VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY
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DUONG THI KIM HUYEN

STABILITY OF SOME CONSTRAINT SYSTEMS AND OPTIMIZATION PROBLEMS

DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY IN MATHEMATICS

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DOCTOR OF PHILOSOPHY IN MATHEMATICS

Supervisor: Prof. Dr.Sc. NGUYEN DONG YEN

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Confirmation

This dissertation was written on the basis of my research works carried out at the Institute of Mathematics, Vietnam Academy of Science and Technology, under the guidance of Prof. Nguyen Dong Yen. All results presented in this dissertation have never been published by others.

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The author

Duong Thi Kim Huyen

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Contents

Table of Notations

$\mathbb{R}$	the set of real numbers
$\bar{\mathbb{R}}$	the set of extended real numbers
$\emptyset$	the empty set
$\mathbb{R}^n$	the n-dimensional Euclidean vector space
$\langle x, y \rangle$	the scalar product in an Euclidean space
$\|\|x\|\|$	the norm of a vector x
$B(x, \rho)$	the open ball centered x with radius ρ
$\bar{B}(x, \rho)$	the closed ball centered x with radius ρ
$\mathbb{B}_X$	the open unit ball of X
$\mathcal{N}(\bar{x})$	the family of the neighborhoods of $\bar{x}$
$\mathbb{R}^n_+$	the nonnegative orthant in $\mathbb{R}^n$
$\mathbb{R}^n_-$	the nonpositive orthant in $\mathbb{R}^n$
$\mathbb{R}^{m \times n}$	the vector space of $m \times n$ real matrices
$\det A$	the determinant of matrix A
A^T	the transpose of matrix A
$\ker A$	the kernel of matrix A (i.e., the null space of the operator corresponding to matrix A)
E	the unit matrix
$\operatorname{rank} C$	the rank of matrix C
$C \preceq 0$	a negative semidefinite matrix
X^*	the dual space of a Banach space X
X^{**}	the dual space of X^*
$A^* : Y^* \to X^*$	the adjoint operator of a bounded linear operator $A : X \to Y$
$d(x, \Omega)$	the distance from x to a set Ω
$\widehat{N}(\bar{x}; \Omega)$ or $\widehat{N}_\Omega(\bar{x})$	the Fréchet normal cone of Ω at $\bar{x}$
$N(\bar{x}; \Omega)$ or $N_\Omega(\bar{x})$	the Mordukhovich normal cone of Ω at $\bar{x}$
$x \xrightarrow{\Omega} \bar{x}$	$x \to \bar{x}$ and $x \in \Omega$

Limsup	the Painlevé-Kuratowski upper limit
$\nabla f(\bar{x})$	the Fréchet derivative of $f : X \to Y$ at $\bar{x}$
$\nabla^2 f(\bar{x})$	the Hessian matrix of $f : X \to \mathbb{R}$ at $\bar{x}$
$\nabla_x \psi(\bar{x}, \bar{y})$	the partial derivative of $\psi : X \times Y \to Z$ in x at $(\bar{x}, \bar{y})$
epi$f$	the epigraph of a function $f : X \to \mathbb{R}$
$\partial f(x)$	the Mordukhovich subdifferential of f at x
$\partial^\infty f(x)$	the singular subdifferential of f at x
$\partial^2 f(\bar{x}, \bar{y})$	the second-order subdifferential of f at $\bar{x}$ in direction $\bar{y} \in \partial f(\bar{x})$
$\partial_x \psi(\bar{x}, \bar{y})$	the partial subdifferential of $\psi : X \times Y \to \mathbb{R}$ in x at $(\bar{x}, \bar{y})$
$g \circ f$	the composite function of g and f
$F : X \rightrightarrows Y$	a set-valued map between X and Y
$\operatorname{gph} F$	the graph of F
$\widehat{D}^* F(\bar{x}, \bar{y})(\cdot)$	the Fréchet coderivative of F at $(\bar{x}, \bar{y})$
$D^* F(\bar{x}, \bar{y})(\cdot)$	the Mordukhovich coderivative of F at $(\bar{x}, \bar{y})$
$\operatorname{int} D$	the topological interior of D
$L^\perp$	the orthogonal complement of a set L
L^*	the polar cone of L
$\operatorname{cone} D$	the cone generated by D
resp.	respectively
$\operatorname{diag}[M_{\alpha\alpha}, M_{\beta\beta}, M_{\gamma\gamma}]$	a block diagonal matrix
LCP	linear complementarity problem
AVI	affine variational inequality
TRS	the trust-region subproblem
MFCQ	The Mangasarian-Fromovitz Constraint Qualification

Introduction

Many real problems lead to formulating equations and solving them. These equations may contain parameters like initial data or control variables. The *solution set* of a parametric equation can be considered as a *multifunction* (that is, a point-to-set function) of the parameters involved. The latter can be called an *implicit multifunction.* A natural question is that *"What properties can the implicit multifunction possess?"*.

Under suitable differentiability assumptions, classical implicit function theorems have addressed thoroughly the above question from finite-dimensional settings to infinite-dimensional settings.

Nowadays, the models of interest (for instance, constrained optimization problems) outrun equations. Thus, Variational Analysis (see, e.g., [50, 80]) has appeared to meet the need of this increasingly strong development.

J.-P. Aubin, J.M. Borwein, A.L. Dontchev, B.S. Mordukhovich, H.V. Ngai, S.M. Robinson, R.T. Rockafellar, M. Théra, Q.J. Zhu, and other authors, have studied implicit multifunctions and qualitative aspects of optimization and equilibrium problems by different approaches. In particular, with the two-volume book *"Variational Analysis and Generalized Differentiation"* (see [50, 51]) and a series of research papers, Mordukhovich has given basic tools (*coderivatives, subdiffentials, normal cones*, and calculus rules), fundamental results, and advanced techniques for qualitative studies of optimization and equilibrium problems. Especially, the fourth chapter of the book is entirely devoted to such important properties of the solution set of parametric problems as the *Lipschitz stability* and *metric regularity.* These properties indicate good behaviors of the multifunction in question. The two models considered in that chapter of Mordukhovich's book bear the names *parametric constraint system* and *parametric variational system.* More discussions and references on implicit multifunction theorems can be found in the books by Borwein and Zhu [10], Dontchev and Rockafellar [19], and Klatte and Kummer [35].

Let us briefly review some contents of the book *"Implicit Functions and Solution Mappings"* [19] of Dontchev and Rockafellar. The first chapter of this book is devoted to functions defined implicitly by equations and the authors begin with classical inverse function theorem and classical implicit function theorem. The book presents a very deep view from Variational Analysis on solution maps. The authors have investigated many properties of solution maps such as calmness, Lipschitz continuity, outer Lipschitz continuity, Aubin property, metric regularity, linear openness, strong regularity and their applications to Numerical Analysis. The main tools that have been used in the book are graphical differentiation and coderivetive.

Within this dissertation we use coderivative to study three properties of solution maps in finite-dimensional settings, which include *Aubin property (Lipschitz-like property)*, *metric regularity*, and *the Robinson stability* of solution maps of constraint and variational systems. Results on these stability properties are applied to studying the solution stability of linear complementarity problem, affine variational inequalities, and a typical parametric optimization problem.

Introduced by Aubin [5, p. 98] under the name pseudo-Lipschitz property, the *Lipschitz-like property* of multifunctions is a fundamental concept in stability and sensitivity analysis of optimization and equilibrium problems. The Lipschitz-like property guarantees the local convergence of some variants of Newton's method for generalized equations [12, 17, 19]. In particular, from [19, Theorem 6C.1, p. 328] it follows that, if a mild approximation condition is satisfied and the solution map under right-hand-side perturbations is Lipschitz-like around a point in question, then there exists an iterative sequence Q-linearly converging to the solution. Moreover, as shown by Dontchev [17, Theorem 1], the Newton method applied to a generalized equation in a Banach space is locally convergent uniformly in the canonical parameter if and only if the solution map of this equation is Lipschitz-like around the reference point. In addition, if the derivative of the base map is locally Lipschitz, then the Lipschitz-likeness implies the existence of a Q-quadratically convergent Newton sequence (see [17, Theorem 2]).

Metric regularity (in the classical sense) is another fundamental property of set-valued mappings. We refer to the survey of A.D. Ioffe [32, 33] on this property and its applications. Borwein and Zhuang [11] and Penot [63] have shown that the Lipschitz-like property of a set-valued mapping $F : X \rightrightarrows Y$ between Banach spaces around a point $(\bar{x}, \bar{y})$ in the graph

$$\operatorname{gph} F := \{(x, y) \in X \times Y \ : \ y \in F(x)\}$$

of F is equivalent to the metric regularity of the inverse map $F^{-1} : Y \rightrightarrows X$ around $(\bar{y}, \bar{x})$. It is also known (see Mordukhovich [49]) that the properties just mentioned are equivalent to the *openness with linear rate* of F around $(\bar{x}, \bar{y})$.

Let $G : X \rightrightarrows Y$ be an implicit multifunction defined by

$$G(x) := \{y \in Y \ : \ 0 \in F(x, y)\} \quad (x \in X), \tag{1}$$

where $F : X \times Y \rightrightarrows Z$ is a multifunction, X, Y, and Z are Banach spaces. Then the concept of *Robinson stability* of G at $(\bar{x}, \bar{y}, 0) \in \operatorname{gph} F$ can be defined. This property of an implicit multifunction, which has been called the metric regularity in the sense of Robinson by several authors, was introduced by Robinson [75]. It is a type of *uniform local error bounds* and it has numerous applications in optimization theory and theory of equilibrium problems.

Stability properties like lower semicontinuity, upper semicontinuity, Hausdorff semicontinuity/continuity, Hölder continuity of solution maps and of approximate solution maps can be studied for very general optimization problems and equilibrium problems (for example, vector optimization problems, vector variational inequalities, vector equilibrium problems). The locally convex Hausdorff topological vector spaces setting can be also adopted. Here, it is not necessary to use the tools from variational analysis and generalized differentiation. We refer to the works by P.Q. Khanh, L.Q. Anh, and their coauthors [1–4] for some typical results in this direction.

The dissertation has four chapters and a list of references.

Chapter 1 collects some basic concepts from Set-Valued Analysis and Variational Analysis and gives a first glance at some properties of multifunctions and key results on implicit multifunctions.

In Chapter 2, we investigate the Lipschitz-like property and the Robinson stability of the solution map of a parametric linear constraint system by means of normal coderivative, the Mordukhovich criterion, and a related theorem due to Levy and Mordukhovich [41]. Among other things, the obtained results yield uniform local error bounds and traditional local error bounds for the linear complementarity problem and the general affine variational inequality problem, as well as verifiable sufficient conditions for the Lipschitz-like property of the solution map of the linear complementarity problem and a class of affine variational inequalities, where all components of the problem data are subject to perturbations.

Chapter 3 shows analogues of the results of the previous chapter for the case where the linear constraint system undergoes linear perturbations.

Finally, in Chapter 4, we analyze the sensitivity of the stationary point set map of a C^2-smooth parametric optimization problem with one C^2-smooth functional constraint under total perturbations by applying some results of Levy and Mordukhovich [41], and Yen and Yao [88]. We not only show necessary and sufficient conditions for the Lipschitz-like property of the stationary point set map, but also sufficient conditions for its Robinson stability. These results lead us to new insights into the preceding deep investigations of Levy and Mordukhovich [41] and of Qui [71, 72] and allow us to revisit and extend several stability theorems in indefinite quadratic programming.

The dissertation is written on the basis of four published articles: paper [31] in *SIAM Journal on Optimization*, paper [28] in *Journal of Set-Valued and Variational Analysis*, and papers [29, 30] in *Journal of Optimization Theory and Applications*.

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Chapter 1

Preliminaries

In this chapter, several concepts and tools from Variational Analysis are recalled. As a preparation for the investigations in Chapters 2–4, we present lower and upper estimates for coderivatives of implicit multifunctions given by Levy and Mordukhovich [41], Lee and Yen [39], as well as the sufficient conditions of Yen and Yao [88] for the Robinson stability property of implicit multifunctions.

The concepts and tools discussed in this chapter can be found in the monographs of Mordukhovich [50, 52] and the classical work of Rockafellar and Wets [80].

1.1 Basic Concepts from Variational Analysis

Introduced by Mordukhovich [48] in 1980, the limiting coderivative is a basic concept of generalized differentiation and it has a very important role in Variational Analysis and applications. One can compare the role of the limiting coderivative, which helps to develop the dual-space approach to optimization and equiliblium problems, with that of derivative in classical Mathematical Analysis. We are going to describe the finite-dimensional version of the concept. The reader is referred to [52, Chapter 1] for a comprehensive treatment of limiting coderivative and related notions.

The *Fréchet normal cone* (also called the *prenormal cone*, or the *regular* *normal cone*) to a set $\Omega \subset \mathbb{R}^s$ at $\bar v \in \Omega$ is given by

$$\widehat{N}_\Omega(\bar v) = \Big\{ v' \in \mathbb{R}^s \,:\, \limsup_{v \xrightarrow{\Omega} \bar v} \frac{\langle v', v - \bar v\rangle}{\|v - \bar v\|} \le 0 \Big\},$$

where $v \xrightarrow{\Omega} \bar v$ means $v \to \bar v$ with $v \in \Omega$. By convention, $\widehat{N}_\Omega(\bar v) := \emptyset$ when $\bar v \notin \Omega$. Provided that Ω is locally closed around $\bar v \in \Omega$, one calls

$$\begin{aligned} N_\Omega(\bar v) &= \operatorname*{Limsup}_{v \to \bar v} \widehat{N}_\Omega(v) \\ &:= \big\{ v' \in \mathbb{R}^s \,:\, \exists \text{ sequences } v_k \to \bar v,\ v'_k \to v', \\ &\qquad\qquad \text{with } v'_k \in \widehat{N}_\Omega(v_k) \ \text{ for all } \ k = 1, 2, \dots \big\} \end{aligned}$$

the *Mordukhovich* (or *limiting/basic*) *normal cone* to Ω at $\bar v$. If $\bar v \notin \Omega$, then one puts $N_\Omega(\bar v) = \emptyset$.

A multifunction $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be *locally closed* around a point $\bar z = (\bar x, \bar y)$ from $\operatorname{gph} \Phi := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \ :\ y \in \Phi(x)\}$ if $\operatorname{gph} \Phi$ is locally closed around $\bar z$. Here, the product space $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ is equipped with the topology generated by the sum norm $\|(x, y)\| = \|x\| + \|y\|$.

For any $\bar z = (\bar x, \bar y) \in \operatorname{gph} \Phi$, the *Fréchet coderivative* of Φ at $\bar z$ is the multifunction $\widehat{D}^*\Phi(\bar z) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with the values

$$\widehat{D}^*\Phi(\bar z)(y') := \big\{ x' \in \mathbb{R}^n \,:\, (x', -y') \in \widehat{N}_{\operatorname{gph}\Phi}(\bar z) \big\} \quad (y' \in \mathbb{R}^m).$$

Similarly, the *Mordukhovich coderivative* (limiting coderivative) of Φ at $\bar z$ is the multifunction $D^*\Phi(\bar z) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with the values

$$D^*\Phi(\bar z)(y') := \big\{ x' \in \mathbb{R}^n \,:\, (x', -y') \in N_{\operatorname{gph}\Phi}(\bar z) \big\} \quad (y' \in \mathbb{R}^m).$$

One says that Φ is *graphically regular* at $\bar z$ if $\widehat{D}^*\Phi(\bar z)(y') = D^*\Phi(\bar z)(y')$ for any $y' \in \mathbb{R}^m$. If Φ is single-valued, then we use the notions $\widehat{D}^*\Phi(\bar x)(y')$ and $D^*\Phi(\bar x)(y')$, respectively, instead of $\widehat{D}^*\Phi(\bar z)(y')$ and $D^*\Phi(\bar z)(y')$, where $\bar z = (\bar x, \Phi(\bar x))$. In the case where Φ is strictly Fréchet differentiable at $\bar x$, by [50, Theorem 1.38] we have

$$\widehat{D}^*\Phi(\bar x)(y') = D^*\Phi(\bar x)(y') = \{\nabla \Phi(\bar x)^*(y')\}$$

for any $y' \in \mathbb{R}^m$. In particular, Φ is graphically regular at $\bar z = (\bar x, \Phi(\bar x))$.

Suppose that X, Y, and Z are finite-dimensional Euclidean spaces. Consider a function $\psi : X \to \bar{\mathbb{R}}$, where $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, and suppose that $|\psi(\bar x)| < \infty$. The set

$$\partial \psi(\bar x) := \{x' \in X^* \,:\, (x', -1) \in N_{\operatorname{epi}\psi}(\bar x, \psi(\bar x))\}$$

is the *Mordukhovich subdifferential* of ψ at $\bar{x}$. If $|\psi(\bar{x})| = \infty$, then we put $\partial\psi(\bar{x}) = \emptyset$. The set

$$\partial^\infty\psi(\bar{x}) := \{x^* \in X^* \,:\, (x^*, 0) \in N_{\operatorname{epi}\psi}(\bar{x}, \psi(\bar{x}))\}$$

is the *singular subdifferential* of ψ at $\bar{x}$. For a set $\Omega \subset X$ and a point $\bar{x} \in \Omega$, we have

$$N(\bar{x}, \Omega) = \partial\delta_\Omega(\bar{x}) = \partial^\infty\delta_\Omega(\bar{x}),$$

where $\delta_\Omega(\bar{x})$ is the indicator function of Ω; see [50, Proposition 1.79]. If ψ depends on two variables x and y, and $|\psi(\bar{x}, \bar{y})| < \infty$, then $\partial_x\psi(\bar{x}, \bar{y})$ denotes the Mordukhovich subdifferential of $\psi(., \bar{y})$ at $\bar{x}$. For any $\bar{v} \in \partial\psi(\bar{x})$,

$$\partial^2\psi(\bar{x}|\bar{v})(u) := D^*(\partial\psi)(\bar{x}|\bar{v})(u) \quad (u \in X^{**} = X)$$

is the *limiting second-order subdifferential* (or the generalized Hessian) of ψ at $\bar{x}$ in direction $\bar{v}$.

1.2 Properties of Multifunctions and Implicit Multifunctions

For set-valued mappings, being *Lipschitz-like* around a point in the graph is a very nice behavior. Maps with this property are considered locally stable in a strong sense. For sum rules, chain rules, etc., Lipschitz-likeness plays a role of constraint qualification. This property was originally defined by J.-P. Aubin who called it the *pseudo-Lipschitz property* [5, p. 98]. It is also known under other names: *the Aubin continuity property* [18, p. 1089], and *the sub-Lipschitzian property* [79]. A characterization of the Lipschitz-like property via the local Lipschitz property of a distance function was given by Rockafellar [79].

A multifunction $G : Y \rightrightarrows X$ is said to be *Lipschitz-like* around a point $(\bar{y}, \bar{x}) \in \operatorname{gph} G$ if there exist a constant $\ell > 0$ and neighborhoods U of $\bar{x}$, V of $\bar{y}$ such that

$$G(y') \cap U \subset G(y) + \ell\|y' - y\|\bar{B}_X \quad \forall y, y' \in V,$$

where $\bar{B}_X$ denotes the closed unit ball in X. The infimum of all such moduli ℓ is called the *exact Lipschitzian bound* of G around $(\bar{y}, \bar{x})$ (see [50, Definition 1.40]).

Theorem 1.1 (Mordukhovich Criterion 1) (see [49], [80, Theorem 9.40], and [50, Theorem 4.10]) *If G is locally closed around $(\bar{y}, \bar{x})$, then G is Lipschitz-like around $(\bar{y}, \bar{x})$ if and only if*

$$D^*G(\bar{y}|\bar{x})(0) = \{0\}.$$

As in [49, Definition 4.1], we say that a multifunction $F : X \rightrightarrows Y$ is *metrically regular* around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with modulus $r > 0$ if there exist neighborhoods U of $\bar{x}$, V of $\bar{y}$, and a number $\gamma > 0$ such that

$$d(x, F^{-1}(y)) \leq r\, d(y, F(x)) \tag{1.1}$$

for any $(x, y) \in U \times V$ with $d(y, F(x)) < \gamma$.

The condition $d(y, F(x)) < \gamma$ can be omitted when F is *inner semicontinuous* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. (This concept can be found on page 42 of the monograph [50].) Indeed, the latter means that for every neighborhood V' of $\bar{y}$, there exits a neighborhood U' of $\bar{x}$ such that $F(x) \cap V' \neq \emptyset$ for all $x \in U'$. Hence, for every neighborhood V' of $\bar{y}$, there exists $\gamma' > 0$ such that $d(y, F(x)) < \gamma'$ for all $x \in U'$ and $y \in V'$. So, if (1.1) holds true with constants r, γ and neighborhoods U and V, then for a number $\gamma'' \in (0, \gamma']$, we can find neighborhoods U'' of $\bar{x}$ and V'' of $\bar{y}$ with the property (1.1). Replacing U by $U \cap U''$, and V by $V \cap V''$, we have the inequality in (1.1). Thus, if F is inner semicontinuous at $(\bar{x}, \bar{y})$, then F is metrically regular around at $(\bar{x}, \bar{y})$ with modulus $r > 0$ if and only if there exist neighborhoods V of $\bar{y}$, U of $\bar{x}$ such that

$$d(x, F^{-1}(y)) \leq r\, d(y, F(x))$$

for any $(x, y) \in U \times V$.

Theorem 1.2 (Mordukhovich Criterion 2) (see [49] and also [19, Theorem 4H.1, p. 246]) *If F is locally closed around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, then F is metrically regular around $(\bar{x}, \bar{y})$ if and only if*

$$0 \in D^*F(\bar{x}|\bar{y})(v') \implies v' = 0.$$

Given a multifunction $F : X \times Y \rightrightarrows Z$ and a pair $(\bar{x}, \bar{y}) \in X \times Y$ satisfying $0 \in F(\bar{x}, \bar{y})$. We say that the implicit multifunction $G : Y \rightrightarrows X$ given by

$$G(y) = \{x \in X \,:\, 0 \in F(x, y)\} \tag{1.2}$$

has the *Robinson stability* at $\omega_0 = (\bar{x}, \bar{y}, 0)$ if there exist constants $r > 0$, $\gamma > 0$, and neighborhoods U of $\bar{x}$, V of $\bar{y}$ such that

$$d(x, G(y)) \leq rd(0, F(x, y)) \tag{1.3}$$

for any $(x, y) \in U \times V$ with $d(0, F(x, y)) < \gamma$. The infimum of all such moduli r is called the *exact Robinson regularity bound* of the implicit multifunction G at $\omega_0 = (\bar{x}, \bar{y}, 0)$.

By suggesting two examples, Jeyakumar and Yen [34, p. 1119] have proved that the Robinson stability of G at $(\bar{x}, \bar{y}, 0) \in \operatorname{gph} F$ is not equivalent to the Lipschitz-like property of G around $(\bar{x}, \bar{y})$. We refer to [14] for a discussion on the relationships between the Robinson stability and the Lipschitz-like behavior of implicit multifunctions.

Recently, Gfrerer and Mordukhovich [21] have given first-order and second-order sufficient conditions for this stability property of a parametric constraint system and put it in the relationships with other properties, such as the classical metric regularity and the Lipschitz-like property.

Note that, in (1.3), the condition $d(0, F(x, y)) < \gamma$ can be omitted if F is inner semicontinuous at $(\bar{x}, \bar{y}, 0)$. Indeed, the latter means that for every $\mu > 0$ there exist neighborhoods U_μ of $\bar{x}$, V_μ of $\bar{y}$ such that

$$F(x, y) \cap B(0, \mu) \neq \emptyset \quad \forall (x, y) \in U_\mu \times V_\mu. \tag{1.4}$$

So, if (1.3) is satisfied with positive constants r, γ and neighborhoods U and V, then for a value $\mu \in (0, \gamma]$ we can find neighborhoods U_μ of $\bar{x}$, V_μ of $\bar{y}$ with the property (1.4). Replacing U by $U \cap U_\mu$, and V by $V \cap V_\mu$, we see that the inequality in (1.3) is fulfilled because, by virtue of (1.4), $d(0, F(x, y)) < \mu \leq \gamma$ for every $(x, y) \in U_\mu \times V_\mu$. Thus, if F is inner semicontinuous at $(\bar{x}, \bar{y}, 0)$, then G has the Robinson stability at $\omega_0 = (\bar{x}, \bar{y}, 0)$ if and only if there exist $r > 0$ and neighborhoods U of $\bar{x}$, V of $\bar{y}$ such that

$$d(x, G(y)) \leq r d(0, F(x, y)) \quad \forall (x, y) \in U \times V.$$

1.3 An Overview on Implicit Function Theorems for Multifunctions

Consider an implicit multifunction of the form

$$S(w) = \{x \in \mathbb{R}^n \, : \, 0 \in G(x, w) + M(x, w)\}, \tag{1.5}$$

with $G : \mathbb{R}^{n+d} \to \mathbb{R}^m$ being a continuously Fréchet differentiable function and $M : \mathbb{R}^{n+d} \rightrightarrows \mathbb{R}^m$ a multifunction with closed graph. Let $(\bar{w}, \bar{x}) \in \operatorname{gph} S$ and $\bar{\tau} = (\bar{w}, \bar{x}, -G(\bar{x}, \bar{w}))$.

Theorem 1.3 (see [41, Theorem 2.1]) *If the constraint qualification*

$$0 \in \nabla G(\bar{x}, \bar{w})^* v_1' + D^* M(\bar{\tau})(v_1') \implies v_1' = 0 \tag{\textbf{C1}}$$

is satisfied, then the upper estimate

$$D^* S(\bar{w}|\bar{x})(x') \subset \Gamma(x'),$$

where

$$\Gamma(x') := \bigcup_{v_1' \in \mathbb{R}^n} \{w' \in \mathbb{R}^d \,:\, (-x', w') \in \nabla G(\bar{x}, \bar{w})^* v_1' + D^* M(\bar{\tau})(v_1')\},$$

is valid for any $x' \in \mathbb{R}^n$. If, in addition, either M is graphically regular at $\bar{\tau}$, or $M = M(x)$ and $\nabla_w G(\bar{x}, \bar{w})$ has full rank, then

$$D^* S(\bar{w}|\bar{x})(x') = \Gamma(x').$$

Theorem 1.4 (see [39, Theorem 3.4]) *The lower estimates*

$$\widehat{\Gamma}(x') \subset \widehat{D}^* S(\bar{w}|\bar{x})(x') \subset D^* S(\bar{w}|\bar{x})(x'), \tag{1.6}$$

where

$$\widehat{\Gamma}(x') := \bigcup_{v_1' \in \mathbb{R}^n} \left\{w' \in \mathbb{R}^d \,:\, (-x', w') \in \nabla G(\bar{x}, \bar{w})^* v_1' + \widehat{D}^* M(\bar{\tau})(v_1')\right\}, \tag{1.7}$$

hold for any $x' \in \mathbb{R}^n$.

Put $\widetilde{M}(x, w) = G(x, w) + M(x, w)$. From (1.5) we have

$$S(w) = \{x \in \mathbb{R}^n \,:\, 0 \in \widetilde{M}(x, w)\}. \tag{1.8}$$

By the Fréchet coderivative sum rule with equalities [50, Theorem 1.62],

$$\widehat{D}^* \widetilde{M}(\omega_0)(v_1') = \nabla G(\bar{x}, \bar{w})^* v_1' + \widehat{D}^* M(\bar{\tau})(v_1')$$

for any $v_1' \in \mathbb{R}^n$, where $\omega_0 := (\bar{x}, \bar{w}, 0) \in \operatorname{gph} \widetilde{M}$. Therefore, we can write

$$\widehat{\Gamma}(x') = \bigcup_{v_1' \in \mathbb{R}^n} \left\{w' \in \mathbb{R}^d \,:\, (-x', w') \in \widehat{D}^* \widetilde{M}(\omega_0)(v_1')\right\}.$$

The first estimate in (1.6) was obtained by Ledyaev and Zhu [36, Proposition 3.7] for a Banach space setting under a set of conditions. Latter, by giving a simple proof, Lee and Yen [39] have showed that the estimate holds for a Banach space setting and the closedness of gph M is an extra assumption. (See [39, Remark 3.2] for comments on lower estimate for the values of the Fréchet coderivative of implicit multifunctions.)

Yen and Yao [88] gave a couple of conditions guaranteeing the Robinson stability of implicit multifunctions. In Chapters 2 and 3, we will show that, for the linear constraint systems, these conditions are also necessary.

Theorem 1.5 (see [88, Theorem 3.1]) *Let S be the implicit multifunction defined by* (1.8). *If* gph $\widetilde{M}$ *is locally closed around the point* $\omega_0 := (\bar{x}, \bar{w}, 0)$ *and*

(a) $\ker D^*\widetilde{M}(\bar{\tau}) = \{0\}$,

(b) $\left\{w' \in \mathbb{R}^d : \exists v_1' \in \mathbb{R}^n \text{ with } (0, w') \in D^*\widetilde{M}(\omega_0)(v_1')\right\} = \{0\}$,

then S has the Robinson stability around ω_0.

Now we are going to find out how the above implicit multifunction theorems can be used to obtain our desired results.

Chapter 2

Linear Constraint Systems under Total Perturbations

The present chapter is devoted to stability analysis of linear constraint systems, linear complementarity systems, and affine variational inequalities under total perturbations. It is written on the basis of the paper [31], where a new concept of linear constraint system was proposed. In that paper, the first time, the concept "uniform local error bounds" for linear complementarity problems and affine variational inequality has been defined. Recently, the paper has been cited by C. Li and K.F. Ng (see [43]).

2.1 An Introduction to Parametric Linear Constraint Systems

In this chapter, we study the Lipschitz-like property and the Robinson stability of the solution map of a *parametric linear constraint system* in the form

$$Ax + b \in K, \tag{2.1}$$

with $A \in \mathbb{R}^{m\times n}$ being an $m \times n$ matrix, $b \in \mathbb{R}^m$ a vector, and $K \subset \mathbb{R}^m$ a closed set. When K is a cone, (2.1) can be formally rewritten as

$$Ax + b \geq_K 0, \tag{2.2}$$

where $v \geq_K u$ means that $v - u \in K$. In addition, if K is convex then the partial order "$\geq_K$" is transitive. For $K = \mathbb{R}^m_+$, where $\mathbb{R}^m_+$ denotes the nonnegative orthant in $\mathbb{R}^m$, (2.2) is a standard linear inequality system.

Unlike the traditional considerations (see, e.g., [9, 75, 87]), *here K needs not to be convex.* This small change, seemingly, brings us a lot of benefits in using theoretical results.

The multifunction $S : \mathbb{R}^{m\times n} \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with

$$S(A, b) := \{x \in \mathbb{R}^n : Ax + b \in K\}$$

is said to be the *solution map* of (2.1). We interpret the pair (A, b) as a parameter. With K being fixed, in the sequel, we will allow *both the linear part* (that is vector b) *and the nonlinear part* (matrix A) *of the data set* $\{A, b\}$ *of* (2.1) *to change.* It is easy to see that the solution map $(A, b) \mapsto S(A, b)$ is a special case of the implicit multifunction $y \mapsto G(y)$ defined by (1.2).

The aim of this chapter will be achieved by using the Mordukhovich Criterion 1 and a formula for computing exactly the limiting coderivative of implicit multifunctions obtained by Levy and Mordukhovich (Theorem 1.3), as well as a result from Yen and Yao on the Robinson stability of implicit multifunctions (Theorem 1.5).

The abstract stability results of (2.1) can be effectively applied to

(a) traditional inequality systems,

(b) linear complementarity problems,

(c) affine variational inequalities

to yield necessary and sufficient conditions for the Lipschitz-like property and the Robinson stability of the related solution maps as well as uniform local error bounds and traditional local error bounds.

According to the classification in [50, Chapter 4], (a) is a class of constraint systems, while (b) and (c) are two classes of variational systems. Note that various sufficient conditions for the Lipschitz-likeness of the solution map of parametric constraint systems and variational systems were given by B. S. Modukhovich and other authors; see [50, Chapter 4], [53], and the references therein. It is well known that qualitative studies of variational systems are more difficult than those of constraint systems. Interestingly, *despite to the fact that* (2.1) *itself is a constraint system, the model can be employed to investigate special variational systems like* (b) *and* (c).

Although stability properties of (a)–(c) and related models have been studied intensively by many authors with various tools (see, e.g., [16, 22–24, 37, 38, 62, 69–71, 74, 75]), the results obtained herein are new. Namely, in this chapter we are able to establish the equivalence between the Lipschitz-like property and the Robinson stability of the solution map of (2.1) and provide a verifiable regularity condition which completely characterizes the two properties. In addition, using the obtained results, we give necessary and sufficient conditions for the Lipschitz-like property and the Robinson stability of the solution map of traditional generalized linear inequality systems under nonlinear perturbations. By reducing linear complementarity problems to the linear constraint system (2.1), we give a new result on the Lipschitz-like property of their solution maps under nonlinear perturbations as well as two related local error bounds. Similarly, at the end of the chapter, we show regularity conditions which guarantee two local error bounds and the solution map of a broad class affine variational inequalities being Lipschitz-like at the reference point when both the basic operator and the constraint system undergo nonlinear perturbations.

Since the linear complementarity problem is a type of affine variational inequality and since the major applications of our theoretical results are related to these models, we now give a brief survey of the preceding results on the solution stability of affine variational inequalities.

Dontchev and Rockafellar [18, Theorem 1] showed the equivalence between the Lipschitz-like property of the solution map of a affine variational inequality under canonical perturbation and others such as the semicontinuity property and the strong regularity. Yao and Yen [85,86] studied the Lipschitz-like property of the solution map of affine variational inequalities under linear perturbations. Henrion, Mordukhovich and Nam [27] gave a comprehensive second-oder analysis of polyhedral systems in finite and infinite dimensions and applications to robust stability of variational inequalities, including the affine problems. In a series of papers, Qui [65,67,70] investigated the stability of the solution map of the parametric affine variational inequality with the matrix M being fixed. Later, Qui [68] derived new results on solution stability of parametric affine variational inequality under nonlinear perturbations.

In a recent paper [26], Henrion *et al.* have computed Fréchet coderivative of the solution map of a parametric variational inequality, whose constraint set is fixed. Note that the constraint qualification condition, denoted by CRCQ (Constant Rank Constraint Qualification), automatically holds for inequality systems given by affine functions. Hence, using [26, Theorem 3.2], one obtains necessary conditions for the Lipschitz-like property of the solution map of a parametric affine variational inequality *with the constraint set being fixed.* Sufficient conditions for the Lipschitz-like property of the solution map in question can be derived from the results of [18, 57].

Since we will focus mainly on parametric affine variational inequalities with the constraint sets being perturbed, our results not only differ from those of [18, 26, 57], but also from other existing results in [27, 65–68, 70, 85, 86].

2.2 The Solution Maps of Parametric Linear Constraint Systems

Let $K \subset \mathbb{R}^m$ be a fixed closed set. For any pair $(A, b) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m$, we consider the parametric linear constraint system (2.1). Put $W = \mathbb{R}^{m\times n} \times \mathbb{R}^m$. For every $w = (A, b) \in W$, we set $G(x, w) = -Ax - b$, $M(x, w) = K$, and $\widetilde{M}(x, w) = G(x, w) + M(x, w)$. Then, the solution map of (2.1) is given by

$$S(w) = \{x \in \mathbb{R}^n \,|\, 0 \in \widetilde{M}(x, w)\}.$$

From now on, let us fix an element $\bar{w} = (\bar{A}, \bar{b})$ *and suppose that* $\bar{x} \in S(\bar{w})$ *with* $\bar{x} = \left(\bar{x}_1, \dots \bar{x}_n\right)^T$. Here and in the sequel, the superscript ^{T} denotes matrix transposition.

We will investigate the Lipschitz-like property of S around the point $(\bar{w}, \bar{x})$ in the graph of S and the Robinson stability of S at $(\bar{x}, \bar{w}, 0)$.

Since $G : \mathbb{R}^n \times W \to \mathbb{R}^m$ is a continuously differentiable mapping, the coderivative of G is the conjugate operator of its derivative by [50, Theorem 1.38, Vol. 1, p. 45]. We can determine the operator

$$\nabla G(\bar{x}, \bar{w})^* : \mathbb{R}^m \to \mathbb{R}^n \times W^*$$

by some arguments used in [40] (here we have $W^* = W$). Note that Lee and Yen [40] have considered the mapping $G(x, w) = Ax$, while here we have $G(x, w) = -Ax - b$.

Lemma 2.1 *For any $v' = \left(v'_1, \ldots, v'_m\right)^T \in \mathbb{R}^m$, it holds that*

$$\nabla G(\bar{x}, \bar{w})^*(v') = \{-\bar{A}^T v'\} \times \{-(v'_i \bar{x}_j)\} \times \{-v'\}, \tag{2.3}$$

where $(v'_i \bar{x}_j)$ is the $m \times n$ matrix whose $(i,j)-$th element is $v'_i \bar{x}_j$.

Proof. The formula $T(x, w) = -A\bar{x} - \bar{A}x - b$, where $x \in \mathbb{R}^n$ and $w = (A, b)$ belongs to W, defines a linear operator $T : \mathbb{R}^n \times W \to \mathbb{R}^m$. We have

$$\begin{aligned}
&\lim_{(x,w)\to(0,0)} \frac{G((\bar{x}, \bar{w}) + (x, w)) - G(\bar{x}, \bar{w}) - T(x, w)}{\|(x, w)\|} \\
&= \lim_{(x,w)\to(0,0)} \frac{-(\bar{A} + A)(\bar{x} + x) - (\bar{b} + b) + \bar{A}\bar{x} + \bar{b} + A\bar{x} + \bar{A}x + b}{\|x\| + \|w\|} \\
&= \lim_{(x,w)\to(0,0)} \frac{-Ax}{\|x\| + \|w\|}.
\end{aligned} \tag{2.4}$$

Since

$$\frac{\|Ax\|}{\|x\| + \|w\|} \le \frac{\|Ax\|}{\|A\|} \le \frac{\|A\|\|x\|}{\|A\|} = \|x\|$$

and x tends to 0, from (2.4) we can deduce that T is the Fréchet derivative of G at $(\bar{x}, \bar{w})$, i.e., $\nabla G(\bar{x}, \bar{w}) = T$.

