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Stability Analysis of an Approximate Method for Biharmonic Equation*

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Abstract. In this paper we prove that the errors committed at each iteration of the solution process for biharmonic type equation do not accumulate or exaggerate. Hence, for obtaining an approximate solution of BVP for biharmonic type equation we can use approximate methods at iterations.

1. Introduction

For solving boundary value problems (BVPs) in geometrically complicated domains - a direction of research intensively developed in two recent decades, many authors have proposed domain decomposition methods for reducing the original problems to a sequence of problems in simple domains (see e.g. [2, 4, 10, 12, 13, 15]). The convergence of the iterative processes on continuous level is established. Another class of BVPs is also lead to a sequence of BVPs for second order differential equations. They are BVPs for biharmonic type equation. Several researchers proposed iterative methods for reducing BVPs for high order equations to the solution of second order problems with the aim to use wealthy available efficient algorithms for the latter ones (see e.g. [1, 3, 5 - 9, 14]). But an important problem that has not evolved in the above mentioned iterative processes, where at each iteration the simpler BVPs should be solved, is the accumulation of errors committed in the approximate solution of the second order problems at iterations. It means that whether we will obtain a good approximate solution of the original problem if at each iteration we have only approximate solution of

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the reduced problems.

In this paper we prove that the errors committed at each iteration of the solution process for biharmonic type equation do not accumulate or exaggerate. Hence, for obtaining an approximate solution of BVP for biharmonic type equation we can use approximate methods at iterations.

2. Recalling an Iterative Process for Biharmonic Equation

For simplicity of presentation, we shall consider below the iterative process for the Dirichlet problem for biharmonic equation

$$\Delta^2 u = f(x), \quad x \in \Omega, \tag{1}$$

$$u|_\Gamma = 0, \ \frac{\partial u}{\partial \nu}\Big|_\Gamma = 0, \tag{2}$$

where Δ is the Laplace operator, Ω is a bounded domain in the m-dimensional Euclidean space R^m with Lipschitz boundary Γ, ν is the outward normal to Γ. This problem is a particular case of the equation $\Delta^2 u - a\Delta u + bu = f$ considered in [5] when $a = b = 0$. The approximate solution of Problem (1), (2) is obtained by extrapolation of the solutions of the problems

$$\Delta^2 u_\delta = f(x), \quad x \in \Omega, \tag{3}$$

$$u_\delta|_\Gamma = 0, \ \Big(\delta \Delta u_\delta + \frac{\partial u_\delta}{\partial \nu}\Big)\Big|_\Gamma = 0, \tag{4}$$

where $\delta > 0$ is a small parameter. With the help of the operator B defined on boundary functions v_0 by the formula

$$B v_0 = \frac{\partial u}{\partial \nu}\Big|_\Gamma, \tag{5}$$

where u is found from the problems

$$\begin{aligned} &\Delta v = 0, \ x \in \Omega, \quad v|_\Gamma = v_0, \\ &\Delta u = v, \ x \in \Omega, \quad u|_\Gamma = 0, \end{aligned} \tag{6}$$

the perturbed problem (3)–(4) is reduced to the operator equation

$$(B + \delta I) v_{\delta 0} = F \tag{7}$$

for

$$v_{\delta 0} = \Delta u_\delta|_\Gamma. \tag{8}$$

Here F is given by the formula

$$F = -\frac{\partial u_2}{\partial \nu}\Big|_\Gamma, \tag{9}$$

u_2 being the solution of the problem

$$\begin{aligned} &\Delta v_2 = f, \ x \in \Omega, \quad v_2|_\Gamma = 0, \\ &\Delta u_2 = v_2, \ x \in \Omega, \quad u_2|_\Gamma = 0. \end{aligned} \tag{10}$$

For solving Problem (3)–(4) we use the following iterative process, where for brief the index δ of u_δ is omitted.

