operatorname{int} Y_+$. Another sufficient condition for the continuity of $v$ is the following: $u$ is concave and strongly continuous on $X_+$ (then $u$ is both weakly upper semicontinuous and strongly lower semicontinuous on $X_+$). Now, suppose that $v$ is continuous and concave on $\operatorname{int} Y_+$. Then, for every $\bar{p} \in \operatorname{int} Y_+$, $\widehat{\partial}(-v)(\bar{p})$ and bounded in the weak$^*$ topology by [43, Prop. 3, p. 199]. In fact, since the continuity and concavity of $v$ on $\operatorname{int} Y_+$ imply that $-v$ is locally Lipschitz and convex on $\operatorname{int} Y_+$ (see, e.g., [63, Corollary 3.10]), $\widehat{\partial}(-v)(\bar{p})$ is weakly$^*$ compact for every $\bar{p} \in \operatorname{int} Y_+$.

Let $\varphi : X \to \overline{\mathrm{I\!R}}$ be finite at $\bar{x} \in X$. The *Clarke subdifferential* of $\varphi$ at $\bar{x}$ is given by

$$\partial_C\varphi(\bar{x}) := \{x^* \in X \ : \ (x^*, -1) \in N_{\operatorname{epi}\varphi}((\bar{x}, \varphi(\bar{x}))\},$$

where recall that $N_{\operatorname{epi}\varphi}((\bar{x}, \varphi(\bar{x}))$ is denoted for the Clarke normal cone to $\operatorname{epi}\varphi$ at $(\bar{x}, \varphi(\bar{x}))$ (see Section 1.2).

The relationships between the Clarke subdifferential and the Mordukhovich subdifferential is as follows.

**Theorem 2.11** (See [56, Theorem 3.57]) *Suppose that $X$ is an Asplund space and $\varphi : X \to \overline{\mathbb{R}} \setminus \{-\infty\}$ is finite and lower semicontinuous at $\bar{x} \in X$. Then*

$$\partial_C\varphi(\bar{x}) = \text{cl}^*[\text{co}\,\partial\varphi(\bar{x}) + \text{co}\,\partial^\infty\varphi(\bar{x})] = \text{cl}^*\text{co}\,[\partial\varphi(\bar{x}) + \partial^\infty\varphi(\bar{x})], \qquad (2.65)$$

*where* $\text{co}\,A$ *stands for the convex closure of* $A \subset X^*$ *and* $\text{cl}^*A$ *denotes the weak* topological closure of* $A$.

Following [74], we write $(x,\alpha) \downarrow_\varphi \bar{x}$ when $(x,\alpha) \in \text{epi}\,\varphi$, $x \to \bar{x}$, and $\alpha \to \varphi(\bar{x})$. The *upper subderivative* of $\varphi$ at $\bar{x}$ with respect to $v \in X$ (see [74, formula (4.2)] or [73, formula (2.3)]) is defined by

$$\varphi^\uparrow(\bar{x}; v) := \lim_{\varepsilon \to 0^+} \limsup_{\substack{t \to 0^+ \\ (x,\alpha)\downarrow_\varphi \bar{x}}} \inf_{w \in v + \varepsilon B} \frac{\varphi(x+tw) - \alpha}{t}.$$

By [74, Theorem 2], the function $v \to \varphi^\uparrow(\bar{x}; v)$ is convex, homogeneous, lower semicontinuous, not identical $+\infty$. When $\varphi$ is lower semicontinuous at $\bar{x}$, $\varphi^\uparrow(\bar{x}; v)$ can be given by a slightly simpler expression

$$\varphi^\uparrow(\bar{x}; v) := \lim_{\varepsilon \to 0^+} \limsup_{\substack{t \to 0^+ \\ x \xrightarrow{\varphi} \bar{x}}} \inf_{w \in v + \varepsilon B} \frac{\varphi(x+tw) - \varphi(x)}{t}.$$

**Theorem 2.12** (See [20, p. 97]) *Let $\varphi : X \to \overline{\mathbb{R}}$ be finite at $\bar{x}$. One has $\partial_C\varphi(\bar{x}) = \emptyset$ if and only if $\varphi^\uparrow(\bar{x}; 0) = -\infty$. Otherwise, one has*

$$\partial_C\varphi(\bar{x}) = \{x^* \in X^* \ : \ \varphi^\uparrow(\bar{x}; v) \geq \langle x^*, v\rangle, \ \forall v \in X\},$$

*and*

$$\varphi^\uparrow(\bar{x}; v) = \sup\{\langle x^*, v\rangle \ : \ x^* \in \partial_C\varphi(\bar{x})\} \qquad (2.66)$$

*for any $v \in X$.*

The *generalized Clarke derivative* of $\varphi$ at $\bar{x}$ with respect to $v \in X$ (see [74, formula (5.2)]) is

$$\varphi^0(\bar{x}; v) := \limsup_{\substack{t \to 0^+ \\ (x,\alpha)\downarrow_\varphi \bar{x}}} \frac{\varphi(x+tv) - \alpha}{t}.$$

If $\varphi$ is lower semicontinuous at $\bar{x}$, then the above formula becomes

$$\varphi^0(\bar{x}; v) := \limsup_{\substack{t \to 0^+ \\ x \xrightarrow{\varphi} \bar{x}}} \frac{\varphi(x+tv) - \varphi(x)}{t}.$$

The function $\varphi$ is said to be *directionally Lipschitzian* at $\bar{x}$ with respect to $v \in X$ (see [74, formula (5.2)]) if $f$ is finite at $x$ and

$$\limsup_{\substack{w \to v,\, t \to 0^+ \\ (x,\alpha) \downarrow_\varphi \bar{x}}} \frac{\varphi(x + tw) - \alpha}{t} < +\infty,$$

a condition which can be simplified when $f$ is lower semicontinuous at $x$ to

$$\limsup_{\substack{w \to v,\, t \to 0^+ \\ x \xrightarrow{\varphi} \bar{x}}} \frac{\varphi(x + tw) - \varphi(x)}{t} < +\infty.$$

It follows from the definition that $f$ is Lipschitz around $\bar{x}$ if and only if it is directionally Lipschitzian at $\bar{x}$ w.r.t. $v = 0$. One says that $f$ is *directionally Lipschitzian* at $\bar{x}$ if there is at least one $v$, not necessarily 0, such that $f$ is directionally Lipschitzian at $\bar{x}$ w.r.t. $v$. We refer to [74, Sect. 6] for some conditions guaranteeing that $f$ is directionally Lipschitzian at $\bar{x}$.

**Theorem 2.13** (See [74, Theorem 3]) *Let $\varphi : X \to \overline{\mathbb{R}}$ be finite at $\bar{x}$. If $\varphi$ is directionally Lipschitzian at $\bar{x}$ w.r.t. $v \in X$, then $\varphi^{\uparrow}(\bar{x}; v) = \varphi^{0}(\bar{x}; v)$.*

Turning back to the indirect utility function $v$ of (1.3), we suppose that $v$ is finite at $\bar{p} \in \operatorname{int} Y_+$. Combining formulas (2.43) and (2.44) (resp., (2.55) and (2.56)), we obtain an upper estimate for the sum $\partial(-v)(\bar{p}) + \partial^{\infty}(-v)(\bar{p})$. Thus, if $v$ is upper semicontinuous at $\bar{p}$ and $v(p) \neq +\infty$ for all $p \in Y_+$, then by (2.65) we get an upper estimate for the Clarke subdifferential $\partial_C(-v)(\bar{p})$. In other words, *we find a subset $A$ of $X$ in an explicit form such that* $\partial_C(-v)(\bar{p}) \subset A$. Since $0 \in \partial^{\infty}(-v)(\bar{p})$, one has $\partial_C(-v)(\bar{p}) \neq \emptyset$ if and only if $\partial(-v)(\bar{p}) \neq \emptyset$. Therefore, if the latter is valid then, for any $q \in X^*$, by (2.66) one has

$$(-v)^{\uparrow}(\bar{p}; q) = \sup_{\xi \in \partial_C(-v)(\bar{p})} \langle \xi, q \rangle;$$

hence

$$(-v)^{\uparrow}(\bar{p}; q) \leq \sup_{\xi \in A} \langle \xi, q \rangle =: \alpha(q).$$

So, if $v$ is upper semicontinuous at $\bar{p}$, then

$$\lim_{\varepsilon \to 0^+} \limsup_{\substack{t \to 0^+ \\ p \xrightarrow{v} \bar{p}}} \inf_{w \in q + \varepsilon B} \frac{(-v)(p + tw) - (-v)(p)}{t} = \sup_{\xi \in \partial_C(-v)(\bar{p})} \langle \xi, q \rangle, \tag{2.67}$$

and

$$\lim_{\varepsilon \to 0^+} \limsup_{\substack{t \to 0^+ \\ p \xrightarrow{v} \bar{p}}} \inf_{w \in q + \varepsilon B} \frac{(-v)(p + tw) - (-v)(p)}{t} \leq \alpha(q). \tag{2.68}$$

If, in addition, $-v$ is directionally Lipschitzian at $\bar{p}$ w.r.t. $q$, then by Theorem 2.13 one has $(-v)^{\uparrow}(\bar{p}; q) = (-v)^{0}(\bar{p}; q)$. Hence, (2.67) and (2.68) respectively yield

$$\limsup_{\substack{t \to 0^+ \\ p \xrightarrow{v} \bar{p}}} \frac{(-v)(p + tq) - (-v)(p)}{t} = \sup_{\xi \in \partial_C(-v)(\bar{p})} \langle \xi, q \rangle, \tag{2.69}$$

and

$$\limsup_{\substack{t \to 0^+ \\ p \xrightarrow{v} \bar{p}}} \frac{(-v)(p + tq) - (-v)(p)}{t} \leq \alpha(q). \tag{2.70}$$

Formula (2.69) is equivalent to

$$\liminf_{\substack{t \to 0^+ \\ p \xrightarrow{v} \bar{p}}} \frac{v(p + tq) - v(p)}{t} = - \sup_{\xi \in \partial_C(-v)(\bar{p})} \langle \xi, q \rangle,$$

and therefore

$$d^{-}v(\bar{p}; q) := \liminf_{t \to 0^+} \frac{v(\bar{p} + tq) - v(\bar{p})}{t} \geq - \sup_{\xi \in \partial_C(-v)(\bar{p})} \langle \xi, q \rangle,$$

with $d^{-}v(\bar{p}; q)$ denoting the *lower Dini directional derivative* of $v$ at $\bar{p}$ in direction $q$. Similarly, (2.70) is equivalent to

$$\liminf_{\substack{t \to 0^+ \\ p \xrightarrow{v} \bar{p}}} \frac{v(p + tq) - v(p)}{t} \geq -\alpha(q),$$

and thus we have the following lower estimate for the lower Dini directional derivative of $v$ at $\bar{p}$ in direction $q$:

$$d^{-}v(\bar{p}; q) \geq -\alpha(q). \tag{2.71}$$

In the first part of this section, we have explained the economic meaning of an upper estimate for the upper Dini directional derivative $d^{+}v(\bar{p}; q)$. Analogously, *the lower Dini directional derivative $d^{-}v(\bar{p}; q)$ can be interpreted as an lower bound for the instant rate of the change of the maximal satisfaction of the consumer.* Therefore, the estimate given by (2.71) reads as follows: *If the current price is $\bar{p}$ and the price moves forward a direction $q \in X^*$, then the instant rate of the change of the maximal satisfaction of the consumer is bounded below by the real number $-\alpha(q) = \inf_{\xi \in A} \langle -\xi, q \rangle$. Here, the set $A$ is an upper bound for the Clarke subdifferential $\partial_C(-v)(\bar{p})$, which is provided either by Theorem 2.9 or by Theorem 2.10.*

## 2.6 Conclusions

We have studied the differential stability of the budget map and the indirect utility function of the parametric consumer problem. Namely, by utilizing some nice features of the budget map, we have been able to establish formulas for estimating the Fréchet, limiting, and singular subdifferentials of the infimal nuisance function, which is obtained from the indirect utility function by changing its sign. Besides, economic meanings of the obtained subdifferential estimates are explained in details.

# Chapter 3

# Parametric Optimal Control Problems with Unilateral State Constraints

This chapter is written on the basis of the paper [37]. In the present chapter and the following one, a maximum principle for finite horizon optimal control problems with state constraints is analyzed via three parametric examples. The difference among those are in the appearance of state constraints: The first one does not contain state constraints, the second one is a problem with unilateral state constraints, and the third one is a problem with bilateral state constraints. The first two problems are studied herein. The last one with bilateral state constraints will be addressed in the next chapter. These problems have the origin in the studies of a recent paper by Basco, Cannarsa, and Frankowska [12, 12, Example 1] and of the book by Clarke [21, Section 22.2 and Exercise 26.1], where the choices of the objective functions and the differential equations in these problems are motivated by optimal economic growth models (see Chapter 5). Thus, our analysis in this and the next chapter not only helps to understand the maximum principle in depth, but also serves as a sample of applying it to meaningful prototypes of economic optimal growth models.

Throughout Chapters 3–5, we will use the following standard notations. The norm (resp., the inner product) in the $n$-dimensional Euclidean space $\mathbb{R}^n$ is denoted by $\|.\|$ (resp., $\langle \cdot, \cdot \rangle$). For a given segment $[t_0, T]$ of the real line, we denote the $\sigma$-algebra of its Lebesgue measurable subsets (resp., the $\sigma$-algebra of its Borel sets) by $\mathcal{L}$ (resp., $\mathcal{B}$). The $\sigma$-algebra of the Borel sets in $\mathbb{R}^m$ is denoted by $\mathcal{B}^m$. The Sobolev space $W^{1,1}([t_0, T], \mathbb{R}^n)$ is the linear space of the absolutely continuous functions $x : [t_0, T] \to \mathbb{R}^n$ endowed with the norm

$$\|x\|_{W^{1,1}} = \|x(t_0)\| + \int_{t_0}^{T} \|\dot{x}(t)\| dt \tag{3.1}$$

(see, e.g., [43, p. 21] for this and another equivalent norm).

The normal cone $N(\bar{x};\Omega)$ of a subset $\Omega \subset \mathbb{R}^n$ (resp., the subdifferential $\partial\varphi(\bar{x})$ of an extended real-valued function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$) at a point $\bar{x}$ is understood in the sense of the Mordukhovich normal cone (resp., the Mordukhovich subdifferential) given in Section 2.1.

## 3.1 Problem Statement

Let $a > \lambda > 0$, $T > t_0 \geq 0$, and $x_0 \in \mathbb{R}$ be given as five parameters. In this chapter, we consider two finite horizon optimal control problems of the Lagrange type denoted by $(FP_1)$ and $(FP_2)$.

The first problem $(FP_1)$ is the following

$$\text{Minimize } \ J(x,u) = \int_{t_0}^{T} \big[ - e^{-\lambda t}(x(t) + u(t))\big] dt$$

over $x \in W^{1,1}([t_0,T],\mathbb{R})$ and measurable functions $u : [t_0,T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{x}(t) = -au(t), & \text{a.e. } t \in [t_0,T] \\ x(t_0) = x_0 & \\ u(t) \in [-1,1], & \text{a.e. } t \in [t_0,T]. \end{cases} \tag{3.2}$$

The second problem $(FP_2)$ is formed from $(FP_1)$ by adding the requirement $x(t) \leq 1$ for all $t \in [t_0,T]$ to the constraint system (3.2), i.e.,

$$\text{Minimize } \ J(x,u) = \int_{t_0}^{T} \big[ - e^{-\lambda t}(x(t) + u(t))\big] dt$$

over $x \in W^{1,1}([t_0,T],\mathbb{R})$ and measurable functions $u : [t_0,T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{x}(t) = -au(t), & \text{a.e. } t \in [t_0,T] \\ x(t_0) = x_0 & \\ u(t) \in [-1,1], & \text{a.e. } t \in [t_0,T] \\ x(t) \leq 1, & \forall t \in [t_0,T]. \end{cases}$$

Later, we will treat $(FP_1)$ and $(FP_2)$ as problems of the Mayer type by setting $x(t) = (x_1(t), x_2(t))$, where $x_1(t)$ plays the role of the variable $x(t)$ in $(FP_1)$ and $(FP_2)$, and

$$x_2(t) := \int_{t_0}^{t} \big[ - e^{-\lambda\tau}(x_1(\tau) + u(\tau))\big] d\tau \tag{3.3}$$

for all $t \in [0, T]$. Then, $(FP_1)$ is equivalent to the problem

$$\text{Minimize } \; x_2(T)$$

over $x = (x_1, x_2) \in W^{1,1}([t_0, T], \mathrm{I\!R}^2)$ and measurable functions $u : [t_0, T] \to \mathrm{I\!R}$ satisfying

$$\begin{cases} \dot{x}_1(t) = -au(t), & \text{a.e. } t \in [t_0, T] \\ \dot{x}_2(t) = -e^{-\lambda t}(x_1(t) + u(t)), & \text{a.e. } t \in [t_0, T] \\ (x(t_0), x(T)) \in \{(x_0, 0)\} \times \mathrm{I\!R}^2 & \\ u(t) \in [-1, 1], & \text{a.e. } t \in [t_0, T], \end{cases} \tag{3.4}$$

which is abbreviated to $(FP_{1a})$. Similarly, $(FP_2)$ is equivalent to the problem $(FP_{2a})$ described as follows.

$$\text{Minimize } \; x_2(T)$$

over $x = (x_1, x_2) \in W^{1,1}([t_0, T], \mathrm{I\!R}^2)$ and measurable functions $u : [t_0, T] \to \mathrm{I\!R}$ satisfying

$$\begin{cases} \dot{x}_1(t) = -au(t), & \text{a.e. } t \in [t_0, T] \\ \dot{x}_2(t) = -e^{-\lambda t}(x_1(t) + u(t)), & \text{a.e. } t \in [t_0, T] \\ (x(t_0), x(T)) \in \{(x_0, 0)\} \times \mathrm{I\!R}^2 & \\ u(t) \in [-1, 1], & \text{a.e. } t \in [t_0, T] \\ x_1(t) \le 1, & \forall t \in [t_0, T]. \end{cases} \tag{3.5}$$

## 3.2 Auxiliary Concepts and Results

As in [82, p. 321], we consider the following *finite horizon optimal control problem of the Mayer type*, denoted by $\mathcal{M}$,

$$\text{Minimize } \; g(x(t_0), x(T)), \tag{3.6}$$

over $x \in W^{1,1}([t_0, T], \mathbb{R}^n)$ and measurable functions $u : [t_0, T] \to \mathbb{R}^m$ satisfying

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & \text{a.e. } t \in [t_0, T] \\ (x(t_0), x(T)) \in C \\ u(t) \in U(t), & \text{a.e. } t \in [t_0, T] \\ h(t, x(t)) \le 0, & \forall t \in [t_0, T], \end{cases} \tag{3.7}$$

where $[t_0, T]$ is a given interval, $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $f : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $h : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}$ are given functions, $C \subset \mathbb{R}^n \times \mathbb{R}^n$ is a closed set, and $U : [t_0, T] \rightrightarrows \mathbb{R}^m$ is a set-valued map.

A measurable function $u : [t_0, T] \to \mathbb{R}^m$ satisfying $u(t) \in U(t)$ for almost every $t \in [t_0, T]$ is called a *control function*. A *process* $(x, u)$ consists of a control function $u$ and an arc $x \in W^{1,1}([t_0, T]; \mathbb{R}^n)$ that is a solution to the differential equation in (3.7). A *state trajectory* $x$ is the first component of some process $(x, u)$. A process $(x, u)$ is called *feasible* if the state trajectory satisfies the *endpoint constraint* $(x(t_0), x(T)) \in C$ and the *state constraint* $h(t, x(t)) \le 0$ for all $t \in [t_0, T]$.

Due to the appearance of the *state constraint* $h(t, x(t)) \le 0, t \in [t_0, T]$, the problem $\mathcal{M}$ in (3.6)–(3.7) is said to be an *optimal control problem with state constraints* or a *state constrained optimal control problem.* But, if the state constraint is fulfilled for any state trajectory $x$ with $(x(t_0), x(T)) \in C$, i.e., the state constraint can be removed from (3.7), then one says that $\mathcal{M}$ an *optimal control problem without state constraints* or an *unconstrained optimal control problem.*

**Definition 3.1** A feasible process $(\bar{x}, \bar{u})$ is called a $W^{1,1}$ *local minimizer* for $\mathcal{M}$ if there exists $\delta > 0$ such that

$$g(\bar{x}(t_0), \bar{x}(T)) \le g(x(t_0), x(T)) \tag{3.8}$$

for any feasible process $(x, u)$ satisfying $\|\bar{x} - x\|_{W^{1,1}} \le \delta$. If (3.8) holds true for every feasible process $(x, u)$, then $(\bar{x}, \bar{u})$ is called a $W^{1,1}$ *global minimizer* for $\mathcal{M}$.

**Definition 3.2** The *Hamiltonian* $\mathcal{H} : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ of (3.7) is defined by

$$\mathcal{H}(t, x, p, u) := \langle p, f(t, x, u) \rangle = \sum_{i=1}^{n} p_i f_i(t, x, u). \tag{3.9}$$

**Definition 3.3** The *partial hybrid subdifferential* (see [82, p. 329]) $\partial_x^{>} h(t,x)$ of $h(t,x)$ w.r.t. $x$ is given by

$$\partial_x^{>} h(t,x) := \text{co}\,\big\{\xi \,:\, \text{there exists } (t_k,x_k) \xrightarrow{h} (t,x) \text{ such that } h(t_k,x_k) > 0 \text{ for all } k \text{ and } \nabla_x h(t_k,x_k) \to \xi\big\}, \qquad (3.10)$$

where $(t_k,x_k) \xrightarrow{h} (t,x)$ means that $(t_k,x_k) \to (t,x)$ and $h(t_k,x_k) \to h(t,x)$ as $k \to \infty$.

To deal with the state constraint $h(t,x(t)) \le 0$ in $\mathcal{M}$, one has to introduce a multiplier that is an element in the topological dual $C^*([t_0,T];\mathbb{R})$ of the space of continuous functions $C([t_0,T];\mathbb{R})$ with the supremum norm. By the Riesz Representation Theorem (see, e.g., [49, Theorem 6, p. 374] and [53, Theorem 1, pp. 113–115]), any bounded linear functional $f$ on $C([t_0,T];\mathbb{R})$ can be uniquely represented in the form

$$f(x) = \int_{[t_0,T]} x(t)dv(t),$$

where $v$ is *a function of bounded variation* on $[t_0,T]$ which vanishes at $t_0$ and which are continuous from the right at every point $\tau \in (t_0,T)$, and $\displaystyle\int_{[t_0,T]} x(t)dv(t)$ is the Riemann-Stieltjes integral of $x$ with respect to $v$ (see, e.g., [49, p. 364]). The set of the elements of $C^*([t_0,T];\mathbb{R})$ which are given by nondecreasing functions $v$ is denoted by $C^{\oplus}(t_0,T)$.

Every $v \in C^*([t_0,T];\mathbb{R})$ corresponds to *a finite regular measure*, denoted by $\mu_v$, on the $\sigma$-algebra $\mathcal{B}$ of the Borel subsets of $[t_0,T]$ by the formula

$$\mu_v(A) := \int_{[t_0,T]} \chi_A(t)dv(t),$$

where

$$\chi_A(t) := \begin{cases} 1, & \text{if } \ t \in A \\ 0, & \text{if } \ t \notin A. \end{cases}$$

Due to the correspondence $v \mapsto \mu_v$, we call every element $v \in C^*([t_0,T];\mathbb{R})$ a "measure" and identify $v$ with $\mu_v$. Clearly, the measure corresponding to each $v \in C^{\oplus}(t_0,T)$ is nonnegative.

The integrals $\displaystyle\int_{[t_0,t)} \nu(s)d\mu(s)$ and $\displaystyle\int_{[t_0,T]} \nu(s)d\mu(s)$ of a Borel measurable function $\nu$ in the next theorem are understood in the sense of the Lebesgue-Stieltjes integration [49, p. 364].

**Theorem 3.1** (See [82, Theorem 9.3.1]) *Let $(\bar{x}, \bar{u})$ be a $W^{1,1}$ local minimizer for $\mathcal{M}$. Assume that for some $\delta > 0$, the following hypotheses are satisfied:*

(H1) *$f(\cdot, x, \cdot)$ is $\mathcal{L} \times \mathcal{B}^m$ measurable, for fixed $x$. There exists a Borel measurable function $k(\cdot, \cdot) : [t_0, T] \times \mathbb{R}^m \to \mathbb{R}$ such that the function $t \mapsto k(t, \bar{u}(t))$ is integrable and, for almost every $t \in [t_0, T]$,*

$$\|f(t,x,u) - f(t,x',u)\| \leq k(t,u)\|x - x'\|, \forall x, x' \in \bar{x}(t) + \delta \bar{B}_{\mathbb{R}^n}, \forall u \in U(t);$$

(H2) $\operatorname{gph} U$ *is a Borel set in $[t_0, T] \times \mathbb{R}^m$;*

(H3) *$g$ is Lipschitz continuous on the ball $(\bar{x}(t_0), \bar{x}(T)) + \delta \bar{B}_{\mathbb{R}^{2n}}$;*

(H4) *$h$ is upper semicontinuous and there exists $K > 0$ such that*

$$\|h(t,x) - h(t,x')\| \leq K\|x - x'\|, \quad \forall x, x' \in \bar{x}(t) + \delta \bar{B}_{\mathbb{R}^n}, \ \forall t \in [t_0, T].$$

*Then there exist $p \in W^{1,1}([t_0, T]; \mathbb{R}^n)$, $\gamma \geq 0$, $\mu \in C^{\oplus}(t_0, T)$, and a Borel measurable function $\nu : [t_0, T] \to \mathbb{R}^n$ such that $(p, \mu, \gamma) \neq (0, 0, 0)$, and for $q(t) := p(t) + \eta(t)$ with*

$$\eta(t) := \int_{[t_0, t)} \nu(s) d\mu(s), \quad \text{if } t \in [t_0, T)$$

*and*

$$\eta(T) := \int_{[t_0, T]} \nu(s) d\mu(s),$$

*the following holds true:*

(i) $\nu(t) \in \partial_x^{>} h(t, \bar{x}(t))$ *$\mu$-a.e.*;

(ii) $-\dot{p}(t) \in \operatorname{co} \partial_x \mathcal{H}(t, \bar{x}(t), q(t), \bar{u}(t))$ *a.e.;*

(iii) $(p(t_0), -q(T)) \in \gamma \partial g(\bar{x}(t_0), \bar{x}(T)) + N((\bar{x}(t_0), \bar{x}(T)); C)$*;*

(iv) $\mathcal{H}(t, \bar{x}(t), q(t), \bar{u}(t)) = \max_{u \in U(t)} \mathcal{H}(t, \bar{x}(t), q(t), u)$ *a.e.*

Applying Theorem 3.1 to unconstrained optimal control problems, one has the next proposition.

**Proposition 3.1** (See [82, Theorem 6.2.1]) *Suppose that $\mathcal{M}$ is an optimal control problem without state constraints. Let $(\bar{x}, \bar{u})$ be a $W^{1,1}$ local minimizer for $\mathcal{M}$. Assume that for some $\delta > 0$, the following hypotheses are satisfied.*

(H1) *The function* $f(\cdot, x, \cdot) : [t_0, T] \times \mathbb{R}^m \to \mathbb{R}^n$ *is* $\mathcal{L} \times \mathcal{B}^m$ *measurable, for every* $x \in \mathbb{R}^n$. *In addition, there exits a Borel measurable function* $k : [t_0, T] \times \mathbb{R}^m \to \mathbb{R}$ *such that* $t \mapsto k(t, \bar{u}(t))$ *is integrable and, for almost every* $t \in [t_0, T]$,

$$\|f(t, x, u) - f(t, x', u)\| \leq k(t, u)\|x - x'\|, \ \ \forall x, x' \in \bar{x}(t) + \delta \bar{B}_{\mathbb{R}^n}, u \in U(t);$$

(H2) $\operatorname{gph} U$ *is an* $\mathcal{L} \times \mathcal{B}^m$ *measurable set in* $[t_0, T] \times \mathbb{R}^m$;

(H3) $g$ *is locally Lipschitz continuous.*

*Then there exist* $p \in W^{1,1}([t_0, T]; \mathbb{R}^n)$ *and* $\gamma \geq 0$ *such that* $(p, \gamma) \neq (0, 0)$ *and the following holds true:*

(i) $-\dot{p}(t) \in \operatorname{co} \partial_x \mathcal{H}(t, \bar{x}(t), p(t), \bar{u}(t))$ *a.e.;*

(ii) $(p(t_0), -p(T)) \in \gamma \partial g(\bar{x}(t_0), \bar{x}(T)) + N((\bar{x}(t_0), \bar{x}(T)); C)$;

(iii) $\mathcal{H}(t, \bar{x}(t), p(t), \bar{u}(t)) = \max_{u \in U(t)} \mathcal{H}(t, \bar{x}(t), p(t), u)$.

To recall a solution existence theorem for optimal control problems with state constraints of the Mayer type, we will use the notations and concepts given in [18, Section 9.2]. Let $A$ be a subset of $\mathbb{R} \times \mathbb{R}^n$ and $U : A \rightrightarrows \mathbb{R}^m$ be a set-valued map defined on $A$. Let

$$M := \{(t, x, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \ : \ (t, x) \in A, \ u \in U(t, x)\},$$

and $f = (f_1, f_2, \ldots, f_n) : M \to \mathbb{R}^n$ be a single-valued map defined on $M$. Let $B$ be a given subset of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and $g : B \to \mathbb{R}$ be a real function defined on $B$. Consider the optimal control problem of the Mayer type

$$\text{Minimize } \ g(t_0, x(t_0), T, x(T)) \tag{3.11}$$

over $x \in W^{1,1}([t_0, T]; \mathbb{R}^n)$ and measurable functions $u : [t_0, T] \to \mathbb{R}^m$ satisfying

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & \text{a.e. } t \in [t_0, T] \\ (t, x(t)) \in A, & \text{for all } t \in [t_0, T] \\ (t_0, x(t_0), T, x(T)) \in B & \\ u(t) \in U(t, x(t)), & \text{a.e. } t \in [t_0, T], \end{cases} \tag{3.12}$$

where $[t_0, T]$ is a given interval. The problem (3.11)–(3.12) is denoted by $\mathcal{M}_1$.

A *feasible process* for $\mathcal{M}_1$ is a pair of functions $(x, u)$ with $x : [t_0, T] \to \mathbb{R}^n$ being absolutely continuous on $[t_0, T]$, $u : [t_0, T] \to \mathbb{R}^m$ being measurable, such that all the requirements in (3.12) are satisfied. If $(x, u)$ is a feasible process for $\mathcal{M}_1$, then $x$ is said to be a *feasible trajectory*, and $u$ a *feasible control function* for $\mathcal{M}_1$. The set of all feasible processes for $\mathcal{M}_1$ is denoted by $\Omega$.

Let $A_0 = \{t \in \mathbb{R} : \exists x \in \mathbb{R}^n \text{ s.t. } (t, x) \in A\}$, i.e., $A_0$ is the projection of $A$ on the $t$-axis. Set

$$A(t) = \{x \in \mathbb{R}^n \ : \ (t, x) \in A\}, \quad t \in A_0$$

and

$$Q(t, x) = \{z \in \mathbb{R}^n \ : \ z = f(t, x, u), \ u \in U(t, x)\}, \quad (t, x) \in A.$$

The forthcoming statement is called *Filippov's Existence Theorem for Mayer problems.*

**Theorem 3.2** (See [18, Theorem 9.2.i and Section 9.4]) *Suppose that $\Omega$ is nonempty, $B$ is closed, $g$ is lower semicontinuous on $B$, $f$ is continuous on $M$ and, for almost every $t \in [t_0, T]$, the sets $Q(t, x)$, $x \in A(t)$, are convex. Moreover, assume either that $A$ and $M$ are compact or that $A$ is not compact but closed and the following three conditions hold*

(a) *For any $\varepsilon \geq 0$, the set $M_\varepsilon := \{(t, x, u) \in M \ : \ \|x\| \leq \varepsilon\}$ is compact;*

(b) *There is a compact subset $P$ of $A$ such that every feasible trajectory $x$ of $\mathcal{M}_1$ passes through at least one point of $P$;*

(c) *There exists a constant $c \geq 0$ such that $\langle x, f(t, x, u)\rangle \leq c(\|x\|^2 + 1)$ for all $(t, x, u)$ in $M$.*

*Then, $\mathcal{M}_1$ has a $W^{1,1}$ global minimizer.*

Clearly, condition (b) is satisfied if the initial point $(t_0, x(t_0))$ or the end point $(T, x(T))$ is fixed. As shown in [18, p. 317], the following condition yields (c):

($c_0$) *There exists a constant $c \geq 0$ such that $\|f(t, x, u)\| \leq c(\|x\| + 1)$ for all $(t, x, u)$ in $M$.*

## 3.3 Solution Existence

It is not hard to see that $(FP_{1a})$ is a problem of the Mayer form $\mathcal{M}_1$ with $n = 2$, $m = 1$, $A = [t_0, T] \times \mathbb{R}^2$, $U(t,x) = [-1,1]$ for all $(t,x) \in A$, $B = \{t_0\} \times \{(x_0, 0)\} \times \mathbb{R} \times \mathbb{R}^2$, $g(t_0, x(t_0), T, x(T)) = x_2(T)$, $M = A \times [-1,1]$, $f(t,x,u) = (-au, -e^{-\lambda t}(x_1 + u))$ for all $(t,x,u) \in M$. We are going to show that $(FP_{1a})$ satisfies all the assumptions of Theorem 3.2.

Clearly, the pair $(x,u)$, with $u(t) = 0$, $x_1(t) = x_0$, and $x_2(t) = -x_0 \displaystyle\int_{t_0}^{t} e^{-\lambda\tau} d\tau$ for all $t \in [t_0, T]$, is a feasible process for $(FP_{1a})$. Thus, the set $\Omega$ of feasible processes is nonempty. Besides, $B$ is closed, $g$ is lower semicontinuous on $B$, $f$ is continuous on $M$. Moreover, from the formula for $A$, one has $A_0 = [t_0, T]$ and $A(t) = \mathbb{R}^2$ for all $t \in A_0$. In addition, by the formulas for $M$, $U$, and $f$, one gets

$$\begin{aligned} Q(t,x) &= \big\{z \in \mathbb{R}^2 \ : \ z = f(t,x,u),\ u \in U(t,x)\big\} \\ &= \big\{z \in \mathbb{R}^2 \ : \ z = (-au, -e^{-\lambda t}(x_1 + u)),\ u \in [-1,1]\big\} \\ &= \big\{(0, -e^{-\lambda t}x_1)\big\} + \{(-a, -e^{-\lambda t})u \ : \ u \in [-1,1]\big\} \end{aligned}$$

for any $(t,x) \in A$. Thus, for every $t \in [t_0, T]$, the sets $Q(t,x)$, $x \in A(t)$, are line segments; hence they are convex. Since $A$ is closed, but not compact, we have to check the conditions (a)–(c) in Theorem 3.2.

*Condition* (a): For any $\varepsilon \geq 0$, since

$$\begin{aligned} M_\varepsilon &= \{(t,x,u) \in M \ : \ \|x\| \leq \varepsilon\} \\ &= \{(t,x,u) \in [t_0, T] \times \mathbb{R}^2 \times [-1,1] \ : \ \|x\| \leq \varepsilon\} \\ &= [t_0, T] \times \{x \in \mathbb{R}^2 \ : \ \|x\| \leq \varepsilon\} \times [-1,1], \end{aligned}$$

one sees that $M_\varepsilon$ is compact.

*Condition* (b): Obviously, $P := \{t_0\} \times \{(x_0, 0)\}$ is a compact subset of $A$, and every feasible trajectory passes through the unique point of $P$. Thus, condition (b) is fulfilled.

*Condition* (c): Choosing $c = a + 1$, we have

$$\begin{aligned} \|f(t,x,u)\| = \|(-au, -e^{-\lambda t}(x_1 + u))\| &\leq a|u| + e^{-\lambda t}|x_1 + u| \\ &\leq a + |x_1| + 1 \\ &\leq c(\|x\| + 1) \end{aligned}$$

for any $(t,x,u) \in M$, because $u \in [-1,1]$ and $e^{-\lambda t} \le 1$ for $t \ge t_0 \ge 0$. Thus, condition $(c_0)$, which implies (c), is satisfied.

By Theorem 3.2, $(FP_{1a})$ has a $W^{1,1}$ global minimizer. Therefore, $(FP_1)$ has a $W^{1,1}$ global minimizer by the equivalence of $(FP_{1a})$ and $(FP_1)$.

To check that $(FP_{2a})$ is of the form $\mathcal{M}_1$, we choose $n = 2$, $m = 1$,

$$A = [t_0, T] \times (-\infty, 1] \times \mathbb{R},$$

$U(t,x) = [-1,1]$ for all $(t,x) \in A$, $B = \{t_0\} \times \{(x_0, 0)\} \times \mathbb{R} \times \mathbb{R}^2$,

$$g(t_0, x(t_0), T, x(T)) = x_2(T),$$

$M = A \times [-1,1]$, $f(t,x,u) = (-au, -e^{-\lambda t}(x_1 + u))$ for all $(t,x,u) \in M$. In comparison with the problem $(FP_{1a})$, the only change in the formulation of $(FP_{2a})$ is that we have $A = [t_0, T] \times (-\infty, 1] \times \mathbb{R}$ instead of $A = [t_0, T] \times \mathbb{R}^2$. Thus, to show that $(FP_{2a})$ satisfies all the assumptions of Theorem 3.2, we can use the arguments showing that $(FP_{1a})$ has a $W^{1,1}$ global minimizer, except those related to the convexity of the sets $Q(t,x)$ and the compactness of $M_\varepsilon$, which have to be verified in a slightly different manner.

By the above formula for $A$, we have $A_0 = [t_0, T]$ and $A(t) = (-\infty, 1] \times \mathbb{R}$ for all $t \in A_0$, and

$$Q(t,x) = \{(0, -e^{-\lambda t} x_1)\} + \{(-a, -e^{-\lambda t})u \ : \ u \in [-1,1]\}$$

for any $(t,x) \in A$. Thus, the assumption of Theorem 3.2 on the convexity of the sets $Q(t,x)$, $x \in A(t)$, for almost every $t \in [t_0, T]$, is satisfied. Since

$$M = [t_0, T] \times (-\infty, 1] \times \mathbb{R} \times [-1,1],$$

for any $\varepsilon \ge 0$, one has

$$\begin{aligned} M_\varepsilon &= \{(t,x,u) \in M \ : \ \|x\| \le \varepsilon\} \\ &= \{(t,x,u) \in [t_0, T] \times (-\infty, 1] \times \mathbb{R} \times [-1,1] \ : \ \|x\| \le \varepsilon\}. \end{aligned}$$

As $M_\varepsilon$ is closed and contained in the compact set

$$[t_0, T] \times \{x \in \mathbb{R}^2 \ : \ \|x\| \le \varepsilon\} \times [-1,1],$$

it is compact.

It follows from Theorem 3.2 that $(FP_{2a})$ has a $W^{1,1}$ global minimizer. Therefore, by the equivalence of $(FP_2)$ and $(FP_{2a})$, we can assert that $(FP_2)$ has a $W^{1,1}$ global minimizer.

## 3.4 Optimal Processes for Problems without State Constraints

To obtain necessary optimality conditions for $(FP_{1a})$, we note that $(FP_{1a})$ is in the form of $\mathcal{M}$ with $g(x,y) = y_2$, $f(t,x,u) = (-au, -e^{-\lambda t}(x_1+u))$, $C = \{(x_0, 0)\} \times \mathbb{R}^2$, $U(t) = [-1,1]$, and $h(t,x) = 0$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$, $t \in [t_0, T]$, and $u \in \mathbb{R}$. Since $(FP_{1a})$ is an optimal control problem without state constraints, we can apply both Proposition 3.1 and Theorem 3.1 to this problem. In accordance with (3.9), the Hamiltonian of $(FP_{1a})$ is given by

$$\mathcal{H}(t,x,p,u) = -aup_1 - e^{-\lambda t}(x_1+u)p_2, \ \ \forall (t,x,p,u) \in [t_0,T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}, \tag{3.13}$$

while by (3.10) we have $\partial_x^{>} h(t,x) = \emptyset$ for all $(t,x) \in [t_0,T] \times \mathbb{R}^2$. Let $(\bar{x}, \bar{u})$ be a $W^{1,1}$ local minimizer of $(FP_{1a})$.

We first study necessary optimality conditions for $(FP_{1a})$ in terms of Proposition 3.1. It is clear that the assumptions (H1)–(H3) of Proposition 3.1 are satisfied for $(FP_{1a})$. Thus, there exist $p \in W^{1,1}([t_0,T]; \mathbb{R}^2)$ and $\gamma \geq 0$ such that $(p,\gamma) \neq (0,0)$, and conditions (i)–(iii) of Proposition 3.1 hold true. Let us analyze these conditions.

**Condition (i)**: By (3.13), $\mathcal{H}$ is differentiable in $x$ and

$$\partial_x \mathcal{H}(t,x,p,u) = \{(-e^{-\lambda t}p_2, 0)\}, \quad \forall (t,x,p,u) \in [t_0,T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}.$$

Thus, condition (i) implies that $\dot{p}_1(t) = e^{-\lambda t} p_2(t)$ for a.e. $t \in [t_0, T]$ and $p_2(t)$ is a constant for all $t \in [t_0, T]$.

**Condition (ii)**: By the formulas for $g$ and $C$, we have

$$\partial g(\bar{x}(t_0), \bar{x}(T)) = \{(0,0,0,1)\} \quad \text{and} \quad N((\bar{x}(t_0), \bar{x}(T)); C) = \mathbb{R}^2 \times \{(0,0)\}.$$

Thus, condition (ii) implies that

$$(p(t_0), -p(T)) \in \gamma\{(0,0,0,1)\} + \mathbb{R}^2 \times \{(0,0)\};$$

hence $p_1(T) = 0$ and $p_2(T) = -\gamma$. As $p_2(t)$ is a constant function, it follows that $p_2(t) = -\gamma$ for all $t \in [t_0, T]$. Therefore, the above analysis of condition (i) gives $p_1(t) = \dfrac{\gamma}{\lambda}(e^{-\lambda t} - e^{-\lambda T})$ for all $t \in [t_0, T]$. Since $(p, \gamma) \neq (0,0)$, we must have $\gamma > 0$.