To find a formula for the adjoint operator $T^* : \mathbb{R}^m \to \mathbb{R}^n \times W^*$, we fix a vector $v' \in \mathbb{R}^m$. For every $(x', A', b') \in \mathbb{R}^n \times W^*$, observe that

$$\begin{aligned}
\langle T^* v', (x', A', b') \rangle = \langle v', T(x', A', b') \rangle &= \langle v', -A'\bar{x} - \bar{A}x' - b' \rangle \\
&= -\langle \bar{A}^T v', x' \rangle - (v')^T A' \bar{x} - \langle v', b' \rangle.
\end{aligned} \tag{2.5}$$

If $A' = (a'_{ij})$, then

$$\begin{aligned}
(v')^T A' \bar{x} &= v'_1(a'_{11}\bar{x}_1 + \cdots + a'_{1n}\bar{x}_n) + \cdots + v'_m(a'_{m1}\bar{x}_1 + \cdots + a'_{mn}\bar{x}_n) \\
&= \sum_{i=1}^m \sum_{j=1}^n (v'_i \bar{x}_j) a'_{ij}.
\end{aligned}$$

Thus, (2.5) implies that

$$T^* v' = \{-\bar{A}^T v'\} \times \{-(v'_i \bar{x}_j)\} \times \{-v'\}.$$

Since $\nabla G(\bar{x}, \bar{w}) = T$, this etablishes formula (2.3). □

Put $\bar{v} = -G(\bar{x}, \bar{w}) = \bar{A}\bar{x} + \bar{b}$. Then, $\bar{x} \in S(\bar{w})$ if and only if $\bar{v} \in K$. As a consequence, $\bar{\tau} := (\bar{x}, \bar{w}, \bar{v})$ belongs to the graph of the constant multifunction $M : \mathbb{R}^n \times W \rightrightarrows \mathbb{R}^m$. Next, we will calculate the Mordukhovich coderivative $D^* M(\bar{\tau}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times W^*$ of M at $\bar{\tau}$.

Lemma 2.2 *For any $v' \in \mathbb{R}^m$, it holds that*

$$D^*M(\bar{\tau})(v') = \begin{cases} \{0\} & \text{if } v' \in -N(\bar{v}; K) \\ \emptyset & \text{if } v' \notin -N(\bar{v}; K). \end{cases} \tag{2.6}$$

Proof. In accordance with the definitions of coderivatives, we have

$$D^*M(\bar{\tau})(v') = \{(x^*, w^*) \in \mathbb{R}^n \times W^* \,:\, (x^*, w^*, -v') \in N(\bar{\tau}; \operatorname{gph} M)\}. \tag{2.7}$$

Since $\operatorname{gph} M = \mathbb{R}^n \times W \times K$, by [50, Proposition 1.2] we get

$$N(\bar{\tau}; \operatorname{gph} M) = N((\bar{x}, \bar{w}); \mathbb{R}^n \times W) \times N(\bar{v}; K) = \{0_{\mathbb{R}^n \times W}\} \times N(\bar{v}; K).$$

Combining this with (2.7), we obtain (2.6). □

A criterion for the Lipschitz-like property of the solution map S of (2.1) around $(\bar{w}, \bar{x})$ is formulated in the following theorem which is our first main result.

Theorem 2.1 *The mapping S is Lipschitz-like around $(\bar{w}, \bar{x})$ if and only if*

$$\left(\ker \bar{A}^T\right) \cap N(\bar{v}; K) = \{0\}, \tag{2.8}$$

where $\bar{v} = \bar{A}\bar{x} + \bar{b}$ and $\ker \bar{A}^T := \{v' \in \mathbb{R}^m : \bar{A}^T v' = 0\}$ is the kernel of $\bar{A}^T$.

Proof. We first compute coderivative values of S. Note that the multifunction $(x, w) \mapsto \widetilde{M}(x, w)$ has closed graph because G is a continuous single-valued map and K is closed. In addition, $\nabla_w G(\bar{x}, \bar{w})$ is a surjective operator. Therefore, by a result of Levy and Mordukhovich [41, Theorem 2.1], we obtain

$$D^*S(\bar{w}|\bar{x})(x') = \bigcup_{v' \in \mathbb{R}^m} \left\{ w' \in W^* \,:\, (-x', w') \in \nabla G(\bar{x}, \bar{w})^* v' + D^*M(\bar{\tau})(v') \right\}, \tag{2.9}$$

for any $x' \in \mathbb{R}^n$. Thanks to Lemmas 2.1 and 2.2, we get

$$D^*S(\bar{w}|\bar{x})(x') = \left\{ \{-(v'_i \bar{x}_j)\} \times \{-v'\} \,:\, v' \in -N(\bar{v}; K) \text{ with } x' = A^T v' \right\}. \tag{2.10}$$

It is easy to show that $\operatorname{gph} S$ is closed. Then, by virtue of the Mordukhovich Criterion 1 (see Theorem 1.1), S is Lipschitz-like around $(\bar{w}, \bar{x})$ if and only if

$$D^*S(\bar{w}|\bar{x})(0) = \{0\}. \tag{2.11}$$

This is equivalent to

$$\left\{ \{-(v'_i \bar{x}_j)\} \times \{-v'\} \,:\, v' \in -N(\bar{v}; K) \text{ with } 0 = \bar{A}^T v' \right\} = \{0\}.$$

It is clear that $\Big\{\{-(v'_i\bar{x}_j)\} \times \{-v'\}\Big\} = \{0\}$ if and only if $v' = 0$. Thus, S is Lipschitz-like around $(\bar{w}, \bar{x})$ if and only if

$$\ker\bar{A}^T \cap (-N(\bar{v}; K)) = \{0\}.$$

Since $\ker\bar{A}^T$ is a linear subspace of $\mathbb{R}^m$, the last equality is equivalent to (2.8).

□

Equality (2.8) is a necessary and sufficient condition for S to be Lipschitz-like around $(\bar{w}, \bar{x})$. If either $\ker\bar{A}^T = \{0\}$ or $N(\bar{v}; K) = \{0\}$, then (2.8) is fulfilled. In next examples, we consider two simple linear constraint systems where $\ker\bar{A}^T \neq \{0\}$ and $N(\bar{v}; K) \neq \{0\}$ to see how Theorem 2.1 works in practice.

Example 2.1 Consider the case where $n = m = 2$,

$$K = \{(u, v) \in \mathbb{R}^2 \ : \ u.v = 0\},$$

and $W = \mathbb{R}^{2\times 2} \times \mathbb{R}^2$. Let $\bar{A} = \begin{pmatrix} 0 & -0.1 \\ 0 & 1 \end{pmatrix}$, $\bar{b} = \begin{pmatrix} 0 \\ 0.01 \end{pmatrix}$, and $\bar{x} = (0, 0)^T$. Here we have $\bar{v} = (0, 0.01)^T$. Since

$$\ker\bar{A}^T = \{v' = (v'_1, v'_2)^T \in \mathbb{R}^2 \ : \ v'_2 = 0.1v'_1\}$$

and $N(\bar{v}; K) = \{v' = (v'_1, v'_2)^T \in \mathbb{R}^2 \ : \ v'_2 = 0\}$, (2.8) is satisfied. Hence $S(\cdot)$ is Lipschitz-like around $(\bar{w}, \bar{x})$ by Theorem 2.1.

Example 2.2 Suppose that n, m, K are as in Example (2.1) and $S : W \rightrightarrows \mathbb{R}^2$ is defined by $S(w) = \{x \in \mathbb{R}^2 \mid Ax + b \in K\}$, where $w = (A, b) \in W$. Let $\bar{A} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\bar{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and put $\bar{w} := (\bar{A}, \bar{b})$. We pay attention to the point $\bar{x} := (0, 0)^T$ that belongs to $S(\bar{w}) = \mathbb{R}^2$. By taking $w_\varepsilon = (A_\varepsilon, b)$, where $A_\varepsilon = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$ and $\varepsilon > 0$ is small enough, we have

$$S(w_\varepsilon) = \{x = (x_1, x_2)^T \in \mathbb{R}^2 \ : \ \varepsilon x_1.x_2 = 0\} = K.$$

To show that $S(.)$ is not Lipschitz-like around $(\bar{w}, \bar{x})$, we suppose on the contrary that there exist $\ell > 0$, $U \in \mathcal{N}(\bar{w})$, and $V \in \mathcal{N}(\bar{x})$ such that

$$S(w') \cap V \subset S(w) + \ell\|w' - w\|\bar{B}_{\mathbb{R}^2} \quad \forall w, w' \in U. \tag{2.12}$$

Choose $\rho > 0$ as small as $\bar{B}(0,\rho) \subset V$. For $w' = \bar{w}$ and $w = w_\varepsilon$, by (2.12) we get

$$V \subset K + \ell\|\bar{w} - w_\varepsilon\|\bar{B}_{\mathbb{R}^2}$$

or, equivalently,

$$V \subset K + \ell\varepsilon\bar{B}_{\mathbb{R}^2}.$$

Clearly, this conclusion is unavailable for any $\varepsilon \in (0, \frac{\rho}{2\ell})$. We have thus proved that $S(.)$ is not Lipschitz-like around $(\bar{w}, \bar{x})$. Now, let us show that (2.8) is not satisfied. Since $\bar{v} = \bar{A}\bar{x} + \bar{b} = (0,0)^T$, $N(\bar{v}; K) = K$. We have

$$\ker \bar{A}^T = \{v' = (v'_1, v'_2)^T \in \mathbb{R}^2 \,:\, v'_2 = 0\}.$$

Thus, $\ker \bar{A}^T \cap N(\bar{v}; K) \neq \{0\}$.

Remark 2.1 One referee of the paper [31] has shown us another proof for our result in Theorem 2.1. Namely, setting $f(x,p) = -Ax - b$ for all $x \in \mathbb{R}^n$ and $p = (A,b) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m$, $F(x) = K$ for all $x \in \mathbb{R}^n$, we see that our system (1.2) (resp., our solution map (1.4)) reduces to the system (3) (resp., the solution map (4)) in [19, p. 172]. Putting $H(x) = -\bar{A}x - \bar{b} + K$ for $x \in \mathbb{R}^n$, by the Mordukhovich criterion 2 of the classical metric regularity property (see Theorem 1.2) we can assert that H is *metrically regular* at $\bar{x}$ for $0 \in H(\bar{x})$ (in the sense explained in [19, p. 164]) if and only if $\ker D^*H(\bar{x}|0) = \{0\}$. According to Theorem 3F.9 from [19, p. 173], if H is metrically regular at $\bar{x}$ then the solution map $(A,b) \mapsto S(A,b)$ is *Lipschitz-like* around $(\bar{p}, \bar{x})$ with $\bar{p} := (\bar{A}, \bar{b})$ if $\ker D^*H(\bar{x}|0) = \{0\}$. Since $D^*H(\bar{x}|0)(u) = \{-A^T u\}$ for any $u \in -N(\bar{A}\bar{x} + \bar{b}; K)$ and $D^*H(\bar{x}|0)(u) = \emptyset$ for any $u \notin -N(\bar{A}\bar{x} + \bar{b}; K)$, the latter can be rewritten equivalently as (2.8). As $\nabla_p f(\bar{x}, \bar{p})(A,b) = -A\bar{x} - b$, we have $\operatorname{rank}\nabla_p f(\bar{x}, \bar{p})(A,b) = m$. Hence, by the second assertion of Theorem 3F.9 from [19, p. 173], if the solution map $(A,b) \mapsto S(A,b)$ is Lipschitz-like around $(\bar{p}, \bar{x})$ then H is metrically regular at $\bar{x}$. Therefore, the last two properties are equivalent, and they hold if and only if (2.8) is valid. This is the content of Theorem 2.1. The idea of Theorem 3F.9 from [19, p. 173] is to study the Lipschitz-like property of the solution map $S(.)$ via the metric regularity of the multifunction

$$H(x) = f(\bar{x}, \bar{p}) + \nabla_x f(\bar{x}, \bar{p})(x - \bar{x}) + F(x),$$

which is an approximation of the map $x \mapsto f(x, \bar{p}) + F(x)$. The proof of this theorem is rather complicated. Our approach is simpler and more elementary. Namely, instead of using the approximation multifunction $H(x)$, we use a formula for the coderivative of implicit multifunctions from [41, Theorem 2.1]. Note that the latter was obtained just by invoking a basic sum rule and a chain rule for coderivative.

We now turn our attention to the Robinson stability of the solution map S at a given point. Firstly, using the specific structure of the constraint system (2.1), we will prove that Robinson stability implies the Lipschitz-like property of S around the corresponding point.

Lemma 2.3 *If S has the Robinson stability at $\omega_0 := (\bar{w}, \bar{x}, 0)$, then it is Lipschitz-like around $(\bar{w}, \bar{x})$.*

Proof. Suppose S has the Robinson stability at ω_0. It is easy to check that the multifunction $\widetilde{M}(x, w) = -Ax - b + K$, where $w = (A, b)$, has the inner semicontinuity property defined in (1.4) at the point $\omega_0 = (\bar{w}, \bar{x}, 0) \in \operatorname{gph} \widetilde{M}$. As observed in Sect. 1.2, by the Robinson stability of S we can find a constant $r > 0$, and bounded neighborhoods U of $\bar{w}$ and V of $\bar{x}$, such that

$$d(x; S(w)) \leq rd(0; F(x, w)) \tag{2.13}$$

for any $w \in U$ and $x \in V$. In particular, $S(w) \neq \emptyset$ for all $w \in U$.

As V is bounded, we can choose $\alpha > 0$ such that $\|x\| \leq \alpha$ for all $x \in V$. To show that S is Lipschitz-like around $(\bar{w}, \bar{x})$, take two distinct elements $w_1 = (A_1, b_1)$, $w_2 = (A_2, b_2)$ in U, and suppose $x_1 \in S(w_1) \cap V$ is chosen arbitrarily. Then, we have $\|x_1\| \leq \alpha$ and $A_1x_1 + b_1 \in K$. By (2.13),

$$\begin{aligned} d(x_1; S(w_2)) \leq rd(0; \widetilde{M}(x_1, w_2)) &= rd(0; -A_2x_1 - b_2 + K) \\ &= rd(A_2x_1 + b_2; K) \\ &\leq r\|(A_2x_1 + b_2) - (A_1x_1 + b_1)\| \\ &= r\|(A_2 - A_1)x_1 + (b_2 - b_1)\| \\ &\leq r(\|A_2 - A_1\|\|x_1\| + \|b_2 - b_1\|) \\ &= r(\alpha\|A_2 - A_1\| + \|b_2 - b_1\|). \end{aligned}$$

So, setting $\ell = \max\{r,\ r\alpha\}$, we have

$$d(x_1; S(w_2)) \leq \ell\|w_2 - w_1\| \quad \forall x_1 \in S(w_1) \cap V.$$

Since $S(w_2) \subset \mathbb{R}^n$ is a nonempty closed set, for every x_1 there is $x_2 \in S(w_2)$ with $d(x_1; S(w_2)) = d(x_1; x_2)$. Therefore,

$$d(x_1; x_2) \leq \rho\|w_2 - w_1\|;$$

and hence,

$$x_1 \in S(w_2) + \rho \|w_2 - w_1\| \bar{B}_{\mathbb{R}^n}.$$

Consequently, we get

$$S(w_1) \cap V \subset S(w_2) + \rho \|w_2 - w_1\| \bar{B}_{\mathbb{R}^n}.$$

We have thus proved that S is Lipschitz-like around $(\bar{w}, \bar{x})$. □

Secondly, we give a sufficient condition for the Robinson stability of S at $\omega_0 = (\bar{w}, \bar{x}, 0)$ in next lemma.

Lemma 2.4 *If condition* (2.8) *is satisfied, then S has the Robinson stability at $\omega_0 = (\bar{w}, \bar{x}, 0)$.*

Proof. In [88, Theorem 3.1], the authors have proposed two conditions guaranteeing that S has the Robinson stability at ω_0 (see Theorem 1.5). The first one is

$$\ker D^* \widetilde{M}(\omega_0) = \{0\}, \tag{2.14}$$

and the second is

$$\Big\{ w' \in W^* \,:\, \exists v' \in \mathbb{R}^m \text{ with } (0, w') \in D^* \widetilde{M}(\omega_0)(v') \Big\} = \{0\}. \tag{2.15}$$

Due to the sum rule for the Mordukhovich coderivative [50, Theorem 1.62], for any $v' \in \mathbb{R}^m$, we have

$$D^* \widetilde{M}(\omega_0)(v') = \nabla G(\bar{x}, \bar{w})^* v' + D^* M(\bar{\tau})(v').$$

Hence, condition (2.14) becomes

$$\big(0 \in \nabla G(\bar{x}, \bar{w})^* v' + D^* M(\bar{\tau})(v')\big) \implies v' = 0.$$

Let us show that it holds automaticlly here. According to Lemmas 2.1 and 2.2, condition (2.14) becomes

$$\big[v' \in -N(\bar{v}; K) \text{ and } \{-\bar{A}^T v'\} \times \{-(v'_i \bar{x}_j)\} \times \{-v'\} = 0\big] \implies v' = 0.$$

This implication is valid because $\{-\bar{A}^T v'\} \times \{-(v'_i \bar{x}_j)\} \times \{-v'\} = 0$ if and only if $v' = 0$. By (2.9), condition (2.15) means that (2.11) is satisfied. In the proof of Theorem 2.1, we have shown that the latter is equivalent to (2.8). In other words, condition (2.8) guarantees the Robinson stability of S at ω_0.

□

As shown by Jeyakumar and Yen [34, p. 1119], the Lipschitz-like property of an implicit multifunction doesn't imply the Robinson stability, and vice-versa. But, for the solution map S of (2.1), these properties are equivalent. Let us explain why it happens so. First, for a fixed x, the map

$$w = (A, b) \mapsto F(x, w) = -Ax - b$$

is a linear operator. In Example 3.6 from [34], which shows that the Robinson stability does not yield the Lipschitz-like property, the corresponding map is nonlinear. Second, for a fixed $w = (A, b)$, the map

$$x \mapsto F(x, w) = -Ax - b$$

is an affine operator. Meanwhile, in Example 3.7 from [34], which shows that the Lipschitz-like property does not imply the Robinson stability, the corresponding map is again nonlinear. Thus, the equivalence between the two properties in question is available for the solution map of (2.1) because, although the map $(x, w) \mapsto F(x, w) = -Ax - b$ with $w := (A, b)$ is nonlinear, it is *bi-affine* on the variables x and w.

Our second main result is as follows.

Theorem 2.2 *The Lipschitz-like property of S around $(\bar{w}, \bar{x})$ and its Robinson stability at $\omega_0 = (\bar{w}, \bar{x}, 0)$ are equivalent. Moreover, these properties appear if and only if condition* (2.8) *is satisfied.*

Proof. The assertions follow from Theorem 2.1 and Lemmas 2.3 and 2.4. □

Remark 2.2 The equivalence between the Robinson stability and the Lipschitz-like property comes by dint of the intermediate condition (2.1). We don't have any direct proof of the fact that the latter property also implies the former.

Remark 2.3 Since the Mordukhovich coderivative of S at $(\bar{w}, \bar{x})$ can be computed explicitly by (2.10), the exact Lipschitzian bound of S around $(\bar{w}, \bar{x})$ can be estimated by invoking formula (4.5) from [50, Theorem 4.10]. Then, the formula $\ell = \max\{r,\ r\alpha\}$, which has been used in the proof of Lemma 2.3, gives us some idea about the relationships between the exact Lipschitzian bound of S around $(\bar{w}, \bar{x})$ and the exact Robinson regularity bound of S at $\omega_0 = (\bar{w}, \bar{x}, 0)$. A deeper analysis of the two bounds can lead us to an upper estimate for the latter.

Condition (2.1) means that the conjugate operator $\bar{A}^T$ well interacts with the normal cone $N(\bar{v}, K)$. Namely, it requires that the cone $N(\bar{v}, K)$ doesn't have any non-zero common element with the linear subspace $\ker\bar{A}^T$ of $\mathbb{R}^m$. To calculate $\ker\bar{A}^T$, we have to solve a homogeneous system of linear equations, which contains n equations and m variables. If K is a polyhedral convex set, one has an explicit formula for $N(\bar{v}, K)$. If K is defined by finitely many smooth equalities and inequalities, a formula for the cone $N(\bar{v}, K)$ is also available.

2.3 Stability Properties of Generalized Linear Inequality Systems

With K specially being a closed convex cone, applying the results of the previous section to the generalized linear inequality system (2.2), we can describe necessary and sufficient conditions for the Lipschitz-like property and the Robinson stability of the solution map S as follows.

Theorem 2.3 *If K is a closed convex cone, then the following properties are equivalent:*

(a) *S is Lipschitz-like around $(\bar{w}, \bar{x})$;*

(b) *S has the Robinson stability at $\omega_0 = (\bar{w}, \bar{x}, 0)$;*

(c) $\big(\ker \bar{A}^T\big) \cap N(\bar{v}; K) = \{0\}$;

(d) $(\ker \bar{A}^T) \cap K^* \cap (\bar{v})^\perp = \{0\}$, *where* $K^* = \{v' \in \mathbb{R}^m \mid \langle v', v\rangle \le 0, \forall v \in K\}$, $\bar{v} = \bar{A}\bar{x} + b$, *and* $(\bar{v})^\perp := \{v' \in \mathbb{R}^m : \langle v', \bar{v}\rangle = 0\}$;

(e) $0 \in \operatorname{int}(\operatorname{rge} \bar{A} + K - \bar{v})$, *where* $\operatorname{rge} \bar{A} := \bar{A}\,(\mathbb{R}^n)$ *is the range of* $\bar{A}$ *and* $\operatorname{int}\Omega$ *denotes the interior of* Ω;

(f) $\operatorname{rge} \bar{A} + \operatorname{cone}(K - \bar{v}) = \mathbb{R}^m$, *where* $\operatorname{cone} C$ *is the cone generated by* C.

Thus, each one of the following conditions is sufficient for (a) *and* (b) *to hold:*

$$
\begin{aligned}
&(\ker \bar{A}^T) \cap K^* = \{0\};\\
&(\ker \bar{A}^T) \cap (\bar{v})^\perp = \{0\};\\
&K^* \cap (\bar{v})^\perp = \{0\}.
\end{aligned}
$$

Proof. The equivalence of (a), (b), and (c) follows from Theorem 2.2.

To show that (c) is equivalent to (d), we prove $N(\bar{v};K) = K^* \cap (\bar{v})^\perp$. Once $v' \in K^*$ and $v' \in (\bar{v})^\perp$, $v' \in N(\bar{v};K)$ is easy. If $v' \in N(\bar{v};K)$ then $\langle v', v - \bar{v}\rangle \leq 0$ for all $v \in K$. Substituting $v = 0$ and $v = 2\bar{v}$ in turn to the last inequality, we have $\langle v', \bar{v}\rangle \geq 0$ and $\langle v', \bar{v}\rangle \leq 0$. Hence $\langle v', \bar{v}\rangle = 0$, or $v' \in (\bar{v})^\perp$. In addition, for any v in K, replacing v by $v + \bar{v}$ in this inequality, we have $\langle v', v\rangle \leq 0$; therefore, v' belongs to K^*.

To prove (c) implies (e), suppose on the contrary that (c) is valid, but $0 \in \mathbb{R}^m$ is a boundary point of the convex set $C := \operatorname{rge} \bar{A} + K - \bar{v}$. By the separation theorem [78, Corollary 11.6.1, p. 100] , we can find a nonzero vector $x^* \in \mathbb{R}^m$ such that $\langle x^*, y\rangle \leq 0$ for every $y \in C$. Substituting $y = \bar{A}x$ into the last inequality yields

$$\langle \bar{A}^T x^*, x\rangle = \langle x^*, \bar{A}x\rangle \leq 0 \quad \forall x \in \mathbb{R}^n;$$

hence $x^* \in \ker \bar{A}^T$. Now, for $y = v - \bar{v}$, $v \in K$, we get $\langle x^*, v - \bar{v}\rangle \leq 0$ for all $v \in K$. This means that $x^* \in N(\bar{v};K)$. Thus, the property $x^* \neq 0$ contradicts (c).

Since $\operatorname{cone}(\operatorname{rge}\bar{A} + (K - \bar{v})) = \operatorname{rge}\bar{A} + \operatorname{cone}(K - \bar{v})$, (e) implies (f).

To complete the proof, we need to show that (f) yields (c). If x^* belongs to $(\ker \bar{A}^T) \cap N(\bar{v};K)$, then $\langle x^*, \bar{A}x\rangle = 0$ for all $x \in \mathbb{R}^n$ and $\langle x^*, v - \bar{v}\rangle \leq 0$ for all $v \in K$. Therefore,

$$\langle x^*, \bar{A}x + \lambda(v - \bar{v})\rangle \leq 0 \quad \forall x \in \mathbb{R}^n, v \in K, \lambda \geq 0.$$

By (f), we can assert that $x^* = 0$. Thus, (c) holds the true. □

Remark 2.4 The equivalences among (b), (e), and (f) were established by Robinson [74] long time ago.

Remark 2.5 We now look at the case where K is the second-order cone or the positive semidefinite cone because this corresponds to the solution maps of *parametric second-order cone programming problems* (see, e.g., [6]) and *parametric semidefinite programming problems* (see, e.g., [8, pp. 470–496]) which have numerous applications in engineering applications. *The second-order cone* in $\mathbb{R}^m$ with $m \geq 3$ (also called *the Lorentz cone*) is given by

$$K = \left\{ y = (y_1, \ldots, y_m) \in \mathbb{R}^m \, : \, y_m \geq \left(\sum_{j=1}^{m-1} y_j^2 \right)^{1/2} \right\}.$$

Since $K^* = -K$, the regularity condition (d) in Theorem 2.3 can be easily verified. Hence, the result can be applied to programming problems under the constraint (2.1) with K being the second-order cone, or a product of several second-order cones. The latter situation was considered in [6, p. 1732]. Now, consider the *linear semidefinite programming problem* [8, p. 471] of the form

$$\min\Big\{\sum_{i=1}^{n} c_i x_i \ :\ X = (x_1, \ldots, x_n)^T \in \mathbb{R}^n,\ A_0 + \sum_{i=1}^{n} x_i A_i \preceq 0\Big\}, \tag{2.16}$$

where $A_0, A_1, \ldots, A_n \in \mathcal{S}^m$, the space of $m \times m$ symmetric matrices. For a square matrix C, the notation $C \preceq 0$ means that C is negative semidefinite. It is well known that C is negative semidefinite if and only if $-C$ is positive semidefinite. Hence, the constraint $A_0 + \sum_{i=1}^{n} x_i A_i \preceq 0$ is equivalent to

$$-A_0 + \sum_{i=1}^{n} x_i(-A_i) \succeq 0.$$

Then, the feasible region of (2.16) is the set of all $X = (x_1, \ldots, x_n)^T$ satisfying the linear constraint $AX + B \in K$, where K is the cone consisting of all positive semidefinite symmetric $(m \times m)$ matrices, $B = -A_0$, and the linear operator $A : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is given by

$$AX = \sum_{i=1}^{n} x_i(-A_i).$$

Here K is also a closed convex cone. It is not difficult to show that and $K^* = -K$ and $\ker A^T$ is the set of $C \in \mathcal{S}^m$ satisfying the following system of n linear equations

$$\sum_{\alpha,\beta} (A_i)_{\alpha\beta} C_{\alpha\beta} = 0 \quad (i = 1, \ldots, n),$$

where the sum is taken on $\alpha, \beta \in \{1, \ldots, m\}$ and, for a matrix $M \in \mathcal{S}^m$, $M_{\alpha\beta}$ indicates the $\alpha\beta$-element of M. Hence the regularity condition (d) in Theorem 2.3 is verifiable.

2.4 The Solution Maps of Linear Complementarity Problems

In this section, we apply results on the stability of solution map of parametric linear constraint system (2.1) to investigate parametric linear complementarity problems.

Given a vector q in $\mathbb{R}^n$, and a matrix M in $\mathbb{R}^{n\times n}$, the *linear complementarity problem* (LCP) aims at finding a vector x in $\mathbb{R}^n$ such that

$$\begin{cases} Mx + q \geq 0, \; x \geq 0 \\ x^T(Mx+q) = 0. \end{cases}$$

The solution set of this problem is denoted by $\mathrm{Sol}(M,q)$. We will transform LCP into a linear constraint system. Let

$$A = \begin{bmatrix} M \\ E \end{bmatrix} \in \mathbb{R}^{(2n)\times n}, \quad b = \begin{pmatrix} q \\ 0 \end{pmatrix} \in \mathbb{R}^{2n},$$

where $E \in \mathbb{R}^{n\times n}$ is the unit matrix, and

$$K = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^n \; : \; u \geq 0, \; v \geq 0, \; v^T u = 0 \right\}.$$

It is clear that $x \in \mathrm{Sol}(M,q)$ if and only if $Ax+b \in K$. Put $W = \mathbb{R}^{(2n)\times n} \times \mathbb{R}^{2n}$ and consider the multifunction $S : W \rightrightarrows \mathbb{R}^n$ defined by

$$S(w) = \{x \in \mathbb{R}^n \; : \; Ax + b \in K\} \quad \forall w = (A,b) \in W.$$

Fix a pair $(\bar{M}, \bar{q}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n$. Let $\bar{x} \in \mathrm{Sol}(\bar{M}, \bar{q})$. Then $\bar{x} \in S(\bar{w})$, where $\bar{w} := (\bar{A}, \bar{b})$. Put $I = \{1, 2, \ldots, n\}$, $I_1 = \{i \in I \; : \; \bar{x}_i = 0, (\bar{M}\bar{x} + \bar{q})_i > 0\}$, $I_2 = \{i \in I \; : \; \bar{x}_i > 0, (\bar{M}\bar{x} + \bar{q})_i = 0\}$, $I_3 = \{i \in I \; : \; \bar{x}_i = 0, (\bar{M}\bar{x} + \bar{q})_i = 0\}$,

$$\bar{y} = \bar{A}\bar{x} + \bar{b} = \begin{pmatrix} \bar{M}\bar{x} + \bar{q} \\ \bar{x} \end{pmatrix},$$

and note that $\bar{y} \in K$.

Continuity properties of the solution map of LCP have been studied by several authors (see, e.g., [16, 22, 23, 37, 62] and the references therein). In this section, we obtain a new result on the Lipschitz-like property of that solution map, as well as two related local error bounds.

For convenience, we present the above cone K in the form

$$K = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^n \; : \; \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in K_i, \; i \in I \right\}, \tag{2.17}$$

where

$$K_i = \left\{ \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in \mathbb{R}^2 \; : \; u_i \geq 0, \; v_i \geq 0, \; u_i v_i = 0 \right\}. \tag{2.18}$$

Definition 2.1 We say that the solution map Sol(.) of LCP satisfies the *uniform local error bound* at $((\bar{M},\bar{q}),\bar{x})$ if there exist constants $r>0$, $\delta>0$, and a neighborhood V of $\bar{x}$ such that

$$d(x;\mathrm{Sol}(M,q)) \le r\sum_{i=1}^{n} d\left(\begin{pmatrix}(Mx+q)_i\\ x_i\end{pmatrix};K_i\right), \tag{2.19}$$

for any $x\in V$ and (M,q) satisfying $\|M-\bar{M}\|+\|q-\bar{q}\|<\delta$.

From (2.19) we can infer that

$$d(x;\mathrm{Sol}(\bar{M},\bar{q})) \le r\sum_{i=1}^{n} d\left(\begin{pmatrix}(\bar{M}x+\bar{q})_i\\ x_i\end{pmatrix};K_i\right), \tag{2.20}$$

for any $x\in V$. Observe that (2.20) is a *local error bound* in a traditional style: the left-hand-side measures the distance from x to the solution set $S(\bar{M},\bar{q})$ which is unknown, while the right-hand-side is an explicit bound depending on the choice of r. If $x\in\mathrm{Sol}(\bar{M},\bar{q})$ (resp., $x\in\mathrm{Sol}(M,q)$), then the right-hand-side of (2.20) (resp., (2.19)) equals to 0.

Let us consider a regularity condition: *If* $u'=(u'_1,\ldots,u'_n)^T\in\mathbb{R}^n$ *and if*

$$\begin{cases}\textit{For each } i\in I_1,\ u'_i=0;\\ \textit{For each } i\in I_2,\ (\bar{M}^Tu')_i=0;\\ \textit{For each } i\in I_3,\ \textit{either } u'_i=0,\ \textit{or } (\bar{M}^Tu')_i=0,\ \textit{or } u'_i\le 0\\ \textit{and } (\bar{M}^Tu')_i\ge 0,\end{cases} \tag{2.21}$$

then $u'=0$.

The major result of this section reads as follows.

Theorem 2.4 *Suppose that* $\bar{x}\in\mathrm{Sol}(\bar{M},\bar{q})$. *If the regularity condition* (2.21) *is satisfied, then the problem* LCP *has the uniform local error bound* (2.19) *and the traditional local error bound* (2.20) *at* $((\bar{M},\bar{q}),\bar{x})$ *and its solution map* Sol(.) *is Lipschitz-like around* $((\bar{M},\bar{q}),\bar{x})$.

Proof. According to Theorem 2.2, if (2.8) is fulfilled, then there exist a constant $r>0$, a neighborhood U of $\bar{w}$, and a bounded neighborhood V of $\bar{x}$ satisfying

$$d(x;S(w))\le rd(Ax+b;K) \tag{2.22}$$

for all $w=(A,b)\in U$ and $x\in V$. Here we have

$$d(Ax+b;K)=d\left(\begin{pmatrix}Mx+q\\ x\end{pmatrix};K\right).$$

Recall that $x \in S(w)$ if and only if $x \in$ Sol(M, q). Hence, by (2.22) there is $\delta > 0$ such that, for any $x \in V$ and (M, q) with $\|M - \bar{M}\| + \|q - \bar{q}\| < \delta$, one has

$$d(x; \mathrm{Sol}(M, q)) \leq rd\left(\begin{pmatrix} Mx + q \\ x \end{pmatrix}; K\right) = r\sum_{i=1}^{n} d\left(\begin{pmatrix} (Mx + q)_i \\ x_i \end{pmatrix}; K_i\right). \tag{2.23}$$

Thus, under condition (2.8), we obtain (2.19), and then (2.20).

Moreover, similarly as in the proof of Lemma 2.3, from (2.23) we can derive the existence of a constant $\ell = \max\{r, r\alpha\}$ such that

$$\mathrm{Sol}(M_1, q_1) \cap V \subset \mathrm{Sol}(M_2, q_2) + \ell(\|M_2 - M_1\| + \|q_2 - q_1\|)\bar{B}_{\mathbb{R}^n}$$

for all the pairs (M_1, q_1) and (M_2, q_2) satisfying $\|M_1 - \bar{M}\| + \|q_1 - \bar{q}\| < \delta$ and $\|M_2 - \bar{M}\| + \|q_2 - \bar{q}\| < \delta$. Thus, the solution map Sol(.) is Lipschitz-like around $((\bar{M}, \bar{q}), \bar{x})$.

To complete the proof, it suffices to show that (2.8) is equivalent to (2.21). Let

$$\begin{pmatrix} u' \\ v' \end{pmatrix} \in \ker \bar{A}^T \cap N(\bar{y}; K).$$

It is clear that $\begin{pmatrix} u' \\ v' \end{pmatrix} \in \ker \bar{A}^T$ if and only if $\bar{M}^T u' + v' = 0$. Therefore,

$$\ker \bar{A}^T = \left\{\begin{pmatrix} u' \\ v' \end{pmatrix} : u' \in \mathbb{R}^n,\ v' = -\bar{M}^T u'\right\}. \tag{2.24}$$

Meanwhile, using the presentation (2.17) for K where the cones K_i are given in (2.18), we can assert by [50, Proposition 1.2] that

$$N(\bar{y}; K) = N\left(\begin{pmatrix} \bar{M}\bar{x} + \bar{q} \\ \bar{x} \end{pmatrix}; K\right) = \prod_{i=1}^{n} N\left(\begin{pmatrix} (\bar{M}\bar{x} + \bar{q})_i \\ \bar{x}_i \end{pmatrix}; K_i\right)$$

with

$$N\left(\begin{pmatrix} (\bar{M}\bar{x} + \bar{q})_i \\ \bar{x}_i \end{pmatrix}; K_i\right) = \begin{cases} \{0\} \times \mathbb{R} & \text{if } i \in I_1 \\ \mathbb{R} \times \{0\} & \text{if } i \in I_2 \\ (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) \cup \mathbb{R}^2_- & \text{if } i \in I_3. \end{cases}$$

(Here and in the sequel, $\mathbb{R}^n_-$ denotes the nonpositive orthant in $\mathbb{R}^n$.) Thus, by virtue of (2.24), we see that the regularity condition (2.21) can be rewritten in the form (2.8). □

Remark 2.6 If $I_1 = I$, i.e., $\bar{x} = 0, \bar{M}\bar{x} + \bar{q} > 0$, the regularity condition (2.21) holds automatically.

Remark 2.7 If $I_2 = I$, i.e., $\bar{x} > 0, \bar{M}\bar{x} + \bar{q} = 0$, condition (2.21) is equivalent to

$$\bar{M}^T u' = 0 \implies u' = 0 \quad (\forall u' \in \mathbb{R}^n),$$

which means that $\bar{M}$ is nonsingular.

We now give a complete geometrical description of the regularity condition (2.21).

Remark 2.8 Set

$$\begin{aligned} L_1 &= \{u' = (u'_1, \ldots, u'_n) \ : \ u'_i = 0 \ \forall i \in I_1\}, \\ L_2 &= \{u' = (u'_1, \ldots, u'_n) \ : \ (\bar{M}^T u')_i = 0 \ \forall i \in I_2\}, \\ L_3 &= \{u' = (u'_1, \ldots, u'_n) \ : \ \text{either } u'_i = 0, \text{ or } (\bar{M}^T u')_i = 0, \\ &\qquad \text{or } u'_i \le 0 \text{ and } (\bar{M}^T u')_i \ge 0, \ \forall i \in I_3\}, \end{aligned}$$

and observe that L_1 and L_2 are linear subspaces of $\mathbb{R}^n$, while L_3 is an union of finitely many closed cones. In this way, (2.21) is reformed as

$$u' \in L_1 \cap L_2 \cap L_3 \implies u' = 0 \quad (\forall u' \in \mathbb{R}^n).$$

Let us consider two illustrative examples.