Step 1. Given a starting approximation $v_0^{(0)}$, for example,

$$v_0^{(0)} = 0. \tag{11}$$

Step 2. Knowing $v_0^{(k)}$ $(k = 0, 1, ...)$ solve consecutively two problems

$$\begin{aligned} &\Delta v^{(k)} = f,\ x \in \Omega, \qquad v^{(k)}|_\Gamma = v_0^{(k)}, \\ &\Delta u^{(k)} = v^{(k)},\ x \in \Omega, \quad u^{(k)}|_\Gamma = 0. \end{aligned} \tag{12}$$

Step 3. Compute

$$\frac{\partial u^{(k)}}{\partial \nu}\Big|_\Gamma. \tag{13}$$

Step 4. Compute the new approximation

$$v_0^{(k+1)} = v_0^{(k)} - \tau\Big(\frac{\partial u^{(k)}}{\partial \nu}\Big|_\Gamma + \delta v_0^{(k)}\Big), \tag{14}$$

where τ is the iterative parameter, whose optimal value is $2/(2\delta + \|B\|)$.

The above iterative process is a realization of the iterative scheme

$$\frac{v_0^{(k+1)} - v_0^{(k)}}{\tau} + (B + \delta I)v_0^{(k)} = F \tag{15}$$

for Equation (7). In [5] it is proved that this process is convergent with the rate of geometric progression.

3. Problem of Error Accumulation

Suppose that in the performance of the iterative process (11)–(14) at each iteration k knowing the actual approximation $\tilde{v}_0^{(k)}$ of $v_0^{(k)}$ after approximately solving the problems (12) we get $\tilde{v}^{(k)}, \tilde{u}^{(k)}$ instead of $v^{(k)}, u^{(k)}$, that is

$$\begin{aligned} \Delta \tilde{v}^{(k)} = f + \xi^{(k)},\ x \in \Omega, \qquad \tilde{v}^{(k)}|_\Gamma = \tilde{v}_0^{(k)} + \varphi^{(k)}, \\ \Delta \tilde{u}^{(k)} = \tilde{v}^{(k)} + \eta^{(k)},\ x \in \Omega, \qquad \tilde{u}^{(k)}|_\Gamma = \psi^{(k)}. \end{aligned} \tag{16}$$

where $\xi^{(k)}, \varphi^{(k)}, \eta^{(k)}, \psi^{(k)}$ are errors of the approximate method for solving (12).

Furthermore, we assume that these errors do not exceed a prescribed ε, namely,

$$\begin{aligned} \|\xi^{(k)}\|_{L_2(\Omega)} \le \varepsilon, \quad \|\varphi^{(k)}\|_{H^{3/2}(\Gamma)} \le \varepsilon, \\ \|\eta^{(k)}\|_{L_2(\Omega)} \le \varepsilon, \quad \|\psi^{(k)}\|_{H^{3/2}(\Gamma)} \le \varepsilon, \end{aligned} \tag{17}$$

where $H^s(\Gamma)$ is a Sobolev space.

Denote by $\zeta^{(k)}$ the error committed in computing of the derivative $\frac{\partial \tilde{u}^{(k)}}{\partial \nu}$,which also is assumed not exceed ε, i.e.

$$\|\zeta^{(k)}\|_{H^{1/2}(\Gamma)} \le \varepsilon. \tag{18}$$

Therefore, the actual approximation $\tilde{v}_0^{(k+1)}$ of v_0 at the iteration k satisfies the relation

$$\tilde{v}_0^{(k+1)} = \tilde{v}_0^{(k)} - \tau\Big(\frac{\partial \tilde{u}^{(k)}}{\partial \nu} + \delta \tilde{v}_0^{(k)} + \zeta^{(k)}\Big), \tag{19}$$

which is equivalent to

$$\frac{\tilde{v}_0^{(k+1)} - \tilde{v}_0^{(k)}}{\tau} + \frac{\partial \tilde{u}^{(k)}}{\partial \nu} + \delta \tilde{v}_0^{(k)} = -\zeta^{(k)} \tag{20}$$