**Condition (iii)**: Due to (3.13), condition (iii) means that

$$-a\bar{u}(t)p_1(t)-e^{-\lambda t}[x_1(t)+\bar{u}(t)]p_2(t) = \max_{u\in[-1,1]} \left\{-aup_1(t) - e^{-\lambda t}[x_1(t)+u]p_2(t)\right\}$$

for a.e. $t \in [t_0, T]$. Equivalently,

$$[ap_1(t) + e^{-\lambda t}p_2(t)]\bar{u}(t) = \min_{u\in[-1,1]} \left\{[ap_1(t) + e^{-\lambda t}p_2(t)]u\right\}, \quad \text{a.e. } t \in [t_0, T]. \tag{3.14}$$

Setting $\varphi(t) := ap_1(t) + e^{-\lambda t}p_2(t)$ for $t \in [t_0, T]$, we have

$$\varphi(t) = a\frac{\gamma}{\lambda}(e^{-\lambda t} - e^{-\lambda T}) - \gamma e^{-\lambda t} = \gamma(\frac{a}{\lambda} - 1)e^{-\lambda t} - \gamma\frac{a}{\lambda}e^{-\lambda T}, \quad \forall t \in [t_0, T].$$

As $\dfrac{a}{\lambda} > 1$, $\varphi$ is decreasing on $\mathbb{R}$. Besides, it is clear that $\varphi(T) = -\gamma e^{-\lambda T} < 0$, and $\varphi(t) = 0$ if and only if $t = \bar{t}$, where

$$\bar{t} := T - \frac{1}{\lambda}\ln\frac{a}{a-\lambda}.$$

We have the following cases.

Case A: $t_0 \geq \bar{t}$. Then $\varphi(t) < 0$ for all $t \in (t_0, T]$. Therefore, condition (3.14) implies $\bar{u}(t) = 1$ for all $t \in [t_0, T]$. Hence, by (3.4), $\bar{x}_1(t) = x_0 - a(t-t_0)$ for a.e. $t \in [t_0, T]$.

Case B: $t_0 < \bar{t}$. Then $\varphi(t) > 0$ for $t \in [t_0, \bar{t})$ and $\varphi(t) < 0$ for $t \in (\bar{t}, T]$. Thus, (3.14) yields $\bar{u}(t) = -1$ for $t \in [t_0, \bar{t})$ and $\bar{u}(t) = 1$ for a.e. $t \in (\bar{t}, T]$; hence $\bar{x}_1(t) = x_0 + a(t - t_0)$ for every $t \in [t_0, \bar{t}]$ and $\bar{x}_1(t) = x_0 - a(t + t_0 - 2\bar{t})$ for every $t \in (\bar{t}, T]$.

Now, let us examine necessary optimality conditions for $(FP_{1a})$ in terms of Theorem 3.1. Since the assumptions (H1)–(H4) of Theorem 3.1 are satisfied for $(FP_{1a})$, by that theorem one can find $p \in W^{1,1}([t_0, T]; \mathbb{R}^2)$, $\gamma \geq 0$, $\mu \in C^{\oplus}(t_0, T)$, and a Borel measurable function $\nu : [t_0, T] \to \mathbb{R}^2$ such that $(p, \mu, \gamma) \neq (0, 0, 0)$, and for $q(t) := p(t) + \eta(t)$ with $\eta : [t_0, T] \to \mathbb{R}^2$ being given by

$$\eta(t) := \int_{[t_0,t)} \nu(s)d\mu(s), \quad t \in [t_0, T)$$

and

$$\eta(T) := \int_{[t_0,T]} \nu(s)d\mu(s),$$

conditions (i)–(iv) in Theorem 3.1 hold true.

Since $\partial_x^> h(t, \bar{x}(t)) = \emptyset$ for all $t \in [t_0, T]$, the inclusion $\nu(t) \in \partial_x^> h(t, \bar{x}(t))$ is violated at every $t \in [t_0, T]$. Hence, condition (i) forces $\mu = 0$. We see that condition (iv) is fulfilled and the conditions (ii)–(iv) in Theorem 3.1 recover the conditions (i)–(iii) of Proposition 3.1.

Going back to the original problem $(FP_1)$, we can put the obtained results in the following theorem.

**Theorem 3.3** *Given any $a, \lambda$ with $a > \lambda > 0$, define $\rho = \dfrac{1}{\lambda} \ln \dfrac{a}{a - \lambda} > 0$ and $\bar{t} = T - \rho$. Then, problem $(FP_1)$ has a unique local solution $(\bar{x}, \bar{u})$, which is a global solution, where $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ for almost every $t \in [t_0, T]$ and $\bar{x}(t)$ can be described as follows:*

(a) *If $t_0 \geq \bar{t}$ (i.e., $T - t_0 \leq \rho$), then*

$$\bar{x}(t) = x_0 - a(t - t_0), \quad t \in [t_0, T].$$

(b) *If $t_0 < \bar{t}$ (i.e., $T - t_0 > \rho$), then*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}] \\ x_0 - a(t + t_0 - 2\bar{t}), & t \in (\bar{t}, T]. \end{cases}$$

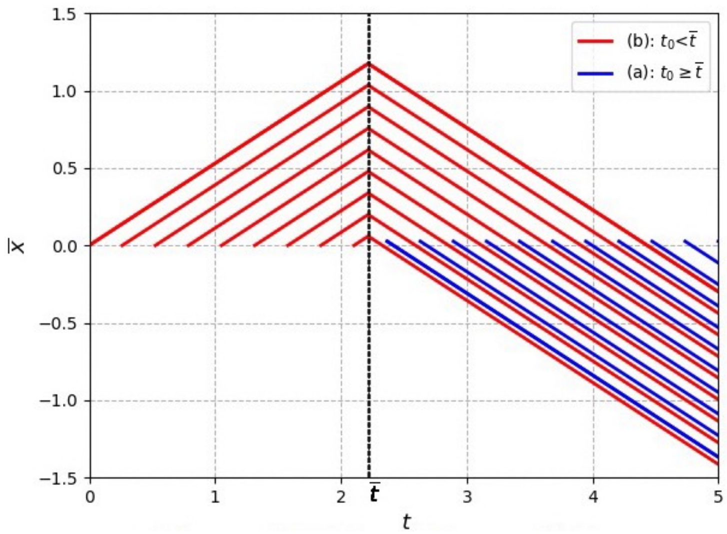


Figure 3.1: The optimal trajectories $\bar{x}(\cdot)$ of $(FP_1)$ w.r.t. parameters $a = 0.53, \lambda = 0.3$, $x_0 = 0$, $T = 5$, and varying $t_0$

**Proof.** The assertions (a) and (b) are straightforward from the results obtained in Case A and Case B, because $\bar{x}_1(t)$ in $(FP_{1a})$ coincides with $\bar{x}(t)$ in $(FP_1)$. $\square$

## 3.5 Optimal Processes for Problems with Unilateral State Constraints

In order to apply Theorem 3.1 for solving $(FP_2)$, we observe that $(FP_{2a})$ is in the form of $\mathcal{M}$ if one defines $g(x,y) = y_2$, $f(t,x,u) = (-au, -e^{-\lambda t}(x_1+u))$, $C = \{(x_0,0)\} \times \mathbb{R}^2$, $U(t) = [-1,1]$, and lets $h(t,x) = x_1 - 1$ for all $t \in [t_0,T]$, $x = (x_1,x_2) \in \mathbb{R}^2$, $y = (y_1,y_2) \in \mathbb{R}^2$ and $u \in \mathbb{R}$.

The forthcoming two propositions describe a fundamental properties of the local minimizers of the problem $(FP_{2a})$, which is obtained from the optimal control problem of the Lagrange type $(FP_2)$ by introducing the artificial variable $x_2$. Similar statements as those in the first proposition are valid for any optimal control problem of the Mayer type, which is obtained from an optimal control problem of the Lagrange type in the same manner. While, the claims in the second proposition hold true for every optimal control problem of the Mayer type, whose objective function does not depend on the initial point.

**Proposition 3.2** *Suppose that $(\bar{x},\bar{u})$ is a $W^{1,1}$ local minimizer for $(FP_{2a})$. Then, for any $\tau_1,\tau_2 \in [t_0,T]$ with $\tau_1 < \tau_2$, the restriction of $(\bar{x},\bar{u})$ on $[\tau_1,\tau_2]$, i.e., the process $(\bar{x}(t),\bar{u}(t))$ with $t \in [\tau_1,\tau_2]$, is a $W^{1,1}$ local minimizer for the following optimal control problem of the Mayer type*

$$\text{Minimize} \;\; x_2(\tau_2)$$

over $x = (x_1,x_2) \in W^{1,1}([\tau_1,\tau_2],\mathbb{R}^2)$ and measurable functions $u : [\tau_1,\tau_2] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{x}_1(t) = -au(t), & \text{a.e. } t \in [\tau_1,\tau_2] \\ \dot{x}_2(t) = -e^{-\lambda t}(x_1(t)+u(t)), & \text{a.e. } t \in [\tau_1,\tau_2] \\ (x(\tau_1),x(\tau_2)) \in \{(\bar{x}_1(\tau_1),\bar{x}_2(\tau_1))\} \times \{\bar{x}_1(\tau_2)\} \times \mathbb{R} & \\ u(t) \in [-1,1], & \text{a.e. } t \in [\tau_1,\tau_2] \\ x_1(t) \le 1, & \forall t \in [\tau_1,\tau_2], \end{cases}$$

*which is denoted by $(FP_{2a})|_{[\tau_1,\tau_2]}$. In other words, for any subsegment $[\tau_1,\tau_2]$ of $[t_0,T]$, the restriction of a $W^{1,1}$ local minimizer for $(FP_{2a})$ on $[\tau_1,\tau_2]$ is a $W^{1,1}$ local minimizer for the Mayer problem $(FP_{2a})|_{[\tau_1,\tau_2]}$, which is obtained from $(FP_{2a})$ by replacing $t_0$ with $\tau_1$, $T$ with $\tau_2$, and $C$ with*

$$\widetilde{C} := \{(\bar{x}_1(\tau_1),\bar{x}_2(\tau_1))\} \times \{\bar{x}_1(\tau_2)\} \times \mathbb{R}.$$

**Proof.** Since $(\bar{x}, \bar{u})$ is a $W^{1,1}$ local minimizer for $(FP_{2a})$, by Definition 3.1 there is $\delta > 0$ such that $(\bar{x}, \bar{u})$ minimizes the quantity $g(x(t_0), x(T)) = x_2(T)$ over all feasible processes $(x, u)$ of $(FP_{2a})$ with $\|\bar{x} - x\|_{W^{1,1}} \leq \delta$.

Clearly, the restriction of $(\bar{x}, \bar{u})$ on $[\tau_1, \tau_2]$ satisfies the conditions given in (3.2). Thus, it is a feasible process for $(FP_{2a})|_{[\tau_1,\tau_2]}$.

Let $(x(t), u(t))$, $t \in [\tau_1, \tau_2]$, be an arbitrary feasible process of $(FP_{2a})|_{[\tau_1,\tau_2]}$ satisfying $\|\bar{x} - x\|_{W^{1,1}([\tau_1,\tau_2],\mathbb{R}^2)} \leq \delta$. Consider the pair of functions $(\widetilde{x}, \widetilde{u})$, where $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2)$, which is given by

$$\widetilde{x}_1(t) := \begin{cases} \bar{x}_1(t), & t \in [t_0, \tau_1] \cup [\tau_2, T] \\ x_1(t), & t \in (\tau_1, \tau_2), \end{cases}$$

$$\widetilde{x}_2(t) := \begin{cases} \bar{x}_2(t), & t \in [t_0, \tau_1] \\ x_2(t), & t \in (\tau_1, \tau_2) \\ x_2(\tau_2) + \displaystyle\int_{\tau_2}^{t} \big[ - e^{-\lambda\tau}(\bar{x}_1(\tau) + \bar{u}(\tau))\big] d\tau, & t \in [\tau_2, T], \end{cases}$$

and

$$\widetilde{u}(t) := \begin{cases} \bar{u}(t), & t \in [t_0, \tau_1] \cup [\tau_2, T] \\ u(t), & t \in (\tau_1, \tau_2). \end{cases}$$

Clearly, $(\widetilde{x}, \widetilde{u})$ is a feasible process of $(FP_{2a})$ satisfying $\|\bar{x} - \widetilde{x}\|_{W^{1,1}([t_0,T],\mathbb{R}^2)} \leq \delta$. Thus, one must have $g(\widetilde{x}(T)) \geq g(\bar{x}(T))$ or, equivalently,

$$x_2(\tau_2) + \int_{\tau_2}^{T} \omega(\tau) d\tau \geq \bar{x}(\tau_2) + \int_{\tau_2}^{T} \omega(\tau) d\tau,$$

where $\omega(\tau) := -e^{-\lambda\tau}(\bar{x}_1(\tau) + \bar{u}(\tau))$. Hence, we get $x_2(\tau_2) \geq \bar{x}(\tau_2)$, which proves that the restriction of the process $(\bar{x}, \bar{u})$ on $[\tau_1, \tau_2]$ is a $W^{1,1}$ local minimizer for $(FP_{2a})|_{[\tau_1,\tau_2]}$. □

**Proposition 3.3** *Suppose that $(\bar{x}, \bar{u})$ is a $W^{1,1}$ local minimizer for $(FP_{2a})$. Then, for any $\tau_1 \in [t_0, T)$, the restriction of the process $(\bar{x}, \bar{u})$ on the time segment $[\tau_1, T]$, i.e., the process $(\bar{x}(t), \bar{u}(t))$ with $t \in [\tau_1, T]$, is a $W^{1,1}$ local minimizer for the following optimal control problem of the Mayer type*

$$\text{Minimize} \;\; x_2(T)$$

over $x = (x_1, x_2) \in W^{1,1}([\tau_1, T], \mathbb{R}^2)$ and measurable functions $u : [\tau_1, T] \to \mathbb{R}$

satisfying

$$\begin{cases} \dot{x}_1(t) = -au(t), & \text{a.e. } t \in [\tau_1, T] \\ \dot{x}_2(t) = -e^{-\lambda t}(x_1(t) + u(t)), & \text{a.e. } t \in [\tau_1, T] \\ (x(\tau_1), x(T)) \in \{(\bar{x}_1(\tau_1), \bar{x}_2(\tau_1))\} \times \mathbb{R}^2 & \\ u(t) \in [-1, 1], & \text{a.e. } t \in [\tau_1, T] \\ x_1(t) \le 1, & \forall t \in [\tau_1, T], \end{cases}$$

*which is denoted by* $(FP_{2b})$. *In other words, for any* $\tau_1 \in [t_0, T)$, *the restriction of a* $W^{1,1}$ *local minimizer for* $(FP_{2a})$ *on the time segment* $[\tau_1, T]$ *is a* $W^{1,1}$ *local minimizer for the Mayer problem* $(FP_{2b})$, *which is obtained from* $(FP_{2a})$ *by replacing* $t_0$ *with* $\tau_1$.

**Proof.** For a fixed $\tau_1 \in [t_0, T)$, let $(FP_{2b})$ be defined as in the formulation of the lemma. It is clear that the process $(\bar{x}(t), \bar{u}(t))$, $t \in [\tau_1, T]$, is feasible for $(FP_{2b})$. Since $(\bar{x}, \bar{u})$ is a $W^{1,1}$ local minimizer of $(FP_{2a})$, it follows from the Definition 3.1 that there exists $\delta > 0$ such that the process $(\bar{x}, \bar{u})$ minimizes the quantity $g(x(t_0), x(T)) = x_2(T)$ over all feasible processes $(x, u)$ of $(FP_{2a})$ with $\|\bar{x} - x\|_{W^{1,1}([t_0,T],\mathbb{R}^2)} \le \delta$. Let $(x(t), u(t))$ with $t \in [\tau_1, T]$, be an arbitrary feasible process of $(FP_{2b})$ satisfying $\|\bar{x} - x\|_{W^{1,1}([\tau_1,T],\mathbb{R}^2)} \le \delta$. Consider the pair of functions $(\widetilde{x}, \widetilde{u})$ given by

$$\widetilde{x}(t) := \begin{cases} \bar{x}(t), & t \in [t_0, \tau_1) \\ x(t), & t \in [\tau_1, T] \end{cases} \quad \text{and} \quad \widetilde{u}(t) := \begin{cases} \bar{u}(t), & t \in [t_0, \tau_1) \\ u(t), & t \in [\tau_1, T]. \end{cases}$$

Clearly, $(\widetilde{x}, \widetilde{u})$ is a feasible process of $(FP_{2a})$ satisfying $\|\bar{x} - \widetilde{x}\|_{W^{1,1}([t_0,T])} \le \delta$. Thus, one must have $g(\widetilde{x}(T)) \ge g(\bar{x}(T))$. Since $\widetilde{x}(T) = x(T)$, one obtains the inequality $g(x(T)) \ge g(\bar{x}(T))$, which justifies our assertion. □

In accordance with (3.9), the Hamiltonian of $(FP_{2a})$ is given by

$$\mathcal{H}(t, x, p, u) = -aup_1 - e^{-\lambda t}(x_1 + u)p_2, \quad \forall (t, x, p, u) \in [t_0, T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}. \tag{3.15}$$

By (3.10), the partial hybrid subdifferential of $h$ at $(t, x) \in [t_0, T] \times \mathbb{R}^2$ is given by

$$\partial_x^{>} h(t, x) = \begin{cases} \emptyset, & \text{if } x_1 < 1 \\ \{(1, 0)\}, & \text{if } x_1 \ge 1. \end{cases} \tag{3.16}$$

*From now on, let* $(\bar{x}, \bar{u})$ *be a* $W^{1,1}$ *local minimizer for* $(FP_{2a})$.

Since the assumptions (H1)–(H4) of Theorem 3.1 are satisfied for $(FP_{2a})$, by that theorem one can find $p \in W^{1,1}([t_0,T];\mathbb{R}^2)$, $\gamma \geq 0$, $\mu \in C^{\oplus}(t_0,T)$, and a Borel measurable function $\nu : [t_0,T] \to \mathbb{R}^2$ such that $(p,\mu,\gamma) \neq (0,0,0)$, and for $q(t) := p(t) + \eta(t)$ with

$$\eta(t) := \int_{[t_0,t)} \nu(\tau)d\mu(\tau), \quad t \in [t_0,T) \tag{3.17}$$

and

$$\eta(T) := \int_{[t_0,T]} \nu(\tau)d\mu(\tau), \tag{3.18}$$

conditions (i)–(iv) in Theorem 3.1 hold true.

**Condition (i)**: Note that

$$\begin{aligned} &\mu\{t \in [t_0,T] \,:\, \nu(t) \notin \partial_x^{>} h(t,\bar{x}(t))\} \\ &= \mu\{t \in [t_0,T] \,:\, \partial_x^{>} h(t,\bar{x}(t)) = \emptyset\} \\ &\quad + \mu\{t \in [t_0,T] \,:\, \partial_x^{>} h(t,\bar{x}(t)) \neq \emptyset,\ \nu(t) \notin \partial_x^{>} h(t,\bar{x}(t))\}. \end{aligned}$$

Since $\bar{x}_1(t) \leq 1$ for every $t$, combining this with (3.16) gives

$$\begin{aligned} &\mu\{t \in [t_0,T] \,:\, \nu(t) \notin \partial_x^{>} h(t,\bar{x}(t))\} \\ &= \mu\{t \in [t_0,T] \,:\, \bar{x}_1(t) < 1\} + \mu\{t \in [t_0,T] \,:\, \bar{x}_1(t) = 1,\ \nu(t) \neq (1,0)\}. \end{aligned}$$

So, from (i) it follows that

$$\mu\{t \in [t_0,T] \,:\, \bar{x}_1(t) < 1\} = 0 \tag{3.19}$$

and

$$\mu\{t \in [t_0,T] \,:\, \bar{x}_1(t) = 1,\ \nu(t) \neq (1,0)\} = 0. \tag{3.20}$$

**Condition (ii)**: By (3.15), $\mathcal{H}$ is differentiable in $x$ and one has

$$\partial_x \mathcal{H}(t,x,p,u) = \{(-e^{-\lambda t}p_2, 0)\}, \quad \forall (t,x,p,u) \in [t_0,T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}.$$

Therefore, (ii) yields $-\dot{p}(t) = (-e^{-\lambda t}q_2(t), 0)$ for a.e. $t \in [t_0,T]$. Hence, $\dot{p}_1(t) = e^{-\lambda t}q_2(t)$ for a.e $t \in [t_0,T]$ and $p_2(t)$ is a constant for all $t \in [t_0,T]$.

**Condition (iii)**: Using the formulas for $g$ and $C$, we can show that $\partial g(\bar{x}(t_0),\bar{x}(T)) = \{(0,0,0,1)\}$ and $N((\bar{x}(t_0),\bar{x}(T));C) = \mathbb{R}^2 \times \{(0,0)\}$. So, (iii) yields $(p(t_0),-q(T)) \in \{(0,0,0,\gamma)\} + \mathbb{R}^2 \times \{(0,0)\}$, which means that $q_1(T) = 0$ and $q_2(T) = -\gamma$.

**Condition (iv)**: By (3.15), from (iv) one gets

$$-a\bar{u}(t)q_1(t) - e^{-\lambda t}[\bar{x}_1(t)+\bar{u}(t)]q_2(t) = \max_{u\in[-1,1]} \{-auq_1(t) - e^{-\lambda t}[\bar{x}_1(t)+u]q_2(t)\}$$

for a.e. $t \in [t_0, T]$ or, equivalently,

$$[aq_1(t) + e^{-\lambda t} q_2(t)]\bar{u}(t) = \min_{u \in [-1,1]} \{[aq_1(t) + e^{-\lambda t} q_2(t)]u\} \text{ a.e. } t \in [t_0, T]. \quad (3.21)$$

Thanks to Proposition 3.2 and the above analysis of Conditions (i)–(iv), we will be able to prove the next statement.

**Proposition 3.4** *Suppose that* $[\tau_1, \tau_2]$ *is a subsegment of* $[t_0, T]$ *such that* $h(t, \bar{x}(t)) < 0$ *for all* $t \in [\tau_1, \tau_2]$. *Then, the curve* $t \mapsto \bar{x}_1(t)$, $t \in [\tau_1, \tau_2]$, *cannot have more than one turning point. To be more precise, the curve must be of one of the following three categories* C1–C3*:*

$$\bar{x}_1(t) = \bar{x}_1(\tau_1) + a(t - \tau_1), \quad t \in [\tau_1, \tau_2], \quad (3.22)$$

$$\bar{x}_1(t) = \bar{x}_1(\tau_1) - a(t - \tau_1), \quad t \in [\tau_1, \tau_2], \quad (3.23)$$

*and*

$$\bar{x}_1(t) = \begin{cases} \bar{x}_1(\tau_1) + a(t - \tau_1), & t \in [\tau_1, t_\zeta] \\ \bar{x}_1(t_\zeta) - a(t - t_\zeta), & t \in (t_\zeta, \tau_2], \end{cases} \quad (3.24)$$

*where* $t_\zeta$ *is a certain point in* $(\tau_1, \tau_2)$ *(see* Figures 3.2–3.4*).*

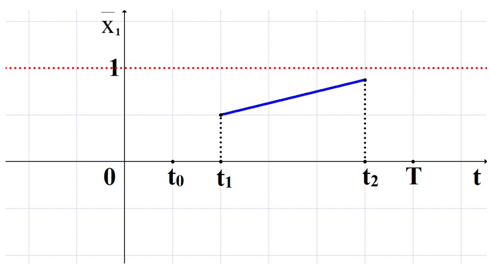


Figure 3.2: Category C1

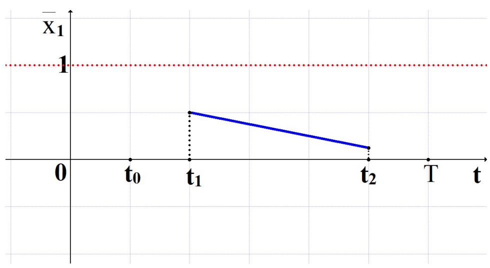


Figure 3.3: Category C2

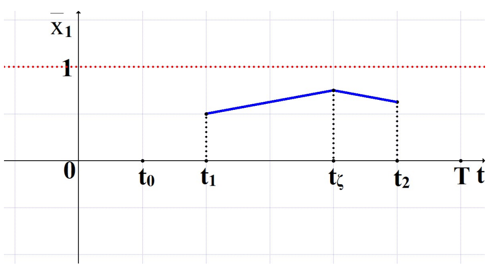


Figure 3.4: Category C3

**Proof.** Let $[\tau_1, \tau_2]$ be a subsegment of $[t_0, T]$ with $h(t, \bar{x}(t)) < 0$ for all $t \in [\tau_1, \tau_2]$, i.e., $\bar{x}_1(t) < 1$ for all $t \in [\tau_1, \tau_2]$. Then, it follows from Proposition 3.2 that the restriction of $(\bar{x}, \bar{u})$ on $[\tau_1, \tau_2]$ is a $W^{1,1}$ local minimizer for $(FP_{2a})|_{[\tau_1,\tau_2]}$. As the latter satisfies the assumptions (H1)–(H4) of Theorem 3.1, by that theorem there exist the multipliers $\widetilde{p} \in W^{1,1}([\tau_1,\tau_2];\mathbb{R}^2)$, $\widetilde{\gamma} \geq 0$, $\widetilde{\mu} \in C^{\oplus}(\tau_1,\tau_2)$, and a Borel measurable function $\widetilde{\nu} : [\tau_1,\tau_2] \to \mathbb{R}^2$ with the property $(\widetilde{p},\widetilde{\mu},\widetilde{\gamma}) \neq (0,0,0)$, and for $\widetilde{q}(t) := \widetilde{p}(t) + \widetilde{\eta}(t)$ with

$$\widetilde{\eta}(t) := \int_{[\tau_1,\tau_2)} \widetilde{\nu}(\tau) d\widetilde{\mu}(\tau), \qquad t \in [\tau_1,\tau_2) \tag{3.25}$$

and

$$\widetilde{\eta}(\tau_2) := \int_{[\tau_1,\tau_2]} \widetilde{\nu}(\tau) d\widetilde{\mu}(\tau), \tag{3.26}$$

the conditions (i)–(iv) in Theorem 3.1 hold true, provided that $t_0, T, p, \mu, \gamma, \nu, \eta$, and $q$ are changed respectively to $\tau_1, \tau_2, \widetilde{p}, \widetilde{\mu}, \widetilde{\gamma}, \widetilde{\nu}, \widetilde{\eta}$, and $\widetilde{q}$.

By Condition (i), one has

$$\begin{cases} \widetilde{\mu}\{t \in [\tau_1,\tau_2] \,:\, \bar{x}_1(t) < 1\} = 0, \\ \widetilde{\mu}\{t \in [\tau_1,\tau_2] \,:\, \bar{x}_1(t) = 1,\ \widetilde{\nu}(t) \neq (1,0)\} = 0. \end{cases} \tag{3.27}$$

By Condition (ii), $\dot{\widetilde{p}}_1(t) = e^{-\lambda t}\widetilde{q}_2(t)$ for a.e. $t \in [\tau_1,\tau_2]$ and $\widetilde{p}_2(t)$ is a constant for all $t \in [\tau_1,\tau_2]$.

Since $N((\bar{x}(\tau_1),\bar{x}(\tau_2));\widetilde{C}) = \mathbb{R}^3 \times \{0\}$, by Condition (iii) one has

$$(\widetilde{p}(\tau_1), -\widetilde{q}(\tau_2)) \in \{(0,0,0,\widetilde{\gamma})\} + \mathbb{R}^3 \times \{0\}.$$

This amounts to saying that $\widetilde{q}_2(\tau_2) = -\widetilde{\gamma}$.

Condition (iv) means that

$$[a\widetilde{q}_1(t) + e^{-\lambda t}\widetilde{q}_2(t)]\bar{u}(t) = \min_{u\in[-1,1]} \left\{[a\widetilde{q}_1(t) + e^{-\lambda t}\widetilde{q}_2(t)]u\right\}, \text{ a.e. } t \in [\tau_1,\tau_2]. \tag{3.28}$$

Since $\bar{x}_1(t) < 1$ for all $t \in [\tau_1,\tau_2]$, it follows from (3.27) that $\widetilde{\mu}([\tau_1,\tau_2]) = 0$, i.e., $\widetilde{\mu} = 0$. Combining this with (3.25) and (3.26), one gets $\widetilde{\eta}(t) = 0$ for all $t \in [\tau_1,\tau_2]$. Thus, the relation $\widetilde{q}(t) = \widetilde{p}(t) + \widetilde{\eta}(t)$ implies that $\widetilde{q}(t) = \widetilde{p}(t)$ for all $t \in [\tau_1,\tau_2]$. Therefore, together with the Lebesgue Theorem [49, Theorem 6, p. 340], the properties of $\widetilde{p}(t)$ and $\widetilde{q}(t)$ established in the above analyses of the conditions (ii) and (iii) give $\widetilde{p}_2(t) = \widetilde{q}_2(t) = -\widetilde{\gamma}$ and $\widetilde{p}_1(t) = \widetilde{q}_1(t) = \dfrac{\widetilde{\gamma}}{\lambda}e^{-\lambda t} + \zeta$ for all $t \in [\tau_1,\tau_2]$, where $\zeta$ is a constant. Substituting these formulas for $\widetilde{q}_1(t)$ and $\widetilde{q}_2(t)$ to (3.28), we have

$$\Big[a(\frac{\widetilde{\gamma}}{\lambda}e^{-\lambda t} + \zeta) - \widetilde{\gamma}e^{-\lambda t}\Big]\bar{u}(t) = \min_{u\in[-1,1]} \Big\{[a(\frac{\widetilde{\gamma}}{\lambda}e^{-\lambda t} + \zeta) - \widetilde{\gamma}e^{-\lambda t}]u\Big\}, \text{ a.e. } t \in [\tau_1,\tau_2]$$

or, equivalently,

$$[\widetilde{\gamma}(\frac{a}{\lambda}-1)e^{-\lambda t}+a\zeta]\bar{u}(t)=\min_{u\in[-1,1]}\left\{[\widetilde{\gamma}(\frac{a}{\lambda}-1)e^{-\lambda t}+a\zeta]u\right\},\ \text{a.e. } t\in[\tau_1,\tau_2]. \tag{3.29}$$

Set $\widetilde{\varphi}(t)=\widetilde{\gamma}(\frac{a}{\lambda}-1)e^{-\lambda t}+a\zeta$ for all $t\in[\tau_1,\tau_2]$.

If $\widetilde{\gamma}=0$, then $\widetilde{\varphi}(t)\equiv a\zeta$ on $[\tau_1,\tau_2]$. Since $a>0$, from the condition $(\widetilde{p},\widetilde{\mu},\widetilde{\gamma})\neq(0,0,0)$ we have $\zeta\neq 0$. If $\zeta>0$, then $\widetilde{\varphi}(t)>0$ for all $t\in[\tau_1,\tau_2]$. If $\zeta<0$, then $\widetilde{\varphi}(t)<0$ for all $t\in[\tau_1,\tau_2]$. Thus, if $\zeta>0$, then (3.29) implies that $\bar{u}(t)=-1$ a.e. $t\in[\tau_1,\tau_2]$. Similarly, if $\zeta<0$, then $\bar{u}(t)=1$ a.e. $t\in[\tau_1,\tau_2]$. Hence, applying the Lebesgue Theorem [49, Theorem 6, p. 340] to the absolutely continuous function $\bar{x}_1(t)$, one has

$$\bar{x}_1(t)=\bar{x}_1(\tau_1)+a(t-\tau_1),\quad \forall t\in[\tau_1,\tau_2] \tag{3.30}$$

in the first case, and

$$\bar{x}_1(t)=\bar{x}_1(\tau_1)-a(t-\tau_1),\quad \forall t\in[\tau_1,\tau_2] \tag{3.31}$$

in the second case.

If $\widetilde{\gamma}>0$ then, due to the assumption $a>\lambda>0$, $\widetilde{\varphi}$ is strictly decreasing on $[\tau_1,\tau_2]$. When there exists $t_\zeta\in(\tau_1,\tau_2)$ such that $\widetilde{\varphi}(t_\zeta)=0$, one has $\widetilde{\varphi}(t)>0$ for $t\in(\tau_1,t_\zeta)$ and $\widetilde{\varphi}(t)<0$ for $t\in(t_\zeta,\tau_2)$. Hence, (3.29) forces $\bar{u}(t)=-1$ a.e. $t\in[\tau_1,t_\zeta]$ and $\bar{u}(t)=1$ a.e. $t\in[t_\zeta,\tau_2]$. Thus, by the cited above Lebesgue Theorem,

$$\bar{x}_1(t)=\begin{cases}\bar{x}_1(\tau_1)+a(t-\tau_1), & t\in[\tau_1,t_\zeta]\\ \bar{x}_1(t_\zeta)-a(t-t_\zeta), & t\in(t_\zeta,\tau_2].\end{cases}$$

As $\bar{x}_1(t)<1$ for all $t\in[\tau_1,\tau_2]$, one has $\bar{x}_1(t_\zeta)<1$, i.e., $t_\zeta<\tau_1+a^{-1}(1-\bar{x}_1(\tau_1))$. When $\widetilde{\varphi}(t)>0$ for all $t\in(\tau_1,\tau_2)$, condition (3.29) implies that $\bar{u}(t)=-1$ a.e. $t\in[\tau_1,\tau_2]$. So, $\bar{x}_1(t)$ is defined by (3.30). When $\widetilde{\varphi}(t)<0$ for all $t\in(\tau_1,\tau_2)$, condition (3.29) implies that $\bar{u}(t)=1$ a.e. $t\in[\tau_1,\tau_2]$. So, $\bar{x}_1(t)$ is defined by (3.31).

In summary, for any $\tau_1,\tau_2$ with $t_0\le\tau_1<\tau_2\le T$ and $\bar{x}_1(t)<1$ for all $t\in[\tau_1,\tau_2]$, the curve $t\mapsto\bar{x}_1(t)$, $t\in[\tau_1,\tau_2]$, cannot have more than one turning point. Namely, the curve must be of one of the three categories (3.22)–(3.24). □

To proceed furthermore, put $\mathcal{T}_1:=\{t\in[t_0,T]\ :\ \bar{x}_1(t)=1\}$. Since $\bar{x}_1(t)$ is a continuous function, $\mathcal{T}_1$ is a compact set (which may be empty).

**Case 1:** $\mathcal{T}_1 = \emptyset$, *i.e.,* $\bar{x}_1(t) < 1$ *for all* $t \in [t_0, T]$. Then, by (3.19) one has $\mu([t_0, T]) = 0$, i.e., $\mu = 0$. Combining this with (3.17) and (3.18), one gets $\eta(t) = 0$ for all $t \in [t_0, T]$. Thus, the relation $q(t) = p(t) + \eta(t)$ allows us to have $q(t) = p(t)$ for every $t \in [t_0, T]$. Therefore, together with the Lebesgue Theorem [49, Theorem 6, p. 340], the properties of $p(t)$ and $q(t)$ established in the above analyses of the conditions (ii) and (iii) give

$$p_2(t) = q_2(T) = -\gamma, \qquad \forall t \in [t_0, T]$$

and

$$p_1(t) = p_1(T) + \int_T^t \dot{p}_1(\tau)d\tau = q_1(T) + \int_T^t \big(-\gamma e^{-\lambda\tau}\big)d\tau = \frac{\gamma}{\lambda}\big(e^{-\lambda t} - e^{-\lambda T}\big)$$

for all $t \in [t_0, T]$. Now, observe that substituting $q(t) = p(t)$ into (3.21) yields

$$[ap_1(t) + e^{-\lambda t}p_2(t)]\bar{u}(t) = \min_{u \in [-1,1]} \big\{[ap_1(t) + e^{-\lambda t}p_2(t)]u\big\} \quad \text{a.e. } t \in [t_0, T]. \tag{3.32}$$

Setting $\varphi(t) = ap_1(t) + e^{-\lambda t}p_2(t)$ for $t \in [t_0, T]$ and using the above formulas of $p_1(t)$ and $p_2(t)$, we have

$$\varphi(t) = a\frac{\gamma}{\lambda}\big(e^{-\lambda t} - e^{-\lambda T}\big) - \gamma e^{-\lambda t} = \gamma\big(\frac{a}{\lambda} - 1\big)e^{-\lambda t} - \gamma\frac{a}{\lambda}e^{-\lambda T}$$

for $t \in [t_0, T]$. Due to the condition $(p, \gamma, \mu) \neq 0$, one must have $\gamma > 0$. Moreover, the assumption $a > \lambda > 0$ implies $\frac{a}{\lambda} > 1$. Thus, the function $\varphi(t)$ is decreasing on $[t_0, T]$. In addition, it is clear that $\varphi(T) = -\gamma e^{-\lambda T} < 0$, and $\varphi(t) = 0$ if and only if $t = \bar{t}$, where

$$\bar{t} := T - \frac{1}{\lambda}\ln\frac{a}{a - \lambda}. \tag{3.33}$$

The assumption $a > \lambda > 0$ implies that $\bar{t} < T$. Note that the number

$$\rho := \frac{1}{\lambda}\ln\frac{a}{a - \lambda} \tag{3.34}$$

does not depend on the initial time $t_0$ and the terminal time $T$.

If $t_0 \geq \bar{t}$, then one has $\varphi(t) < 0$ for all $t \in (t_0, T)$. This situation happens if and only if $T - t_0 \leq \rho$ (the time interval of the optimal control problem is rather small). Clearly, condition (3.32) forces $\bar{u}(t) = 1$ a.e. $t \in [t_0, T]$. Since (3.5) is fulfilled for $x(t) = \bar{x}(t)$ and $u(t) = \bar{u}(t)$, applying the Lebesgue Theorem [49, Theorem 6, p. 340] to the absolutely continuous function $\bar{x}_1(t)$, one has

$$\bar{x}_1(t) = \bar{x}_1(t_0) + \int_{t_0}^t \dot{\bar{x}}_1(\tau)d\tau = \bar{x}_1(t_0) + \int_{t_0}^t (-a\bar{u}(\tau))d\tau = x_0 - a(t - t_0) \tag{3.35}$$

for all $t \in [t_0, T]$. In addition, by (3.3) one finds that

$$\bar{x}_2(t) = \int_{t_0}^{t} [-e^{-\lambda\tau}(\bar{x}_1(\tau)+\bar{u}(\tau))]d\tau = \int_{t_0}^{t} [-e^{-\lambda\tau}\big(x_0-a(\tau-t_0)+1\big)]d\tau \quad (3.36)$$

for all $t \in [t_0, T]$.

If $t_0 < \bar{t}$, then $\varphi(t) > 0$ for $t \in (t_0, \bar{t})$ and $\varphi(t) < 0$ for $t \in (\bar{t}, T)$. This situation happens if and only if $T - t_0 > \rho$ (the time interval of the optimal control problem is large enough). Condition (3.32) yields $\bar{u}(t) = -1$ for a.e. $t \in [t_0, \bar{t}]$ and $\bar{u}(t) = 1$ for a.e. $t \in [\bar{t}, T]$. Hence, by the above-cited Lebesgue Theorem, one has

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}] \\ \bar{x}_1(\bar{t}) - a(t - \bar{t}), & t \in (\bar{t}, T]. \end{cases} \quad (3.37)$$

Therefore, from (3.3), we have

$$\bar{x}_2(t) = \begin{cases} \displaystyle\int_{t_0}^{t} \big[-e^{-\lambda\tau}\big(x_0 + a(\tau - t_0) + 1\big)\big]d\tau, & t \in [t_0, \bar{t}] \\ \displaystyle\int_{t_0}^{t} \big[-e^{-\lambda\tau}\big(\bar{x}_1(\bar{t}) - a(\tau - \bar{t}) + 1\big]d\tau, & t \in (\bar{t}, T]. \end{cases} \quad (3.38)$$

Noting that $\bar{x}_1(t) < 1$ for all $t \in [t_0, T]$ by our assumption, we must have $\bar{x}_1(\bar{t}) < 1$, i.e., $\bar{t} < t_0 + a^{-1}(1 - x_0)$. Since $\bar{t} = T - \rho$, the last inequality is equivalent to $T - t_0 < \rho + a^{-1}(1 - x_0)$.

Thus, if $\mathcal{T}_1 = \emptyset$ and $T - t_0 \le \rho$, then the unique process $(\bar{x}, \bar{u})$ suspected for a $W^{1,1}$ local minimizer of $(FP_{2a})$ is the one with $\bar{u}(t) = 1$ for a.e. $t \in [t_0, T]$ and $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t))$, where $\bar{x}_1(t)$ and $\bar{x}_2(t)$ are given respectively by (3.35) and (3.36). Otherwise, if $\mathcal{T}_1 = \emptyset$ and $\rho < T - t_0 < \rho + a^{-1}(1 - x_0)$, then the unique process $(\bar{x}, \bar{u})$ serving as a $W^{1,1}$ local minimizer of $(FP_{2a})$ is the one with $\bar{u}(t) = -1$ for almost every $t \in [t_0, \bar{t}]$ and $\bar{u}(t) = 1$ for almost every $t \in [\bar{t}, T]$, $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t))$, where $\bar{x}_1(t)$ and $\bar{x}_2(t)$ are defined respectively by (3.37) and (3.38). The situation where $\mathcal{T}_1 = \emptyset$ and $T - t_0 \ge \rho + a^{-1}(1 - x_0)$ cannot occur. The situation where $\mathcal{T}_1 = \emptyset$ and $x_0 \ge 1 - a(\bar{t} - t_0)$ also cannot occur.

Now, suppose that $\mathcal{T}_1 \neq \emptyset$, i.e., there exists $t \in [t_0, T]$ with $\bar{x}_1(t) = 1$. Put

$$\alpha_1 = \min\{t \in [t_0, T] \, : \, \bar{x}_1(t) = 1\}, \quad \alpha_2 = \max\{t \in [t_0, T] \, : \, \bar{x}_1(t) = 1\}.$$

Then one has $t_0 \le \alpha_1 \le \alpha_2 \le T$ and the following situations can occur.

