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STABILITY OF SOME CONSTRAINT SYSTEMS AND OPTIMIZATION PROBLEMS

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SUMMARY

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Introduction

Many real problems lead to formulating equations and solving them. These equations may contain parameters like initial data or control variables. The *solution set* of a parametric equation can be considered as a *multifunction* (that is, a point-to-set function) of the parameters involved. The latter can be called an *implicit multifunction*. A natural question is that *"What properties can the implicit multifunction possess?"*.

Under suitable differentiability assumptions, classical implicit function theorems have addressed thoroughly the above question from finite-dimensional settings to infinite-dimensional settings.

Nowadays, the models of interest (for instance, constrained optimization problems) outrun equations. Thus, Variational Analysis has appeared to meet the need of this increasingly strong development.

J.-P. Aubin, J.M. Borwein, A.L. Dontchev, B.S. Mordukhovich, H.V. Ngai, S.M. Robinson, R.T. Rockafellar, M. Théra, Q.J. Zhu, and other authors, have studied implicit multifunctions and qualitative aspects of optimization and equilibrium problems by different approaches. In particular, with the two-volume book *"Variational Analysis and Generalized Differentiation"* (2006) and a series of research papers, Mordukhovich has given basic tools (*coderivatives, subdiffentials, normal cones*, and calculus rules), fundamental results, and advanced techniques for qualitative studies of optimization and equilibrium problems. Especially, the fourth chapter of the book is entirely devoted to such important properties of the solution set of parametric problems as the *Lipschitz stability* and *metric regularity*. These properties indicate good behaviors of the multifunction in question. The two models considered in that chapter of Mordukhovich's book bear the names *parametric constraint system* and *parametric variational system*. More discussions and references on implicit multifunction theorems can be found in the books by Borwein and Zhu (2015), Dontchev and Rockafellar (2009), and Klatte and Kummer (2002).

Stability properties like lower semicontinuity, upper semicontinuity, Hausdorff semicontinuity/continuity, Hölder continuity of solution maps and of approximate solution maps can be studied for very general optimization problems and equilibrium problems (for example, vector optimization problems, vector variational inequalities, vector equilibrium problems). The locally convex Hausdorff topological vector spaces setting can be also adopted. Here, it is not necessary to use the tools from variational analysis and generalized differentiation. We refer to the works by P.Q. Khanh, L.Q. Anh, and their coauthors for some typical results in this direction.

Within this dissertation we use coderivative to study three properties of solution maps in finite-dimensional settings, which include *Aubin property (Lipschitz-like property)*, *metric regularity*, and *the Robinson stability* of solution maps of constraint and variational systems. Results on these stability properties are applied to studying the solution stability of linear complementarity problem, affine variational inequalities, and a typical parametric optimization problem. The dissertation has four chapters and a list of references.

Chapter 1 collects some basic concepts from Set-Valued Analysis and Variational Analysis and gives a first glance at some properties of multifunctions and key results on implicit multifunctions.

In Chapter 2, we investigate the Lipschitz-like property and the Robinson stability of the solution map of a parametric linear constraint system by means of normal coderivative, the Mordukhovich criterion, and a related theorem due to Levy and Mordukhovich (2004). Among other things, the obtained results yield uniform local error bounds and traditional local error bounds for the linear complementarity problem and the general affine variational inequality problem, as well as verifiable sufficient conditions for the Lipschitz-like property of the solution map of the linear complementarity problem and a class of affine variational inequalities, where all components of the problem data are subject to perturbations.

Chapter 3 shows analogues of the results of the previous chapter for the case where the linear constraint system undergoes linear perturbations.

Finally, in Chapter 4, we analyze the sensitivity of the stationary point set map of a C^2-smooth parametric optimization problem with one C^2-smooth functional constraint under total perturbations by applying some results of Levy and Mordukhovich (2004), and Yen and Yao (2009). We not only show necessary and sufficient conditions for the Lipschitz-like property of the stationary point set map, but also sufficient conditions for its Robinson stability. These results lead us to new insights into the preceding deep investigations of Levy and Mordukhovich (2004) and of Qui (2014, 2016) and allow us to revisit and extend several stability theorems in indefinite quadratic programming.

Chapter 1

Preliminaries

1.1 Basic Concepts from Variational Analysis

In this chapter, several concepts and tools from Variational Analysis are recalled. As a preparation for the investigations in Chapters 2–4, we present lower and upper estimates for coderivatives of implicit multifunctions given by Levy and Mordukhovich (2004), Lee and Yen (2011), as well as the sufficient conditions of Yen and Yao (2009) for the Robinson stability property of implicit multifunctions.

The *Fréchet normal cone* (also called the *prenormal cone*, or the *regular normal cone*) to a set $\Omega \subset \mathbb{R}^s$ at $\bar{v} \in \Omega$ is given by

$$\widehat{N}_\Omega(\bar{v}) = \left\{ v' \in \mathbb{R}^s \, : \, \limsup_{v \xrightarrow{\Omega} \bar{v}} \frac{\langle v', v - \bar{v} \rangle}{\|v - \bar{v}\|} \leq 0 \right\},$$

where $v \xrightarrow{\Omega} \bar{v}$ means $v \to \bar{v}$ with $v \in \Omega$. By convention, $\widehat{N}_\Omega(\bar{v}) := \emptyset$ when $\bar{v} \notin \Omega$.

Provided that Ω is locally closed around $\bar{v} \in \Omega$, one calls

$$\begin{aligned} N_\Omega(\bar{v}) &= \operatorname*{Lim\,sup}_{v \to \bar{v}} \widehat{N}_\Omega(v) \\ &:= \Big\{ v' \in \mathbb{R}^s \,:\, \exists \text{ sequences } v_k \to \bar{v},\ v'_k \to v', \\ &\qquad\qquad \text{with } v'_k \in \widehat{N}_\Omega(v_k) \text{ for all } k = 1, 2, \dots \Big\} \end{aligned}$$

the *Mordukhovich* (or *limiting/basic*) *normal cone* to Ω at $\bar{v}$. If $\bar{v} \notin \Omega$, then one puts $N_\Omega(\bar{v}) = \emptyset$.

A multifunction $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be *locally closed* around a point $\bar{z} = (\bar{x}, \bar{y})$ from $\operatorname{gph} \Phi := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \,:\, y \in \Phi(x)\}$ if $\operatorname{gph} \Phi$ is locally closed around $\bar{z}$. Here, the product space $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ is equipped with the topology generated by the sum norm $\|(x, y)\| = \|x\| + \|y\|$.

For any $\bar{z} = (\bar{x}, \bar{y}) \in \operatorname{gph} \Phi$, the *Fréchet coderivative* of Φ at $\bar{z}$ is the multifunction $\widehat{D}^*\Phi(\bar{z})$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ with the values

$$\widehat{D}^*\Phi(\bar{z})(y') := \big\{ x' \in \mathbb{R}^n \,:\, (x', -y') \in \widehat{N}_{\operatorname{gph} \Phi}(\bar{z}) \big\} \quad (y' \in \mathbb{R}^m).$$

Similarly, the *Mordukhovich coderivative* (limiting coderivative) of Φ at $\bar{z}$ is the multifunction $D^*\Phi(\bar{z}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with the values

$$D^*\Phi(\bar{z})(y') := \big\{ x' \in \mathbb{R}^n \,:\, (x', -y') \in N_{\operatorname{gph} \Phi}(\bar{z}) \big\} \quad (y' \in \mathbb{R}^m).$$

One says that Φ is *graphically regular* at $\bar{z}$ if $\widehat{D}^*\Phi(\bar{z})(y') = D^*\Phi(\bar{z})(y')$ for any $y' \in \mathbb{R}^m$.

Suppose that X, Y, and Z are finite-dimensional Euclidean spaces. Consider a function $\psi : X \to \bar{\mathbb{R}}$, where $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, and suppose that $|\psi(\bar{x})| < \infty$. The set

$$\partial\psi(\bar{x}) := \{x' \in X^* \,:\, (x', -1) \in N_{\operatorname{epi} \psi}(\bar{x}, \psi(\bar{x}))\}$$

is the *Mordukhovich subdifferential* of ψ at $\bar{x}$. If $|\psi(\bar{x})| = \infty$, then we put $\partial\psi(\bar{x}) = \emptyset$. The set

$$\partial^\infty\psi(\bar{x}) := \{x^* \in X^* \,:\, (x^*, 0) \in N_{\operatorname{epi} \psi}(\bar{x}, \psi(\bar{x}))\}$$

is the *singular subdifferential* of ψ at $\bar{x}$. For a set $\Omega \subset X$ and a point $\bar{x} \in \Omega$, we have

$$N(\bar{x}, \Omega) = \partial\delta_\Omega(\bar{x}) = \partial^\infty\delta_\Omega(\bar{x}),$$

where $\delta_\Omega(\bar{x})$ is the indicator function of Ω. If ψ depends on two variables x and y, and $|\psi(\bar{x}, \bar{y})| < \infty$, then $\partial_x\psi(\bar{x}, \bar{y})$ denotes the Mordukhovich subdifferential of $\psi(., \bar{y})$ at $\bar{x}$. For any $\bar{v} \in \partial\psi(\bar{x})$,

$$\partial^2\psi(\bar{x}|\bar{v})(u) := D^*(\partial\psi)(\bar{x}|\bar{v})(u) \quad (u \in X^{**} = X)$$

is the *limiting second-order subdifferential* (or the generalized Hessian) of ψ at $\bar{x}$ in direction $\bar{v}$.