Example 2.3 Choose $n = 2$, and $\bar{M} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$. Consider three situations:

a) $\bar{q} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ and $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Here $I_1 = I = \{1, \ 2\}$.

b) $\bar{q} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$ and $\bar{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Here $I_2 = I = \{1, \ 2\}$ and $\det\bar{M} \neq 0$.

c) $\bar{q} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\bar{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Here $I_1 = \emptyset$, $I_2 = \{2\}$, and $I_3 = \{1\}$. It is easy to check that the intersection $L_1 \cap L_2 \cap L_3$ contains only one element $u' = 0$.

In each one of these situations, by Theorem 2.4 and the above remarks, the solution map Sol(.) of LCP is Lipschitz-like around $((\bar{M}, \bar{q}), \bar{x})$ and it has the uniform local error bound (2.19) at $((\bar{M}, \bar{q}), \bar{x})$.

Example 2.4 Let $\bar{M} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$, $\bar{q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $\bar{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Here we have $I_1 = \emptyset$, $I_2 = \{2\}$, and $I_3 = \{1\}$. Because the intersection

$$L_1 \cap L_2 \cap L_3 = \{u' = (u'_1, u'_2) \,:\, u'_2 = -2u'_1,\ u'_1 \leq 0\}$$

contains nonzero vectors, condition (2.21) is violated. So we can assert nothing about the behavior of the solution map Sol(.) of LCP.

Based on a recent study of Huyen and Yao [28], we now give some remarks on the observation from one referee who says that the sufficient condition (2.21) for the local error bounds (2.19) and (2.20) might be related to the definition of P-matrix.

Recall [16, Definition 3.3.1] that $M \in \mathbb{R}^{n\times n}$ is a *P-matrix* if all its principal minors are positive. The class of such matrices is denoted $\mathcal{P}$. For a subset $\alpha \subset I$, the submatrix $(m_{ij})_{i\in\alpha,j\in\alpha}$ of an $(n \times n)$ matrix $M = (m_{ij})$ is denoted by $M_{\alpha\alpha}$. Given three subsets α, β, γ of I, the 9-block matrix

$$\begin{bmatrix} M_{\alpha\alpha} & 0 & 0 \\ 0 & M_{\beta\beta} & 0 \\ 0 & 0 & M_{\gamma\gamma} \end{bmatrix}$$

is abbreviated to $\mathrm{diag}[M_{\alpha\alpha}, M_{\beta\beta}, M_{\gamma\gamma}]$.

The following sufficient condition for the fulfillment of the regularity condition (2.21) clearly shows that (2.21) is much weaker than the condition $\bar{M} \in \mathcal{P}$.

Theorem 2.5 (See [28, Theorem 4.7]) *If* $\bar{M} = \mathrm{diag}[\bar{M}_{I_1I_1}, \bar{M}_{I_2I_2}, \bar{M}_{I_3I_3}]$ *with* $\bar{M}_{I_2I_2}$ *being a nonsingular matrix and* $\bar{M}_{I_3I_3} \in \mathcal{P}$*, then* (2.21) *is satisfied.*

Remark 2.9 A solution of the issues stated in Remark 2.3 will provide us with explicit upper bounds for the uniform local error bound r in (2.19) and the local error bound r in (2.20). For the case $\bar{M} \in \mathcal{P}$, several local and global error bounds for LCP can be found in [13] and [47, Theorem 1]. Such bounds can be used to construct derivative-free descent methods for solving LCP. For more details, we refer to [44, p. 684] where the derivative-free descent methods for the variational inequality problem, which is more general than the linear complementary problem, are discussed. Similar remarks are applied to the local error bounds for affine variational inequalities, which will be obtained in the sequel. A theory of error bounds for variational inequalities and complementarity problems can be found in [20].

In next section, from Theorem 2.2 we will derive the Lipschitz-like property of the solution map and uniform local error bounds for affine variational inequalities. As LCP is an affine variational inequality, Examples 2.3 and 2.4 can serve well as illustrations for the forthcoming results.

2.5 The Solution Maps of Affine Variational Inequalities

Let $M \in \mathbb{R}^{n\times n}$, $q \in \mathbb{R}^n$, and let $\Delta \subset \mathbb{R}^n$ be a polyhedral convex set defined by

$$\Delta = \{x \in \mathbb{R}^n \,:\, Cx \geq d\},$$

where $C \in \mathbb{R}^{m\times n}$, $d \in \mathbb{R}^m$. The problem finding $\bar{x} \in \Delta$ such that

$$\langle M\bar{x} + q, y - \bar{x}\rangle \geq 0 \quad \forall y \in \Delta,$$

is called an *affine variational inequality* (AVI). Denote the solution set of AVI by $\mathrm{Sol}(M, q, C, d)$. Clearly, taking $m = n$, $d = 0 \in \mathbb{R}^m$, and $C = E$ for AVI, we get the problem LCP.

Fix a vector field $(\bar{M}, \bar{q}, \bar{C}, \bar{d})$. According to [24] (see also [37, Theorem 5.3]), $\bar{x} \in \mathrm{Sol}(\bar{M}, \bar{q}, \bar{C}, \bar{d})$ if and only if there exists a Lagrange multiplier $\lambda \in \mathbb{R}^m$ satisfying

$$\begin{cases} \bar{M}\bar{x} - \bar{C}^T\lambda + \bar{q} = 0 \\ \lambda \geq 0,\ \bar{C}\bar{x} \geq \bar{d} \\ (\bar{C}\bar{x} - \bar{d})^T\lambda = 0. \end{cases}$$

Put

$$\bar{A} = \begin{bmatrix} \bar{M} & -\bar{C}^T \\ 0 & E \\ \bar{C} & 0 \end{bmatrix} \in \mathbb{R}^{(n+m+m)\times(n+m)}, \quad \bar{b} = \begin{pmatrix} \bar{q} \\ 0 \\ -\bar{d} \end{pmatrix} \in \mathbb{R}^{n+m+m},$$

$$K = \left\{(s, u, v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \,:\, s = 0,\ u \geq 0,\ v \geq 0,\ v^Tu = 0\right\},$$

and

$$K_0 = \left\{(u, v) \in \mathbb{R}^m \times \mathbb{R}^m \,:\, u \geq 0,\ v \geq 0,\ v^Tu = 0\right\}.$$

Here E is the unit matrix of order m. Thus, $\bar{x} \in \mathrm{Sol}(\bar{M}, \bar{q}, \bar{C}, \bar{d})$ if and only if there exists $\lambda \in \mathbb{R}^m$ such that

$$\bar{A} \begin{pmatrix} \bar{x} \\ \lambda \end{pmatrix} + \bar{b} \in K.$$

We consider the multifunction $S : W \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ defined by

$$S(w) = \left\{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \,:\, A \begin{pmatrix} x \\ \lambda \end{pmatrix} + b \in K \right\} \quad \forall w = (A, b) \in W,$$

where $W := \mathbb{R}^{(n+m+m)\times(n+m)} \times \mathbb{R}^{n+m+m}$. Put $\bar{w} = (\bar{A}, \bar{b})$ and suppose that $(\bar{x}, \bar{\lambda}) \in S(\bar{w})$. Then, $\bar{x} \in \mathrm{Sol}(\bar{M}, \bar{q}, \bar{C}, \bar{d})$.

Set $I = \{1, 2, \dots, m\}$, $I_1 = \{i \in I : (\bar{C}\bar{x} - \bar{d})_i = 0,\ \bar{\lambda}_i > 0\}$,

$$I_2 = \{i \in I : (\bar{C}\bar{x} - \bar{d})_i > 0,\ \bar{\lambda}_i = 0\},$$

$$I_3 = \{i \in I : (\bar{C}\bar{x} - \bar{d})_i = 0,\ \bar{\lambda}_i = 0\},$$

and

$$\bar{z} = \bar{A}\bar{x} + \bar{b} = \begin{pmatrix} \bar{M}\bar{x} - \bar{C}^T\bar{\lambda} + \bar{q} \\ \bar{\lambda} \\ \bar{C}\bar{x} - \bar{d} \end{pmatrix}.$$

Note that $\bar{M}\bar{x} - \bar{C}^T\bar{\lambda} + \bar{q} = 0$ and $\begin{pmatrix} \bar{\lambda} \\ \bar{C}\bar{x} - \bar{d} \end{pmatrix} \in K_0$. Similarly as it was done for (LCP), we decompose the cone K_0 by writing

$$K_0 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^m \,:\, \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in K_i,\ i \in I \right\},$$

where

$$K_i = \left\{ \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in \mathbb{R}^2 \,:\, u_i \geq 0,\ v_i \geq 0,\ u_i v_i = 0 \right\}.$$

Definition 2.2 We say that the problem AVI has the *uniform local error bound* at $((\bar{M}, \bar{q}, \bar{C}, \bar{d}), \bar{x})$ if there exist $\bar{\lambda} \in \mathbb{R}^m$, positive constants r and δ, and a neighborhood V of $\bar{x}$ such that

$$d(x; \mathrm{Sol}(M, q, C, d)) \leq r \left(\|Mx - C^T\bar{\lambda} + q\| + \sum_{i=1}^{m} d\left(\begin{pmatrix} \bar{\lambda}_i \\ (Cx - d)_i \end{pmatrix} ; K_i \right) \right), \tag{2.25}$$

for any $x \in V$ and (M, q, C, d) with $\|M - \bar{M}\| + \|q - \bar{q}\| + \|C - \bar{C}\| + \|d - \bar{d}\| < \delta$.

Substituting $(M, q, C, d) = (\bar{M}, \bar{q}, \bar{C}, \bar{d})$ into (2.25) yields

$$d(x; \mathrm{Sol}(\bar{M}, \bar{q}, \bar{C}, \bar{d})) \le r \left(\|\bar{M}x - \bar{C}^T\bar{\lambda} + \bar{q}\| + \sum_{i=1}^{m} d\left(\begin{pmatrix} \bar{\lambda}_i \\ (\bar{C}x - \bar{d})_i \end{pmatrix}; K_i \right) \right), \tag{2.26}$$

for any $x \in V$. This is a *local error bound* for the unperturbed AVI in the traditional form.

Later on, we will see that $\bar{\lambda}$ satisfying (2.25) and (2.26) can be chosen from the set of Lagrange multipliers corresponding to $\bar{x}$.

If the problem AVI has the form of LCP then the Lagrange multiplier corresponding to a solution $\bar{x}$ is unique and $\bar{\lambda} = \bar{M}\bar{x} + \bar{q}$. For a general AVI, there may exist many Lagrange multipliers corresponding to one solution. So, the study of AVI is more complicated.

Consider the regularity condition: *If vector* $(z_1', z_3') \in \mathbb{R}^n \times \mathbb{R}^m$ *satisfies* $\bar{M}^T z_1' + \bar{C}^T z_3' = 0$ *and the system*

$$\begin{cases} (\bar{C}z_1')_i = 0 \quad \text{if } i \in I_1; \\ (z_3')_i = 0 \quad \text{if } i \in I_2; \\ \text{either } (\bar{C}z_1')_i = 0, \text{ or } (z_3')_i = 0, \text{ or } (\bar{C}z_1')_i \le 0 \text{ and } (z_3')_i \le 0 \quad \text{if } i \in I_3, \end{cases} \tag{2.27}$$

then $(z_1', z_3') = (0, 0)$.

The next result is applied to a class of AVIs where $\operatorname{rank} C = m$.

Theorem 2.6 *Suppose that* $\bar{x} \in \mathrm{Sol}(\bar{M}, \bar{q}, \bar{C}, \bar{d})$ *and* $\bar{\lambda}$ *is a Lagrange multiplier corresponding to* $\bar{x}$*. If the regularity condition* (2.27) *is satisfied, then there are the local error bounds for* AVI *in the forms* (2.25) *and* (2.26)*. Moreover, if* $\operatorname{rank} \bar{C} = m$ *then the solution map* $\mathrm{Sol}\,(.)$ *of* AVI *is Lipschitz-like around* $((\bar{M}, \bar{q}, \bar{C}, \bar{d}), \bar{x})$*.*

Proof. Let $\bar{x}$ belong to $\mathrm{Sol}(\bar{M}, \bar{q}, \bar{C}, \bar{d})$ and $\bar{\lambda}$ be a Lagrange multiplier corresponding to $\bar{x}$. As in Theorem 2.4, we first suppose that (2.8) is available. Then, by virtue of Theorem 2.2, there exist a constant $r > 0$, a neighborhood U of $\bar{w}$, a bounded neighborhood V of $\bar{x}$, and a bounded neighborhood $V_{\bar{\lambda}}$ of $\bar{\lambda}$ such that

$$d\left(\begin{pmatrix} x \\ \lambda \end{pmatrix}; S(w) \right) \le r d\left(A \begin{pmatrix} x \\ \lambda \end{pmatrix} + b; K \right)$$

for every $w \in U$, $x \in V$, and $\lambda \in V_{\bar{\lambda}}$. Hence,

$$d\left(\begin{pmatrix} x \\ \bar{\lambda} \end{pmatrix}; S(w)\right) \le rd\left(A\begin{pmatrix} x \\ \bar{\lambda} \end{pmatrix} + b; K\right) \tag{2.28}$$

for any $w \in U$ and $x \in V$. In this case, we have

$$\begin{aligned} d\left(A\begin{pmatrix} x \\ \bar{\lambda} \end{pmatrix} + b; K\right) &= d\left(\begin{pmatrix} Mx - C^T\bar{\lambda} + q \\ \bar{\lambda} \\ Cx - d \end{pmatrix}; K\right) \\ &= \|Mx - C^T\bar{\lambda} + q\| + d\left(\begin{pmatrix} \bar{\lambda} \\ Cx - d \end{pmatrix}; K_0\right) \\ &= \|Mx - C^T\bar{\lambda} + q\| + \sum_{i=1}^{n} d\left(\begin{pmatrix} \bar{\lambda}_i \\ (Cx - d)_i \end{pmatrix}; K_i\right). \end{aligned}$$

But

$$\begin{pmatrix} x \\ \lambda \end{pmatrix} \in S(w) \implies x \in \text{ Sol}(M, q, C, d).$$

So, due to (2.28) there is $\delta > 0$ such that, for any $x \in V$ and (M, q, C, d) with $\|M - \bar{M}\| + \|q - \bar{q}\| + \|C - \bar{C}\| + \|d - \bar{d}\| < \delta$, we have

$$\begin{aligned} d\left(x; \text{Sol}(M, q, C, d)\right) &\le d\left(\begin{pmatrix} x \\ \bar{\lambda} \end{pmatrix}; S(w)\right) \\ &\le r\left(\|Mx - C^T\bar{\lambda} + q\| + \sum_{i=1}^{n} d\left(\begin{pmatrix} \bar{\lambda}_i \\ (Cx - d)_i \end{pmatrix}; K_i\right)\right). \end{aligned}$$

Thus, we have (2.25), and also (2.26).

Adding the assumption $\operatorname{rank} \bar{C} = m$, we will prove that the solution map of AVI is Lipschitz-like around the point $((\bar{M}, \bar{q}, \bar{C}, \bar{d}), \bar{x})$. Indeed, once $\bar{\lambda}$ is a Lagrange multiplier corresponding to $\bar{x}$,

$$\bar{M}\bar{x} - C^T\bar{\lambda} + \bar{q} = 0$$

or, equivalently,

$$\bar{C}^T\bar{\lambda} = \bar{M}\bar{x} + \bar{q}. \tag{2.29}$$

As $\operatorname{rank} \bar{C} = m$, we must have $m \le n$. In addition, there exists a nonsingular $m \times m$ submatrix $\bar{C}_J^T$ composed by the j-th rows of $\bar{C}^T$ with $j \in J$, where the index set J has m elements. Then, (2.29) is equivalent to

$$\bar{\lambda} = \left(\bar{C}_J^T\right)^{-1} (\bar{M}\bar{x} + \bar{q})_J,$$

where $(\bar{M}\bar{x}+\bar{q})_J$ denotes the vector composed by j-th component of vector $\bar{M}\bar{x}+\bar{q}$, $j \in J$. This implies that $\bar{\lambda}$ is uniquely defined by the data $\bar{M}, \bar{q}, \bar{C}$, and the solution $\bar{x}$. Now, replacing the above δ by a smaller positive number and V by a smaller neighborhood of $\bar{x}$, if necessary, we can assume that, for any $x \in V$ and (M, q, C, d) with $\|M-\bar{M}\|+\|q-\bar{q}\|+\|C-\bar{C}\|+\|d-\bar{d}\| < \delta$, the submatrix C_J^T is nonsingular, the equation

$$Mx - C_i^T\lambda + q = 0$$

yields the unique solution

$$\lambda = \left(C_J^T\right)^{-1}(Mx+q)_J$$

and, moreover, $\lambda \in V_{\bar{\lambda}}$. Remember that, under the condition (2.8), Theorem 2.2 asserts that $S(.)$ is Lipschitz-like $(\bar{w}, \bar{x})$ with constant $\ell > 0$. Hence, without loss of generality, we can assume that

$$S(w_1) \cap (V \times V_{\bar{\lambda}}) \subset S(w_2) + \ell\|w_2 - w_1\|\bar{B}_{\mathbb{R}^n \times \mathbb{R}^m}, \tag{2.30}$$

for any $w_i = (A_i, b_i)$, $i = 1, 2$, with

$$A_i = \begin{bmatrix} M_i & -C_i^T \\ 0 & E \\ C_i & 0 \end{bmatrix} \in \mathbb{R}^{(n+m+m)\times(n+m)}, \quad b_i = \begin{pmatrix} q_i \\ 0 \\ -d_i \end{pmatrix} \in \mathbb{R}^{n+m+m},$$

and $\|M_i-\bar{M}\|+\|q_i-\bar{q}\|+\|C_i-\bar{C}\|+\|d_i-\bar{d}\| < \delta$. From (2.30) we can see that $S(w_i) \neq \emptyset$, $i = 1, 2$; hence, $\mathrm{Sol}(M_i, q_i, C_i, d_i) \neq \emptyset$, $i = 1, 2$. Suppose that $x_1 \in \mathrm{Sol}\,(M_1, q_1, C_1, d_1) \cap V$. Then, there is only one Lagrange multiplier λ_1 corresponding to x_1, $\lambda_1 \in V_{\bar{\lambda}}$, and

$$\begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix} \in S(w_1) \cap (V \times V_{\bar{\lambda}})\,.$$

By (2.30), there exists $\begin{pmatrix} x_2 \\ \lambda_2 \end{pmatrix} \in S(w_2)$, which means $x_2 \in \mathrm{Sol}\,(M_2, q_2, C_2, d_2)$ and λ_2 is the Lagrange multiplier corresponding to x_2, such that

$$\left\| \begin{pmatrix} x_2 \\ \lambda_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix} \right\| \leq \ell\|w_2 - w_1\|.$$

We have

$$\begin{aligned} \|w_2 - w_1\| &= \|A_2 - A_1\| + \|b_2 - b_1\| \\ &= (\|M_2 - M_1\| + \|C_1^T - C_2^T\| + \|C_2 - C_1\|) \end{aligned}$$

$$
\begin{aligned}
&\quad +(\|q_2 - q_1\| + \|d_1 - d_2\|) \\
&= \|M_2 - M_1\| + \|q_2 - q_1\| + 2\|C_2 - C_1\| + \|d_1 - d_2\| \\
&\le 2(\|M_2 - M_1\| + \|q_2 - q_1\| + \|C_2 - C_1\| + \|d_2 - d_1\|) \\
&= 2\|(M_2, q_2, C_2, d_2) - (M_1, q_1, C_1, d_1)\|.
\end{aligned}
$$

Hence,

$$\|x_2 - x_1\| \le 2\ell\|(M_2, q_2, C_2, d_2) - (M_1, q_1, C_1, d_1)\|.$$

Thus,

$$
\begin{aligned}
\mathrm{Sol}\,(M_1, q_1, C_1, d_1) \cap V \;\subset\; & \mathrm{Sol}\,(M_2, q_2, C_2, d_2) \\
& +2\ell\|(M_2, q_2, C_2, d_2) - (M_1, q_1, C_1, d_1)\|\bar{B}_{\mathbb{R}^n},
\end{aligned}
$$

for any (M_i, q_i, C_i, d_i) satisfying $\|M_i - \bar{M}\| + \|q_i - \bar{q}\| + \|C_i - \bar{C}\| + \|d_i - \bar{d}\| < \delta$, $i = 1, 2$. In other words, the solution map Sol(.) of AVI in case of $\mathrm{rank}\, C = m$ is Lipschitz-like around $((M, q, C, d), \bar{x})$ with $\bar{x} \in \mathrm{Sol}\,(M, q, C, d)$.

To end the proof, we will show that (2.8) is equivalent to the regularity condition (2.27). Suppose

$$z' = \begin{pmatrix} z'_1 \\ z'_2 \\ z'_3 \end{pmatrix} \in \ker \bar{A}^T \cap N(\bar{y}; K).$$

We can see that $z' \in \ker \bar{A}^T$ if and only if

$$\begin{cases} \bar{M}^T z'_1 + \bar{C}^T z'_3 = 0 \\ -\bar{C} z'_1 + z'_2 = 0. \end{cases}$$

Hence,

$$\ker \bar{A}^T = \left\{ \begin{pmatrix} z'_1 \\ z'_2 \\ z'_3 \end{pmatrix} \;:\; z'_1 \in \mathbb{R}^m,\; z'_2 = \bar{C} z'_1,\; \bar{M}^T z'_1 + \bar{C}^T z'_3 = 0 \right\}. \tag{2.31}$$

In addition,

$$N(\bar{z}; K) = \mathbb{R}^n \times N\left(\begin{pmatrix} \bar{\lambda} \\ \bar{C}\bar{x} - \bar{d} \end{pmatrix} ; K_0 \right).$$

By dint of the structure of K_0,

$$N\left(\begin{pmatrix} \bar{\lambda}_i \\ (\bar{C}\bar{x} - \bar{d})_i \end{pmatrix} ; K_i \right) = \begin{cases} \{0\} \times \mathbb{R} \;\text{ if } i \in I_1 \\ \mathbb{R} \times \{0\} \;\text{ if } i \in I_2 \\ (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) \cup \mathbb{R}^2_- \;\text{ if } i \in I_3. \end{cases}$$

By (3.17), z' belongs to $\ker \bar{A}^T \cap N(\bar{y}; K)$ if and only if $\bar{M}^T z_1' + \bar{C}^T z_3' = 0$, $z_2' = \bar{C} z_1'$ and the following is satisfied

$$\begin{cases} (z_2')_i = 0 \quad \textit{if } i \in I_1; \\ (z_3')_i = 0 \quad \textit{if } i \in I_2; \\ \textit{either } (z_2')_i = 0, \ \textit{or } (z_3')_i = 0, \ \textit{or } \ (z_2')_i \leq 0 \ \textit{and } (z_3')_i \leq 0 \quad \textit{if } i \in I_3. \end{cases}$$

Thus, condition (2.8) can be transformed into (2.27). □

Remark 2.10 Calmness of $S(.)$ at $((\bar{A}, \bar{b}), \bar{x})$ is much weaker than the uniform local error bounds in (2.19) for parametric LCPs and (2.25) for parametric AVIs whose constraint sets are subject to perturbations. As far as we know, there are no results similar to ours in the literature. The just mentioned calmness is even weaker than our local error bounds in (2.20) for parametric LCPs and (2.26) for parametric AVIs. To specify, our error bounds allow us to estimate the distance from an arbitrary x in the neighborhood V of $\bar{x}$ to the solution set $S(\bar{A}, \bar{b})$. Meanwhile, if $S(.)$ is calm at $(\bar{A}, \bar{b}, \bar{x})$ (see [15, p. 678]), i.e., there exist a neighborhood V of $\bar{x}$ and real numbers $\ell > 0$, $\delta > 0$ such that

$$\text{Sol}(M, q) \cap V \subset \text{Sol}(\bar{M}, \bar{q}) + \ell(\|M - \bar{M}\| + \|q - \bar{q}\|)\bar{B}_{\mathbb{R}^n} \tag{2.32}$$

for all (M, q) satisfying $\|M - \bar{M}\| + \|q - \bar{q}\| < \delta$, then we only can estimate the distance from those x which are solutions of the problem LCP, with (M, q) being near to $(\bar{M}, \bar{q})$, to the solution set $\text{Sol}(\bar{M}, \bar{q})$. Moreover, condition (2.32) does not guarantee that $\text{Sol}(M, q) \neq \emptyset$.

Remark 2.11 A careful analysis of the error bounds (2.19) and (2.20), as well as the error bounds (2.25) and (2.26), and the calmness in (2.32) assures us that these properties are very different by their nature. We believe that it is impossible to derive our error bounds from calmness.

2.6 Conclusions

We have presented verifiable necessary and sufficient conditions for the Lipschitz-like property as well as the Robinson stability of the parametric linear constraint system $Ax + b \in K$, where the closed set K may be nonconvex. It turned out that the two properties are equivalent.

We have established uniform local error bounds and traditional local error bounds for the linear complementary problem LCP and, more general, for affine variational inequalities. Furthermore, we have obtained the Lipschitz-like property of the solution map of the linear complementarity problem, and a class of affine variational inequalities.

Chapter 3

Linear Constraint Systems under Linear Perturbations

Linear perturbations are just special cases of total perturbations. However, they have their own significance.

This chapter shows analogues of the results of the previous one for the case when the given linear constraint system undergoes linear perturbations. As consequences, we will obtain sufficient conditions for various stability properties of LCPs and also of AVIs under linear perturbations.

The research topics of the present chapter are closely related to, but do not coincide with, those of the preceding works of Dontchev and Rockafellar [18], Facchinei and Pang [20], Gowda [22], Gowda and Pang [23, 24], Henrion, Kruger and Outrata [26], Henrion, Mordukhovich and Nam [27], Lee, Tam and Yen [37], Mathias and Pang [47], Mordukhovich and Outrata [57], Oettli and Yen [62], Qui [66–68], Robinson [74–76], Trang [83, 84], Yao and Yen [85, 86], and others.

This chapter can be considered as a continuation and development of Chapter 2. If in Chapter 2 we have exploited the solution map of a parametric linear constraint system in the term of an implicit multifunction, then in this paper we interpret it as inverse multifunction. Among other things, the regularity conditions guaranteeing the stability of LCPs and AVIs are studied by a matrix decomposition scheme via three related index sets.

The notion of P-matrix and its characterizations from the LCP theory (see, e.g., [16, Chapter 3]) have been proved to be very useful tools for our investigations of parametric LCPs. Here, we present a large class of matrices guaranteeing the uniform local error bounds for LCPs as well as the Lipschitz-like property of the solution map of LCPs. Surprisingly, our result is not limited to familiar classes of matrices, e.g., $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{S}$.

This chapter is written on the basis of the paper [28].

3.1 Stability properties of Linear Constraint Systems under Linear Perturbations

As in Chapter 2, consider the parametric linear constraint system (2.1). When b is subject to change, the solution map $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of (2.1) is defined by

$$S(b) = \{x \in \mathbb{R}^n \,:\, b \in -Ax + K\}.$$

Put $\Psi(x) = -Ax + K$ and note that $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is the inverse multifunction of S. For convenience, we put $T(x) = K$ for all $x \in \mathbb{R}^n$.

In what follows, we fix a vector $\bar{b}$ and a solution $\bar{x} = \left(\bar{x}_1, \ldots \bar{x}_n\right)^T \in S(\bar{b})$. Then, $(\bar{x}, \bar{b})$ belongs to the graph of Ψ. Let $\bar{v} := A\bar{x} + \bar{b}$.

Theorem 3.1 *The following statements are equivalent:*

(a) Ψ *is metrically regular around* $(\bar{x}, \bar{b}) \in \operatorname{gph} \Psi$ *with modulus* ℓ*;*

(b) S *is Lipschitz-like around* $(\bar{b}, \bar{x}) \in \operatorname{gph} S$ *with modulus* ℓ*;*

(c) $(\ker A^T) \cap N(\bar{v}; K) = \{0\}$.

Moreover, when K *is a closed convex cone, these statements are equivalent to each one of the following:*

(d) $(\ker A^T) \cap K^* \cap (\bar{v})^\perp = \{0\}$*, where* $K^* = \{v' \in \mathbb{R}^m \,:\, \langle v', v\rangle \le 0, \forall v \in K\}$*,* $\bar{v} = A\bar{x} + b$*, and* $(\bar{v})^\perp := \{v' \in \mathbb{R}^m \,:\, \langle v', \bar{v}\rangle = 0\}$;

(e) $0 \in \operatorname{int}(\operatorname{rge} A + K - \bar{v})$*, where* $\operatorname{rge} A := A\left(\mathbb{R}^n\right)$ *is the range of* A *and* $\operatorname{int} \Omega$ *denotes the interior of* Ω*;*

(f) $\operatorname{rge} A + \operatorname{cone}(K - \bar{v}) = \mathbb{R}^m$*, where* $\operatorname{cone} C$ *is the cone generated by* C*.*

Proof. The equivalence between (a) and (b) follows from the well known results [11, 63] (see also [64]). We will prove that (a) is equivalent to (c). First, by the coderivative sum rule [50, Theorem 1.62],

$$D^*\Psi(\bar{x},\bar{b})(v') = -A^T v' + D^*T(\bar{x}, A\bar{x}+\bar{b})(v') \quad \forall v' \in \mathbb{R}^m.$$

By definition,

$$D^*T(\bar{x}, A\bar{x}+\bar{b})(v') = \{u' \in \mathbb{R}^n : (u', -v') \in N((\bar{x}, A\bar{x}+\bar{b}); \operatorname{gph} T)\}.$$

Since $\operatorname{gph} T = \mathbb{R}^n \times K$, using [50, Proposition 1.2] we have

$$N((\bar{x}, A\bar{x}+\bar{b}); \operatorname{gph} T) = N(\bar{x}; \mathbb{R}^n) \times N(A\bar{x}+\bar{b}; K) = \{0_{\mathbb{R}^n}\} \times N(A\bar{x}+\bar{b}; K).$$

Hence, $D^*T(\bar{x}, A\bar{x}+\bar{b})(v') = \{0\}$ if $v' \in -N(A\bar{x}+\bar{b}; K)$ and

$$D^*T(\bar{x}, A\bar{x}+\bar{b})(v') = \emptyset$$

otherwise. Since $\operatorname{gph}\Psi$ is closed, due to the Mordukhovich criterion cited above, Ψ is metrically regular around $(\bar{x},\bar{b}) \in \operatorname{gph}\Psi$ if and only if

$$0 \in D^*\Psi(\bar{x},\bar{b})(v') \implies v' = 0.$$

Clearly, this condition is satisfied if and only if

$$[v' \in -N(A\bar{x}+\bar{b}; K) \text{ and } -A^T v' = 0] \implies v' = 0$$

or, equivalently,

$$\left(\ker A^T\right) \cap N(\bar{v}; K) = \{0\}. \tag{3.1}$$

Similarly as it was done in [31], condition (3.1) can be equivalently rewritten in the forms (d), (e), and (f). □

Remark 3.1 Condition (e) in Theorem 3.1 is the Robinson regularity condition (see [8, 74, 75], where the inequality systems in Banach spaces have been considered).

Remark 3.2 One can also prove Theorem 3.1 by using the Mordukhovich criterion for the Lipschitz-likeness of S (see [49], [80, Theorem 9.40], [50, Theorem 4.10]), which asserts that S is Lipschitz-like around $(\bar{b}, \bar{x})$ if and only if $D^*S(\bar{b}, \bar{x})(0) = \{0\}$.

Remark 3.3 In modern terminology, the metrically regular of Ψ can be considered as uniform metric subreguarity of Ψ with respect to x; see [19, p. 183]. Ioffe's recent survey [32, 33] on metric regularity really brings us a deeper knowledge about metric regularity, metric subregularity, and error bounds as well.

3.2 Solution Stability of Linear Complementarity Problems under Linear Perturbations

Following the approach adopted in Chapter 2, we transform LCP into the linear constraint system in the form (2.1) by setting

$$A = \begin{bmatrix} M \\ E \end{bmatrix} \in \mathbb{R}^{(2n)\times n}, \quad b = \begin{pmatrix} q \\ 0 \end{pmatrix} \in \mathbb{R}^{2n},$$

and

$$K = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^n \,:\, u \geq 0,\ v \geq 0,\ v^T u = 0 \right\}.$$

It is clear that $x \in \mathrm{Sol}_M(q)$ if and only if $Ax + b \in K$. Unlike Chapter 2, here we consider the case where only vector q of the problem LCP is subject to change. Hence, in the corresponding linear constraint system (2.1), only vector b is perturbed. Thus various results on the solution stability of LCP under perturbations of q can be obtained by considering the multifunction $S : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$, $S(b) = \{x \in \mathbb{R}^n \,:\, Ax + b \in K\}$, where A and K have been defined above.

Fix a vector $\bar{b} = \begin{pmatrix} \bar{q} \\ 0 \end{pmatrix}$ and suppose that $\bar{x} \in S(\bar{b})$ or $\bar{x} \in \mathrm{Sol}_M(\bar{q})$. For convenience, we put $I = \{1, 2, \ldots, n\}$, $I_1 = \{i \in I \,:\, \bar{x}_i = 0,\ (M\bar{x} + \bar{q})_i > 0\}$, $I_2 = \{i \in I \,:\, \bar{x}_i > 0,\ (M\bar{x} + \bar{q})_i = 0\}$, $I_3 = \{i \in I \,:\, \bar{x}_i = 0,\ (M\bar{x} + \bar{q})_i = 0\}$ and

$$\bar{y} = A\bar{x} + \bar{b} = \begin{pmatrix} M\bar{x} + \bar{q} \\ \bar{x} \end{pmatrix}.$$

Recall that $\bar{y} \in K$. Denoting

$$K_i = \left\{ \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in \mathbb{R} \times \mathbb{R} \,:\, u_i \geq 0,\ v_i \geq 0,\ u_i v_i = 0 \right\},$$

we have

$$K = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^n \,:\, \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in K_i,\ i \in I \right\}. \tag{3.2}$$

Definition 3.1 (see [31]) We say that the problem LCP satisfies the *uniform local error bound* at $(\bar{q}, \bar{x})$ if there exist constants $\ell > 0$, $\delta > 0$ and a neighborhood V of $\bar{x}$ such that

$$d(x;\mathrm{Sol}_M(q)) \leq \ell \sum_{i=1}^{n} d\left(\begin{pmatrix}(Mx+q)_i\\ x_i\end{pmatrix}; K_i\right), \tag{3.3}$$

for any $x \in V$ and q satisfying $\|q - \bar{q}\| < \delta$.

From (3.3) we can infer that

$$d(x;\mathrm{Sol}_M(\bar{q})) \leq \ell \sum_{i=1}^{n} d\left(\begin{pmatrix}(Mx+\bar{q})_i\\ x_i\end{pmatrix}; K_i\right), \tag{3.4}$$

for any $x \in V$. Note that (3.4) is a *local error bound* in a traditional style: the left-hand-side measures the distance from x to the solution set $\mathrm{Sol}_M(\bar{q})$ which is unknown, while the right-hand-side is an explicit bound depending on the choice of r. Moreover, if $x \in \mathrm{Sol}_M(\bar{q})$, then the right-hand-side of (3.4) equals to 0.

Concerning the parametric local error bound (3.3), we observe that its left-hand-side is the distance from x to the parametric solution set $\mathrm{Sol}_M(q)$ and its right-hand-side of (3.4) equals to 0 if $x \in \mathrm{Sol}_M(q)$.

Now, we introduce a verifiable regularity condition which plays a central role in our study of LCP: *If* $u' = (u'_1, \ldots, u'_n)^T \in \mathbb{R}^n$ *and if*

$$\begin{cases} \text{For each } i \in I_1,\ u'_i = 0; \\ \text{For each } i \in I_2,\ (M^T u')_i = 0; \\ \text{For each } i \in I_3,\ \text{either } u'_i = 0,\ \text{or } (M^T u')_i = 0, \\ \text{or } (u'_i \leq 0 \text{ and } (M^T u')_i \geq 0), \end{cases} \tag{3.5}$$

then $u' = 0$.

Note that $x \in \mathrm{Sol}_M(q)$ if and only if $0 \in Mx + q + N(x;\mathbb{R}^n_+)$. Hence,

$$\mathrm{Sol}_M(q) = \{x \in \mathbb{R}^n \,:\, q \in -Mx - N(x;\mathbb{R}^n_+)\}.$$

Consider multifunction $\Psi_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by $\Psi_1(x) = -Mx - N(x;\mathbb{R}^n_+)$. Clearly, Ψ has closed graph and it is the inverse multifunction of $\mathrm{Sol}_M(.)$.

Now, using (3.5), we can formulate the first result of this section as follows.

Theorem 3.2 *Suppose that* $\bar{x} \in \mathrm{Sol}_M(\bar{q})$. *If the regularity condition* (3.5) *is satisfied, then the problem* LCP *satisfies the uniform local error bounds* (3.3) *and the traditional local error bound* (3.4) *at* $(\bar{q}, \bar{x})$ *and its solution map* $\mathrm{Sol}_M(.)$ *is Lipschitz-like around* $(\bar{q}, \bar{x})$.

Proof. Due to Theorem 3.1, if (3.1) is available, then the inverse multifunction Ψ_1 of S is metrically regular around $(\bar{x}, \bar{b})$. Moreover, since Ψ_1 is inner semicontinuous, there exist a constant $\ell > 0$, a bounded neighborhood U of $\bar{b}$, and a neighborhood V of $\bar{x}$ such that

$$d(x; S(b)) \leq \ell\, d(b; \Psi_1(x)) \tag{3.6}$$

for any $b \in U$ and $x \in V$. This property implies

$$d(x; S(b)) \leq \ell\, d(Ax + b; K) \tag{3.7}$$

for any $b \in U$ and $x \in V$. Here,

$$d(Ax + b; K) = d\left(\begin{pmatrix} Mx + q \\ x \end{pmatrix}; K\right).$$

We know that x belongs to $S(b)$ if and only if $x \in \mathrm{Sol}_M(q)$. So, by (3.7) there exists $\delta > 0$ such that

$$d(x; \mathrm{Sol}_M(q)) \leq \ell\, d\left(\begin{pmatrix} Mx + q \\ x \end{pmatrix}; K\right) = \ell \sum_{i=1}^{n} d\left(\begin{pmatrix} (Mx + q)_i \\ x_i \end{pmatrix}; K_i\right)$$

for all $x \in V$ and $q \in \mathbb{R}^n$ with $\|q - \bar{q}\| < \delta$. Thus, condition (3.1) allows us to obtain (3.3) as well as (3.4). On the other hand, from Theorem 3.1, condition (3.1) guarantees the Lipschitz-likeness of $S(.)$ around $(\bar{b}, \bar{x})$ with modulus ℓ. The latter means that there exist a neighborhood V of $\bar{x}$ and $\delta > 0$ such that

$$S(b_1) \cap V \subset S(b_2) + \ell(\|b_2 - b_1\|)\bar{B}_{\mathbb{R}^n},$$

for every b_1, b_2 with $\|b_1 - \bar{b}\| < \delta$ and $\|b_2 - \bar{b}\| < \delta$. Consequently, there exits $\delta' > 0$ with $\delta' < \delta$ such that

$$\mathrm{Sol}_M(q_1) \cap V \subset \mathrm{Sol}_M(q_2) + \ell(\|q_2 - q_1\|)\bar{B}_{\mathbb{R}^n},$$

for every q_1, q_2 with $\|q_1 - \bar{q}\| < \delta'$ and $\|q_2 - \bar{q}\| < \delta'$. Hence, the solution map $\mathrm{Sol}_M(.)$ of LCP is Lipschitz-like around $(\bar{q}, \bar{x})$.