Next we denote by $\bar{v}^{(k)}$ and $\bar{u}^{(k)}$ the solutions of the problems

$$\begin{aligned} &\Delta \bar{v}^{(k)} = f,\ x \in \Omega, \quad \bar{v}^{(k)}|_\Gamma = \tilde{v}_0^{(k)}, \\ &\Delta \bar{u}^{(k)} = \bar{v}^{(k)},\ x \in \Omega, \quad \bar{u}^{(k)}|_\Gamma = 0. \end{aligned} \tag{21}$$

Then from (16) and (21) we have

$$\begin{aligned} &\Delta(\tilde{v}^{(k)} - \bar{v}^{(k)}) = \xi^{(k)},\ x \in \Omega, \quad (\tilde{v}^{(k)} - \bar{v}^{(k)})|_\Gamma = \varphi^{(k)}, \\ &\Delta(\tilde{u}^{(k)} - \bar{u}^{(k)}) = (\tilde{v}^{(k)} - \bar{v}^{(k)}) + \eta^{(k)},\ x \in \Omega, \quad (\tilde{u}^{(k)} - \bar{u}^{(k)})|_\Gamma = \psi^{(k)}. \end{aligned}$$

Using the estimate for the solution of the above Dirichlet problems [11] and taking into account (17) we obtain

$$\|\tilde{u}^{(k)} - \bar{u}^{(k)}\|_{H^2(\Omega)} \le C_1 \varepsilon,$$

where and hereafter C_i $(i = 1, 2, ...)$ are constants independent of k.

By the well-known imbedding theorem [11]

$$\Big\|\frac{\partial u}{\partial \nu}\Big\|_{H^{1/2}(\Gamma)} \le C_2 \|u\|_{H^2(\Omega)},\ \forall u \in H^2(\Omega)$$

we have

$$\Big\|\frac{\partial \tilde{u}^{(k)}}{\partial \nu} - \frac{\partial \bar{u}^{(k)}}{\partial \nu}\Big\|_{H^{1/2}(\Gamma)} \le C_1 C_2 \varepsilon \tag{22}$$

From the definition of the operator B (5)–(6) and from (21) it follows that

$$B\tilde{v}_0^{(k)} = F + \frac{\partial \bar{u}^{(k)}}{\partial \nu},$$

where F is given by (9)–(10).

Taking into account the above equality from (20) we get

$$\frac{\tilde{v}_0^{(k+1)} - \tilde{v}_0^{(k)}}{\tau} + (B + \delta I)\tilde{v}_0^{(k)} = F + \theta^{(k)}, \tag{23}$$

where

$$\theta^{(k)} = \zeta^{(k)} - \Big(\frac{\partial \tilde{u}^{(k)}}{\partial \nu} - \frac{\partial \bar{u}^{(k)}}{\partial \nu}\Big).$$

Hence, using the estimates (18) and (22) we obtain

$$\|\theta^{(k)}\|_{H^{1/2}(\Gamma)} \le (1 + C_1 C_2)\varepsilon. \tag{24}$$

Now, we shall estimate the deviation of $\tilde{v}_0^{(k+1)}$ from $v_0^{(k+1)}$ exactly computed by the algorithm (11)–(14). For this reason we set

$$z^{(k+1)} = \tilde{v}_0^{(k+1)} - v_0^{(k+1)}. \tag{25}$$

From (15) and (23) it follows that

$$\frac{z^{(k+1)} - z^{(k)}}{\tau} + (B + \delta I)z^{(k)} = \theta^{(k)}.$$

Consequently,

$$z^{(k+1)} = (I - \tau B_\delta)z^{(k)} + \tau\theta^{(k)}, \tag{26}$$

where $B_\delta = B + \delta I$. Since $B = B^* > 0$ (see [5]) we have $B_\delta = B_\delta^* \geq \delta I$. Hence, it is possible to choose τ such that $\|I - \tau B_\delta\| \leq \rho < 1$. Then from (26) it follows that

$$\|z^{(k+1)}\|_{H^{1/2}(\Gamma)} \leq \rho\|z^{(k)}\|_{H^{1/2}(\Gamma)} + C_3\tau\varepsilon$$