1.2 Properties of Multifunctions and Implicit Multifunctions

A multifunction $G : Y \rightrightarrows X$ is said to be *Lipschitz-like* around a point $(\bar{y}, \bar{x}) \in \operatorname{gph} G$ if there exist a constant $\ell > 0$ and neighborhoods U of $\bar{x}$, V of $\bar{y}$ such that

$$G(y') \cap U \subset G(y) + \ell\|y' - y\|\bar{B}_X \quad \forall y, y' \in V,$$

where $\bar{B}_X$ denotes the closed unit ball in X. The infimum of all such moduli ℓ is called the *exact Lipschitzian bound* of G around $(\bar{y}, \bar{x})$.

Theorem 1.1 (Mordukhovich Criterion 1) *If G is locally closed around $(\bar{y}, \bar{x})$, then G is Lipschitz-like around $(\bar{y}, \bar{x})$ if and only if*

$$D^*G(\bar{y}|\bar{x})(0) = \{0\}.$$

We say that a multifunction $F : X \rightrightarrows Y$ is *metrically regular* around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with modulus $r > 0$ if there exist neighborhoods U of $\bar{x}$, V of $\bar{y}$, and a number $\gamma > 0$ such that

$$d(x, F^{-1}(y)) \le r\, d(y, F(x)) \tag{1.1}$$

for any $(x, y) \in U \times V$ with $d(y, F(x)) < \gamma$.

The condition $d(y, F(x)) < \gamma$ can be omitted when F is *inner semicontinuous* at $(\bar{x}, \bar{y})$ in the graph of F.

Theorem 1.2 (Mordukhovich Criterion 2) *If F is locally closed around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, then F is metrically regular around $(\bar{x}, \bar{y})$ if and only if*

$$0 \in D^*F(\bar{x}|\bar{y})(v') \implies v' = 0.$$

Given a multifunction $F : X \times Y \rightrightarrows Z$ and a pair $(\bar{x}, \bar{y}) \in X \times Y$ satisfying $0 \in F(\bar{x}, \bar{y})$. We say that the implicit multifunction $G : Y \rightrightarrows X$ given by

$$G(y) = \{x \in X \,:\, 0 \in F(x, y)\} \tag{1.2}$$

has the *Robinson stability* at $\omega_0 = (\bar{x}, \bar{y}, 0)$ if there exist constants $r > 0$, $\gamma > 0$, and neighborhoods U of $\bar{x}$, V of $\bar{y}$ such that

$$d(x, G(y)) \le r d(0, F(x, y)) \tag{1.3}$$

for any $(x, y) \in U \times V$ with $d(0, F(x, y)) < \gamma$. The infimum of all such moduli r is called the *exact Robinson regularity bound* of the implicit multifunction G at $\omega_0 = (\bar{x}, \bar{y}, 0)$.

Note that, in (1.3), the condition $d(0, F(x, y)) < \gamma$ can be omitted if F is inner semicontinuous at $(\bar{x}, \bar{y}, 0)$.

1.3 An Overview on Implicit Function Theorems for Multifunctions

Consider an implicit multifunction of the form

$$S(w) = \{x \in \mathbb{R}^n \,:\, 0 \in G(x, w) + M(x, w)\}, \tag{1.4}$$

with $G : \mathbb{R}^{n+d} \to \mathbb{R}^m$ being a continuously Fréchet differentiable function and M a multifunction with closed graph from $\mathbb{R}^{n+d}$ to $\mathbb{R}^m$. Let $(\bar{w}, \bar{x}) \in \operatorname{gph} S$ and $\bar{\tau} = (\bar{w}, \bar{x}, -G(\bar{x}, \bar{w}))$.

Theorem 1.3 (see Levy and Mordukhovich (2004)) *If the constraint qualification*

$$0 \in \nabla G(\bar{x}, \bar{w})^* v_1' + D^*M(\bar{\tau})(v_1') \implies v_1' = 0 \tag{\textbf{C1}}$$

is satisfied, then the upper estimate

$$D^*S(\bar{w}|\bar{x})(x') \subset \Gamma(x'),$$

where

$$\Gamma(x') := \bigcup_{v_1' \in \mathbb{R}^n} \left\{ w' \in \mathbb{R}^d \,:\, (-x', w') \in \nabla G(\bar{x}, \bar{w})^* v_1' + D^* M(\bar{\tau})(v_1') \right\},$$

is valid for any $x' \in \mathbb{R}^n$*. If, in addition, either* M *is graphically regular at* $\bar{\tau}$*, or* $M = M(x)$ *and* $\nabla_w G(\bar{x}, \bar{w})$ *has full rank, then*

$$D^* S(\bar{w}|\bar{x})(x') = \Gamma(x').$$

Theorem 1.4 (see Lee and Yen (2011)) *The lower estimates*

$$\widehat{\Gamma}(x') \subset \widehat{D}^* S(\bar{w}|\bar{x})(x') \subset D^* S(\bar{w}|\bar{x})(x'), \tag{1.5}$$

where

$$\widehat{\Gamma}(x') := \bigcup_{v_1' \in \mathbb{R}^n} \left\{ w' \in \mathbb{R}^d \,:\, (-x', w') \in \nabla G(\bar{x}, \bar{w})^* v_1' + \widehat{D}^* M(\bar{\tau})(v_1') \right\}, \tag{1.6}$$

hold for any $x' \in \mathbb{R}^n$*.*

Yen and Yao (2009) gave a couple of conditions guaranteeing the Robinson stability of implicit multifunctions. In Chapters 2 and 3, it is shown that, for the linear constraint systems, these conditions are also necessary.

Theorem 1.5 (see Yen and Yao (2009)) *Let* S *be the implicit multifunction defined by*

$$S(w) = \{x \in \mathbb{R}^n \,:\, 0 \in \widetilde{M}(x, w)\}. \tag{1.7}$$

If $\operatorname{gph} \widetilde{M}$ *is locally closed around the point* $\omega_0 := (\bar{x}, \bar{w}, 0)$ *and*

(a) $\ker D^* \widetilde{M}(\bar{\tau}) = \{0\}$,

(b) $\left\{ w' \in \mathbb{R}^d \,:\, \exists v_1' \in \mathbb{R}^n \text{ with } (0, w') \in D^* \widetilde{M}(\omega_0)(v_1') \right\} = \{0\}$,

then S *has the Robinson stability around* ω_0*.*

Thanks to the sum rule for the Mordukhovich coderivative (see Mordukhovich (2006)), for any $v' \in \mathbb{R}^m$, we have

$$D^* \widetilde{M}(\omega_0)(v') = \nabla G(\bar{x}, \bar{w})^* v' + D^* M(\bar{\tau})(v').$$

Chapter 2

Linear Constraint Systems under Total Perturbations

The present chapter is devoted to stability analysis of linear constraint systems, linear complementarity systems, and affine variational inequalities under total perturbations. It is written on the basis of the paper of Huyen and Yen (2016), where a new concept of linear constraint system was proposed. In that paper, the first time, the concept "uniform local error bounds" for linear complementarity problems and affine variational inequality has been defined. Recently, the paper has been cited by C. Li and K.F. Ng (2018).

2.1 An Introduction to Parametric Linear Constraint Systems

In this chapter, we study the Lipschitz-like property and the Robinson metric regularity of the solution map of a *parametric linear constraint system* in the form

$$Ax + b \in K, \tag{2.1}$$

with $A \in \mathbb{R}^{m\times n}$ being an $m \times n$ matrix, $b \in \mathbb{R}^m$ a vector, and $K \subset \mathbb{R}^m$ a closed set. When K is a cone, (2.1) can be formally rewritten as

$$Ax + b \geq_K 0, \tag{2.2}$$

where $v \geq_K u$ means that $v - u \in K$. In addition, if K is convex then the partial order "$\geq_K$" is transitive. For $K = \mathbb{R}^m_+$, where $\mathbb{R}^m_+$ denotes the nonnegative orthant in $\mathbb{R}^m$, (2.2) is a standard linear inequality system.