To finish the proof, we need to show that (3.1) is equivalent to (3.5). Suppose that

$$\begin{pmatrix} u' \\ v' \end{pmatrix} \in \ker A^T \cap N(\bar{y}; K).$$

Then, $\begin{pmatrix} u' \\ v' \end{pmatrix} \in N(\bar{y}; K)$ and $M^T u' + v' = 0$. In addition, by applying (3.2) for K and by [50, Proposition 1.2], we have

$$N(\bar{y}; K) = N\left(\begin{pmatrix} M\bar{x} + \bar{q} \\ \bar{x} \end{pmatrix}; K\right) = \prod_{i=1}^{n} N\left(\begin{pmatrix} (M\bar{x} + \bar{q})_i \\ \bar{x}_i \end{pmatrix}; K_i\right),$$

where

$$N\left(\begin{pmatrix} (M\bar{x} + \bar{q})_i \\ \bar{x}_i \end{pmatrix}; K_i\right) = \begin{cases} \{0\} \times \mathbb{R} & \text{if } i \in I_1 \\ \mathbb{R} \times \{0\} & \text{if } i \in I_2 \\ (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) \cup \mathbb{R}^2_- & \text{if } i \in I_3. \end{cases}.$$

It is easy to see that $\ker A^T \cap N(\bar{y}; K) = \{0\}$ if and only if (3.5) is satisfied.

□

Theorem 3.2 shows that (3.5) is sufficient for both the local error bound (3.3) at $(\bar{q}, \bar{x})$ and the Lipschitz-likeness of $\mathrm{Sol}_M(.)$ around $(\bar{q}, \bar{x})$. Now, following a suggestion from one of the referees of this paper, we will improve the claim about the Lipschitz-like property of the solution map. Namely, by a direct proof based on the Mordukhovich criterion and coderivative calculation, we will show that our regularity condition (3.5) is not only sufficient, but also necessary for the Lipschitz-like property of the solution map $\mathrm{Sol}_M(.)$ of the linear complementarity problem.

Proposition 3.1 *The solution map* $\mathrm{Sol}_M(.)$ *of* LCP *is Lipschitz-like around* $(\bar{q}, \bar{x})$ *if and only if the regularity condition* (3.5) *is satisfied.*

Proof. Note that $x \in \mathrm{Sol}_M(q)$ if and only if $0 \in Mx + q + N\big(x; \mathbb{R}^n_+\big)$. Hence,

$$\mathrm{Sol}_M(q) = \{x \in \mathbb{R}^n \,:\, q \in -Mx - N\big(x; \mathbb{R}^n_+\big)\}.$$

Clearly, the multifuntion $\Psi_1(x) = -Mx - N\big(x; \mathbb{R}^n_+\big)$ has closed graph and it is the inverse multifunction of $\mathrm{Sol}_M(.)$. Similarly as it has been done in proving Theorem 3.1, we will show that Ψ_1 is metrically regular around $(\bar{x}, \bar{q}) \in \mathrm{gph}\, \Psi_1$ if and only if (3.5) is fulfilled. Putting $T(x) = -N\big(x; \mathbb{R}^n_+\big)$ and using the coderivative sum rule [50, Theorem 1.62], we have

$$D^*\mathrm{Sol}_M(\bar{x}, \bar{q})(u') = -M^T u' + D^*T(\bar{x}, M\bar{x} + \bar{q})(u') \quad \forall u' \in \mathbb{R}^n.$$

By definition,

$$D^*T(\bar{x}, M\bar{x} + \bar{q})(u') = \{w' \in \mathbb{R}^n \,:\, (w', -u') \in N((\bar{x}, M\bar{x} + \bar{q}); \mathrm{gph}\, T)\}.$$

Here,

$$\begin{aligned}\operatorname{gph} T &= \{(x,v) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, v \in -N(x;\mathbb{R}^n_+)\}\\ &= \{(x,v) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, v \in (-N(x_1;\mathbb{R}_+)) \times \cdots \times (-N(x_n;\mathbb{R}_+))\}\\ &= \{(x,v) \,:\, (x_i,v_i) \in K_i \ \ \forall i \in \{1,\ldots,n\}\},\end{aligned}$$

where $x = (x_1,\ldots,x_n)$, $v = (v_1,\ldots,v_n)$, and

$$K_i = \{(\xi,\eta) \in \mathbb{R} \times \mathbb{R}, : \xi > 0 \text{ and } \eta = 0, \text{ or } \ \xi = 0 \text{ and } \eta \geq 0\}.$$

Recall that $I = \{1,\ldots,n\}$, $I_1 = \{i \in I \,:\, \bar{x}_i = 0, \ (M\bar{x}+\bar{q})_i > 0\}$, and

$I_2 = \{i \in I \,:\, \bar{x}_i > 0, \ (M\bar{x}+\bar{q})_i = 0\}, I_3 = \{i \in I \,:\, \bar{x}_i = 0, \ (M\bar{x}+\bar{q})_i = 0\}$.

Hence, for any $i \in I_1$, $N((\bar{x}_i,(M\bar{x}+\bar{q})_i);K_i) = \mathbb{R} \times \{0\}$. Meanwhile, for any $i \in I_2$,

$$N((\bar{x}_i,(M\bar{x}+\bar{q})_i);K_i) = \{0\} \times \mathbb{R},$$

and

$$N((\bar{x}_i,(M\bar{x}+\bar{q})_i);K_i) = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R}) \cup (\mathbb{R}_- \times \mathbb{R}_-)$$

for any $i \in I_3$. Hence, the coderivative value $D^*T(\bar{x}, M\bar{x}+\bar{q})(u')$ consists of all $w' \in \mathbb{R}^n$ satisfying the following condition

$$\begin{cases}(w'_i,-u'_i) \in \mathbb{R} \times \{0\} \ \text{ if } i \in I_1,\\ (w'_i,-u'_i) \in \{0\} \times \mathbb{R} \ \text{ if } i \in I_2,\\ (w'_i,-u'_i) \in (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R}) \cup (\mathbb{R}_- \times \mathbb{R}_-) \ \text{ if } i \in I_3.\end{cases}$$

$D^*T(\bar{x}, M\bar{x}+\bar{q})(u') = \emptyset$ if there exists $i \in I_1$ with $u'_i \neq 0$. Consequently, if $u'_i = 0$ for all $i \in I_1$, then $D^*T(\bar{x}, -M\bar{x}-\bar{q})(u')$ consists of all $w' \in \mathbb{R}^n$ satisfying

$$\begin{cases}(w'_i,-u'_i) \in \{0\} \times \mathbb{R} \ \text{ if } i \in I_2,\\ (w'_i,-u'_i) \in (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R}) \cup (\mathbb{R}_- \times \mathbb{R}_-) \ \text{ if } i \in I_3.\end{cases}$$

By the Mordukhovich criterion 2, Ψ_1 is metrically regular around the point $(\bar{x},\bar{q}) \in \operatorname{gph} \Psi$ if and only if

$$0 \in D^*\Psi_1(\bar{x},\bar{q})(u') \implies u' = 0$$

or, equivalently,

$$\big(M^T u' \in D^*T(\bar{x}, M\bar{x}+\bar{q})(u')\big) \implies u' = 0.$$

From the above arguments, Ψ_1 is metrically regular around $(\bar{x}, \bar{q})$ if and only if the next condition is satisfied: *If* $u' = (u'_1, \ldots, u'_n)^T \in \mathbb{R}^n$ *and if*

$$\begin{cases} \textit{for each } i \in I_1, \ u'_i = 0 \textit{ and } (M^T u')_i \in \mathbb{R} \\ \textit{for each } i \in I_2, \ (M^T u')_i \in \{0\} \\ \textit{for each } i \in I_3, \ u'_i = 0 \textit{ and } (M^T u')_i \in \mathbb{R}, \textit{ or } \ (M^T u')_i \in \{0\}, \\ \ \textit{or } \ u'_i \geq 0 \textit{ and } (M^T u')_i \in \mathbb{R}_-, \end{cases}$$

then $u' = 0$.
Clearly, this condition is equivalent to (3.5). Thus, (3.5) is necessary and sufficient for the Lipschitz-likeness of $\mathrm{Sol}_M(.)$ around $(\bar{q}, \bar{x})$. □

Using a detailed guidance of one of the referees, we can prove the following interesting fact.

Proposition 3.2 *The uniform local error bound* (3.3) *implies that* $\mathrm{Sol}_M(.)$ *is Lipschitz-like around* $(\bar{q}, \bar{x})$.

Proof. Recall that $\Psi_1(x) = -Mx - N\big(x; \mathbb{R}^n_+\big)$ for any $x \in \mathbb{R}^n$. We have

$$\begin{aligned} [\mathrm{Sol}_M]^{-1}(x) &= \{q \, : \, 0 \in Mx + q + N\big(x; \mathbb{R}^n_+\big)\} \\ &= \{q \, : \, q \in -Mx - N\big(x; \mathbb{R}^n_+\big)\} \\ &= \Psi_1(x). \end{aligned}$$

Note that $(u_i, v_i) \in K_i$ if and only if $-v_i \in N(u_i; \mathbb{R}_+)$. In addition, for vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ from $\mathbb{R}^n$, the inclusions $-v_i \in N(u_i; \mathbb{R}_+)$, $i = 1, \ldots, n$, can be rewritten equivalently as $-v \in N(u; \mathbb{R}^n_+)$. Also, the last inclusion is equivalent to $-u \in N(v; \mathbb{R}^n_+)$. Therefore, if the uniform local error bound (4.2) is satisfied at $(\bar{q}, \bar{x})$, then

$$\begin{aligned} & d(x; \mathrm{Sol}_M(q)) \\ & \leq \ell \sum_{i=1}^{n} d\left(\begin{pmatrix} (Mx + q)_i \\ x_i \end{pmatrix} ; K_i \right) \\ & \leq \ell \sum_{i=1}^{n} \inf \big\{ |(Mx+q)_i - u_i| + |x_i - v_i| \, : \, (u_i, v_i) \in \mathbb{R}^2, \ -v_i \in N(u_i; \mathbb{R}_+) \big\} \\ & = \ell \inf \big\{ \|(Mx+q) - u\| + \|x - v\| \, : \, (u, v) \in \mathbb{R}^n \times \mathbb{R}^n, \ -v \in N(u; \mathbb{R}^n_+) \big\} \\ & \leq \ell \inf \big\{ \|Mx + q - u\| + \|x - x\| \, : \, u \in \mathbb{R}^n, \ -x \in N(u; \mathbb{R}^n_+) \big\} \\ & = \ell \inf \big\{ \|Mx + q - u\| \, : \, -u \in N(x; \mathbb{R}^n_+) \big\} \\ & = \ell \inf \big\{ \|q - (-Mx + u)\| \, : \, u \in -N(x; \mathbb{R}^n_+) \big\} \\ & = \ell \, d(q, \Psi_1(x)). \end{aligned}$$

The inequality $d(x; \mathrm{Sol}_M(q)) \le \ell\, d(q, \Psi_1(x))$ verifies the metric regularity of Ψ_1 around $(\bar{x}, \bar{q})$ and thus the Lipschitz-like property of $\mathrm{Sol}_M(.)$ at $(\bar{q}, \bar{x})$. □

Combining Theorem 3.2 with Propositions 3.1 and 3.2 gives the next complete characterization of two fundamental stability properties of LCP on the basis of our regularity condition (3.5).

Theorem 3.3 *Suppose that $\bar{x} \in \mathrm{Sol}_M(\bar{q})$. The following statements are equivalent:*

(a) *The problem* LCP *satisfies the uniform local error bound* (3.3) *at $(\bar{q}, \bar{x})$;*

(b) *The solution map $\mathrm{Sol}_M(.)$ is Lipschitz-like around $(\bar{q}, \bar{x}) \in \mathrm{gph}\, S$;*

(c) *The regularity condition* (3.5) *holds.*

Remark 3.4 If $I = I_1$, then (3.5) is automatically satisfied. If $I = I_2$, then (3.5) holds if and only if M is nonsingular.

The notion of P-matrix is fundamental in theory of linear complementarity problems; see, e.g., [16, Chapters 3 and 7] and [60]. To proceed furthermore, we need to recall this notion.

Definition 3.2 (See [16, Definition 3.3.1]) One says that $M \in \mathbb{R}^{n\times n}$ is a *P-matrix* if all its principal minors are positive. The class of such matrices is denoted by $\mathcal{P}$.

It is clear that every positive definite matrix is a P-matrix. But, as shown in [16, Example 3.3.2], the converse is not true in general.

The following characterization of P-matrices is a part of Theorem 3.3.4 from [16].

Lemma 3.1 *An element $M \in \mathbb{R}^{n\times n}$ is a P-matrix if and only if, for any $z = (z_1, \ldots, z_n)^T \in \mathbb{R}^n$,*

$$[z_i(Mz)_i \le 0 \ \ \textit{for all } i \in \{1, \ldots, n\}] \implies z = 0.$$

Proposition 3.3 *If $I_3 = I$, then the regularity condition* (3.5) *is satisfied if and only if M is P-matrix.*

Proof. *Necessity:* Suppose that $M \in \mathcal{P}$. Then, M^T is also a P-matrix. By Lemma 3.1, the following implication is valid:

$$\left[u'_i(M^T u')_i \le 0 \ \text{ for all } i \in \{1, \ldots, n\}\right] \implies u' = 0. \tag{3.8}$$

Since $I_3 = I$, we have $I_1 = I_2 = \emptyset$. If (3.5) is violated, then there exists $u' \in \mathbb{R}^n \setminus \{0\}$ such that, for each $i \in I_3$, either $u'_i = 0$, or $(M^T u')_i = 0$, or $u'_i \leq 0$ and $(M^T u')_i \geq 0$. Therefore, $u'_i(M^T u')_i \leq 0$ for all $i \in \{1, \ldots, n\}$. This contradicts (3.8) because $u' \neq 0$.

Sufficiency: On the contrary, suppose that (3.5) is fulfilled, but $M \notin \mathcal{P}$. Due to Lemma 3.1, the latter implies the existence of $u' \in \mathbb{R}^n \setminus \{0\}$ with $u'_i(M^T u')_i \leq 0$ for all $i \in \{1, \ldots, n\}$. Then, for any $i \in \{1, \ldots, n\}$, either $u'_i = 0$, or $(M^T u')_i = 0$, or $u'_i \leq 0$ and $(M^T u')_i \geq 0$, or $u'_i \geq 0$ and $(M^T u')_i \leq 0$. Since $I_3 = I$ and $u' \neq 0$, this obviously contradicts (3.5). □

For a subset $\alpha \subset I$, the submatrix $(m_{ij})_{i \in \alpha, j \in \alpha}$ of $M = (m_{ij})$ is denoted by $M_{\alpha\alpha}$.

Given three subsets α, β, γ of I, we abbreviate the 9-block matrix

$$\begin{bmatrix} M_{\alpha\alpha} & 0 & 0 \\ 0 & M_{\beta\beta} & 0 \\ 0 & 0 & M_{\gamma\gamma} \end{bmatrix}$$

to $\text{diag}[M_{\alpha\alpha}, M_{\beta\beta}, M_{\gamma\gamma}]$.

We now obtain a sufficient condition for the fulfillment of (3.5). Then, by virtue of Theorem 3.2, we are able to provide a verifiable sufficient condition for several stability properties of LCPs under linear perturbations.

Theorem 3.4 *If* $M = \text{diag}[M_{I_1I_1}, M_{I_2I_2}, M_{I_3I_3}]$ *with* $M_{I_2I_2}$ *being a nonsingular matrix and* $M_{I_3I_3} \in \mathcal{P}$*, then* (3.5) *is satisfied.*

Proof Suppose that $M_{I_2I_2}$ is nonsingular and $M_{I_3I_3} \in \mathcal{P}$. By reordering the index set I, if necessary, we can assume that $i_1 \leq i_2 \leq i_3$ for every $i_1 \in I_1$, $i_2 \in I_2$, and $i_3 \in I_3$. For any $u' = (u'_1, u'_2, ..., u'_n)^T \in \mathbb{R}^n$,

$$(M^T u')_i = \sum_{j=1}^{n} m_{ji} u'_j, \quad \forall i \in \{1, \ldots, n\}. \tag{3.9}$$

To verify (3.5), suppose that $u'_{I_1} = 0$, $(M^T u')_i = 0$ for all $i \in I_2$, and for each $i \in I_3$, either $u'_i = 0$, or $(M^T u')_i = 0$, or $u'_i \leq 0$ and $(M^T u')_i \geq 0$. From (3.9) we have

$$\sum_{j \in I_2} m_{ji} u'_j = 0, \quad \forall i \in I_2.$$

This means that $M_{I_2I_2} u'_{I_2} = 0$. As $\det M_{I_2I_2} = 0$, the last equality implies

$u'_{I_2} = 0$. By the assumption made on u',

$$u'_i (M^T u')_i \le 0, \quad \forall i \in I_3.$$

Since $M_{I_3 I_3} \in \mathcal{P}$, combining the last property with Lemma 3.1 yields $u'_{I_3} = 0$; hence $u' = 0$. We have thus shown that (3.5) is fulfilled. □

The next example is designed to clarify the applicability of Theorem 3.4.

Example 3.1 Consider the problem LCP with $n = 3$, $M = \mathrm{diag}\,[\alpha_1, \alpha_2, \alpha_3]$, where $\alpha_2 < 0$ and $\alpha_3 > 0$, $\bar{q} = (\bar{q}_1, \bar{q}_2, 0)^T$, $\bar{q}_1 > 0$ and $\bar{q}_2 > 0$. It is clear that $\bar{x} := (0, -\bar{q}_2 \alpha_2^{-1}, 0)^T$ belongs to $\mathrm{Sol}_M(\bar{q})$, $I_1 = \{1\}$, $I_2 = \{2\}$, and $I_3 = \{3\}$. For any $u' = (u'_1, u'_2, u'_3)^T \in \mathbb{R}^3$, we have $(M^T u')_2 = \alpha_2 u'_2$ and $(M^T u')_3 = \alpha_3 u'_3$. It is easy to check that if

$$\begin{cases} u'_1 = 0, \\ \alpha_2 u'_2 = 0, \\ \text{either } \; u'_3 = 0, \; \text{ or } \; \alpha_3 u'_3 = 0, \; \text{ or } \; (u'_3 \le 0 \; \text{ and } \; \alpha_3 u'_3 \ge 0), \end{cases}$$

then $u' = (0, 0, 0)^T$. This means that (3.5) is satisfied.

It is worth to stress that the sufficient condition given by Theorem 3.4 is not a necessary one. This means that there exists $\bar{q}$, and $\bar{x} \in \mathrm{Sol}_M(\bar{q})$ such that (3.5) is satisfied, but M needs not to have the form

$$M = \mathrm{diag}[M_{I_1 I_1}, M_{I_2 I_2}, M_{I_3 I_3}]$$

with $M_{I_2 I_2}$ being a nonsingular matrix and $M_{I_3 I_3} \in \mathcal{P}$. Let us consider two examples of this type.

Example 3.2 Choose $n = 3$, $M = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $\bar{q} = \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \bar{q}_3 \end{pmatrix}$, where $\bar{q}_1 > 0$, $\bar{q}_2 > 0$, and $\bar{q}_3 = -\bar{q}_2$. Note that M is not a P-matrix and $\bar{x} := (0, \bar{q}_2, 0)^T$ is an element of $\mathrm{Sol}_M(\bar{q})$. Here, $I_1 = \{1\}$, $I_2 = \{2\}$, $I_3 = \{3\}$. For any $u' = (u'_1, u'_2, u'_3)^T \in \mathbb{R}^3$,

$$(M^T u')_2 = -u'_2 + u'_3, \quad (M^T u')_3 = u'_3.$$

Suppose that $u'_1 = 0$ and $(M^T u')_2 = 0$. If $u'_3 = 0$, then we have $u'_1 = u'_2 = 0$ and $u' = 0$. If $(M^T u')_3 = 0$, which means that $u'_3 = 0$, we also obtain $u' = 0$ by the condition $(M^T u')_2 = 0$. Finally, if $u'_3 \le 0$ and $(M^T u')_3 \ge 0$, then $u'_3 = 0$. Hence, again by the condition $(M^T u')_2 = 0$, we can assert that $u' = 0$. Thus, we have shown that (3.5) is satisfied.

Example 3.3 Let $n = 2$, $M = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, $\bar{q} = \begin{pmatrix} \bar{q}_1 \\ 0 \end{pmatrix}$ with $\bar{q}_1 > 0$, and $\bar{x} = (\bar{q}_1, -\bar{q}_1)^T$. Clearly, $M \notin \mathcal{P}$, $\bar{x} \in \mathrm{Sol}_M(\bar{q})$, $I_1 = \emptyset$, $I_2 = \{1\}$, and $I_3 = \{2\}$. For any $u' = (u'_1, u'_2)^T \in \mathbb{R}^2$,

$$(M^T u')_1 = -u'_1 + u'_2, \quad (M^T u')_2 = u'_2.$$

Suppose that $(M^T u')_1 = 0$ or, equivalently, $-u'_1 + u'_2 = 0$. Hence, if $u'_2 = 0$, then $u'_1 = u'_2 = 0$. If $(M^T u')_2 = 0$, then $u'_2 = 0$. So we have $u'_1 = 0$ by the condition $-u'_1 + u'_2 = 0$. If $u'_2 \leq 0$ and $(M^T u')_2 \geq 0$, then we have $u'_2 = 0$ and thereby $u'_1 = 0$. Thus, condition (3.5) holds.

In the theory of linear complementarity problems, besides $\mathcal{P}$, there are two other very important classes of matrices whose definitions [16, p. 145 and p. 140] are recalled now. One says that $M \in \mathbb{R}^{n\times n}$ is a *Q-matrix*, and writes $M \in \mathcal{Q}$, if the problem $\mathrm{Sol}_M(q)$ has a solution for all vectors q. One calls an element $M \in \mathbb{R}^{n\times n}$ a *Stiemke matrix* and write $M \in \mathcal{S}$ if there exists a vector $z > 0$ satisfying $Mz > 0$. It is well known that $\mathcal{P} \subset \mathcal{Q} \subset \mathcal{S}$. For the matrix M given in Example 3.2, we have seen that $M \notin \mathcal{P}$. To show that $M \notin \mathcal{Q}$, it suffices to observe that $\mathrm{Sol}(M, q) = \emptyset$ for all $q = (q_1, q_2, q_3)^T$ with $q_1 < 0$. Moreover, $M \notin \mathcal{S}$ because the system of the strict inequalities $z > 0$ and $Mz > 0$ has no solution. For the matrix M given in Example 3.3, we also have $M \notin \mathcal{S}$.

Remark 3.5 According to [18, Theorem 3], the following are equivalent:

(a) $\mathrm{Sol}_M(.)$ is Lipschitz-like around $(\bar{q}, \bar{x})$;

(b) $\mathrm{Sol}_M(.)$ is locally single-valued and Lipschitz continuous around $(\bar{q}, \bar{x})$;

(c) The variational system $0 \in Mx + q + N\big(x; \mathbb{R}^n_+\big)$ is strongly regular at $(\bar{q}, \bar{x})$.

Here, the strong regularity is understood in the sense of Robinson [76]. Note that Dontchev and Rockafellar [19] have provided several characterizations for the strong regularity.

3.3 Solution Stability of Affine Variational Inequalities under Linear Perturbations

In Chapter 2, we have established a verifiable sufficient condition for the existence of local error bounds for problem AVI as well as the Lipschitz-likeness of its solution map around a given point when the total initial data witness perturbations. However, there was no example to illustrate its efficiency. Here, in the case of linear perturbations, we will give not only a similar sufficient condition, but also a large class of problems to show how the condition works in practice.

As it has been noted in Chapter 2, $x \in \mathrm{Sol}_{M,C}(q,d)$ if and only if there exists a Lagrange multiplier $\lambda \in \mathbb{R}^m$ such that

$$A\begin{pmatrix} x \\ \lambda \end{pmatrix} + b \in K,$$

where

$$A = \begin{bmatrix} M & -C^T \\ 0 & E \\ C & 0 \end{bmatrix} \in \mathbb{R}^{(n+m+m)\times(n+m)}, \ \ b = \begin{pmatrix} q \\ 0 \\ -d \end{pmatrix} \in \mathbb{R}^{n+m+m},$$

and

$$K = \left\{(s,u,v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m : s = 0, \ u \geq 0, \ v \geq 0, \ v^T u = 0\right\}.$$

Here, as before, E is the unit matrix of order m. We will study the multifunction $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ defined by

$$S(b) = \left\{(x,\lambda) \in \mathbb{R}^n \times \mathbb{R}^m \ : \ A\begin{pmatrix} x \\ \lambda \end{pmatrix} + b \in K\right\}.$$

In what follows, we fix a pair $(\bar{q},\bar{d})$ and suppose that $\bar{x} \in \mathrm{Sol}_{M,C}(\bar{q},\bar{d})$. Put $\bar{b} = \begin{pmatrix} \bar{q} \\ 0 \\ -\bar{d} \end{pmatrix}$ and suppose that $\bar{\lambda}$ is a Lagrange multiplier corresponding to $\bar{x}$. Then, $(\bar{x},\bar{\lambda}) \in S(\bar{b})$. Set $I = \{1,\ldots,m\}$ and consider the index sets

$$\begin{aligned} I_1 &:= \{i \in I \ : \ (C\bar{x} - \bar{d})_i = 0, \ \bar{\lambda}_i > 0\}, \\ I_2 &:= \{i \in I \ : \ (C\bar{x} - \bar{d})_i > 0, \ \bar{\lambda}_i = 0\}, \\ I_3 &:= \{i \in I \ : \ (C\bar{x} - \bar{d})_i = 0, \ \bar{\lambda}_i = 0\}. \end{aligned} \tag{3.10}$$

Let

$$\bar{z} = A\bar{x} + \bar{b} = \begin{pmatrix} M\bar{x} - C^T\bar{\lambda} + \bar{q} \\ \bar{\lambda} \\ C\bar{x} - \bar{d} \end{pmatrix}.$$

Note that $M\bar{x} - C^T\bar{\lambda} + \bar{q} = 0$ and $\begin{pmatrix} \bar{\lambda} \\ C\bar{x} - \bar{d} \end{pmatrix} \in K_0$, where K_0 is defined by

$$K_0 = \left\{(u, v) \in \mathbb{R}^m \times \mathbb{R}^m \,:\, u \geq 0,\ v \geq 0,\ v^T u = 0\right\}.$$

The set K_0 can be decomposed by writing

$$K_0 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^m \,:\, \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in K_i,\ i \in I \right\},$$

where

$$K_i = \left\{ \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in \mathbb{R}^2 \,:\, u_i \geq 0,\ v_i \geq 0,\ u_i v_i = 0 \right\}.$$

Definition 3.3 (see [31]) We say that the problem AVI has the *uniform local error bound* at $((\bar{q}, \bar{d}), \bar{x})$ if there exist $\bar{\lambda} \in \mathbb{R}^m$, positive constants ℓ and δ, and a neighborhood V of $\bar{x}$ such that

$$d(x; \mathrm{Sol}_{M,C}(q, d)) \leq \ell \left(\|Mx - C^T\bar{\lambda} + q\| + \sum_{i=1}^{m} d\left(\begin{pmatrix} \bar{\lambda}_i \\ (Cx - d)_i \end{pmatrix}; K_i \right) \right), \quad (3.11)$$

for any $x \in V$ and (q, d) with $\|q - \bar{q}\| + \|d - \bar{d}\| < \delta$.

Substituting $(q, d) = (\bar{q}, \bar{d})$ into (3.11) yields

$$d(x; \mathrm{Sol}_{M,C}(\bar{q}, \bar{d})) \leq \ell \left(\|Mx - C^T\bar{\lambda} + \bar{q}\| + \sum_{i=1}^{m} d\left(\begin{pmatrix} \bar{\lambda}_i \\ (Cx - \bar{d})_i \end{pmatrix}; K_i \right) \right), \quad (3.12)$$

for any $x \in V$. This is a *local error bound* for the unperturbed AVI in the traditional form.

We now consider the regularity condition: *If vector* $(u', \eta') \in \mathbb{R}^n \times \mathbb{R}^m$ *satisfies* $M^T u' + C^T \eta' = 0$ *and the system*

$$\begin{cases} (Cu')_i = 0 \quad \text{if } i \in I_1; \\ \eta'_i = 0 \quad \text{if } i \in I_2; \\ \text{either } (Cu')_i = 0,\ \text{or } \eta'_i = 0,\ \text{or } ((Cu')_i \leq 0 \text{ and } \eta'_i \leq 0) \quad \text{if } i \in I_3, \end{cases} \quad (3.13)$$

then $(u', \eta') = (0, 0)$.

Note that for LCP, a special case of AVI, the Lagrange multiplier corresponding to a solution $\bar{x}$ is unique. To be more precise, from the conditions $C = E$ and $M\bar{x} - C^T\bar{\lambda} + \bar{q} = 0$, it follows that $\bar{\lambda} = M\bar{x} + \bar{q}$. For a general AVI, there may exist many Lagrange multipliers corresponding to one solution. So, to study the problem AVI, we need to impose an additional condition on the matrix C.

Theorem 3.5 *Suppose that* $\bar{x} \in \mathrm{Sol}_{M,C}(\bar{q}, \bar{d})$ *and* $\bar{\lambda}$ *is a Lagrange multiplier corresponding to* $\bar{x}$*. If the regularity condition* (3.13) *is satisfied, then the local error bounds for* AVI *in the forms* (3.11) *and* (3.12) *are valid. Moreover, if* $\operatorname{rank} C = m$*, then the solution map* $(q, d) \mapsto \mathrm{Sol}_{M,C}(q, d)$ *is Lipschitz-like around* $((\bar{q}, \bar{d}), \bar{x})$*.*

Proof. Let $\bar{x} \in \mathrm{Sol}_{M,C}(\bar{q}, \bar{d})$ and $\bar{\lambda}$ be a Lagrange multiplier corresponding to $\bar{x}$. Due to Theorem 3.1, if (3.1) is available, then there exist $\ell > 0$, a bounded neighborhood U of $\bar{b}$, a neighborhood V of $\bar{x}$, and a neighborhood $V_{\bar{\lambda}}$ of $\bar{\lambda}$ such that

$$d\left(\begin{pmatrix} x \\ \lambda \end{pmatrix}; S(b)\right) \le \ell\, d\left(A\begin{pmatrix} x \\ \lambda \end{pmatrix} + b; K\right)$$

for every $b \in U$, $x \in V$, and $\lambda \in V_{\bar{\lambda}}$. Hence,

$$d\left(\begin{pmatrix} x \\ \bar{\lambda} \end{pmatrix}; S(b)\right) \le \ell\, d\left(A\begin{pmatrix} x \\ \bar{\lambda} \end{pmatrix} + b; K\right) \tag{3.14}$$

for every $b \in U$ and $x \in V$. Here, we have

$$\begin{aligned} d\left(A\begin{pmatrix} x \\ \bar{\lambda} \end{pmatrix} + b; K\right) &= d\left(\begin{pmatrix} Mx - C^T\bar{\lambda} + q \\ \bar{\lambda} \\ Cx - d \end{pmatrix}; K\right) \\ &= \|Mx - C^T\bar{\lambda} + q\| + d\left(\begin{pmatrix} \bar{\lambda} \\ Cx - d \end{pmatrix}; K_0\right) \\ &= \|Mx - C^T\bar{\lambda} + q\| + \sum_{i=1}^{n} d\left(\begin{pmatrix} \bar{\lambda}_i \\ (Cx - d)_i \end{pmatrix}; K_i\right). \end{aligned}$$

Notice that

$$\begin{pmatrix} x \\ \bar{\lambda} \end{pmatrix} \in S(b) \implies x \in \mathrm{Sol}_{M,C}(q, d).$$

So, by (3.14) there exists $\delta > 0$ such that, for any $x \in V$ and (M, q, C, d) with $\|q - \bar{q}\| + \|d - \bar{d}\| < \delta$, we have

$$\begin{aligned} d\left(x; \mathrm{Sol}_{M,C}(q, d)\right) &\leq d\left(\begin{pmatrix} x \\ \bar{\lambda} \end{pmatrix}; S(b)\right) \\ &\leq \ell\left(\|Mx - C^T\bar{\lambda} + q\| + \sum_{i=1}^{n} d\left(\begin{pmatrix} \bar{\lambda}_i \\ (Cx - d)_i \end{pmatrix}; K_i\right)\right). \end{aligned}$$

Therefore, (3.11) and (3.12) are valid.

Under the assumption $\operatorname{rank} C = m$, the multifunction $(q, d) \mapsto \mathrm{Sol}_{M,C}(q, d)$ is Lipschitz-like around $((\bar{q}, \bar{d}), \bar{x})$. Indeed, as $\bar{\lambda}$ is a Lagrange multiplier corresponding to $\bar{x}$,

$$M\bar{x} - C^T\bar{\lambda} + \bar{q} = 0$$

or, equivalently,

$$C^T\bar{\lambda} = M\bar{x} + \bar{q}. \tag{3.15}$$

Since $\operatorname{rank} C = m$, we have $m \leq n$. Furthermore, there is a nonsingular $m \times m$ submatrix C_J^T composed by the j-th rows of C^T with $j \in J$, where the index set J has m elements. Then, (3.15) is equivalent to

$$\bar{\lambda} = \left(C_J^T\right)^{-1} (M\bar{x} + \bar{q})_J,$$

where $(M\bar{x} + \bar{q})_J$ denotes the vector composed by j-th component of vector $M\bar{x} + \bar{q}$, $j \in J$. Hence, $\bar{\lambda}$ is uniquely defined by the data $M, \bar{q}, C$, and the solution $\bar{x}$. Now, replacing the above δ by a smaller positive number and V by a smaller neighborhood of $\bar{x}$, if necessary, we can assume that, for any $x \in V$ and for any (q, d) satisfying $\|q - \bar{q}\| + \|d - \bar{d}\| < \delta$, the submatrix C_J^T is nonsingular, the equation

$$Mx - C_i^T\lambda + q = 0$$

gives us the unique solution

$$\lambda = \left(C_J^T\right)^{-1} (Mx + q)_J$$

and, moreover, $\lambda \in V_{\bar{\lambda}}$. We know that if condition (3.1) is satisfied, then $S(\cdot)$ is Lipschitz-like around $(\bar{b}, \bar{x})$ with constant $r > 0$ (see Theorem 3.1). Without loss of generality, we can assume that

$$S(b_1) \cap (V \times V_{\bar{\lambda}}) \subset S(b_2) + \ell\|b_2 - b_1\|\bar{B}_{\mathbb{R}^n \times \mathbb{R}^m}, \tag{3.16}$$

for any $b_i, i = 1, 2$, with

$$b_i = \begin{pmatrix} q_i \\ 0 \\ -d_i \end{pmatrix} \in \mathbb{R}^{n+m+m},$$

and $\|q_i - \bar{q}\| + \|d_i - \bar{d}\| < \delta$. Hence, $S(b_i) \neq \emptyset$, $i = 1, 2$; and therefore, $\mathrm{Sol}_{M,C}(q_i, d_i) \neq \emptyset$, $i = 1, 2$. Suppose that $x_1 \in \mathrm{Sol}_{M,C}(q_1, d_1) \cap V$. Then, there exits a unique Lagrange multiplier λ_1 corresponding to x_1, $\lambda_1 \in V_{\bar{\lambda}}$, and

$$\begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix} \in S(b_1) \cap (V \times V_{\bar{\lambda}}).$$

By (3.16) there is a pair $\begin{pmatrix} x_2 \\ \lambda_2 \end{pmatrix} \in S(w_2)$, which means $x_2 \in \mathrm{Sol}_{M,C}(q_2, d_2)$ and λ_2 is the Lagrange multiplier corresponding to x_2, such that

$$\left\| \begin{pmatrix} x_2 \\ \lambda_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix} \right\| \leq \ell \|b_2 - b_1\|.$$

Moreover,

$$\|b_2 - b_1\| = \|q_2 - q_1\| + \|d_2 - d_1\|$$

Hence,

$$\|x_2 - x_1\| \leq \ell \|(q_2, d_2) - (q_1, d_1)\|.$$

Thus,

$$\mathrm{Sol}_{M,C}(q_1, d_1) \cap V \subset \mathrm{Sol}_{M,C}(q_2, d_2) + \ell \|(q_2, d_2) - (q_1, d_1)\| \bar{B}_{\mathbb{R}^n},$$

for any (q_i, d_i) satisfying $\|q_i - \bar{q}\| + \|d_i - \bar{d}\| < \delta$, $i = 1, 2$. In other words, in case of $\operatorname{rank} C = m$, the solution map $\mathrm{Sol}_{M,C}(.)$ of AVI is Lipschitz-like around $((\bar{q}, \bar{d}), \bar{x})$ with $\bar{x} \in \mathrm{Sol}_{M,C}(\bar{q}, \bar{d})$.