**Case 2:** $t_0 = \alpha_1 = \alpha_2 < T$, *i.e.,* $x_0 = 1$ *and* $\bar{x}_1(t) < 1$ *for* $t \in (t_0, T]$. Let $\bar{\varepsilon} > 0$ be such that $t_0 + \bar{\varepsilon} < T$. For any $k \in I\!N$ with $k^{-1} \in (0, \bar{\varepsilon})$, by Proposition 3.3 we know that the restriction of $(\bar{x}, \bar{u})$ on $[t_0 + k^{-1}, T]$ is a $W^{1,1}$ local minimizer for the Mayer problem $(FP_{2b})$, which is obtained from $(FP_{2a})$ by replacing $t_0$ with $t_0 + k^{-1}$. As $\bar{x}_1(t) < 1$ for all $t \in [t_0 + k^{-1}, T]$, repeating the arguments used in Case 1, we get that either

$$\bar{x}_1(t) = \bar{x}_1(t_0 + k^{-1}) - a(t - t_0 - k^{-1}), \quad t \in [t_0 + k^{-1}, T]$$

or

$$\bar{x}_1(t) = \begin{cases} \bar{x}_1(\bar{t}) + a(t - \bar{t}), & t \in [t_0 + k^{-1}, \bar{t}] \\ \bar{x}_1(\bar{t}) - a(t - \bar{t}), & t \in (\bar{t}, T] \end{cases}$$

with $\bar{t} = T - \rho$, $\bar{t} \in [t_0 + k^{-1}, T]$, and $\bar{x}_1(\bar{t}) < 1$. By the Dirichlet principle, there must exist an infinite number of indexes $k$ with $k^{-1} \in (0, \bar{\varepsilon})$ such that $\bar{x}_1(t)$ has the first form (resp., the second form). Without loss of generality, we may assume that this happens for all $k$ with $k^{-1} \in (0, \bar{\varepsilon})$. If the first situation occurs, then by letting $k \to \infty$ we can assert that $\bar{x}_1(t) = 1 - a(t - t_0)$ for all $t \in [t_0, T]$. If the second situation occurs, then we have

$$x_0 = \lim_{k\to\infty} \bar{x}_1(t_0 + k^{-1}) = \lim_{k\to\infty} \left[\bar{x}_1(\bar{t}) + a(t_0 + k^{-1} - \bar{t})\right] = \bar{x}_1(\bar{t}) + a(t_0 - \bar{t}).$$

Since $\bar{x}_1(\bar{t}) + a(t_0 - \bar{t}) \le \bar{x}_1(\bar{t}) < 1$ and $x_0 = 1$, we have arrived at a contradiction.

**Case 3:** $t_0 \le \alpha_1 < \alpha_2 \le T$. If $\alpha_2 < T$, to find a formula for $(\bar{x}, \bar{u})$ on $[\alpha_2, T]$, we observe from Proposition 3.3 that the restriction of $(\bar{x}, \bar{u})$ on $[\alpha_2, T]$ is a $W^{1,1}$ local minimizer for the Mayer problem obtained from $(FP_{2a})$ by replacing $t_0$ with $\alpha_2$. Therefore, applied to $(\bar{x}(t), \bar{u}(t))$, $t \in [\alpha_2, T]$, the result in Case 2 implies that

$$\bar{x}_1(t) = 1 - a(t - \alpha_2), \quad t \in [\alpha_2, T]$$

and

$$\bar{x}_2(t) = \int_{\alpha_2}^{t} \left[ - e^{-\lambda\tau}(2 - a(\tau - \alpha_2))\right] d\tau, \quad t \in [\alpha_2, T].$$

By the first differential equation in (3.5), we have $\bar{u}(t) = 1$ for a.e. $t \in [\alpha_2, T]$.

Now, if $t_0 < \alpha_1$, then to find a formula for $\bar{x}_1(\cdot)$ on $[t_0, \alpha_1]$, we temporarily fix a value $\alpha \in (t_0, \alpha_1)$ (later, we will let $\alpha$ converge to $\alpha_1$). As $\bar{x}_1(t) < 1$ for all $[t_0, \alpha]$, applying Proposition 3.4 with $\tau_1 := t_0$ and $\tau_2 := \alpha$, we can assert that the restriction of $\bar{x}_1(\cdot)$ on $[t_0, \alpha]$ is defined by one of the next three formulas:

$$\bar{x}_1(t) = x_0 + a(t - t_1), \quad t \in [t_0, \alpha], \tag{3.39}$$

$$\bar{x}_1(t) = x_0 - a(t - t_1), \quad t \in [t_0, \alpha], \tag{3.40}$$

and

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_1), & t \in [t_0, t_\zeta] \\ \bar{x}_1(t_\zeta) - a(t - t_\zeta), & t \in (t_\zeta, \alpha], \end{cases} \tag{3.41}$$

where $t_\zeta \in (t_0, \alpha)$. Hence, the graph of $\bar{x}_1(\cdot)$ on $[t_0, \alpha]$ is of one of the following types: C1$\alpha$) *Going up* as in Fig. 3.5; C2$\alpha$) *Going down* as in Fig. 3.6; C3$\alpha$) *Going up first and then going down* as in Fig. 3.7.

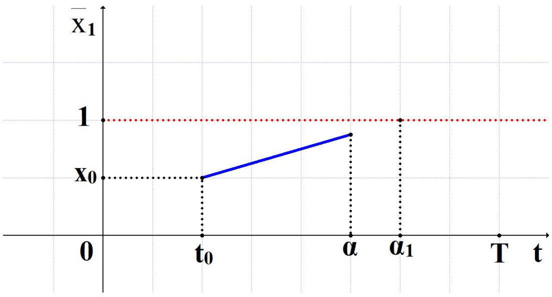


Figure 3.5: Category C1$\alpha$

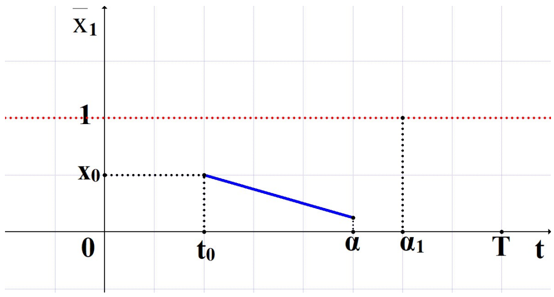


Figure 3.6: Category C2$\alpha$

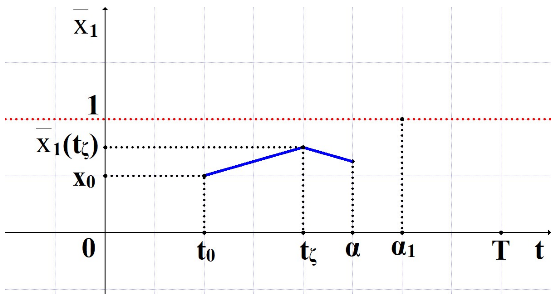


Figure 3.7: Category C3$\alpha$

Let $\alpha = \alpha^{(k)}$ with $\alpha^{(k)} := \alpha_1 - \dfrac{1}{k}$, where $k \in I\!N$ is as large as $\alpha \in (t_0, \alpha_1)$. Since for each $k$ the restriction of the graph of $\bar{x}_1(\cdot)$ on $[t_0, \alpha^{(k)}]$ must be of one of the three types C1$\alpha$–C3$\alpha$, by the Dirichlet principle we can find a subsequence $\{k'\}$ of $\{k\}$ such that the corresponding graphs belong to a fixed category. If the latter is C2$\alpha$, then by (3.40) and the continuity of $\bar{x}_1(\cdot)$ one has

$$\bar{x}_1(\alpha_1) = \lim_{k' \to \infty} \bar{x}_1(\alpha^{(k')}) = \lim_{k' \to \infty} \left[ x_0 - a(\alpha^{(k')} - t_0) \right] = x_0 - a(\alpha_1 - t_0).$$

This is impossible, because $\bar{x}_1(\alpha_1) = 1$. Similarly, the situation where the fixed category is C3$\alpha$ can be excluded by using (3.41). If the graphs belong to the category C1$\alpha$, from (3.39) we deduce that

$$\bar{x}_1(t) = x_0 + a(t - t_0), \quad \forall t \in [t_0, \alpha_1]. \tag{3.42}$$

By (3.42) and the first differential equation in (3.5), we have $\bar{u}(t) = -1$ for a.e. $t \in [t_0, \alpha_1]$. Clearly, the condition $\bar{x}_1(\alpha_1) = 1$ is satisfied if and only if

$$\alpha_1 = t_0 + a^{-1}(1 - x_0). \tag{3.43}$$

Finally, we are going to obtain a formula for $(\bar{x}, \bar{u})$ on $[\alpha_1, \alpha_2]$. Suppose that there exists $\hat{t} \in (\alpha_1, \alpha_2)$ such that $\bar{x}_1(\hat{t}) < 1$. Then, the constants $\hat{\alpha}_1 := \max\{t \in [\alpha_1, \hat{t}] \,:\, \bar{x}_1(t) = 1\}$ and $\hat{\alpha}_2 := \min\{t \in [\hat{t}, \alpha_2] \,:\, \bar{x}_1(t) = 1\}$ are well defined by the continuity of $\bar{x}_1(\cdot)$. Note that $\hat{t} \in \big(\hat{\alpha}_1, \hat{\alpha}_2\big)$ and $\bar{x}_1(t) < 1$ for every $t \in (\hat{\alpha}_1, \hat{\alpha}_2)$. If $\varepsilon > 0$ is small enough, then $\hat{\alpha}_1 + \varepsilon \in \big(\hat{\alpha}_1, \hat{t}\big)$. Repeating the arguments used for establishing a formula for $\bar{x}_1(\cdot)$ on $[t_0, \alpha_1]$ when $t_0 < \alpha_1$ (with $\hat{\alpha}_1 + \varepsilon$ playing the role of $t_0$ and $\hat{\alpha}_2$ taking the place of $\alpha_1$), one finds that

$$\bar{x}_1(t) = \bar{x}_1(\hat{\alpha}_1 + \varepsilon) + a(t - \hat{\alpha}_1 - \varepsilon), \quad \forall t \in [\hat{\alpha}_1 + \varepsilon, \hat{\alpha}_2].$$

Clearly, $\bar{x}_1(\cdot)$ is strictly increasing on $[\hat{\alpha}_1 + \varepsilon, \hat{\alpha}_2]$. As $\hat{t} \in \big(\alpha_1 + \varepsilon, \hat{\alpha}_2\big)$, this implies that $\bar{x}_1(\hat{\alpha}_1 + \varepsilon) < \bar{x}_1(\hat{t})$. Then, by the continuity of $\bar{x}_1(t)$ we get

$$\bar{x}_1(\hat{\alpha}_1) = \lim_{\varepsilon \to 0^+} \bar{x}_1(\hat{\alpha}_1 + \varepsilon) \le \bar{x}_1(\hat{t}) < 1.$$

Since $\bar{x}_1(\hat{\alpha}_1) = 1$, we have arrived at a contradiction. Thus, we must have $\bar{x}_1(t) = 1$ for all $t \in [\alpha_1, \alpha_2]$. The latter implies that $\bar{u}(t) = 0$ for almost every $t \in [\alpha_1, \alpha_2]$. Noting that $\alpha_1$ has already been given by (3.43), we are now proving that $\alpha_2 = \bar{t}$.

Since $(\bar{x}, \bar{u})$ is a $W^{1,1}$ local minimizer of $(FP_{2a})$, by Definition 3.1 we can find $\delta > 0$ such that the process $(\bar{x}, \bar{u})$ minimizes $g(x(t_0), x(T)) = x_2(T)$ over all feasible processes $(x, u)$ of $(FP_{2a})$ with $\|\bar{x} - x\|_{W^{1,1}} \le \delta$. Fix a number $\alpha \in (\alpha_1, \alpha_2]$ and consider the pair of functions $(\widetilde{x}^\alpha, \widetilde{u}^\alpha)$ with

$$\widetilde{x}_1^\alpha(t) := \begin{cases} \bar{x}_1(t), & t \in [t_0, \alpha) \\ 1 - a(t - \alpha), & t \in [\alpha, T] \end{cases} \quad \text{and} \quad \widetilde{u}^\alpha(t) := \begin{cases} \bar{u}(t), & t \in [t_0, \alpha) \\ 1, & t \in [\alpha, T]. \end{cases}$$

Clearly, $(\widetilde{x}^\alpha, \widetilde{u}^\alpha)$ is a feasible process of $(FP_{2a})$. Set $\Delta(\alpha) = \bar{x}_2(T) - \widetilde{x}_2^\alpha(T)$ and observe that $\widetilde{x}_1^\alpha(t) = \bar{x}_1(t) + a(\alpha - \alpha_2)$ for every $t \in [\alpha_2, T]$. Using the formula for integration by parts, we have

$$\begin{aligned}
\Delta(\alpha) &= \int_{t_0}^{T}\Big[-e^{-\lambda\tau}\Big(\big(\bar{x}_1(\tau)-\widetilde{x}_1^{\alpha}(\tau)\big)+\big(\bar{u}(\tau)-\widetilde{u}^{\alpha}(\tau)\big)\Big)\Big]d\tau\\
&= \int_{\alpha}^{\alpha_2}\Big[-e^{-\lambda\tau}\big(a(\tau-\alpha)-1\big)\Big]d\tau+a(\alpha_2-\alpha)\int_{\alpha_2}^{T}\Big(-e^{-\lambda\tau}\Big)d\tau\\
&= \frac{a}{\lambda}\int_{\alpha}^{\alpha_2}\tau d\Big(e^{-\lambda\tau}\Big)+(a\alpha+1)\int_{\alpha}^{\alpha_2}e^{-\lambda\tau}d\tau+\frac{a}{\lambda}(\alpha_2-\alpha)\big[e^{-\lambda T}-e^{-\lambda\alpha_2}\big]\\
&= \frac{a}{\lambda}\Big(\big[\alpha_2e^{-\lambda\alpha_2}-\alpha e^{-\lambda\alpha}\big]-\int_{\alpha}^{\alpha_2}e^{-\lambda\tau}d\tau\Big)+(a\alpha+1)\int_{\alpha}^{\alpha_2}e^{-\lambda\tau}d\tau\\
&\quad+\frac{a}{\lambda}(\alpha_2-\alpha)\big[e^{-\lambda T}-e^{-\lambda\alpha_2}\big]\\
&= \frac{a}{\lambda}\big[\alpha_2e^{-\lambda\alpha_2}-\alpha e^{-\lambda\alpha}\big]+\big(a\alpha+1-\frac{a}{\lambda}\big)\int_{\alpha}^{\alpha_2}e^{-\lambda\tau}d\tau\\
&\quad+\frac{a}{\lambda}(\alpha_2-\alpha)\big[e^{-\lambda T}-e^{-\lambda\alpha_2}\big]\\
&= \frac{a}{\lambda}\big[\alpha_2e^{-\lambda\alpha_2}-\alpha e^{-\lambda\alpha}\big]-\frac{1}{\lambda}\big(a\alpha+1-\frac{a}{\lambda}\big)\big[e^{-\lambda\alpha_2}-e^{-\lambda\alpha}\big]\\
&\quad+\frac{a}{\lambda}(\alpha_2-\alpha)\big[e^{-\lambda T}-e^{-\lambda\alpha_2}\big]\\
&= \frac{a}{\lambda}\alpha_2e^{-\lambda\alpha_2}-\frac{1}{\lambda}\big(a\alpha+1-\frac{a}{\lambda}\big)e^{-\lambda\alpha_2}+\big(\frac{1}{\lambda}-\frac{a}{\lambda^2}\big)e^{-\lambda\alpha}\\
&\quad+\frac{a}{\lambda}(\alpha_2-\alpha)\big[e^{-\lambda T}-e^{-\lambda\alpha_2}\big].
\end{aligned}$$

Therefore, $\Delta(\alpha_2)=0$ and one has for every $\alpha\in(\alpha_1,\alpha_2)$ the following:

$$\dot{\Delta}(\alpha)=-\frac{a}{\lambda}e^{-\lambda\alpha_2}-\big(1-\frac{a}{\lambda}\big)e^{-\lambda\alpha}-\frac{a}{\lambda}\big[e^{-\lambda T}-e^{-\lambda\alpha_2}\big]=\frac{a}{\lambda}\Big[\big(1-\frac{\lambda}{a}\big)e^{-\lambda\alpha}-e^{-\lambda T}\Big].$$

So, due to the fact that $a>\lambda>0$, we have $\dot{\Delta}(\alpha)<0$ if and only if

$$\alpha>T+\frac{1}{\lambda}\ln\big(1-\frac{\lambda}{a}\big).$$

This is equivalent to saying that

$$\alpha>\bar{t}, \tag{3.44}$$

where the value $\bar{t}=T-\dfrac{1}{\lambda}\ln\dfrac{a}{a-\lambda}$ has been defined in (3.33).

If $\alpha_2>\bar{t}$, then we have (3.44) for all $\alpha\in(\alpha_1,\alpha_2)$ sufficiently close to $\alpha_2$. So, we can find $\varepsilon>0$ such that the function $\Delta(\alpha)$ is strictly decreasing on the subsegment $[\alpha_2-\varepsilon,\alpha_2]$ of $[\alpha_1,\alpha_2]$. Since $\Delta(\alpha_2)=0$, we can assert that $\Delta(\alpha)>0$ for every $\alpha\in(\alpha_2-\varepsilon,\alpha_2)$. As $\lim\limits_{\alpha\to\alpha_2}\|\bar{x}-\widetilde{x}^{\alpha}\|_{W^{1,1}}=0$, one has $\|\bar{x}-\widetilde{x}^{\alpha}\|_{W^{1,1}}\leq\delta$ for all $\alpha\in(\alpha_1,\alpha_2)$ sufficiently close to $\alpha_2$. Hence, the fact that $\bar{x}_2(T)-\widetilde{x}_2^{\alpha}(T)=\Delta(\alpha)>0$ contradicts the assumed $W^{1,1}$ local optimality of the process $(\bar{x},\bar{u})$. Thus, we must have $\alpha_2\leq\bar{t}$.

To prove that $\alpha_2 = \bar{t}$, we suppose on the contrary that $\alpha_2 < \bar{t}$. Consider the pair of functions $(\hat{x}, \hat{u})$ with

$$\hat{x}_1(t) := \begin{cases} \bar{x}_1(t), & t \in [t_0, \alpha_2) \\ 1, & t \in [\alpha_2, \bar{t}) \\ 1 - a(t - \bar{t}), & t \in [\bar{t}, T] \end{cases} \quad \text{and} \quad \hat{u}(t) := \begin{cases} \bar{u}(t), & t \in [t_0, \alpha_2) \\ 0, & t \in [\alpha_2, \bar{t}) \\ 1, & t \in [\bar{t}, T]. \end{cases}$$

Clearly, $(\hat{x}, \hat{u})$ is a feasible process of $(FP_{2a})$. Fix a number $\alpha \in (\alpha_1, \bar{t}]$ and consider the new pair of functions $(\widetilde{x}^\alpha, \widetilde{u}^\alpha)$ with

$$\widetilde{x}^\alpha_1(t) := \begin{cases} \hat{x}_1(t), & t \in [t_0, \alpha) \\ 1 - a(t - \alpha), & t \in [\alpha, T] \end{cases} \quad \text{and} \quad \widetilde{u}^\alpha(t) := \begin{cases} \hat{u}(t), & t \in [t_0, \alpha) \\ 1, & t \in [\alpha, T]. \end{cases}$$

Note that $(\widetilde{x}^\alpha, \widetilde{u}^\alpha)$ is a feasible process of $(FP_{2a})$. Set $\Delta_1(\alpha) = \hat{x}_2(T) - \widetilde{x}^\alpha_2(T)$ and apply the above integration and differentiation calculations with $\Delta_1(\alpha)$, $\bar{t}$, and $(\hat{x}, \hat{u})$ playing the roles of $\Delta(\alpha)$, $\alpha_2$, and $(\bar{x}, \bar{u})$, respectively. In the result, we have

$$\dot{\Delta}_1(\alpha) = \frac{a}{\lambda}\Big[(1 - \frac{\lambda}{a})e^{-\lambda\alpha} - e^{-\lambda T}\Big]$$

for every $\alpha \in (\alpha_1, \bar{t})$. Since $\alpha < \bar{t}$, this implies that $\dot{\Delta}_1(\alpha) > 0$ for every $\alpha \in (\alpha_1, \bar{t})$. So, the function $\Delta_1(\alpha) = \hat{x}_2(T) - \widetilde{x}^\alpha_2(T)$ is strictly increasing on the segment $[\alpha_1, \bar{t}]$. Thus, the function $\alpha \mapsto \widetilde{x}^\alpha_2(T)$ is strictly decreasing on the segment $[\alpha_1, \bar{t}]$. As one has $\alpha_2 \in (\alpha_1, \bar{t})$ and the process $(\widetilde{x}^{\alpha_2}, \widetilde{u}^{\alpha_2})$ coincides with the process $(\bar{x}, \bar{u})$, the latter cannot be a $W^{1,1}$ local minimizer of $(FP_{2a})$. We have arrived at a contradiction, which completes the proof of the equality $\alpha_2 = \bar{t}$.

We have obtained formula for $(\bar{x}, \bar{u})$ on the whole time segment $[t_0, T]$. Namely, one has

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \alpha_1) \\ 1, & t \in [\alpha_1, \bar{t}) \\ 1 - a(t - \alpha_2), & t \in [\bar{t}, T] \end{cases} \quad \text{and} \quad \bar{u}(t) = \begin{cases} -1, & \text{a.e. } t \in [t_0, \alpha_1) \\ 0, & \text{a.e. } t \in [\alpha_1, \bar{t}) \\ 1, & \text{a.e. } t \in [\bar{t}, T] \end{cases}$$

with $\alpha_1$ being given by (3.43).

**Case 4:** $t_0 < \alpha_1 = \alpha_2 < T$. Here we have $\bar{x}_1(\alpha_2) = 1$ and $\bar{x}_1(t) < 1$ for all $t \in [t_0, T] \setminus \{\alpha_2\}$. The formulas for $(\bar{x}, \bar{u})$ on $[t_0 < \alpha_1]$ and on $[\alpha_2, T]$ are obtained similarly as in Case 3. We now show that $\alpha_2 = \bar{t}$. If $\alpha_2 < \bar{t}$, then we can have a contradiction by the arguments used in Case 3 for proving

$\alpha_2 = \bar{t}$. Now, suppose that $\alpha_2 > \bar{t}$, i.e., $\alpha_1 > \bar{t}$. Then, for an arbitrarily given $\alpha \in (\bar{t}, \alpha_1)$, we consider the problem $(FP_{1b})$ (resp., the problem $(FP_{2b})$) which is obtained from the problem $(FP_{1a})$ in Section 3.4 (resp., from the above problem $(FP_{2a})$) by letting $\alpha$ play the role of the initial time $t_0$. Since $\alpha > \bar{t}$, it follows from Theorem 3.3 that $(FP_{1b})$ has a unique global solution $(\bar{x}^\alpha, \bar{u}^\alpha)$, where $\bar{u}(t) = 1$ for almost everywhere $t \in [\alpha, T]$, $\bar{x}_1^\alpha(t) = \bar{x}_1(\alpha) - a(t - \alpha)$ for all $t \in [\alpha, T]$, and $\bar{x}_2^\alpha(t) = \int_\alpha^t \big[ - e^{-\lambda\tau}(x_1(\tau) + u(\tau))\big] d\tau$ for all $t \in [\alpha, T]$. Clearly, the restriction of $(\bar{x}, \bar{u})$ on $[\alpha, T]$ is a feasible process for $(FP_{1b})$. Thus, we have

$$\bar{x}_2^\alpha(T) < \bar{x}_2(T). \tag{3.45}$$

Besides, by Proposition 3.3, the restriction of $(\bar{x}, \bar{u})$ on $[\alpha, T]$ is a $W^{1,1}$ local solution for $(FP_{2b})$. So, there exits $\delta > 0$ such that the restriction of $(\bar{x}, \bar{u})$ on $[\alpha, T]$ minimizes the quantity $x_2(T)$ over all feasible processes $(x, u)$ of $(FP_{2b})$ with $\|x - \bar{x}\|_{W^{1,1}([\alpha,T];\mathbb{R}^n)} \le \delta$. Clearly, $(\bar{x}^\alpha, \bar{u}^\alpha)$ is a feasible process of $(FP_{2b})$. Therefore, since $\|\bar{x}^\alpha - \bar{x}\|_{W^{1,1}([\alpha,T];\mathbb{R}^n)} \le \delta$ for all $\alpha$ sufficiently close to $\alpha_1$, we have $\bar{x}_2^\alpha(T) \ge \bar{x}_2(T)$ for those $\alpha$. This contradicts (3.45). We have thus proved that $\alpha_1 = \alpha_2 = \bar{t}$. Therefore, the formulas for $(\bar{x}, \bar{u})$ on the whole time segment $[t_0, T]$ is given by

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}) \\ 1 - a(t - \bar{t}), & t \in [\bar{t}, T] \end{cases} \quad \text{and} \quad \bar{u}(t) = \begin{cases} -1, & \text{a.e. } t \in [t_0, \bar{t}) \\ 1, & \text{a.e. } t \in [\bar{t}, T]. \end{cases}$$

**Case 5:** *$t_0 < \alpha_1 = \alpha_2 = T$, i.e., $\bar{x}_1(t) < 1$ for $t \in [t_0, T)$ and $\bar{x}_1(T) = 1$.* Repeating the arguments used to find the formula for $\bar{x}_1(\cdot)$ on $[t_0, \alpha_1]$ in Case 3 with $T$ playing the role of $\alpha_1$, we get $\bar{u}(t) = -1$ for almost every $t \in [t_0, T]$ and $\bar{x}_1(t) = x_0 + a(t - t_0)$ for all $t \in [t_0, T]$. But this cannot happen. Indeed, since $(\bar{x}, \bar{u})$ is a $W^{1,1}$ local minimizer of $(FP_{2a})$, by Definition 3.1 we can find $\delta > 0$ such that the process $(\bar{x}, \bar{u})$ minimizes the quantity $x_2(T)$ over all feasible processes $(x, u)$ of $(FP_{2a})$ with $\|\bar{x} - x\|_{W^{1,1}} \le \delta$. For a number $\varepsilon > 0$ such that $T - \varepsilon \in (t_0, T)$, consider the pair of functions $(\widetilde{x}^\varepsilon, \widetilde{u}^\varepsilon)$ with

$$\widetilde{x}_1^\varepsilon(t) := \begin{cases} \bar{x}_1(t), & t \in [t_0, T - \varepsilon] \\ \bar{x}_1(T - \varepsilon) - a(t - T + \varepsilon), & t \in (T - \varepsilon, T] \end{cases}$$

and

$$\widetilde{u}^\varepsilon(t) := \begin{cases} \bar{u}(t), & t \in [t_0, T - \varepsilon] \\ 1, & t \in (T - \varepsilon, T]. \end{cases}$$

Clearly, $(\widetilde{x}^\varepsilon, \widetilde{u}^\varepsilon)$ is a feasible process of $(FP_{2a})$. For $\Delta_\varepsilon := \bar{x}_2(T) - \widetilde{x}_2^\varepsilon(T)$, one has

$$\begin{aligned}
\Delta_\varepsilon &= \int_{t_0}^{T} \big[-e^{-\lambda t}(\bar{x}_1(t) + \bar{u}(t))\big]dt - \int_{t_0}^{T} \big[-e^{-\lambda \tau}(\widetilde{x}_1^\varepsilon(t) + \widetilde{u}^\varepsilon(t))\big]dt \\
&= \int_{T-\varepsilon}^{T} \big[\widetilde{x}_1^\varepsilon(t) - \bar{x}_1(t) + \widetilde{u}^\varepsilon(t) - \bar{u}(t)\big]e^{-\lambda t}dt \\
&= \int_{T-\varepsilon}^{T} \big[\bar{x}_1(T-\varepsilon) - a(t - T + \varepsilon) - (x_0 + a(t-t_0)) + 1 - (-1)\big]e^{-\lambda t}dt \\
&= \int_{T-\varepsilon}^{T} \big[2a(T-\varepsilon-t) + 2\big]e^{-\lambda t}dt = \frac{-1}{\lambda}\int_{T-\varepsilon}^{T} \big[2a(T-\varepsilon-t) + 2\big]d(e^{-\lambda t})
\end{aligned}$$

Thus, using the formula for integration by parts, we can deduce that

$$\begin{aligned}
-\lambda\Delta_\varepsilon &= \Big(\big[2a(T-\varepsilon-t)+2\big]e^{-\lambda t}\Big)\Big|_{T-\varepsilon}^{T} - \int_{T-\varepsilon}^{T} e^{-\lambda t}d[2a(T-\varepsilon-t)+2] \\
&= (2 - 2a\varepsilon)e^{-\lambda T} - 2e^{-\lambda(T-\varepsilon)} - \frac{2a}{\lambda}(e^{-\lambda T} - e^{-\lambda(T-\varepsilon)}) \\
&= 2\Big(1 - \frac{a}{\lambda}\Big)(e^{-\lambda T} - e^{-\lambda(T-\varepsilon)}) - 2a\varepsilon e^{-\lambda T}.
\end{aligned}$$

Since $\lambda > 0$, one has $\Delta_\varepsilon > 0$ if and only if $-\lambda\Delta_\varepsilon < 0$. The latter means that

$$2\Big(1 - \frac{a}{\lambda}\Big)(e^{-\lambda T} - e^{-\lambda(T-\varepsilon)}) - 2a\varepsilon e^{-\lambda T} < 0.$$

Dividing both sides of this inequality by $e^{-\lambda(T-\varepsilon)}$, one gets

$$2\Big(1 - \frac{a}{\lambda}\Big)(e^{-\lambda\varepsilon} - 1) - 2a\varepsilon e^{-\lambda\varepsilon} < 0.$$

Let $z(t) := 2\Big(1 - \frac{a}{\lambda}\Big)(e^{-\lambda t} - 1) - 2ate^{-\lambda t}$ for $t \in \mathbb{R}$. Clearly, for any $t \in [0, a^{-1})$ one has

$$\dot{z}(t) = 2(a-\lambda)e^{-\lambda t} - 2ae^{-\lambda t} + 2a\lambda te^{-\lambda t} = 2\lambda(at-1)e^{-\lambda t} < 0.$$

Hence the function $z(\cdot)$ is strictly decreasing on $[0, a^{-1})$. This and the fact that $z(0) = 0$ yield $z(\varepsilon) < 0$ for every $\varepsilon \in (0, a^{-1})$. The latter means that $\Delta_\varepsilon > 0$, i.e., $\bar{x}_2(T) > \widetilde{x}_2^\varepsilon(T)$ for every $\varepsilon \in (0, a^{-1})$. Since $\lim\limits_{\varepsilon\to 0} \|\bar{x} - \widetilde{x}^\varepsilon\|_{W^{1,1}} = 0$, one has $\|\bar{x} - \widetilde{x}^\varepsilon\|_{W^{1,1}} \leq \delta$ for all $\varepsilon$ sufficiently close to 0. This contradicts the assumed $W^{1,1}$ local optimality of the process $(\bar{x}, \bar{u})$. We have thus proved that the situation described in this case is excluded.

Going back to the original problem $(FP_2)$, we can summarize the results of this section in the next theorem.

**Theorem 3.4** *Given any $a, \lambda$ with $a > \lambda > 0$, define $\rho = \dfrac{1}{\lambda} \ln \dfrac{a}{a - \lambda} > 0$, $\bar{t} = T - \rho$, $\bar{x}_0 = 1 - a(\bar{t} - t_0)$, and $\alpha_1 = t_0 + a^{-1}(1 - x_0)$. Then, $(FP_2)$ has a unique local solution $(\bar{x}, \bar{u})$, which is a global solution, where $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ for almost every $t \in [t_0, T]$ and $\bar{x}(t)$ is described as follows:*

(a) *If $t_0 \geq \bar{t}$ (i.e, $T - t_0 \leq \rho$), then*

$$\bar{x}(t) = x_0 - a(t - t_0), \quad t \in [t_0, T].$$

(b) *If $t_0 < \bar{t}$ and $x_0 < \bar{x}_0$ (i.e, $\rho < T - t_0 < \rho + a^{-1}(1 - x_0)$), then*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}] \\ x_0 - a(t + t_0 - 2\bar{t}), & t \in (\bar{t}, T]. \end{cases}$$

(c) *If $t_0 < \bar{t}$ and $x_0 = \bar{x}_0$ (i.e, $T - t_0 = \rho + a^{-1}(1 - x_0)$), then*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}) \\ 1 - a(t - \bar{t}), & t \in [\bar{t}, T]. \end{cases}$$

(d) *If $t_0 < \bar{t}$ and $x_0 > \bar{x}_0$ (i.e, $T - t_0 > \rho + a^{-1}(1 - x_0)$), then*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \alpha_1) \\ 1, & t \in [\alpha_1, \bar{t}) \\ 1 - a(t - \bar{t}), & t \in [\bar{t}, T]. \end{cases}$$

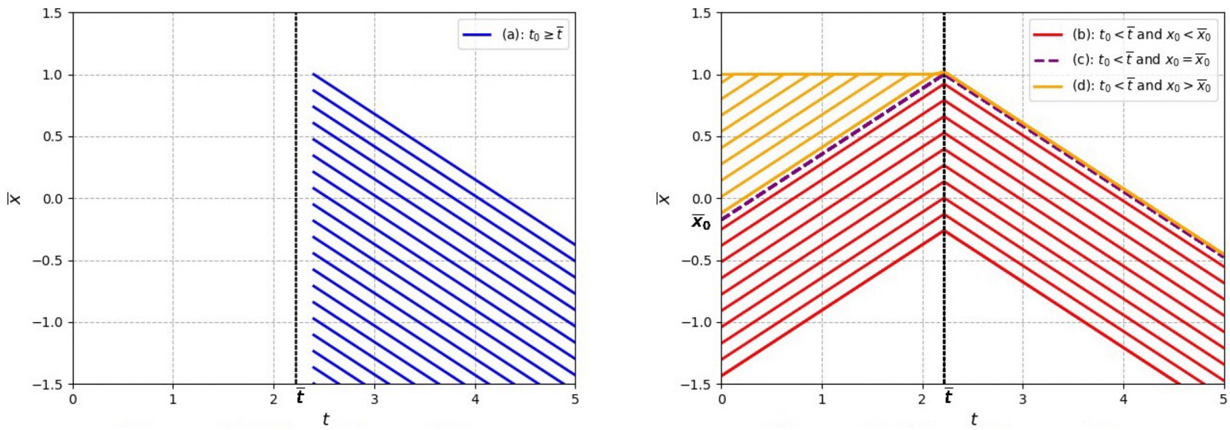


Figure 3.8: The optimal trajectories $\bar{x}(\cdot)$ of $(FP_2)$ w.r.t. parameters $a = 0.53, \lambda = 0.3, T = 5$, and varying $t_0, x_0$

**Proof.** To prove the assertions (a)–(d), it suffices to combine the results formulated in the above Cases 1–5, having in mind that $\bar{x}_1(t)$ in $(FP_{2a})$ plays the role of $\bar{x}(t)$ in $(FP_2)$. □

Geometrically, depending on the parameters tube $(\lambda, a, x_0, t_0, T)$, by Theorem 3.4 we know that the unique optimal trajectory $\{(t, \bar{x}(t)) : t \in [t_0, T]\}$ of $(FP_2)$ must be of one the following four types:

(a) "*interior straight trajectory*" – a line segment, which does not touch the boundary line $x = 1$;

(b) "*interior triangular trajectory*" – the union of two line segments, which does not touch the boundary line $x = 1$ and which has a turning point at the time moment $t = \bar{t}$;

(c) "*boundary triangular trajectory*" – the union of two line segments, which touches the boundary line $x = 1$ and has a turning point at the time moment $t = \bar{t}$;

(d) "*boundary trapezoidal trajectory*" – the union of three line segments, which moves on the boundary $x = 1$ from the time moment $t = \alpha_1$ to the time moment $t = \bar{t}$ and has two turning points at the moments $\alpha_1$ and $\bar{t}$.

Correspondingly, the optimal control function $\bar{u}(\cdot)$ may have no switching point (in situation (a)), one switching point (in the situations (b) and (c)) or two switching points (in situation (d)).

## 3.6 Conclusions

We have analyzed a maximum principle for finite horizon state constrained problems via two parametric examples of optimal control problems of the Lagrange type, which have five parameters. These problems resemble the optimal economic growth problems in macroeconomics. The first example is related to control problems without state constraints. The second one belongs to the class of irregular control problems with unilateral state constraints. We have proved that the control problem in each example has a unique local solution, which is a global solution. Moreover, we have presented an explicit description of the optimal process with respect to the five parameters.

The obtained results allow us to have a deep understanding of the maximum principle in question.

We believe that economic optimal growth models can be studied by advanced tools from functional analysis and optimal control theory via the approach adopted in this chapter.

# Chapter 4

# Parametric Optimal Control Problems with Bilateral State Constraints

The present chapter is written on the basis of the paper [38]. Here, finite horizon optimal control problems with bilateral state constraints will be considered.

## 4.1 Problem Statement

By $(FP_3)$ we denote the finite horizon optimal control problem of the Lagrange type

$$\text{Minimize } \; J(x,u) = \int_{t_0}^{T} \big[ -e^{-\lambda t}(x(t)+u(t))\big]\,dt \tag{4.1}$$

over $x \in W^{1,1}([t_0,T],\mathbb{R})$ and measurable functions $u : [t_0,T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{x}(t) = -au(t), & \text{a.e. } t \in [t_0,T] \\ x(t_0) = x_0 & \\ u(t) \in [-1,1], & \text{a.e. } t \in [t_0,T] \\ -1 \le x(t) \le 1, & \forall t \in [t_0,T] \end{cases} \tag{4.2}$$

with $a > \lambda > 0$, $T > t_0 \ge 0$, and $-1 \le x_0 \le 1$ being given as five parameters. Thus, it is instantly seen that the only difference among $(FP_3)$ and the problems $(FP_1)$ and $(FP_2)$ is the appearance of the bilateral state constraints $-1 \le x(t) \le 1$, for all $t \in [t_0,T]$.

We want to know about the change of the optimal solutions when one more state constraint is added. We will follow the approach adopted in the preceding chapter for dealing with $(FP_1)$ and $(FP_2)$. This means that we will first show the solution existence and then study the necessary conditions provided by the maximum principle [82, Theorem 9.3.1]. It turns out that, in comparison with previous chapter, herein we have to prove a series of delicate lemmas. Moreover, the synthesis of finitely many processes suspected for being local minimizers is rather sophisticated, and it requires a lot of refined arguments.

Similarly as it was done in the case of the problems $(FP_1)$ and $(FP_2)$, we now treat $(FP_3)$ in (4.1)–(4.2) as a problem of the Mayer type by setting $x(t) = (x_1(t), x_2(t))$, where $x_1(t)$ plays the role of $x(t)$ in $(FP_3)$ and

$$x_2(t) := \int_{t_0}^{t} \big[ -e^{-\lambda\tau}(x_1(\tau) + u(\tau))\big] d\tau, \quad t \in [0, T]. \tag{4.3}$$

Thus, $(FP_3)$ is equivalent to the problem

$$\text{Minimize } \ x_2(T) \tag{4.4}$$

over $x = (x_1, x_2) \in W^{1,1}([t_0, T], \mathbb{R}^2)$ and measurable functions $u : [t_0, T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{x}_1(t) = -au(t), & \text{a.e. } t \in [t_0, T] \\ \dot{x}_2(t) = -e^{-\lambda t}(x_1(t) + u(t)), & \text{a.e. } t \in [t_0, T] \\ (x(t_0), x(T)) \in \{(x_0, 0)\} \times \mathbb{R}^2 & \\ u(t) \in [-1, 1], & \text{a.e. } t \in [t_0, T] \\ -1 \le x_1(t) \le 1, & \forall t \in [t_0, T]. \end{cases} \tag{4.5}$$

The problem (4.4)–(4.5) is abbreviated to $(FP_{3a})$.

## 4.2 Solution Existence

To verify that $(FP_{3a})$ is of the form $\mathcal{M}_1$ (see Section 3.2), one can choose $n = 2$, $m = 1$, $A = [t_0, T] \times [-1, 1] \times \mathbb{R}$, $U(t, x) = [-1, 1]$ for all $(t, x) \in A$, $B = \{t_0\} \times \{(x_0, 0)\} \times \mathbb{R} \times \mathbb{R}^2$, $g(t_0, x(t_0), T, x(T)) = x_2(T)$,

$$M = A \times [-1, 1], \quad f(t, x, u) = (-au, -e^{-\lambda t}(x_1 + u))$$

for all $(t, x, u) \in M$. To show that $(FP_{3a})$ satisfies all the assumptions of Theorem 3.2, we can use the arguments showing that $(FP_{1a})$ has a $W^{1,1}$

global minimizer (see Section 3.3), except those related to the convexity of the sets $Q(t,x)$ and the compactness of $M_\varepsilon$, which will be verified as follows.

By the formula for $A$, one has $A_0 = [t_0, T]$ and $A(t) = [-1,1] \times \mathbb{R}$ for all $t \in A_0$. Thus, the requirement in Theorem 3.2 on the convexity of the sets $Q(t,x)$, $x \in A(t)$, for almost every $t \in [t_0, T]$ is satisfied. Since

$$M = [t_0, T] \ \times [-1,1] \times \mathbb{R} \times [-1,1],$$

for any $\varepsilon \geq 0$, one has the expression

$$M_\varepsilon = \{(t,x,u) \in [t_0,T] \ \times [-1,1] \times \mathbb{R} \times [-1,1] \ : \ \|x\| \leq \varepsilon\},$$

which justifies the compactness of $M_\varepsilon$.

Theorem 3.2 tells us that $(FP_{3a})$ has a $W^{1,1}$ global minimizer. Thus, by the equivalence of $(FP_3)$ and $(FP_{3a})$, we can assert that $(FP_3)$ has a $W^{1,1}$ global solution.

## 4.3 Preliminary Investigations of the Optimality Condition

To solve problem $(FP_3)$ by applying Theorem 3.1, note that $(FP_{3a})$ is in the form of $\mathcal{M}$ with $g(x,y) = y_2$,

$$f(t,x,u) = (-au, -e^{-\lambda t}(x_1 + u)),$$

$C = \{(x_0, 0)\} \times \mathbb{R}^2$, $U(t) = [-1,1]$, and $h(t,x) = |x_1| - 1$ for all $t \in [t_0, T]$, $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$ and $u \in \mathbb{R}$. According to (3.9), the Hamiltonian of $(FP_{3a})$ is the function

$$\mathcal{H}(t,x,p,u) = -aup_1 - e^{-\lambda t}(x_1 + u)p_2, \quad \forall (t,x,p,u) \in [t_0,T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}. \tag{4.6}$$

By (3.10), the partial hybrid subdifferential of $h$ at $(t,x) \in [t_0,T] \times \mathbb{R}^2$ is the set

$$\partial_x^> h(t,x) = \begin{cases} \{(-1,0)\}, & \text{if } x_1 \leq -1 \\ \emptyset, & \text{if } |x_1| < 1 \\ \{(1,0)\}, & \text{if } x_1 \geq 1. \end{cases} \tag{4.7}$$

Let $(\bar{x}, \bar{u})$ be a $W^{1,1}$ local minimizer for $(FP_{3a})$. Since the assumptions (H1)–(H4) of Theorem 3.1 are satisfied for $(FP_{3a})$, applying that theorem one can find $p \in W^{1,1}([t_0,T];\mathbb{R}^2)$, $\gamma \geq 0$, $\mu \in C^{\oplus}(t_0,T)$, and a Borel measurable function $\nu : [t_0,T] \to \mathbb{R}^2$ such that $(p,\mu,\gamma) \neq (0,0,0)$, and for $q(t) := p(t)+\eta(t)$ with

$$\eta(t) := \int_{[t_0,t)} \nu(\tau)d\mu(\tau), \quad t \in [t_0,T)$$

and

$$\eta(T) := \int_{[t_0,T]} \nu(\tau)d\mu(\tau),$$

conditions (i)–(iv) in Theorem 3.1 hold true.