Unlike the traditional considerations, *here K needs not to be convex.*

The multifunction $S : \mathbb{R}^{m\times n} \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with

$$S(A, b) := \{x \in \mathbb{R}^n \mid Ax + b \in K\}$$

is said to be the *solution map* of (2.1). We interpret the pair (A, b) as a parameter. With K being fixed, in the sequel, we will allow *both the linear part* (that is vector b) *and the nonlinear part* (matrix A) *of the data set* $\{A, b\}$ *of* (2.1) *to change.* It is easy to see that the solution map $(A, b) \mapsto S(A, b)$ is a special case of the implicit multifunction $y \mapsto G(y)$ defined by (1.2).

The aim of this chapter will be achieved by using the Mordukhovich Criterion 1 and a formula for computing exactly the limiting coderivative of implicit multifunctions obtained by

Levy and Mordukhovich (Theorem 1.3), as well as a result from Yen and Yao on the Robinson stability of implicit multifunctions (Theorem 1.5).

The abstract stability results of (2.1) can be effectively applied to

(a) traditional inequality systems,

(b) linear complementarity problems,

(c) affine variational inequalities

to yield necessary and sufficient conditions for the Lipschitz-like property and the Robinson stability of the related solution maps as well as uniform local error bounds and traditional local error bounds.

2.2 The Solution Maps of Parametric Linear Constraint Systems

Let $K \subset \mathbb{R}^m$ be a fixed closed set. For any pair $(A, b) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m$, we consider the parametric linear constraint system (2.1). Put $W = \mathbb{R}^{m\times n} \times \mathbb{R}^m$. For every $w = (A, b) \in W$, we set $G(x, w) = -Ax - b$, $M(x, w) = K$, and $\widetilde{M}(x, w) = G(x, w) + M(x, w)$. Then, the solution map of (2.1) is given by

$$S(w) = \{x \in \mathbb{R}^n \,|\, 0 \in \widetilde{M}(x, w)\}.$$

From now on, let us fix an element $\bar{w} = (\bar{A}, \bar{b})$ *and suppose that* $\bar{x} = \big(\bar{x}_1, \dots \bar{x}_n\big)^T$ *belongs to* $S(\bar{w})$. Here and in the sequel, the superscript ^{T} denotes matrix transposition.

Theorem 2.1 *The mapping S is Lipschitz-like around $(\bar{w}, \bar{x})$ if and only if*

$$\big(\ker \bar{A}^T\big) \cap N(\bar{v}; K) = \{0\}, \tag{2.3}$$

where $\bar{v} = \bar{A}\bar{x} + \bar{b}$ *and* $\ker \bar{A}^T := \{v' \in \mathbb{R}^m \mid \bar{A}^T v' = 0\}$ *is the kernel of* $\bar{A}^T$.

Until now, no criterion for the Robinson stability has been found. However, with two following lemmas, we can show that the Lipschitz-like property and the Robinson stability of for our parametric linear constraint systems are equivalent.

Lemma 2.1 *If S has the Robinson stability at $\omega_0 := (\bar{w}, \bar{x}, 0)$, then it is Lipschitz-like around $(\bar{w}, \bar{x})$.*

Lemma 2.2 *If condition* (2.3) *is satisfied, then S has the Robinson stability at $\omega_0 = (\bar{w}, \bar{x}, 0)$.*

The above results lead us to the following main theorem of this chapter.

Theorem 2.2 *The Lipschitz-like property of S around $(\bar{w}, \bar{x})$ and its Robinson stability at $\omega_0 = (\bar{w}, \bar{x}, 0)$ are equivalent. Moreover, these properties appear if and only if condition* (2.3) *is satisfied.*

2.3 Stability Properties of Generalized Linear Inequality Systems

With K specially being a closed convex cone, applying the results of the previous section to the generalized linear inequality system (2.2), we can describe necessary and sufficient conditions for the Lipschitz-like property and the Robinson stability of the solution map S as follows.

Theorem 2.3 *If K is a closed convex cone, then the following properties are equivalent:*

(a) *S is Lipschitz-like around $(\bar{w}, \bar{x})$;*

(b) *S has the Robinson stability at $\omega_0 = (\bar{w}, \bar{x}, 0)$;*

(c) $\left(\ker \bar{A}^T\right) \cap N(\bar{v}; K) = \{0\}$*;*

(d) $(\ker \bar{A}^T) \cap K^* \cap (\bar{v})^\perp = \{0\}$*, where* $K^* = \{v' \in \mathbb{R}^m \mid \langle v', v\rangle \leq 0, \forall v \in K\}$*,* $\bar{v} = \bar{A}\bar{x} + b$*, and* $(\bar{v})^\perp := \{v' \in \mathbb{R}^m \mid \langle v', \bar{v}\rangle = 0\}$;

(e) $0 \in \operatorname{int}(\operatorname{rge} \bar{A} + K - \bar{v})$*, where* $\operatorname{rge} \bar{A} := \bar{A}(\mathbb{R}^n)$ *is the range of* $\bar{A}$ *and* $\operatorname{int} \Omega$ *denotes the interior of* Ω*;*

(f) $\operatorname{rge} \bar{A} + \operatorname{cone}(K - \bar{v}) = \mathbb{R}^m$*, where* $\operatorname{cone} C$ *is the cone generated by* C*.*

Thus, each one of the following conditions is sufficient for (a) *and* (b) *to hold:*

$$(\ker \bar{A}^T) \cap K^* = \{0\};$$
$$(\ker \bar{A}^T) \cap (\bar{v})^\perp = \{0\};$$
$$K^* \cap (\bar{v})^\perp = \{0\}.$$

2.4 The Solution Maps of Linear Complementarity Problems

In this section, we apply results on the stability of solution map of parametric linear constraint system (2.1) to investigate parametric linear complementarity problems.

Given a vector q in $\mathbb{R}^n$, and a matrix M in $\mathbb{R}^{n\times n}$, the *linear complementarity problem* (LCP) aims at finding a vector x in $\mathbb{R}^n$ such that

$$\begin{cases} Mx + q \geq 0,\ x \geq 0 \\ x^T(Mx + q) = 0. \end{cases}$$

The solution set of this problem is denoted by $\operatorname{Sol}(M, q)$.

Let

$$A = \begin{bmatrix} M \\ E \end{bmatrix} \in \mathbb{R}^{(2n)\times n}, \quad b = \begin{pmatrix} q \\ 0 \end{pmatrix} \in \mathbb{R}^{2n},$$

where $E \in \mathbb{R}^{n\times n}$ is the unit matrix, and

$$K = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^n \ :\ u \geq 0,\ v \geq 0,\ v^T u = 0 \right\}.$$

It is clear that $x \in \mathrm{Sol}(M, q)$ if and only if $Ax + b \in K$. Put $W = \mathbb{R}^{(2n)\times n} \times \mathbb{R}^{2n}$ and consider the multifunction $S : W \rightrightarrows \mathbb{R}^n$ defined by

$$S(w) = \{x \in \mathbb{R}^n \,:\, Ax + b \in K\} \quad \forall w = (A, b) \in W.$$

Fix a pair $(\bar{M}, \bar{q}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n$. Let $\bar{x} \in \mathrm{Sol}(\bar{M}, \bar{q})$. Then $\bar{x} \in S(\bar{w})$, where $\bar{w} := (\bar{A}, \bar{b})$. Put $I = \{1, 2, \ldots, n\}$, $I_1 = \{i \in I \,:\, \bar{x}_i = 0, (\bar{M}\bar{x} + \bar{q})_i > 0\}$, $I_2 = \{i \in I \,:\, \bar{x}_i > 0, (\bar{M}\bar{x} + \bar{q})_i = 0\}$, $I_3 = \{i \in I \,:\, \bar{x}_i = 0, (\bar{M}\bar{x} + \bar{q})_i = 0\}$,

$$\bar{y} = \bar{A}\bar{x} + \bar{b} = \begin{pmatrix} \bar{M}\bar{x} + \bar{q} \\ \bar{x} \end{pmatrix},$$

and note that $\bar{y} \in K$.

We present the above cone K in the form

$$K = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^n \,:\, \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in K_i,\ i \in I \right\}, \tag{2.4}$$

where

$$K_i = \left\{ \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in \mathbb{R}^2 \,:\, u_i \geq 0,\ v_i \geq 0,\ u_i v_i = 0 \right\}. \tag{2.5}$$

Definition 2.1 We say that the solution map Sol(.) of LCP satisfies the *uniform local error bound* at $((\bar{M}, \bar{q}), \bar{x})$ if there exist constants $r > 0$, $\delta > 0$, and a neighborhood V of $\bar{x}$ such that

$$d(x; \mathrm{Sol}(M, q)) \leq r \sum_{i=1}^{n} d\left(\begin{pmatrix} (Mx + q)_i \\ x_i \end{pmatrix}; K_i \right), \tag{2.6}$$

for any $x \in V$ and (M, q) satisfying $\|M - \bar{M}\| + \|q - \bar{q}\| < \delta$.