To complete the proof, we need to show that (3.1) is equivalent to (3.13). Let

$$\begin{pmatrix} u' \\ v' \\ \eta' \end{pmatrix} \in \ker A^T \cap N(\bar{y}; K).$$

Clearly, a pair (u', v', η') belongs to $\ker A^T$ if and only if

$$\begin{cases} M^T u' + C^T \eta' = 0 \\ -Cu' + v' = 0. \end{cases}$$

Hence,

$$\ker A^T = \left\{ \begin{pmatrix} u' \\ v' \\ \eta' \end{pmatrix} : u' \in \mathbb{R}^n,\ v' = Cu',\ \eta' \in \mathbb{R}^m,\ M^T u' + C^T \eta' = 0 \right\}. \quad (3.17)$$

In addition,

$$N(\bar{z}; K) = \mathbb{R}^n \times N\left(\begin{pmatrix} \bar{\lambda} \\ C\bar{x} - \bar{d} \end{pmatrix}; K_0 \right).$$

By virtue of the structure of K_0,

$$N\left(\begin{pmatrix} \bar{\lambda}_i \\ (C\bar{x} - \bar{d})_i \end{pmatrix}; K_i \right) = \begin{cases} \{0\} \times \mathbb{R} & \text{if } i \in I_1 \\ \mathbb{R} \times \{0\} & \text{if } i \in I_2 \\ (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) \cup \mathbb{R}^2_- & \text{if } i \in I_3. \end{cases}$$

Due to (3.17), a vector $(u', v', \eta')^T \in \mathbb{R}^m$ belongs to $\ker A^T \cap N(\bar{y}; K)$ if and only if $M^T u' + C^T \eta' = 0$, $v' = Cu'$, and the following is satisfied

$$\begin{cases} v'_i = 0 \text{ if } i \in I_1; \\ \eta'_i = 0 \text{ if } i \in I_2; \\ \text{either } v'_i = 0, \text{ or } \eta'_i = 0, \text{ or } (v'_i \le 0 \text{ and } \eta'_i \le 0) \text{ if } i \in I_3. \end{cases}$$

Now it is clear that (3.1) is equivalent to (3.13). □

Thanks to the hints of one referee of [28], we are able to show that, under the assumption $\operatorname{rank} C = m$, (3.13) is not only sufficient but also necessary for the Lipschitz-likeness $\mathrm{Sol}_{M,C}(.)$ around $(\bar{q}, \bar{d}, \bar{x})$.

Proposition 3.4 *If* $\operatorname{rank} C = m$, *then the solution map* $(q, d) \mapsto \mathrm{Sol}_{M,C}(q, d)$ *is Lipschitz-like around* $((\bar{q}, \bar{d}), \bar{x})$ *if and only if* (3.13) *is satisfied.*

Proof. Since AVI can be written as the variational system

$$0 \in Mx + q + N\big(x; \Delta(d)\big),$$

we have

$$\mathrm{Sol}_{M,C}(q, d) = \{x \in \mathbb{R}^n : 0 \in q + Mx + N_{\Delta(d)}(x)\}.$$

Define $\Psi_2(x, d) = Mx + N_{\Delta(d)}(x)$ and $g(x, d) = -Cx + d$. By setting

$$f(x, d) = \theta(g(x, d)) = (\theta \circ g)(x, d),$$

where $\theta(y) = \delta_{\mathbb{R}^m_-}(y)$, i.e., $\theta(y) = 0$ for $y \in \mathbb{R}^m_-$ and $\theta(y) = +\infty$ for $y \notin \mathbb{R}^m_-$, we can show that $N_{\Delta(d)}(x) = \partial_x f(x, d)$ for any $d \in \mathbb{R}^m$ (see [50, Subsections 1.3.1 and 2.3.2] for detailed definitions and relationships). Hence

$$\mathrm{Sol}_{M,C}(q, d) = \{x \in \mathbb{R}^n \,:\, 0 \in q + Mx + \partial_x f(x, d)\}.$$

Since $(q', d', x') \in \mathrm{gph}\,\mathrm{Sol}_{M,C}$ if and only if $(x', d', -q') \in \mathrm{gph}\,\Psi_2$, we have the following equivalence:

$$\begin{aligned}
& (q^*, d^*) \in D^*\mathrm{Sol}_{M,C}(\bar{q}, \bar{d}, \bar{x})(x^*) \\
\Leftrightarrow\ & (q^*, d^*, -x^*) \in N\left((\bar{q}, \bar{d}, \bar{x}); \mathrm{gph}\,\mathrm{Sol}_{M,C}\right) \\
\Leftrightarrow\ & (-x^*, d^*, -q^*) \in N\left((\bar{d}, \bar{x}, -\bar{q}); \mathrm{gph}\,\Psi_2\right) \\
\Leftrightarrow\ & (-x^*, d^*) \in D^*\Psi_2(\bar{x}, \bar{d}, -\bar{q})(q^*).
\end{aligned}$$

By the coderivative sum rule [50, Theorem 1.62] we have

$$D^*\Psi_2(\bar{x}, \bar{d}, -\bar{q})(q^*) = (M^T q^*, 0) + D^*\partial_x f(\bar{x}, \bar{d}, -\bar{q} - M\bar{x})(q^*).$$

So, $(-x^*, d^*) \in D^*\Psi_2(\bar{x}, \bar{d}, -\bar{q})(q^*)$ if and only if

$$(-x^* - M^T q^*, d^*) \in D^*\partial_x f(\bar{x}, \bar{d}, -\bar{q} - M\bar{x})(q^*).$$

According to the Mordukhovich criterion 1, $\mathrm{Sol}_{M,C}$ is Lipschitz-like around $((\bar{q}, \bar{d}), \bar{x})$ if and only if $D^*\mathrm{Sol}_{M,C}(\bar{q}, \bar{d}, \bar{x})(0) = \{0\}$ or, equivalently,

$$(-M^T q^*, d^*) \in D^*\partial_x f(\bar{x}, \bar{d}, -\bar{q} - M\bar{x})(q^*) \implies (q^*, d^*) = (0, 0). \tag{3.18}$$

Let us show that (3.18) is fulfilled if and only if (3.13) holds. Due to $\mathrm{rank}\, C = m$, we can apply the formula for the extended partial second-order subdifferential in [58, Theorem 3.1] to obtain

$$\begin{aligned}
D^*\partial_x f(\bar{x}, \bar{d}, -\bar{q} - M\bar{x})(q^*) &= (\nabla_x g(\bar{x}, \bar{d}), \nabla_d g(\bar{x}, \bar{d}))^* \partial^2\theta(\bar{z}, \bar{\lambda})(\nabla_x g(\bar{x}, \bar{d}) q^*) \\
&= \{(-C^T\xi, \xi) \,:\, \xi \in \partial^2\theta(\bar{z}, \bar{\lambda})(\nabla_x g(\bar{x}, \bar{d}) q^*)\}.
\end{aligned} \tag{3.19}$$

Here $\bar{z} = g(\bar{x}, \bar{d})$, and $\bar{\lambda} \in \mathbb{R}^m$ is the unique vector satisfying $C^T\bar{\lambda} = M\bar{x} + q$. We have known that $\bar{\lambda}$ is the Lagrange multiplier corresponding to $\bar{x}$. Note that

$$\begin{aligned}
\partial^2\theta(\bar{z}, \bar{\lambda})(\nabla_x g(\bar{x}, \bar{d}) q^*) &= D^*\partial\theta(\bar{z}, \bar{\lambda})(\nabla_x g(\bar{x}, \bar{d}) q^*) \\
&= \{\xi \in \mathbb{R}^m \,:\, (\xi, -\nabla_x g(\bar{x}, \bar{d}) q^*) \in N((\bar{z}, \bar{\lambda}); \mathrm{gph}\,\partial\theta)\} \\
&= \{\xi \in \mathbb{R}^m \,:\, (\xi, C q^*) \in N((\bar{z}, \bar{\lambda}); \mathrm{gph}\,\partial\theta)\}.
\end{aligned}$$

For any $z \in \mathbb{R}^m_-$, $\partial\theta(z) = N(z; \mathbb{R}^m_-) = N(z_1; \mathbb{R}_-) \times \cdots \times N(z_m; \mathbb{R}_-)$. With each i belonging to $I = \{1, \ldots, m\}$, one has $N(z_i; \mathbb{R}_-) = \{0\}$ if $z_i < 0$ and

$N(z_i;\mathbb{R}_-) = \mathbb{R}_+$ if $z_i = 0$. So,

$$\begin{aligned} \operatorname{gph} \partial\theta &= \{(z, v') \in \mathbb{R}^m \times \mathbb{R}^m \,:\, v' \in N(z;\mathbb{R}^m_-)\} \\ &= \{(z, v') \in \mathbb{R}^m \times \mathbb{R}^m \,:\, v'_i \in N(z_i;\mathbb{R}_-) \text{ for any } i \in I\} \\ &= \{(z, v') \in \mathbb{R}^m \times \mathbb{R}^m \,:\, (z_i, v'_i) \in \Omega_i \text{ for any } i \in I\}. \end{aligned}$$

Here, $\Omega_i := \mathbb{R}_- \times \{0\} \cup \{0\} \times \mathbb{R}_+$ for all $i \in I$. Hence

$$N((\bar{z}, \bar{\lambda}); \operatorname{gph} \partial\theta) = N((\bar{z}_1, \bar{\lambda}_1); \Omega_1) \times \cdots \times N((\bar{z}_m, \bar{\lambda}_m); \Omega_m).$$

Recall that $\bar{z} = g(\bar{x}, \bar{d}) = -Cx + d$, $I_1 = \{i \in I \,:\, (C\bar{x} - \bar{d})_i = 0,\ \bar{\lambda}_i > 0\}$, $I_2 = \{i \in I \,:\, (C\bar{x} - \bar{d})_i > 0,\ \bar{\lambda}_i = 0\}$, and

$$I_3 = \{i \in I \,:\, (C\bar{x} - \bar{d})_i = 0,\ \bar{\lambda}_i = 0\}.$$

Then, $(\xi, Cq^*) \in N((\bar{z}, \bar{\lambda}); \operatorname{gph} \partial\theta)$ if and only if $(\xi_i, (Cq^*)_i) \in \Gamma_i$ for any $i \in I$, where

$$\Gamma_i := \begin{cases} \mathbb{R} \times \{0\} & \text{if } i \in I_1; \\ \{0\} \times \mathbb{R} & \text{if } i \in I_2; \\ \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R} \cup \mathbb{R}_+ \times \mathbb{R}_- & \text{if } i \in I_3. \end{cases}$$

Combining this with (3.19), we can see that (3.18) is satisfied if and only if

$$\begin{cases} -M^T q^* + C^T \xi = 0; \\ d^* = \xi; \\ \text{for any } i \in I,\ (\xi_i, (Cq^*)_i) \in \Gamma_i \end{cases}$$

forces $(q^*, d^*) = (0, 0)$. Thus, (3.18) is equivalent to the following: *If a vector (q^*, d^*) from $\mathbb{R}^n \times \mathbb{R}^m$ satisfies the equation $M^T q^* - C^T d^* = 0$ and the system*

$$\begin{cases} d^*_i = 0 & \textit{if } i \in I_1; \\ (Cq^*)_i = 0 & \textit{if } i \in I_2; \\ \textit{either } d^*_i = 0, \textit{ or } (Cq^*)_i = 0, \textit{ or } (Cq^*)_i \le 0 \textit{ and } (d^*_i \ge 0) & \textit{if } i \in I_3, \end{cases}$$

then $(q^*, d^*) = (0, 0)$. Putting $u' = q^*$ and $\eta' = -d^*$, we see at once that this condition coincides with (3.13). □

Remark 3.6 If $I = I_2$, then (3.13) is satisfied if and only if M is nonsingular. In this case, $(\bar{x}, \bar{d})$ is an interior point of the domain

$$D := \{(d, x) \in \mathbb{R}^n \times \mathbb{R}^m \,:\, Cx \ge d\}.$$

The next proposition gives a set of general conditions for the validity of (3.13).

Proposition 3.5 *Let the index sets I_1, I_2, and I_3 be defined by* (3.10). *If $M = \mathrm{diag}[M_{I_1I_1}, M_{I_2I_2}, M_{I_3I_3}]$ and $C = \mathrm{diag}[C_{I_1I_1}, C_{I_2I_2}, C_{I_3I_3}]$, where $M_{I_2I_2}$ and $C_{I_1I_1}$ are nonsingular, $M_{I_3I_3} \in \mathcal{P}$, and $C_{I_3I_3}$ is a diagonal matrix with positive diagonal elements, then* (3.13) *is fulfilled.*

Proof Suppose that $(u', \eta') \in \mathbb{R}^n \times \mathbb{R}^m$ and

$$M^T u' + C^T \eta' = 0. \tag{3.20}$$

Suppose in addition that $(Cu')_{I_1} = 0$, $\eta'_{I_2} = 0$ and

$$\forall i \in I_3, \text{ either } (Cu')_i = 0, \text{ or } \eta'_i = 0, \text{ or } ((Cu')_i \leq 0 \text{ and } \eta'_i \leq 0). \tag{3.21}$$

Since $C = \mathrm{diag}[C_{I_1I_1}, C_{I_2I_2}, C_{I_3I_3}]$, from $(Cu')_{I_1} = 0$ it follows that $C_{I_1I_1}u'_{I_1} = 0$. As $C_{I_1I_1}$ is nonsingular, we must have $u'_{I_1} = 0$. By (3.20) and the formula $M = \mathrm{diag}[M_{I_1I_1}, M_{I_2I_2}, M_{I_3I_3}]$, this implies that $C^T_{I_1I_1}\eta'_{I_1} = 0$. Since $\det C_{I_1I_1} \neq 0$, we have $\eta'_{I_1} = 0$. Moreover, by the equality $\eta'_{I_2} = 0$ and (3.20), we have $M^T_{I_2I_2}u'_{I_2} = 0$. Then we get $u'_{I_2} = 0$ due to the nonsingularity of $M_{I_2I_2}$.

It remains to prove that $u'_{I_3} = 0$ and $\eta'_{I_3} = 0$. From (3.20) we have

$$\left((M_{I_3I_3})^T u'_{I_3}\right)_i + \left((C_{I_3I_3})^T \eta'_{I_3}\right)_i = 0 \tag{3.22}$$

for every $i \in I_3$. Combining this with (3.21) and the assumption that $C_{I_3I_3}$ is a diagonal matrix with positive diagonal elements, we obtain

$$u'_i(M^T_{I_3I_3}u'_{I_3})_i \leq 0, \quad \forall i \in I_3.$$

Since $M_{I_3I_3} \in \mathcal{P}$, by Lemma 3.1 we can assert that $u'_{I_3} = 0$. Hence, by (3.22), we obtain $\eta'_{I_3} = 0$ and complete the proof. □

Remark 3.7 The assumption $\mathrm{rank}\, C = m$ in Theorem 3.5 means that C is of full row rank. Note that inactive constraints do not have any role in criteria of local stability because they are automatically fulfilled under small perturbations. So, for a given data $(\bar{q}, \bar{d})$ and a solution $\bar{x} \in \mathrm{Sol}_{M,C}(\bar{q}, \bar{d})$, we only need to suppose that $C_{I(\bar{x},\bar{d})}$, where $I(\bar{x}, \bar{d}) := \{i \in I : (C\bar{x} - \bar{d})_i = 0\}$, is of full row rank. Thus, in this sense, the condition $\mathrm{rank}\, C = m$ is equivalent to the linear independence constraint qualification (LICQ) in [61, 83, 84]. Under the assumption (LICQ), Nam gave a criterion for the Lipschitz-like property of the solution map $S(.)$ (which coincides with $\mathrm{Sol}_{M,C}(.)$ if we apply his model to AVIs); see [61, Theorem 5.3].

Remark 3.8 Following an idea from one of referees of using Lagrange multipliers to transform a parametric AVI to a complementarity problem, we note

x is a solution of our problem AVI if and only if it is a solution of the linear program

$$\min\{\varphi(y) := \langle Mx + q, y\rangle \,:\, x \in \Delta(d)\}$$

where $\Delta(d) = \{x \in \mathbb{R}^n : Cx \geq d\}$. The Lagrange function of this optimization problem is

$$L(y, \lambda) = \varphi(y) + \langle \lambda, -Cy + d\rangle \quad (\lambda \in \mathbb{R}^m_+).$$

So, x is a solution of AVI if and only if there exists a multiplier λ satisfying

$$\nabla_y L(., \lambda)(x) = 0, \;\; Cx \geq d, \;\; \lambda \geq 0, \;\; \langle \lambda, -Cx + d\rangle = 0.$$

This system can be written as

$$\begin{cases} Mx - C^T\lambda + q = 0; \\ Cx \geq d, \; \lambda \geq 0; \\ \langle \lambda, Cx - d\rangle = 0 \end{cases} \tag{3.23}$$

(see, e.g., [37, Theorem 5.3]). Denote the solution set of (3.23) by $F(q, d)$. Then we have

$$\mathrm{Sol}_{M,C}(q, d) = \{x \in \mathbb{R}^n \,:\, \exists \lambda \in \mathbb{R}^m \text{ s.t. } (x, \lambda) \in F(q, d)\} = \mathrm{pr}_{\mathbb{R}^n} F(q, d). \tag{3.24}$$

Due to the first equation in (3.23), we think that it is impossible to interpret the system as an LCP (introducing an artificial variable may not help). So, the results and arguments of Section 4 cannot be used here. Theorems 5 and 6 in [18] characterize certain properties of the multifunction $(q, d) \mapsto F(q, d)$. Since $\mathrm{Sol}_{M,C}(q, d)$ is computed via $F(q, d)$ by (3.24), the characterizations and equivalences obtained by Dontchev and Rockafellar [18, Section 5] cannot yield similar results for the solution map $\mathrm{Sol}_{M,C}(.)$.

3.4 Conclusions

Sufficient conditions for various stability properties of linear complementarity problems and also of affine variational inequalities under linear perturbations have been obtained in this chapter. Thus, we have shown that the approach to variational systems under linear perturbations via linear constraint systems is interesting and effective. Besides, the approach allows us to look deeper into some fundamental variational systems.

It is well known that the linear complementarity problem LCP(M, q) has a unique solution for every vector q if and only if the involved matrix M is P-matrix. One result in this chapter shows that, under a weaker regularity, the problem LCP(M, q) has a unique solution for every vector q near an initial vector $\bar{q}$.

Chapter 4

Sensitivity Analysis of a Stationary Point Set Map under Total Perturbations

This chapter studies coderivative estimates and stability properties of the stationary point set map of a smooth optimization problem under total perturbation.

The coderivative analysis of composite constraint functions of Levy and Mordukhovich [41] is based on the rich generalized differentiation calculus in [80, Chapter 10]. Among other things, it uses the properties of amenable functions and strongly amenable functions, and the extended chain rule for subdifferentials [80, Theorem 10.49]. The analysis allows us to derive sharp upper estimates for the Mordukhovich coderivative of the stationary point set map of a smooth optimization problem under total perturbation, where the limiting second-order subdifferential is used.

To get lower estimates for the Fréchet and the Mordukhovich coderivatives of the stationary point set map, we combine the lower estimates of Lee and Yen [39] with some results of Qui [71, 72].

With the above upper and lower coderivative estimates, we can use the *Mordukhovich criterion* for the Lipschitz-like property of locally closed multifunctions to obtain both *necessary and sufficient conditions* for this property of the stationary point set map. Here, we do not need an additional technical assumption of Qui. Besides, by invoking a result of [88], we are able to show that these sufficient conditions also guarantee the Robinson stability of the stationary point set map.

Our conditions are easy to verify and can be effectively applied to nonconvex quadratic programming under a possibly nonconvex quadratic constraint. The results on quadratic programming in this chapter extend the preceding ones of Lee and Yen [40] and Qui and Yen [73] to a broader class of quadratic programs.

Optimization problems under total perturbations have been studied in [54–56, 59] by different approaches and concepts. But, our results are very different from those of the cited works.

Solution stability of variational inequalities on fixed or linearly perturbed polyhedral convex sets, which is closely related to that of optimization problems under linear constraints, has been investigated intensively; see [7, 18, 27, 46, 61, 65–67, 77, 84–86], and the references therein. The case of nonlinearly perturbed polyhedral convex sets has been considered in [68].

It is well known that *calmness* is weaker than the Lipschitz-like property. Calmness of the stationary point set map of a general parametric optimization problem has been considered, e.g., in [25, Sect. 4].

This chapter is written on the basis of the papers [29, 30].

4.1 Problem Formulation

Let f_0 and F be twice continuously differentiable real-valued functions (C^2-functions for brevity) defined on the product $\mathbb{R}^n \times \mathbb{R}^d$ of two Euclidean spaces. For every $w \in \mathbb{R}^d$, we consider the parametric optimization problem

$$(P_w) \qquad \text{Minimize } \ f_0(x, w) \ \text{ subject to } x \in \mathbb{R}^n \ \text{ and } \ F(x, w) \leq 0.$$

The constraint set of (P_w) is $C(w) := \{x \in \mathbb{R}^n \,:\, F(x, w) \leq 0\}$. The stationary point set of (P_w) is defined by

$$S(w) = \{x \in \mathbb{R}^n \,:\, 0 \in \nabla_x f_0(x, w) + N_{C(w)}(x)\}. \tag{4.1}$$

When w varies on $\mathbb{R}^d$, one has a multifunction $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ with $S(w)$ being calculated by (4.1). Setting $f(x, w) = g(F(x, w)) = (g \circ F)(x, w)$, where $g(y) = \delta_{\mathbb{R}_-}(y)$, i.e., $g(y) = 0$ for $y \leq 0$ and $g(y) = +\infty$ for $y > 0$, we can rewrite (4.1) as

$$S(w) = \{x \in \mathbb{R}^n \, : \, 0 \in \nabla_x f_0(x, w) + \partial_x f(x, w)\}. \tag{4.2}$$

Fix a vector $w = \bar{w} \in \mathbb{R}^d$ and suppose that $\bar{x} \in S(\bar{w})$. Note that $(P_{\bar{w}})$ has a single smooth inequality constraint. The Mangasarian-Fromovitz Constraint Qualification is fulfilled at $\bar{x} \in C(\bar{w})$ if and only if

$$\text{If } F(\bar{x}, \bar{w}) = 0, \text{ then } \nabla_x F(\bar{x}, \bar{w}) \neq 0. \tag{\textbf{MFCQ}}$$

In what follows, we assume that **(MFCQ)** is valid.

To study the stability of the stationary point set map S around the point $(\bar{w}, \bar{x})$ in the graph of S, we compute the Mordukhovich and the Fréchet coderivatives of the partial subdifferential map $\partial_x f : \mathbb{R}^n \times \mathbb{R}^d \rightrightarrows \mathbb{R}^n$. In general, there is no explicit formula for the coderivatives of such maps. However, the results of [41] provide us with some tools which allow us to estimate the coderivative value $D^*S(\bar{w}|\bar{x})(x')$ for every $x' \in \mathbb{R}^n$.

4.2 Auxiliary Results

The fulfillment of **(MFCQ)** at $(\bar{x}, \bar{w})$ implies that $g(x, w) = g(F(x, w))$ is a strongly amenable in x at $\bar{x}$ with compatible parameterization in w at $\bar{w}$. Then, by [80, Theorem 10.49], for (x, w) near $(\bar{x}, \bar{w})$, we have

$$\partial f(x, w) = \nabla F(x, w)^*(\partial g(F(x, w))) \tag{4.3}$$

and

$$\partial_x f(x, w) = \nabla_x F(x, w)^*(\partial g(F(x, w))); \tag{4.4}$$

see [41, formulas (14) and (15)].

In order to estimate the limiting second-order subdifferential of f, we need the following result.

Lemma 4.1 (see [41, Theorem 3.1]) *Suppose that $\bar{v} \in \partial f(\bar{x}, \bar{w})$. Then, for any $v' \in \mathbb{R}^n \times \mathbb{R}^d$,*

$$\partial^2 f((\bar{x}, \bar{w})|\bar{v})(v')$$
$$\subset \bigcup_{\substack{\bar{y} \in \partial g(F(\bar{x}, \bar{w})) \text{ with} \\ \nabla F(\bar{x}, \bar{w})^* \bar{y} = \bar{v}}} \Big(\nabla^2(\bar{y} \cdot F)(\bar{x}, \bar{w})v' + D^*(\partial g \circ F)(\bar{x}, \bar{w})|\bar{y})(\nabla F(\bar{x}, \bar{w})v') \Big),$$

where the function $\bar{y} \cdot F : \mathbb{R}^{n+d} \to \mathbb{R}$ *is defined by* $(\bar{y} \cdot F)(x, w) := \bar{y}F(x, w)$. *If, in addition, at every* $\bar{y} \in \partial g(F(\bar{x}, \bar{w}))$ *with* $\nabla F(\bar{x}, \bar{w})^*\bar{y} = \bar{v}$*, one has the second-order constraint qualification*

$$\partial^2 g(F(\bar{x}, \bar{w})|\bar{y})(0) \cap \ker\nabla F(\bar{x}, \bar{w})^* = \{0\}, \tag{4.5}$$

then the estimate above for the second-order subdifferential can be refined by replacing the coderivative of the multifunction $\partial g \circ F$ *via the inclusion*

$$D^*(\partial g \circ F)((\bar{x}, \bar{w})|\bar{y})(\nabla F(\bar{x}, \bar{w})v') \subset \nabla F(\bar{x}, \bar{w})^*\partial^2 g(F(\bar{x}, \bar{w})|\bar{y})(\nabla F(\bar{x}, \bar{w})v').$$

In our problem (P_w), condition (4.5) can be omitted. Indeed, $\bar{y} \in \partial g(F(\bar{x}, \bar{w}))$ if and only if $\bar{y} \in N_{\mathbb{R}_-}(F(\bar{x}, \bar{w}))$. Hence, $\bar{y} \geq 0$. Clearly,

$$\operatorname{gph} \partial g = (\mathbb{R}_- \times \{0\}) \cup (\{0\} \times \mathbb{R}_+).$$

If $F(\bar{x}, \bar{w}) < 0$, then $\bar{y} = 0$ and $N_{\operatorname{gph} \partial g}(F(\bar{x}, \bar{w}), \bar{y}) = \{0\} \times \mathbb{R}$. It follows that

$$\begin{aligned} \partial^2 g(F(\bar{x}, \bar{w})|\,\bar{y})(0) &= D^*(\partial g(F(\bar{x}, \bar{w})|\bar{y}))(0) \\ &= \{u' \in \mathbb{R} : (u', 0) \in N_{\operatorname{gph} \partial g}(F(\bar{x}, \bar{w}), \bar{y})\} = \{0\}. \end{aligned}$$

If $F(\bar{x}, \bar{w}) = 0$, then **(MFCQ)** implies $\nabla F(\bar{x}, \bar{w}) \neq 0$. Hence the linear operator $\nabla F(\bar{x}, \bar{w}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is surjective. By [50, Lemma 1.18], we have $\ker \nabla F(\bar{x}, \bar{w})^* = \{0\}$. Thus, (4.5) is fulfilled. Now, we reformulate Lemma 4.1 for (P_w) as follows: *For any* $\bar{v} \in \partial f(\bar{x}, \bar{w})$ *and* $v' \in \mathbb{R}^n \times \mathbb{R}^d$,

$$\partial^2 f((\bar{x}, \bar{w})|\bar{v})(v') \subset \bigcup_{\substack{\bar{y} \in \partial g(F(\bar{x}, \bar{w})) \text{ with} \\ \nabla F(\bar{x}, \bar{w})^*\bar{y} = \bar{v}}} \Big(\nabla^2(\bar{y} \cdot F)(\bar{x}, \bar{w})v' + \Omega_1(\bar{y}, v')\Big), \tag{4.6}$$

where

$$\Omega_1(\bar{y}, v') := \nabla F(\bar{x}, \bar{w})^*\partial^2 g(F(\bar{x}, \bar{w})|\bar{y})(\nabla F(\bar{x}, \bar{w})v').$$

Remark 4.1 Concerning the paper [58], observe that the set $\partial^2 f((\bar{x}, \bar{w})|\bar{v})(v')$ in formula (4.6) is analogous to the set $\widetilde{\varphi}^2_x(\bar{x}, \bar{w}, \bar{y})(u)$ (a value of the extended partial second-order subdifferential) in formula (3.4) of that work. A careful checking shows that equality (3.4) of [58] implies the upper estimate (4.6).

In what follows, for any $\bar{v} = (\bar{v}_x, \bar{v}_w) \in \mathbb{R}^n \times \mathbb{R}^d$, we put $\operatorname{proj}_1\bar{v} = \bar{v}_x$. The upper estimation for the coderivative values of the stationary point set map S given by Levy and Mordukhovich [41] requires the following *regularity* *condition*: *For any* $v_1' \in \mathbb{R}^n$,

$$0 \in \nabla^2 f_0(\bar{x}, \bar{w})^*(v_1', 0) + \bigcup_{\substack{\bar{v} \in \partial f(\bar{x}, \bar{w}) \text{ with} \\ \text{proj}_1 \bar{v} = -\nabla_x f_0(\bar{x}, \bar{w})}} \partial^2 f((\bar{x}, \bar{w})|\bar{v})(v_1', 0) \implies v_1' = 0 \tag{4.7}$$

(see [41, formula (11)]). For our problem (P_w), by the assumption **(MFCQ)** and formula (4.3), we have $\partial f(\bar{x}, \bar{w}) = \nabla F(\bar{x}, \bar{w})^*(\partial g(\bar{x}, \bar{w}))$. In addition, it is easy to show that, for every $\bar{y} \in \partial g(\bar{x}, \bar{w})$, $\text{proj}_1 (\nabla F(\bar{x}, \bar{w})^* \bar{y}) = \nabla_x F(\bar{x}, \bar{w})^* \bar{y}$. Hence

$$\begin{aligned} &\bigcup_{\substack{\bar{v} \in \partial f(\bar{x}, \bar{w}) \text{ with} \\ \text{proj}_1 \bar{v} = -\nabla_x f_0(\bar{x}, \bar{w})}} \partial^2 f((\bar{x}, \bar{w})|\bar{v})(v_1', 0) \\ &= \bigcup_{\substack{\bar{y} \in \partial g(F(\bar{x}, \bar{w})) \text{ with} \\ \nabla_x F(\bar{x}, \bar{w})^* \bar{y} = -\nabla_x f_0(\bar{x}, \bar{w})}} \partial^2 f((\bar{x}, \bar{w})|\nabla F(\bar{x}, \bar{w})^* \bar{y})(v_1', 0). \end{aligned} \tag{4.8}$$

So (4.7) is equivalent to the following condition:

$$0 \in \nabla^2 f_0(\bar{x}, \bar{w})^*(v_1', 0) + \Omega_2(v_1') \implies v_1' = 0, \tag{\textbf{C0}}$$

where

$$\Omega_2(v_1') := \bigcup_{\substack{\bar{y} \in \partial g(F(\bar{x}, \bar{w})) \text{ with} \\ \nabla_x F(\bar{x}, \bar{w})^* \bar{y} = -\nabla_x f_0(\bar{x}, \bar{w})}} \partial^2 f((\bar{x}, \bar{w})|\nabla F(\bar{x}, \bar{w})^* \bar{y})(v_1', 0). \tag{4.9}$$

The next result from [41] provides us with an upper estimation for the values of the coderivative map $D^* S(\bar{w}|\bar{x}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^d$.

Lemma 4.2 (see [41, Corollary 3.1]) *If the regularity condition* (**C0**) *holds then, for each* $x' \in \mathbb{R}^n$, *the coderivative value* $D^* S(\bar{w}|\bar{x})(x')$ *is contained in the set of* $w' \in \mathbb{R}^d$ *for which there exists a vector* $v_1' \in \mathbb{R}^n$ *with*

$$(-x', w') - \nabla^2 f_0(\bar{x}, \bar{w})^*(v_1', 0) \in \Omega_2(v_1').$$

Although it is rather difficult to compute the set $\Omega_2(v_1')$, we can still estimate it by using (4.6).

Upper estimates for the limiting coderivative values of S can be derived from a result of Levy and Mordukhovich, but, a constraint qualification must be imposed to have these estimates (see Theorem 1.3). Interestingly, due to a result of Lee and Yen (see Theorem 1.4), sharp *lower estimates* for the Fréchet coderivative values of S can be given without any condition. Put

$G(x,w) = \nabla_x f_0(x,w)$ and $M(x,w) = \partial_x f(x,w)$. Then, S can be presented in the form (1.5), i.e.,

$$S(w) = \{x \in \mathbb{R}^n \,:\, 0 \in G(x,w) + M(x,w)\}.$$

Since $\bar{x} \in S(\bar{w})$, $\bar{\tau} := (\bar{x}, \bar{w}, -\nabla_x f_0(\bar{x}, \bar{w}))$ belongs to the graph of M. Note that $\operatorname{gph} M$ is locally closed around $\bar{\tau}$.

Now, let us look back at the implicit multifunction theorems presented in Chapter 1. From Theorem 1.4, for any $x' \in \mathbb{R}^n$, $\widehat{\Gamma}(x') \subset \widehat{D}^*S(\bar{w}|\bar{x})(x') \subset D^*S(\bar{w}|\bar{x})(x')$. This implies that $\widehat{\Gamma}(0) \subset \widehat{D}^*S(\bar{w}|\bar{x})(0) \subset D^*S(\bar{w}|\bar{x})(0)$. Note that $0 \in \widehat{\Gamma}(0)$. According to the Mordukhovich criterion 1, if S is Lipschitz-like around $(\bar{w}, \bar{x})$, then $D^*S(\bar{w}|\bar{x})(0) = \{0\}$ and $\widehat{\Gamma}(0) = \{0\}$ as a result. In addition, if the constraint qualification (**C1**) is fulfilled, then Theorem 1.3 yields

$$D^*S(\bar{w}|\bar{x})(x') \subset \Gamma(x')$$

for any $x' \in \mathbb{R}^n$. In particular, $D^*S(\bar{w}|\bar{x})(0) \subset \Gamma(0)$. Hence, if (**C1**) is valid and $\Gamma(0) = \{0\}$, then

$$D^*S(\bar{w}|\bar{x})(0) = \{0\}.$$

So, due to the Mordukhovich criterion 1, S is Lipschitz-like around $(\bar{w}, \bar{x})$. This idea has been presented in [39] and we will follow it throughout this chapter.

Put $\mathcal{D} = \{(x,w) \in \mathbb{R}^n \times \mathbb{R}^d \,:\, F(x,w) \le 0\}$. If $F(\bar{x}, \bar{w}) < 0$, then $(\bar{x}, \bar{w})$ is an *interior point* of $\mathcal{D}$. If $F(\bar{x}, \bar{w}) = 0$, then $(\bar{x}, \bar{w})$ is a *boundary point* of $\mathcal{D}$. In the next two sections, we will consider separately these two possibilities of the reference point $(\bar{x}, \bar{w})$. Remind that $\bar{w} \in \mathbb{R}^d$ and $\bar{x} \in S(\bar{w})$ are fixed and all the notations of this section are kept unchanged.

4.3 Lipschitzian Stability of the Stationary Point Set Map

4.3.1 Interior Points

Suppose that $F(\bar{x}, \bar{w}) < 0$, i.e., $(\bar{x}, \bar{w})$ is an *interior point* of $\mathcal{D}$. A point $\bar{x}$ belongs to $S(\bar{w})$ if and only if

$$0 \in \nabla_x f_0(\bar{x}, \bar{w}) + N_{C(\bar{w})}(\bar{x}),$$

where $C(\bar{w}) = \{x \in \mathbb{R}^n : F(x, \bar{w}) \le 0\}$. Since $F(\bar{x}, \bar{w}) < 0$, the continuity of $F(., \bar{w})$ implies that $\bar{x} \in \operatorname{int} C(\bar{w})$. This yields $N_{C(\bar{w})}(\bar{x}) = \{0\}$. Thus, $\bar{x} \in S(\bar{w})$ if and only if $\nabla_x f_0(\bar{x}, \bar{w}) = 0$.

The inequality $F(\bar{x}, \bar{w}) < 0$ implies that $\partial g(F(\bar{x}, \bar{w})) = \{0\}$. So, $\bar{y} = 0$ is the unique element of $\partial g(F(\bar{x}, \bar{w}))$. Since $\operatorname{gph} \partial g = (\mathbb{R}_- \times \{0\}) \cup (\{0\} \times \mathbb{R}_+)$,

$$N_{\operatorname{gph} \partial g}(F(\bar{x}, \bar{w}), \bar{y}) = \{0\} \times \mathbb{R}.$$

Hence, for any $v' \in \mathbb{R}^n \times \mathbb{R}^d$,

$$\begin{aligned} &\partial^2 g(F(\bar{x}, \bar{w})|\, \bar{y})(\nabla F(\bar{x}, \bar{w})v') \\ &= D^*(\partial g(F(\bar{x}, \bar{w})|\bar{y}))(\nabla F(\bar{x}, \bar{w})v') \\ &= \{u' \in \mathbb{R} : (u', -\nabla F(\bar{x}, \bar{w})v') \in N_{\operatorname{gph} \partial g}(F(\bar{x}, \bar{w}), \bar{y})\} = \{0\}. \end{aligned}$$

Therefore, $\Omega_1(\bar{y}, v') = \{0\}$ for any $v' \in \mathbb{R}^n \times \mathbb{R}^d$. So, from (4.6) it follows that

$$\partial^2 f((\bar{x}, \bar{w})|\bar{v})(v') \subset \{0\}$$

for any $\bar{v} \in \partial f(\bar{x}, \bar{w})$ and $v' \in \mathbb{R}^n \times \mathbb{R}^d$. Since $\nabla_x f(\bar{x}, \bar{w}) = 0$, invoking (4.3) and the fact that $\partial g(F(\bar{x}, \bar{w})) = \{0\}$, we get

$$\Omega_2(v_1') = \partial^2 f((\bar{x}, \bar{w})|0)(v_1', 0) \subset \{0\}$$

for any $v_1' \in \mathbb{R}^n$. Then, condition (**C0**) is fulfilled if

$$0 \in \nabla^2 f_0(\bar{x}, \bar{w})^*(v_1', 0) \implies v_1' = 0.$$

By the symmetry of $\nabla^2_{xx} f_0(\bar{x}, \bar{w})$ and the equality $\nabla^2_{xw} f_0(\bar{x}, \bar{w})^T = \nabla^2_{wx} f_0(\bar{x}, \bar{w})$, this is equivalent to

$$\begin{cases} \nabla^2_{xx} f_0(\bar{x}, \bar{w})v_1' = 0 \\ \nabla^2_{wx} f_0(\bar{x}, \bar{w})v_1' = 0 \end{cases} \implies v_1' = 0.$$

Clearly, the latter means that

$$\ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \cap \ker \nabla^2_{wx} f_0(\bar{x}, \bar{w}) = \{0\}. \tag{4.10}$$

We now suppose that condition (4.4), which guarantees the validity of (**C0**), is satisfied. Then, by Lemma 4.2,

$$D^*S(\bar{w}|\bar{x})(x') \subset \bigcup_{v_1' \in \mathbb{R}^n} \{w' \in \mathbb{R}^d : (-x', w') - \nabla^2 f_0(\bar{x}, \bar{w})^*(v_1', 0) \in \Omega_2(v_1')\},$$

for any $x' \in \mathbb{R}^n$. Since $\Omega_2(v_1') \subset \{0\}$, we have

$$\begin{aligned} &D^*S(\bar{w}|\bar{x})(x') \\ &\subset \Gamma_1(x') := \bigcup_{v_1' \in \mathbb{R}^n} \left\{w' \in \mathbb{R}^d : \nabla^2_{xx} f_0(\bar{x}, \bar{w})v_1' = -x',\ w' = \nabla^2_{wx} f_0(\bar{x}, \bar{w})v_1'\right\}. \end{aligned}$$

Note that $0 \in D^*S(\bar{w}|\bar{x})(0)$. So, if $\Gamma_1(0) = \{0\}$, then $D^*S(\bar{w}|\bar{x})(0) = \{0\}$; as a result, S is Lipschitz-like around $(\bar{w}, \bar{x})$ by the Mordukhovich Criterion 1. We have $\Gamma_1(0) = \{0\}$ if and only if

$$\{w' \in \mathbb{R}^d \,:\, \exists v_1' \in \mathbb{R}^n \text{ with } \nabla^2_{xx} f_0(\bar{x}, \bar{w})v_1' = 0,\ w' = \nabla^2_{wx} f_0(\bar{x}, \bar{w})v_1'\} = \{0\}.$$

This can be rewritten equivalently as

$$\ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \subset \ker \nabla^2_{wx} f_0(\bar{x}, \bar{w}). \tag{4.11}$$

It is easy to show that (4.11) and (4.10) hold simultaneously if and only if

$$\ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) = \{0\}. \tag{4.12}$$

In particular, (4.12) is a *sufficient condition* for S being Lipschitz-like around $(\bar{w}, \bar{x})$.