**Condition (i)**: Note that

$$\begin{aligned}
&\mu\{t \in [t_0,T] \,:\, \nu(t) \notin \partial_x^{>} h(t,\bar{x}(t))\} \\
&= \ \mu\{t \in [t_0,T] \,:\, \partial_x^{>} h(t,\bar{x}(t)) = \emptyset\} \\
&\quad + \mu\{t \in [t_0,T] \,:\, \partial_x^{>} h(t,\bar{x}(t)) \neq \emptyset,\ \nu(t) \notin \partial_x^{>} h(t,\bar{x}(t))\}.
\end{aligned}$$

Since $-1 \leq x_1(t) \leq 1$ for every $t$, combining this with (4.7) gives

$$\begin{aligned}
&\mu\{t \in [t_0,T] \,:\, \nu(t) \notin \partial_x^{>} h(t,\bar{x}(t))\} \\
&= \mu\{t \in [t_0,T] \,:\, -1 < \bar{x}_1(t) < 1\} \\
&\quad + \mu\{t \in [t_0,T] \,:\, \bar{x}_1(t) = 1,\ \nu(t) \neq (1,0)\} \\
&\quad + \mu\{t \in [t_0,T] \,:\, \bar{x}_1(t) = -1,\ \nu(t) \neq (-1,0)\}.
\end{aligned}$$

So, from (i) it follows that

$$\mu\{t \in [t_0,T] \,:\, -1 < \bar{x}_1(t) < 1\} = 0, \tag{4.8}$$

$$\mu\{t \in [t_0,T] \,:\, \bar{x}_1(t) = 1,\ \nu(t) \neq (1,0)\} = 0, \tag{4.9}$$

$$\mu\{t \in [t_0,T] \,:\, \bar{x}_1(t) = -1,\ \nu(t) \neq (-1,0)\} = 0. \tag{4.10}$$

**Condition (ii)**: By (4.6), $\mathcal{H}$ is differentiable in $x$ and

$$\partial_x \mathcal{H}(t,x,p,u) = \{(-e^{-\lambda t}p_2, 0)\}, \quad \forall (t,x,p,u) \in [t_0,T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}.$$

Thus, (ii) implies that $-\dot{p}(t) = (-e^{-\lambda t}q_2(t), 0)$ for a.e. $t \in [t_0,T]$. Hence, $\dot{p}_1(t) = e^{-\lambda t}q_2(t)$ for a.e. $t \in [t_0,T]$ and $p_2(t)$ is a constant for all $t \in [t_0,T]$.

**Condition (iii)**: Using the formulas for $g$ and $C$, we can show that $\partial g(\bar{x}(t_0),\bar{x}(T)) = \{(0,0,0,1)\}$ and $N((\bar{x}(t_0),\bar{x}(T));C) = \mathbb{R}^2 \times \{(0,0)\}$. So, (iii) yields

$$(p(t_0), -q(T)) \in \{(0,0,0,\gamma)\} + \mathbb{R}^2 \times \{(0,0)\},$$

which means that $q_1(T) = 0$ and $q_2(T) = -\gamma$.

**Condition (iv)**: By (4.6), from (iv) one gets

$$-a\bar{u}(t)q_1(t) - e^{-\lambda t}[\bar{x}_1(t) + \bar{u}(t)]q_2(t) = \max_{u\in[-1,1]} \{-auq_1(t) - e^{-\lambda t}[\bar{x}_1(t) + u]q_2(t)\}$$

for almost every $t \in [t_0, T]$. or, equivalently,

$$[aq_1(t) + e^{-\lambda t}q_2(t)]\bar{u}(t) = \min_{u\in[-1,1]} \{[aq_1(t) + e^{-\lambda t}q_2(t)]u\} \text{ a.e. } t \in [t_0, T].$$

## 4.4 Basic Lemmas

If the curve $\bar{x}_1(t)$ remains in the interior of the domain $[-1, 1]$ for all $t$ from an open interval $(\tau_1, \tau_2)$ of the time axis and touches the boundary of the domain at the moments $\tau_1$ and $\tau_2$, then it must have some special form. A formal formulation of this observation is as follows.

**Lemma 4.1** *Suppose that $[\tau_1, \tau_2]$, $\tau_1 < \tau_2$, is a subsegment of $[t_0, T]$ with $\bar{x}_1(t) \in (-1, 1)$ for all $t \in (\tau_1, \tau_2)$. Then, next statements hold true.*

S1) *If $\bar{x}_1(\tau_1) = -1$ and $\bar{x}_1(\tau_2) = 1$, then $\tau_2 - \tau_1 = 2a^{-1}$ and*

$$\bar{x}_1(t) = -1 + a(t - \tau_1), \quad t \in [\tau_1, \tau_2].$$

S2) *If $\bar{x}_1(\tau_1) = 1$ and $\bar{x}_1(\tau_2) = -1$, then $\tau_2 - \tau_1 = 2a^{-1}$ and*

$$\bar{x}_1(t) = 1 - a(t - \tau_1), \quad t \in [\tau_1, \tau_2].$$

S3) *If $\bar{x}_1(\tau_1) = \bar{x}_1(\tau_2) = -1$, then $\tau_2 - \tau_1 < 4a^{-1}$ and*

$$\bar{x}_1(t) = \begin{cases} -1 + a(t - \tau_1), & t \in [\tau_1, \hat{t}] \\ -1 - a(t - \tau_2), & t \in (\hat{t}, \tau_2], \end{cases}$$

*where $\hat{t} := (\tau_1 + \tau_2)/2$.*

S4) *The situation where $\bar{x}_1(\tau_1) = \bar{x}_1(\tau_2) = 1$ cannot happen.*

**Proof.** Choose $\varepsilon_1 >$ and $\varepsilon_2 > 0$ small enough so as $[\tau_1 + \varepsilon_1, \tau_2 - \varepsilon_2] \subset [\tau_1, \tau_2]$. Then, $\bar{x}_1(t) \in (-1, 1)$ for all $t \in [\tau_1 + \varepsilon_1, \tau_2 - \varepsilon_2]$, i.e., $h(t, \bar{x}(t)) < 0$ for all $t \in [\tau_1 + \varepsilon_1, \tau_2 - \varepsilon_2]$. Thus, applying Proposition 3.4 in the previous chapter with $(FP_{3a})$ in the place of $(FP_{2a})$ in its formulation, one finds that the formula for $\bar{x}_1(\cdot)$ on $[\tau_1+\varepsilon_1, \tau_2-\varepsilon_2]$ belongs to one of the following categories C1–C3:

$$\bar{x}_1(t) = \bar{x}_1(\tau_1+\varepsilon_1) + a(t-\tau_1-\varepsilon_1), \quad t \in [\tau_1+\varepsilon_1, \tau_2-\varepsilon_2],$$

$$\bar{x}_1(t) = \bar{x}_1(\tau_1+\varepsilon_1) - a(t-\tau_1-\varepsilon_1), \quad t \in [\tau_1+\varepsilon_1, \tau_2-\varepsilon_2],$$

and

$$\bar{x}_1(t) = \begin{cases} \bar{x}_1(\tau_1+\varepsilon_1) + a(t-\tau_1-\varepsilon_1), & t \in [\tau_1+\varepsilon_1, t_\zeta] \\ \bar{x}_1(t_\zeta) - a(t-t_\zeta), & t \in (t_\zeta, \tau_2-\varepsilon_2], \end{cases}$$

where $t_\zeta$ is some point in $(\tau_1+\varepsilon_1, \tau_2-\varepsilon_2)$.

To prove the statement S1, let $\varepsilon_2 = k^{-1}$ with $k$ being a positive integer, as large as $k^{-1} \in (\tau_1+\varepsilon_1, \tau_2)$. Since the formula for $\bar{x}_1(\cdot)$ on $[\tau_1+\varepsilon_1, \tau_2-k^{-1}]$ for each $k$ must be of the three types C1–C3, by the Dirichlet principle there must exist a subsequence $\{k'\}$ of $\{k\}$ such that the corresponding formulas belong to a fixed category. If the latter is happens to be C2, then by the continuity of $\bar{x}_1(\cdot)$ one has

$$\begin{aligned}\bar{x}_1(\tau_2) = \lim_{k'\to\infty} \bar{x}_1(\tau_2 - \frac{1}{k'}) &= \lim_{k'\to\infty} \Big[\bar{x}_1(\tau_1+\varepsilon_1) - a(\tau_2 - \frac{1}{k'} - \tau_1 - \varepsilon_1)\Big] \\ &= \bar{x}_1(\tau_1+\varepsilon_1) - a(\tau_2-\tau_1-\varepsilon_1).\end{aligned}$$

This is impossible, because $\bar{x}_1(\tau_2) = 1$. Similarly, the situation where the fixed category is C3 must also be excluded. In the case where the formulas for $\bar{x}_1(\cdot)$ belong to the category C1, we have

$$\bar{x}_1(t) = \bar{x}_1(\tau_1+\varepsilon_1) + a(t-\tau_1-\varepsilon_1), \quad t \in [\tau_1+\varepsilon_1, \tau_2].$$

Now, letting $\varepsilon_1$ tend to zero and using continuity of $\bar{x}_1(\cdot)$, we obtain

$$\bar{x}_1(t) = \bar{x}_1(\tau_1) + a(t-\tau_1), \quad t \in [\tau_1, \tau_2].$$

As $\bar{x}_1(\tau_1) = -1$, the statement S1 is proved.

The statements S2 and S3 are proved similarly.

To prove the assertion S4, it suffices to apply the arguments of the second part (dealing with the segment $[\hat{\alpha}_1, \hat{\alpha_2}]$) of the analysis of Subcase 3a in Section 3.5 □

The forthcoming technical lemma will be in use very frequently.

**Lemma 4.2** *Given any $t_1, t_2 \in [t_0, T]$, $t_1 < t_2$, one puts*

$$J(x,u)|_{[t_1,t_2]} := \int_{t_1}^{t_2} \big[ - e^{-\lambda t}\big(x_1(t) + u(t)\big)\big]dt \tag{4.11}$$

*for any feasible process $(x, u)$ of $(FP_{3a})$. If $(\widetilde{x}, \widetilde{u})$ and $(\check{x}, \check{u})$ are feasible processes for $(FP_{3a})$ with $\widetilde{x}_1(t) = 1$ for all $t \in [t_1, t_2]$ and*

$$\check{x}_1(t) = \begin{cases} 1 - a(t - t_1), & t \in [t_1, \check{t}] \\ 1 + a(t - t_2), & t \in (\check{t}, t_2], \end{cases}$$

*where $\check{t} := 2^{-1}(t_1 + t_2)$, then one has*

$$J(\check{x}, \check{u})|_{[t_1,t_2]} - J(\widetilde{x}, \widetilde{u})|_{[t_1,t_2]} = \frac{1}{\lambda}\Big(\frac{a}{\lambda} - 1\Big)\Delta(t_1, t_2) \tag{4.12}$$

*with*

$$\Delta(t_1, t_2) := e^{-\lambda t_1} - 2e^{-\frac{1}{2}\lambda(t_1+t_2)} + e^{-\lambda t_2}. \tag{4.13}$$

*Besides, it holds that $\Delta(t_1, t_2) > 0$ and $J(\check{x}, \check{u})|_{[t_1,t_2]} > J(\widetilde{x}, \widetilde{u})|_{[t_1,t_2]}$.*

**Proof.** Using the equation $\dot{x}_1(t) = -au(t)$ in (4.5), which is fulfilled for almost all $t \in [t_0, T]$, and the assumed properties of the processes $(\widetilde{x}, \widetilde{u})$ and $(\check{x}, \check{u})$, we have $\widetilde{u}(t) = 0$ for almost all $t \in [t_1, t_2]$ and

$$\check{u}(t) = \begin{cases} 1, & \text{a.e. } t \in [t_1, \check{t}] \\ -1, & \text{a.e. } t \in (\check{t}, t_2]. \end{cases}$$

As $\check{x}(\cdot)$ is a feasible trajectory for $(FP_{3a})$, $\check{x}(\check{t}) \geq -1$, i.e., $t_2 - t_1 \leq 4a^{-1}$.

By the formulas for $\widetilde{x}_1$ and $\widetilde{u}$ on $[t_1, t_2]$,

$$J(\widetilde{x}, \widetilde{u})|_{[t_1,t_2]} = \int_{t_1}^{t_2} \big[ - e^{-\lambda t}\big(\widetilde{x}_1(t) + \widetilde{u}(t)\big)\big]dt = -\int_{t_1}^{t_2} e^{-\lambda t}dt = \frac{1}{\lambda}e^{-\lambda t_2} - \frac{1}{\lambda}e^{-\lambda t_1}.$$

Similarly, from the formulas for $\check{x}_1$ and $\check{u}$ on $[t_1, t_2]$ it follows that

$$\begin{aligned} J(\check{x}, \check{u})|_{[t_1,t_2]} &= \int_{t_1}^{t_2} \big[ - e^{-\lambda t}\big(\check{x}_1(t) + \check{u}(t)\big)\big]dt \\ &= \int_{t_1}^{\check{t}} \big[ - e^{-\lambda t}\big((1 - a(t - t_1)) + 1\big)\big]dt \\ &\quad + \int_{\check{t}}^{t_2} \big[ - e^{-\lambda t}\big((1 + a(t - t_2)) - 1\big)\big]dt \\ &= \int_{t_1}^{\check{t}} e^{-\lambda t}\big[a(t - t_1) - 2\big]dt - \int_{\check{t}}^{t_2} e^{-\lambda t}a(t - t_2)dt. \end{aligned}$$

We have $J(\check{x},\check{u})|_{[t_1,t_2]} = I_1 - I_2$ by denoting the last two integrals respectively by $I_1$ and $I_2$. By regrouping and applying the formula for integration by parts, one has

$$\begin{aligned} I_1 &= -\frac{a}{\lambda}\int_{t_1}^{\check{t}}(t-t_1)d(e^{-\lambda t}) - 2\int_{t_1}^{\check{t}} e^{-\lambda t}dt \\ &= -\frac{a}{\lambda}\big[(t-t_1)e^{-\lambda t}\big]\Big|_{t_1}^{\check{t}} - \int_{t_1}^{\check{t}} e^{-\lambda t}dt\big] - 2\int_{t_1}^{\check{t}} e^{-\lambda t}dt \\ &= (\frac{a}{\lambda}-2)\int_{t_1}^{\check{t}} e^{-\lambda t}dt - \frac{a}{2\lambda}(t_2-t_1)e^{-\lambda\check{t}} \\ &= (\frac{2}{\lambda}-\frac{a}{\lambda^2})\big(e^{-\lambda\check{t}}-e^{-\lambda t_1}\big) - \frac{a}{2\lambda}(t_2-t_1)e^{-\lambda\check{t}}. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= -\frac{a}{\lambda}\int_{\check{t}}^{t_2}(t-t_2)d(e^{-\lambda t}) \\ &= \frac{a}{\lambda}\big[(t-t_2)e^{-\lambda t}\big]\Big|_{t_2}^{\check{t}} + \frac{a}{\lambda}\big[\int_{\check{t}}^{t_2} e^{-\lambda t}dt\big] \\ &= -\frac{a}{2\lambda}(t_2-t_1)e^{-\lambda\check{t}} - \frac{a}{\lambda^2}\big[e^{-\lambda t_2}-e^{-\lambda\check{t}}\big]. \end{aligned}$$

Thus,

$$\begin{aligned} J(\check{x},\check{u})|_{[t_1,t_2]} &= (\frac{2}{\lambda}-\frac{a}{\lambda^2})\big(e^{-\lambda\check{t}}-e^{-\lambda t_1}\big) - \frac{a}{2\lambda}(t_2-t_1)e^{-\lambda\check{t}} + \frac{a}{2\lambda}(t_2-t_1)e^{-\lambda\check{t}} \\ &\quad + \frac{a}{\lambda^2}\big[e^{-\lambda t_2}-e^{-\lambda\check{t}}\big] \\ &= (\frac{2}{\lambda}-\frac{2a}{\lambda^2})e^{-\lambda\check{t}} + (\frac{a}{\lambda^2}-\frac{2}{\lambda})e^{-\lambda t_1} + \frac{a}{\lambda^2}e^{-\lambda t_2}. \end{aligned}$$

Therefore,

$$\begin{aligned} J(\check{x},\check{u})|_{[t_1,t_2]} - J(\widetilde{x},\widetilde{u})|_{[t_1,t_2]} &= (\frac{2}{\lambda}-\frac{2a}{\lambda^2})e^{-\lambda\check{t}} + (\frac{a}{\lambda^2}-\frac{1}{\lambda})e^{-\lambda t_1} + (\frac{a}{\lambda^2}-\frac{1}{\lambda})e^{-\lambda t_2} \\ &= \frac{1}{\lambda}(\frac{a}{\lambda}-1)\big(e^{-\lambda t_1}-2e^{-\lambda\check{t}}+e^{-\lambda t_2}\big). \end{aligned} \tag{4.14}$$

Thus, formula (4.12) is proved. To obtain the second assertion of the lemma, put $\psi(t) = e^{-\lambda t}$ for all $t \in \mathbb{R}$. Since $\psi''(t) > 0$ for every $t$, the function $\psi$ is strictly convex. So,

$$\psi(\frac{1}{2}t_1 + \frac{1}{2}t_2) < \frac{1}{2}\psi(t_1) + \frac{1}{2}\psi(t_2).$$

It follows that $\Delta(t_1,t_2) > 0$ for any $t_1 < t_2$. Combining this with (4.14) and the inequality $\frac{a}{\lambda} - 1 > 0$, we obtain $J(\check{x},\check{u})|_{[t_1,t_2]} > J(\widetilde{x},\widetilde{u})|_{[t_1,t_2]}$. □

The following analogue of Lemma 4.2 will be used latter on.

**Lemma 4.3** *Let $t_1, t_2$ be as in Lemma 4.2. Let $J(x,u)|_{[t_1,t_2]}$ and $\Delta(t_1,t_2)$ be defined, respectively, by* (4.11) *and* (4.13)*. If $(\widetilde{x},\widetilde{u})$ and $(\hat{x},\hat{u})$ are feasible processes for $(FP_{3a})$ with $\widetilde{x}_1(t)=-1$ for all $t\in[t_1,t_2]$ and*

$$\hat{x}_1(t)=\begin{cases}-1+a(t-t_1), & t\in[t_1,\hat{t}]\\ -1-a(t-t_2), & t\in(\hat{t},t_2],\end{cases}$$

*where $\hat{t}:=2^{-1}(t_1+t_2)$, then one has*

$$J(\hat{x},\hat{u})|_{[t_1,t_2]}-J(\widetilde{x},\widetilde{u})|_{[t_1,t_2]}=-\frac{1}{\lambda}(\frac{a}{\lambda}-1)\Delta(t_1,t_2).$$

*Therefore, $J(\hat{x},\hat{u})|_{[t_1,t_2]}<J(\widetilde{x},\widetilde{u})|_{[t_1,t_2]}$.*

**Proof.** By (4.5), from our assumptions it follows that $\widetilde{u}(t)=0$ for almost every $t\in[t_1,t_2]$ and

$$\hat{u}(t)=\begin{cases}-1, & \text{a.e. } t\in[t_1,\hat{t}]\\ 1, & \text{a.e. } t\in(\hat{t},t_2].\end{cases}$$

Since $\hat{x}(\cdot)$ is a feasible trajectory for $(FP_{3a})$, $\hat{x}(\hat{t})\le 1$, i.e., $t_2-t_1\le 4a^{-1}$. One has

$$J(\widetilde{x},\widetilde{u})|_{[t_1,t_2]}=\int_{t_1}^{t_2}\big[-e^{-\lambda t}\big(\widetilde{x}_1(t)+\widetilde{u}(t)\big)\big]dt=\int_{t_1}^{t_2}e^{-\lambda t}dt=-\frac{1}{\lambda}e^{-\lambda t_2}+\frac{1}{\lambda}e^{-\lambda t_1}.$$

Besides, the formulas for $\hat{x}_1$ and $\hat{u}$ on $[t_1,t_2]$ imply that

$$\begin{aligned}J(\hat{x},\hat{u})|_{[t_1,t_2]}&=\int_{t_1}^{\hat{t}}\big[-e^{-\lambda t}\big((-1+a(t-t_1))-1\big)\big]dt\\&\quad+\int_{\hat{t}}^{t_2}\big[-e^{-\lambda t}\big((-1-a(t-t_2))+1\big)\big]dt\\&=-\int_{t_1}^{\hat{t}}e^{-\lambda t}\big[a(t-t_1)-2\big]dt+\int_{\hat{t}}^{t_2}e^{-\lambda t}a(t-t_2)dt.\end{aligned}$$

Thus, changing the sign of the expression $J(\hat{x},\hat{u})|_{[t_1,t_2]}-J(\widetilde{x},\widetilde{u})|_{[t_1,t_2]}$ we get the expression on the left-hand-side of (4.12). So, the desired results follow from Lemma 4.2. □

We will need two more lemmas.

**Lemma 4.4** *Consider the function $\Delta:\mathbb{R}^2\to\mathbb{R}$ defined by* (4.13)*. For any $t_1,t_2\in\mathbb{R}$ with $t_1<t_2$ and for any $\bar{\varepsilon}\in(0,t_2-t_1)$, one has*

$$\Delta(t_1+\bar{\varepsilon},t_2)<\Delta(t_1,t_2) \tag{4.15}$$

*and*

$$\Delta(t_1,t_2) > \Delta(t_1,t_1+\bar\varepsilon) + \Delta(t_1+\bar\varepsilon,t_2). \tag{4.16}$$

**Proof.** Fix a value $\bar\varepsilon \in (0, t_2-t_1)$. To obtain (4.15), consider the function $\psi_1(\varepsilon) := \Delta(t_1+\varepsilon, t_2)$ of the variable $\varepsilon \in \mathbb{R}$. Since

$$\psi_1(\varepsilon) = e^{-\lambda(t_1+\varepsilon)} - 2e^{-\frac{1}{2}\lambda(t_1+\varepsilon+t_2)} + e^{-\lambda t_2},$$

one sees that $\psi_1(\cdot)$ is continuously differentiable on $\mathbb{R}$ and

$$\psi_1'(\varepsilon) = \lambda\big(e^{-\frac{1}{2}\lambda(t_1+\varepsilon+t_2)} - e^{-\lambda(t_1+\varepsilon)}\big).$$

As the function $r(t) := e^{-\lambda t}$ is strictly decreasing on $\mathbb{R}$, the last equality implies that $\psi_1'(\varepsilon) < 0$ for every $\varepsilon \in [0, t_2-t_1)$. Hence, the function $\psi_1(\cdot)$ is strictly decreasing on $[0, t_2-t_1)$. So, the inequality (4.15) is valid.

To obtain (4.16), observe from (4.13) that

$$\begin{aligned}
&\Delta(t_1,t_2) - \Delta(t_1,t_1+\bar\varepsilon) - \Delta(t_1+\bar\varepsilon,t_2)\\
&= \ e^{-\lambda t_1} - 2e^{-\lambda\frac{t_1+t_2}{2}} + e^{-\lambda t_2} - \big(e^{-\lambda t_1} - 2e^{-\lambda(t_1+\frac{\bar\varepsilon}{2})} + e^{-\lambda(t_1+\bar\varepsilon)}\big)\\
&\quad - \big(e^{-\lambda(t_1+\bar\varepsilon)} - 2e^{-\lambda(\frac{t_1+t_2}{2}+\frac{\bar\varepsilon}{2})} + e^{-\lambda t_2}\big)\\
&= \ 2\big[e^{-\lambda(\frac{t_1+t_2}{2}+\frac{\bar\varepsilon}{2})} - e^{-\lambda\frac{t_1+t_2}{2}}\big] - 2\big[e^{-\lambda(t_1+\bar\varepsilon)} - e^{-\lambda(t_1+\frac{\bar\varepsilon}{2})}\big]
\end{aligned}$$

Applying the classical mean value theorem to the differentiable function $r(t) = e^{-\lambda t}$, one can find $\tau_1 \in (t_1+\frac{\bar\varepsilon}{2}, t_1+\bar\varepsilon)$ and $\tau_2 \in (\frac{t_1+t_2}{2}, \frac{t_1+t_2}{2}+\frac{\bar\varepsilon}{2})$ such that

$$e^{-\lambda(t_1+\bar\varepsilon)} - e^{-\lambda(t_1+\frac{\bar\varepsilon}{2})} = \frac{\bar\varepsilon}{2}(-\lambda)e^{-\lambda\tau_1},$$

$$e^{-\lambda(\frac{t_1+t_2}{2}+\frac{\bar\varepsilon}{2})} - e^{-\lambda\frac{t_1+t_2}{2}} = \frac{\bar\varepsilon}{2}(-\lambda)e^{-\lambda\tau_2}.$$

Thus, $\Delta(t_1,t_2)-\Delta(t_1,t_1+\bar\varepsilon)-\Delta(t_1+\bar\varepsilon,t_2) = \bar\varepsilon\lambda\big[e^{-\lambda\tau_1}-e^{-\lambda\tau_2}\big]$. As the function $r(t)$ is strictly decreasing on $\mathbb{R}$ and $\tau_1 < \tau_2$, one gets $e^{-\lambda\tau_1} - e^{-\lambda\tau_2} > 0$; hence the inequality (4.16) is proved. □

**Lemma 4.5** *Let there be given $t_1, t_2 \in [t_0, T]$, $t_1 < t_2$, and $\xi > 0$. Suppose that $(\widetilde{x}^\xi, \widetilde{u}^\xi)$ and $(\check{x}^\xi, \check{u}^\xi)$ are feasible processes for $(FP_{3a})$ with $\widetilde{x}_1^\xi(t) = \xi$ for all $t \in [t_1, t_2]$ and*

$$\check{x}_1^\xi(t) = \begin{cases} \xi - a(t-t_1), & t \in [t_1, \check{t}] \\ \xi + a(t-t_2), & t \in (\check{t}, t_2], \end{cases}$$

*where* $\check{t} := 2^{-1}(t_1 + t_2)$. *Then one has*

$$J(\check{x}^\xi, \check{u}^\xi)|_{[t_1,t_2]} - J(\widetilde{x}^\xi, \widetilde{u}^\xi)|_{[t_1,t_2]} = \frac{1}{\lambda}(\frac{a}{\lambda} - 1)\Delta(t_1, t_2),$$

*with* $J(x, u)|_{[t_1,t_2]}$ *and* $\Delta(t_1, t_2)$ *being defined respectively by* (4.11) *and* (4.13). *Besides, the strict inequality* $J(\check{x}^\xi, \check{u}^\xi)|_{[t_1,t_2]} > J(\widetilde{x}^\xi, \widetilde{u}^\xi)|_{[t_1,t_2]}$ *is valid.*

**Proof.** The proof is similar to that of Lemma 4.2. □

The following two lemmas are crucial for describing the behavior of the local solutions of $(FP_{3a})$.

**Lemma 4.6** *The situation where* $\bar{x}_1(t) = -1$ *for all* $t$ *from a subsegment* $[t_1, t_2]$ *of* $[t_0, T]$ *with* $t_1 < t_2$ *cannot happen.*

**Proof.** Since $(\bar{x}, \bar{u})$ is a $W^{1,1}$ local minimizer of $(FP_{3a})$, by Definition 3.1 there exists $\delta > 0$ such that the process $(\bar{x}, \bar{u})$ minimizes the quantity $x_2(T)$ over all feasible processes $(x, u)$ of $(FP_{3a})$ with $\|\bar{x} - x\|_{W^{1,1}([t_0,T];\mathbb{R}^2)} \leq \delta$.

To prove our assertion, suppose on the contrary that there are $t_1, t_2$ with $t_0 \leq t_1 < t_2 \leq T$ such that $\bar{x}_1(t) = -1$ for all $t \in [t_1, t_2]$. Fixing a number $\varepsilon \in (0, t_2 - t_1)$, we consider the pair of functions $(\hat{x}^\varepsilon, \hat{u}^\varepsilon)$, where

$$\hat{x}_1^\varepsilon(t) := \begin{cases} \bar{x}_1(t), & t \in [t_0, t_1) \cup (t_1 + \varepsilon, T] \\ -1 + a(t - t_1), & t \in [t_1, t_1 + 2^{-1}\varepsilon] \\ -1 - a(t - t_1 - \varepsilon), & t \in (t_1 + 2^{-1}\varepsilon, t_1 + \varepsilon] \end{cases}$$

and

$$\hat{u}^\varepsilon(t) := -a^{-1}\frac{d\hat{x}_1^\varepsilon(t)}{dt}, \quad t \in [t_0, T].$$

Clearly, $(\hat{x}^\varepsilon, \hat{u}^\varepsilon)$ is a feasible process of $(FP_{3a})$. By (4.3), (4.11), and the definition of $\hat{x}_1^\varepsilon(\cdot)$, we have

$$\bar{x}_2(T) - \hat{x}_2^\varepsilon(T) = J(\bar{x}, \bar{u})_{|[t_1,t_1+\varepsilon]} - J(\hat{x}^\varepsilon, \hat{u}^\varepsilon)_{|[t_1,t_1+\varepsilon]}. \tag{4.17}$$

Besides, it follows from Lemma 4.3 and the constructions of $\bar{x}$ and $\hat{x}^\varepsilon$ on $[t_1, t_1 + \varepsilon]$ that

$$J(\bar{x}, \bar{u})_{|[t_1,t_1+\varepsilon]} - J(\hat{x}^\varepsilon, \hat{u}^\varepsilon)_{|[t_1,t_1+\varepsilon]} > 0.$$

Combining this with (4.17) yields $\bar{x}_2(T) > \hat{x}_2^\varepsilon(T)$, which contradicts the $W^{1,1}$ local optimality of $(\bar{x}, \bar{u})$, because $\|\bar{x} - \hat{x}^\varepsilon\|_{W^{1,1}([t_0,T];\mathbb{R}^2)} \leq \delta$ for $\varepsilon > 0$ small enough. □

**Lemma 4.7** *One must have $\bar{x}_1(t) > -1$ for all $t \in (t_0, T)$.*

**Proof.** By our standing assumption, $(\bar{x}, \bar{u})$ is a $W^{1,1}$ a local minimizer for $(FP_{3a})$. Let $\delta > 0$ be chosen as in the proof of Lemma 4.6. If the assertion is false, there would exist $\check{t} \in (t_0, T)$ with $\bar{x}_1(\check{t}) = -1$.

If there exist two positive numbers $\varepsilon_1$ and $\varepsilon_2$ such that $\bar{x}_1(t) > -1$ for all $t \in (\check{t}-\varepsilon_1, \check{t}) \cup (\check{t}, \check{t}+\varepsilon_2)$. Then, thanks to the continuity of $\bar{x}_1(\cdot)$, by shrinking $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ (if necessary) one may assume that $\bar{x}_1(t) \in (-1, 1)$ for all $t \in (\check{t} - \varepsilon_1, \check{t}) \cup (\check{t}, \check{t} + \varepsilon_2)$. Then, since the curve $\bar{x}_1(\cdot)$ cannot have more than one turning on the interval $(\check{t} - \varepsilon_1, \check{t})$ (resp., on the interval $(\check{t}, \check{t} + \varepsilon_2)$) by the observation given at the beginning of the proof of Lemma 4.1. So, replacing $\varepsilon_1$ (resp., $\varepsilon_2$) by a smaller positive number, one may assume that

$$\bar{x}_1(t) = \begin{cases} -1 - a(t - \check{t}), & t \in [\check{t} - \varepsilon_1, \check{t}] \\ -1 + a(t - \check{t}), & t \in (\check{t}, \check{t} + \varepsilon_2]. \end{cases}$$

To get a contradiction, we can apply the construction given in Lemma 4.5. Namely, choose $\varepsilon > 0$ as small as $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$ and define a feasible process $(\widetilde{x}^\varepsilon, \widetilde{u}^\varepsilon)$ for $(FP_{3a})$ by setting

$$\widetilde{u}^\varepsilon(t) = \begin{cases} 0, & t \in [\check{t} - \varepsilon, \check{t} + \varepsilon] \\ \bar{u}(t), & t \in [t_0, \check{t} - \varepsilon) \cup (\check{t} + \varepsilon, T] \end{cases}$$

and

$$\widetilde{x}^\varepsilon(t) = \begin{cases} \bar{x}_1(\check{t} - \varepsilon), & t \in [\check{t} - \varepsilon, \check{t} + \varepsilon] \\ \bar{x}(t), & t \in [t_0, \check{t} - \varepsilon) \cup (\check{t} + \varepsilon, T]. \end{cases}$$

Then, by Lemma 4.5 one has $J(\bar{x}, \bar{u}) > J(\widetilde{x}^\varepsilon, \widetilde{u}^\varepsilon)$. This contradicts the $W^{1,1}$ local optimality of $(\bar{x}, \bar{u})$, because $\|\bar{x} - \hat{x}^\varepsilon\|_{W^{1,1}([t_0,T];\mathbb{R}^2)} \leq \delta$ for $\varepsilon > 0$ small enough.

Since one cannot find $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that the strict inequality $\bar{x}_1(t) > -1$ holds for all $t \in (\check{t} - \varepsilon_1, \check{t}) \cup (\check{t}, \check{t} + \varepsilon_2)$, there must exist a sequence $\{t_k\}$ in $(t_0, T)$ converging to $\check{t}$ such that either $t_k < \check{t}$ for all $k$ or $t_k > \check{t}$ for all $k$, and $\bar{x}_1(t_k) = -1$ for each $k$. It suffices to consider the case $t_k < \check{t}$ for all $k$, as the other case can be treated similarly. By considering a subsequence (if necessary), we may assume that $t_k < t_{k+1}$ for all $k$.

Choose $\bar{k}$ as large as

$$\check{t} - t_{\bar{k}} < \min\{2\delta a^{-1},\, 4a^{-1}\}. \tag{4.18}$$

This choice of $\bar{k}$ guarantees that $\bar{x}_1(t) < 1$ for every $t \in [t_{\bar{k}}, \check{t}]$. Indeed, otherwise there is some $\alpha \in (t_{\bar{k}}, \check{t})$ with $\bar{x}_1(\alpha) = 1$. Setting

$$\alpha_1 = \min\big\{t \in [t_{\bar{k}}, \alpha] \,:\, \bar{x}_1(t) = 1\big\},\ \ \alpha_2 = \max\big\{t \in [\alpha, \check{t}] \,:\, \bar{x}_1(t) = 1\big\},$$

one has $\alpha_1 \leq \alpha_2$, $[\alpha_1, \alpha_2] \subset [t_{\bar{k}}, \check{t}]$, and $\bar{x}_1(t) \in (-1, 1)$ for all $t$ in intervals $(t_{\bar{k}}, \alpha_1)$ and $(\alpha_2, \check{t})$. Then, it follows from assertion S1 of Lemma 4.1 that $\alpha_1 - t_{\bar{k}} = 2a^{-1}$. Similarly, by assertion S2 in that lemma, one has $\check{t} - \alpha_2 = 2a^{-1}$. So, one gets $\check{t} - t_{\bar{k}} \geq 4a^{-1}$, which comes in conflict with (4.18).

By Lemma 4.6, one cannot have $\bar{x}_1(t) = -1$ for all $t \in [t_{\bar{k}}, t_{\bar{k}+1}]$. Thus, there is some $\tau \in (t_{\bar{k}}, t_{\bar{k}+1})$ with $\bar{x}_1(\tau) > -1$. Setting

$$\tau_1 = \max\big\{t \in [t_{\bar{k}}, \tau] \,:\, \bar{x}_1(t) = -1\big\},\ \ \tau_2 = \min\big\{t \in [\tau, t_{\bar{k}+1}] \,:\, \bar{x}_1(t) = -1\big\},$$

one has $\tau_1 < \tau_2$, $[\tau_1, \tau_2] \subset [t_{\bar{k}}, t_{\bar{k}+1}]$, and $\bar{x}_1(t) \in (-1, 1)$ for all $t \in (\tau_1, \tau_2)$. Hence, replacing $t_{\bar{k}}$ (resp., $t_{\bar{k}+1}$) by $\tau_1$ (resp., $\tau_2$), one sees that all the above-described properties of the sequence $\{t_k\}$ remain and, in addition,

$$\bar{x}_1(t) \in (-1, 1), \quad \forall t \in (t_{\bar{k}}, t_{\bar{k}+1}). \tag{4.19}$$

Let $F := \{t \in [t_{\bar{k}}, \check{t}] \,:\, \bar{x}_1(t) = -1\}$ and $E := [t_{\bar{k}}, \check{t}] \setminus F$. Since $F$ is a closed subset of $\mathbb{R}$ and $E = (t_{\bar{k}}, \check{t}) \setminus F$, $E$ is an open subset of $\mathbb{R}$. So, $E$ is the union of a countable family of disjoint open intervals (see [75, Proposition 9, p. 17]). Since $t_k \notin E$ for all $k$, we have a representation $E = \bigcup_{i=1}^{\infty} E_i$, where the intervals $E_i = (\tau_1^{(i)}, \tau_2^{(i)})$, $i \in \mathbb{N}$, are nonempty and disjoint. Thanks to (4.19), one may suppose that $E_1 = (\tau_1^{(1)}, \tau_2^{(1)}) = (t_{\bar{k}}, t_{\bar{k}+1})$. Note also that, for any $i \in \mathbb{N}$, $\bar{x}_1(t) \in (-1, 1)$ for all $t \in E_i$. Since $\bar{x}_1(\tau_1^{(i)}) = \bar{x}_1(\tau_2^{(i)}) = -1$, by assertion S3 of Lemma 4.1 one gets

$$\bar{x}_1(t) = \begin{cases} -1 + a(t - \tau_1^{(i)}), & t \in [\tau_1^{(i)}, 2^{-1}(\tau_1^{(i)} + \tau_2^{(i)})] \\ -1 - a(t - \tau_2^{(i)}), & t \in (2^{-1}(\tau_1^{(i)} + \tau_2^{(i)}), \tau_2^{(i)}]. \end{cases}$$

If the set $F_1 := F \setminus \{t_{\bar{k}}\}$ has an isolated point in the induced topology of $[t_{\bar{k}}, \check{t}]$, says, $\bar{t}$. Then, one must have $\bar{t} \in [t_{\bar{k}+1}, \check{t})$. So, there exists $\varepsilon > 0$ such that $(\bar{t} - \varepsilon, \bar{t} + \varepsilon) \subset (t_{\bar{k}}, \check{t})$ and $\bar{x}_1(t) \in (-1, 1)$ for all $t \in (\bar{t} - \varepsilon, \bar{t}) \cup (\bar{t}, \bar{t} + \varepsilon)$. Applying the construction given in the first part of this proof, we find a feasible process $(\widetilde{x}^\varepsilon, \widetilde{u}^\varepsilon)$ for $(FP_{3a})$ with the property $J(\bar{x}, \bar{u}) > J(\widetilde{x}^\varepsilon, \widetilde{u}^\varepsilon)$. This contradicts the $W^{1,1}$ local optimality of $(\bar{x}, \bar{u})$, because (4.18) assures that $\|\bar{x} - \hat{x}^\varepsilon\|_{W^{1,1}([t_0,T];\mathbb{R}^2)} \leq \delta$.

Now, suppose that every point in the compact set $F_1$ is a limit point of this set in the induced topology of $[t_{\bar{k}}, \check{t}]$. Then, if the Lebesgue measure $\mu_L(F_1)$ of $F_1$ is null, then the structure of $F_1$ is similar to that of *the Cantor set*[1], constructed from the segment $[t_{\bar{k}+1}, \check{t}] \subset \mathbb{R}$. If $\mu_L(F_1) > 0$, the structure of $F_1$ is similar to that of a *fat Cantor set*, which is also called *a Smith-Volterra-Cantor set*[2].

Putting

$$\widetilde{u}(t) = \begin{cases} 0, & t \in [t_{\bar{k}}, \check{t}] \\ \bar{u}(t), & t \in [t_0, t_{\bar{k}}) \cup (\check{t}, T] \end{cases} \tag{4.20}$$

and

$$\widetilde{x}_1(t) = \begin{cases} -1, & t \in [t_{\bar{k}}, \check{t}] \\ \bar{x}_1(t), & t \in [t_0, t_{\bar{k}}) \cup (\check{t}, T], \end{cases} \tag{4.21}$$

we see that $(\widetilde{x}, \widetilde{u})$ is a feasible process for $(FP_{3a})$. Similarly, define

$$u(t) = \begin{cases} -1, & t \in [t_{\bar{k}+1}, 2^{-1}(t_{\bar{k}+1} + \check{t})] \\ 1, & t \in (2^{-1}(t_{\bar{k}+1} + \check{t}), \check{t}] \\ \bar{u}(t), & t \in [t_0, t_{\bar{k}+1}) \cup (\check{t}, T] \end{cases} \tag{4.22}$$

and

$$x_1(t) = \begin{cases} -1 + a(t - t_{\bar{k}+1}), & t \in [t_{\bar{k}+1}, 2^{-1}(t_{\bar{k}} + \check{t})] \\ -1 - a(t - \check{t}), & t \in (2^{-1}(t_{\bar{k}+1} + \check{t}), \check{t}] \\ \bar{x}_1(t), & t \in [t_0, t_{\bar{k}+1}) \cup (\check{t}, T], \end{cases} \tag{4.23}$$

and observe that $(x, u)$ is a feasible process for $(FP_{3a})$. Using (4.18), it is easy to verify that $\|x - \bar{x}\|_{W^{1,1}([t_0,T];\mathbb{R}^2)} \le \delta$. Thus, if it can be shown that

$$J(x, u) < J(\bar{x}, \bar{u}), \tag{4.24}$$

then we get a contradiction to the $W^{1,1}$ local optimality of $(\bar{x}, \bar{u})$. Hence, the proof of the lemma will be completed.

By (4.20)–(4.23) and Lemma 4.3, one has

$$J(\widetilde{x}, \widetilde{u}) - J(x, u) = J(\widetilde{x}, \widetilde{u})|_{[t_{\bar{k}}, \check{t}]} - J(x, u)|_{[t_{\bar{k}}, \check{t}]}.$$

Therefore,

$$J(\widetilde{x}, \widetilde{u}) - J(x, u) = \frac{1}{\lambda}\Big(\frac{a}{\lambda} - 1\Big)\big[\Delta(t_{\bar{k}}, t_{\bar{k}+1}) + \Delta(t_{\bar{k}+1}, \check{t})\big], \tag{4.25}$$

[1] https://en.wikipedia.org/wiki/Cantor_set.