From (2.6) we can infer that

$$d(x; \mathrm{Sol}(\bar{M}, \bar{q})) \leq r \sum_{i=1}^{n} d\left(\begin{pmatrix} (\bar{M}x + \bar{q})_i \\ x_i \end{pmatrix}; K_i \right), \tag{2.7}$$

for any $x \in V$.

Let us consider a regularity condition: *If* $u' = (u'_1, \ldots, u'_n)^T \in \mathbb{R}^n$ *and if*

$$\begin{cases} \text{For each } i \in I_1,\ u'_i = 0; \\ \text{For each } i \in I_2,\ (\bar{M}^T u')_i = 0; \\ \text{For each } i \in I_3,\ \text{either } u'_i = 0,\ \text{or } (\bar{M}^T u')_i = 0,\ \text{or } u'_i \leq 0 \\ \text{and } (\bar{M}^T u')_i \geq 0, \end{cases} \tag{2.8}$$

then $u' = 0$.

The major result of this section reads as follows.

Theorem 2.4 *Suppose that* $\bar{x} \in \mathrm{Sol}(\bar{M}, \bar{q})$. *If the regularity condition* (2.8) *is satisfied, then the problem* LCP *has the uniform local error bound* (2.6) *and the traditional local error bound* (2.7) *at* $((\bar{M}, \bar{q}), \bar{x})$ *and its solution map* Sol(.) *is Lipschitz-like around* $((\bar{M}, \bar{q}), \bar{x})$.

2.5 The Solution Maps of Affine Variational Inequalities

Let $M \in \mathbb{R}^{n\times n}$, $q \in \mathbb{R}^n$, and let $\Delta \subset \mathbb{R}^n$ be a polyhedral convex set defined by

$$\Delta = \{x \in \mathbb{R}^n \,:\, Cx \geq d\},$$

where $C \in \mathbb{R}^{m\times n}$, $d \in \mathbb{R}^m$. The problem finding $\bar{x} \in \Delta$ such that

$$\langle M\bar{x} + q, y - \bar{x}\rangle \geq 0 \quad \forall y \in \Delta,$$

is called an *affine variational inequality* (AVI). Denote the solution set of AVI by $\mathrm{Sol}(M, q, C, d)$.

Fix a vector field $(\bar{M}, \bar{q}, \bar{C}, \bar{d})$. According to Gowda and Pang (1994) (see also Theorem 5.3 of Lee and Yen (2005)), $\bar{x} \in \mathrm{Sol}(\bar{M}, \bar{q}, \bar{C}, \bar{d})$ if and only if there exists a Lagrange multiplier $\lambda \in \mathbb{R}^m$ satisfying

$$\begin{cases} \bar{M}\bar{x} - \bar{C}^T\lambda + \bar{q} = 0 \\ \lambda \geq 0,\, \bar{C}\bar{x} \geq \bar{d} \\ (\bar{C}\bar{x} - \bar{d})^T\lambda = 0. \end{cases}$$

Put

$$\bar{A} = \begin{bmatrix} \bar{M} & -\bar{C}^T \\ 0 & E \\ \bar{C} & 0 \end{bmatrix} \in \mathbb{R}^{(n+m+m)\times(n+m)},\ \ \bar{b} = \begin{pmatrix} \bar{q} \\ 0 \\ -\bar{d} \end{pmatrix} \in \mathbb{R}^{n+m+m},$$

$$K = \left\{(s, u, v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \,:\, s = 0,\ u \geq 0,\ v \geq 0,\ v^T u = 0\right\},$$

and

$$K_0 = \left\{(u, v) \in \mathbb{R}^m \times \mathbb{R}^m \,:\, u \geq 0,\ v \geq 0,\ v^T u = 0\right\}.$$

Here E is the unit matrix of order m. Thus, $\bar{x} \in \mathrm{Sol}(\bar{M}, \bar{q}, \bar{C}, \bar{d})$ if and only if there exists $\lambda \in \mathbb{R}^m$ such that

$$\bar{A}\begin{pmatrix} \bar{x} \\ \lambda \end{pmatrix} + \bar{b} \in K.$$

We consider the multifunction $S : W \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ defined by

$$S(w) = \left\{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \,:\, A\begin{pmatrix} x \\ \lambda \end{pmatrix} + b \in K\right\} \quad \forall w = (A, b) \in W,$$

where $W := \mathbb{R}^{(n+m+m)\times(n+m)} \times \mathbb{R}^{n+m+m}$. Put $\bar{w} = (\bar{A}, \bar{b})$ and suppose that $(\bar{x}, \bar{\lambda}) \in S(\bar{w})$. Then, $\bar{x} \in \mathrm{Sol}(\bar{M}, \bar{q}, \bar{C}, \bar{d})$. Set $I = \{1, 2, \ldots, m\}$, $I_1 = \{i \in I \mid (\bar{C}\bar{x} - \bar{d})_i = 0,\ \bar{\lambda}_i > 0\}$, $I_2 = \{i \in I \mid (\bar{C}\bar{x} - \bar{d})_i > 0,\ \bar{\lambda}_i = 0\}$, $I_3 = \{i \in I \mid (\bar{C}\bar{x} - \bar{d})_i = 0,\ \bar{\lambda}_i = 0\}$,

$$\bar{z} = \bar{A}\bar{x} + \bar{b} = \begin{pmatrix} \bar{M}\bar{x} - \bar{C}^T\bar{\lambda} + \bar{q} \\ \bar{\lambda} \\ \bar{C}\bar{x} - \bar{d} \end{pmatrix},$$

and note that $\bar{M}\bar{x} - \bar{C}^T\bar{\lambda} + \bar{q} = 0$, and $\begin{pmatrix} \bar{\lambda} \\ \bar{C}\bar{x} - \bar{d} \end{pmatrix} \in K_0$. Similarly as it was done for (LCP), we decompose the cone K_0 by writing

$$K_0 = \left\{\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^m \,:\, \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in K_i,\ i \in I\right\},$$

where

$$K_i = \left\{ \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in \mathbb{R}^2 \; : \; u_i \geq 0, \; v_i \geq 0, \; u_i v_i = 0 \right\}.$$

Definition 2.2 We say that the problem AVI has the *uniform local error bound* at the point $((\bar{M}, \bar{q}, \bar{C}, \bar{d}), \bar{x})$ if there exist $\bar{\lambda} \in \mathbb{R}^m$, positive constants r and δ, and a neighborhood V of $\bar{x}$ such that

$$d(x; \mathrm{Sol}(M, q, C, d)) \leq r \left(\|Mx - C^T \bar{\lambda} + q\| + \sum_{i=1}^{m} d \left(\begin{pmatrix} \bar{\lambda}_i \\ (Cx - d)_i \end{pmatrix} ; K_i \right) \right), \tag{2.9}$$

for any $x \in V$ and (M, q, C, d) with $\|M - \bar{M}\| + \|q - \bar{q}\| + \|C - \bar{C}\| + \|d - \bar{d}\| < \delta$.

Substituting $(M, q, C, d) = (\bar{M}, \bar{q}, \bar{C}, \bar{d})$ into (2.9) yields

$$d(x; \mathrm{Sol}(\bar{M}, \bar{q}, \bar{C}, \bar{d})) \leq r \left(\|\bar{M}x - \bar{C}^T \bar{\lambda} + \bar{q}\| + \sum_{i=1}^{m} d \left(\begin{pmatrix} \bar{\lambda}_i \\ (\bar{C}x - \bar{d})_i \end{pmatrix} ; K_i \right) \right), \tag{2.10}$$

for any $x \in V$. This is a *local error bound* for the unperturbed AVI in the traditional form.

Consider the regularity condition: *If vector* $(z_1', z_3') \in \mathbb{R}^n \times \mathbb{R}^m$ *satisfies* $\bar{M}^T z_1' + \bar{C}^T z_3' = 0$ *and the system*

$$\begin{cases} (\bar{C}z_1')_i = 0 \quad \text{if } i \in I_1; \\ (z_3')_i = 0 \quad \text{if } i \in I_2; \\ \text{either } (\bar{C}z_1')_i = 0, \; \text{or } (z_3')_i = 0, \; \text{or } (\bar{C}z_1')_i \leq 0 \text{ and } (z_3')_i \leq 0 \quad \text{if } i \in I_3, \end{cases} \tag{2.11}$$

then $(z_1', z_3') = (0, 0)$.

The next result is applied to a class of AVIs where $\operatorname{rank} C = m$.

Theorem 2.5 *Suppose that* $\bar{x} \in \mathrm{Sol}(\bar{M}, \bar{q}, \bar{C}, \bar{d})$ *and* $\bar{\lambda}$ *is a Lagrange multiplier corresponding to* $\bar{x}$. *If the regularity condition* (2.11) *is satisfied, then there are the local error bounds for* AVI *in the forms* (2.9) *and* (2.10). *Moreover, if* $\operatorname{rank} \bar{C} = m$ *then the solution map* Sol (.) *of* AVI *is Lipschitz-like around* $((\bar{M}, \bar{q}, \bar{C}, \bar{d}), \bar{x})$.