Example 4.1 Consider the problem (P_w) with $f_0(x, w) = \frac{1}{2}x^T Dx + c^T x$ and $F(x, w) = \|x\|^2 - \rho^2$, where $w = (D, c, \rho)$ with D being a $n \times n$ symmetric matrix, $c \in \mathbb{R}^n$, and $\rho > 0$. Suppose that $\bar{x} \in S(\bar{w})$ with $\bar{w} := (\bar{D}, \bar{c}, \bar{\rho})$ and $\|\bar{x}\| < \bar{\rho}$. If $\det \bar{D} \neq 0$, then S is Lipschitz-like around $(\bar{w}, \bar{x})$ because (4.12) is satisfied.

Having the sufficient condition (4.12) for the Lipschitz-likeness of S around $(\bar{w}, \bar{x})$, we want to find a *necessary condition* for this property. We know that if S is Lipschitz-like around $(\bar{w}, \bar{x})$, then $\widehat{\Gamma}(0) = \{0\}$, where the sets have been defined in (1.7). Since $F(\bar{x}, \bar{w}) < 0$ and F is continuous, there exit neighborhoods U of $\bar{x}$ and W of $\bar{w}$ such that $F(x, w) < 0$ for any $(x, w) \in U \times W$. It follows that, for each $w \in W$, the inclusion $x \in \operatorname{int} C(w)$ holds for every $x \in U$. Hence, for each $w \in W$, $N_{C(w)}(x) = \{0\}$ for all $x \in U$. This means that $M(x, w) = \{0\}$ for (x, w) in a neighborhood of $(\bar{x}, \bar{w})$. Therefore, from (4.1),

$$S(w) \cap U = \{x \in \mathbb{R}^n \,:\, \nabla_x f_0(x, w) = 0\} \cap U \quad (\forall w \in W). \tag{4.13}$$

Since $\widehat{D}^* M(\bar{\tau})(v_1') = \{0\}$ for any $v_1' \in \mathbb{R}^n$, one has

$$\begin{aligned}
\widehat{\Gamma}(x') &= \bigcup_{v_1' \in \mathbb{R}^n} \{w' \in \mathbb{R}^d \,:\, (-x', w') \in \nabla G(\bar{x}, \bar{w})^* v_1'\} \\
&= \bigcup_{v_1' \in \mathbb{R}^n} \{w' \in \mathbb{R}^d \,:\, (-x', w') \in (\nabla^2_{xx} f_0(\bar{x}, \bar{w})v_1', \nabla^2_{wx} f_0(\bar{x}, \bar{w})v_1')\}.
\end{aligned}$$

It follows that

$$\widehat{\Gamma}(0) = \bigcup_{v_1' \in \mathbb{R}^n} \{w' \in \mathbb{R}^d \,:\, (0, w') \in (\nabla^2_{xx} f_0(\bar{x}, \bar{w})v_1', \nabla^2_{wx} f_0(\bar{x}, \bar{w})v_1')\}.$$

Then, $\widehat{\Gamma}(0) = \{0\}$ if and only if (4.11) holds. Thus, condition (4.11) is necessary for S being Lipschitz-like around $(\bar{w}, \bar{x})$.

Let us consider an illustrative example, where the objective function is bilinear and the inequality constraint is polynomial.

Example 4.2 Consider the problem (P_w) with $f_0(x, w) = \sum_{i=1}^{n} x_i w_i$ and

$$F(x, w) = \sum_{i=1}^{n} (1 - w_i^2) x_i^2 - 1$$

for all $(x, w) \in \mathbb{R}^n \times \mathbb{R}^n$. The stationary point set of this problem is given by

$$S(w) = \{x \in \mathbb{R}^n \, : \, 0 \in \nabla_x f_0(x, w) + N_{C(w)}(x)\} = \{x \in \mathbb{R}^n \, : \, -w \in N_{C(w)}(x)\}. \tag{4.14}$$

In particular, for $\bar{w} = 0$ one has $S(\bar{w}) = \{x \in \mathbb{R}^n \, : \, \|x\| \leq 1\}$. Let $\bar{x} \in S(\bar{x})$ and $\|\bar{x}\| < 1$. Note that

$$\nabla^2_{xx} f_0(\bar{x}, \bar{w}) = 0, \;\; \nabla^2_{wx} f_0(\bar{x}, \bar{w}) = E,$$

where E stands for the unit matrix in $\mathbb{R}^{n \times n}$. Since $\ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) = \mathbb{R}^n$, the sufficient condition (4.12) fails. However, we can assert that S is not Lipschitz-like around $(\bar{w}, \bar{x})$ because the necessary condition (4.11) is not satisfied. In fact, we directly prove that S is not Lipschitz-like around $(\bar{w}, \bar{x})$. Indeed, by (4.14) one can find neighborhoods W of $\bar{w}$ and U of $\bar{x}$ satisfying $S(w) \cap V = \emptyset$ for all $w \in W \setminus \{\bar{w}\}$. This leads us to the desired result.

Remark 4.2 The above arguments show that if $(\bar{x}, \bar{w}) \in \operatorname{int} D$, then

$$D^* M(\bar{\tau})(v_1') = \widehat{D}^* M(\bar{\tau})(v_1') = \{0\} \tag{4.15}$$

for any $v_1' \in \mathbb{R}^n$. This implies that M is graphically regular at $\bar{\tau}$. According to Theorems 1.3 and 1.4, if the constraint qualification (**C1**) is valid, then $\widehat{\Gamma}(x') = D^* S(\bar{w}|\bar{x})(x') = \Gamma(x')$ for any $x' \in \mathbb{R}^n$. Suppose that (**C1**) is fulfilled. Then,

$$D^* S(\bar{w}|\bar{x})(x') = \bigcup_{v_1' \in \mathbb{R}^n} \{w' \in \mathbb{R}^d \, : \, (-x', w') \in \nabla G(\bar{x}, \bar{w})^* v_1'\}$$

for any $x' \in \mathbb{R}^n$. In particular,

$$D^* S(\bar{w}|\bar{x})(0) = \left\{w' \in \mathbb{R}^d \, : \, \exists v_1' \in \mathbb{R}^n \text{ with } (0, w') = \nabla G(\bar{x}, \bar{w})^* v_1'\right\}.$$

According to the Mordukhovich Criterion 1, S is Lipschitz-like around $(\bar{w}, \bar{x})$ if and only if $D^*S(\bar{w}|\bar{x})(0) = \{0\}$. The latter means that

$$\{w' \in \mathbb{R}^d : \exists v'_1 \in \mathbb{R}^n \text{ with } \nabla^2_{xx} f_0(\bar{x}, \bar{w})v'_1 = 0,\ w' = \nabla^2_{wx} f_0(\bar{x}, \bar{w})v'_1\} = \{0\}.$$

Clearly, this is fulfilled if and only if (4.11) is satisfied. Now let us verify the constraint qualification (**C1**). Due to (4.15), (**C1**) becomes

$$0 \in \nabla G(\bar{x}, \bar{w})^* v'_1 \implies v'_1 = 0,$$

or, equivalently,

$$0 \in \nabla^2 f_0(\bar{x}, \bar{w})^*(v'_1, 0) \implies v'_1 = 0.$$

The last condition has been proved to be equivalent to (4.10). Thus, if (4.10) is satisfied, then S is Lipschitz-like around $(\bar{w}, \bar{x})$ if and only if (4.11) holds.

The following theorem summarizes our results for the case of interior points.

Theorem 4.1 *Suppose that $F(\bar{x}, \bar{w}) < 0$. The following assertions are valid:*

(a) *If S is Lipschitz-like around $(\bar{w}, \bar{x})$, then condition* (4.11) *holds;*

(b) *If conditions* (4.10) *and* (4.11) *are simultaneously fulfilled, i.e.,* (4.12) *is fulfilled, then S is Lipschitz-like around $(\bar{w}, \bar{x})$;*

(c) *If condition* (4.10) *is satisfied, then S is Lipschitz-like around $(\bar{w}, \bar{x})$ if and only if condition* (4.11) *holds.*

Thus, if condition (4.10) fails we can assert nothing about the Lipschitz-likeness of S around $(\bar{w}, \bar{x})$. The next example shows that S can be Lipschitz-like around $(\bar{w}, \bar{x})$ when (4.10) is not satisfied.

Example 4.3 Consider (P_w) with $f_0(x, w) = \frac{1}{3}x^3 - w^2 x$ and

$$F(x, w) = x^2 + w^2 - 1$$

for all $(x, w) \in \mathbb{R} \times \mathbb{R}$. Put $\bar{w} = 0$. The point $\bar{x} = 0$ belongs to $S(\bar{w})$ because $(\bar{x}, \bar{w}) \in \operatorname{int} D$ and $\nabla_x f_0(\bar{x}, \bar{w}) = 0$. Since $\nabla^2_{xx} f_0(\bar{x}, \bar{w}) = 0$ and

$$\nabla^2_{wx} f_0(\bar{x}, \bar{w}) = 0,$$

condition (4.10) is invalid. We have known that S is locally defined by

$$\begin{aligned} S(w) &= \{x \in \mathbb{R} : \nabla_x f_0(x, w) = 0\}. \\ &= \{x \in \mathbb{R} : |x| = |w|\}. \end{aligned}$$

Then, for any $x' \in \mathbb{R}$,

$$\begin{aligned} D^*S(\bar{w}|\bar{x})(x') &= \{w' \in \mathbb{R} \,:\, (w', -x') \in N_{\mathrm{gph}\, S}(0,0)\}. \\ &= \{w' \in \mathbb{R} \,:\, |w'| = |x'|\}. \end{aligned}$$

It follows that $D^*S(\bar{w}|\bar{x})(0) = \{0\}$. Thus, S is Lipschitz-like around $(\bar{w}, \bar{x})$.

One referee of the paper [29] asks: *Whether the condition* (4.11) *alone can ensure the Lipschitz-likeness of S around* $(\bar{w}, \bar{x})$. Answering the question, we construct the next example to demonstrate that, even for polynomial optimization problems, (4.11) is not sufficient for the later property to hold.

Example 4.4 Consider (P_w) with $f_0(x, w) = \frac{1}{4}x^4 - w^2 x$ and

$$F(x, w) = x - w - 1$$

for all $(x, w) \in \mathbb{R} \times \mathbb{R}$. Put $\bar{w} = 0$. The point $\bar{x} := 0$ belongs to $S(\bar{w})$ because $F(\bar{x}, \bar{w}) < 0$ and $\nabla_x f_0(\bar{x}, \bar{w}) = 0$. Since $\nabla^2_{xx} f_0(\bar{x}, \bar{w}) = \nabla^2_{wx} f_0(\bar{x}, \bar{w}) = 0$, we see that (4.10) fails, but (4.11) is valid. As

$$S(w) = \{x \in \mathbb{R} \,:\, x^3 - w^2 = 0\} = \{w^{\frac{2}{3}}\},$$

the stationary point set map is not Lipschitz-like around $(\bar{w}, \bar{x}) = (0, 0)$.

Remark 4.3 The second assertion of Theorem 4.1 can be obtained by the classical implicit function theorem. Indeed, if (4.10) and (4.11) are simultaneously fulfilled, i.e., (4.12) is satisfied, then by [19, Theorem 1B.1] (see also [81, Theorem 9.28]) the implicit multifunction $w \mapsto \{x \in \mathbb{R}^n \,:\, \nabla_x f_0(x, w) = 0\}$ defined by the equation $\nabla_x f_0(x, w) = 0$ has a *single-valued localization* [19, p. 4] around $\bar{w}$ for $\bar{x}$ which is continuously differentiable in a neighborhood of $\bar{w}$. This means that there exist a neighborhood W of $\bar{w}$ and a neighborhood U of $\bar{x}$ such that for each $w \in W$ there is a unique vector $x = s(w)$ in U satisfying the equation $\nabla_x f_0(x, w) = 0$ and $s : W \to U$ is continuously differentiable. Without loss of generality, we can assume that $F(x, w) < 0$ for all $(x, w) \in U \times W$. So, by (4.13), $S(w) \cap U = \{s(w)\}$ for all $w \in W$. Hence, S is Lipschitz-like around $(\bar{w}, \bar{x})$.

Remark 4.4 Consider the extended stationary point set map $(w, z) \mapsto \widetilde{S}(w, z)$ of (P_w), which is defined by

$$\widetilde{S}(w, z) = \{x \in \mathbb{R}^n \,:\, z \in \nabla_x f_0(x, w) + N_{C(w)}(x)\}.$$

If $F(\bar{x},\bar{w}) < 0$, then around the point $((\bar{w},0),\bar{x}) \in \operatorname{gph}\widetilde{S}$ one can represent $\bar{S}$ locally as

$$\widetilde{S}(w,z) = \{x \in \mathbb{R}^n \,:\, \widetilde{G}(x,w,z) = 0\},$$

where $\widetilde{G}(x,w,z) := \nabla_x f_0(x,w) - z$. Since $\nabla_{(w,z)}\widetilde{G}(\bar{x},\bar{w},0)$ has full rank, by [41, Theorem 2.1] one has

$$D^*\widetilde{S}((\bar{w},0)|\bar{x})(x') = \{(w',z') \in \mathbb{R}^d \times \mathbb{R}^n \quad : \quad w' = -\big(\nabla^2_{wx} f_0(\bar{x},\bar{w})\big)^T z', \\ x' = \nabla^2_{xx} f_0(\bar{x},\bar{w}) z'\}.$$

Hence,

$$D^*\widetilde{S}((\bar{w},0)|\bar{x})(0) = \{(w',z') \in \mathbb{R}^d \times \mathbb{R}^n \quad : \quad \nabla^2_{xx} f_0(\bar{x},\bar{w}) z' = 0 \\ w' = -\big(\nabla^2_{wx} f_0(\bar{x},\bar{w})\big)^T z'\}.$$

Therefore, $D^*\widetilde{S}((\bar{w},0),\bar{x})(0) = \{0\}$ if and only if $\ker \nabla^2_{xx} f_0(\bar{x},\bar{w}) = \{0\}$. Thanks to Mordukhovich's criterion, this shows that condition (4.12) is necessary and sufficient for the Lipschitz-like property of the the extended stationary point set map $\widetilde{S}$ around $((\bar{w},0),\bar{x})$.

4.3.2 Boundary Points

Suppose that $F(\bar{x},\bar{w}) = 0$, i.e., $(\bar{x},\bar{w})$ is a *boundary point* of $\mathcal{D}$. To obtain a sufficient condition for the Lipschitz-like property of S around $(\bar{w},\bar{x})$, we will follow the scheme which has been used in the case of interior points. We first need to find out a condition which guarantees the fulfillment of (**C0**). Note that **(MFCQ)** yields $\nabla_x F(\bar{x},\bar{w}) \neq 0$. Since $\partial g(F(\bar{x},\bar{w})) \subset \mathbb{R}$ and

$$\nabla_x F(\bar{x},\bar{w})^*\gamma = \gamma \nabla_x F(\bar{x},\bar{w})$$

for any $\gamma \in \mathbb{R}$, there exists only one element $\lambda \in \partial g(F(\bar{x},\bar{w}))$ satisfying

$$\nabla_x f_0(\bar{x},\bar{w}) + \lambda \nabla_x F(\bar{x},\bar{w}) = 0. \tag{4.16}$$

This element λ is the unique *Lagrange multiplier* for the stationary point $\bar{x}$ of the minimization problem $(P_{\bar{w}})$. Due to $\lambda \in \partial g(F(\bar{x},\bar{w}))$, we have $\lambda \geq 0$. Let us consider two possibilities of the Lagrange multiplier.

Case 1: The Nondegenerate Case

Suppose that $\lambda > 0$. Clearly, the equality $\operatorname{gph}\partial g = (\mathbb{R}_- \times \{0\}) \cup (\{0\} \times \mathbb{R}_+)$ yields

$$N_{\operatorname{gph}\partial g}(F(\bar{x},\bar{w}),\lambda) = N_{\operatorname{gph}\partial g}(0,\lambda) = \mathbb{R} \times \{0\}.$$

By definition, for any $v' \in \mathbb{R}^n \times \mathbb{R}^d$,

$$\begin{aligned}&\partial^2 g(F(\bar{x},\bar{w})|\,\lambda)(\nabla F(\bar{x},\bar{w})v')\\ &= D^*(\partial g)(F(\bar{x},\bar{w})|\lambda)(\nabla F(\bar{x},\bar{w})v')\\ &= \{u' \in \mathbb{R} \,:\, (u', -\nabla F(\bar{x},\bar{w})v') \in N_{\mathrm{gph}\,\partial g}(F(\bar{x},\bar{w}),\lambda)\}.\end{aligned}$$

Hence

$$\partial^2 g(F(\bar{x},\bar{w})|\,\lambda)(\nabla F(\bar{x},\bar{w})v') = \begin{cases}\mathbb{R}, & \text{if } \nabla F(\bar{x},\bar{w})v' = 0,\\ \emptyset, & \text{if } \nabla F(\bar{x},\bar{w})v' \neq 0.\end{cases}$$

So,

$$\Omega_1(\lambda, v') = \begin{cases}\{\gamma \nabla F(\bar{x},\bar{w}) \,:\, \gamma \in \mathbb{R}\}, & \text{if } \nabla F(\bar{x},\bar{w})v' = 0,\\ \emptyset, & \text{if } \nabla F(\bar{x},\bar{w})v' \neq 0.\end{cases}$$

For $\bar{v} := \nabla F(\bar{x},\bar{w})^*\lambda$, the conditions $\bar{y} \in \partial g(F(\bar{x},\bar{w}))$ and $\nabla F(\bar{x},\bar{w})^*\bar{y} = \bar{v}$ force $\bar{y} = \lambda$. So, by (4.6) we get

$$\begin{aligned}\partial^2 f((\bar{x},\bar{w})|\bar{v})(v') &\subset \lambda \nabla^2 F(\bar{x},\bar{w})v' + \Omega_1(\lambda, v')\\ &= \lambda \nabla^2 F(\bar{x},\bar{w})v' + \{\gamma \nabla F(\bar{x},\bar{w}) \,:\, \gamma \in \mathbb{R}\}\end{aligned}$$

if $\nabla F(\bar{x},\bar{w})v' = 0$, and $\partial^2 f((\bar{x},\bar{w})|\bar{v})(v') = \emptyset$ if $\nabla F(\bar{x},\bar{w})v' \neq 0$. Since $\bar{y} = \lambda$ is the unique element satisfying the conditions $\bar{y} \in \partial g(F(\bar{x},\bar{w}))$ and

$$\nabla F(\bar{x},\bar{w})^*\bar{y} = -\nabla_x f_0(\bar{x},\bar{w}),$$

from (4.9) it follows that

$$\Omega_2(v_1') \subset \left\{\lambda \nabla^2 F(\bar{x},\bar{w})(v_1', 0) + \gamma \nabla F(\bar{x},\bar{w}) \,:\, \gamma \in \mathbb{R}\right\}, \tag{4.17}$$

for any v_1' with $\nabla_x F(\bar{x},\bar{w})v_1' = 0$. Besides, if $\nabla_x F(\bar{x},\bar{w})v_1' \neq 0$, then the set $\Omega_2(v_1')$ is empty. Thus, (**C0**) is fulfilled if the following is satisfied: *for any* $v_1' \in \mathbb{R}^n$, *if*

$$\begin{cases}-\nabla^2 f_0(\bar{x},\bar{w})^*(v_1', 0) \in \{\lambda \nabla^2 F(\bar{x},\bar{w})(v_1', 0) + \gamma \nabla F(\bar{x},\bar{w}) \,:\, \gamma \in \mathbb{R}\}\\ \nabla_x F(\bar{x},\bar{w})v_1' = 0,\end{cases}$$

then $v_1' = 0$. Equivalently, *for any* $v_1' \in \mathbb{R}^n$, *if*

$$\begin{cases}0 = \nabla^2_{xx} f_0(\bar{x},\bar{w})v_1' + \lambda \nabla^2_{xx} F(\bar{x},\bar{w})v_1' + \gamma \nabla_x F(\bar{x},\bar{w}),\\ 0 = \nabla^2_{wx} f_0(\bar{x},\bar{w})v_1' + \lambda \nabla^2_{wx} F(\bar{x},\bar{w})v_1' + \gamma \nabla_w F(\bar{x},\bar{w}),\\ \nabla_x F(\bar{x},\bar{w})v_1' = 0,\ \gamma \in \mathbb{R},\end{cases}$$

then $v_1' = 0$. Putting

$$A_1 = \left[\begin{array}{cc} \nabla^2_{xx} f_0(\bar{x}, \bar{w}) + \lambda \nabla^2_{xx} F(\bar{x}, \bar{w}) & \nabla_x F(\bar{x}, \bar{w}) \end{array}\right] \in \mathbb{R}^{n \times (n+1)}$$

and

$$A_2 = \left[\begin{array}{cc} \nabla^2_{wx} f_0(\bar{x}, \bar{w}) + \lambda \nabla^2_{wx} F(\bar{x}, \bar{w}) & \nabla_w F(\bar{x}, \bar{w}) \end{array}\right] \in \mathbb{R}^{d \times (n+1)},$$

where $\nabla_x F(\bar{x}, \bar{w})$ and $\nabla_w F(\bar{x}, \bar{w})$ are interpreted as column vectors, we can rewrite the last condition equivalently as follows:

$$\begin{cases} A_1 \begin{pmatrix} v_1' \\ \gamma \end{pmatrix} = 0, \quad A_2 \begin{pmatrix} v_1' \\ \gamma \end{pmatrix} = 0 \\ \nabla_x F(\bar{x}, \bar{w}) v_1' = 0, \quad \gamma \in \mathbb{R} \end{cases} \qquad \Longrightarrow \quad v_1' = 0.$$

Since $\nabla_x F(\bar{x}, \bar{w}) \neq 0$, the latter is equivalent to saying that

$$\ker A_1 \cap \ker A_2 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) = \{(0, 0)\}. \tag{4.18}$$

We now suppose that condition (4.18) is satisfied. Then, by Lemma 4.2, for any $x' \in \mathbb{R}^n$,

$$D^* S(\bar{w}|\bar{x})(x') \subset \bigcup_{v_1' \in \mathbb{R}^n} \left\{ w' \in \mathbb{R}^d \,:\, (-x', w') - \nabla^2 f_0(\bar{x}, \bar{w})^* (v_1', 0) \in \Omega_2(v_1') \right\},$$

with $\Omega_2(v_1')$ admitting the upper estimation (4.17). So, for any $x' \in \mathbb{R}^n$,

$$D^* S(\bar{w}|\bar{x})(x') \subset \Gamma_2(x'), \tag{4.19}$$

where $\Gamma_2(x')$ consists of vectors $w' \in \mathbb{R}^d$ such that

$$\begin{cases} w' = \left[\nabla^2_{wx} f_0(\bar{x}, \bar{w}) + \lambda \nabla^2_{wx} F(\bar{x}, \bar{w})\right] v_1' + \gamma \nabla_w F(\bar{x}, \bar{w}), \\ \left[\nabla^2_{xx} f_0(\bar{x}, \bar{w}) + \lambda \nabla^2_{xx} F(\bar{x}, \bar{w})\right] v_1' + \gamma \nabla_x F(\bar{x}, \bar{w}) = -x', \\ \nabla_x F(\bar{x}, \bar{w}) v_1' = 0, \ v_1' \in \mathbb{R}^n, \ \gamma \in \mathbb{R}. \end{cases}$$

To obtain a lower estimate for the Fréchet coderivative values of S, we will use some results of Qui [71]. For any $v_1' \in \mathbb{R}^n$ satisfying $\nabla_x F(\bar{x}, \bar{w}) v_1' = 0$, the arguments given in [71, pp. 410–412] provide us with the inclusion

$$\widehat{\Omega}_M(v_1') \subset \widehat{D}^* M(\bar{\tau})(v_1'), \tag{4.20}$$

where $M(x, w) = \partial_x f(x, w)$ and

$$\begin{aligned} \widehat{\Omega}_M(v_1') := \ \{(x', w') \in \mathbb{R}^n \times \mathbb{R}^d \,:\, & x' = \gamma \nabla_x F(\bar{x}, \bar{w}) + \hat{y} \nabla^2_{xx} F(\bar{x}, \bar{w}) v_1', \\ & w' = \gamma \nabla_w F(\bar{x}, \bar{w}) + \hat{y} \nabla^2_{wx} F(\bar{x}, \bar{w}) v_1', \ \gamma \in \mathbb{R}\}. \end{aligned}$$

In addition, if $\nabla_x F(\bar{x},\bar{w})v_1' \neq 0$, then by the reasoning in [71, pp. 405–406] we have $\widehat{D}^* M(\bar{\tau})(v_1') = \emptyset$. Note that the Lagrange multiplier λ here coincides with the constant μ in [71, Theorem 3.2] and *the above assertions do not require* $\nabla_w F(\bar{x},\bar{w}) \neq 0$. Then, by (1.7) and (4.20),

$$\widehat{\Gamma}(x') \supset \bigcup_{\substack{v_1' \in \mathbb{R}^n, \\ \nabla_x F(\bar{x},\bar{w})v_1' = 0}} \left\{ w' \in \mathbb{R}^d \,:\, (-x', w') \in \nabla G(\bar{x},\bar{w})^* v_1' + \widehat{\Omega}_M(v_1') \right\}.$$

Since $\nabla G(\bar{x},\bar{w})^* v_1' = (\nabla^2_{xx} f_0(\bar{x},\bar{w})v_1', \nabla^2_{wx} f_0(\bar{x},\bar{w})v_1')$, one can easily show that the right-hand-side set equals to $\Gamma_2(x')$. Therefore, if (4.18) is satisfied, then by (4.19) we have

$$\widehat{\Gamma}(x') = \widehat{D}^* S(\bar{w}|\bar{x})(x') = D^* S(\bar{w}|\bar{x})(x') = \Gamma_2(x').$$

Thus, under the assumption (4.18), S is Lipschitz-like around $(\bar{w},\bar{x})$ if and only if $\Gamma_2(0) = \{0\}$. Clearly, the set $\Gamma_2(0)$ consists of vectors $w' \in \mathbb{R}^d$ such that

$$\begin{cases} w' = [\nabla^2_{xw} f_0(\bar{x},\bar{w})^T + \lambda \nabla^2_{wx} F(\bar{x},\bar{w})]\, v_1' + \gamma \nabla_w F(\bar{x},\bar{w}), \\ [\nabla^2_{xx} f_0(\bar{x},\bar{w}) + \lambda \nabla^2_{xx} F(\bar{x},\bar{w})]\, v_1' + \gamma \nabla_x F(\bar{x},\bar{w}) = 0, \\ \nabla_x F(\bar{x},\bar{w})v_1' = 0,\ v_1' \in \mathbb{R}^n,\ \gamma \in \mathbb{R}. \end{cases}$$

Equivalently, $w' \in \mathbb{R}^d$ belongs to $\Gamma_2(0)$ iff

$$\begin{cases} w' = A_2 \begin{pmatrix} v_1' \\ \gamma \end{pmatrix}, \\ A_1 \begin{pmatrix} v_1' \\ \gamma \end{pmatrix} = 0, \quad \nabla_x F(\bar{x},\bar{w})v_1' = 0, \\ v_1' \in \mathbb{R}^n,\ \gamma \in \mathbb{R}. \end{cases}$$

So, $\Gamma_2(0) = \{0\}$ if and only if $A_2 \begin{pmatrix} v_1' \\ \gamma \end{pmatrix} = 0$ for any pair $(v_1', \gamma) \in \mathbb{R}^n \times \mathbb{R}$ satisfying

$$A_1 \begin{pmatrix} v_1' \\ \gamma \end{pmatrix} = 0, \quad \nabla_x F(\bar{x},\bar{w})v_1' = 0.$$

The latter happens if and only if

$$\ker A_1 \cap (\ker \nabla_x F(\bar{x},\bar{w}) \times \mathbb{R}) \subset \ker A_2. \tag{4.21}$$

To sum up, we state the following theorem.

Theorem 4.2 *Suppose that $F(\bar{x},\bar{w})=0$ and the Lagrange multiplier λ corresponding to the stationary point $\bar{x}\in S(\bar{w})$ is positive. If* (4.18) *holds, then S is Lipschitz-like around $(\bar{w},\bar{x})$ if and only if* (4.21) *is satisfied.*

Remark 4.5 Combining the conditions (4.18) and (4.21), we obtain a sufficient condition for S being Lipschitz-like around $(\bar{w},\bar{x})$, that is

$$\ker A_1 \cap (\ker \nabla_x F(\bar{x},\bar{w}) \times \mathbb{R}) = \{0\}. \tag{4.22}$$

Example 4.5 Consider the problem (P_w) with $f_0(x,w)=-x^2+(w-1)x$ and $F(x,w)=x^2+w^2-2$ for any $(x,w)\in\mathbb{R}\times\mathbb{R}$. Then, by (4.2), the stationary point set map of (P_w) is defined by

$$S(w)=\{x\in\mathbb{R}\,:\,-2x+w-1+\partial_x f(x,w)\},$$

with $f(x,w)=(g\circ F)(x,w)$ and $g(y)=\delta_{\mathbb{R}_-}(y)$ for any $y\in\mathbb{R}$. Let $\bar{w}=1$ and $\bar{x}=1$. Since $F(\bar{x},\bar{w})=0$ and $\nabla_x F(\bar{x},\bar{w})=2$, condition (**MFCQ**) is valid. Hence, from (4.4) we have

$$\begin{aligned}\partial_x f(\bar{x},\bar{w}) &= \nabla_x F(\bar{x},\bar{w})^* N_{\mathbb{R}_-}(F(\bar{x},\bar{w}))\\ &= \nabla_x F(\bar{x},\bar{w})^* \mathbb{R}_+ = \mathbb{R}_+.\end{aligned}$$

Now, it easy to show that $\bar{x}\in S(\bar{w})$. We have $\nabla_x f_0(\bar{x},\bar{w})=-2$ and $\lambda=1$ due to (4.5). Hence, $A_1=[0\ \ 2]$ and $\ker A_1=\mathbb{R}\times\{0\}$. Thus, (4.22) is fulfilled and consequently S is Lipschitz-like around $(\bar{w},\bar{x})$.

Remark 4.6 For any stationary point $\bar{x}\in S(\bar{w})$ satisfying **(MFCQ)**, the corresponding unique multiplier λ is defined by the equation (4.5). This fact justifies to the following assertion: *Given any $w\in\mathbb{R}^d$, if $\nabla_x F(x,w)\neq 0$ for all x with $F(x,w)=0$, then one has*

$$\begin{aligned}S(w)=\big\{x\in\mathbb{R}^n :\ & F(x,w)\le 0,\ \exists\lambda\ge 0 \text{ s.t. } \lambda F(x,w)=0,\\ & \nabla_x f_0(x,w)+\lambda\nabla_x F(x,w)=0\big\}.\end{aligned}$$

One referee of the paper [29] asks: *Whether the condition* (4.21) *alone can ensure the Lipschitz-likeness of S around $(\bar{w},\bar{x})$.* To answer the question, let us consider the next example showing that, even for polynomial optimization problems, (4.21) is not sufficient for S to be Lipschitz-like around $(\bar{w},\bar{x})$.

Example 4.6 Consider (P_w) with $n=2$, $d=1$, $f_0(x,w)=\frac{1}{4}wx_1^4-wx_1-x_2$, and $F(x,w)=wx_1+x_2-w$ for all $(x,w)=(x_1,x_2,w)\in\mathbb{R}^2\times\mathbb{R}$. Choose $\bar{x}=(0,0)$ and $\bar{w}=0$. Using Remark 4.4, one can show that $S(\bar{w})=\mathbb{R}\times\{0\}$

and $S(w) = \{(0,w)\}$ for every $w \neq 0$. Hence the stationary map $S(.)$ is not Lipschitz-like around $(\bar{w}, \bar{x})$. Furthermore, since the unique Lagrange multiplier corresponding to $\bar{x} \in S(\bar{w})$ is $\lambda = 1$, one can find that $\ker A_1 = \mathbb{R} \times \mathbb{R} \times \{0\}$, $\ker A_2 = \mathbb{R} \times \mathbb{R} \times \{0\}$, and $\ker \nabla_x F(\bar{w}, \bar{x}) = \mathbb{R} \times \{0\}$. So (4.21) is satisfied, but (4.18) fails to hold.

Case 2: The Degenerate Case

Consider the second possibility: $\lambda = 0$. We have

$$N_{\mathrm{gph}\,\partial g}(F(\bar{x}, \bar{w}), \lambda) = N_{\mathrm{gph}\,\partial g}(0,0) = (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) \cup (\mathbb{R}_+ \times \mathbb{R}_-).$$

Hence, for any $v' \in \mathbb{R}^n \times \mathbb{R}^d$,

$$\partial^2 g(F(\bar{x}, \bar{w})|\,\lambda)(\nabla F(\bar{x}, \bar{w})v') = \begin{cases} \mathbb{R}, & \text{if } \nabla F(\bar{x}, \bar{w})v' = 0, \\ \mathbb{R}_{+,} & \text{if } \nabla F(\bar{x}, \bar{w})v' > 0, \\ \{0\}, & \text{if } \nabla F(\bar{x}, \bar{w})v' < 0. \end{cases}$$

Consequently,

$$\Omega_1(\lambda, v') = \begin{cases} \{\gamma \nabla F(\bar{x}, \bar{w}) \,:\, \gamma \in \mathbb{R}\}, & \text{if } \nabla F(\bar{x}, \bar{w})v' = 0, \\ \{\gamma \nabla F(\bar{x}, \bar{w}) \,:\, \gamma \in \mathbb{R}_+\}, & \text{if } \nabla F(\bar{x}, \bar{w})v' > 0, \\ \{0\}, & \text{if } \nabla F(\bar{x}, \bar{w})v' < 0. \end{cases}$$

In this case, $\bar{y} = \lambda = 0$ is the unique element satisfying $\bar{y} \in \partial g(F(\bar{x}, \bar{w}))$ and $\nabla_x F(\bar{x}, \bar{w})^* \bar{y} = -\nabla_x f_0(\bar{x}, \bar{w})$. So, by (4.9) we have

$$\Omega_2(v_1') = \partial^2 f((\bar{x}, \bar{w})|0)(v_1', 0).$$

Clearly, the conditions $\bar{y} \in \partial g(F(\bar{x}, \bar{w}))$ and $\nabla F(\bar{x}, \bar{w})^* \bar{y} = 0$ imply $\bar{y} = \lambda = 0$. So, for $\bar{v} = 0$, from (4.6) we get

$$\partial^2 f((\bar{x}, \bar{w})|0)(v') \subset \Omega_1(\lambda, v')$$

for all $v' \in \mathbb{R}^n \times \mathbb{R}^d$. It follows that

$$\Omega_2(v_1') \subset \begin{cases} \{\gamma \nabla F(\bar{x}, \bar{w}) \,:\, \gamma \in \mathbb{R}\}, & \text{if } \nabla_x F(\bar{x}, \bar{w})v_1' = 0, \\ \{\gamma \nabla F(\bar{x}, \bar{w}) \,:\, \gamma \in \mathbb{R}_+\}, & \text{if } \nabla_x F(\bar{x}, \bar{w})v_1' > 0, \\ \{0\}, & \text{if } \nabla_x F(\bar{x}, \bar{w})v_1' < 0. \end{cases} \tag{4.23}$$

Hence, the condition (**C0**) is satisfied if the following holds: *For any* $v_1' \in \mathbb{R}^n$,

(i) *if* $\nabla_x F(\bar{x}, \bar{w}) v_1' = 0$ *and*

$$\big(-\nabla^2_{xx} f_0(\bar{x}, \bar{w}) v_1', -\nabla^2_{wx} f_0(\bar{x}, \bar{w}) v_1'\big) \in \{\gamma \nabla F(\bar{x}, \bar{w}) \,:\, \gamma \in \mathbb{R}\},$$

then $v_1' = 0$;

(ii) *if* $\nabla_x F(\bar{x}, \bar{w}) v_1' > 0$, *then*

$$\big(-\nabla^2_{xx} f_0(\bar{x}, \bar{w}) v_1', -\nabla^2_{wx} f_0(\bar{x}, \bar{w}) v_1'\big) \notin \{\gamma \nabla F(\bar{x}, \bar{w}) \,:\, \gamma \in \mathbb{R}_+\};$$

(iii) *if* $\nabla_x F(\bar{x}, \bar{w}) v_1' < 0$, *then* $(\nabla^2_{xx} f_0(\bar{x}, \bar{w}) v_1', \nabla^2_{wx} f_0(\bar{x}, \bar{w}) v_1') \neq 0$.