[2] https://en.wikipedia.org/wiki/Smith-Volterra-Cantor_set.

where $\Delta(t_1,t_2)$, for any $t_1,t_2$ with $t_1<t_2$, is given by (4.13). In addition, using (4.20), (4.21), the decomposition $[t_{\bar{k}+1},\check{t}]=\big(\bigcup_{i=2}^{\infty}E_i\big)\cup F_1$, and the sum rule [49, Theorem 1', p. 297] and the decomposition formula [49, Theorem 4, p. 298] for the Lebesgue integrals, one gets

$$\begin{aligned}J(\bar{x},\bar{u})-J(\widetilde{x},\widetilde{u}) &= J(\bar{x},\bar{u})|_{[t_{\bar{k}},\check{t}]}-J(\widetilde{x},\widetilde{u})|_{[t_{\bar{k}},\check{t}]}\\ &=\int_{[t_{\bar{k}},\check{t}]}\Big[-e^{-\lambda t}\big([\bar{x}_1(t)+\bar{u}(t)]-[\widetilde{x}_1(t)+\widetilde{u}(t)]\big)\Big]dt\\ &=\sum_{i=2}^{\infty}\int_{E_i}\Big[-e^{-\lambda t}\big([\bar{x}_1(t)+\bar{u}(t)]-[\widetilde{x}_1(t)+\widetilde{u}(t)]\big)\Big]dt\\ &+\int_{F_1}\Big[-e^{-\lambda t}\big([\bar{x}_1(t)+\bar{u}(t)]-[\widetilde{x}_1(t)+\widetilde{u}(t)]\big)\Big]dt.\end{aligned}$$

Hence, it holds that

$$J(\bar{x},\bar{u})-J(\widetilde{x},\widetilde{u})=-\frac{1}{\lambda}\Big(\frac{a}{\lambda}-1\Big)\sum_{i=2}^{\infty}\Delta\big(\tau_1^{(i)},\tau_2^{(i)}\big)+I, \tag{4.26}$$

where $I:=\int_{F_1}\Big[-e^{-\lambda t}\big([\bar{x}_1(t)+\bar{u}(t)]-[\widetilde{x}_1(t)+\widetilde{u}(t)]\big)\Big]dt$. Given any $t\in F_1$, we observe that $\bar{x}_1(t)=\widetilde{x}_1(t)=-1$ and $\widetilde{u}(t)=0$. Since every point in $F_1$ is a limit point of this set in the induced topology of $[t_{\bar{k}},\check{t}]$, we can find a sequence $\{\xi_j^t\}$ in $F_1$ satisfying $\lim_{j\to\infty}\xi_j^t=t$. As the derivative $\bar{x}_1(t)$ exists a.e. on $[t_0,T]$, it exists a.e. on $F_1$. In combination with the first differential equation in (4.5), this yields $\dot{\bar{x}}_1(t)=-a\bar{u}(t)$ a.e. $t\in F_1$. Since $\bar{x}_1(t)=-1$ for all $t\in F_1$, for a.e. $t\in F_1$ it holds that

$$\bar{u}(t)=-\frac{1}{a}\dot{\bar{x}}_1(t)=-\frac{1}{a}\lim_{j\to\infty}\frac{\bar{x}_1(\xi_j^t)-\bar{x}_1(t)}{\xi_j^t-t}=0.$$

We have thus shown that $\big[\bar{x}_1(t)+\bar{u}(t)\big]-\big[\widetilde{x}_1(t)+\widetilde{u}(t)\big]=0$ for a.e. $t\in F_1$. This implies that $I=0$. Now, adding (4.25) (4.26), we get

$$\begin{aligned}&J(\bar{x},\bar{u})-J(x,u)\\ &=\frac{1}{\lambda}\Big(\frac{a}{\lambda}-1\Big)\Big[\Delta(t_{\bar{k}},t_{\bar{k}+1})+\Delta(t_{\bar{k}+1},\check{t})-\textstyle\sum_{i=2}^{\infty}\Delta\big(\tau_1^{(i)},\tau_2^{(i)}\big)\Big].\end{aligned} \tag{4.27}$$

We have

$$\sum_{i=2}^{\infty}\Delta\big(\tau_1^{(i)},\tau_2^{(i)}\big)\leq\Delta(t_{\bar{k}+1},\check{t}). \tag{4.28}$$

To establish this inequality, we first show that

$$\sum_{i=2}^{m}\Delta\big(\tau_1^{(i)},\tau_2^{(i)}\big)<\Delta(t_{\bar{k}+1},\check{t}) \tag{4.29}$$

for any integer $m \geq 2$. Taking account of the fact that every point in $F_1$ is a limit point of this set in the induced topology of $[t_{\bar{k}}, \check{t}]$, by reordering the intervals $\big(\tau_1^{(i)}, \tau_2^{(i)}\big)$ for $i = 2, \ldots, m$, we may assume that

$$t_{\bar{k}+1} < \tau_1^{(2)} < \tau_2^{(2)} < \tau_1^{(3)} < \tau_2^{(3)} < \cdots < \tau_1^{(m)} < \tau_2^{(m)} < \check{t}.$$

Then, by Lemma 4.4 and by induction, we have

$$\begin{aligned}
\sum_{i=2}^{m} \Delta\big(\tau_1^{(i)}, \tau_2^{(i)}\big) &< \Big[\Delta\big(t_{\bar{k}+1}, \tau_1^{(2)}\big) + \Delta\big(\tau_1^{(2)}, \tau_2^{(2)}\big)\Big] + \sum_{i=3}^{m} \Delta\big(\tau_1^{(i)}, \tau_2^{(i)}\big) \\
&< \Big[\Delta\big(t_{\bar{k}+1}, \tau_2^{(2)}\big) + \Delta\big(\tau_2^{(2)}, \tau_1^{(3)}\big)\Big] + \sum_{i=3}^{m} \Delta\big(\tau_1^{(i)}, \tau_2^{(i)}\big) \\
&\vdots \\
&< \Delta\big(t_{\bar{k}+1}, \tau_2^{(m)}\big) + \Delta\big(\tau_2^{(m)}, \check{t}\big) \\
&< \Delta(t_{\bar{k}+1}, \check{t}).
\end{aligned}$$

Thus, (4.29) is valid. Since $\Delta\big(\tau_1^{(i)}, \tau_2^{(i)}\big) > 0$ for all $i = 2, 3, \ldots,$ the estimate (4.29) shows that the series $\sum_{i=2}^{\infty} \Delta\big(\tau_1^{(i)}, \tau_2^{(i)}\big)$ is convergent. Letting $m \to \infty$, from (4.29) one obtains (4.28). Since $\Delta(t_{\bar{k}}, t_{\bar{k}+1}) > 0$, the equality (4.27) and the inequality (4.28) imply (4.24).

The proof is complete. □

## 4.5 Synthesis of the Optimal Processes

To continue, using the parameters tube $(\lambda, a, x_0, t_0, T)$ of $(FP_{3a})$, we define

$$\rho = \frac{1}{\lambda} \ln \frac{a}{a - \lambda} > 0 \quad \text{and} \quad \bar{t} = T - \rho.$$

Besides, for a given $x_0 \in [-1, 1]$, let

$$\rho_1 := a^{-1}(1 + x_0) \quad \text{and} \quad \rho_2 := a^{-1}(1 - x_0). \tag{4.30}$$

As $x_0 \in [-1, 1]$, one has $\rho_1 \in [0, 2a^{-1}]$ and $\rho_2 \in [0, 2a^{-1}]$. Moreover, since $\bar{x}_1(t)$ is a continuous function, the set

$$\mathcal{T}_1 := \{t \in [t_0, T] \,:\, \bar{x}_1(t) = 1\}$$

is a compact set (which may be empty). If $\mathcal{T}_1$ is nonempty, then we consider the numbers

$$\alpha_1 := \min\{t \,:\, t \in \mathcal{T}_1\} \quad \text{and} \alpha_2 := \max\{t \,:\, t \in \mathcal{T}_1\}.$$

By Lemma 4.7, one of the next four cases must occur.

**Case 1:** $x_0 > -1$ *and* $\bar{x}_1(t) > -1$ *for all* $t \in (t_0, T]$. Then, condition (i) means that (4.8) and (4.9) are satisfied, while conditions (ii)–(iv) remain the same as those in Section 3.5 of Chapter 3. Moreover, it is clear that $(\bar{x}, \bar{u})$ is a $W^{1,1}$ local minimizer of the problem considered therein. So, the curve $\bar{x}_1(t)$ must be of one of the four types (a)–(d) depicted in Theorem 3.4, where we let $\bar{x}_1(t)$ play the role of $\bar{x}(t)$. Of course, the condition $\bar{x}_1(t) > -1$ for all $t \in [t_0, T]$ must be satisfied. With respect to the just mentioned four types of $\bar{x}(t)$, we have the following four subcases.

<u>*Subcase 1a*</u>: $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = x_0 - a(t - t_0), \quad t \in [t_0, T]. \tag{4.31}$$

By statement (a) of Theorem 3.4, this situation happens when $T - t_0 \leq \rho$ (i.e., $\bar{t} \leq t_0$). By (4.31), the condition $\bar{x}_1(t) > -1$ for all $t \in (t_0, T]$ is equivalent to $\bar{x}_1(T) > -1$ or, equivalently, $T - t_0 < \rho_1$. Therefore, if either $\rho < \rho_1$ and $T - t_0 \leq \rho$ or $\rho \geq \rho_1$ and $T - t_0 < \rho_1$, then $\bar{x}_1(t)$ is given by (4.31).

<u>*Subcase 1b*</u>: $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}] \\ x_0 - a(t + t_0 - 2\bar{t}), & t \in (\bar{t}, T]. \end{cases} \tag{4.32}$$

Accordingly, statement (b) of Theorem 3.4 requires that $\rho < T - t_0 < \rho + \rho_2$. By (4.32), the condition $\bar{x}_1(t) > -1$ for all $t \in (t_0, T]$ is equivalent to the requirement $\bar{x}_1(T) > -1$, which means that $T - t_0 > 2\rho - \rho_1$. Thus, if $\max\{\rho; 2\rho - \rho_1\} < T - t_0 < \rho + \rho_2$, then $\bar{x}_1(t)$ is given by (4.32).

<u>*Subcase 1c*</u>: $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T]. \end{cases} \tag{4.33}$$

Then, statement (c) of Theorem 3.4 requires that $\rho < T - t_0 = \rho + \rho_2$. By (4.33), the condition $\bar{x}_1(t) > -1, \forall t \in (t_0, T]$ is equivalent to the requirement $\bar{x}_1(T) > -1$, which means that $\rho < 2a^{-1}$. Thus, if $\rho < T - t_0 = \rho + \rho_2$ and $\rho < 2a^{-1}$, then $\bar{x}_1(t)$ is given by (4.33).

<u>*Subcase 1d*</u>: $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_0 + \rho_2) \\ 1, & t \in [t_0 + \rho_2, \bar{t}) \\ 1 - a(t - \bar{t}), & t \in [\bar{t}, T]. \end{cases} \tag{4.34}$$

In this situation, statement (d) of Theorem 3.4 requires that $\rho + \rho_2 < T - t_0$. Meanwhile, by (4.34), the condition $\bar{x}_1(t) > -1$, $\forall t \in (t_0, T]$ is equivalent to the requirement $\bar{x}_1(T) > -1$, which means that $\rho < 2a^{-1}$. So, if $\rho + \rho_2 < T - t_0$ and $\rho < 2a^{-1}$, then $\bar{x}_1(t)$ is given by (4.34).

**Case 2:** $x_0 = -1$ *and* $\bar{x}_1(t) > -1$ *for all* $t \in (t_0, T]$. Let $\bar{\varepsilon} > 0$ be such that $t_0 + \bar{\varepsilon} < T$. For any $k \in I\!N$ with $k^{-1} \in (0, \bar{\varepsilon})$, based on the comments before Propositions 3.2 and 3.3, the restriction of $(\bar{x}, \bar{u})$ on $[t_0 + k^{-1}, T]$ is a $W^{1,1}$ local minimizer for the Mayer problem obtained from $(FP_{3a})$ by replacing $t_0$ with $t_0 + k^{-1}$. Since $\bar{x}_1(t) > -1$ for every $t \in [t_0 + k^{-1}, T]$, repeating the arguments already used in Case 1 yields a formula for $\bar{x}_1(t)$ on $[t_0 + k^{-1}, T]$. With the numbers $\rho_1(k) := a^{-1}[1 + \bar{x}_1(t_0 + k^{-1})]$ and $\rho_2(k) := a^{-1}[1 - \bar{x}_1(t_0 + k^{-1})]$, for every $k \in I\!N$ we see that the function $\bar{x}_1(t)$ on $[t_0 + k^{-1}, T]$ must belong to one of the following four categories, which correspond to the four forms of the function $\bar{x}_1(t)$ in Case 1.

(C1) $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \bar{x}_1(t_0 + k^{-1}) - a(t - t_0 - k^{-1}), \quad t \in [t_0 + k^{-1}, T],$$

provided that $\rho < \rho_1(k)$ and $T - t_0 - k^{-1} \leq \rho$ or that $\rho \geq \rho_1(k)$ and $T - t_0 - k^{-1} < \rho_1(k)$.

(C2) $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} \bar{x}_1(t_0 + k^{-1}) + a(t - t_0 - k^{-1}), & t \in [t_0 + k^{-1}, \bar{t}] \\ \bar{x}_1(t_0 + k^{-1}) - a(t + t_0 + k^{-1} - 2\bar{t}), & t \in (\bar{t}, T], \end{cases}$$

provided that $\max\{\rho; 2\rho - \rho_1(k)\} < T - t_0 - k^{-1} < \rho + \rho_2(k)$.

(C3) $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} \bar{x}_1(t_0 + k^{-1}) + a(t - t_0 - k^{-1}), & t \in [t_0 + k^{-1}, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T], \end{cases}$$

provided that $\rho < T - t_0 - k^{-1} = \rho + \rho_2(k)$ and $\rho < 2a^{-1}$.

(C4) $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} \bar{x}_1(t_0 + k^{-1}) + a(t - t_0 - k^{-1}), & t \in [t_0 + k^{-1}, t_0 + \rho_2(k)] \\ 1, & t \in (t_0 + \rho_2(k), \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T], \end{cases}$$

provided that $\rho + \rho_2(k) < T - t_0 - k^{-1}$ and $\rho < 2a^{-1}$.

By the Dirichlet principle, there exist an infinite number of indexes $k$ with $k^{-1} \in (0, \bar{\varepsilon})$ such that the formula for $\bar{x}_1(t)$ is given in the category C1 (resp., C2, C3 or C4). By considering a subsequence if necessary, we may assume that this happens for all $k$ with $k^{-1} \in (0, \bar{\varepsilon})$.

If the first situation occurs, then $\bar{x}_1(t) = -1 - a(t - t_0)$ for all $t \in [t_0, T]$ by letting $k \to \infty$. This is impossible since the requirement $\bar{x}_1(t) > -1$ for all $t \in (t_0, T]$ is violated.

If the second situation occurs, then by letting $k \to \infty$ we have

$$\bar{x}_1(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, \bar{t}] \\ -1 - a(t + t_0 - 2\bar{t}), & t \in (\bar{t}, T], \end{cases}$$

provided that $\max\{\rho; 2\rho - \rho_1\} \le T - t_0 \le \rho + \rho_2$. Since $\bar{x}_1(t) > -1$ for all $t \in (t_0, T]$, one must have $\bar{x}_1(T) > -1$; hence $2\rho < T - t_0$. Since $x_0 = -1$, by (4.30) one has $\rho_1 = 0$ and $\rho_2 = 2a^{-1}$. Thus, this situation happens when $2\rho < T - t_0 \le \rho + 2a^{-1}$.

If the third situation occurs, then $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T], \end{cases}$$

provided that $T - t_0 = \rho + 2a^{-1}$, and $\rho < 2a^{-1}$.

If the fourth situation occurs, then $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} \bar{x}_1(t_0) + a(t - t_0), & t \in [t_0, t_0 + 2a^{-1}] \\ 1, & t \in (t_0 + 2a^{-1}, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T], \end{cases}$$

provided that $\rho + 2a^{-1} < T - t_0$ and $\rho < 2a^{-1}$.

**Case 3:** $\bar{x}_1(T) = -1$ *and* $\bar{x}_1(t) > -1$ *for all* $t \in [t_0, T)$.

*Subcase 3a*: $\mathcal{T}_1 = \emptyset$. Then $\bar{x}_1(t) \in (-1, 1)$ for all $t \in [t_0, T)$ and $\bar{x}_1(T) = -1$. By some arguments similar to those of the proof of Lemma 4.1, one can show that formula for $\bar{x}_1(.)$ on $[t_0, T]$ is one of the following two types:

$$\bar{x}_1(t) = x_0 - a(t - t_0), \quad t \in [t_0, T]; \tag{4.35}$$

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_\zeta] \\ -1 - a(t - T), & t \in (t_\zeta, T] \end{cases} \tag{4.36}$$

with $t_\zeta \in (t_0, T)$.

If $\bar{x}_1(.)$ is given by (4.35), then $\bar{x}_1(T) = -1$ if and only if $T - t_0 = \rho_1$. Since $x_0 \in (-1, 1]$, the latter yields $0 < T - t_0 = \rho_1 \leq 2a^{-1}$.

If $\bar{x}_1(.)$ is of the form (4.36), then $t_\zeta = 2^{-1}[T + t_0 - \rho_1]$ as $\bar{x}_1(T) = -1$. Since $t_\zeta > t_0$, one must have $T - t_0 > \rho_1$. Meanwhile, by (4.36) and our standing assumption in this subcase, $\bar{x}_1(t_\zeta) < 1$. So, $T - t_0 < \rho_1 + 2\rho_2 = a^{-1}(3 - x_0)$. Combining this and the inequality $T - t_0 > \rho_1$ yields $\rho_1 < T - t_0 < a^{-1}(3 - x_0)$. Our results in this subcase can be summarized as follows:

- $\bar{x}_1(.)$ is given by (4.35), provided that $T - t_0 = \rho_1$.
- $\bar{x}_1(.)$ is given by (4.36), provided that $\rho_1 < T - t_0 < a^{-1}(3 - x_0)$.

*Subcase 3b*: $\mathcal{T}_1 \neq \emptyset$. Then we have $t_0 \leq \alpha_1 \leq \alpha_2 < T$. It follows from assertion S2 of Lemma 4.1 that $T - \alpha_2 = 2a^{-1}$ and $\bar{x}_1(t) = 1 - a(t - \alpha_2)$ for all $t \in [\alpha_2, T]$. Thus, we have $\alpha_2 = T - 2a^{-1}$ and $\bar{x}_1(t) = 1 - a(t - T + 2a^{-1})$ for all $t \in [T - 2a^{-1}, T]$.

If $\alpha_1 < \alpha_2$, then $\bar{x}_1(t) = 1$ for all $t \in [\alpha_1, \alpha_2]$. Indeed, suppose on the contrary that there exists $\bar{t} \in (\alpha_1, \alpha_2)$ satisfying $\bar{x}_1(\bar{t}) < 1$. Set

$$\bar{\alpha}_1 = \max\{t \in [\alpha_1, \bar{t}] \ : \ \bar{x}_1(t) = 1\} \quad \text{and} \quad \bar{\alpha}_2 = \min\{t \in [\bar{t}, \alpha_2] \ : \ \bar{x}_1(t) = 1\}.$$

Clearly, $[\bar{\alpha}_1, \bar{\alpha}_2] \subset [\alpha_1, \alpha_2] \subset [t_0, T)$ and $\bar{x}_1(t) < 1$ for all $t \in (\bar{\alpha}_1, \bar{\alpha}_2)$. This and the condition $\bar{x}_1(t) > -1$ for all $t \in [t_0, T)$ imply that $\bar{x}_1(t) \in (-1, 1)$ for all $t \in (\bar{\alpha}_1, \bar{\alpha}_2)$. So, by assertion S4 of Lemma 4.1, we obtain a contradiction. Our claim has been proved.

If $t_0 < \alpha_1$, then $\bar{x}_1(t) \in (-1, 1)$ for all $t \in [t_0, \alpha_1)$ and $\bar{x}_1(\alpha_1) = 1$. Thus, repeating the arguments in the proof of assertion S1 of Lemma 4.1, we find that $\bar{x}_1(t) = x_0 + a(t - t_0)$ for all $t \in [t_0, \alpha_1]$. As $\bar{x}_1(\alpha_1) = 1$, we have $\alpha_1 = t_0 + \rho_2$. Consequently, the inequality as $T - t_0 \geq (\alpha_1 - t_0) + (T - \alpha_2)$ implies that $T - t_0 \geq \rho_2 + 2a^{-1} = a^{-1}(3 - x_0)$. Our results in this subcase can be summarized as follows:

• $\bar{x}_1(.)$ is given by

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, T - 2a^{-1}] \\ -1 - a(t - T), & t \in (T - 2a^{-1}, T], \end{cases}$$

provided that $T - t_0 = a^{-1}(3 - x_0)$.

• $\bar{x}_1(.)$ is given by

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_0 + \rho_2] \\ 1, & t \in (t_0 + \rho_2, T - 2a^{-1}] \\ -1 - a(t - T), & t \in (T - 2a^{-1}, T], \end{cases}$$

provided that $T - t_0 > a^{-1}(3 - x_0)$.

**Case 4:** $\bar{x}_1(t_0) = \bar{x}_1(T) = -1$ *and* $\bar{x}_1(t) > -1$ *for all* $t \in (t_0, T)$.

*Subcase 4a*: $\mathcal{T}_1 = \emptyset$. Then, $\bar{x}_1(t) \in (-1, 1)$ for all $t \in (t_0, T)$. Thus, by assertion S3 of Lemma 4.1 one has $T - t_0 < 4a^{-1}$ and

$$\bar{x}_1(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, 2^{-1}(t_0 + T)] \\ -1 - a(t - T), & t \in (2^{-1}(t_0 + T), T]. \end{cases}$$

*Subcase 4b*: $\mathcal{T}_1 \neq \emptyset$. Then, the numbers $\alpha_1$ and $\alpha_2$ exist and we have $t_0 < \alpha_1 \leq \alpha_2 < T$. It follows from statements S1 and S2 of Lemma 4.1 that $\alpha_1 - t_0 = T - \alpha_2 = 2a^{-1}$, $\bar{x}_1(t) = -1 + a(t - t_0)$ on the segment $[t_0, \alpha_1]$, and $\bar{x}_1(t) = 1 - a(t - \alpha_2)$ on the segment $[\alpha_2, T]$. Thus, $\alpha_1 = t_0 + 2a^{-1}$, $\alpha_2 = T - 2a^{-1}$, $\bar{x}_1(t) = -1 + a(t - t_0)$ for every $t \in [t_0, t_0 + 2a^{-1}]$, and $\bar{x}_1(t) = 1 - a(t - T + 2a^{-1})$ for all $t \in [T - 2a^{-1}, T]$. Note that one must have $T - t_0 \geq 4a^{-1}$ in this subcase as $T - t_0 \geq (\alpha_1 - t_0) + (T - \alpha_2)$.

If $T - t_0 > 4a^{-1}$, i.e., $\alpha_1 < \alpha_2$, then by the result given in Subcase 3b we have $\bar{x}_1(t) = 1$ for all $t \in [t_0 + 2a^{-1}, T - 2a^{-1}]$.

Our results in this case can be summarized as follows:

• $\bar{x}_1(.)$ is given by

$$\bar{x}_1(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, 2^{-1}(t_0 + T)] \\ -1 - a(t - T), & t \in (2^{-1}(t_0 + T), T], \end{cases}$$

provided that $T - t_0 \leq 4a^{-1}$.

• $\bar{x}_1(.)$ is given by

$$\bar{x}_1(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, t_0 + 2a^{-1}] \\ 1, & t \in (t_0 + 2a^{-1}, T - 2a^{-1}] \\ -1 - a(t - T), & t \in (T - 2a^{-1}, T], \end{cases}$$

provided that $T - t_0 > 4a^{-1}$.

Now we turn our attention back to the original problem $(FP_3)$. In Section 4.2, it was observed that $(FP_3)$ has a global solution. So, the set of the local solutions of $(FP_3)$ is nonempty. Consequently, given the parameters tube $(\lambda, a, x_0, t_0, T)$, if we can show that for any local solution $(\bar{x}, \bar{u})$ of $(FP_3)$, $\bar{x}$ is described by a unique formula, then we can assert that the pair $(\bar{x}, \bar{u})$ is a unique local solution of $(FP_3)$, which is also a unique global solution of the problem.

Using the given constants $a, \lambda$ with $a > \lambda > 0$, we define

$$\rho = \frac{1}{\lambda} \ln \frac{a}{a - \lambda} > 0 \quad \text{and} \quad \bar{t} = T - \rho.$$

The number $\rho$ is a characteristic constant of $(FP_3)$. From the analysis given in the present section we can obtain a complete synthesis of optimal processes. Due to the complexity of the possible trajectories, we prefer to present our results in six separate theorems. The first one deals with the situation where $\rho \geq 2a^{-1}$, while the other five treat the situation where $\rho < 2a^{-1}$.

Based on the results obtained in Cases 1–4, we will provide a complete synthesis of the global solutions of $(FP_3)$. Recall that $\bar{x}_1(t)$ in $(FP_{3a})$ plays the role of $\bar{x}(t)$ in $(FP_3)$.

**Theorem 4.1** *If $\rho \geq 2a^{-1}$, then problem $(FP_3)$ has a unique local solution $(\bar{x}, \bar{u})$, which is a unique global solution, where $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ for almost every $t \in [t_0, T]$ and $\bar{x}(t)$ can be described as follows:*

(a) *If $T - t_0 \leq a^{-1}(1 + x_0)$, then*

$$\bar{x}(t) = x_0 - a(t - t_0), \quad t \in [t_0, T].$$

(b) *If $a^{-1}(1 + x_0) < T - t_0 < a^{-1}(3 - x_0)$, then*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_\zeta] \\ -1 - a(t - T), & t \in (t_\zeta, T] \end{cases}$$

*with $t_\zeta := 2^{-1}[T + t_0 - a^{-1}(1 + x_0)]$.*

(c) *If* $T - t_0 \geq a^{-1}(3 - x_0)$, *then*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_0 + a^{-1}(1 - x_0)] \\ 1, & t \in (t_0 + a^{-1}(1 - x_0), T - 2a^{-1}] \\ -1 - a(t - T), & t \in (T - 2a^{-1}, T]. \end{cases}$$

**Proof.** Suppose that $\rho \geq 2a^{-1}$. Let $\rho_1, \rho_2$ be defined as in (4.30). Then, one has $\rho \geq \rho_1$, $2\rho - \rho_1 \geq \rho + \rho_2$, $\rho + \rho_2 \geq 2a^{-1} + \rho_2$, $2\rho \geq \rho + 2a^{-1}$, and $\rho + 2a^{-1} \geq 4a^{-1}$. Thus, the above Case 2 and the situations in Subcase 1b, Subcase 1c, Subcase 1d of Case 1 cannot happen. The situation in Subcase 1a happens when $T - t_0 < \rho_1$. Combining this with the results formulated in Case 3 and Case 4, we obtain the assertions of the theorem. □

If $\rho < 2a^{-1}$, then the locally optimal processes of $(FP_3)$ *depend greatly on the relative position of* $x_0$ in the segment $[-1, 1]$. In the forthcoming theorems, we distinguish five alternatives (one instance must occur, and any instance excludes others):

(i) $x_0 = -1$;

(ii) $x_0 > -1$ and $a^{-1}(1 + x_0) \leq \rho$;

(iii) $x_0 > -1$, $\rho < a^{-1}(1 + x_0)$, and $a^{-1}(1 + x_0) < \rho + a^{-1}(1 - x_0)$;

(iv) $x_0 > -1$, $\rho < a^{-1}(1 + x_0)$, and $a^{-1}(1 + x_0) = \rho + a^{-1}(1 - x_0)$;

(v) $x_0 > -1$, $\rho < a^{-1}(1 + x_0)$, and $\rho + a^{-1}(1 - x_0) < a^{-1}(1 + x_0)$.

It is worthy to stress that to describe the possibilities (i)–(v) we have used just the parameters $a, \lambda$, and $x_0$. In each one of the situations (i)–(v), the synthesis of the trajectories suspected for local minimizers of $(FP_3)$ is obtained by considering *the position of the number* $T - t_0 > 0$ *on the half-line* $[0, +\infty)$, which is divided into sections by the values $\rho$, $2\rho$, $\rho + 2a^{-1}$, $4a^{-1}$, and other constants appeared in (i)–(v).

**Theorem 4.2** *If* $\rho < 2a^{-1}$ *and* $x_0 = -1$, *then any local solution of problem* $(FP_3)$ *must have the form* $(\bar{x}, \bar{u})$, *where* $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ *for a.e.* $t \in [t_0, T]$ *and* $\bar{x}(t)$ *is described as follows:*

(a) *If* $T - t_0 \leq 2\rho$, *then*

$$\bar{x}(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, 2^{-1}(t_0 + T)] \\ -1 - a(t - T), & t \in (2^{-1}(t_0 + T), T]. \end{cases} \tag{4.37}$$

*In this situation, $(\bar{x}, \bar{u})$ is a unique local solution of $(FP_3)$, which is also a unique global solution of the problem.*

(b) *If $2\rho < T - t_0 < \rho + 2a^{-1}$, then $\bar{x}(t)$ is given by either*

$$\bar{x}(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, \bar{t}] \\ -1 - a(t + t_0 - 2\bar{t}), & t \in (\bar{t}, T] \end{cases} \tag{4.38}$$

*or* (4.37).

(c) *If $T - t_0 = \rho + 2a^{-1}$, then $\bar{x}(t)$ is given by either*

$$\bar{x}(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T] \end{cases} \tag{4.39}$$

*or* (4.37).

(d) *If $\rho + 2a^{-1} < T - t_0 \leq 4a^{-1}$, then $\bar{x}(t)$ is given by either*

$$\bar{x}(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, t_0 + 2a^{-1}] \\ 1, & t \in (t_0 + 2a^{-1}, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T] \end{cases} \tag{4.40}$$

*or* (4.37).

(e) *If $T - t_0 > 4a^{-1}$, then $\bar{x}(t)$ is given by either* (4.40) *or*

$$\bar{x}(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, t_0 + 2a^{-1}] \\ 1, & t \in (t_0 + 2a^{-1}, T - 2a^{-1}] \\ -1 - a(t - T), & t \in (T - 2a^{-1}, T]. \end{cases}$$

*In situations* (b)–(e), *the unique global solution of the problem $(FP_3)$ is given correspondingly by* (4.38), (4.39), (4.40), *and* (4.40), *where the last switching time of the optimal control function $\bar{u}(\cdot)$ is $\bar{t}$.*

**Proof.** Suppose that $\rho < 2a^{-1}$ and $x_0 = -1$. Since the above Case 1 and Case 3 are excluded, to obtain the assertions (a)–(e) we just need to combine the results formulated in Case 2 and Case 4. Here we have

$$2\rho < \rho + 2a^{-1} < 4a^{-1}.$$

In what follows, we will consider the position of the number $T - t_0$ on the half-line $[0, +\infty)$ marked by the values $2\rho$, $\rho + 2a^{-1}$, and $4a^{-1}$.

First, consider situation (a), where $T - t_0 \le 2\rho$. Since all the three possibilities depicted in the conclusion part of Case 2 are excluded, Case 4 must occur. Since $T - t_0 \le 2\rho < 4a^{-1}$, from the results in Case 4 it follows that $\bar{x}$ is described by (4.37). Thus, assertion (a) of the theorem is proved.

Now, consider situations (b)–(e). Here we have $T - t_0 > 2\rho$. So, our assertions follow from the results formulated in Case 2 and Case 4.

The fact that $(FP_3)$ has a global solution has been observed before. To prove the last assertion of the theorem about the uniqueness of the global solution of $(FP_3)$ in situations (b)–(e), we suppose on the contrary that the set of the global optimal processes of $(FP_3)$ is not a singleton. In each situation, by assertions (b)–(e) we know that the set of the local optimal processes contains no more than two elements. Hence, our supposition means that both distinctly feasible processes depicted therein in each situation are global ones. Observe that, in situations (b)–(c), $2^{-1}(t_0 + T) < \bar{t}$ because $T - t_0 > 2\rho$ and $\bar{t} = T - \rho$. Moreover, the formulas given in each situation show that the state trajectories of the two global processes coincide on $[t_0, 2^{-1}(t_0 + T)]$. Consider the problem denoted by $(FP_2)|_{[2^{-1}(t_0+T),T]}$, which is obtained from $(FP_3)$ by replacing the initial time $t_0$ and the bilateral state constraints $-1 \le x(t) \le 1$ respectively by $2^{-1}(t_0 + T)$ and the unilateral state constraints $x(t) \le 1$. As $2^{-1}(t_0 + T) < \bar{t}$, it follows from Theorem 3.4 that $(FP_2)|_{[2^{-1}(t_0+T),T]}$ has a unique global process, in which the switching time of the optimal control is $\bar{t}$. Consequently, $(FP_3)$ has a unique global process, in which the switching time of the optimal control function is $\bar{t}$, a contradiction. Similarly, observing that $2^{-1}(t_0 + T) \le t_0 + 2a^{-1} < \bar{t}$ in situation (d) (resp., $T - 2a^{-1} < \bar{t}$ in situation (e)) and using the previous arguments, we will arrive at a contradiction. □

**Theorem 4.3** *If $\rho < 2a^{-1}$, $x_0 > -1$, and $a^{-1}(1 + x_0) \le \rho$, then any local solution of problem $(FP_3)$ must have the form $(\bar{x}, \bar{u})$, where $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ for almost everywhere $t \in [t_0, T]$ and $\bar{x}(t)$ can be described as follows:*

(a) *If $T - t_0 \le a^{-1}(1 + x_0)$, then $\bar{x}(t)$ is given by*

$$\bar{x}(t) = x_0 - a(t - t_0), \quad t \in [t_0, T]. \tag{4.41}$$

*In this situation, $(\bar{x}, \bar{u})$ is a unique local solution of $(FP_3)$, which is also a unique global solution of the problem.*

(b) *If* $a^{-1}(1+x_0) < T - t_0 \le 2\rho - a^{-1}(1+x_0)$*, then* $\bar{x}(t)$ *is given by*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_\zeta] \\ -1 - a(t - T), & t \in (t_\zeta, T] \end{cases} \tag{4.42}$$

*with* $t_\zeta := 2^{-1}[T + t_0 - a^{-1}(1+x_0)]$*. In this situation,* $(\bar{x}, \bar{u})$ *is a unique local solution of* $(FP_3)$*, which is also a unique global solution of the problem.*

(c) *If* $2\rho - a^{-1}(1+x_0) < T - t_0 < \rho + a^{-1}(1-x_0)$*, then* $\bar{x}(t)$ *is given by either*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}] \\ x_0 - a(t + t_0 - 2\bar{t}), & t \in (\bar{t}, T] \end{cases} \tag{4.43}$$

*or* (4.42).

(d) *If* $T - t_0 = \rho + a^{-1}(1-x_0)$*, then* $\bar{x}(t)$ *is given by either*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T] \end{cases} \tag{4.44}$$

*or* (4.42).

(e) *If* $\rho + a^{-1}(1-x_0) < T - t_0 < a^{-1}(3-x_0)$*, then* $\bar{x}(t)$ *is given by either*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_0 + a^{-1}(1-x_0)) \\ 1, & t \in [t_0 + a^{-1}(1-x_0), \bar{t}) \\ 1 - a(t - \bar{t}), & t \in [\bar{t}, T] \end{cases} \tag{4.45}$$

*or* (4.42).

(f) *If* $T - t_0 = a^{-1}(3-x_0)$*, then* $\bar{x}(t)$ *is given by either* (4.45) *or*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, T - 2a^{-1}] \\ -1 - a(t - T), & t \in (T - 2a^{-1}, T]. \end{cases} \tag{4.46}$$

(g) *If* $T - t_0 > a^{-1}(3-x_0)$*, then* $\bar{x}(t)$ *is given by either* (4.45) *or*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_0 + a^{-1}(1-x_0)] \\ 1, & t \in (t_0 + a^{-1}(1-x_0), T - 2a^{-1}] \\ -1 - a(t - T), & t \in (T - 2a^{-1}, T]. \end{cases}$$

*In situations* (c)–(g)*, the unique global solution of the problem* $(FP_3)$ *is the one in which the last switching time of the optimal control function* $\bar{u}(\cdot)$ *is* $\bar{t}$*.*

**Proof.** Suppose that $\rho < 2a^{-1}$, $x_0 > -1$, $a^{-1}(1+x_0) \le \rho$, and let $\rho_1, \rho_2$ be given by (4.30). Then, it is easy to verify that $\max\{\rho, 2\rho - \rho_1\} = 2\rho - \rho_1$ and

$$\rho_1 \le 2\rho - \rho_1 < \rho + \rho_2 < a^{-1}(3 - x_0).$$

The above Case 2 and Case 4 are excluded by the condition $x_0 > -1$. So, to obtain the desired assertions (a)–(g), it suffices to combine the results formulated in Case 1 and Case 3 with an observation on the position of the number $T - t_0$ on the half-line $[0, +\infty)$, which is marked by the values $\rho_1$, $2\rho - \rho_1$, $\rho + \rho_2$, and $a^{-1}(3 - x_0)$.

Recall that $(FP_3)$ has a global solution. To prove the last assertion of the theorem about the uniqueness of the global solution of $(FP_3)$ in situations (c)–(g), we suppose on the contrary that the set of the global optimal processes of $(FP_3)$ is not a singleton. In each situation, by assertions(c)–(g) we know that the set of the local optimal processes contains no more than two elements. Hence, our supposition means that both distinctly feasible processes depicted in each situation are global ones. In situations (c)–(g), as $T - t_0 > 2\rho - \rho_1 \ge \rho$ and $\bar{t} = T - \rho$, we have $t_0 < \bar{t}$. Thus, by Theorem 3.4, the problem $(FP_2)$ obtained from $(FP_3)$ by replacing the constraint $-1 \le x(t) \le 1$ by $x(t) \le 1$ has a unique global solution, which is the one where the last switching time of the optimal control function $\bar{u}(\cdot)$ is $\bar{t}$. Consequently, $(FP_3)$ has a unique global solution, in which the switching time of the optimal control function is $\bar{t}$, a contradiction. □

**Theorem 4.4** *If $\rho < 2a^{-1}$, $x_0 > -1$, $\rho < a^{-1}(1 + x_0)$, and*

$$a^{-1}(1 + x_0) < \rho + a^{-1}(1 - x_0), \tag{4.47}$$

*then any local solution of problem $(FP_3)$ must have the form $(\bar{x}, \bar{u})$, where $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ for a.e. $t \in [t_0, T]$ and $\bar{x}(t)$ is described as follows:*

(a) *If $T - t_0 \le \rho$, then $\bar{x}(t)$ is given by* (4.41)*. In this situation, $(\bar{x}, \bar{u})$ is a unique local solution of $(FP_3)$, which is also a unique global solution of the problem.*

(b) *If $\rho < T - t_0 < a^{-1}(1 + x_0)$, then $\bar{x}(t)$ is given by* (4.43)*. In this situation, $(\bar{x}, \bar{u})$ is a unique local solution of $(FP_3)$, which is also a unique global solution of the problem.*

(c) *If $T - t_0 = a^{-1}(1 + x_0)$, then $\bar{x}(t)$ is given by either* (4.43) *or* (4.41)*.*

(d) *If* $a^{-1}(1+x_0) < T - t_0 < \rho + a^{-1}(1-x_0)$*, then* $\bar{x}(t)$ *is given by either* (4.43) *or* (4.42).

(e) *If* $T - t_0 = \rho + a^{-1}(1-x_0)$*, then* $\bar{x}(t)$ *is given by either* (4.44) *or* (4.42).

(f) *If* $\rho + a^{-1}(1-x_0) < T - t_0 < a^{-1}(3-x_0)$*, then* $\bar{x}(t)$ *is given by either* (4.45) *or* (4.42).

(g) *If* $T - t_0 = a^{-1}(3-x_0)$*, then* $\bar{x}(t)$ *is given by either* (4.45) *or* (4.46).

(h) *If* $T - t_0 > a^{-1}(3-x_0)$*, then* $\bar{x}(t)$ *is given by either* (4.45) *or* (4.3).

*In situations* (c)–(h)*, the unique global solution of* $(FP_3)$ *is the one in which the last switching time of the optimal control function* $\bar{u}(\cdot)$ *is* $\bar{t}$.

**Proof.** Suppose that $\rho < 2a^{-1}$, $x_0 > -1$, $\rho < a^{-1}(1+x_0)$, and the inequality (4.47) holds. Let $\rho_1, \rho_2$ be defined as in (4.30). By the assumptions made, $\max\{\rho, 2\rho - \rho_1\} = \rho$ and $\rho < \rho_1 < \rho + \rho_2 < a^{-1}(3-x_0)$. Due to the condition $x_0 > -1$, the above Case 2 and Case 4 are excluded. So, to obtain the assertions in (a)–(h), it suffices to combine the results formulated in Case 1 and Case 3 and observe the position of the number $T - t_0$ on the half-line $[0, +\infty)$, which is marked by the values $\rho$, $\rho_1$, $\rho + \rho_2$, and $a^{-1}(3-x_0)$.

The last assertion of the theorem about the uniqueness of the global solution of $(FP_3)$ in situations (c)–(h) is proved similarly as in the second part of the proof of Theorem 4.3. □

**Theorem 4.5** *If* $\rho < 2a^{-1}$, $x_0 > -1$, $\rho < a^{-1}(1+x_0)$*, and*

$$a^{-1}(1+x_0) = \rho + a^{-1}(1-x_0), \tag{4.48}$$

*then any local solution of problem* $(FP_3)$ *must have the form* $(\bar{x}, \bar{u})$*, where* $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ *for a.e.* $t \in [t_0, T]$ *and* $\bar{x}(t)$ *is described as follows:*

(a) *If* $T - t_0 \le \rho$*, then* $\bar{x}(t)$ *is given by* (4.41)*. In this situation,* $(\bar{x}, \bar{u})$ *is a unique local solution of* $(FP_3)$*, which is also a unique global solution of the problem.*

(b) *If* $\rho < T - t_0 < a^{-1}(1+x_0)$*, then* $\bar{x}(t)$ *is given by* (4.43)*. In this situation,* $(\bar{x}, \bar{u})$ *is a unique local solution of* $(FP_3)$*, which is also a unique global solution of the problem.*

(c) *If* $T - t_0 = a^{-1}(1+x_0)$*, then* $\bar{x}(t)$ *is given by either* (4.44) *or* (4.41).

(d) *If* $a^{-1}(1+x_0) < T - t_0 < a^{-1}(3-x_0)$*, then* $\bar{x}(t)$ *is given by either* (4.45) *or* (4.42).

(e) *If* $T - t_0 = a^{-1}(3-x_0)$*, then* $\bar{x}(t)$ *is given by either* (4.45) *or* (4.46)

(f) *If* $T - t_0 > a^{-1}(3-x_0)$*, then* $\bar{x}(t)$ *is given by either* (4.45) *or* (4.3).

*In situations* (c)–(f)*, the unique global solution of the problem* $(FP_3)$ *is the one in which the last switching time of the optimal control function* $\bar{u}(\cdot)$ *is* $\bar{t}$*.*

**Proof.** Suppose that $\rho < 2a^{-1}$, $x_0 > -1$, $\rho < a^{-1}(1+x_0)$, and (4.48) holds. Let $\rho_1, \rho_2$ be as in (4.30). Then, we have $\max\{\rho, 2\rho - \rho_1\} = \rho$ and

$$\rho < \rho_1 = \rho + \rho_2 < a^{-1}(3-x_0).$$

Since $x_0 > -1$, the above Case 2 and Case 4 are excluded. Hence, the desired assertions (a)–(g) follow from combining the results formulated in Case 1 and Case 3 with an observation on the position of the number $T - t_0$ on the half-line $[0, +\infty)$, which is marked by the values $\rho$, $\rho_1$, and $a^{-1}(3-x_0)$.

The assertion on the uniqueness of the global solution of $(FP_3)$ in situations (c)–(g) is proved similarly as in the second part of the proof of Theorem 4.3.
□

Finally, consider the situation where $\rho < 2a^{-1}$, $x_0 > -1$, $\rho < a^{-1}(1+x_0)$, and $\rho + a^{-1}(1-x_0) < a^{-1}(1+x_0)$. As $x_0 \le 1$, we have $\rho \le \rho + a^{-1}(1-x_0)$. Combining the latter with the inequality $\rho + a^{-1}(1-x_0) < a^{-1}(1+x_0)$ yields $\rho < a^{-1}(1+x_0)$. So, the last inequality can be omitted in the formulation of the following theorem.

**Theorem 4.6** *If* $\rho < 2a^{-1}$*,* $x_0 > -1$*, and* $\rho + a^{-1}(1-x_0) < a^{-1}(1+x_0)$*, then any local solution of problem* $(FP_3)$ *must have the form* $(\bar{x}, \bar{u})$*, where* $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ *for a.e.* $t \in [t_0, T]$ *and* $\bar{x}(t)$ *is described as follows:*

(a) *If* $T - t_0 \le \rho$*, then* $\bar{x}(t)$ *is given by* (4.41).

(b) *If* $\rho < T - t_0 < \rho + a^{-1}(1-x_0)$*, then* $\bar{x}(t)$ *is given by* (4.43).

(c) *If* $T - t_0 = \rho + a^{-1}(1-x_0)$*, then* $\bar{x}(t)$ *is given by* (4.44).

(d) *If* $\rho + a^{-1}(1-x_0) < T - t_0 < a^{-1}(1+x_0)$*, then* $\bar{x}(t)$ *is given by* (4.45).

(e) *If* $T - t_0 = a^{-1}(1+x_0)$*, then* $\bar{x}(t)$ *is given by either* (4.45) *or* (4.41).

(f) *If* $a^{-1}(1+x_0) < T - t_0 < a^{-1}(3-x_0)$*, then* $\bar{x}(t)$ *is given by either* (4.45) *or* (4.42)*.*

(h) *If* $T - t_0 = a^{-1}(3-x_0)$*, then* $\bar{x}(t)$ *is given by either* (4.45) *or* (4.46)*.*

(g) *If* $T - t_0 > a^{-1}(3-x_0)$*, then* $\bar{x}(t)$ *is given by* (4.45) *or* (4.3)*.*

*In situations* (a)–(d)*,* $(\bar{x}, \bar{u})$ *is a unique local solution of* $(FP_3)$*, which is also a unique global solution of the problem. In situations* (e)–(g)*, the unique global solution of* $(FP_3)$ *is the one in which the last switching time of the optimal control function* $\bar{u}(\cdot)$ *is* $\bar{t}$*.*

**Proof.** Suppose that $\rho < 2a^{-1}$, $x_0 > -1$, and $\rho + a^{-1}(1-x_0) < a^{-1}(1+x_0)$. With $\rho_1, \rho_2$ being defined (4.30), one has $\max\{\rho, 2\rho - \rho_1\} = \rho$ and

$$\rho \leq \rho + \rho_2 < \rho_1 \leq a^{-1}(3 - x_0).$$

Since $x_0 > -1$, to obtain the desired assertions (a)–(g) we just need to combine the results formulated in Case 1 and Case 3 with an observation on the position of the number $T - t_0$ on the half-line $[0, +\infty)$, which is marked by the values $\rho$, $\rho + \rho_2$, $\rho_1$, and $a^{-1}(3 - x_0)$.

The uniqueness of the global solution of $(FP_3)$ in situations (a)–(d) is obvious. The uniqueness of the global solution of $(FP_3)$ in situations (e)–(g) can be proved similarly as in the final part of the proof of Theorem 4.3. □

An anonymous referee of the paper [38] noticed that we may face with *the degeneracy phenomenon of the maximum principle* in our analysis of the optimal control problems with state constraints in question. Although we have overcome the phenomenon by employing the special structure of the problem and several technical arguments, the next section is devoted to a discussion on the degeneracy phenomenon of the maximum principle, which sheds new lights on the synthesis of the optimal processes given in this and previous chapter as well as to highlight the tools one may use to deal with the degeneracy phenomenon.

## 4.6 On the Degeneracy Phenomenon of the Maximum Principle

Regarding maximum principles for optimal control problems with state constraints, there is a so-called *degeneracy phenomenon*, which has been widely discussed in the literature (see, e.g., the books [5, 82], the papers [6, 27–29, 46], and the references therein). In our notations, when the left endpoint $t_0$ is fixed and the initial state $x_0$ lies in the boundary of the state constraint, i.e., $h(t_0, x_0) = 0$, then standard variants of Pontryagin's maximum principle may be degenerate. This means that such maximum principles are satisfied by *trivial multipliers* (see [6, 27] and the following paragraph for details); hence no useful information is obtained. For problem $(FP_{3a})$, the analysis given after Lemma 4.7 may face with the degeneracy phenomenon in Case 2 and Case 4. We have overcome the phenomenon by employing the special structure of $(FP_{3a})$ and several technical arguments (consider the restrictions of the trajectories in question on a sequence of a strict subintervals of $[t_0, T]$ and classify their shapes into four categories, apply the Dirichlet principle, and use the valuable observation in Lemma 4.1 for a trajectory which remains in the interior of the domain $[-1, 1]$ for all $t$ from an open interval $(\tau_1, \tau_2)$ of the time axis and touches the boundary of the domain at the moments $\tau_1$ and $\tau_2$).

As shown in [6, p. 932 and Section 5] and [5, Chapter 2][3], the degeneracy phenomenon can be dealt with by *new versions of the maximum principle.* It can be also effectively studied by using the concept of *hybrid subdifferential.* Namely, as mentioned in [82, Remark (b), pp. 330–331], regarding a $W^{1,1}$ local minimizer $(\bar{x}, \bar{u})$ for $\mathcal{M}$, the maximum principle formulated in Theorem 3.1 for the problem $\mathcal{M}$ is of interest only when the state constraint is *nondegenerate*, in the sense that $0 \notin \partial_x^{>} h(t, \bar{x}(t))$ *for every* $t$ *satisfying* $h(t, \bar{x}(t)) = 0$. The reason is that the necessary conditions (i)–(iv) automatically hold if

$$0 \in \partial_x^{>} h(t', \bar{x}(t')) \quad \text{and} \quad h(t', \bar{x}(t')) = 0$$

for some time $t'$, with the choice of multipliers $\mu = \delta_{\{t'\}}$ (the unit measure concentrated on $\{t'\}$), $p(t) \equiv 0$, $\nu : [t_0, T] \to \mathbb{R}^n$ is a Borel measurable function with the property that $\nu(t') = 0$, and $\gamma = 0$. (It is worthy to

[3]See also Theorem 10.6.1 and Corollary 10.6.2 in the book by Vinter [82], which are based on the preceding works of A. V. Arutyunov, S. M. Aseev, and V. I. Blagodat.skikh on extra conditions to eliminate degeneracy.

stress that our problem $(FP_{3a})$ is nondegenerate. Indeed, applied to any feasible process $(\bar{x}, \bar{u})$ of $(FP_{3a})$, formula (4.7) shows that $0 \notin \partial_x^{>} h(t, \bar{x}(t))$ for every $t$ satisfying $h(t, \bar{x}(t)) = 0$. Furthermore, if the state constraint of $\mathcal{M}$ is nondegenerate w.r.t. a $W^{1,1}$ local minimizer $(\bar{x}, \bar{u})$, then one can apply some regularity conditions to assure that $\gamma > 0$ (in that case, not only the necessary conditions (i)–(iv) are meaningful, but the objective function $g$ is taken into full account in condition (iii)). The reader is referred to [27–29] for further information in this direction. As concerning our analysis in Cases 1–4 above, we have not used these regularity conditions.

## 4.7 Conclusions

We have analyzed a maximum principle for finite horizon optimal control problems with state constraints via one parametric example, which resembles the optimal economic growth problems in macroeconomics. This example is an optimal control problem with bilateral state constraints of the Lagrange type and has five parameters. We have proved that the optimal control problem can have at most two local optimal processes. Moreover, we have obtained explicit descriptions of the unique global optimal process with respect to all possible configurations of the five parameters.

# Chapter 5

# Finite Horizon Optimal Economic Growth Problems

Written on the basis of the papers [39, 40], this chapter establishes several theorems on solution existence for optimal economic growth problems in general forms as well as in some typical ones and a synthesis of optimal processes for one of such typical problems. Some open questions and conjectures about the uniqueness and regularity of the global solutions of optimal economic growth problems are formulated herein.

## 5.1 Optimal Economic Growth Models

Following Takayama [79, Sections C and D in Chapter 5], we consider the problem of *optimal growth of an aggregative economy.* Suppose that the economy can be characterized by one sector, which produces the *national product* $Y(t)$ at time $t$. Suppose that $Y(t)$ depends on two factors, the *labor* $L(t)$ and the *capital* $K(t)$, and the dependence is described by a *production function* $F$. Namely, one has

$$Y(t) = F(K(t), L(t)), \quad \forall t \geq 0.$$

It is assumed that $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a function defined on the nonnegative orthant $\mathbb{R}^2_+$ of $\mathbb{R}^2$ having nonnegative real values, and that it exhibits constant returns to scale, i.e.,

$$F(\alpha K, \alpha L) = \alpha F(K, L) \tag{5.1}$$

for any $(K, L) \in \mathbb{R}^2_+$ and $\alpha > 0$.

For every $t \geq 0$, by $C(t)$ and $I(t)$, respectively, we denote the *consumption amount* and the *investment amount* of the economy. The *equilibrium relation* in the output market is depicted by

$$Y(t) = C(t) + I(t), \quad \forall t \geq 0. \tag{5.2}$$

The relationship between the capital $K(t)$ and the investment amount $I(t)$ is given by the differential equation

$$\dot{K}(t) = I(t), \quad \forall t \geq 0, \tag{5.3}$$

where $\dot{K}(t) = \dfrac{dK(t)}{dt}$ denotes the Fréchet derivative of $K(\cdot)$ at time instance $t$ (see, e.g., [19, pp. 465–466]). If the investment function $I(\cdot)$ is continuous, then one can compute the capital stock $K(t)$ at time $t$ by the formula

$$K(t) = K(0) + \int_0^t I(\tau)d\tau,$$

where the integral is Riemannian and $K(0)$ signifies the initial capital stock. In particular, the rate of increase of the capital stock $\dot{K}(t)$ at every time moment $t$ exists and it is finite.

If the initial labor amount is $L_0 > 0$ and the *rate of labor force* is a constant $\sigma > 0$ (i.e., $\dot{L}(t) = \sigma L(t)$ for all $t \geq 0$), then the labor amount at time moment $t$ is

$$L(t) = L_0 e^{\sigma t}, \quad \forall t \geq 0. \tag{5.4}$$

For any $t \geq 0$, as $L(t) > 0$, from (5.1) we have

$$\frac{Y(t)}{L(t)} = F\Big(\frac{K(t)}{L(t)}, 1\Big), \quad \forall t \geq 0.$$

By introducing the *capital-to-labor ratio* $k(t) := \dfrac{K(t)}{L(t)}$ and the function $\phi(k) := F(k, 1)$ for $k \geq 0$, from the last equality we have

$$\phi(k(t)) = \frac{Y(t)}{L(t)}, \quad \forall t \geq 0. \tag{5.5}$$

Due to (5.5), one calls $\phi(k(t))$ the *output per capita* at time $t$ and $\phi(\cdot)$ the *per capita production function.* Since $F$ has nonnegative values, so does $\phi$. Combining the continuous differentiability of $K(\cdot)$ and $L(\cdot)$, which is guaranteed by (5.3) and (5.4), with the equality defining the capital-to-labor ratio, one can assert that $k(\cdot)$ is continuously differentiable. Thus, from the relation $K(t) = k(t)L(t)$ one obtains

$$\dot{K}(t) = \dot{k}(t)L(t) + k(t)\dot{L}(t), \quad \forall t \geq 0.$$

Dividing both sides of the above equality by $L(t)$ and invoking $\dot{L}(t) = \sigma L(t)$, we get

$$\frac{\dot{K}(t)}{L(t)} = \dot{k}(t) + \sigma k(t), \quad \forall t \geq 0. \tag{5.6}$$

Similarly, dividing both sides of the equality in (5.3) by $L(t)$ and using (5.2), we have

$$\frac{\dot{K}(t)}{L(t)} = \frac{Y(t)}{L(t)} - \frac{C(t)}{L(t)}, \quad \forall t \geq 0.$$

So, by considering the *per capita consumption* $c(t) := \dfrac{C(t)}{L(t)}$ of the economy at time $t$ and invoking (5.5), one obtains

$$\frac{\dot{K}(t)}{L(t)} = \phi(k(t)) - c(t), \quad \forall t \geq 0.$$

Combining this with (5.6) yields

$$\dot{k}(t) = \phi(k(t)) - \sigma k(t) - c(t), \quad \forall t \geq 0. \tag{5.7}$$

The amount of consumption at time $t$ is

$$C(t) = (1 - s(t))Y(t), \quad \forall t \geq 0, \tag{5.8}$$

with $s(t) \in [0, 1]$ being the *propensity to save* at time $t$ (thus, $1 - s(t)$ is the *propensity to consume* at time $t$). Then, by dividing both sides of (5.8) by $L(t)$ and referring to (5.5), one gets

$$c(t) = (1 - s(t))\phi(k(t)), \quad \forall t \geq 0. \tag{5.9}$$

Thanks to (5.9), one can rewrite (5.7) equivalently as

$$\dot{k}(t) = s(t)\phi(k(t)) - \sigma k(t), \quad \forall t \geq 0. \tag{5.10}$$

In the special case where $s(\cdot)$ is a constant function, i.e., $s(t) = s > 0$ for all $t \geq 0$, relation (5.10) is the fundamental equation of the *neo-classical aggregate growth model* of Solow [77].

One major concern of the planners is to choose a pair of functions $(k, c)$ (or $(k, s)$) defined on a planning interval $[t_0, T] \subset [0, +\infty]$, that satisfies (5.7) (or (5.10)) and the initial condition $k(t_0) = k_0$, to maximize a certain target of consumption. Here $k_0 > 0$ is a given value. As the target function one may choose is $\int_{t_0}^{T} c(t)dt$, which is the total amount of per capita consumption on the time period $[t_0, T]$. A more general kind of the target function is $\int_{t_0}^{T} \omega(c(t))e^{-\lambda t}dt$, where $\omega : \mathbb{R}_+ \to \mathbb{R}$ is a *utility function* associated with the representative individual consumption $c(t)$ in the society and $e^{-\lambda t}$ is the time discount factor. The number $\lambda \geq 0$ is called the *real interest rate*. Clearly, the former target function is a particular case of the latter one with $\omega(c) = c$ being a linear utility function and the real interest rate $\lambda = 0$. For more discussions about the length of the planning interval, the choice the utility function $\omega(\cdot)$ (*it must be linear, or it can be nonlinear?*), as well the choice of the real interest rate (*one must have $\lambda = 0$, or one can have $\lambda > 0$?*), we refer the reader to [79, pp. 445–447].

The just mentioned planning task is a *state constrained optimal control problem*. Interpreting $k(t)$ as the state trajectory and $s(t)$ as the control function, we can formulate the problem as follows.

Let there be given a production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ satisfying (5.1) for any $(K, L)$ from $\mathbb{R}^2_+$ and $\alpha > 0$. Define the function $\phi(k)$ on $\mathbb{R}_+$ by setting $\phi(k) = F(k, 1)$. Assume that a finite time interval $[t_0, T]$ with $T > t_0 \geq 0$, a utility function $\omega : \mathbb{R}_+ \to \mathbb{R}$, and a real interest rate $\lambda \geq 0$ are given. Since $c(t) = (1 - s(t))\phi(k(t))$ by (5.9), the target function can be expressed via $k(t)$ and $s(t)$ as

$$\int_{t_0}^{T} \omega(c(t))e^{-\lambda t}dt = \int_{t_0}^{T} \omega[(1 - s(t))\phi(k(t))]e^{-\lambda t}dt.$$

So, the problem of finding an optimal growth process for an aggregative economy is the following one:

$$\text{Maximize } \ I(k, s) := \int_{t_0}^{T} \omega[(1 - s(t))\phi(k(t))]e^{-\lambda t}dt \tag{5.11}$$

over $k \in W^{1,1}([t_0, T], \mathbb{R})$ and measurable functions $s : [t_0, T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{k}(t) = s(t)\phi(k(t)) - \sigma k(t), & \text{a.e. } t \in [t_0, T] \\ k(t_0) = k_0 & \\ s(t) \in [0, 1], & \text{a.e. } t \in [t_0, T] \\ k(t) \geq 0, & \forall t \in [t_0, T]. \end{cases} \tag{5.12}$$

This problem has five parameters: $t_0$, $T$, $\lambda \geq 0$, $\sigma > 0$, and $k_0 > 0$.

The optimal control problem in (5.11)–(5.12) will be denoted by $(GP)$. According to [18], $(GP)$ is a finite horizon optimal control problem of the Lagrange type. In addition, due to the requirement that $k(t) \geq 0$ for all $t \in [t_0, T]$, this is an optimal control problem with state constraints.

To make $(GP)$ competent with the given modeling presentation, one has to explain why the state trajectory can be sought in $W^{1,1}([t_0, T], \mathbb{R})$ and the control function is just required to be measurable. If one assumes that the investment function $I(\cdot)$ is continuous on $[t_0, T]$, then (5.3) implies that $K(\cdot)$ is continuously differentiable; hence so is $k(\cdot)$. However, in practice, the investment function $I(\cdot)$ can be discontinuous at some points $t \in [t_0, T]$ (say, the policy has a great change, and the government decides to allocate a large amount of money into the production field, or to cancel a large amount of money from it). Thus, the requirement that $k(\cdot)$ is differentiable at these points may not be fulfilled. To deal with this situation, it is reasonable to assume that the state trajectory $k(\cdot)$ belongs to the space of continuous, piecewise continuously differentiable functions on $[t_0, T]$, which is endowed with the norm $\|k\| = \max\limits_{t \in [t_0, T]} |k(t)|$. Since the latter space is incomplete one embeds it into the space $W^{1,1}([t_0, T], \mathbb{R})$, which possesses many good properties (see [49]). In that way, tools from the Lebesgue integration theory and results from the conventional optimal control theory can be used for $(GP)$. Now, concerning the control function $s(\cdot)$, one has the following observation. Since the derivative $\dot{k}(t)$ exists almost everywhere on $[t_0, T]$ and $\dot{k}(\cdot)$ is a measurable function, for the fulfillment of the relation $\dot{k}(t) = s(t)\phi(k(t)) - \sigma k(t)$ almost everywhere on $[t_0, T]$, it suffices to assume that $s(\cdot)$ is a measurable function. Recall that a function $\varphi : [t_0, T] \to \mathbb{R}$ is said to be *measurable* if for any $\alpha \in \mathbb{R}$ the set $\{t \in [t_0, T] \, : \, \varphi \in (-\infty, \alpha)\}$ is Lebesgue measurable. This is equivalent to saying that the inverse $\varphi^{-1}(B) := \{t \in [t_0, T] \, : \, \varphi \in B\}$ of any Borel set $B$ in $\mathbb{R}$ via $\varphi$ is Lebesgue measurable.

## 5.2 Auxiliary Concepts and Results

To recall a solution existence theorem for finite horizon optimal control problems with state constraints of the Bolza type, we will use the notations and concepts given in the monograph of Cesari [18, Sections 9.2, 9.3, and

9.5]. Let $A$ be a subset of $\mathbb{R} \times \mathbb{R}^n$ and $U : A \rightrightarrows \mathbb{R}^m$ be a set-valued map defined on $A$. Let

$$M := \{(t, x, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \ : \ (t, x) \in A, \ u \in U(t, x)\},$$

$f_0(t, x, u)$ and $f(t, x, u) = (f_1, f_2, \ldots, f_n)$ be functions defined on $M$. Let $B$ be a given subset of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and $g(t_1, x_1, t_2, x_2)$ be a real valued function defined on $B$. Let there be given an interval $[t_0, T] \subset \mathbb{R}$. Consider the problem of the Bolza type

$$\text{Minimize } \ I(x, u) := g(t_0, x(t_0), T, x(T)) + \int_{t_0}^{T} f_0(t, x(t), u(t))dt \tag{5.13}$$

over pairs of functions $(x, u)$ with $x(\cdot) \in W^{1,1}([t_0, T], \mathbb{R}^n)$, $u(\cdot) : [t_0, T] \to \mathbb{R}^m$ being measurable, $f_0(., x(\cdot), u(\cdot)) : [t_0, T] \to \mathbb{R}$ being Lebesgue integrable on $[t_0, T]$, and satisfying

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & \text{a.e. } t \in [t_0, T] \\ (t, x(t)) \in A, & \forall t \in [t_0, T] \\ (t_0, x(t_0), T, x(T)) \in B & \\ u(t) \in U(t, x(t)), & \text{a.e. } t \in [t_0, T]. \end{cases} \tag{5.14}$$

Such a pair $(x, u)$ is called a *feasible process*. The problem (5.13)–(5.14) will be denoted by $\mathcal{B}$.

If $(x, u)$ is a feasible process for $\mathcal{B}$, then $x$ is said to be a *feasible trajectory*, and $u$ a *feasible control*. The set of all the feasible processes for $\mathcal{B}$ is denoted by $\Omega$. A feasible process $(\bar{x}, \bar{u})$ is said to be a *global minimizer* for $\mathcal{B}$ if one has $I(\bar{x}, \bar{u}) \leq I(x, u)$ for any feasible process $(x, u)$.

Let $A_0$ be the projection of $A$ on the $t$-axis, i.e.,

$$A_0 = \big\{t \ : \ \exists x \in \mathbb{R}^n \text{ s.t. } (t, x) \in A\big\}.$$

Set $A(t) = \big\{x \in \mathbb{R}^n \ : \ (t, x) \in A\big\}$ for every $t \in A_0$ and

$$\widetilde{Q}(t, x) = \big\{(z^0, z) \in \mathbb{R}^{n+1} : z^0 \geq f_0(t, x, u), z = f(t, x, u) \text{ for some } u \in U(t, x)\big\}$$

for every $(t, x) \in A$.

The forthcoming statement is known as *Filippov's Existence Theorem for Bolza problems*. It is an analogue of Theorem 3.2 addressing the solution existence for Mayer problems, which has been used repeatedly in the preceding two chapters.

**Theorem 5.1** (See [18, Theorem 9.3.i, p. 317, and Section 9.5]) *It is supposed that $\Omega$ is nonempty, $B$ is closed, $g$ is lower semicontinuous on $B$, $f_0$ and $f$ is continuous on $M$ and, for almost every $t \in [t_0, T]$, the sets $\widetilde{Q}(t, x)$, $x \in A(t)$, are convex. Moreover, assume either that $A$ and $M$ are compact or that $A$ is not compact but closed and contained in a slab $[t_1, t_2] \times \mathbb{R}^n$ with $t_1$ and $t_2$ being finite, and the following conditions are fulfilled:*

(a) *For any $\varepsilon \geq 0$, the set $M_\varepsilon := \{(t, x, u) \in M \ : \ \|x\| \leq \varepsilon\}$ is compact;*

(b) *There is a compact subset $P$ of $A$ such that every feasible trajectory $x$ of $\mathcal{B}$ passes through at least one point of $P$;*

(c) *There exists a constant $c \geq 0$ such that $\langle x, f(t, x, u)\rangle \leq c(\|x\|^2 + 1)$ for all $(t, x, u) \in M$.*

*Then, $\mathcal{B}$ has a global minimizer.*

Clearly, condition (b) is satisfied if the initial point $(t_0, x(t_0))$ or the end point $(T, x(T))$ is fixed. As shown in [18, p. 317], the following condition implies (c):

($c_0$) *There exists $c \geq 0$ such that $\|f(t, x, u)\| \leq c(\|x\|+1)$ for all $(t, x, u) \in M$.*

In the next two sections, several results on the solution existence of optimal economic growth problems will be derived from Theorem 5.1.

## 5.3 Existence Theorems for General Problems

Our first result on the solution existence of the finite horizon optimal economic growth problem $(GP)$ in (5.11)–(5.12) is stated as follows.

**Theorem 5.2** *For the problem $(GP)$, suppose that $\omega(\cdot)$ and $\phi(\cdot)$ are continuous on $\mathbb{R}_+$. If, in addition, $\omega(\cdot)$ is concave on $\mathbb{R}_+$ and the function $\phi(\cdot)$ satisfies the condition*

($c_1$) *There exists $c \geq 0$ such that $\phi(k) \leq (c - \sigma)k + c$ for all $k \in \mathbb{R}_+$,*

*then $(GP)$ has a global solution.*

**Proof.** To apply Theorem 5.1, we have to interpret $(GP)$ in the form of $\mathcal{B}$. For doing so, we let the variable $k$ (resp., the variable $s$) play the role of the phase variable $x$ in $\mathcal{B}$ (resp., the control variable $u$ in $\mathcal{B}$). Then, $(GP)$ has the form of $\mathcal{B}$ with $n = m = 1$, $A = [t_0, T] \times \mathbb{R}_+$, $U(t,k) = [0,1]$ for all $(t,k) \in A$, $B = \{t_0\} \times \{k_0\} \times \{T\} \times \mathbb{R}$, $M = [t_0, T] \times \mathbb{R}_+ \times [0,1]$, $g \equiv 0$ on $B$, $f_0(t,k,s) = -\omega((1-s)\phi(k))e^{-\lambda t}$, and $f(t,k,s) = s\phi(k) - \sigma k$ for all $(t,k,s) \in M$.

Setting $s(t) = 0$ and $k(t) = k_0 e^{-\sigma(t-t_0)}$ for all $t \in [t_0, T]$, one can easily verify that the pair $(k(\cdot), s(\cdot))$ is a feasible process for $(GP)$. Thus, the set $\Omega$ of the feasible processes is nonempty. It is clear that $B$ is closed, $g$ is continuous on $B$ and, by the assumed continuity of $\omega(\cdot)$ and $\phi(\cdot)$, $f_0$ and $f$ are continuous on $M$. Besides, the formula for $A$ implies that $A_0 = [t_0, T]$ and $A(t) = \mathbb{R}_+$ for all $t \in A_0$. In addition, by the formulas for $f_0$, $f$ and $U$, one has for any $(t,k) \in A$ the following:

$$\begin{aligned}\widetilde{Q}(t,k) &= \big\{(z^0, z) \in \mathbb{R}^2 : z^0 \geq f_0(t,k,s),\ \ z = f(t,k,s) \text{ for some } s \in U(t,k)\big\} \\ &= \big\{(z^0, z) \in \mathbb{R}^2 : \exists s \in [0,1] \text{ s.t. } \ \ z^0 \geq -\omega((1-s)\phi(k))e^{-\lambda t}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ z = s\phi(k) - \sigma k\big\}.\end{aligned}$$

Let us show that, for any $t \in [t_0, T]$ and $k \in A(t) = \mathbb{R}_+$, the set $\widetilde{Q}(t,k)$ is convex. Indeed, given any $(z_1^0, z_1)$, $(z_2^0, z_2) \in \widetilde{Q}(t,k)$ and $\mu \in [0,1]$, one can find $s_1, s_2 \in [0,1]$ such that

$$\begin{aligned}z_1^0 &\geq -\omega((1-s_1)\phi(k))e^{-\lambda t},\ \ z_1 = s_1\phi(k) - \sigma k, \\ z_2^0 &\geq -\omega((1-s_2)\phi(k))e^{-\lambda t},\ \ z_2 = s_2\phi(k) - \sigma k.\end{aligned}$$

Therefore, it holds that

$$\mu z_1^0 + (1-\mu)z_2^0 \geq -\mu\omega((1-s_1)\phi(k))e^{-\lambda t} - (1-\mu)\omega((1-s_2)\phi(k))e^{-\lambda t} \quad (5.15)$$

and

$$\mu z_1 + (1-\mu)z_2 = \mu[s_1\phi(k) - \sigma k] + (1-\mu)[s_2\phi(k) - \sigma k]. \quad (5.16)$$

Setting $s_\mu = \mu s_1 + (1-\mu)s_2$, one has $s_\mu \in [0,1]$ and it follows from (5.16) that

$$\mu z_1 + (1-\mu)z_2 = s_\mu\phi(k) - \sigma k. \quad (5.17)$$

Clearly, the concavity of $\omega(\cdot)$ on $\mathbb{R}_+$ yields

$$\begin{aligned}&-\mu\omega((1-s_1)\phi(k)) - (1-\mu)\omega((1-s_2)\phi(k)) \\ &\geq -\omega[\mu(1-s_1)\phi(k) + (1-\mu)(1-s_2)\phi(k)] \\ &= -\omega((1-s_\mu)\phi(k)).\end{aligned}$$

Hence, by (5.15) we obtain

$$\mu z_1^0 + (1-\mu)z_2^0 \geq -\omega[(1-s_\mu)\phi(k)]e^{-\lambda t},$$

which together with (5.17) implies that $\mu(z_1^0, z_1) + (1-\mu)(z_2^0, z_2) \in \widetilde{Q}(t,k)$.

Now, although $A = [t_0, T] \times \mathbb{R}_+$ is noncompact, the fact that $A$ is closed and contained in a slab $[t_1, t_2] \times \mathbb{R}$ with $t_1$ and $t_2$ being finite is clear. It remains to check the conditions (a)–(c) in Theorem 5.1.

For any $\varepsilon \geq 0$, the set $M_\varepsilon$ is compact because

$$\begin{aligned} M_\varepsilon &= \{(t,k,s) \in [t_0,T] \times \mathbb{R}_+ \times [0,1] \ : \ |k| \leq \varepsilon\} \\ &= [t_0, T] \times [0, \varepsilon] \times [0,1]. \end{aligned}$$

So, condition (a) is satisfied. As $P := \{(t_0, k_0)\}$ is a compact subset of $A$, and every feasible trajectory of $(GP)$ passes through $(t_0, k_0)$, condition (b) is fulfilled. Applied to the case of $(GP)$, where $f(t,k,s) = s\phi(k) - \sigma k$ and $M = [t_0, T] \times \mathbb{R}_+ \times [0,1]$ as explained above, condition (c) in Theorem 5.1 can be rewritten as

(c′) *There exists* $c \geq 0$ *such that* $sk\phi(k) \leq (c+\sigma)k^2 + c$ *for all* $(k,s)$ *in* $\mathbb{R}_+ \times [0,1]$.

By the comment given after Theorem 5.1, (c) is valid if ($\mathrm{c}_0$) holds. As $f(t,k,s) = s\phi(k) - \sigma k$ and $M = [t_0, T] \times \mathbb{R} \times [0,1]$, the latter can be stated as

($\mathrm{c}_0'$) *There exists* $c \geq 0$ *such that* $|s\phi(k) - \sigma k| \leq c(k+1)$ *for all* $(k,s)$ *in* $\mathbb{R}_+ \times [0,1]$.

To prove ($\mathrm{c}_0'$), observe that the estimates

$$|s\phi(k) - \sigma k| \leq s\phi(k) + \sigma k \leq \phi(k) + \sigma k \tag{5.18}$$

hold for any $(k,s) \in \mathbb{R}_+ \times [0,1]$. Furthermore, thanks to the assumption ($\mathrm{c}_1$), we can find a constant $c \geq 0$ such that $\phi(k) \leq (c-\sigma)k + c$ for all $k \in \mathbb{R}_+$. Since the last inequality can be rewritten as $\phi(k) + \sigma k \leq c(k+1)$, from (5.18) we get ($\mathrm{c}_0'$).

Since our problem $(GP)$ in the interpretation given above satisfies all the assumptions of Theorem 5.1, we conclude that it has a global solution. □

In Theorem 5.2, it is not required that $\phi(\cdot)$ is concave on $\mathbb{R}_+$. It turns out that if the concavity of $\phi(\cdot)$ is available, then there is no need to check

($c_1$). Since the assumption saying that the per capita production function $\phi(k) := F(k, 1)$ is concave on $\mathbb{R}_+$ is reasonable in practice, the next theorem seems to be interesting.

**Theorem 5.3** *If both functions $\omega(\cdot)$ and $\phi(\cdot)$ are continuous and concave on $\mathbb{R}_+$, then $(GP)$ has a global solution.*

**Proof.** Set $\psi = -\phi$ and put $\psi(k) = +\infty$ for every $k \in (-\infty, 0)$. Then, $\psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a proper convex function and the effective domain $\operatorname{dom}\psi$ of $\psi$ is $\mathbb{R}_+$. Select any $\bar{k} > 0$. Since $\bar{k}$ belongs to the interior of $\operatorname{dom}\psi$, by [72, Theorem 23.4] we know that the *subdifferential* (see, e.g., [72, p. 215]) $\partial\psi(\bar{k})$ of $\psi$ at $\bar{k}$ is nonempty. Thus, taking an element $a \in \partial\psi(\bar{k})$, one has

$$\psi(k) - \psi(\bar{k}) \geq a(k - \bar{k}), \quad \forall k \geq 0,$$

or, equivalently,

$$\phi(k) \leq -ak + a\bar{k} + \phi(\bar{k}), \quad \forall k \geq 0. \tag{5.19}$$

For $c := \max\{0, \sigma - a, \phi(\bar{k}) + a\bar{k}\}$, one has $c \geq 0$ and

$$-ak + a\bar{k} + \phi(\bar{k}) \leq (c - \sigma)k + c, \quad \forall k \geq 0. \tag{5.20}$$

Combining (5.19) and (5.20), one can assert that condition ($c_1$) in Theorem 5.2 is fulfilled. Thus, the assumed continuity of $\omega(\cdot)$ and $\phi(\cdot)$ together with the concavity of $\omega(\cdot)$ allows us to apply Theorem 5.2 to conclude that $(GP)$ has a global solution. □

The next proposition reveals the nature of condition ($c_1$), which is essential for the validity of Theorem 5.2.

**Proposition 5.1** *Condition* ($c_1$) *and the conditions* ($c'$) *and* ($c'_0$)*, which were formulated in the proof of Theorem 5.2, are equivalent. Moreover, each of these conditions is equivalent to the condition*

$$\limsup_{k \to +\infty} \frac{\phi(k)}{k} < +\infty \tag{5.21}$$

*on the asymptotic behavior of $\phi$.*

**Proof.** The implications ($c_1$) $\Rightarrow$ ($c'_0$) and ($c'_0$) $\Rightarrow$ ($c'$) were obtained in the proof of Theorem 5.2. So, the proposition will be proved if we can show that ($c'$) implies (5.21) and (5.21) implies ($c_1$).

To get the implication (c′) ⇒ (5.21), suppose that (c′) holds. Then, there exists $c \geq 0$ satisfying $sk\phi(k) \leq (c+\sigma)k^2 + c$ for all $(k,s) \in \mathbb{R}_+ \times [0,1]$. Thus, choosing $s = 1$, one has

$$\frac{\phi(k)}{k} \leq c + \sigma + \frac{c}{k^2}, \quad \forall k \in (0, +\infty).$$

By taking the limsup on both sides of the last inequality when $k$ tends to infinity, one gets (5.21).

Now, to obtain the implication (5.21) ⇒ (c′), suppose that (5.21) holds. Then, there exist $\gamma_1 > 0$ and $\mu > 0$ such that $\dfrac{\phi(k)}{k} \leq \gamma_1$ for every $k > \mu$. Thanks to the continuity of $\phi$ at $k = 0$, one can find $\gamma_2 > 0$ and $\varepsilon \in (0, \mu)$ such that $\phi(k) \leq \gamma_2$ for all $k \in [0, \varepsilon)$. Moreover, by the continuity of the function $k \mapsto \phi(k)/k$ on the compact interval $[\varepsilon, \mu]$, the number

$$\gamma_3 := \max\left\{\frac{\phi(k)}{k} \ : \ k \in [\varepsilon, \mu]\right\}$$

is well defined. Thus, for any $c \geq \max\{\gamma_1 + \sigma,\ \gamma_2,\ \gamma_3 + \sigma\}$, it holds that

$$\phi(k) \leq \gamma_2 \leq c \leq (c - \sigma)k + c \quad \forall k \in [0, \varepsilon),$$
$$\phi(k) \leq \gamma_3 k \leq (c - \sigma)k + c \quad \forall k \in [\varepsilon, \mu],$$

and

$$\phi(k) \leq \gamma_1 k \leq (c - \sigma)k + c \quad \forall k \in (\mu, +\infty).$$

Thus, one has $\phi(k) \leq (c - \sigma)k + c$ for every $k \geq 0$, which justifies ($c_1$). □

**Remark 5.1** There are many continuous functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ that are nonconcave on $\mathbb{R}_+$ but satisfy condition ($c_1$) in Theorem 5.2. Indeed, suppose that the values $\bar{k} > 0$, $\phi_0 \geq 0$, and $a > 0$ are given arbitrarily. Setting

$$\phi(k) = \begin{cases} \phi_0, & \text{if } k \in [0, \bar{k}] \\ a(k - \bar{k}) + \phi_0, & \text{if } k \in (\bar{k}, +\infty), \end{cases}$$

one has a function $\phi$, that is continuous and nonconcave on $\mathbb{R}_+$. But, since the coercivity condition (5.21) is fulfilled, this $\phi$ satisfies ($c_1$). More generally, the continuous function

$$\phi(k) = \begin{cases} \phi_1(k), & \text{if } k \in [0, \bar{k}] \\ a(k - \bar{k})^\alpha + \phi_1(\bar{k}), & \text{if } k \in (\bar{k}, +\infty), \end{cases}$$

where $\alpha \in (0, 1]$ is a constant and $\phi : [0, \bar{k}] \to \mathbb{R}_+$ is a continuous function, also satisfies ($c_1$) because (5.21) is fulfilled. Clearly, there are many ways to choose $\phi_1(k)$ such that this function $\phi$ in nonconcave on $\mathbb{R}_+$.

Economic growth problems with utility functions $\omega(\cdot)$ and production functions $F(\cdot)$ of two typical types will be the subject of our consideration in the next section.

## 5.4 Solution Existence for Typical Problems

As observed by Takayama [79, p. 450], the production function given by

$$F(K,L) = \frac{1}{a}K, \quad \forall (K,L) \in \mathbb{R}^2_+, \tag{5.22}$$

where $a > 0$ is a constant representing the *capital-to-output ratio*, is of a great importance. This function is in the form of the *AK function* (see, e.g., [11, Subsection 1.3.2]) with *the diminishing returns to capital being absent*, which is a key property of endogenous growth models. The function in (5.22) is also referred to in connection with the Harrod-Domar model of which a main assumption is that the labor factor is not explicitly involved in the production function (see, e.g., [79, Footnote 5, p. 464]). In the notations of Section 5.1, by (5.22) one has

$$\phi(k) = \frac{1}{a}k, \quad \forall k \geq 0.$$

So, the differential equation in (5.12) becomes

$$\dot{k}(t) = \frac{1}{a}s(t)k(t) - \sigma k(t), \quad \text{a.e. } t \in [t_0, T].$$

Another popular type of the production function $F$ is the *Cobb-Douglas function* (see, e.g., [11, p. 29]), which is given by

$$F(K,L) = AK^{\alpha}L^{1-\alpha}, \quad \forall (K,L) \in \mathbb{R}^2_+ \tag{5.23}$$

with $A > 0$ and $\alpha \in (0,1)$ being constants. In the terminology of economics, the exponent $\alpha$ (resp., $1-\alpha$) refers to the *output elasticity of capital* (resp., the *output elasticity of labor*), which represents the share of the contribution of the capital (resp., of the labor) to the total product $F(K,L)$. Meanwhile, the coefficient $A$ expresses the *total factor productivity*[1] (TFP). This measure of economic efficiency is calculated by dividing output by the weighted average of labour and capital input. TFP represents the increase in total production which is in excess of the increase that results from increase in inputs and depends on some intangible factors such as technological

[1]See, e.g., https://en.wikipedia.org/wiki/Total_factor_productivity.

change, education, research and development, etc. As $\alpha \in (0,1)$, $F$ *exhibits diminishing returns to capital and labor* (see, e.g., [79, p. 433]). The latter means that the *marginal products* of both capital and labor are diminishing (see, e.g., [1, p. 29]). The presence of diminishing returns to capital, which plays a very important role in many results of the basic growth model (see, e.g., [1, p. 29]), distinguishes the production given by (5.23) with the one in (5.22). The per capita production function corresponding to (5.23) is

$$\phi(k) = Ak^\alpha, \quad \forall k \geq 0. \tag{5.24}$$

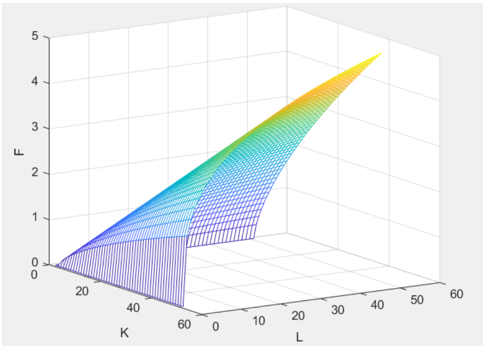


Figure 5.1: Cobb-Douglas production function

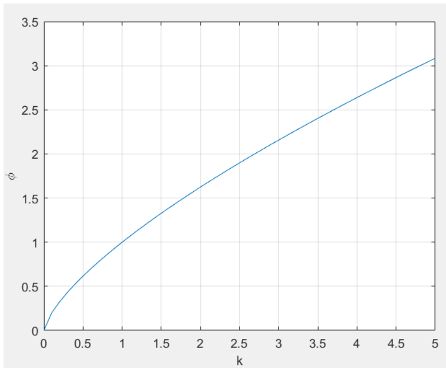


Figure 5.2: Cobb-Douglas per capita function

Therefore, (5.12) collapses to

$$\dot{k}(t) = As(t)k^\alpha(t) - \sigma k(t), \quad \text{a.e. } t \in [t_0, T]. \tag{5.25}$$

Since (5.22) can be written in the form of (5.23) with $\alpha := 1$ and $A := 1/a$, one can combine the above two types of production functions in a general one by considering (5.23) with $A > 0$ and $\alpha \in (0,1]$. This means that one has deal with the model (5.24)–(5.25), where $A > 0$ and $\alpha \in (0,1]$ are given constants. In the same manner, concerning the utility function $\omega(\cdot)$, the formula

$$\omega(c) = c^\beta, \quad \forall c \geq 0 \tag{5.26}$$

with $\beta \in (0,1]$ can be considered. For $\beta = 1$, $\omega(\cdot)$ is a linear function. For $\beta \in (0,1)$, it is a Cobb-Douglas function.