Chapter 3

Linear Constraint Systems under Linear Perturbations

Linear perturbations are just special cases of total perturbations. However, they have their own significance.

As in Chapter 2, consider the parametric linear constraint system (2.1). When b is subject to change, the solution map $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of (2.1) is defined by

$$S(b) = \{x \in \mathbb{R}^n \,:\, b \in -Ax + K\}.$$

Put $\Psi(x) = -Ax + K$ and note that $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is the inverse multifunction of S.

In what follows, we fix a vector $\bar{b}$ and a solution $\bar{x} = \big(\bar{x}_1, \dots \bar{x}_n\big)^T \in S(\bar{b})$. Then, $(\bar{x}, \bar{b})$ belongs to the graph of Ψ. Let $\bar{v} := A\bar{x} + \bar{b}$.

Theorem 3.1 *The following statements are equivalent:*

(a) Ψ *is metrically regular around* $(\bar{x}, \bar{b}) \in \operatorname{gph} \Psi$ *with modulus* ℓ*;*

(b) S *is Lipschitz-like around* $(\bar{b}, \bar{x}) \in \operatorname{gph} S$ *with modulus* ℓ*;*

(c) $\big(\ker A^T\big) \cap N(\bar{v}; K) = \{0\}$.

Moreover, when K *is a closed convex cone, these statements are equivalent to each one of the following:*

(d) $(\ker A^T) \cap K^* \cap (\bar{v})^{\perp} = \{0\}$*, where* $K^* = \{v' \in \mathbb{R}^m \,:\, \langle v', v\rangle \le 0, \forall v \in K\}$*,* $\bar{v} = A\bar{x} + b$*, and* $(\bar{v})^{\perp} := \{v' \in \mathbb{R}^m \,:\, \langle v', \bar{v}\rangle = 0\}$*;*

(e) $0 \in \operatorname{int}(\operatorname{rge} A + K - \bar{v})$*, where* $\operatorname{rge} A := A(\mathbb{R}^n)$ *is the range of* A *and* $\operatorname{int} \Omega$ *denotes the interior of* Ω*;*

(f) $\operatorname{rge} A + \operatorname{cone}(K - \bar{v}) = \mathbb{R}^m$*, where* $\operatorname{cone} C$ *is the cone generated by* C*.*

3.1 Stability properties of Linear Constraint Systems under Linear Perturbations

Following the approach adopted in Chapter 2, we transform LCP into the linear constraint system in the form (2.1) by setting

$$A = \begin{bmatrix} M \\ E \end{bmatrix} \in \mathbb{R}^{(2n)\times n}, \quad b = \begin{pmatrix} q \\ 0 \end{pmatrix} \in \mathbb{R}^{2n},$$

and

$$K = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^n \,:\, u \ge 0,\ v \ge 0,\ v^T u = 0 \right\}.$$

It is clear that $x \in \operatorname{Sol}_M(q)$ if and only if $Ax + b \in K$.

Unlike Chapter 2, here we consider the case where only vector q of the problem LCP is subject to change. Hence, in the corresponding linear constraint system (2.1), only vector b is perturbed. Thus various results on the solution stability of LCP under perturbations of q can be obtained by considering the multifunction $S : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$, $S(b) = \{x \in \mathbb{R}^n \,:\, Ax + b \in K\}$, where A and K have been defined above.

Fix a vector $\bar{b} = \begin{pmatrix} \bar{q} \\ 0 \end{pmatrix}$ and suppose that $\bar{x} \in S(\bar{b})$ or $\bar{x} \in \operatorname{Sol}_M(\bar{q})$. For convenience, we put $I = \{1, 2, \dots, n\}$, $I_1 = \{i \in I \,:\, \bar{x}_i = 0,\ (M\bar{x} + \bar{q})_i > 0\}$, $I_2 = \{i \in I \,:\, \bar{x}_i > 0,\ (M\bar{x} + \bar{q})_i = 0\}$,

$I_3 = \{i \in I \,:\, \bar{x}_i = 0,\ (M\bar{x} + \bar{q})_i = 0\}$ and

$$\bar{y} = A\bar{x} + \bar{b} = \begin{pmatrix} M\bar{x} + \bar{q} \\ \bar{x} \end{pmatrix}.$$

Recall that $\bar{y} \in K$. Denoting

$$K_i = \left\{ \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in \mathbb{R} \times \mathbb{R} \,:\, u_i \geq 0,\ v_i \geq 0,\ u_i v_i = 0 \right\},$$

we have

$$K = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^n \,:\, \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in K_i,\ i \in I \right\}. \tag{3.1}$$

Definition 3.1 (see Chapter 2) We say that the problem LCP satisfies the *uniform local error bound* at $(\bar{q}, \bar{x})$ if there exist constants $\ell > 0$, $\delta > 0$ and a neighborhood V of $\bar{x}$ such that

$$d(x; \mathrm{Sol}_M(q)) \leq \ell \sum_{i=1}^{n} d\left(\begin{pmatrix} (Mx + q)_i \\ x_i \end{pmatrix}; K_i \right), \tag{3.2}$$

for any $x \in V$ and q satisfying $\|q - \bar{q}\| < \delta$.

From (3.2) we can infer that

$$d(x; \mathrm{Sol}_M(\bar{q})) \leq \ell \sum_{i=1}^{n} d\left(\begin{pmatrix} (Mx + \bar{q})_i \\ x_i \end{pmatrix}; K_i \right), \tag{3.3}$$

for any $x \in V$.

The following verifiable regularity condition plays a central role in our study of LCP: *If $u' = (u'_1, \ldots, u'_n)^T \in \mathbb{R}^n$ and if*

$$\begin{cases} \textit{For each } i \in I_1,\ u'_i = 0; \\ \textit{For each } i \in I_2,\ (M^T u')_i = 0; \\ \textit{For each } i \in I_3,\ \textit{either } u'_i = 0,\ \textit{or } (M^T u')_i = 0, \\ \textit{or } (u'_i \leq 0 \textit{ and } (M^T u')_i \geq 0), \end{cases} \tag{3.4}$$

then $u' = 0$.

The linear perturbations permit us to go to a further result as follows.

Theorem 3.2 *Suppose that $\bar{x} \in \mathrm{Sol}_M(\bar{q})$. The following statements are equivalent:*

(a) *The problem* LCP *satisfies the uniform local error bound* (3.2) *at $(\bar{q}, \bar{x})$;*

(b) *The solution map $\mathrm{Sol}_M(.)$ is Lipschitz-like around $(\bar{q}, \bar{x}) \in \mathrm{gph}\, S$;*

(c) *The regularity condition* (3.4) *holds.*

Looking back to the regularity condition (3.4), we can see that If $I = I_1$, then (3.4) is automatically satisfied; If $I = I_2$, then (3.4) holds if and only if M is nonsingular. The following result helps us to open condition (3.4).

Proposition 3.1 *If $I_3 = I$, then* (3.4) *is satisfied if and only if M is P-matrix.*

We now obtain a sufficient condition for the fulfillment of (3.4). This result can be presented similarly for the case of total perturbations considered in Chapter 2.

Theorem 3.3 *If* $M = \mathrm{diag}[M_{I_1I_1}, M_{I_2I_2}, M_{I_3I_3}]$ *with* $M_{I_2I_2}$ *being a nonsingular matrix and* $M_{I_3I_3} \in \mathcal{P}$*, then* (3.4) *is satisfied.*

3.2 Solution Stability of Affine Variational Inequalities under Linear Perturbations

As it has been noted in Chapter 2, $x \in \mathrm{Sol}_{M,C}(q,d)$ if and only if there exists a Lagrange multiplier $\lambda \in \mathbb{R}^m$ such that

$$A\begin{pmatrix} x \\ \lambda \end{pmatrix} + b \in K,$$

where

$$A = \begin{bmatrix} M & -C^T \\ 0 & E \\ C & 0 \end{bmatrix} \in \mathbb{R}^{(n+m+m)\times(n+m)}, \quad b = \begin{pmatrix} q \\ 0 \\ -d \end{pmatrix} \in \mathbb{R}^{n+m+m},$$

and

$$K = \left\{(s,u,v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \mid s = 0,\ u \geq 0,\ v \geq 0,\ v^T u = 0\right\}.$$

Here, as before, E is the unit matrix of order m. We are going to study the multifunction $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ defined by

$$S(b) = \left\{(x,\lambda) \in \mathbb{R}^n \times \mathbb{R}^m \ :\ A\begin{pmatrix} x \\ \lambda \end{pmatrix} + b \in K\right\}.$$