It follows that (**C0**) is fulfilled if the following holds: *for any* $v_1' \in \mathbb{R}^n$, *if*

$$\big(-\nabla^2_{xx} f_0(\bar{x}, \bar{w}) v_1', -\nabla^2_{wx} f_0(\bar{x}, \bar{w}) v_1'\big) \in \{\gamma \nabla F(\bar{x}, \bar{w}) \,:\, \gamma \in \mathbb{R}\},$$

then $v_1' = 0$. The latter can be written equivalently as

$$\begin{cases} A_1' \begin{pmatrix} v_1' \\ \gamma \end{pmatrix} = 0, \quad A_2' \begin{pmatrix} v_1' \\ \gamma \end{pmatrix} = 0 \\ v_1' \in \mathbb{R}^n, \quad \gamma \in \mathbb{R} \end{cases} \Longrightarrow \quad v_1' = 0,$$

where

$$A_1' := \begin{bmatrix} \nabla^2_{xx} f_0(\bar{x}, \bar{w}) & \nabla_x F(\bar{x}, \bar{w}) \end{bmatrix} \in \mathbb{R}^{n \times (n+1)},$$

and

$$A_2' := \begin{bmatrix} \nabla^2_{wx} f_0(\bar{x}, \bar{w}) & \nabla_w F(\bar{x}, \bar{w}) \end{bmatrix} \in \mathbb{R}^{d \times (n+1)}.$$

Since $\nabla_x F(\bar{x}, \bar{w}) \neq 0$, this condition is equivalent to

$$\ker A_1' \cap \ker A_2' = \{0\}. \tag{4.24}$$

We now suppose that (4.24) is valid. Then, by Lemma 4.2 and the upper estimate (4.23), for any $x' \in \mathbb{R}^n$,

$$D^* S(\bar{w}|\bar{x})(x') \subset \Gamma_3(x'), \tag{4.25}$$

where $\Gamma_3(x')$ consists of vectors $w' \in \mathbb{R}^d$ for which there exists $v_1' \in \mathbb{R}^n$ with

$$\begin{cases} (-x' - \nabla^2_{xx} f_0(\bar{x}, \bar{w}) v_1', w' - \nabla^2_{wx} f_0(\bar{x}, \bar{w}) v_1') \in \{\gamma \nabla F(\bar{x}, \bar{w}) \,:\, \gamma \in \mathbb{R}\}, \\ \nabla_x F(\bar{x}, \bar{w}) v_1' = 0, \end{cases}$$

or

$$\begin{cases} (-x' - \nabla^2_{xx} f_0(\bar{x}, \bar{w}) v_1', w' - \nabla^2_{wx} f_0(\bar{x}, \bar{w}) v_1') \in \{\gamma \nabla F(\bar{x}, \bar{w}) \,:\, \gamma \in \mathbb{R}_+\}, \\ \nabla_x F(\bar{x}, \bar{w}) v_1' > 0, \end{cases}$$

or

$$\begin{cases} -x' - \nabla^2_{xx} f_0(\bar{x}, \bar{w})v'_1 = 0, \;\; w' - \nabla^2_{wx} f_0(\bar{x}, \bar{w})v'_1 = 0, \\ \nabla_x F(\bar{x}, \bar{w})v'_1 < 0. \end{cases}$$

By putting

$$\Delta_1 = \{(v'_1, \gamma) \in \mathbb{R}^n \times \mathbb{R} \,:\, \nabla_x F(\bar{x}, \bar{w})v'_1 > 0,\, \gamma \geq 0\}$$

and

$$\Delta_2 = \{v'_1 \in \mathbb{R}^n \,:\, \nabla_x F(\bar{x}, \bar{w})v'_1 < 0\},$$

we see that $\Gamma_3(x')$ consists of vectors $w' \in \mathbb{R}^d$ such that

$$\begin{cases} w' = \nabla^2_{wx} f_0(\bar{x}, \bar{w})v'_1 + \gamma \nabla_w F(\bar{x}, \bar{w}), \\ \nabla^2_{xx} f_0(\bar{x}, \bar{w})v'_1 + \gamma \nabla_x F(\bar{x}, \bar{w}) = -x', \\ (v'_1, \gamma) \in \ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}, \end{cases}$$

or

$$\begin{cases} w' = \nabla^2_{wx} f_0(\bar{x}, \bar{w})v'_1 + \gamma \nabla_w F(\bar{x}, \bar{w}), \\ \nabla^2_{xx} f_0(\bar{x}, \bar{w})v'_1 + \gamma \nabla_x F(\bar{x}, \bar{w}) = -x', \\ (v'_1, \gamma) \in \Delta_1, \end{cases}$$

or

$$\begin{cases} w' = \nabla^2_{wx} f_0(\bar{x}, \bar{w})v'_1, \\ \nabla^2_{xx} f_0(\bar{x}, \bar{w})v'_1 = -x', \\ v'_1 \in \Delta_2. \end{cases}$$

In particular, the set $\Gamma_3(0)$ consists of vectors $w' \in \mathbb{R}^d$ such that

$$\begin{cases} w' = A'_2 \begin{pmatrix} v'_1 \\ \gamma \end{pmatrix}, \;\; A'_1 \begin{pmatrix} v'_1 \\ \gamma \end{pmatrix} = 0, \\ (v'_1, \gamma) \in \ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}. \end{cases}$$

or

$$\begin{cases} w' = A'_2 \begin{pmatrix} v'_1 \\ \gamma \end{pmatrix}, \;\; A'_1 \begin{pmatrix} v'_1 \\ \gamma \end{pmatrix} = 0, \\ (v'_1, \gamma) \in \Delta_1. \end{cases}$$

or

$$\begin{cases} w' = \nabla^2_{wx} f_0(\bar{x}, \bar{w})v'_1, \\ \nabla^2_{xx} f_0(\bar{x}, \bar{w})v'_1 = 0,\; v'_1 \in \Delta_2. \end{cases}$$

Therefore, $\Gamma_3(0) = \{0\}$ if and only if the following three conditions are simultaneously fulfilled

$$\ker A'_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) \subset \ker A'_2, \tag{4.26}$$

$$\ker A'_1 \cap \Delta_1 \subset \ker A'_2, \tag{4.27}$$

and

$$\ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \cap \Delta_2 \subset \ker \nabla^2_{wx} f_0(\bar{x}, \bar{w}). \tag{4.28}$$

Remember that under condition (4.24) we have (4.25). Therefore, the fulfillment of (4.24), (4.26), (4.27), and (4.28) implies $D^*S(\bar{w}|\bar{x})(0) = \{0\}$. Applying the Mordukhovich Criterion 1, we obtain the following *sufficient condition* for the Lipschitz-likeness of S around $(\bar{w}, \bar{x})$:

$$\begin{cases} \ker A'_1 \cap \ker A'_2 = \{0\}, \\ \ker A'_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) \subset \ker A'_2, \\ \ker A'_1 \cap \Delta_1 \subset \ker A'_2, \\ \ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \cap \Delta_2 \subset \ker \nabla^2_{wx} f_0(\bar{x}, \bar{w}). \end{cases} \tag{4.29}$$

Now, by the arguments of Qui [71, pp. 414–416], the Fréchet coderivative values of the multifunction $M(x, w) = \partial_x f(x, w)$ admit the lower estimate

$$\widehat{D}^* M(\bar{\tau})(v'_1) \supset \big\{\gamma\big(\nabla_x F(\bar{x}, \bar{w}), \nabla_w F(\bar{x}, \bar{w})\big) \,:\, \gamma \geq 0\big\}$$

for any v'_1 with $\nabla_x F(\bar{x}, \bar{w}) v'_1 \geq 0$. If $\nabla_x F(\bar{x}, \bar{w}) v'_1 < 0$, then $\widehat{D}^* M(\bar{\tau})(v'_1) = \emptyset$; see [71, p. 412]. (It is important to stress that *we don't need the condition* $\nabla_w F(\bar{x}, \bar{w}) \neq 0$ *here.*) So, by (1.7) one has $\widehat{\Gamma}(x') \supset \widehat{\Gamma}_1(x')$ for any $x' \in \mathbb{R}^n$, where

$$\widehat{\Gamma}_1(x') := \bigcup_{\substack{v'_1 \in \mathbb{R}^n, \\ \nabla_x F(\bar{x}, \bar{w}) v'_1 \geq 0}} \Big\{ w' \in \mathbb{R}^d \,:\, (-x', w') \in \nabla G(\bar{x}, \bar{w})^* v'_1 + \big\{\gamma\big(\nabla_x F(\bar{x}, \bar{w}), \nabla_w F(\bar{x}, \bar{w})\big) \,:\, \gamma \geq 0\big\} \Big\}.$$

Choosing $v'_1 = 0$, we have $0 \in \widehat{\Gamma}_1(0)$. In combination with the results in Theorems 1.3 and 1.4, this yields

$$\{0\} \subset \widehat{\Gamma}_1(0) \subset \widehat{\Gamma}(0) \subset \widehat{D}^* S(\bar{w}|\bar{x})(0) \subset D^* S(\bar{w}|\bar{x})(0).$$

According to the Mordukhovich criterion, if S is Lipschitz-like around $(\bar{w}, \bar{x})$, then $D^*S(\bar{w}|\bar{x})(0) = \{0\}$ and $\widehat{\Gamma}_1(0) = \{0\}$ as a result. Since

$$\nabla G(\bar{x}, \bar{w})^* v'_1 = (\nabla^2_{xx} f_0(\bar{x}, \bar{w}) v'_1, \nabla^2_{wx} f_0(\bar{x}, \bar{w}) v'_1),$$

the latter means that

$$\begin{cases} w' = \nabla^2_{wx} f_0(\bar{x}, \bar{w}) v'_1 + \gamma \nabla_w F(\bar{x}, \bar{w}) \\ \nabla^2_{xx} f_0(\bar{x}, \bar{w}) v'_1 + \gamma \nabla_x F(\bar{x}, \bar{w}) = 0 \\ \nabla_x F(\bar{x}, \bar{w}) v'_1 \geq 0, \;\; v'_1 \in \mathbb{R}^n, \;\; \gamma \in \mathbb{R}_+ \end{cases} \Longrightarrow \;\; w' = 0.$$

For $\Delta_3 := \{(v'_1, \gamma) \in \mathbb{R}^n \times \mathbb{R} \; : \; \nabla_x F(\bar{x}, \bar{w}) v'_1 \geq 0, \gamma \geq 0\}$, this condition becomes

$$\begin{cases} w' = A'_2 \begin{pmatrix} v'_1 \\ \gamma \end{pmatrix} \\ A'_1 \begin{pmatrix} v'_1 \\ \gamma \end{pmatrix} = 0 \\ (v'_1, \gamma) \in \Delta_3 \end{cases} \Longrightarrow \;\; w' = 0.$$

Clearly, the last property is equivalent to

$$\ker A'_1 \cap \Delta_3 \subset \ker A'_2. \tag{4.30}$$

Thus, we have shown that (4.30) is a *necessary condition* for S being Lipschitz-like around $(\bar{w}, \bar{x})$.

Condition (4.29) implies (4.30). Indeed, suppose that (4.29) is fulfilled and $(v'_1, \gamma) \in \ker A'_1 \cap \Delta_3$. If $\nabla_x F(\bar{x}, \bar{w}) v'_1 = 0$, then

$$(v'_1, \gamma) \in \ker A'_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R})\, ;$$

so $(v'_1, \gamma) \in \ker A'_2$ by the second condition in (4.29). If $\nabla_x F(\bar{x}, \bar{w}) v'_1 > 0$, then $(v'_1, \gamma) \in \ker A'_1 \cap \Delta_1$; hence $(v'_1, \gamma) \in \ker A'_2$ by the third condition in (4.29). We have thus proved that (4.29) yields (4.30).

The above elementary analysis clearly shows how the sufficient condition for the Lipschitz-likeness of S around $(\bar{w}, \bar{x})$ in the degenerate case is stronger than the necessary one.

We can summarize our results for the degenerate case as follows.

Theorem 4.3 *Suppose that $F(\bar{x}, \bar{w}) = 0$ and the Lagrange multiplier λ corresponding to the stationary point $\bar{x} \in S(\bar{w})$ is null. The following assertions are true:*

(a) *If S is Lipschitz-like around $(\bar{w}, \bar{x})$, then condition* (4.30) *holds;*

(b) *If condition* (4.29) *is fulfilled, then* S *is Lipschitz-like around* $(\bar{w}, \bar{x})$.

Let us consider a simple example to see how Theorem 4.3 works for concrete optimization problems.

Example 4.7 Let $f_0(x, w) = x^2(w-2)$ and $F(x, w) = w(x-1)$ for all $(x, w) \in \mathbb{R} \times \mathbb{R}$. The stationary point set of (P_w) is given by

$$S(w) = \{x \in \mathbb{R} \, : \, 0 \in 2x(w-2) + \partial_x f(x, w)\},$$

with $f(x, w) = (g \circ F)(x, w)$ and $g(y) = \delta_{\mathbb{R}_-}(y)$ for all $y \in \mathbb{R}$. Let $\bar{w} = 1$. Then, the point $\bar{x} = 1$ belongs to $S(\bar{w})$. Indeed, since $F(\bar{x}, \bar{w}) = 0$ and $\nabla_x F(\bar{x}, \bar{w}) = 1$, condition **(MFCQ)** is valid. Hence, from (4.4) we have

$$\begin{aligned} \partial_x f(\bar{x}, \bar{w}) &= \nabla_x F(\bar{x}, \bar{w})^* N_{\mathbb{R}_-}(F(\bar{x}, \bar{w})) \\ &= \nabla_x F(\bar{x}, \bar{w})^* \mathbb{R}_+ = \mathbb{R}_+. \end{aligned}$$

Now it is not difficult to check that $\bar{x} \in S(\bar{w})$. Here we have $A_1' = [\,-4 \;\; 1\,]$, $A_2' = [\,2 \;\; 0\,]$, and $\ker \nabla_x F(\bar{x}, \bar{w}) = \{0\}$. Thus, condition (4.29) is fulfilled; as a result, S is Lipschitz-like around $(\bar{w}, \bar{x})$.

Remark 4.7 Looking back to Example 4.7, we see that

$$\nabla_w F(\bar{x}, \bar{w}) = \bar{x} - 1 = 0.$$

So, the preceding results of Qui [72, Theorem 4.2] cannot be applied for the boundary point $(\bar{x}, \bar{w})$ of the set

$$\begin{aligned} \mathcal{D} &= \{(x, w) \, : \, F(x, w) \leq 0\} \\ &= \{(x, w) \, : \, x \leq 1, \, w \geq 0\} \cup \{(x, w) \, : \, x \leq 1, \, w \geq 0\}, \end{aligned}$$

as it requires that $\nabla_w F(\bar{x}, \bar{w}) \neq 0$.

4.4 The Robinson Stability of the Stationary Point Set Map

Now we turn attention to the Robinson stability of the stationary point set map S of the problem (P_w). As in the preceding sections, we assume the fulfillment of the condition **(MFCQ)**, which requires that $\nabla_x F(\bar{x}, \bar{w}) \neq 0$ whenever $F(\bar{x}, \bar{w}) = 0$.

The sum rule in [50, Theorem 1.62] implies

$$D^*\widetilde{M}(\omega_0)(v_1') = \nabla G(\bar{x},\bar{w})^* v_1' + D^*M(\bar{\tau})(v_1') \tag{4.31}$$

for any $v_1' \in \mathbb{R}^n$. So, condition (**C1**) can be rewritten as

$$\ker D^*\widetilde{M}(\omega_0) = \{0\}.$$

By Theorem 1.5, S has the Robinson stability at $\omega_0 = (\bar{x},\bar{w},0) \in \operatorname{gph}\widetilde{M}$ if (**C1**) and the condition

$$\Big\{w' \in \mathbb{R}^d \,:\, \exists v_1' \in \mathbb{R}^n \text{ with } (0,w') \in D^*\widetilde{M}(\omega_0)(v_1')\Big\} = \{0\}, \tag{\textbf{C2}}$$

is fulfilled. By (4.31) we can rewrite (**C2**) equivalently as

$$\{w' \in \mathbb{R}^d \,:\, \exists v_1' \in \mathbb{R}^n \text{ with } (0,w') \in \nabla G(\bar{x},\bar{w})^* v_1' + D^*M(\bar{\tau})(v_1')\} = \{0\}.$$

With $\Gamma(x')$ defined by (1.3), we can assert that (**C2**) is equivalent to the requirement $\Gamma(0) = \{0\}$. In the proof of [41, Corollary 2.2], the authors have commented that the constraint qualification (4.7), which is equivalent to (**C0**), is stronger than (**C1**). Now we go back to three cases considered in Sect. 4.3.

First, for the case $(\bar{x},\bar{w}) \in \operatorname{int}\mathcal{D}$, we have shown in Sect. 4.3.1 that $D^*M(\bar{\tau})(v_1') = \{0\}$ for any $v_1' \in \mathbb{R}^n$. So, condition (**C2**) becomes

$$\{w' \in \mathbb{R}^d \,:\, \exists v_1' \in \mathbb{R}^n \text{ with } (0,w') \in \nabla G(\bar{x},\bar{w})^* v_1'\} = \{0\}.$$

Since $\nabla G(\bar{x},\bar{w})^* v_1' = (\nabla^2_{xx} f_0(\bar{x},\bar{w})v_1', \nabla^2_{wx} f_0(\bar{x},\bar{w})v_1')$, this is equivalent to

$$\{w' \in \mathbb{R}^d \,:\, \exists v_1' \in \mathbb{R}^n \text{ with } \nabla^2_{xx} f_0(\bar{x},\bar{w})v_1' = 0,\ w' = \nabla^2_{wx} f_0(\bar{x},\bar{w})v_1'\} = \{0\}.$$

The latter can be rewritten as

$$\ker \nabla^2_{xx} f_0(\bar{x},\bar{w}) \subset \ker \nabla^2_{wx} f_0(\bar{x},\bar{w}).$$

This condition has been denoted by (4.11) Besides, if the condition (4.10), i.e.,

$$\ker \nabla^2_{xx} f_0(\bar{x},\bar{w}) \cap \ker \nabla^2_{wx} f_0(\bar{x},\bar{w}) = \{0\}$$

is fulfilled, then (**C0**) is valid and (**C1**) is also valid as a result. Thus, *if* (4.10) *and* (4.11) *are simultaneously satisfied, then S has the Robinson stability at* ω_0.

Let us move to the next case where $F(\bar{x},\bar{w}) = 0$ and the Lagrange multiplier λ corresponding to the stationary point $\bar{x} \in S(\bar{w})$ is positive. First, it is worth to stress that for (P_w), the assumptions (i), (ii), and (10) in [41, Proposition 2.1] are fulfilled. So, from [41, Corollary 2.1], for any $v_1' \in \mathbb{R}^n$,

$$D^*M(\bar{\omega})(v_1') \subset \bigcup_{\substack{\bar{v}\in\partial f(\bar{x},\bar{w}) \text{ with} \\ \text{proj}_1\bar{v}=-\nabla_x f_0(\bar{x},\bar{w})}} \partial^2 f((\bar{x},\bar{w})|\bar{v})(v_1',0).$$

With $\Omega_2(v_1')$ defined by (4.9), using (4.8) we have

$$\Omega_2(v_1') = \bigcup_{\substack{\bar{v}\in\partial f(\bar{x},\bar{w}) \text{ with} \\ \text{proj}_1\bar{v}=-\nabla_x f_0(\bar{x},\bar{w})}} \partial^2 f((\bar{x},\bar{w})|\bar{v})(v_1',0).$$

Hence, $D^*M(\bar{\omega})(v_1') \subset \Omega_2(v_1')$ for any $v_1' \in \mathbb{R}^n$. Therefore, from formula (1.3) and the presentation $\nabla G(\bar{x},\bar{w})^*(v_1') = \nabla^2 f_0(\bar{x},\bar{w})^*(v_1',0)$, we have

$$\Gamma(x') \subset \left\{w' \in \mathbb{R}^d : \exists v_1' \in \mathbb{R}^n \text{ with } (-x',w') - \nabla^2 f_0(\bar{x},\bar{w})^*(v_1',0) \in \Omega_2(v_1')\right\} \tag{4.32}$$

for any $x' \in \mathbb{R}^n$. This implies that $\Gamma(x')$ is contained in $\Gamma_2(x')$ which is defined in the first case discussed in Sect. 4.3.2. In particular, $\Gamma(0) \subset \Gamma_2(0)$. So, if $\Gamma_2(0) = \{0\}$, then $\Gamma(0) = \{0\}$. We have shown that $\Gamma_2(0) = \{0\}$ if and only if the inclusion

$$\ker A_1 \cap (\ker \nabla_x F(\bar{x},\bar{w}) \times \mathbb{R}) \subset \ker A_2. \tag{4.33}$$

is valid. Therefore, if (4.33) is satisfied, then $\Gamma(0) = \{0\}$ which implies the fulfillment of (**C2**). Let

$$A_1 = \begin{bmatrix} \nabla^2_{xx} f_0(\bar{x},\bar{w}) + \lambda\nabla^2_{xx}F(\bar{x},\bar{w}) & \nabla_x F(\bar{x},\bar{w}) \end{bmatrix} \in \mathbb{R}^{n\times(n+1)} \tag{4.34}$$

and

$$A_2 = \begin{bmatrix} \nabla^2_{wx} f_0(\bar{x},\bar{w}) + \lambda\nabla^2_{wx}F(\bar{x},\bar{w}) & \nabla_w F(\bar{x},\bar{w}) \end{bmatrix} \in \mathbb{R}^{d\times(n+1)}, \tag{4.35}$$

where $\nabla_x F(\bar{x},\bar{w})$ and $\nabla_w F(\bar{x},\bar{w})$ are interpreted as column vectors. If the equality

$$\ker A_1 \cap \ker A_2 \cap (\ker \nabla_x F(\bar{x},\bar{w}) \times \mathbb{R}) = \{(0,0)\}. \tag{4.36}$$

is satisfied then, as shown in the first case discussed in Set. 4.3.2, (**C0**) is fulfilled; consequently, (**C1**) is valid. Thus, *in the case under our consideration, once* (4.36) *and* (4.33) *are simultaneously satisfied, S has the Robinson stability at ω_0.*

Finally, we consider the case where $F(\bar{x},\bar{w}) = 0$ and the Lagrange multiplier λ corresponding to the stationary point $\bar{x} \in S(\bar{w})$ equals to zero. In this case, if

$$\ker A_1' \cap \ker A_2' = \{0\}, \tag{4.37}$$

where

$$A_1' := \left[\begin{array}{cc} \nabla^2_{xx} f_0(\bar{x}, \bar{w}) & \nabla_x F(\bar{x}, \bar{w}) \end{array} \right] \in \mathbb{R}^{n \times (n+1)},$$

and

$$A_2' := \left[\begin{array}{cc} \nabla^2_{wx} f_0(\bar{x}, \bar{w}) & \nabla_w F(\bar{x}, \bar{w}) \end{array} \right] \in \mathbb{R}^{d \times (n+1)},$$

is valid, then (**C0**) holds (see the second case discussed in Sect. 4.3.2). So, (4.37) guarantees the validity of (**C1**). Concerning condition (**C2**), we will show that if the conditions

$$\ker A_1' \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) \subset \ker A_2', \tag{4.38}$$

$$\ker A_1' \cap \Delta_1 \subset \ker A_2', \tag{4.39}$$

and

$$\ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \cap \Delta_2 \subset \ker \nabla^2_{wx} f_0(\bar{x}, \bar{w}). \tag{4.40}$$

are satisfied, then (**C2**) is fulfilled.

Let $\Gamma_3(x')$ be the set of vectors $w' \in \mathbb{R}^d$ for which there exists $v_1' \in \mathbb{R}^n$ with

$$\begin{cases} (-x' - \nabla^2_{xx} f_0(\bar{x}, \bar{w}) v_1', w' - \nabla^2_{wx} f_0(\bar{x}, \bar{w}) v_1') \in \{\gamma \nabla F(\bar{x}, \bar{w}) : \gamma \in \mathbb{R}\}, \\ \nabla_x F(\bar{x}, \bar{w}) v_1' = 0, \end{cases}$$

or

$$\begin{cases} (-x' - \nabla^2_{xx} f_0(\bar{x}, \bar{w}) v_1', w' - \nabla^2_{wx} f_0(\bar{x}, \bar{w}) v_1') \in \{\gamma \nabla F(\bar{x}, \bar{w}) : \gamma \in \mathbb{R}_+\}, \\ \nabla_x F(\bar{x}, \bar{w}) v_1' > 0, \end{cases}$$

or

$$\begin{cases} -x' - \nabla^2_{xx} f_0(\bar{x}, \bar{w}) v_1' = 0, \;\; w' - \nabla^2_{wx} f_0(\bar{x}, \bar{w}) v_1' = 0, \\ \nabla_x F(\bar{x}, \bar{w}) v_1' < 0. \end{cases}$$

As it has been proved in the second case discussed in Sect. 4.3.2,

$$\Omega_2(v_1') \subset \begin{cases} \{\gamma \nabla F(\bar{x}, \bar{w}) : \gamma \in \mathbb{R}\} & \text{if } \nabla_x F(\bar{x}, \bar{w}) v_1' = 0, \\ \{\gamma \nabla F(\bar{x}, \bar{w}) : \gamma \in \mathbb{R}_+\} & \text{if } \nabla_x F(\bar{x}, \bar{w}) v_1' > 0, \\ \{0\} & \text{if } \nabla_x F(\bar{x}, \bar{w}) v_1' < 0. \end{cases} \tag{4.41}$$

From (4.32) and the inclusion (4.41) we have $\Gamma(x') \subset \Gamma_3(x')$ for any $x' \in \mathbb{R}^n$. In particular, $\Gamma(0) \subset \Gamma_3(0)$. Hence, if $\Gamma_3(0) = \{0\}$, then $\Gamma(0) = \{0\}$. We have shown that $\Gamma_3(0) = \{0\}$ holds if and only if the system (4.38)–(4.40) is satisfied. Thus, the validity of (4.38)–(4.40) implies $\Gamma(0) = \{0\}$ which yields the fulfillment of (**C2**). Therefore, *if* (4.37) *and the system* (4.38)–(4.40) *are simultaneously satisfied, then* S *has the Robinson stability at* ω_0.

We have thus shown that the sufficient conditions for S being Lipschitz-like around $(\bar{w}, \bar{x})$ in each case also guarantee for S having the Robinson stability at ω_0.

Our results on the Robinson stability of S are summarized as follows.

Theorem 4.4 *The stationary point set map* S *of* (P_w) *has the Robinson stability at* $\omega_0 = (\bar{x}, \bar{w}, 0)$ *if one of the following is valid:*

(a) $F(\bar{x}, \bar{w}) < 0$ *and the condition* (4.12) *holds, i.e,*

$$\ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) = \{0\};$$

(b) $F(\bar{x}, \bar{w}) = 0$*, the Lagrange multiplier* λ *corresponding to the stationary point* $\bar{x} \in S(\bar{w})$ *is positive, and condition* (4.22) *is satisfied, i.e.,*

$$\ker A_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times I\!R) = \{0\};$$

(c) $F(\bar{x}, \bar{w}) = 0$*, the Lagrange multiplier* λ *corresponding to the stationary point* $\bar{x} \in S(\bar{w})$ *equals to zero, and condition* (4.29) *satisfied, i.e,*

$$\begin{cases} \ker A_1' \cap \ker A_2' = \{0\}, \\ \ker A_1' \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times I\!R) \subset \ker A_2', \\ \ker A_1' \cap \Delta_1 \subset \ker A_2', \\ \ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \cap \Delta_2 \subset \ker \nabla^2_{wx} f_0(\bar{x}, \bar{w}). \end{cases}$$

It is worthy to stress that the Robinson stability of S at ω_0 is available for the examples of the previous section where our sufficient conditions for the Lipschitz-likeness of S around $(\bar{w}, \bar{x})$ are fulfilled.

4.5 Applications to Quadratic Programming

In this section, the above general results are applied to a class of nonconvex quadratic programming problems. Namely, we will consider the problems of minimizing a linear-quadratic function under one linear-quadratic functional constraint. Special cases of such problems have been considered in, e.g., [38], [40], and [73]. Nonconvex quadratic programming under linear constraints was studied by many authors; see, e.g., the dissertation of Tam [82], and the book by Lee, Tam, and Yen [37], and the references therein.

Denote by $\mathcal{S}_n$ the space of $n \times n$ symmetric matrices. Let $D, A \in \mathcal{S}_n$, c and b be vectors in $\mathbb{R}^n$, and α a real number. Put $w = (w_1, w_2)$ with $w_1 := (D, c)$ and $w_2 := (A, b, \alpha)$. Denote the problem (P_w) with $f_0(x, w) = \frac{1}{2}x^T Dx + c^T x$ and $F(x, w) = \frac{1}{2}x^T Ax + b^T x + \alpha$ by (QP_w). For convenience, we put $W_1 = \mathcal{S}_n \times \mathbb{R}^n$, $W_2 = \mathcal{S}_n \times \mathbb{R}^n \times \mathbb{R}$, and $W = W_1 \times W_2$. Fix a vector $\bar{w} = (\bar{w}_1, \bar{w}_2) \in W$ with $\bar{w}_1 = (\bar{D}, \bar{c})$, $\bar{w}_2 = (\bar{A}, \bar{b}, \bar{\alpha})$, and suppose that a stationary point $\bar{x} \in S(\bar{w})$ is given.

To ease the description of certain second order differential operators, sometimes we will present the matrices D and A in the following column forms

$$D = \begin{pmatrix} d_1^T \\ d_2^T \\ \vdots \\ d_n^T \end{pmatrix}, \quad A = \begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{pmatrix},$$

where $d_i = (d_{i1} \dots d_{in})$ and $a_i = (a_{i1} \dots a_{in})$ are, respectively, the i-th row of D and the i-th row of A. We have $\nabla_x f_0(\bar{x}, \bar{w}) = \bar{D}\bar{x} + \bar{c}$,

$$\nabla_{w_1} f_0(\bar{x}, \bar{w}) = \left(\tfrac{1}{2}\bar{x}_1\bar{x}_1 \ \dots \ \tfrac{1}{2}\bar{x}_1\bar{x}_n \ \dots \ \tfrac{1}{2}\bar{x}_n\bar{x}_1 \ \dots \ \tfrac{1}{2}\bar{x}_n\bar{x}_n \ \bar{x}_1 \ \dots \ \bar{x}_n \right)^T,$$

$\nabla^2_{xx} f_0(\bar{x}, \bar{w}) = \bar{D}$, $\nabla^2_{w_2 x} f_0(\bar{x}, \bar{w}) = 0_{W_2}$, and

$$\nabla^2_{w_1 x} f_0(\bar{x}, \bar{w}) = \begin{pmatrix} \bar{X} & \dots & 0 \\ & \ddots & \\ 0 & \dots & \bar{X} \\ 1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & 1 \end{pmatrix}.$$

Here, $\bar{X} := \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix}$ is an $n \times 1$ matrix. Similarly, $\nabla_x F(\bar{x}, \bar{w}) = \bar{A}\bar{x} + \bar{b}$, $\nabla^2_{xx} F(\bar{x}, \bar{w}) = \bar{A}$,

$$\nabla_{w_2} F(\bar{x}, \bar{w}) = \left(\tfrac{1}{2}\bar{x}_1\bar{x}_1 \ \dots \ \tfrac{1}{2}\bar{x}_1\bar{x}_n \ \dots \ \tfrac{1}{2}\bar{x}_n\bar{x}_1 \ \dots \ \tfrac{1}{2}\bar{x}_n\bar{x}_n \ \bar{x}_1 \ \dots \ \bar{x}_n \ 1 \right)^T,$$

$\nabla^2_{w_1x}F(\bar{x},\bar{w}) = 0_{W_1}$, and

$$\nabla^2_{w_2x}F(\bar{x},\bar{w}) = \begin{pmatrix} \bar{X} & \dots & 0 \\ & \ddots & \\ 0 & \dots & \bar{X} \\ 1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & 1 \\ 0 & \dots & 0 \end{pmatrix}.$$

We have

$$\nabla^2_{wx}f_0(\bar{x},\bar{w}) = \left(\nabla^2_{w_1x}f_0(\bar{x},\bar{w}) \;\; \nabla^2_{w_2x}f_0(\bar{x},\bar{w})\right).$$

Since $\nabla^2_{w_2x}f_0(\bar{x},\bar{w}) = 0$,

$$\ker\nabla^2_{wx}f_0(\bar{x},\bar{w}) = \left\{v_1' \in \mathbb{R}^n : \nabla^2_{w_1x}f_0(\bar{x},\bar{w})v_1' = 0\right\} = \{0\}.$$

First, we consider the case of interior points $(\bar{x},\bar{w})$, i.e., $F(\bar{x},\bar{w}) < 0$. The conditions (4.4), (4.4), and (4.12) are equivalent due to $\ker\nabla^2_{wx}f_0(\bar{x},\bar{w}) = \{0\}$. Thus, by Theorem 4.1, the stationary point set map S of (P_w) is Lipschitz-like around $(\bar{w},\bar{x})$ if and only if $\ker\nabla^2_{xx}f_0(\bar{x},\bar{w}) = \{0\}$, or $\ker\bar{D} = \{0\}$. In other words, S is Lipschitz-like around $(\bar{w},\bar{x})$ if and only if matrix $\bar{D}$ is nonsingular. By Theorem 4.4, this condition is sufficient for S having the Robinson stability at ω_0.

Next, consider the second case where $(\bar{x},\bar{w})$ is a boundary point of $\mathcal{D}$ and the Lagrange multiplier λ corresponding to $\bar{x} \in S(\bar{w})$ is positive. As in Part 1, λ is defined by

$$\nabla_x f_0(\bar{x},\bar{w}) + \lambda\nabla_x F(\bar{x},\bar{w}) = 0,$$

which is rewritten as

$$\lambda(\bar{A}\bar{x} + \bar{b}) = -(\bar{D}\bar{x} + \bar{c}). \tag{4.42}$$

We have $\nabla^2_{xx} f_0(\bar{x}, \bar{w}) + \lambda \nabla^2_{xx} F(\bar{x}, \bar{w}) = \bar{D} + \lambda \bar{A}$ and

$$\nabla^2_{wx} f_0(\bar{x}, \bar{w}) + \lambda \nabla^2_{wx} F(\bar{x}, \bar{w}) = \begin{pmatrix} \bar{X} & \dots & 0 \\ & \ddots & \\ 0 & \dots & \bar{X} \\ 1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & 1 \\ \lambda\bar{X} & \dots & 0 \\ & \ddots & \\ 0 & \dots & \lambda\bar{X} \\ \lambda & \dots & 0 \\ & \ddots & \\ 0 & \dots & \lambda \\ 0 & \dots & 0 \end{pmatrix}.$$

Now, the matrices A_1 and A_2 defined in Sect. 3 are described as follows

$$A_1 = \begin{bmatrix} \bar{D} + \lambda\bar{A} & \bar{A}\bar{x} + \bar{b} \end{bmatrix}$$

and

$$A_2 = \begin{pmatrix} \bar{X} & \dots & 0 & 0 \\ & \ddots & & \vdots \\ 0 & \dots & \bar{X} & 0 \\ 1 & \dots & 0 & 0 \\ & \ddots & & \vdots \\ 0 & \dots & 1 & 0 \\ \lambda\bar{X} & \dots & 0 & \frac{1}{2}\bar{x}_1\bar{x}_1 \\ & \ddots & & \vdots \\ 0 & \dots & \lambda\bar{X} & \frac{1}{2}\bar{x}_1\bar{x}_n \\ \lambda & \dots & 0 & \bar{x}_1 \\ & \ddots & & \vdots \\ 0 & \dots & \lambda & \bar{x}_n \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

Hence, $\ker A_2 = \{0\}$. This implies that (4.18) is automatically satisfied. So, according to Theorem 4.2, S is Lipschitz-like around $(\bar{w}, \bar{x})$ if and only if (4.21) is fulfilled. Note that

$$\ker A_1 = \{(v_1', \gamma) \in \mathbb{R}^n \times \mathbb{R} : (\bar{D} + \lambda\bar{A})v_1' + \gamma(\bar{A}\bar{x} + \bar{b}) = 0\}$$

and

$$\ker \nabla_x F(\bar{x}, \bar{w}) = \{v_1' \in \mathbb{R}^n : (\bar{A}\bar{x} + \bar{b})^T v_1' = 0\}.$$

Hence, (4.21) holds if and only if

$$\begin{cases} (\bar{D} + \lambda \bar{A})v_1' + \gamma(\bar{A}\bar{x} + \bar{b}) = 0 \\ (\bar{A}\bar{x} + \bar{b})^T v_1' = 0 \end{cases} \qquad \Longrightarrow \quad \begin{cases} v_1' = 0 \\ \gamma = 0 \end{cases}$$

or, equivalently,

$$\det \begin{pmatrix} \bar{D} + \lambda \bar{A} & \bar{A}\bar{x} + \bar{b} \\ (\bar{A}\bar{x} + \bar{b})^T & 0 \end{pmatrix} \neq 0. \tag{4.43}$$

Thus, S is Lipschitz-like around $(\bar{w}, \bar{x})$ if and only if (4.43) is satisfied. Moreover, by Theorem 4.4, (4.43) guarantees for S having the Robinson stability at ω_0.