In the rest of this section, for the problem $(GP)$, we assume that $\phi(\cdot)$ and $\omega(\cdot)$ are given respectively by (5.24) and (5.26). Then, the target function of $(GP)$ is

$$I(k,s) = \int_{t_0}^{T} [1 - s(t)]^\beta \phi^\beta(k(t)) e^{-\lambda t} dt = A^\beta \int_{t_0}^{T} [1 - s(t)]^\beta k^{\alpha\beta}(t) e^{-\lambda t} dt.$$

Thus, we have to solve the following equivalent problem:

$$\text{Maximize} \quad \int_{t_0}^{T} [1 - s(t)]^{\beta} k^{\alpha\beta}(t) e^{-\lambda t} dt \tag{5.27}$$

over $k \in W^{1,1}([t_0, T], \mathbb{R})$ and measurable functions $s : [t_0, T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{k}(t) = Ak^{\alpha}(t)s(t) - \sigma k(t), & \text{a.e. } t \in [t_0, T] \\ k(t_0) = k_0 & \\ s(t) \in [0, 1], & \text{a.e. } t \in [t_0, T] \\ k(t) \geq 0, & \forall\, t \in [t_0, T] \end{cases} \tag{5.28}$$

with $T > t_0 \geq 0$, $\lambda \geq 0$, $A > 0$, $\sigma > 0$, and $k_0 > 0$ being given parameters. We denote the problem (5.27)–(5.28) by $(GP_1)$.

The forthcoming result is a consequence of Theorem 5.3.

**Theorem 5.4** *For any constants $\alpha \in (0, 1]$ and $\beta \in (0, 1]$, the optimal economic growth problem $(GP_1)$ possesses a global solution.*

**Proof.** By the assumptions $A > 0$, $\alpha \in (0, 1]$, and $\beta \in (0, 1]$, the functions $\phi(k) = Ak^{\alpha}$ and $\omega(c) = c^{\beta}$ are continuous on $\mathbb{R}_+$. The concavity of $\phi(\cdot)$ on $(0, +\infty)$ follows from the fact that $\phi''(k) = A\alpha(\alpha - 1)k^{\alpha-2} < 0$ for all $k \geq 0$ (see, e.g., [72, Theorem 4.4]). As $\phi(\cdot)$ is continuous at 0, we can assert that $\phi(\cdot)$ is concave on $\mathbb{R}_+$. The concavity of $\phi(\cdot)$ on $\mathbb{R}_+$ is verified similarly. Since both $\omega(\cdot)$ and $\phi(\cdot)$ are continuous and concave on $\mathbb{R}_+$, Theorem 5.3 assures the solution existence for the problem (5.27)–(5.28).

**Remark 5.2** Depending on the displacement of $\alpha$ and $\beta$ on $(0, 1]$, we have four types of the model (5.27)–(5.28):

(T1) "Linear-linear": $\phi(k) = Ak$ and $\omega(c) = c$ (both the per capita production function and the utility function are linear);

(T2) "Linear-nonlinear": $\phi(k) = Ak$ and $\omega(c) = c^{\beta}$ with $\beta \in (0, 1)$ (the per capita production function is linear, but the utility function is nonlinear);

(T3) "Nonlinear-linear": $\phi(k) = Ak^{\alpha}$ and $\omega(c) = c$ with $\alpha \in (0, 1)$ (the per capita production function is nonlinear, but the utility function is linear);

(T4) "Nonlinear-nonlinear": $\phi(k) = Ak^\alpha$ and $\omega(c) = c^\beta$ with $\alpha \in (0,1)$ and $\beta \in (0,1)$ (both the per capita production function and the utility function are nonlinear).

Although the problem in question of each type has a global solution by Theorem 5.4, the above classification arranges the difficulties of solving (5.27)–(5.28), say, by the Maximum Principle given in [82, Theorem 9.3.1]. Obviously, problems of the first type (T1) are the easiest ones, while those of the fourth type (T4) are the most difficult ones.

In the next section, we will discuss some assumptions used for getting Theorems 5.2 and 5.3.

## 5.5 The Asymptotic Behavior of $\phi$ and Its Concavity

The results in Section 5.3 were obtained under certain assumptions on the per capita production function $\phi$, which is defined via the production function $F(K, L)$ by the formula

$$\phi(k) = F(k, 1) = \frac{F(K, L)}{L} \tag{5.29}$$

with $k := \dfrac{K}{L}$ signifying the capital-to-labor ratio. We want to know: *How the assumptions made on $\phi$ can be traced back to $F$?*

**Proposition 5.2** *The per capita production function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies condition $(c_1)$ if and only if the production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ has the following property:*

($c_1'$) *There exists $c \geq 0$ such that $F(K, L) \leq (c - \sigma)K + cL$ for all $K \geq 0$ and $L > 0$.*

**Proof.** Suppose that $(c_1)$ is satisfied, i.e., there exists $c \geq 0$ such that $\phi(k) \leq (c - \sigma)k + c$ for all $k \in \mathbb{R}_+$. Then, given any $K \geq 0$ and $L > 0$, by substituting $k = \dfrac{K}{L}$ into the last inequality and using (5.29), one gets

$$\frac{F(K, L)}{L} \leq (c - \sigma)\frac{K}{L} + c.$$

This justifies $(c_1')$. Conversely, suppose that $F(K,L) \le (c-\sigma)K + cL$ holds for all $K \ge 0$ and $L > 0$, where $c \ge 0$ is a constant. Then, letting $L = 1$ and $K = k$, where $k \ge 0$ is given arbitrarily, one gets the inequality

$$\phi(k) \le (c-\sigma)k + c.$$

Thus, $(c_1)$ is fulfilled.

**Proposition 5.3** *The function $\phi$ satisfies* (5.21) *if and only if $F$ fulfills the following inequality:*

$$\limsup_{\frac{K}{L}\to+\infty} \frac{F(K,L)}{K} < +\infty. \tag{5.30}$$

**Proof.** By (5.29), for any $K > 0$ and $L > 0$, one has

$$\frac{F(K,L)}{K} = \frac{L^{-1}F(K,L)}{L^{-1}K} = \frac{\phi(k)}{k}$$

with $k := \dfrac{K}{L}$. Thus, the equivalence between (5.21) and (5.30) is straightforward.

Propositions 5.2 and 5.3 show that the assumption made on $\phi$ in Theorem 5.2 and its equivalent representations given in Proposition 5.1 can be checked directly on the original function $F$.

**Proposition 5.4** *The per capita production function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is concave on $\mathbb{R}_+$ if and only if the production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is concave on $\mathbb{R}_+ \times (0,+\infty)$.*

**Proof.** Firstly, suppose that $F$ is concave on $\mathbb{R}_+ \times (0,+\infty)$. Let $k_1, k_2 \in \mathbb{R}_+$ and $\lambda \in [0,1]$ be given arbitrarily. The concavity of $F$ and (5.29) yield

$$F(\lambda(k_1,1)+(1-\lambda)(k_2,1)) \ge \lambda F(k_1,1)+(1-\lambda)F(k_2,1) = \lambda\phi(k_1)+(1-\lambda)\phi(k_2).$$

Since $F(\lambda(k_1,1)+(1-\lambda)(k_2,1)) = F(\lambda k_1+(1-\lambda)k_2, 1)$, combining this with (5.29), one obtains $\phi(\lambda k_1 + (1-\lambda)k_2) \ge \lambda\phi(k_1) + (1-\lambda)\phi(k_2)$. This justifies the concavity of $\phi$.

Now, suppose that $\phi$ is concave on $\mathbb{R}_+$. If $F$ is not concave on $\mathbb{R}_+ \times (0,+\infty)$, then there exist $(K_1,L_1), (K_2,L_2)$ in $\mathbb{R}_+ \times (0,+\infty)$ and $\lambda \in (0,1)$ such that

$$F(\lambda K_1 + (1-\lambda)K_2, \lambda L_1 + (1-\lambda)L_2) < \lambda F(K_1,L_1) + (1-\lambda)F(K_2,L_2).$$

By (5.29), it holds that $F(K,L) = L\phi(\frac{K}{L})$ for any $(K,L) \in \mathbb{R}_+ \times (0,+\infty)$. Therefore, we have

$$[\lambda L_1 + (1-\lambda)L_2]\phi\Big(\frac{\lambda K_1 + (1-\lambda)K_2}{\lambda L_1 + (1-\lambda)L_2}\Big) < \lambda L_1\phi\Big(\frac{K_1}{L_1}\Big) + (1-\lambda)L_2\phi\Big(\frac{K_2}{L_2}\Big).$$

Dividing both sides of this inequality by $\lambda L_1 + (1-\lambda)L_2$ gives

$$\phi\Big(\frac{\lambda K_1 + (1-\lambda)K_2}{\lambda L_1 + (1-\lambda)L_2}\Big) < \frac{\lambda L_1}{\lambda L_1 + (1-\lambda)L_2}\phi\Big(\frac{K_1}{L_1}\Big) + \frac{(1-\lambda)L_2}{\lambda L_1 + (1-\lambda)L_2}\phi\Big(\frac{K_2}{L_2}\Big). \tag{5.31}$$

Setting $\mu = \dfrac{\lambda L_1}{\lambda L_1 + (1-\lambda)L_2}$, one has $1-\mu = \dfrac{(1-\lambda)L_2}{\lambda L_1 + (1-\lambda)L_2}$, $\mu \in (0,1)$, and

$$\mu\frac{K_1}{L_1} + (1-\mu)\frac{K_2}{L_2} = \frac{\lambda K_1 + (1-\lambda)K_2}{\lambda L_1 + (1-\lambda)L_2}.$$

Thus, (5.31) means that

$$\phi\Big(\mu\frac{K_1}{L_1} + (1-\mu)\frac{K_2}{L_2}\Big) < \mu\phi\Big(\frac{K_1}{L_1}\Big) + (1-\mu)\phi\Big(\frac{K_2}{L_2}\Big),$$

a contradiction to the assumed concavity of $\phi$. The proof is complete. □

We have seen that the assumption on the concavity of $\phi$ used in Theorem 5.3 can be verified directly on $F$.

Now, we will look deeper into Theorems 5.2 and 5.3 and the typical optimal economic growth problems in Section 5.4 by raising some open questions and conjectures about the *uniqueness* and the *regularity* of the global solutions of $(GP)$.

## 5.6 Regularity of Optimal Processes

Solution regularity is an important concept which helps one to look deeper into the structure of the problem in question. One may have deal with Lipschitz continuity, Hölder continuity, and degree of differentiability of the obtained solutions. We refer to [82, Chapter 11] for a solution regularity theory in optimal control and to [48, Theorem 9.2, p. 140] for a result on the solution regularity for variational inequalities.

The results of Sections 5.3 and 5.4 assure that, if some mild assumptions on the per capital function and the utility function are satisfied, then $(GP)$

has a global solution $(\bar{k}, \bar{s})$ with $\bar{k}(\cdot)$ being absolutely continuous on $[t_0, T]$ and $\bar{s}(\cdot)$ being measurable. Since the saving policy $\bar{s}(\cdot)$ on the time segment $[t_0, T]$ cannot be implemented if it has an infinite number of discontinuities, the following concept of regularity of the solutions of the optimal economic growth problem $(GP)$ appears in a natural way.

**Definition 5.1** A global solution $(\bar{k}, \bar{s})$ of $(GP)$ is said to be *regular* if the propensity to save function $\bar{s}(\cdot)$ only has finitely many discontinuities of first type on $[t_0, T]$. This means that there is a positive integer $m$ such that the segment $[t_0, T]$ can be divided into $m$ subsegments $[\tau_i, \tau_{i+1}]$, $i = 0, \ldots, m-1$, with $\tau_0 = t_0$, $\tau_m = T$, $\tau_i < \tau_{i+1}$ for all $i$, $\bar{s}(\cdot)$ is continuous on each open interval $(\tau_i, \tau_{i+1})$, and the one-sided limit $\lim\limits_{t \to \tau_i{}^+} \bar{s}(t)$ (resp., $\lim\limits_{t \to \tau_i{}^-} \bar{s}(t)$) exists for each $i \in \{0, 1, \ldots m-1\}$ (resp., for each $i \in \{1, \ldots m\}$).

In Definition 5.1, as $\bar{s}(t) \in [0, 1]$ for every $t \in [t_0, T]$, the one-sided limit $\lim\limits_{t \to \tau_i{}^+} \bar{s}(t)$ (resp., $\lim\limits_{t \to \tau_i{}^-} \bar{s}(t)$) must be finite for each $i \in \{0, 1, \ldots m-1\}$ (resp., for each $i \in \{1, \ldots m\}$).

**Proposition 5.5** *Suppose that the function $\phi$ is continuous on $[t_0, T]$. If $(\bar{k}, \bar{s})$ is a regular global solution of $(GP)$, then the capital-to-labor ratio $\bar{k}(t)$ is a continuous, piecewise continuously differentiable function on the segment $[t_0, T]$. In particular, the function $\bar{k}(\cdot)$ is Lipschitz on $[t_0, T]$.*

**Proof.** Since $(\bar{k}, \bar{s})$ is a regular global solution of $(GP)$, there is a positive integer $m$ such that the segment $[t_0, T]$ can be divided into $m$ subsegments $[\tau_i, \tau_{i+1}]$, $i = 0, \ldots, m-1$, and all the requirements stated in Definition 5.1 are fulfilled. Then, for each $i \in \{0, \ldots, m-1\}$, from the first relation in (5.12) we have

$$\dot{\bar{k}}(t) = \bar{s}(t)\phi(\bar{k}(t)) - \sigma\bar{k}(t), \quad \text{a.e. } t \in (\tau_i, \tau_{i+1}). \tag{5.32}$$

Hence, by the continuity of $\phi$ on $[t_0, T]$ and the continuity of $\bar{s}(\cdot)$ on $(\tau_i, \tau_{i+1})$, we can assert that the derivative $\dot{\bar{k}}(t)$ exists for every $t \in (\tau_i, \tau_{i+1})$. Indeed, fixing any point $\bar{t} \in (\tau_i, \tau_{i+1})$ and using the Lebesgue Theorem [49, Theorem 6, p. 340] for the absolutely continuous function $\bar{k}(\cdot)$, we have

$$\bar{k}(t) = \int_{\bar{t}}^{t} \dot{\bar{k}}(\tau)d\tau, \quad \forall t \in (\tau_i, \tau_{i+1}), \tag{5.33}$$

where integral on the right-hand-side of the equality is understood in the the Lebesgue sense. Since the Lebesgue integral does not change if one modifies the integrand on a set of zero measure, thanks to (5.32) we have

$$\bar{k}(t) = \int_{\bar{t}}^{t} [\bar{s}(\tau)\phi(\bar{k}(\tau)) - \sigma\bar{k}(\tau)]d\tau. \tag{5.34}$$

As the integrand of the last integral is a continuous function on $(\tau_i, \tau_{i+1})$, the integration in the Lebesgue sense coincides with that in the Riemanian sense, (5.34) proves our claim that the derivative $\dot{\bar{k}}(t)$ exists for every $t \in (\tau_i, \tau_{i+1})$. Moreover, taking derivative of both sides of the equality (5.33) yields

$$\dot{\bar{k}}(t) = \bar{s}(t)\phi(\bar{k}(t)) - \sigma\bar{k}(t), \quad \forall t \in (\tau_i, \tau_{i+1}). \tag{5.35}$$

So, the function $\bar{k}(\cdot)$ is continuously differentiable of $(\tau_i, \tau_{i+1})$. In addition, the relation (5.35) and the existence of the finite one-sided limit $\lim\limits_{t\to\tau_i^+} \bar{s}(t)$ (resp., $\lim\limits_{t\to\tau_i^-} \bar{s}(t)$) for each $i \in \{0, 1, \ldots m-1\}$ (resp., for each $i \in \{1, \ldots m\}$) implies that the one-sided limit $\lim\limits_{t\to\tau_i^+} \dot{\bar{k}}(t)$ (resp., $\lim\limits_{t\to\tau_i^-} \dot{\bar{k}}(t)$) is finite for each $i \in \{0, 1, \ldots m-1\}$ (resp., for each $i \in \{1, \ldots m\}$. Thus, the restriction of $\bar{k}(\cdot)$ on each segment $[\tau_i, \tau_{i+1}]$, $i = 0, \ldots, m-1$, is a continuously differentiable function. We have shown that the capital-to-labor ratio $\bar{k}(t)$ is a continuous, piecewise continuously differentiable function on the segment $[t_0, T]$.

We omit the proof of the Lipschitz property of on $[t_0, T]$ of $\bar{k}(\cdot)$, which follows easily from the continuity and piecewise continuously differentiablity of the function by using the classical mean value theorem. □

We conclude this section by two open questions and three independent conjectures, whose solutions or partial solutions will reveal more the beauty of the optimal economic growth model $(GP)$.

**Open question 1:** *The assumptions of Theorem 5.2 are not enough to guarantee that $(GP)$ has a regular global solution?*

**Open question 2:** *The assumptions of Theorem 5.3 are enough to guarantee that every global solution of $(GP)$ is a regular one?*

**Conjectures:** *The assumptions of Theorem 5.4 guarantee that*

(a) *$(GP)$ has a unique global solution*;

(b) *Any global solution of $(GP)$ is a regular one*;

(c) *If $(\bar{k}, \bar{s})$ is a regular global solution of $(GP)$, then the optimal propensity to save function $\bar{s}(\cdot)$ can have at most one discontinuity on the time segment $[t_0, T]$.*

## 5.7 Optimal Processes for a Typical Problem

To apply Theorem 3.1 for finding optimal processes for $(GP_1)$, we have to interpret $(GP_1)$ in the form of the Mayer problem $\mathcal{M}$ in Section 3.2. For doing so, we set $x(t) = (x_1(t), x_2(t))$, where $x_1(t)$ plays the role of $k(t)$ in (5.27)–(5.28) and

$$x_2(t) := -\int_{t_0}^{t} [1 - s(\tau)]^{\beta} x_1^{\alpha\beta}(\tau) e^{-\lambda\tau} d\tau \tag{5.36}$$

for all $t \in [0, T]$. Thus, $(GP_1)$ is equivalent to the following problem:

$$\text{Minimize } \ x_2(T) \tag{5.37}$$

over $x = (x_1, x_2) \in W^{1,1}([t_0, T], \mathbb{R}^2)$ and measurable functions $s : [t_0, T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{x}_1(t) = A x_1^{\alpha}(t) s(t) - \sigma x_1(t), & \text{a.e. } t \in [t_0, T] \\ \dot{x}_2(t) = -[1 - s(t)]^{\beta} x_1^{\alpha\beta}(t) e^{-\lambda t}, & \text{a.e. } t \in [t_0, T] \\ (x(t_0), x(T)) \in \{(k_0, 0)\} \times \mathbb{R}^2 & \\ s(t) \in [0, 1], & \text{a.e. } t \in [t_0, T] \\ x_1(t) \geq 0, & \forall t \in [t_0, T]. \end{cases} \tag{5.38}$$

The optimal control problem in (5.37)–(5.38) is denoted by $(GP_{1a})$.

To see $(GP_{1a})$ in the form of $\mathcal{M}$, we choose $n = m = 1$, $C = \{(k_0, 0)\} \times \mathbb{R}^2$, $U(t) = [0, 1]$ for all $t \in [t_0, T]$, $g(x, y) = y_2$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$, $h(t, x) = -x_1$ for every $(t, x) \in [t_0, T] \times \mathbb{R}^2$. When it comes to the function $f$, for any $(t, x, s) \in [t_0, T] \times \mathbb{R}^2 \times \mathbb{R}$, one lets $f(t, x, s) = (A x_1^{\alpha} s - \sigma x_1, -(1 - s)^{\beta} x_1^{\alpha\beta} e^{-\lambda t})$ if $x_1 \geq 0$ and $s \in [0, 1]$, and defines $f(t, x, s)$ in a suitable way if $x_1 \notin \mathbb{R}_+$, or $s \notin [0, 1]$.

Let $(\bar{x}, \bar{s})$ be a $W^{1,1}$ local minimizer for $(GP_{1a})$. To satisfy the assumption (H1) in Theorem 3.1, for any $s \in [0, 1]$, the function $f(t, \cdot, s)$ must be locally Lipschitz around $\bar{x}(t)$ for almost every $t \in [t_0, T]$. This requirement cannot be satisfied if $\alpha \in (0, 1)$ and the set of $t \in [t_0, T]$ when the curve $\bar{x}_1(t)$ hits the lower bound $x_1 = 0$ of the state constraint $x_1(t) \geq 0$ has a positive measure. To overcome this situation, we may use one of the following two additional assumptions:

(A1) $\alpha = 1$;

(A2) $\alpha \in (0, 1)$ *and the set* $\{t \in [t_0, T] \ : \ \bar{x}_1(t) = 0\}$ *has the Lebesgue measure 0*, i.e., $\bar{x}_1(t) > 0$ for almost every $t \in [t_0, T]$.

Regarding the exponent $\beta \in (0,1]$ in the formula of $\omega(\cdot)$, we distinguish two cases:

(B1) $\beta = 1$;

(B2) $\beta \in (0,1)$.

***From now on, we will consider problem*** $(GP_{1a})$ ***under the conditions*** (A1) ***and*** (B1). Thanks to these assumptions, we have

$$f(t,x,s) = (Ax_1^\alpha s - \sigma x_1, -(1-s)^\beta x_1^{\alpha\beta} e^{-\lambda t}) = ((As-\sigma)x_1, (s-1)x_1 e^{-\lambda t})$$

if $x_1 \in \mathbb{R}_+$ and $s \in [0,1]$. Clearly, the most natural extension of the function $f$ from the domain $[t_0,T] \times \mathbb{R}_+ \times \mathbb{R} \times [0,1]$ to $[t_0,T] \times \mathbb{R}^2 \times \mathbb{R}$, which is the domain of variables required by Theorem 3.1, is as follows:

$$f(t,x,s) = ((As-\sigma)x_1, (s-1)x_1 e^{-\lambda t}), \quad \forall (t,x,s) \in [t_0,T] \times \mathbb{R}^2 \times \mathbb{R}. \tag{5.39}$$

In accordance with (3.9) and (5.39), the Hamiltonian of $(GP_{1a})$ is given by

$$\mathcal{H}(t,x,p,s) = (As-\sigma)x_1 p_1 + (s-1)x_1 e^{-\lambda t} p_2 \tag{5.40}$$

for every $(t,x,p,s) \in [t_0,T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$. Since the function in (5.40) is continuously differentiable in $x$, we have

$$\partial_x \mathcal{H}(t,x,p,u) = \big\{((As-\sigma)p_1 + (s-1)e^{-\lambda t}p_2, 0)\big\} \tag{5.41}$$

for all $(t,x,p,s) \in [t_0,T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$. By (3.10), the partial hybrid subdifferential of $h$ at $(t,x) \in [t_0,T] \times \mathbb{R}^2$ is given by

$$\partial_x^> h(t,x) = \begin{cases} \emptyset, & \text{if } x_1 > 0 \\ \{(-1,0)\}, & \text{if } x_1 \le 0. \end{cases} \tag{5.42}$$

The relationships between a control function $s(\cdot)$ and the corresponding trajectory $x(\cdot)$ of (5.38) can be described as follows.

**Lemma 5.1** *For each measurable function* $s : [t_0,T] \to \mathbb{R}$ *with* $s(t) \in [0,1]$*, there exists a unique trajectory* $x = (x_1,x_2) \in W^{1,1}([t_0,T],\mathbb{R}^2)$ *such that* $(x,s)$ *is a feasible process of* (5.38). *Moreover, for every* $\tau \in [t_0,T]$*, one has*

$$x_1(t) = x_1(\tau) e^{\int_\tau^t (As(z)-\sigma)dz}, \quad \forall t \in [t_0,T]. \tag{5.43}$$

*In particular,* $x_1(t) > 0$ *for all* $t \in [t_0,T]$.

**Proof.** Given a function $s$ satisfying the assumptions of the proposition, we suppose that $x = (x_1,x_2) \in W^{1,1}([t_0,T],\mathbb{R}^2)$ such that $(x,s)$ is a feasible process of (5.38). Then, the condition $\alpha = 1$ implies that

$$\begin{cases} \dot{x}_1(t) = [As(t) - \sigma]x_1(t), & \text{a.e. } t \in [t_0, T] \\ x_1(t_0) = k_0. \end{cases} \tag{5.44}$$

As $s(\cdot)$ is measurable and bounded on $[t_0, T]$, so is the function $t \mapsto As(t) - \sigma$. In particular, the latter is Lebesgue integrable on $[t_0, T]$. Hence, by the lemma in [2, pp. 121–122] on the solution existence and uniqueness of the Cauchy problem for linear differential equations, one knows that (5.44) has a unique solution. Thus, $x_1(\cdot)$ is defined uniquely via $s(\cdot)$. This and the equality $x_2(t) = -\displaystyle\int_{t_0}^{t} [1 - s(\tau)]x_1(\tau)e^{-\lambda\tau}d\tau$, which follows from (5.36) together with the conditions $\alpha = 1$ and $\beta = 1$, imply the uniqueness of $x_2(\cdot)$. To prove the second assertion, put

$$\Omega(t, \tau) = e^{\int_\tau^t (As(z) - \sigma)dz}, \quad \forall t, \tau \in [t_0, T]. \tag{5.45}$$

By the Lebesgue integrability of the function $t \mapsto As(t) - \sigma$ on $[t_0, T]$, $\Omega(t, \tau)$ is well defined on $[t_0, T] \times [t_0, T]$, and by [49, Theorem 8, p. 324] one has

$$\frac{d}{dt}\left(\int_\tau^t (As(z) - \sigma)dz\right) = As(t) - \sigma, \quad \text{a.e. } t \in [t_0, T]. \tag{5.46}$$

Therefore, from (5.45) and (5.46) it follows that $\Omega(\cdot, \tau)$ is the solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\Omega(t, \tau) = (As(t) - \sigma)\Omega(t, \tau), & \text{a.e. } t \in [t_0, T] \\ \Omega(\tau, \tau) = 1. \end{cases}$$

In other words, the real-valued function $\Omega(t, \tau)$ of the variables $t$ and $\tau$ is the *principal matrix solution* (see [2, p. 123]) specialized to the homogeneous differential equation in (5.44). Hence, by the theorem in [2, p. 123] on the solution of linear differential equations, we obtain (5.43). As $x_1(t_0) = k_0 > 0$, applying (5.43) for $\tau = t_0$ implies that $x_1(t) > 0$ for all $t \in [t_0, T]$. □

The next two remarks are aimed at clarifying the tool used to solve $(GP_{1a})$.

**Remark 5.3** By Lemma 5.1, any process satisfying the first four conditions in (5.38) automatically satisfies the state constraint $x_1(t) \geq 0$ for all $t \in [t_0, T]$. Thus, the latter can be omitted in the problem formulation. This means that, for the case $\alpha = 1$, instead of the maximum principle in Theorem 3.1 for problems with state constraints one can apply the one in

Proposition 3.1 for problems without state constraints. Note that both Theorem 3.1 and Proposition 3.1 yield the same necessary optimality conditions in such a situation (see Section 3.4).

**Remark 5.4** For the case $\alpha \in (0,1)$, one cannot claim that any process satisfying the first four conditions in (5.38) automatically satisfies the state constraint $x_1(t) \geq 0$ for all $t \in [t_0, T]$. Thus, if we consider problem $(GP_{1a})$ under the conditions (A2) and (B1), or (A2) and (B2), then we have to rely on Theorem 3.1. Referring to the classification of optimal economic growth models given in Remark 5.2, we can say that models of the types "Nonlinear-linear" and "Nonlinear-nonlinear" may require the use of Theorem 3.1. For this reason, we prefer to present the latter in this paper to prepare a suitable framework for dealing with $(GP_{1a})$ under different sets of assumptions.

*Recall that $(\bar{x}, \bar{s})$ is a $W^{1,1}$ local minimizer for $(GP_{1a})$.* It is easy to show that, for any $\delta > 0$, there are constants $M_1 > 0$ and $M_2 > 0$ such that $k(t,x) := M_1 + M_2 e^{-\lambda t}$ satisfies the conditions described in the hypothesis (H1) of Theorem 3.1. The fulfillment of the hypotheses (H2)–(H4) is obvious. Applying Theorem 3.1, we can find $p \in W^{1,1}([t_0,T]; \mathbb{R}^2)$, $\gamma \geq 0$, $\mu \in C^{\oplus}(t_0,T)$, and a Borel measurable function $\nu : [t_0,T] \to \mathbb{R}^2$ such that $(p, \mu, \gamma) \neq (0,0,0)$, and for $q(t) := p(t) + \eta(t)$ with

$$\eta(t) := \int_{[t_0,t)} \nu(\tau) d\mu(\tau), \qquad t \in [t_0, T) \tag{5.47}$$

and

$$\eta(T) := \int_{[t_0,T]} \nu(\tau) d\mu(\tau), \tag{5.48}$$

conditions (i)–(iv) in Theorem 3.1 hold true.

Let us expose the meanings of the conditions (i)–(iv) in Theorem 3.1.

**Condition (i)**: Note that

$$\begin{aligned} &\mu\{t \in [t_0,T] \,:\, \nu(t) \notin \partial_x^{>} h(t, \bar{x}(t))\} \\ &= \mu\{t \in [t_0,T] \,:\, \partial_x^{>} h(t, \bar{x}(t)) = \emptyset\} \\ &\quad + \mu\{t \in [t_0,T] \,:\, \partial_x^{>} h(t, \bar{x}(t)) \neq \emptyset,\ \nu(t) \notin \partial_x^{>} h(t, \bar{x}(t))\}. \end{aligned}$$

Since $\bar{x}_1(t) \geq 0$ for every $t$, combining this with (5.42) gives

$$\begin{aligned} &\mu\{t \in [t_0,T] \,:\, \nu(t) \notin \partial_x^{>} h(t, \bar{x}(t))\} \\ &= \mu\{t \in [t_0,T] \,:\, \bar{x}_1(t) > 0\} + \mu\{t \in [t_0,T] \,:\, \bar{x}_1(t) = 0,\ \nu(t) \neq (-1,0)\}. \end{aligned}$$

So, from (i) it follows that

$$\mu\{t \in [t_0, T] \,:\, \bar{x}_1(t) > 0\} = 0 \tag{5.49}$$

and $\mu\{t \in [t_0, T] \,:\, \bar{x}_1(t) = 0,\ \nu(t) \neq (-1, 0)\} = 0$.

**Condition (ii)**: By (5.41), (ii) implies that

$$-\dot{p}(t) = ((A\bar{s}(t) - \sigma)q_1(t) + (\bar{s}(t) - 1)e^{-\lambda t}q_2(t), 0), \quad \text{a.e. } t \in [t_0, T].$$

Hence, $p_2(t)$ is a constant for all $t \in [t_0, T]$ and

$$\dot{p}_1(t) = -(A\bar{s}(t) - \sigma)q_1(t) + (1 - \bar{s}(t))e^{-\lambda t}q_2(t), \quad \text{a.e. } t \in [t_0, T].$$

**Condition (iii)**: Using the formulas for $g$ and $C$, we can show that $\partial g(\bar{x}(t_0), \bar{x}(T)) = \{(0, 0, 0, 1)\}$ and $N((\bar{x}(t_0), \bar{x}(T)); C) = \mathbb{R}^2 \times \{(0, 0)\}$. So, (iii) yields

$$(p(t_0), -q(T)) \in \{(0, 0, 0, \gamma)\} + \mathbb{R}^2 \times \{(0, 0)\},$$

which means that $q_1(T) = 0$ and $q_2(T) = -\gamma$.

**Condition (iv)**: By (5.40), from (iv) one gets

$$\begin{aligned}(A\bar{s}(t) - \sigma)\bar{x}_1(t)q_1(t) \;+\; & (\bar{s}(t) - 1)\bar{x}_1(t)e^{-\lambda t}q_2(t) \\ & = \max_{s \in [0,1]} \left\{(As - \sigma)\bar{x}_1(t)q_1(t) + (s - 1)\bar{x}_1(t)e^{-\lambda t}q_2(t)\right\}\end{aligned}$$

for almost every $t \in [t_0, T]$. Equivalently, we have

$$\big(Aq_1(t) + e^{-\lambda t}q_2(t)\big)\bar{x}_1(t)\bar{s}(t) = \max_{s \in [0,1]} \left\{\big(Aq_1(t) + e^{-\lambda t}q_2(t)\big)\bar{x}_1(t)s\right\}, \text{ a.e. } t \in [t_0, T].$$

Since $\bar{x}_1(t) > 0$ for all $t \in [t_0, T]$, it follows that

$$\big(Aq_1(t) + e^{-\lambda t}q_2(t)\big)\bar{s}(t) = \max_{s \in [0,1]} \left\{\big(Aq_1(t) + e^{-\lambda t}q_2(t)\big)s\right\}, \text{ a.e. } t \in [t_0, T]. \tag{5.50}$$

To prove that the optimal control problem in question has a unique optimal solution under a mild condition imposed on the data tube $(A, \sigma, \lambda)$, we have to deepen the above analysis of the conditions (i)–(iv). As $\bar{x}_1(t) > 0$ for all $t \in [t_0, T]$ by Lemma 5.1, the equality (5.49) implies that $\mu([t_0, T]) = 0$, i.e., $\mu = 0$. Combining this with (5.47) and (5.48), one gets $\eta(t) = 0$ for all $t \in [t_0, T]$. Thus, the relation $q(t) = p(t) + \eta(t)$ allows us to have $q(t) = p(t)$ for every $t \in [t_0, T]$. Therefore, the properties of $p(t)$ and $q(t)$ established in the above analysis of the conditions (ii) and (iii) imply that $p_2(t) = -\gamma$ for every $t \in [t_0, T]$, $p_1(T) = 0$, and

$$\dot{p}_1(t) = -(A\bar{s}(t) - \sigma)p_1(t) + \gamma(\bar{s}(t) - 1)e^{-\lambda t}, \quad \text{a.e. } t \in [t_0, T]. \tag{5.51}$$

Now, by substituting $q_1(t) = p_1(t)$ and $q_2(t) = -\gamma$ into (5.50), we have

$$\left(Ap_1(t) - \gamma e^{-\lambda t}\right)\bar{s}(t) = \max_{s\in[0,1]} \left\{\left(Ap_1(t) - \gamma e^{-\lambda t}\right)s\right\}, \quad \text{a.e. } t \in [t_0, T]. \quad (5.52)$$

Describing the adjoint trajectory $p$ corresponding to $(\bar{x}, \bar{s})$ in (5.51), the next lemma is an analogue of Lemma 5.1.

**Lemma 5.2** *The Cauchy problem defined by the differential equation* (5.51) *and the condition* $p_1(T) = 0$ *possesses a unique solution* $p_1(\cdot) : [t_0, T] \to \mathbb{R}$*,*

$$p_1(t) = -\int_t^T c(z)\bar{\Omega}(z,t)dz, \quad \forall t \in [t_0, T], \quad (5.53)$$

*where* $\bar{\Omega}(t,\tau)$ *is defined by* (5.45) *for* $s(t) = \bar{s}(t)$*, i.e.,*

$$\bar{\Omega}(t,\tau) := e^{\int_\tau^t (A\bar{s}(z)-\sigma)dz}, \quad t, \tau \in [t_0, T], \quad (5.54)$$

*and*

$$c(t) := \gamma(\bar{s}(t) - 1)e^{-\lambda t}, \quad t \in [t_0, T]. \quad (5.55)$$

*In addition, for any fixed value* $\tau \in [t_0, T]$*, one has*

$$p_1(t) = p_1(\tau)\bar{\Omega}(\tau, t) - \int_t^\tau c(z)\bar{\Omega}(z,t)dz, \quad \forall t \in [t_0, T]. \quad (5.56)$$

**Proof.** Since $\bar{s}(\cdot)$ is measurable and bounded, the function $t \mapsto c(t)$ defined by (5.55) is also measurable and bounded on $[t_0, T]$. Moreover, the function $t \mapsto A\bar{s}(t) - \sigma$ is also measurable and bounded on $[t_0, T]$. In particular, both functions $c(\cdot)$ and $A\bar{s}(\cdot) - \sigma$ are Lebesgue integrable on $[t_0, T]$. Hence, by the lemma in [2, pp. 121–122] we can assert that, for any $\tau \in [t_0, T]$ and $\eta \in \mathbb{R}$, the Cauchy problem defined by the linear differential equation (5.51) and the initial condition $p_1(\tau) = \eta$ has a unique solution $p_1(\cdot) : [t_0, T] \to \mathbb{R}$. As shown in the proof of Lemma 5.1, $\bar{\Omega}(t,\tau)$ given in (5.54) is the principal solution of the homogeneous equation

$$\dot{\bar{x}}_1(t) = (A\bar{s}(t) - \sigma)\bar{x}_1(t), \quad \text{a.e. } t \in [t_0, T].$$

Besides, by the form of (5.51) and by the theorem in [2, p. 123], the solution of (5.51) is given by (5.56). Especially, applying this formula for the case $\tau = T$ and note that $p_1(T) = 0$, we obtain (5.53). □

In Theorem 3.1, the objective function $g$ plays a role in condition (iii) only if $\gamma > 0$. In such a situation, the maximum principle is said to be *normal*. Investigations on the normality of maximum principles for optimal control problems are available in [27–29]. For the problem $(GP_{1a})$, by using (5.53)–(5.55) and the property $(p, \mu, \gamma) \neq (0, 0, 0)$, we now show that the situation $\gamma = 0$ cannot happen.

**Lemma 5.3** *One must have $\gamma > 0$.*

**Proof.** Suppose on the contrary that $\gamma = 0$. Then, $c(t) \equiv 0$ by (5.55). Hence, from (5.53) it follows that $p_1(t) \equiv 0$. Combining this with the facts that $p_2(t) = -\gamma = 0$ for all $t \in [t_0, T]$ and $\mu = 0$, we get a contradiction to the requirement $(p, \mu, \gamma) \neq (0, 0, 0)$ in Theorem 3.1. □

In accordance with (5.52), to define the control value $\bar{s}(t)$, it is important to know the sign of the real-valued function

$$\psi(t) := Ap_1(t) - \gamma e^{-\lambda t} \tag{5.57}$$

for each $t \in [t_0, T]$. Namely, one has $\bar{s}(t) = 1$ whenever $\psi(t) > 0$ and $\bar{s}(t) = 0$ whenever $\psi(t) < 0$. Hence $\bar{s}(\cdot)$ is a constant function on each segment where $\psi(\cdot)$ has a fixed sign. The forthcoming lemma gives formulas for $\bar{x}_1(\cdot)$ and $p_1(\cdot)$ on such a segment.

**Lemma 5.4** *Let $[t_1, t_2] \subset [t_0, T]$ and $\tau \in [t_1, t_2]$ be given arbitrarily.*

(a) *If $\bar{s}(t) = 1$ for a.e. $t \in [t_1, t_2]$, then*

$$\bar{x}_1(t) = \bar{x}_1(\tau)e^{(A-\sigma)(t-\tau)}, \quad \forall t \in [t_1, t_2] \tag{5.58}$$

*and*

$$p_1(t) = p_1(\tau)e^{-(A-\sigma)(t-\tau)}, \quad \forall t \in [t_1, t_2]. \tag{5.59}$$

(b) *If $\bar{s}(t) = 0$ for a.e. $t \in [t_1, t_2]$, then*

$$\bar{x}_1(t) = \bar{x}_1(\tau)e^{-\sigma(t-\tau)}, \quad \forall t \in [t_1, t_2] \tag{5.60}$$

*and*

$$p_1(t) = p_1(\tau)e^{\sigma(t-\tau)} + \frac{\gamma}{\sigma+\lambda}e^{\sigma t}\big[e^{-(\sigma+\lambda)t} - e^{-(\sigma+\lambda)\tau}\big], \quad \forall t \in [t_1, t_2]. \tag{5.61}$$

**Proof.** If $\bar{s}(t) = 1$ for a.e. $t \in [t_1, t_2]$, then (5.58) is obtained from (5.43) with $x_1(\cdot) = \bar{x}_1(\cdot)$ and $s(\cdot) = \bar{s}(\cdot)$. Besides, as $\bar{s}(\cdot) \equiv 1$ a.e. on $[t_1, t_2]$, the function $c(t)$ defined in (5.55) equals 0 a.e. on $[t_1, t_2]$, which implies that the integral in (5.56) vanishes. In addition, substituting the formulas for $\bar{s}(\cdot)$ and

$\bar{x}_1(\cdot)$ on $[t_1, t_2]$ to (5.54), we get $\bar{\Omega}(\tau, t) = e^{-(A-\sigma)(t-\tau)}$ for all $t \in [t_1, t_2]$. Thus, (5.59) follows from (5.56).

If $\bar{s}(t) = 0$ for a.e. $t \in [t_1, t_2]$, then we get (5.60) by applying (5.43) with $x_1(\cdot) = \bar{x}_1(\cdot)$ and $s(\cdot) = \bar{s}(\cdot)$. To prove (5.61), we use (5.56) and the formulas for $\bar{s}(\cdot)$ and $\bar{x}_1(\cdot)$ on $[t_1, t_2]$. Namely, we have $\bar{\Omega}(\tau, t) = e^{\sigma(t-\tau)}$, $\bar{\Omega}(z, t) = e^{\sigma(t-z)}$, and $c(z) = -\gamma e^{-\lambda z}$ for all $t, z \in [t_1, t_2]$. Substituting these formulas to (5.56) yields

$$\begin{aligned} p_1(t) &= p_1(\tau)e^{\sigma(t-\tau)} - \int_t^\tau (-\gamma e^{-\lambda z})(e^{\sigma(t-z)})dz \\ &= p_1(\tau)e^{\sigma(t-\tau)} + \gamma e^{\sigma t} \int_t^\tau e^{-(\sigma+\lambda)z}dz \\ &= p_1(\tau)e^{\sigma(t-\tau)} - \frac{\gamma}{\sigma+\lambda}e^{\sigma t}\big[e^{-(\sigma+\lambda)\tau} - e^{-(\sigma+\lambda)t}\big] \end{aligned}$$

for all $t \in [t_1, t_2]$. This shows that (5.61) is valid. □

For any $t \in [t_0, T]$, if $\psi(t) = 0$, then (5.52) holds automatically no matter what $\bar{s}(t)$ is. Thus, by (5.52) we can assert nothing about the control function $\bar{s}(\cdot)$ at this $t$. Motivated by this observation, we consider the set

$$\Gamma = \{t \in [t_0, T] : \psi(t) = 0\}.$$

As the functions $p_1(\cdot)$ is absolutely continuous on $[t_0, T]$, so is $\psi(\cdot)$. It follows that $\Gamma$ is a compact set. Besides, since $p_1(T) = 0$ and $\gamma > 0$, the equality $\psi(T) = Ap_1(T) - \gamma e^{-\lambda T}$ implies that $\psi(T) < 0$. Thus, $T \notin \Gamma$.

First, consider the situation where $\Gamma = \emptyset$. Then we have $\psi(t) < 0$ on the whole segment $[t_0, T]$. Indeed, otherwise we would find a point $\tau \in [t_0, T)$ such that $\psi(\tau) > 0$. Since $\psi(\tau)\psi(T) < 0$, by the continuity of $\psi(\cdot)$ on $[t_0, T]$ we can assert that $\Gamma \cap (\tau, T) \neq \emptyset$. This contradicts our assumption that $\Gamma = \emptyset$. Now, as $\psi(t) < 0$ for all $t \in [t_0, T]$, from (5.52) we have $\bar{s}(t) = 0$ for a.e. $t \in [t_0, T]$. Applying Lemma 5.4 for $t_1 = t_0$, $t_2 = T$, and $\tau = t_0$, we get $\bar{x}_1(t) = k_0 e^{-\sigma(t-t_0)}$ for all $t \in [t_0, T]$.

Now, consider the situation where $\Gamma \neq \emptyset$. Let

$$\alpha_1 := \min\{t \,:\, t \in \Gamma\} \quad \text{and} \quad \alpha_2 := \max\{t \,:\, t \in \Gamma\}. \tag{5.62}$$

Since $\psi(T) < 0$, we see that $t_0 \leq \alpha_1 \leq \alpha_2 < T$. Moreover, by the continuity of $\psi(\cdot)$, and the fact that $\psi(T) < 0$, we have $\psi(t) < 0$ for every $t \in (\alpha_2, T]$. This and (5.52) imply that $\bar{s}(t) = 0$ for almost every $t \in [\alpha_2, T]$. Invoking

Lemma 5.4 for $t_1 = \alpha_2$, $t_2 = T$, and $\tau = \alpha_2$, we obtain $\bar{x}_1(t) = \bar{x}_1(\alpha_2)e^{-\sigma(t-\alpha_2)}$ for all $t \in [\alpha_2, T]$. If $t_0 < \alpha_1$, then to find $\bar{s}(\cdot)$ and $\bar{x}_1(\cdot)$ on $[t_0, \alpha_1]$, we will use the following observation.

**Lemma 5.5** *Suppose that $t_0 < \alpha_1$. If $\psi(t_0) < 0$, then $\bar{s}(t) = 0$ for a.e. $t \in [t_0, \alpha_1]$ and $\bar{x}_1(t) = k_0 e^{-\sigma(t-t_0)}$ for all $t \in [t_0, \alpha_1]$. If $\psi(t_0) > 0$, then $\bar{s}(t) = 1$ for a.e. $t \in [t_0, \alpha_1]$ and $\bar{x}_1(t) = k_0 e^{(A-\sigma)(t-t_0)}$ for all $t \in [t_0, \alpha_1]$.*

**Proof.** As $t_0 < \alpha_1$, one has $\psi(t_0)\psi(t) > 0$ for every $t \in [t_0, \alpha_1)$. Indeed, otherwise there is some $\tau \in (t_0, \alpha_1)$ satisfying $\psi(t_0)\psi(\tau) < 0$, which together with the continuity of $\psi(\cdot)$ implies that there is some $\bar{t} \in \Gamma$ with $\bar{t} < \alpha_1$. This contradicts the definition of $\alpha_1$. If $\psi(t_0) < 0$, then $\psi(t) < 0$ for all $t \in [t_0, \alpha_1)$. Hence, by (5.52), $\bar{s}(t) = 0$ for a.e. $t \in [t_0, \alpha_1]$. If $\psi(t_0) > 0$, then $\psi(t) > 0$ for all $t \in [t_0, \alpha_1)$. In this situation, by (5.52) we have $\bar{s}(t) = 1$ for a.e. $t \in [t_0, \alpha_1]$. Thus, in both situations, applying Lemma 5.4 for $t_1 = t_0$, $t_2 = \alpha_1$, and $\tau = t_0$, we obtain the desired formulas for $\bar{x}_1(\cdot)$ on $[t_0, \alpha_1]$. □

If $\alpha_1 \neq \alpha_2$, then we must have a complete understanding of the behavior of the function $\psi(t)$ on the whole interval $[\alpha_1, \alpha_2]$. Towards that aim, we are going to establish three lemmas.

**Lemma 5.6** *There does not exist any subinterval $[t_1, t_2]$ of $[t_0, T]$ with $t_1 < t_2$ such that $\psi(t_1) = \psi(t_2) = 0$, and $\psi(t) > 0$ for every $t \in (t_1, t_2)$.*

**Proof.** On the contrary, suppose that there is a subinterval $[t_1, t_2]$ of $[t_0, T]$ with $t_1 < t_2$ such that $\psi(t) > 0$ for all $t \in (t_1, t_2)$ and $\psi(t_1) = \psi(t_2) = 0$. Then, by (5.52) we have $\bar{s}(t) = 1$ almost everywhere on $[t_1, t_2]$. So, using claim (a) in Lemma 5.4 with $\tau = t_1$, we have $p_1(t) = p_1(t_1)e^{-(A-\sigma)(t-t_1)}$ for all $t \in [t_1, t_2]$. The condition $\psi(t_1) = 0$ implies that $p_1(t_1) = \dfrac{\gamma}{A}e^{-\lambda t_1}$. Thus, $p_1(t) = \dfrac{\gamma}{A}e^{-\lambda t_1}e^{-(A-\sigma)(t-t_1)}$ for all $t \in [t_1, t_2]$. As $\gamma e^{-\lambda t} > 0$ for all $t \in [t_0, T]$, the function $\psi_1(t) := \dfrac{\psi(t)}{\gamma e^{-\lambda t}}$ is well defined on $[t_1, t_2]$. By the definition of $\psi(\cdot)$, the above formulas for $\bar{x}_1(\cdot)$ and $p_1(\cdot)$ on $[t_1, t_2]$, we have

$$\psi_1(t) = \frac{Ap_1(t)}{\gamma e^{-\lambda t}} - 1 = \frac{\gamma e^{-\lambda t_1}e^{-(A-\sigma)(t-t_1)}}{\gamma e^{-\lambda t}} - 1 = e^{(\sigma+\lambda-A)(t-t_1)} - 1$$

for all $t \in [t_1, t_2]$. If $\sigma + \lambda - A \neq 0$, then it is easy to see that the equation $\psi_1(t) = 0$ has a unique solution $t_1$ on $[t_1, t_2]$. Hence $\psi(t_2) \neq 0$, and we have arrived at a contradiction. If $\sigma+\lambda-A = 0$, then $\psi_1(t) = 0$ for every $t \in (t_1, t_2)$.

This implies that $\psi(t) = 0$ for every $t \in (t_1, t_2)$. The latter contradicts our assumption on $\psi(t)$.

The proof is complete. $\square$

**Lemma 5.7** *There does not exist a subinterval $[t_1, t_2]$ of $[t_0, T]$ with $t_1 < t_2$ such that $\psi(t_1) = \psi(t_2) = 0$ and $\psi(t) < 0$ for all $t \in (t_1, t_2)$.*

**Proof.** To argue by contradiction, suppose that there is a subinterval $[t_1, t_2]$ of $[t_0, T]$ with $t_1 < t_2$, $\psi(t) < 0$ for all $t \in (t_1, t_2)$, and $\psi(t_1) = \psi(t_2) = 0$. Then, by (5.52) we have $\bar{s}(t) = 0$ almost everywhere on $[t_1, t_2]$. Therefore, using claim (b) in Lemma 5.4 with $\tau = t_1$, we obtain

$$p_1(t) = p_1(t_1)e^{\sigma(t-t_1)} + \frac{\gamma}{\sigma+\lambda}e^{\sigma t}\big[e^{-(\sigma+\lambda)t} - e^{-(\sigma+\lambda)t_1}\big], \quad \forall t \in [t_1, t_2].$$

The assumption $\psi(t_1) = 0$ yields $p_1(t_1) = \dfrac{\gamma}{A}e^{-\lambda t_1}$. Thus,

$$p_1(t) = \frac{\gamma}{A}e^{-\lambda t_1}e^{\sigma(t-t_1)} + \frac{\gamma}{\sigma+\lambda}e^{\sigma t}\big[e^{-(\sigma+\lambda)t} - e^{-(\sigma+\lambda)t_1}\big], \quad \forall t \in [t_1, t_2].$$

By the definition of $\psi(\cdot)$ and the formulas for $\bar{x}_1(\cdot)$ and $p_1(\cdot)$ on $[t_1, t_2]$, we have

$$\psi(t) = \gamma e^{-\lambda t_1}e^{\sigma(t-t_1)} + \frac{A\gamma}{\sigma+\lambda}e^{\sigma t}\big[e^{-(\sigma+\lambda)t} - e^{-(\sigma+\lambda)t_1}\big] - \gamma e^{-\lambda t}, \quad \forall t \in [t_1, t_2].$$

Consider the function $\psi_2(t) := \dfrac{\psi(t)}{\gamma e^{\sigma t}}$, which is well defined for every $t \in [t_1, t_2]$. Then, by an elementary calculation one has

$$\psi_2(t) = \left(\frac{A}{\sigma+\lambda} - 1\right)\big[e^{-(\sigma+\lambda)t} - e^{-(\sigma+\lambda)t_1}\big], \quad \forall t \in [t_1, t_2]. \tag{5.63}$$

If $\dfrac{A}{\sigma+\lambda} - 1 = 0$, then $\psi_2(t) = 0$ for all $t \in [t_1, t_2]$. This yields $\psi(t) = 0$ for all $t \in [t_1, t_2]$, a contradiction to our assumption that $\psi(t) < 0$ for all $t \in (t_1, t_2)$. If $\dfrac{A}{\sigma+\lambda} - 1 \neq 0$, then by (5.63) one can assert that $\psi_2(t) = 0$ if and only if $t = t_1$. Equivalently, $\psi(t) = 0$ if and only if $t = t_1$. The latter contradicts the conditions $\psi(t_2) = 0$ and $t_2 \neq t_1$. $\square$

**Lemma 5.8** *If the condition*

$$A \neq \sigma + \lambda \tag{5.64}$$

*is fulfilled, then we cannot have $\psi(t) = 0$ for all $t$ from an open subinterval $(t_1, t_2)$ of $[t_0, T]$ with $t_1 < t_2$.*

**Proof.** Suppose that (5.64) is valid. If the claim is false, then we would find $t_1, t_2 \in [t_0, T]$ with $t_1 < t_2$ such that $\psi(t) = 0$ for $t \in (t_1, t_2)$. So, from (5.57) it follows that

$$p_1(t) = \frac{\gamma}{A} e^{-\lambda t}, \quad \forall t \in (t_1, t_2). \tag{5.65}$$

Therefore, one has $\dot{p}_1(t) = -\dfrac{\lambda\gamma}{A} e^{-\lambda t}$ for almost every $t \in (t_1, t_2)$. This and (5.51) imply that

$$-(A\bar{s}(t) - \sigma)p_1(t) + \gamma(\bar{s}(t) - 1)e^{-\lambda t} = -\frac{\lambda\gamma}{A} e^{-\lambda t}, \quad \text{a.e.}\, t \in (t_1, t_2).$$

Combining this with (5.65) yields

$$-(A\bar{s}(t) - \sigma)\frac{\gamma}{A} e^{-\lambda t} + \gamma(\bar{s}(t) - 1)e^{-\lambda t} = -\frac{\lambda\gamma}{A} e^{-\lambda t}, \quad \text{a.e.}\, t \in (t_1, t_2).$$

for almost every $t \in (t_1, t_2)$. Since $\gamma > 0$, simplifying the last equality yields $A = \sigma + \lambda$. This contradicts a (5.64). $\square$

Under a mild condition, the constants $\alpha_1$ and $\alpha_2$ defined by (5.62) coincide. Namely, the following statement holds true.

**Lemma 5.9** *If* (5.64) *is fulfilled, then the situation $\alpha_1 \neq \alpha_2$ cannot occur.*

**Proof.** Suppose on the contrary that (5.64) is satisfied, but $\alpha_1 \neq \alpha_2$. Then, we cannot have $\psi(t) = 0$ for all $t \in (\alpha_1, \alpha_2)$ by Lemma 5.8. This means that there exists $\bar{t} \in (\alpha_1, \alpha_2)$ such that $\psi(\bar{t}) \neq 0$. Put

$$\bar{\alpha}_1 = \max\{t \in [\alpha_1, \bar{t}] \,:\, \psi(t) = 0\} \quad \text{and} \quad \bar{\alpha}_2 = \min\{t \in [\bar{t}, \alpha_2] \,:\, \psi(t) = 0\}.$$

It is not hard to see that $\psi(\bar{\alpha}_1) = \psi(\bar{\alpha}_2) = 0$ and $\psi(\bar{t})\psi(t) > 0$ for all $t \in (\bar{\alpha}_1, \bar{\alpha}_2)$. This is impossible by either Lemma 5.6 when $\psi(\bar{t}) > 0$ or Lemma 5.7 when $\psi(\bar{t}) < 0$. $\square$

We are now in a position to formulate and prove the main result of this section.

**Theorem 5.5** *Suppose that the assumptions* (A1) *and* (B1) *are satisfied. If*

$$A < \sigma + \lambda, \tag{5.66}$$

*then* $(GP_{1a})$ *has a unique $W^{1,1}$ local minimizer $(\bar{x}, \bar{s})$, which is a global minimizer, where $\bar{s}(t) = 0$ for a.e. $t \in [t_0, T]$ and $\bar{x}_1(t) = k_0 e^{-\sigma(t-t_0)}$ for all $t \in [t_0, T]$. This means that the problem $(GP_1)$ has a unique solution $(\bar{k}, \bar{s})$, where $\bar{s}(t) = 0$ for a.e. $t \in [t_0, T]$ and $\bar{k}(t) = k_0 e^{-\sigma(t-t_0)}$ for all $t \in [t_0, T]$.*

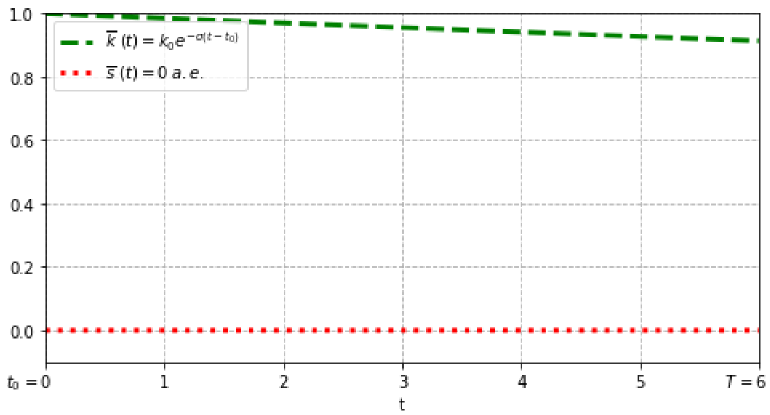


Figure 5.3: The optimal process $(\bar{k}, \bar{s})$ of $(GP_1)$ corresponding to parameters $\alpha = 1$, $\beta = 1$, $A = 0.045$, $\sigma = 0.015$, $\lambda = 0.034$, $k_0 = 1$, $t_0 = 0$, and $T = 6$

**Proof.** Suppose that (A1), (B1), and the condition (5.66) are satisfied. According to Theorem 5.4, $(GP_1)$ has a global solution. Hence $(GP_{1a})$ also has a global solution.

Let $(\bar{x}, \bar{s})$ be a $W^{1,1}$ local minimizer of $(GP_{1a})$. As it has already been explained in this section, applying Theorem 3.1, we can find $p \in W^{1,1}([t_0, T]; \mathbb{R}^2)$, $\gamma \geq 0$, $\mu \in C^{\oplus}(t_0, T)$, and a Borel measurable function $\nu : [t_0, T] \to \mathbb{R}^2$ such that $(p, \mu, \gamma) \neq (0, 0, 0)$ and conditions (i)–(iv) in Theorem 3.1 hold true for $q(t) := p(t) + \eta(t)$ with $\eta(t)$ (resp., $\eta(T)$) being given by (5.47) for $t \in [t_0, T)$ (resp., by (5.48)). In the above notations, we consider the set $\Gamma = \{t \in [t_0, T] : \psi(t) = 0\}$.

In the case $\Gamma = \emptyset$, we have shown that $\bar{s}(t) = 0$ for a.e. $t \in [t_0, T]$ and $\bar{x}_1(t) = k_0 e^{-\sigma(t-t_0)}$ for all $t \in [t_0, T]$ (see the arguments given after Lemma 5.4).

In the case $\Gamma \neq \emptyset$, we define the numbers $\alpha_1$ and $\alpha_2$ by (5.62). Thanks to the condition (5.66), which implies (5.64), by Lemma 5.9 we have $\alpha_2 = \alpha_1$. Then, as it was shown before Lemma 5.5, we must have $\bar{s}(t) = 0$ for a.e. $t \in [\alpha_1, T]$ and $\bar{x}_1(t) = \bar{x}_1(\alpha_1)e^{-\sigma(t-\alpha_1)}$ for all $t \in [\alpha_1, T]$. If $t_0 = \alpha_1$, then we obtain the desired formulas for $\bar{s}(\cdot)$ and $\bar{x}_1(\cdot)$.

Suppose that $t_0 < \alpha_1$. If $\psi(t_0) < 0$, then we can get the desired formulas for $\bar{s}(\cdot)$ and $\bar{x}_1(\cdot)$ on $[t_0, T]$ from the formulas for $\bar{s}(\cdot)$ and $\bar{x}_1(\cdot)$ on $[t_0, \alpha_1]$ in Lemma 5.5 and the just mentioned formulas for $\bar{s}(\cdot)$ and $\bar{x}_1(\cdot)$ on $[\alpha_1, T]$. If $\psi(t_0) > 0$, by Lemma 5.5 one has $\bar{s}(t) = 1$ for a.e. $t \in [t_0, \alpha_1]$. Then we have

$$\bar{s}(t) = \begin{cases} 1, & \text{a.e.} \quad t \in [t_0, \alpha_1] \\ 0, & \text{a.e.} \quad t \in (\alpha_1, T] \end{cases} \quad \text{and} \quad \bar{x}_1(t) = \begin{cases} k_0 e^{(A-\sigma)(t-t_0)}, & t \in [t_0, \alpha_1] \\ \bar{x}_1(\alpha_1) e^{-\sigma(t-\alpha_1)}, & t \in (\alpha_1, T]. \end{cases}$$

To proceed furthermore, fix an arbitrary number $\varepsilon \in (0, \alpha_1 - t_0]$ and put $t_\varepsilon = \alpha_1 - \varepsilon$. Consider the control function $s^\varepsilon(t)$ defined by setting $s_\varepsilon(t) = 1$ for all $t \in [t_0, t_\varepsilon]$ and $s^\varepsilon(t) = 0$ for all $t \in (t_\varepsilon, T]$. Denote the trajectory corresponding to $s^\varepsilon(\cdot)$ by $x^\varepsilon(\cdot)$. Then one has

$$x_1^\varepsilon(t) = \begin{cases} k_0 e^{(A-\sigma)(t-t_0)}, & t \in [t_0, t_\varepsilon] \\ x_1^\varepsilon(t_\varepsilon) e^{-\sigma(t-t_\varepsilon)}, & t \in (t_\varepsilon, T]. \end{cases}$$

Note that

$$\begin{aligned} \bar{x}_2(T) &= -\int_{t_0}^{T} [1 - \bar{s}(\tau)]\bar{x}_1(\tau) e^{-\lambda\tau} d\tau \\ &= -\int_{\alpha_1}^{T} \bar{x}_1(\tau) e^{-\lambda\tau} d\tau \\ &= -\int_{\alpha_1}^{T} \bar{x}_1(\alpha_1) e^{-\sigma(\tau-\alpha_1)} e^{-\lambda\tau} d\tau \\ &= \frac{\bar{x}_1(\alpha_1) e^{\sigma\alpha_1}}{\sigma + \lambda} \big[e^{-(\sigma+\lambda)T} - e^{-(\sigma+\lambda)\alpha_1}\big]. \end{aligned}$$

Since $\bar{x}_1(\alpha_1) = k_0 e^{(A-\sigma)(\alpha_1 - t_0)}$, it follows that

$$\bar{x}_2(T) = \frac{k_0}{\sigma + \lambda} e^{(\sigma - A)t_0} e^{A\alpha_1} \big[e^{-(\sigma+\lambda)T} - e^{-(\sigma+\lambda)\alpha_1}\big].$$

Similarly, one gets

$$x_2^\varepsilon(T) = \frac{k_0}{\sigma + \lambda} e^{(\sigma - A)t_0} e^{At_\varepsilon} \big[e^{-(\sigma+\lambda)T} - e^{-(\sigma+\lambda)t_\varepsilon}\big].$$

Therefore, one gets

$$\begin{aligned} \bar{x}_2(T) - x_2^\varepsilon(T) &= \frac{k_0 e^{(\sigma-A)t_0}}{\sigma+\lambda} \times \Big\{ e^{A\alpha_1}\big[e^{-(\sigma+\lambda)T} - e^{-(\sigma+\lambda)\alpha_1}\big] \\ &\qquad - e^{At_\varepsilon}\big[e^{-(\sigma+\lambda)T} - e^{-(\sigma+\lambda)t_\varepsilon}\big] \Big\} \\ &= \frac{k_0 e^{(\sigma-A)t_0}}{\sigma+\lambda} \times \Big\{ e^{-(\sigma+\lambda)T}\big[e^{A\alpha_1} - e^{At_\varepsilon}\big] \\ &\qquad + \big[e^{(A-\sigma-\lambda)t_\varepsilon} - e^{(A-\sigma-\lambda)\alpha_1}\big] \Big\}. \end{aligned}$$

Since $t_\varepsilon \in [t_0, \alpha_1)$, we have $e^{A\alpha_1} - e^{At_\varepsilon} > 0$. In addition, as $A - \sigma - \lambda < 0$ by (5.66), we get $e^{(A-\sigma-\lambda)t_\varepsilon} - e^{(A-\sigma-\lambda)\alpha_1} > 0$. Combining these inequalities with the above expression for $\bar{x}_2(T) - x_2^\varepsilon(T)$, we conclude that $x_2^\varepsilon(T) < \bar{x}_2(T)$. By using (3.1), it is not difficult to show that the norm $\|\bar{x} - x^\varepsilon\|_{W^{1,1}}$ tends to 0 as $\varepsilon$ goes to 0. So, the inequality $x_2^\varepsilon(T) < \bar{x}_2(T)$, which holds for every $\varepsilon \in (0, \alpha_1 - t_0]$, implies that the process $(\bar{x}, \bar{s})$ under our consideration cannot be a $W^{1,1}$ local minimizer of $(GP_{1a})$ (see Definition 3.1).

Summing up the above analysis and taking into account the fact that $(GP_{1a})$ has a global minimizer, we can conclude that $(GP_{1a})$ has a unique $W^{1,1}$ local minimizer $(\bar{x}, \bar{s})$, which is a global minimizer, where $\bar{s}(t) = 0$ for a.e. $t \in [t_0, T]$ and $\bar{x}_1(t) = k_0 e^{-\sigma(t-t_0)}$ for all $t \in [t_0, T]$. □

## 5.8 Some Economic Interpretations

Needless to say that investigations on the solution existence of any optimization problem, including finite horizon optimal economic growth problems, are important. However, it is worthy to state clearly some economic interpretation of Theorem 5.5.

Recall that $\sigma$ and $\lambda$ are the *rate of labor force* and the *real interest rate*, respectively (see Section 5.1) and that $A$ is the *total factor productivity* (see Section 5.4). Therefore, the result in Theorem 5.5 can be interpreted as follows: If the total factor productivity $A$ is smaller than the sum of the rate of labor force $\sigma$ and the real interest rate $\lambda$, then optimal strategy is to keep the saving equal to 0. In other words, *if the total factor productivity $A$ is relatively small, then an expansion of the production facility does not lead to a higher total consumption satisfaction of the society.*

**Remark 5.5** The rate of labor force $\sigma$ is around 1.5%. The real interest rate $\lambda$ is in general 3.4%. Hence $\sigma + \lambda = 0.049$. Thus, roughly speaking, the assumption $A < \sigma + \lambda$ in Theorem 5.5 means that $A < 0.05$. Since weak and very weak economies do exist, the latter assumption is acceptable. Theorem 5.5 is meaningful as here the barrier $A = \sigma + \lambda$ for the total factor productivity appears for the first time. Due to Theorem 5.5, the notions of weak economy (with $A < \sigma + \lambda$) and strong economy (with $A > \sigma + \lambda$) can have exact meanings. Moreover, the behaviors of a weak economy and of a strong economy might be very different.

**Remark 5.6** By Theorem 5.5 we have solved the problem $(GP_1)$ in the situation where $A < \sigma + \lambda$. A natural question arises: *What happens if $A > \sigma + \lambda$?* The latter condition means that if the total factor productivity $A$ is relatively large. In this situation, it is likely that the optimal strategy requires to make the maximum saving until a special time $\bar{t} \in (t_0, T)$, which depends on the data tube $(A, \sigma, \lambda)$, then switch the saving to minimum. Further investigations in this direction are going on.

## 5.9 Conclusions

We have studied the solution existence of finite horizon optimal economic growth problems. Several existence theorems have been obtained not only for general problems but also for typical ones with the production function and the utility function being either the $AK$ function or the Cobb–Douglas one. Besides, we have raised some open questions and conjectures about the regularity of the global solutions of finite horizon optimal economic growth problems. Moreover, we have solved one of the above-mentioned typical problems and stated the economic interpretation for this obtained results.

# General Conclusions

In this dissertation, we have applied different tools from set-valued analysis, variational analysis, optimization theory, and optimal control theory to study qualitative properties (solution existence, optimality conditions, stability, and sensitivity) of some optimization problems arisen in consumption economics, production economics, optimal economic growths and their prototypes in the form of parametric optimal control problems.

The main results of the dissertation include:

1) Sufficient conditions for: the upper continuity, the lower continuity, and the continuity of the budget map, the indirect utility function, and the demand map; the Robinson stability and the Lipschitz-like property of the budget map; the Lipschitz property of the indirect utility function; the Lipschitz-Hölder property of the demand map.

2) Formulas for computing the Fréchet/limitting coderivatives of the budget map; the Fréchet/limitting subdifferentials of the infimal nuisance function, upper and lower estimates for the upper and the lower Dini directional derivatives of the indirect utility function.

3) The syntheses of finitely many processes suspected for being local minimizers for parametric optimal control problems without/with state constraints.

4) Three theorems on solution existence for optimal economic growth problems in general forms as well as in some typical ones, and the synthesis of optimal processes for one of such typical problems.

5) Interpretations of the economic meanings for most of the obtained results.

# List of Author's Related Papers

1. Vu Thi Huong, Jen-Chih Yao, Nguyen Dong Yen, *On the stability and solution sensitivity of a consumer problem*, Journal of Optimization Theory and Applications, **175** (2017), 567–589. (SCI)

2. Vu Thi Huong, Jen-Chih Yao, Nguyen Dong Yen, *Differentiability properties of a parametric consumer problem*, Journal of Nonlinear and Convex Analysis, **19** (2018), 1217–1245. (SCI-E)

3. Vu Thi Huong, Jen-Chih Yao, Nguyen Dong Yen, *Analyzing a maximum principle for finite horizon state constrained problems via parametric examples. Part 1: Unilateral state constraints*, Journal of Nonlinear and Convex Analysis **21** (2020), 157–182. (SCI-E)

4. Vu Thi Huong, Jen-Chih Yao, Nguyen Dong Yen, *Analyzing a maximum principle for finite horizon state constrained problems via parametric examples. Part 2: Bilateral state constraints*, preprint, 2019. (https://arxiv.org/abs/1901.09718; submitted)

5. Vu Thi Huong, *Solution existence theorems for finite horizon optimal economic growth problems*, preprint, 2019. (https://arxiv.org/abs/2001.03298; submitted)

6. Vu Thi Huong, Jen-Chih Yao, Nguyen Dong Yen, *Optimal processes in a parametric optimal economic growth model*, Taiwanese Journal of Mathematics, https://doi.org/10.11650/tjm/200203 (2020). (SCI)

# References

[1] D. Acemoglu, *Introduction to Modern Economic Growth*, Princeton University Press, 2009.

[2] V. M. Alekseev, V. M. Tikhomirov, S. V. Fomin, *Optimal control*, Consultants Bureau, New York, 1987.

[3] W. B. Allen, N. A. Doherty, K. Weigelt, E. Mansfield, *Managerial Economics. Theory, Applications, and Cases,* 6th edition, W. W. Norton and Company, New York, 2005.

[4] B. K. Ane, A. M. Tarasyev, C. Watanabe, *Construction of nonlinear stabilizer for trajectories of economic growth*, J. Optim. Theory Appl. **134** (2007), 303–320.

[5] A. V. Arutyunov, *Optimality Conditions. Abnormal and Degenerate Problems*, translated from the Russian by S. A. Vakhrameev, Kluwer Academic Publishers, Dordrecht, 2000.

[6] A. V. Arutyunov, S. M. Aseev, *Investigation of the degeneracy phenomenon of the maximum principle for optimal control problems with state constraints*, SIAM J. Control Optim. **35** (1997), 930–952.

[7] S. M. Aseev, A. V. Kryazhimskii, *The Pontryagin maximum principle and problems of optimal economic growth* (in Russian), Tr. Mat. Inst. Steklova Vol. 257 (2007); translation in Proc. Steklov Inst. Math. **257** (2007), 1–255.

[8] J.-P. Aubin, *Lipschitz behavior of solutions to convex minimization problems*, Math. Oper. Res., **9** (1984), 87–111.

[9] J.-P. Aubin, H. Frankowska, *Set-Valued Analysis*, reprint of the 1990 edition, Birkhäuser, Boston-Basel-Berlin, 2009.

[10] E. J. Balder, *An existence result for optimal economic growth problems*, J. Math. Anal. Appl. **95** (1983), 195–213.

[11] R. J. Barro, X. Sala-i-Martin, *Economic Growth*, MIT Press, 2004.

[12] V. Basco, P. Cannarsa, H. Frankowska, *Necessary conditions for infinite horizon optimal control problems with state constraints*, Math. Control Relat. Fields **8** (2018), 535–555.

[13] A. Beggs, *Sensitivity analysis of boundary equilibria,* Econom. Theory **66** (2018), 763–786.

[14] J. F. Bonnans, A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer, New York, 2000.

[15] J. M. Borwein, *Stability and regular points of inequality systems*, J. Optim. Theory Appl. **48** (1986), 9–52.

[16] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

[17] D. Cass, *Optimum growth in an aggregative model of capital accumulation*, Rev. Econ. Stud. **32** (1965), 233–240.

[18] L. Cesari, *Optimization Theory and Applications*, 1st edition, Springer-Verlag, New York, 1983.

[19] A. C. Chiang, K. Wainwright, *Fundamental Methods of Mathematical Economics*, 4th edition, McGraw-Hill, New York, 2005.

[20] F. H. Clarke, *Optimization and Nonsmooth Analysis*, 2nd edition, McGraw-Hill, SIAM, Philadelphia, 2002.

[21] F. H. Clarke, *Functional Analysis, Calculus of Variations and Optimal Control*, Springer, London, 2013.

[22] J.-P. Crouzeix, *Duality between direct and indirect utility functions. Differentiability properties*, J. Math. Econom. **12** (1983), 149–165.

[23] J.-P. Crouzeix, *On twice differentiable direct and indirect utility functions and the monotonicity of the demand*, Optimization **57** (2008), 419–433.

[24] H. d'Albis, P. Gourdel, C. Le Van, *Existence of solutions in continuous-time optimal growth models*, Econom. Theory **37** (2008), 321–333.

[25] W. E. Diewert, *Applications of duality theory*, In "Frontiers of Quantitative Economics", Vol. II, pp. 106–206, North-Holland, Amsterdam, 1974.

[26] E. D. Domar, *Capital expansion, rate of growth, and employment*, Econometrica **14** (1946), 137–147.

[27] M. M. A. Ferreira, R. B. Vinter, *When is the maximum principle for state constrained problems nondegenerate?*, J. Math. Anal. Appl. **187** (1994), 438–467.

[28] F. A. C. C. Fontes, H. Frankowska, *Normality and nondegeneracy for optimal control problems with state constraints*, J. Optim. Theory Appl. **166** (2015), 115–136.

[29] H. Frankowska, *Normality of the maximum principle for absolutely continuous solutions to Bolza problems under state constraints*, Control Cybernet. **38** (2009), 1327–1340.

[30] H. Gfrerer, B. S. Mordukhovich, *Robinson stability of parametric constraint systems via variational analysis,* SIAM J. Optim. **27** (2017), 438–465.

[31] N. Hadjisavvas, S. Komlósi, S. Schaible, *Handbook of Generalized Convexity and Generalized Monotonicity*, Springer, New York, 2005.

[32] N. Hadjisavvas, J.-P. Penot, *Revisiting the problem of integrability in utility theory*, Optimization **64** (2015), 2495–2509.

[33] R. F. Harrod, *An essay in dynamic theory*, Econ. J. **49** (1939), 14–33.

[34] R. F. Hartl, S. P. Sethi, and R. G. Vickson, *A survey of the maximum principles for optimal control problems with state constraints*, SIAM Rev. **37** (1995), 181–218.

[35] V. T. Huong, J.-C. Yao, N. D. Yen, *On the stability and solution sensitivity of a consumer problem*, J. Optim. Theory Appl. **175** (2017), 567–589.

[36] V. T. Huong, J.-C. Yao, N. D. Yen, *Differentiability properties of a parametric consumer problem*, J. Nonlinear Convex Anal. **19** (2018), 1217–1245.

[37] V. T. Huong, J.-C. Yao, N. D. Yen, *Analyzing a maximum principle for finite horizon state constrained problems via parametric examples. Part 1:*

*Unilateral state constraints*, J. Nonlinear Convex Anal. **21** (2020), 157–182.

[38] V. T. Huong, J.-C. Yao, N. D. Yen, *Analyzing a maximum principle for finite horizon state constrained problems via parametric examples. Part 2: Bilateral state constraints*, preprint, 2019. (Submitted)

[39] V. T. Huong, *Solution existence theorems for finite horizon optimal economic growth problems*, preprint, 2019. (https://arxiv.org/abs/2001.03298; submitted)

[40] V. T. Huong, J.-C. Yao, N. D. Yen, *Optimal processes in a parametric optimal economic growth model*, Taiwanese J. Math., https://doi.org/10.11650/tjm/200203 (2020).

[41] D. T. K. Huyen, N. D. Yen, *Coderivatives and the solution map of a linear constraint system*, SIAM J. Optim. **26** (2016), 986–1007.

[42] M. D. Intriligator, *Mathematical Optimization and Economic Theory*, 2nd edition, SIAM, Philadelphia, 2002.

[43] A. D. Ioffe, V. M. Tihomirov, *Theory of extremal problems*, North-Holland Publishing Co., Amsterdam-New York, 1979.

[44] F. Jarrahi, W. Abdul-Kader, *Performance evaluation of a multi-product production line: An approximation method*, Appl. Math. Model. **39** (2015), 3619–3636.

[45] V. Jeyakumar, N. D. Yen, *Solution stability of nonsmooth continuous systems with applications to cone-constrained optimization*, SIAM J. Optim. **14** (2004), 1106–1127.

[46] D. Karamzin, F. L. Pereira, *On a few questions regarding the study of state-constrained problems in optimal control*, J. Optim. Theory Appl. **180** (2019), 235–255.

[47] G. A. Keskin, S. I. Omurca, N. Aydin, E. Ekinci, *A comparative study of production-inventory model for determining effective production quantity and safety stock level*, Appl. Math. Model. **39** (2015), 6359–6374.

[48] D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

[49] A. N. Kolmogorov and S. V. Fomin, *Introductory Real Analysis,* revised English edition, translated from the Russian and edited by R. A. Silverman, Dovers Publications, Inc., New York, 1970.

[50] T. C. Koopmans, *On the concept of optimal economic growth*, in The Econometric Approach to Development Planning, North-Holland, Amsterdam, pp. 225–295, 1965.

[51] A. A. Krasovskii, A. M. Tarasyev, *Construction of nonlinear regulators in economic growth models*, Proc. Steklov Inst. Math. **268** (2010), suppl. 1, S143–S154.

[52] E. B. Lee, L. Markus, *Foundations of Optimal Control Theory*, 2nd edition, Robert E. Krieger Publishing Co., Inc., Melbourne, FL, 1986.

[53] D. G. Luenberger, *Optimization by Vector Space Methods.* John Wiley & Sons, New York, 1969.

[54] J.-E. Martínez-Legaz, M. S. Santos, *Duality between direct and indirect preferences*, Econ. Theory **3** (1993), 335–351.

[55] K. Miyagishima, *Implementability and equity in production economies with unequal skills*, Rev. Econ. Design **19** (2015), 247–257.

[56] B. S. Mordukhovich, *Variational analysis and generalized differentiation*, Vol. I: Basic Theory, Springer, New York, 2006.

[57] B. S. Mordukhovich, *Variational analysis and generalized differentiation*, Vol. II: Applications, Springer, New York, 2006.

[58] B. S. Mordukhovich, *Coderivative analysis of variational systems*, J. Global Optim., **28** (2004), 347–362.

[59] B. S. Mordukhovich, *Variational Analysis and Applications*, Springer, Berlin, Switzerland, 2018.

[60] B. S. Mordukhovich, N. M. Nam, N. D. Yen, *Subgradients of marginal functions in parametric mathematical programming*, Math. Program., **116** (2009), no. 1-2, Ser. B, 369–396.

[61] W. Nicholson, C. Snyder, *Microeconomic Theory: Basic Principles and Extension*, 7th edition, South-Western, Cengage Learning, 2012.

[62] M. S. Nikolskii, *Study of an optimal control problem related to the Solow control model*, Proc. Steklov Inst. Math. **292** (2016), S231–S237.

[63] J.-P. Penot, *Calculus without Derivatives*, Springer, New York, 2013.

[64] J.-P. Penot, *Variational analysis for the consumer theory*, J. Optim. Theory Appl. **159** (2013), 769–794.

[65] J.-P. Penot, *Some properties of the demand correspondence in the consumer theory*, J. Nonlinear Convex Anal. **15** (2014), 1071–1085.

[66] H. X. Phu, *A solution method for regular optimal control problems with state constraints*, J. Optim. Theory Appl. **62** (1989), 489–513.

[67] H. X. Phu, *Investigation of a macroeconomic model by the method of region analysis*, J. Optim. Theory Appl. **72** (1992), 319–332.

[68] N. V. T. Pierre, *Introductory Optimization Dynamics. Optimal Control with Economics and Management Science Applications*, Springer-Verlag, Berlin, 1984.

[69] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*, John Wiley & Sons, Inc., New York–London, 1962.

[70] F. P. Ramsey, *A mathematical theory of saving*, Econ. J. **l38** (1928), 543–559.

[71] S. Rasmussen, *Production Economics: The Basic Theory of Production Optimisation*, 2nd edition, Springer, Berlin-Heidelberg, 2013.

[72] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.

[73] R. T. Rockafellar, *Directionally lipschitzian functions and subdifferential calculus*, Proc. London Math. Soc. **39** (1979), 331–355.

[74] R. T. Rockafellar, *Generalized directional derivatives and subgradients of nonconvex functions*, Canad. J. Math. **32** (1980), 257–280.

[75] H. L. Royden, P. M. Fitzpatrick, *Real Analysis*, 4th edition, China Machine Press, 2010.

[76] W. Rudin, *Functional Analysis*, 2nd edition, McGraw-Hill, Inc., New York, 1991.

[77] R. M. Solow, *A contribution to the theory of economic growth*, Quart. J. Econom. **70** (1956), 65–94.

[78] T. W. Swan, *Economic growth and capital accumulation*, Economic Record **32** (1956), 334–361.

[79] A. Takayama, *Mathematical Economics*, The Dryden Press, Hinsdale, Illinois, 1974.

[80] Y.-Ch. Tsao, *A piecewise nonlinear optimization for a production-inventory model under maintenance, variable setup costs, and trade credits*, Ann. Oper. Res. **233** (2015) 465– 481.

[81] F. P. Vasilev, *Numerical Methods for Solving Extremal Problems* (in Russian), 2nd edition, Nauka, Moscow, 1988.

[82] R. Vinter, *Optimal Control*, Birkhäuser, Boston, 2000.

[83] J.-C. Yao, *Variational inequalities with generalized monotone operators*, Math. Oper. Res. **19** (1994), 691–705.

[84] J.-C. Yao,*Multi-valued variational inequalities with K-pseudomonotone operators*, J. Optim. Theory Appl. **80** (1994), 63–74.

[85] J.-C. Yao, O. Chadli, *Pseudomonotone complementarity problems and variational in equalities*, in: "Handbook of Generalized Convexity and Generalized Monotonicity" (N. Hadjisavvas, S. Komlósi, and S. Schaible, Eds.), pp. 501–558, Springer, 2005.

[86] N. D. Yen, *Hölder continuity of solutions to a parametric variational inequality*, Applied Math. Optim. **31** (1995), 245–255.

[87] N. D. Yen, *Implicit function theorems for set-valued maps*, Acta Math. Vietnam. **12** (1987), 17–28.

[88] N. D. Yen, *Stability of the solution set of perturbed nonsmooth inequality systems and application*, J. Optim. Theory Appl. **93** (1997), 199–225.

[89] E. Zakon, *Basic concepts of mathematics*, in "The Zakon Series on Mathematical Analysis", The Trillia Group, West Lafayette, Indiana, USA, 2017.