In what follows, we fix a pair $(\bar{q}, \bar{d})$ and suppose that $\bar{x} \in \mathrm{Sol}_{M,C}(\bar{q}, \bar{d})$. Put $\bar{b} = \begin{pmatrix} \bar{q} \\ 0 \\ -\bar{d} \end{pmatrix}$ and suppose that $\bar{\lambda}$ is a Lagrange multiplier corresponding to $\bar{x}$. Then, $(\bar{x}, \bar{\lambda}) \in S(\bar{b})$. Set $I = \{1, \ldots, m\}$ and consider the index sets

$$\begin{aligned} I_1 &:= \{i \in I \ :\ (C\bar{x} - \bar{d})_i = 0,\ \bar{\lambda}_i > 0\}, \\ I_2 &:= \{i \in I \ :\ (C\bar{x} - \bar{d})_i > 0,\ \bar{\lambda}_i = 0\}, \\ I_3 &:= \{i \in I \ :\ (C\bar{x} - \bar{d})_i = 0,\ \bar{\lambda}_i = 0\}. \end{aligned} \tag{3.5}$$

Let

$$\bar{z} = A\bar{x} + \bar{b} = \begin{pmatrix} M\bar{x} - C^T\bar{\lambda} + \bar{q} \\ \bar{\lambda} \\ C\bar{x} - \bar{d} \end{pmatrix}.$$

Note that $M\bar{x} - C^T\bar{\lambda} + \bar{q} = 0$ and $\begin{pmatrix} \bar{\lambda} \\ C\bar{x} - \bar{d} \end{pmatrix} \in K_0$, where K_0 is defined by

$$K_0 = \left\{(u,v) \in \mathbb{R}^m \times \mathbb{R}^m \ :\ u \geq 0,\ v \geq 0,\ v^T u = 0\right\}.$$

The set K_0 can be decomposed by writing

$$K_0 = \left\{\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^m \ :\ \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in K_i,\ i \in I\right\},$$

where

$$K_i = \left\{ \begin{pmatrix} u_i \\ v_i \end{pmatrix} \in \mathbb{R}^2 \,:\, u_i \geq 0,\ v_i \geq 0,\ u_i v_i = 0 \right\}.$$

Definition 3.2 (see Chapter 2) We say that the problem AVI has the *uniform local error bound* at $((\bar{q}, \bar{d}), \bar{x})$ if there exist $\bar{\lambda} \in \mathbb{R}^m$, positive constants ℓ and δ, and a neighborhood V of $\bar{x}$ such that

$$d(x; \mathrm{Sol}_{M,C}(q,d)) \leq \ell \left(\|Mx - C^T\bar{\lambda} + q\| + \sum_{i=1}^{m} d\left(\begin{pmatrix} \bar{\lambda}_i \\ (Cx-d)_i \end{pmatrix}; K_i \right) \right), \tag{3.6}$$

for any $x \in V$ and (q,d) with $\|q - \bar{q}\| + \|d - \bar{d}\| < \delta$.

Substituting $(q,d) = (\bar{q}, \bar{d})$ into (3.6) yields

$$d(x; \mathrm{Sol}_{M,C}(\bar{q},\bar{d})) \leq \ell \left(\|Mx - C^T\bar{\lambda} + \bar{q}\| + \sum_{i=1}^{m} d\left(\begin{pmatrix} \bar{\lambda}_i \\ (Cx-\bar{d})_i \end{pmatrix}; K_i \right) \right), \tag{3.7}$$

for any $x \in V$. This is a *local error bound* for the unperturbed AVI in the traditional form.

We now consider the regularity condition: *If vector* $(u', \eta') \in \mathbb{R}^n \times \mathbb{R}^m$ *satisfies* equality $M^T u' + C^T \eta' = 0$ *and the system*

$$\begin{cases} (Cu')_i = 0 \quad \text{if } i \in I_1; \\ \eta'_i = 0 \quad \text{if } i \in I_2; \\ \text{either } (Cu')_i = 0, \ \text{ or } \eta'_i = 0, \ \text{ or } \ ((Cu')_i \leq 0 \ \text{ and } \eta'_i \leq 0) \quad \text{if } i \in I_3, \end{cases} \tag{3.8}$$

then $(u', \eta') = (0,0)$.

Theorem 3.4 *Suppose that* $\bar{x} \in \mathrm{Sol}_{M,C}(\bar{q}, \bar{d})$ *and* $\bar{\lambda}$ *is a Lagrange multiplier corresponding to* $\bar{x}$. *If the regularity condition* (3.8) *is satisfied, then the local error bounds for* AVI *in the forms* (3.6) *and* (3.7) *are valid. Moreover, if* $\operatorname{rank} C = m$, *then the solution map*

$$(q,d) \mapsto \mathrm{Sol}_{M,C}(q,d)$$

is Lipschitz-like around $((\bar{q}, \bar{d}), \bar{x})$.

The next result appears by the linear perturbations.

Proposition 3.2 *If* $\operatorname{rank} C = m$, *then the solution map* $(q,d) \mapsto \mathrm{Sol}_{M,C}(q,d)$ *is Lipschitz-like around* $((\bar{q}, \bar{d}), \bar{x})$ *if and only if* (3.8) *is satisfied.*

The next proposition gives a set of general conditions for the validity of (3.8).

Proposition 3.3 *Let the index sets* I_1, I_2, *and* I_3 *be defined by* (3.5). *If*

$$M = \mathrm{diag}[M_{I_1 I_1}, M_{I_2 I_2}, M_{I_3 I_3}]$$

and $C = \mathrm{diag}[C_{I_1 I_1}, C_{I_2 I_2}, C_{I_3 I_3}]$, *where* $M_{I_2 I_2}$ *and* $C_{I_1 I_1}$ *are nonsingular,* $M_{I_3 I_3} \in \mathcal{P}$, *and* $C_{I_3 I_3}$ *is a diagonal matrix with positive diagonal elements, then* (3.8) *is fulfilled.*

Chapter 4

Sensitivity Analysis of a Stationary Point Set Map under Total Perturbations

4.1 Problem Formulation

Let f_0 and F be twice continuously differentiable real-valued functions (C^2-functions for brevity) defined on the product $\mathbb{R}^n \times \mathbb{R}^d$ of two Euclidean spaces. For every $w \in \mathbb{R}^d$, we consider the parametric optimization problem

$$(P_w) \qquad \text{Minimize } f_0(x,w) \text{ subject to } x \in \mathbb{R}^n \text{ and } F(x,w) \le 0.$$

The constraint set of (P_w) is $C(w) := \{x \in \mathbb{R}^n : F(x,w) \le 0\}$. The stationary point set of (P_w) is defined by

$$S(w) = \{x \in \mathbb{R}^n : 0 \in \nabla_x f_0(x,w) + N_{C(w)}(x)\}. \tag{4.1}$$

When w varies on $\mathbb{R}^d$, one has a multifunction $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ with $S(w)$ being calculated by (4.1). Setting $f(x,w) = g(F(x,w)) = (g \circ F)(x,w)$, where $g(y) = \delta_{\mathbb{R}_-}(y)$, i.e., $g(y) = 0$ for $y \le 0$ and $g(y) = +\infty$ for $y > 0$, we can rewrite (4.1) as

$$S(w) = \{x \in \mathbb{R}^n : 0 \in \nabla_x f_0(x,w) + \partial_x f(x,w)\}. \tag{4.2}$$

Fix a vector $w = \bar{w} \in \mathbb{R}^d$ and suppose that $\bar{x} \in S(\bar{w})$. Note that $(P_{\bar{w}})$ has a single smooth inequality constraint. The Mangasarian-Fromovitz Constraint Qualification is fulfilled at the point $\bar{x} \in C(\bar{w})$ if and only if

$$\text{If } F(\bar{x},\bar{w}) = 0, \text{ then } \nabla_x F(\bar{x},\bar{w}) \neq 0. \tag{\textbf{MFCQ}}$$

In what follows, we assume that **(MFCQ)** is valid.

Put $\mathcal{D} = \{(x,w) \in \mathbb{R}^n \times \mathbb{R}^d : F(x,w) \le 0\}$. If $F(\bar{x},\bar{w}) < 0$, then $(\bar{x},\bar{w})$ is an *interior point* of $\mathcal{D}$. If $F(\bar{x},\bar{w}) = 0$, then $(\bar{x},\bar{w})$ is a *boundary point* of $\mathcal{D}$.