Due to $z^{(0)} = 0$ we have

$$\|z^{(k+1)}\|_{H^{1/2}(\Gamma)} \leq \frac{C_3\tau}{1-\rho}\varepsilon \tag{27}$$

From the representation

$$\tilde{v}_0^{(k+1)} - v_0 = v_0^{(k+1)} - v_0 + z^{(k+1)}$$

using (27) we obtain

$$\|v_0^{(k+1)} - v_0\|_{H^{1/2}(\Gamma)} \leq \|v_0^{(k+1)} - v_0\|_{H^{1/2}(\Gamma)} + C_4\varepsilon \tag{28}$$

Next we estimate $\tilde{u}^{(k)} - u$, where u is the exact solution of Problem (3)–(4) and $\tilde{u}^{(k)}$ is the actual approximate solution obtained by the algorithm (11)-(14). From (12) and (16) we have

$$\begin{aligned}
&\Delta(\tilde{v}^{(k)} - v^{(k)}) = \xi^{(k)},\ x \in \Omega, \quad (\tilde{v}^{(k)} - v^{(k)})|_\Gamma = \tilde{v}_0^{(k)} - v_0^{(k)} + \varphi^{(k)},\\
&\Delta(\tilde{u}^{(k)} - u^{(k)}) = (\tilde{v}^{(k)} - v^{(k)}) + \eta^{(k)},\ x \in \Omega, \quad (\tilde{u}^{(k)} - u^{(k)})|_\Gamma = \psi^{(k)}.
\end{aligned}$$

From here and (17) it is easy to get

$$\|\tilde{u}^{(k)} - u^{(k)}\|_{H^2(\Omega)} \leq C_5\|\tilde{v}_0^{(k)} - v_0^{(k)}\|_{H^{1/2}(\Gamma)} + 4C_5\varepsilon$$

Taking into account (25) and (27) we obtain

$$\|\tilde{u}^{(k)} - u^{(k)}\|_{H^2(\Omega)} \leq C_6\varepsilon$$

Hence,

$$\|\tilde{u}^{(k)} - u\|_{H^2(\Omega)} \leq \|\tilde{u}^{(k)} - u^{(k)}\|_{H^2(\Omega)} + \|u^{(k)} - u\|_{H^2(\Omega)} \leq \|u^{(k)} - u\|_{H^2(\Omega)} + C_6\varepsilon$$

Thus, we have proved the following

Theorem. *The errors committed at iterations of the computational process by the algorithm* (11)–(14) *are not accumulated, namely, for the actual approximations* $\tilde{v}_0^{(k)}$ *and* $\tilde{u}^{(k)}$ *we have the estimates*

$$\begin{aligned}
\|\tilde{v}_0^{(k)} - v_0\|_{H^{1/2}(\Gamma)} &\leq \|v_0^{(k)} - v_0\|_{H^{1/2}(\Gamma)} + C_4\varepsilon,\\
\|\tilde{u}^{(k)} - u\|_{H^2(\Omega)} &\leq \|u^{(k)} - u\|_{H^2(\Omega)} + C_6\varepsilon,
\end{aligned}$$

where $v_0^{(k)}$ *and* $u^{(k)}$ *are exactly computed by the algorithm* (11)–(14), u *is the exact solution of Problem* (3) $-$ (4), $v_0 = \Delta u|_\Gamma$, ε *is the error of the method for solving* (12) *and computing* (13) *(see* (17)–(18)*), and* C_4 *and* C_6 *are constants independent of* k *.*

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