Let us consider the last case where $(\bar{x}, \bar{w})$ is a boundary point of $\mathcal{D}$ and the Lagrange multiplier λ corresponding to $\bar{x} \in S(\bar{w})$ equals to zero. The matrices A_1' and A_2' defined in Sect. 3 are described as $A_1' = \begin{bmatrix} \bar{D} & \bar{A}\bar{x} + \bar{b} \end{bmatrix}$ and

$$A_2' = \begin{pmatrix} \bar{X} & \dots & 0 & 0 \\ & \ddots & & \vdots \\ 0 & \dots & \bar{X} & 0 \\ 1 & \dots & 0 & 0 \\ & \ddots & & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & \frac{1}{2}\bar{x}_1\bar{x}_1 \\ & \ddots & & \vdots \\ 0 & \dots & 0 & \frac{1}{2}\bar{x}_1\bar{x}_n \\ 0 & \dots & 0 & \bar{x}_1 \\ & \ddots & & \vdots \\ 0 & \dots & 0 & \bar{x}_n \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

Since $\ker A_2' = \{0\}$, using the equality $\ker \nabla^2_{wx} f_0(\bar{x}, \bar{w}) = \{0\}$ we can rewrite (4.29) as

$$\begin{cases} \ker A_1' \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) = \{0\}, \\ \ker A_1' \cap \Delta_1 = \{0\}, \\ \ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \cap \Delta_2 = \{0\}. \end{cases}$$

This condition holds if and only if the following conditions are simultaneously satisfied:

$$\begin{cases} \bar{D}v_1' + \gamma(\bar{A}\bar{x} + \bar{b}) = 0 \\ (\bar{A}\bar{x} + \bar{b})^T v_1' = 0 \end{cases} \implies \begin{cases} v_1' = 0 \\ \gamma = 0, \end{cases}$$

$$\begin{cases} \bar{D}v_1' + \gamma(\bar{A}\bar{x} + \bar{b}) = 0 \\ (\bar{A}\bar{x} + \bar{b})^T v_1' > 0, \ \gamma \geq 0 \end{cases} \implies \begin{cases} v_1' = 0 \\ \gamma = 0, \end{cases}$$

and

$$\begin{cases} \bar{D}v_1' = 0 \\ (\bar{A}\bar{x} + \bar{b})^T v_1' < 0 \end{cases} \implies v_1' = 0.$$

These implications can be rewritten respectively as

$$\det \begin{pmatrix} \bar{D} & \bar{A}\bar{x} + \bar{b} \\ (\bar{A}\bar{x} + \bar{b})^T & 0 \end{pmatrix} \neq 0, \tag{4.44}$$

$$[\bar{D}v_1' + \gamma(\bar{A}\bar{x} + \bar{b}) = 0, \ \gamma \geq 0] \implies (\bar{A}\bar{x} + \bar{b})^T v_1' \leq 0, \tag{4.45}$$

and

$$\bar{D}v_1' = 0 \implies (\bar{A}\bar{x} + \bar{b})^T v_1' = 0. \tag{4.46}$$

Thus, in accordance with Theorem 4.3, S is Lipschitz-like around $(\bar{w}, \bar{x})$ if (4.44)–(4.46) are satisfied. Moreover, by Theorem 4.4, the fulfillment of (4.44)–(4.46) is sufficient for S having the Robinson stability at ω_0. Let us consider the necessary condition

$$\ker A_1' \cap \Delta_3 \subset \ker A_2'.$$

for the Lipschitz-like property of S around $(\bar{w}, \bar{x})$, which is now reduced to $\ker A_1' \cap \Delta_3 = \{0\}$. Clearly, this condition is equivalent to

$$\begin{cases} \bar{D}v_1' + \gamma(\bar{A}\bar{x} + \bar{b}) = 0 \\ (\bar{A}\bar{x} + \bar{b})^T v_1' \geq 0, \ \gamma \geq 0 \end{cases} \implies \begin{cases} v_1' = 0 \\ \gamma = 0. \end{cases} \tag{4.47}$$

By [29, Theorem 4.1], (4.47) is a necessary condition for S being Lipschitz-like around $(\bar{w}, \bar{x})$.

The obtained results can be formulated as follows.

Theorem 4.5 *The following assertions are true:*

(a) *If $F(\bar{x}, \bar{w}) < 0$, then the stationary point set map S of (QP_w) is Lipschitz-like around $(\bar{w}, \bar{x})$ if and only if* $\det \bar{D} \neq 0$. *Moreover, under this condition, S has the Robinson stability at ω_0;*

(b) *If $F(\bar{x},\bar{w})=0$ and the Lagrange multiplier λ corresponding to $\bar{x}\in S(\bar{w})$ is positive, then S is Lipschitz-like around $(\bar{w},\bar{x})$ if and only if* (4.43) *is fulfilled. This condition is sufficient for S having the Robinson stability at ω_0;*

(c) *If $F(\bar{x},\bar{w})=0$ and the Lagrange multiplier λ corresponding to $\bar{x}\in S(\bar{w})$ is zero, then* (4.47) *is necessary for S being Lipschitz-like around $(\bar{w},\bar{x})$. Meanwhile, the fulfillment of* (4.44)–(4.46) *is sufficient for the Lipschitz-like property of S around $(\bar{w},\bar{x})$, as well as for the Robinson stability of S at ω_0.*

To show how these results can work, we revisit some examples from [73].

Example 4.8 (see [73, Example 4.1]) Consider the problem (QP_w) in the case $n=2$. Choosing $\bar{w}=(\bar{D},\bar{c},\bar{A},\bar{b},\bar{\alpha})$ with

$$\bar{D}=\begin{pmatrix}0 & 0\\ 0 & -8\end{pmatrix},\quad \bar{c}=\begin{pmatrix}1\\0\end{pmatrix}$$

and

$$\bar{A}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \bar{b}=\begin{pmatrix}0\\0\end{pmatrix},\quad \bar{\alpha}=-1,$$

one has $f_0(x,\bar{w})=-4x_2^2+x_1$ and $F(x,\bar{w})=x_1^2+x_2^2-1$. To show that $\bar{x}:=(-\frac{1}{8},\frac{\sqrt{63}}{8})^T$ is a stationary point of $(P_{\bar{w}})$, we note by (4.2) that

$$S(\bar{w})=\{x\in\mathbb{R}\,:\,0\in\nabla_x f_0(\bar{x},\bar{w})+\partial_x f(\bar{x},\bar{w})\},$$

with $f(x,w)=(g\circ F)(x,w)$ and $g(y)=\delta_{\mathbb{R}_-}(y)$ for any $y\in\mathbb{R}$. As $F(\bar{x},\bar{w})=0$ and $\nabla_x F(\bar{x},\bar{w})\neq 0$, condition **(MFCQ)** is valid. So, from (4.4) we have

$$\begin{aligned}\partial_x f(\bar{x},\bar{w}) &= \nabla_x F(\bar{x},\bar{w})^* N_{\mathbb{R}_-}(F(\bar{x},\bar{w}))\\ &= \nabla_x F(\bar{x},\bar{w})^*\mathbb{R}_+\\ &= \left\{(-\tfrac{1}{4}\gamma,\tfrac{\sqrt{63}}{4}\gamma)\,:\,\gamma\in\mathbb{R}_+\right\}.\end{aligned}$$

Besides, $\nabla_x f_0(\bar{x},\bar{w})=(1,-\sqrt{63})^T$. Now, it is clear that $\bar{x}\in S(\bar{w})$. From (4.42), the Lagrange multiplier corresponding to $\bar{x}$ is $\lambda=8$. Hence,

$$\det\begin{pmatrix}\bar{D}+\lambda\bar{A} & \bar{A}\bar{x}+\bar{b}\\ (\bar{A}\bar{x}+\bar{b})^T & 0\end{pmatrix}=\det\begin{pmatrix}8 & 0 & \frac{1}{8}\\ 0 & 0 & -\frac{\sqrt{63}}{8}\\ -\frac{1}{8} & \frac{\sqrt{63}}{8} & 0\end{pmatrix}=\frac{63}{8}.$$

So, (4.43) is fulfilled. Thus, by Theorem 4.5, the stationary point set map S of (P_w) not only is Lipschitz-like around $(\bar{w},\bar{x})$ but also has the Robinson stability at $\omega_0=(\bar{x},\bar{w},0)$. Similarly, we can show that $\bar{x}=(-\frac{1}{8},-\frac{\sqrt{63}}{8})^T$ and $\bar{x}=(-1,0)^T$ belong to $S(\bar{w})$ and (4.43) is also valid for them.

Example 4.9 (see [73, Example 4.2]) Consider the problem (QP_w) in the case $n = 3$ and choose $\bar{w} = (\bar{D}, \bar{c}, \bar{A}, \bar{b}, \bar{\alpha})$ with

$$\bar{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix}, \quad \bar{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\bar{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{\alpha} = -1.$$

Here, $f_0(x, \bar{w}) = -4(x_2^2 + x_3^2) + x_1$ and $F(x, \bar{w}) = x_1^2 + x_2^2 + x_3^2 - 1$. Arguments similar to those in the previous example show that $\bar{x} := (-1, 0, 0)^T$ is a stationary point of $(P_{\bar{w}})$ with the associated Lagrange multiplier $\lambda = 1$. It is easy to check that (4.43) is satisfied. So, by Theorem 4.5, the stationary point set map S of (P_w) is Lipschitz-like around $(\bar{w}, \bar{x})$ and it has the Robinson stability at $\omega_0 = (\bar{x}, \bar{w}, 0)$. However, for the stationary points

$$\bar{x}_t := \left(-\frac{1}{8}, \left(\frac{\sqrt{63}}{8}\right)\sin t, \left(\frac{\sqrt{63}}{8}\right)\cos t\right)^T,$$

with $t \in [0, 2\pi)$, which share the common associated Lagrange multiplier $\lambda = 8$, (4.43) is violated because

$$\det\begin{pmatrix} \bar{D} + \lambda\bar{A} & \bar{A}\bar{x} + \bar{b} \\ (\bar{A}\bar{x} + \bar{b})^T & 0 \end{pmatrix} = \det\begin{pmatrix} 8 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & -\frac{\sqrt{63}}{8}\sin t \\ 0 & 0 & 0 & -\frac{\sqrt{63}}{8}\cos t \\ -\frac{1}{8} & \frac{\sqrt{63}}{8}\sin t & \frac{\sqrt{63}}{8}\cos t & 0 \end{pmatrix} = 0.$$

Thus, by Theorem 4.5, S is not Lipschitz-like around $(\bar{w}, \bar{x})$.

The *parametric trust-region subproblem* (TRS) considered in [38, 39, 73] is a special case of our quadratic programming problem (QP_w), where A is the unit matrix, $b = 0$, and $\alpha < 0$.

For TRS, in the case where $F(\bar{x}, \bar{w}) = 0$ and the Lagrange multiplier λ corresponding to $\bar{x} \in S(\bar{w})$ is positive, the matrix in (4.43) coincides with the matrix $Q(.)$ in [40, Theorem 5.1] and [73, Theorem 4.2], called the *stability matrix* (see [40, p. 200]). Therefore, Theorem 4.2 in [73], which only discusses the Lipschitz-like property, is a consequence of the assertions (a) and (b) of Theorem 4.5.

In the case where $F(\bar{x}, \bar{w}) = 0$ and the Lagrange multiplier λ corresponding to $\bar{x} \in S(\bar{w})$ equals to zero, the matrix in (4.44) coincides with the *stability matrix* $Q_1(.)$ in [73, Theorem 4.3]. So, condition (4.10) in [73] coincides with our condition (4.44). The *sufficient condition* for the Lipschitz-like property in [73, Theorem 4.3] also requires $\det\bar{D} \neq 0$. Under this assumption, (4.45) and (4.46) are valid if and only if $\bar{x}^T\bar{D}^{-1}\bar{x} \geq 0$. However, our conditions (4.44)–(4.46) do not require $\det\bar{D} \neq 0$. Thus, for TRS, the sufficient conditions in Theorem 4.5(c) and in [73, Theorem 4.3(ii)] are independent results. Finally, note that the *necessary condition* (4.9) in [73] for the Lipschitz-like property coincides with our condition (4.47).

4.6 Results Obtained by Another Approach

Following the detailed hints of one referee of this paper, we will compare our results with those which can be obtained by using the theory of strongly regular generalized equations of Robinson [76].

Suppose that $\bar{x} \in S(\bar{w})$ and the condition **(MFCQ)** is satisfied. It is not difficult to show that, thanks to **(MFCQ)**, there exist a neighborhood W_0 of $\bar{w}$ and a neighborhood U_0 of $\bar{x}$ such that for every $(x, w) \in U_0 \times W_0$ one has $N_{C(w)}(x) = \{\lambda\nabla_x F(x, w) : \lambda \geq 0\}$ when $F(x, w) = 0$ and $N_{C(w)}(x) = \{0\}$ when $F(x, w) < 0$. Hence, for every $(x, w) \in U_0 \times W_0$, the condition

$$0 \in \nabla_x f_0(x, w) + N_{C(w)}(x)$$

is equivalent to the existence of a Lagrange multiplier $\alpha \in \mathbb{R}$ such that

$$0 \in \begin{pmatrix} \nabla_x \mathcal{L}(x, \alpha, w) \\ -F(x, w) \end{pmatrix} + N_{\mathbb{R}^n \times \mathbb{R}_+}(x, \alpha),$$

where $\mathcal{L}(x, \alpha, w) := f_0(x, w) + \alpha F(x, w)$. Setting $g(x, \alpha, w) = \begin{pmatrix} \nabla_x \mathcal{L}(x, \alpha, w) \\ -F(x, w) \end{pmatrix}$, we consider the *parametric generalized equation*

$$0 \in g(x, \alpha, w) + N_{\mathbb{R}^n \times \mathbb{R}_+}(x, \alpha) \quad (w \in \mathbb{R}^d) \tag{4.48}$$

and denote its solution set by $\widehat{S}(w)$. Then,

$$\widehat{S}(w) = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R}^d \,:\, 0 \in g(x, \alpha, w) + N_{\mathbb{R}^n \times \mathbb{R}_+}(x, \alpha)\}$$

and $\widehat{S}(.)$ is the *implicit multifunction* defined by (4.50). (The writing of the necessary optimality condition of a constrained smooth mathematical programming problem in a form similar to (4.50) has been used by Robinson [76, p. 54].) From what has been said we have

$$S(w) \cap U_0 = \{x \in U_0 \,:\, \exists\, \alpha \text{ s.t. } (x, \alpha) \in \widehat{S}(w)\} \quad (\forall w \in W_0). \tag{4.49}$$

As in preceding sections, we will denote by λ the unique multiplier corresponding to $\bar{x} \in S(\bar{w})$. Consider the following three cases.

Case 1: $F(\bar{x}, \bar{w}) < 0$. This case has been analyzed in Remark 4.3.

Case 2: $F(\bar{x}, \bar{w}) = 0$ *and* $\lambda > 0$. In accordance with [76, p. 45], the unperturbed generalized equation

$$0 \in g(x, \alpha, \bar{w}) + N_{\mathbb{R}^n \times \mathbb{R}_+}(x, \alpha) \tag{4.50}$$

is said to be *strongly regular* at $(\bar{x}, \lambda)$ if there exist a constant $\ell_0 > 0$ and neighborhoods U of the origin in $\mathbb{R}^n \times \mathbb{R}$ and V of $(\bar{x}, \lambda)$ such that for every $(x', \alpha') \in U$ one can find a unique vector (x, α) in V, denoted by $s_0(x', \alpha')$, satisfying

$$\begin{pmatrix} x' \\ \alpha' \end{pmatrix} \in g(\bar{x}, \lambda, \bar{w}) + \nabla_{(x,\alpha)} g(\bar{x}, \lambda, \bar{w})((x, \alpha) - (\bar{x}, \lambda)) + N_{\mathbb{R}^n \times \mathbb{R}_+}(x, \alpha)$$

and the mapping $s_0 : U \to V$ is Lipschitzian on U with modulus ℓ_0. Using the condition $\lambda > 0$ and the results of Dontchev and Rockafellar [18], one can prove next lemma; see Sect. 4.7 for details.

Lemma 4.3 *The generalized equation* (4.50) *is strongly regular at* $(\bar{x}, \lambda)$ *iff the matrix*

$$\begin{pmatrix} \nabla^2_{xx} \mathcal{L}(\bar{x}, \lambda, \bar{w}) & \nabla_x F(\bar{x}, \bar{w}) \\ \nabla_x F(\bar{x}, \bar{w})^T & 0 \end{pmatrix} \tag{4.51}$$

is nonsingular.

The condition formulated in Lemma 4.3 is equivalent to condition (4.22). Indeed, by (4.34) one has

$$A_1 = \left[\; \nabla^2_{xx} \mathcal{L}(\bar{x}, \lambda, \bar{w}) \quad \nabla_x F(\bar{x}, \bar{w}) \;\right].$$

Hence, $(x', \tau') \in \ker A_1 \cap \big(\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}\big)$ iff

$$\begin{pmatrix} \nabla^2_{xx} \mathcal{L}(\bar{x}, \lambda, \bar{w}) & \nabla_x F(\bar{x}, \bar{w}) \\ \nabla_x F(\bar{x}, \bar{w})^T & 0 \end{pmatrix} \begin{pmatrix} x' \\ \tau' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, the matrix in (4.51) is nonsingular if (4.22) is valid. Now, applying Theorem 2.1 from [76] to the parametric generalized equation (4.50), we can assert that if (4.50) is strongly regular at $(\bar{x}, \lambda)$, then the implicit multifunction $\widehat{S}(.)$ has a *single-valued localization* [19, p. 4] around $\bar{w}$ for $(\bar{x}, \lambda)$ which is Lipschitz continuous in a neighborhood of $\bar{w}$. This means that there exist $\ell > 0$, a neighborhood W of $\bar{w}$, a neighborhood U of $\bar{x}$, and neighborhood V of $\bar{x}$ such that for each $w \in W$ there is a unique vector $(x(w), \alpha(w))$, denoted by $\hat{s}(w)$, in $U \times V$ satisfying the equation (4.50) and $\|\hat{s}(w_2) - \hat{s}(w_1)\| \leq \ell \|w_2 - w_1\|$ for any $w_1, w_2 \in W$. Therefore, thanks to (4.49), we obtain the following result.

Proposition 4.1 *Suppose that* $F(\bar{x}, \bar{w}) = 0$ *and the Lagrange multiplier* λ *corresponding to the stationary point* $\bar{x} \in S(\bar{w})$ *is positive. If condition* (4.22) *is satisfied, then* S *has a Lipschitz continuous single-valued localization around* $\bar{w}$ *for* $\bar{x}$.

Clearly, Proposition 4.1 encompasses Remark 4.5, which gives a sufficient condition for the Lipschitz-like property of S around $(\bar{w}, \bar{x})$.

Case 3: $F(\bar{x}, \bar{w}) = 0$ *and* $\lambda = 0$. In this case, using the results of Dontchev and Rockafellar [18] one can verify the following lemma; see Sect. 4.8 for details.

Lemma 4.4 *The generalized equation* (4.50) *is strongly regular at* $(\bar{x}, \lambda)$ *iff the matrix* $\nabla^2_{xx}\mathcal{L}(\bar{x}, \lambda, \bar{w}) = \nabla^2_{xx} f_0(\bar{x}, \bar{w})$ *is nonsingular and*

$$\nabla_x F(\bar{x}, \bar{w})^T \nabla^2_{xx} f_0(\bar{x}, \bar{w})^{-1} \nabla_x F(\bar{x}, \bar{w}) > 0. \tag{4.52}$$

The sufficient condition for S to be Lipschitz-like around $(\bar{w}, \bar{x})$ in assertion (b) of Theorem 4.3 is (4.29), which reads as

$$\ker A_1' \cap \ker A_2' = \{0\} \tag{4a}$$

$$\ker A_1' \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) \subset \ker A_2' = \{0\} \tag{4b}$$

$$\ker A_1' \cap \Delta_1 \subset \ker A_2' \tag{4c}$$

$$\ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \cap \Delta_2 \subset \ker \nabla^2_{wx} f_0(\bar{x}, \bar{w}) \tag{4d}$$

where

$$A_1' = \begin{pmatrix} \nabla^2_{xx} f_0(\bar{x}, \bar{w}) & \nabla_x F(\bar{x}, \bar{w}) \end{pmatrix},$$
$$A_2' = \begin{pmatrix} \nabla^2_{wx} f_0(\bar{x}, \bar{w}) & \nabla_w F(\bar{x}, \bar{w}) \end{pmatrix},$$

$$\Delta_1 = \{(v_1', \gamma) : \nabla_x F(\bar{x}, \bar{w})^T v_1' > 0,\ \gamma \geq 0\},$$
$$\Delta_2 = \{(v_1', \gamma) : \nabla_x F(\bar{x}, \bar{w})^T v_1' < 0\}.$$

Now conditions (4a)–(4c) imply

$$\ker A_1' \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) = \{0\}, \tag{4.53}$$

$$\ker A_1' \cap \Delta_1 = \emptyset. \tag{4.54}$$

Indeed, by (4b) one sees that the linear subspace $\ker A_1' \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R})$ is contained in $\ker A_1' \cap \ker A_2'$. So, by (4a), the subspace just consists of the origin. This justifies (4.53). Similarly, by (4c), the set $\ker A_1' \cap \Delta_1$ is contained in $\ker A_1' \cap \ker A_2'$. Hence, from (4a) it follows that $\ker A_1' \cap \Delta_1 \subset \{0\}$. As $0 \notin \Delta_1$, condition (4.54) is satisfied.

Condition (4.53) implies that $\det \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \neq 0$. Indeed, if there existed $x' \in \mathbb{R}^n \setminus \{0\}$, then by choosing $\tau' = 0$ we would have

$$(x', \tau') \in \ker A_1' \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}).$$

This contradicts (4.53).

To see that (4.53) and (4.54) yield (4.52), put $v' = \nabla^2_{xx} f_0(\bar{x}, \bar{w})^{-1} \nabla_x F(\bar{x}, \bar{w}))$. We have to show that $\nabla_x F(\bar{x}, \bar{w})^T v' > 0$. If $\nabla_x F(\bar{x}, \bar{w})^T v' = 0$, then

$$(v', -1) \in \ker A_1' \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}).$$

This contradicts (4.53). If $\nabla_x F(\bar{x}, \bar{w})^T v' < 0$, then for $v_1' := -v'$ one has $\nabla_x F(\bar{x}, \bar{w})^T v_1' > 0$. Choosing $\gamma = 1$, by direct calculation we can verify that $(v_1', \gamma) \in \ker A_1' \cap \Delta_1$. This is a contraction to (4.54).

We have thus proved that the conditions $\det \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \neq 0$ and (4.52) follow from (4a)–(4c). Hence, if the conditions (4a)–(4c) are satisfied, then (4.50) is strongly regular at $(\bar{x}, \lambda) = (\bar{x}, 0)$. Therefore, invoking Theorem 2.1 from [76] to the parametric generalized equation (4.50), we can assert that if (4a)–(4c) are satisfied, then the implicit multifunction $\widehat{S}(.)$ has a Lipschitz continuous single-valued localization around $\bar{w}$ for $(\bar{x}, \lambda) = (\bar{x}, 0)$. Thus, thanks to (4.49), we have the following result.

Proposition 4.2 *Suppose that $F(\bar{x}, \bar{w}) = 0$ and the Lagrange multiplier λ corresponding to the stationary point $\bar{x} \in S(\bar{w})$ is zero. If* (4a)–(4c) *are satisfied, then S has a Lipschitz continuous single-valued localization around $\bar{w}$ for $\bar{x}$.*

The result stated in Proposition 4.2 is better than assertion (b) of Theorem 4.3, which says that if (4.29) is fulfilled, i.e., (4a)–(4d) are valid, then S is Lipschitz-like around $(\bar{w}, \bar{x})$.

4.7 Proof of Lemma 4.3

By the definition of Robinson [76, p. 45], the strong regularity of (4.50) at $(\bar{x}, \lambda)$ is identical to the strong regularity of the affine variational inequality

$$0 \in A \begin{pmatrix} x \\ \alpha \end{pmatrix} + \bar{q} + N_{\mathbb{R}^n \times \mathbb{R}_+}(x, \alpha), \tag{4.55}$$

where

$$A := \nabla_{(x,\alpha)} g(\bar{x}, \lambda, \bar{w}) = \begin{pmatrix} \nabla^2_{xx}\mathcal{L}(\bar{x}, \lambda, \bar{w}) & \nabla_x F(\bar{x}, \bar{w}) \\ -\nabla_x F(\bar{x}, \bar{w})^T & 0 \end{pmatrix} \tag{4.56}$$

and

$$\bar{q} := g(\bar{x}, \lambda, \bar{w}) - A \begin{pmatrix} \bar{x} \\ \lambda \end{pmatrix},$$

at $(\bar{x}, \lambda)$.

According to [18, Theorem 1], the affine variational inequality (4.55) is strongly regular at $(\bar{x}, \lambda)$ if and only if the multifunction $L : \mathbb{R}^n \times \mathbb{R} \rightrightarrows \mathbb{R}^n \times \mathbb{R}$ with

$$L(q) := \left\{ \begin{pmatrix} x \\ \alpha \end{pmatrix} : 0 \in A \begin{pmatrix} x \\ \alpha \end{pmatrix} + q + N_{\mathbb{R}^n \times \mathbb{R}_+}(x, \alpha) \right\}$$

is Lipschitz-like around $(\bar{q}, (\bar{x}, \lambda))$. Furthermore, applying [18, Theorem 2], we can assert that the latter is valid iff the *critical face condition* holds at $(\bar{q}, (\bar{x}, \lambda))$, i.e., *for any choice of closed faces F_1 and F_2 of the critical cone K_0 with $F_1 \supset F_2$,*

$$\big[u \in F_1 - F_2, \;\; A^T u \in (F_1 - F_2)^*\big] \implies u = 0. \tag{4.57}$$

Here,

$$K_0 = K((\bar{x}, \lambda), v_0) := \big\{(x', \alpha') \in T_{\mathbb{R}^n \times \mathbb{R}_+}(\bar{x}, \lambda) : (x', \alpha') \perp v_0\big\},$$

with

$$v_0 := -A \begin{pmatrix} \bar{x} \\ \lambda \end{pmatrix} - \bar{q} \in N_{\mathbb{R}^n \times \mathbb{R}_+}(\bar{x}, \lambda).$$

Recall that a convex subset F of a convex set $C \subset \mathbb{R}^p$ is a *face* of C if every closed line segment in C with a relative interior point in F has both endpoints in F. When K_0 is a linear subspace of $\mathbb{R}^n \times \mathbb{R}$, it has a unique closed face, namely itself. Then, the critical face condition is reduced to

$$\left[u \in K_0,\ \ A^T u \perp K_0\right] \implies u = 0. \tag{4.58}$$

In the case $\lambda > 0$, the critical face is equivalent to the nonsingularity of the matrix in (4.51). Indeed, the condition $\lambda > 0$ implies $N_{\mathbb{R}^n \times \mathbb{R}_+}(\bar{x}, \lambda) = \{(0,0)\}$, $T_{\mathbb{R}^n \times \mathbb{R}_+}(\bar{x}, \lambda) = \mathbb{R}^n \times \mathbb{R}$, and $v_0 = (0,0)$. It follows that $K_0 = \mathbb{R}^n \times \mathbb{R}$. So, the critical face is reduced to (4.58), which is

$$A^T u = 0 \implies u = 0.$$

The latter means that A is nonsingular; or, equivalently, the matrix (4.51) is nonsingular.

Thus, we have proved that the generalized equation (4.50) is strongly regular at $(\bar{x}, \lambda)$ iff the matrix (4.51) is nonsingular. □

4.8 Proof of Lemma 4.4

The arguments described in the beginning of the proof of Lemma 4.3 in the preceding section show that the generalized equation (4.50) is strongly regular at $(\bar{x}, \lambda)$ iff the critical face condition holds at $(\bar{q}, (\bar{x}, \lambda))$, i.e., for any choice of closed faces F_1 and F_2 of the critical cone K_0 with $F_1 \supset F_2$ the condition (4.57) is fulfilled.

Since $\lambda = 0$, $N_{\mathbb{R}^n \times \mathbb{R}_+}(\bar{x}, \lambda) = \{0\} \times \mathbb{R}_-$, and

$$T_{\mathbb{R}^n \times \mathbb{R}_+}(\bar{x}, \lambda) = \mathbb{R}^n \times \mathbb{R}_+.$$

As $v_0 \in N_{\mathbb{R}^n \times \mathbb{R}_+}(\bar{x}, \lambda)$, there are two situations: (a) $v_0 = (0, \beta)$ with $\beta < 0$; (b) $v_0 = (0,0)$. If (a) occurs, then $K_0 = \mathbb{R}^n \times \{0\}$. Since K_0 is a linear subspace, the critical face condition is reduced to (4.58). Using the formula for A in (4.56), one can easily show that (4.58) is equivalent to the requirement that the matrix $\nabla^2_{xx}\mathcal{L}(\bar{x}, \lambda, \bar{w})$ is nonsingular. As $\lambda = 0$, one has $\nabla^2_{xx}\mathcal{L}(\bar{x}, \lambda, \bar{w}) = \nabla^2_{xx} f_0(\bar{x}, \bar{w})$. So, (4.58) is also equivalent to the condition saying that the matrix $\nabla^2_{xx} f_0(\bar{x}, \bar{w})$ is nonsingular. Now, suppose that the situation (b) occurs. Then,

$$K_0 = K((\bar{x}, \lambda), v_0) = \mathbb{R}^n \times \mathbb{R}_+.$$

Obviously, K_0 has only two nonempty faces: $\mathbb{R}^n \times \{0\}$ and $\mathbb{R}^n \times \mathbb{R}_+$.

For $F_1 = F_2 = \mathbb{R}^n \times \{0\}$, one has $F_1 - F_2 = \mathbb{R}^n \times \{0\}$. Then,

$$(F_1 - F_2)^* = \{0\} \times \mathbb{R}$$

and (4.57) is satisfied iff, for any $u' \in \mathbb{R}^n$,

$$\nabla^2_{xx}\mathcal{L}(\bar{x}, \lambda, \bar{w})u' = 0 \implies u' = 0.$$

As $\lambda = 0$, it holds that $\nabla^2_{xx}\mathcal{L}(\bar{x}, \lambda, \bar{w}) = \nabla^2_{xx} f_0(\bar{x}, \bar{w})$. Therefore, (4.57) is valid iff, for any $u' \in \mathbb{R}^n$,

$$\nabla^2_{xx} f_0(\bar{x}, \bar{w})u' = 0 \implies u' = 0.$$

This is equivalent to saying that $\nabla^2_{xx} f_0(\bar{x}, \bar{w})$ is nonsingular.

For $F_1 = F_2 = \mathbb{R}^n \times \mathbb{R}_+$, $F_1 - F_2 = \mathbb{R}^n \times \mathbb{R}$. Then, $(F_1 - F_2)^* = \{0\} \times \{0\}$ and (4.57) is satisfied iff the matrix

$$A^T = \begin{pmatrix} \nabla^2_{xx} f_0(\bar{x}, \bar{w}) & -\nabla_x F(\bar{x}, \bar{w}) \\ \nabla_x F(\bar{x}, \bar{w})^T & 0 \end{pmatrix}$$

is nonsingular, or, A is nonsingular.

For $F_1 = \mathbb{R}^n \times \mathbb{R}_+$ and $F_2 = \mathbb{R}^n \times \{0\}$,

$$F_1 - F_2 = \mathbb{R}^n \times \mathbb{R}_+, \ (F_1 - F_2)^* = \{0\} \times \mathbb{R}_-.$$

Then, (4.57) is fulfilled iff

$$\begin{cases} \nabla^2_{xx} f_0(\bar{x}, \bar{w})u' - \gamma \nabla_x F(\bar{x}, \bar{w}) = 0 \\ \nabla_x F(\bar{x}, \bar{w})^T u' \le 0 \\ u' \in \mathbb{R}^n, \ \gamma \ge 0 \end{cases} \implies \begin{cases} u' = 0 \\ \gamma = 0. \end{cases} \tag{4.59}$$

The proof of the "necessity part" of Lemma 4.4 will be completed if we can show that (4.52) is valid. If (4.52) does not hold, then by putting

$$u' = \nabla^2_{xx} f_0(\bar{x}, \bar{w})^{-1} \nabla_x F(\bar{x}, \bar{w}),$$

we have

$$\nabla_x F(\bar{x}, \bar{w})^T u' = \nabla_x F(\bar{x}, \bar{w})^T \nabla^2_{xx} f_0(\bar{x}, \bar{w})^{-1} \nabla_x F(\bar{x}, \bar{w}) \le 0.$$

So, for $\gamma = 1$, one has

$$\begin{cases} \nabla^2_{xx} f_0(\bar{x}, \bar{w})u' - \gamma \nabla_x F(\bar{x}, \bar{w}) = 0 \\ \nabla_x F(\bar{x}, \bar{w})^T u' \le 0 \\ u' \in \mathbb{R}^n, \ \gamma \ge 0. \end{cases}$$

This contradicts (4.59). We have thus proved (4.52) is valid.

To prove the "sufficiency part" of Lemma 4.4, we suppose that the matrix $\nabla^2_{xx}\mathcal{L}(\bar{x}, \lambda, \bar{w}) = \nabla^2_{xx} f_0(\bar{x}, \bar{w})$ is nonsingular and (4.52) is fulfilled. To verify the fulfillment of the critical face condition at $(\bar{q}, (\bar{x}, \lambda))$, we need only to show that the matrix A is nonsingular and the implication (4.59) holds.

To obtain the nonsingularity of A, suppose to the contrary that there exists a pair $(u', \gamma) \neq (0, 0)$ satisfying

$$\begin{cases} \nabla^2_{xx} f_0(\bar{x}, \bar{w})u' - \gamma \nabla_x F(\bar{x}, \bar{w}) = 0 \\ \nabla_x F(\bar{x}, \bar{w})^T u' = 0. \end{cases} \tag{4.60}$$

If $\gamma = 0$, then the first equation of (4.60) forces $u' = 0$, because the matrix $\nabla^2_{xx} f_0(\bar{x}, \bar{w})$ is nonsingular. So, we must have $\gamma \neq 0$. From the first equation of (4.60) we deduce that

$$u' = \gamma \nabla^2_{xx} f_0(\bar{x}, \bar{w})^{-1} \nabla_x F(\bar{x}, \bar{w}).$$

Hence, by the second equation of (4.60), we obtain

$$\gamma \nabla_x F(\bar{x}, \bar{w})^T \nabla^2_{xx} f_0(\bar{x}, \bar{w})^{-1} \nabla_x F(\bar{x}, \bar{w}) = 0.$$

This obviously contradicts (4.52). Thus, A is nonsingular.

Finally, to obtain the implication (4.59), let $u' \in \mathbb{R}^n$ and $\gamma \geq 0$ be such that

$$\begin{cases} \nabla^2_{xx} f_0(\bar{x}, \bar{w})u' - \gamma \nabla_x F(\bar{x}, \bar{w}) = 0 \\ \nabla_x F(\bar{x}, \bar{w})^T u' \leq 0 \end{cases} \tag{4.61}$$

Multiplying both sides of the equation in (4.61) from the left with the $1 \times n$ matrix $\nabla_x F(\bar{x}, \bar{w})^T \nabla^2_{xx} f_0(\bar{x}, \bar{w})^{-1}$, one obtains

$$\nabla_x F(\bar{x}, \bar{w})^T u' - \gamma \nabla_x F(\bar{x}, \bar{w})^T \nabla^2_{xx} f_0(\bar{x}, \bar{w})^{-1} \nabla_x F(\bar{x}, \bar{w}) = 0. \tag{4.62}$$

Due to (4.52) and the condition $\gamma \geq 0$,

$$-\gamma \nabla_x F(\bar{x}, \bar{w})^T \nabla^2_{xx} f_0(\bar{x}, \bar{w})^{-1} \nabla_x F(\bar{x}, \bar{w}) \leq 0.$$

Combining this with the inequality $\nabla_x F(\bar{x}, \bar{w})^T u' \leq 0$ from (4.61), by (4.62) one has

$$-\gamma \nabla_x F(\bar{x}, \bar{w})^T \nabla^2_{xx} f_0(\bar{x}, \bar{w})^{-1} \nabla_x F(\bar{x}, \bar{w}) = 0.$$

Due to (4.52), $\gamma = 0$. Then, the first equation in (4.61) implies equality $\nabla^2_{xx} f_0(\bar{x}, \bar{w})u' = 0$. As $\nabla^2_{xx} f_0(\bar{x}, \bar{w})$ is nonsingular, one has $u' = 0$. Thus, (4.59) is valid.

The proof is complete. $\square$

4.9 Conclusions

In this chapter, we have analyzed the Lipschitz-like property and the Robinson stability of the stationary point set map of a smooth parametric optimization problem with one smooth functional constraint under total perturbations in two different cases: interior points and boundary points.

The interior point case, which somewhat means that the inequality constraint could be neglected in considering the stationary point set map locally, has a strong connection with the classical implicit function theorem.

The boundary point case is much more involved. Our detailed consideration has been given for the nondegenerate subcase (where the corresponding Lagrange multiplier is positive) and degenerate subcase (where the corresponding Lagrange multiplier is zero).

General Conclusions

The main results of this dissertation include:

1) Criterion for the Lipschitz-like property and the Robinson stability of the solution map of a parametric linear constraint system under total perturbations and applications to the linear complementarity problems and affine variational inequality problems;

2) Analogues of the above results for the case when the linear constraint system undergoes linear perturbations;

3) The Lipschitz-like property of the stationary point set map of a smooth parametric optimization problem with one smooth functional constraint under total perturbations;

4) The Robinson stability of the above stationary point set map and applications to quadratic programming.

Recently, we have developed a part of the results in Chapter 4, which were obtained for an optimization problem with just one inequality constraint, for the case where finitely many equality and inequality constraints are involved.

List of Author's Related Papers

1. D.T.K. Huyen and N.D. Yen, *Coderivatives and the solution map of a linear constraint system,* SIAM Journal on Optimization, **26** (2016), pp. 986–1007. (SCI)
2. D.T.K. Huyen and J.-C. Yao, *Solution stability of a linearly perturbed constraint system and applications,* Set-Valued and Variational Analysis, **27** (2019), pp. 169–189. (SCIE)
3. D.T.K. Huyen, J.-C. Yao, and N.D. Yen, *Sensitivity analysis of an optimization problem under total perturbations. Part 1: Lipschitzian stability,* Journal of Optimization Theory and Applications, **180** (2019), pp. 91–116. (SCI)
4. D.T.K. Huyen, J.-C. Yao, and N.D. Yen, *Sensitivity analysis of an optimization problem under total perturbations. Part 2: Robinson stability,* Journal of Optimization Theory and Applications, **180** (2019), pp. 117–139. (SCI)

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