4.2 Lipschitzian Stability of the Stationary Point Set Map

4.2.1 Interior Points

Suppose that $F(\bar{x},\bar{w}) < 0$. Consider a couple of conditions:

$$\ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \cap \ker \nabla^2_{wx} f_0(\bar{x}, \bar{w}) = \{0\} \tag{4.3}$$

and

$$\ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \subset \ker \nabla^2_{wx} f_0(\bar{x}, \bar{w}). \tag{4.4}$$

Theorem 4.1 *Suppose that $F(\bar{x}, \bar{w}) < 0$. The following assertions are valid:*

(a) *If S is Lipschitz-like around $(\bar{w}, \bar{x})$, then condition* (4.4) *holds;*

(b) *If conditions* (4.3) *and* (4.4) *are simultaneously fulfilled, i.e.,*

$$\ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) = \{0\}, \tag{4.5}$$

then S is Lipschitz-like around $(\bar{w}, \bar{x})$;

(c) *If condition* (4.3) *is satisfied, then S is Lipschitz-like around $(\bar{w}, \bar{x})$ if and only if condition* (4.4) *holds.*

4.2.2 Boundary Points

Suppose that $F(\bar{x}, \bar{w}) = 0$.

Case 1: The Nondegenerate Case ($\lambda > 0$)

Consider a couple of conditions:

$$\ker A_1 \cap \ker A_2 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) = \{(0, 0)\} \tag{4.6}$$

and

$$\ker A_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) \subset \ker A_2, \tag{4.7}$$

where

$$A_1 := \begin{bmatrix} \nabla^2_{xx} f_0(\bar{x}, \bar{w}) + \lambda \nabla^2_{xx} F(\bar{x}, \bar{w}) & \nabla_x F(\bar{x}, \bar{w}) \end{bmatrix} \in \mathbb{R}^{n \times (n+1)}$$

and

$$A_2 := \begin{bmatrix} \nabla^2_{wx} f_0(\bar{x}, \bar{w}) + \lambda \nabla^2_{wx} F(\bar{x}, \bar{w}) & \nabla_w F(\bar{x}, \bar{w}) \end{bmatrix} \in \mathbb{R}^{d \times (n+1)}.$$

Theorem 4.2 *Suppose that $F(\bar{x}, \bar{w}) = 0$ and the Lagrange multiplier λ corresponding to the stationary point $\bar{x} \in S(\bar{w})$ is positive. Then the following assertions are true:*

(a) *S is Lipschitz-like around $(\bar{w}, \bar{x})$ if* (4.6) *and* (4.7) *hold simultaneously, i.e.,*

$$\ker A_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) = \{0\}. \tag{4.8}$$

(b) *If* (4.6) *holds, then S is Lipschitz-like around $(\bar{w}, \bar{x})$ if and only if* (4.7) *is satisfied.*

Case 2: The Degenerate Case ($\lambda = 0$)

Consider the system of conditions:

$$\begin{cases} \ker A'_1 \cap \ker A'_2 = \{0\}, \\ \ker A'_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) \subset \ker A'_2, \\ \ker A'_1 \cap \Delta_1 \subset \ker A'_2, \\ \ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \cap \Delta_2 \subset \ker \nabla^2_{wx} f_0(\bar{x}, \bar{w}) \end{cases} \tag{4.9}$$

and

$$\ker A'_1 \cap \Delta_3 \subset \ker A'_2. \tag{4.10}$$

Here

$$A'_1 := \left[\begin{array}{cc} \nabla^2_{xx} f_0(\bar{x}, \bar{w}) & \nabla_x F(\bar{x}, \bar{w}) \end{array} \right] \in \mathbb{R}^{n \times (n+1)},$$

$$A'_2 := \left[\begin{array}{cc} \nabla^2_{wx} f_0(\bar{x}, \bar{w}) & \nabla_w F(\bar{x}, \bar{w}) \end{array} \right] \in \mathbb{R}^{d \times (n+1)},$$

$$\Delta_1 := \{(v'_1, \gamma) \in \mathbb{R}^n \times \mathbb{R} \,:\, \nabla_x F(\bar{x}, \bar{w}) v'_1 > 0,\, \gamma \geq 0\},$$

$$\Delta_2 := \{v'_1 \in \mathbb{R}^n \,:\, \nabla_x F(\bar{x}, \bar{w}) v'_1 < 0\},$$

and

$$\Delta_3 := \{(v'_1, \gamma) \in \mathbb{R}^n \times \mathbb{R} \,:\, \nabla_x F(\bar{x}, \bar{w}) v'_1 \geq 0,\, \gamma \geq 0\}.$$

Theorem 4.3 *Suppose that $F(\bar{x}, \bar{w}) = 0$ and the Lagrange multiplier λ corresponding to the stationary point $\bar{x} \in S(\bar{w})$ is zero. The following assertions are true:*

(a) *If S is Lipschitz-like around $(\bar{w}, \bar{x})$, then condition* (4.10) *holds;*

(b) *If condition* (4.9) *is fulfilled, then S is Lipschitz-like around $(\bar{w}, \bar{x})$.*

4.3 The Robinson Stability of the Stationary Point Set Map

We have shown that the sufficient conditions for S being Lipschitz-like around $(\bar{w}, \bar{x})$ in each case also guarantee for S having the Robinson stability at ω_0.

Theorem 4.4 *The stationary point set map S of (P_w) has the Robinson stability at the point $\omega_0 = (\bar{x}, \bar{w}, 0)$ if one of the following is valid:*

(a) *$F(\bar{x}, \bar{w}) < 0$ and the condition* (4.5) *holds, i.e,*

$$\ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) = \{0\};$$

(b) *$F(\bar{x}, \bar{w}) = 0$, the Lagrange multiplier λ corresponding to the stationary point $\bar{x} \in S(\bar{w})$ is positive, and condition* (4.8) *is satisfied, i.e.,*

$$\ker A_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) = \{0\};$$

(c) $F(\bar{x},\bar{w}) = 0$, *the Lagrange multiplier* λ *corresponding to the stationary point* $\bar{x} \in S(\bar{w})$ *equals to zero, and condition* (4.9) *satisfied, i.e,*

$$\begin{cases} \ker A_1' \cap \ker A_2' = \{0\}, \\ \ker A_1' \cap (\ker \nabla_x F(\bar{x},\bar{w}) \times \mathbb{R}) \subset \ker A_2', \\ \ker A_1' \cap \Delta_1 \subset \ker A_2', \\ \ker \nabla^2_{xx} f_0(\bar{x},\bar{w}) \cap \Delta_2 \subset \ker \nabla^2_{wx} f_0(\bar{x},\bar{w}). \end{cases}$$

4.4 Applications to Quadratic Programming

In this section, the above general results are applied to a class of nonconvex quadratic programming problems. Namely, we will consider the problems of minimizing a linear-quadratic function under one linear-quadratic functional constraint. Special cases of such problems have been considered by Lee, Tam, and Yen (2012), Lee and Yen (2014), Qui and Yen (2014), etc. Nonconvex quadratic programming under linear constraints was studied by many authors; see, e.g., the dissertation of N.N. Tam (2000), and the book by Lee, Tam, and Yen (2005) and the references therein.

Denote by $\mathcal{S}_n$ the space of $n \times n$ symmetric matrices. Let $D, A \in \mathcal{S}_n$, c and b be vectors in $\mathbb{R}^n$, and α a real number. Put $w = (w_1, w_2)$ with $w_1 := (D, c)$ and $w_2 := (A, b, \alpha)$. Denote the problem (P_w) with $f_0(x,w) = \frac{1}{2}x^T Dx + c^T x$ and $F(x,w) = \frac{1}{2}x^T Ax + b^T x + \alpha$ by (QP_w). For convenience, we put $W_1 = \mathcal{S}_n \times \mathbb{R}^n$, $W_2 = \mathcal{S}_n \times \mathbb{R}^n \times \mathbb{R}$, and $W = W_1 \times W_2$. Fix a vector $\bar{w} = (\bar{w}_1, \bar{w}_2) \in W$ with $\bar{w}_1 = (\bar{D}, \bar{c})$, $\bar{w}_2 = (\bar{A}, \bar{b}, \bar{\alpha})$, and suppose that a stationary point $\bar{x} \in S(\bar{w})$ is given.

Consider the following conditions:

$$\det \begin{pmatrix} \bar{D} + \lambda \bar{A} & \bar{A}\bar{x} + \bar{b} \\ (\bar{A}\bar{x} + \bar{b})^T & 0 \end{pmatrix} \neq 0, \tag{4.11}$$

$$\det \begin{pmatrix} \bar{D} & \bar{A}\bar{x} + \bar{b} \\ (\bar{A}\bar{x} + \bar{b})^T & 0 \end{pmatrix} \neq 0, \tag{4.12}$$

$$[\bar{D}v_1' + \gamma(\bar{A}\bar{x} + \bar{b}) = 0,\ \gamma \geq 0] \implies (\bar{A}\bar{x} + \bar{b})^T v_1' \leq 0, \tag{4.13}$$

$$\bar{D}v_1' = 0 \implies (\bar{A}\bar{x} + \bar{b})^T v_1' = 0, \tag{4.14}$$

and

$$\begin{cases} \bar{D}v_1' + \gamma(\bar{A}\bar{x} + \bar{b}) = 0 \\ (\bar{A}\bar{x} + \bar{b})^T v_1' \geq 0,\ \gamma \geq 0 \end{cases} \implies \begin{cases} v_1' = 0 \\ \gamma = 0. \end{cases} \tag{4.15}$$

Theorem 4.5 *The following assertions are true:*

(a) *If* $F(\bar{x},\bar{w}) < 0$, *then* S *is Lipschitz-like around* $(\bar{w},\bar{x})$ *if and only if* $\det \bar{D} \neq 0$. *Moreover, under this condition,* S *has the Robinson stability at* ω_0*;*

(b) *If* $F(\bar{x},\bar{w}) = 0$ *and the Lagrange multiplier* λ *corresponding to* $\bar{x} \in S(\bar{w})$ *is positive, then* S *is Lipschitz-like around* $(\bar{w},\bar{x})$ *if and only if* (4.11) *is fulfilled. This condition is sufficient for* S *having the Robinson stability at* ω_0*;*

(c) *If $F(\bar{x},\bar{w}) = 0$ and the Lagrange multiplier λ corresponding to $\bar{x} \in S(\bar{w})$ is zero, then* (4.15) *is necessary for S being Lipschitz-like around $(\bar{w},\bar{x})$. Meanwhile, the fulfillment of* (4.12)–(4.14) *is sufficient for the Lipschitz-like property of S around $(\bar{w},\bar{x})$, as well as for the Robinson stability of S at ω_0.*

4.5 Results Obtained by Another Approach

Following the detailed hints of one referee of this paper, we will compare our results with those which can be obtained by using the theory of strongly regular generalized equations of Robinson (1980).

Suppose that $\bar{x} \in S(\bar{w})$ and the condition **(MFCQ)** is satisfied. It is not difficult to show that, thanks to **(MFCQ)**, there exist a neighborhood W_0 of $\bar{w}$ and a neighborhood U_0 of $\bar{x}$ such that for every $(x,w) \in U_0 \times W_0$ one has $N_{C(w)}(x) = \{\lambda \nabla_x F(x,w) \mid \lambda \geq 0\}$ when $F(x,w) = 0$ and $N_{C(w)}(x) = \{0\}$ when $F(x,w) < 0$. Hence, for every $(x,w) \in U_0 \times W_0$, the condition

$$0 \in \nabla_x f_0(x,w) + N_{C(w)}(x)$$

is equivalent to the existence of a Lagrange multiplier $\alpha \in \mathbb{R}$ such that

$$0 \in \begin{pmatrix} \nabla_x \mathcal{L}(x,\alpha,w) \\ -F(x,w) \end{pmatrix} + N_{\mathbb{R}^n \times \mathbb{R}_+}(x,\alpha),$$

where $\mathcal{L}(x,\alpha,w) := f_0(x,w) + \alpha F(x,w)$. Setting $g(x,\alpha,w) = \begin{pmatrix} \nabla_x \mathcal{L}(x,\alpha,w) \\ -F(x,w) \end{pmatrix}$, we consider the *parametric generalized equation*

$$0 \in g(x,\alpha,w) + N_{\mathbb{R}^n \times \mathbb{R}_+}(x,\alpha) \quad (w \in \mathbb{R}^d) \tag{4.16}$$

and denote its solution set by $\widehat{S}(w)$. Then,

$$\widehat{S}(w) = \{(x,\alpha) \in \mathbb{R}^n \times \mathbb{R}^d \,:\, 0 \in g(x,\alpha,w) + N_{\mathbb{R}^n \times \mathbb{R}_+}(x,\alpha)\}$$

and $\widehat{S}(.)$ is the *implicit multifunction* defined by (4.16). From what has been said we have

$$S(w) \cap U_0 = \{x \in U_0 \,:\, \exists\, \alpha \text{ s.t. } (x,\alpha) \in \widehat{S}(w)\} \quad (\forall w \in W_0). \tag{4.17}$$

As in preceding sections, we will denote by λ the unique multiplier corresponding to $\bar{x}$ belonging to $S(\bar{w})$. Consider the following three cases.

CASE 1: $F(\bar{x},\bar{w}) < 0$. This case has been analyzed in Subsection 4.2.1.

Thanks to a criterion for the strong regularity of the generalized equation (4.16), we obtain results for the cases of boundary points.

CASE 2: $F(\bar{x},\bar{w}) = 0$ *and* $\lambda > 0$.

Proposition 4.1 *Suppose that $F(\bar{x},\bar{w}) = 0$ and the Lagrange multiplier λ corresponding to the stationary point $\bar{x} \in S(\bar{w})$ is positive. If condition* (4.8) *is satisfied, then S has a Lipschitz continuous single-valued localization around $\bar{w}$ for $\bar{x}$.*

Clearly, Proposition 4.1 encompasses claim (a) in Theorem 4.2, which gives a sufficient condition for the Lipschitz-like property of S around $(\bar{w},\bar{x})$.

Case 3: $F(\bar{x},\bar{w})=0$ *and* $\lambda=0$.

Consider the four conditions:

$$\ker A_1' \cap \ker A_2' = \{0\}, \tag{4a}$$

$$\ker A_1' \cap (\ker \nabla_x F(\bar{x},\bar{w}) \times \mathbb{R}) \subset \ker A_2' = \{0\}, \tag{4b}$$

and

$$\ker A_1' \cap \Delta_1 \subset \ker A_2', \tag{4c}$$

and

$$\ker \nabla^2_{xx} f_0(\bar{x},\bar{w}) \cap \Delta_2 \subset \ker \nabla^2_{wx} f_0(\bar{x},\bar{w}). \tag{4d}$$

Proposition 4.2 *Suppose that $F(\bar{x},\bar{w})=0$ and the Lagrange multiplier λ corresponding to the stationary point $\bar{x}\in S(\bar{w})$ is zero. If* (4a)–(4c) *are satisfied, then S has a Lipschitz continuous single-valued localization around $\bar{w}$ for $\bar{x}$.*

The result stated in Proposition 4.2 is better than assertion (b) of Theorem 4.3, which says that if (4.9) is fulfilled, i.e., (4a)–(4d) are valid, then S is Lipschitz-like around $(\bar{w},\bar{x})$.

General Conclusions

The main results of this dissertation include:

1) Criterion for the Lipschitz-like property and the Robinson stability of the solution map of a parametric linear constraint system under total perturbations and applications to the linear complementarity problems and affine variational inequality problems;

2) Analogues of the above results for the case when the linear constraint system undergoes linear perturbations;

3) The Lipschitz-like property of the stationary point set map of a smooth parametric optimization problem with one smooth functional constraint under total perturbations;

4) The Robinson stability of the above stationary point set map and applications to quadratic programming.

Recently, we have developed a part of the results in Chapter 4, which were obtained for an optimization problem with just one inequality constraint, for the case where finitely many equality and inequality constraints are involved.

List of Author's Related Papers

1. D.T.K. Huyen and N.D. Yen, *Coderivatives and the solution map of a linear constraint system,* SIAM Journal on Optimization **26** (2016), pp. 986–1007. (SCI)
2. D.T.K. Huyen and J.-C. Yao, *Solution stability of a linearly perturbed constraint system and applications,* Set-Valued and Variational Analysis, **27** (2019), pp. 169–189. (SCIE)
3. D.T.K. Huyen, J.-C. Yao, and N.D. Yen, *Sensitivity analysis of an optimization problem under total perturbations. Part 1: Lipschitzian stability,* Journal of Optimization Theory and Applications, **180** (2019), pp. 91–116. (SCI)
4. D.T.K. Huyen, J.-C. Yao, and N.D. Yen, *Sensitivity analysis of an optimization problem under total perturbations. Part 2: Robinson stability,* Journal of Optimization Theory and Applications, **180** (2019), pp. 117–139. (SCI)

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- The weekly seminar of the Department of Numerical Analysis and Scientific Computing, Institute of Mathematics, Vietnam Academy of Science and Technology;

- Workshop "International Workshop on Nonlinear and Variational Analysis" (August 7–9, 2015, Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung, Taiwan);

- "Taiwan-Vietnam 2015 Winter Mini-Workshop on Optimization" (November 17, 2015, National Cheng Kung University, Tainan, Taiwan);

- The 15$^{\text{th}}$ Workshop on "Optimization and Scientific Computing" (April 21–23, 2016, Ba Vi, Hanoi);

- Seminar of Prof. Xiao-qi Yang's research group (June 2016, Department of Applied Mathematics, Hong Kong Polytechnic University, Hong Kong);

- "Vietnam-Korea Workshop on Selected Topics in Mathematics" (February 20–24, 2017, Danang, Vietnam);

- "Taiwan-Vietnam Workshop on Mathematics" (May 9–11, 2018, Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan).