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*Survey*

# Some Recent Trends in Calibrated Geometries*

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**Abstract.** This paper is a survey on some modern trends in calibrated geometries such as: computing the maximal directions of a $k$-form; the Cartesian products problem; the classification problem; system of calibrations for Steiner networks.

## 1. Calibrated Geometries

A one-dimensional curve, whose tangent vectors are in a constant direction (i.e. a straightline segment), is the shortest path between two end points. We can see that, the constant direction is just the (unique) direction, on which the dual form attains its maximum.

Generalize this fact to higher dimensional cases, where $k$-dimensional surfaces are considered instead of curves and the sets of all maximal ($k$-dimensional) directions of differential $k$-forms are considered instead of constant directions, one gets a very nice mathematical idea called after the Principle of calibrations:

*Let $M$ be a Riemannian manifold of dimension $n$, $\mathcal{S}$ a compact oriented $k$-dimensional submanifold of $M(k < n)$, $\varphi$ a closed differential $k$-form on $M$, satisfying*

$$\varphi(\xi) \leq 1, \tag{1}$$

*for any unit tangent simple $k$-covector $\xi$ of $M$ and*

$$\varphi(\xi) = 1 \tag{2}$$

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*whenever $\xi$ is tangent to $\mathcal{S}$. Then $\mathcal{S}$ is volume minimizing in its homological class.*

We call $\varphi$ a calibration, $(M, \varphi)$ a calibrated manifold, $\mathcal{S}$ a $\varphi$-submanifold or $\mathcal{S}$ calibrated by $\varphi$. The proof is very simple. Suppose $\mathcal{S}'$ is another submanifold of $M$ in the same homological class as $\mathcal{S}$. Then

$$Vol_k(\mathcal{S}) = \int_{\mathcal{S}} \varphi = \int_{\mathcal{S}'} \varphi \leq Vol_k(\mathcal{S}'),$$

by the equality in (2), by Stokes' Theorem, and by the inequality in (1), respectively.

Suppose $\varphi$ is a $k$-covector on $\mathbb{R}^n$. Denote by

$$\|\varphi\|^* = \sup\{\varphi(\xi) \mid \xi \in G(k, \mathbb{R}^n)\}$$

the comass of $\varphi$, and

$$G(\varphi) = \{\xi \in G(k, \mathbb{R}^n) \mid \varphi(\xi) = \|\varphi\|^*\}$$

the set of all maximal directions of $\varphi$. $G(\varphi)$ is just the intersection of $G(k, \mathbb{R}^n)$ and the hyperplane $\{\xi \in \bigwedge^k(\mathbb{R}^n) \mid \varphi(\xi) = 1\}$. So we also call $G(\varphi)$ the face of $\varphi$, or $\varphi$-Grassmannian.

If $w$ is a differential $k$-form on a Riemannian manifold $M$, then the comass of $w$ is defined by

$$\|w\|^* = \sup\{\|w_x\|^* \mid x \in M\},$$

while the set of all maximal directions (the face) of $w$ is

$$G(w) = \{\xi \in G(k, T_xM) \mid w(\xi) = \|w\|^*\} = \bigcup_{\|w_x\|^* = \|w\|^*} G(w_x).$$

It is easy to see that the conditions (1) and (2) are equivalent to $\|w\|^* = 1$ and $\xi \in G(w)$ respectively.

Calibrated geometries are a mathematical trend developed rapidly in the past twenty years to make good use of Principle of calibrations for studying of globally minimal surfaces on Riemannian manifolds.

Let

$$\Omega_p = \frac{1}{p!}\,\Omega^p = \frac{1}{p!}\,\underbrace{\Omega \wedge \Omega \wedge \ldots \wedge \Omega}_{p},$$

where $\Omega$ is the Kǎhler form on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ defined by

$$\Omega = \sum (\frac{i}{2}) dz_j \wedge d\overline{z_j}.$$

By Wirtinger's inequality $\|\Omega_p\|^* = 1$ and

$$G(\Omega_p) = \{\xi \in G(2p, \mathbb{C}^n = \mathbb{R}^{2n}) \mid \xi \text{ is a complex } p\text{-plane}\}.$$

In 1965, Federer [14] used $\Omega_p$ as calibrations to show that every complex submanifold of $\mathbb{C}^n$ is area-minimizing in its homological class.

In 1972, Berger [1] used suitably normalized powers of the quaternionic form in the same way to prove that, every quaternionic submanifold of a Kăhler quaternionic manifold is area-minimizing.

In 1977, 1978 Thi [66 - 71] used invariant forms on Lie groups and symmetric spaces to investigate the global minimality of Lie subgroups, totally geodesic submanifolds, the Pontryagin cycles.... By using the fundamental 3-form

$$\tau(X, Y, Z) = \frac{1}{2}\langle [X, Y], Z\rangle = -trXYZ$$

as a calibration, he proved that $SU(2)$ is homologically area-minimizing.

In 1982 Lawson and Harvey coined the term " calibration" in their foundational paper "Calibrated Geometries" [29]. They discovered interesting (parallel) calibrations, including

$$\varphi = \mathrm{Re}dZ = \mathrm{Re}dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n$$

on $\mathbb{C}^n \cong \mathbb{R}^{2n}$,

$$\begin{aligned} \Phi_{ASSOC}(x \wedge y \wedge z) &= \langle x, yz\rangle; \\ \Phi_{COASSOC}(x \wedge y \wedge z \wedge w) &= \langle x, [x, y, z]\rangle \\ &= \frac{1}{2}\langle x, (yz)w - y(zw)\rangle \end{aligned}$$

on $\mathrm{Im}\mathbb{O} \cong \mathbb{R}^7$, and

$$\begin{aligned} \Phi_{CAYLEY}(x \wedge y \wedge z \wedge w) &= \langle x, y \times z \times w\rangle \\ &= \frac{1}{2}\langle x, y(\overline{z}w) - w(\overline{z}y)\rangle \end{aligned}$$

on $\mathbb{O} \cong \mathbb{R}^8$, which are called the special Lagrangian, the associative, the coassociative and the Cayley calibrations, respectively.

They proved that $G(\mathrm{Re}dZ)$ is just the $SU(n)$ orbit of $\xi_0$, where $\xi_0$ is the real part of $\mathbb{C}^n \cong \mathbb{R}^n \oplus i\mathbb{R}^n$. So the term special Lagrangian was used because the $U(n)$ orbit of $\xi_0$ is the set of Lagrangian planes.

They also used the terms associative and coassociative because $G(\Phi_{ASSOC}) \cup G(-\Phi_{ASSOC})$ is the set of all $\xi = x \wedge y \wedge z$, where $x, y, z$ are orthonormal vectors satisfying the equality $(xy)z - x(yz) = 0$, and $\Phi_{COASSOC} = *\Phi_{ASSOC}$.

It is easy to see that

$$\Phi_{CAYLEY} = 1^* \wedge \Phi_{ASSOC} + \Phi_{COASSOC},$$

and by choosing a suitably complex structure on $\mathbb{R}^8$, we have

$$\begin{aligned} \Phi_{ASSOC} &= (Je_1)^* \wedge \Omega + \mathrm{Re}dZ, \\ \Phi_{COASSOC} &= (Je_1)^* \wedge \mathrm{Im}dZ - \Omega_2, \end{aligned}$$

At present, calibrated geometries have flourished with many applications to global problems in Riemannian geometry and interest many mathematicians: Morgan, Dadok, Lawlor, Gluck, Mackenzie, Tasaki, Bryant, Haskins, Matessi, McLean, Gang Tian, Joyce, Hitchin, Sema Salur, Van, Dung, Huan, Quang, Binh...

Every well described calibration is often associated with some strong structure of manifolds or of the form itself. The Kắhler and the special Lagrangian calibrations are associated with the complex structure, the associative; the coassociative and the Cayley calibrations are associated with the octonionic structure, the quaternionic calibration is associated with the Kắhler quaternionic structure, while the fundamental 3-form $\tau$ is defined on Lie groups. Some nice calibrations on symmetric spaces, Grassmannian manifolds were discovered in [19 - 20, 64 - 71, 76 78],...

It seems very hard to discover calibrations on a general Riemannian manifold, because of the difficulty of computing the comass and describing the faces of $k$-covectors. Therefore, the application of the Principle of calibrations to discover minimal surfaces runs into obstacles.

Some authors try to enlarge the method of calibrations to the problem of minimizing the functional given by a Lagrangian [12, 13, 71]. In this case, Theorem 3.6 in [71] shows that, the Principle of calibrations, which applies to the case of the (real) currents, is also the necessary condition, i.e. a current $S$ is globally minimal with respect to the integrand $J$ given by a Lagrangian $l$ if and only if it is calibrated by some calibration.

In another direction, some suitable systems of calibrations were applied to soap films, immiscible fluids [17, 58] or to globally minimal Steiner networks [59 - 61, 73].

Very recently, many mathematicians (Bryant, Haskins, Matessi, Joyce, Hit -chin, Sema Salur...) study on special Lagrangian geometries (geometries determined by the special Lagrangian calibrations) [5 - 8, 40, 42 - 52, 63 - 64]. Many new examples on area-minimizing surfaces calibrated by special Lagrangian calibrations have been constructed.

In this paper we want to mention some study trends in calibrated geometries including:

1. Computing the comass and describing the face of a $k$-covector.
2. The Cartesian products problem.
3. The classification problem.
4. System of calibrations and the globally length-minimizing Steiner networks.

## 2. Computing the Comass and Describing the Face of a $k$-Covector

Computing the comass and describing the face of a differential $k$-form $w$ is the first obstacle for using $w$ as a calibration. In order to know the comass and the face of a differential $k$-form $w$, first one needs to know the comass and the face of (the $k$-covector) $w_x$ for every $x$. Even in that case, the computation is still quite difficult.

In this section, we present some recent results on computing the comass and describing the faces of some classes of $k$-covectors. Every method for computing the comass seems to be useful only for some classes of $k$-covectors. The method in [31] is useful for 3-covectors on $\mathbb{R}^7$, the method in Subsec. 2.1 for some classes of 3-covectors in low dimensional spaces, while Theorem 3 can be used for $k$-covectors invariant under transitive actions and Theorem 2 for some cases in general.

### 2.1. A Method for Computing the Comass and Describing the Faces of 3-Covectors

Let $\varphi$ be a $k$-covector on $\mathbb{R}^n$, and $\{e_1, e_2, \dots, e_n\}$ an orthonormal basis of $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$, set

$$\varphi_x = x \lrcorner\ \varphi,$$
$$\overline{\varphi} = (\varphi_{e_1}, \varphi_{e_2}, \dots, \varphi_{e_n}),$$

i.e. $\varphi_x(\eta) = \varphi(x \wedge \eta)$ and $\overline{\varphi}(\eta) = \Sigma \varphi_{e_i}(\eta) e_i$ for every $(k-1)$-vector $\eta$.

We can prove that, $\overline{\varphi}$ does not depend on the chosen basis and

$$\varphi(x_1 \wedge x_2 \wedge \ldots \wedge x_k) = \langle x_1, \overline{\varphi}(x_2 \wedge x_3 \wedge \ldots \wedge x_k)\rangle.$$

Some results concerning $\overline{\varphi}$ are (see [33−34]):

(1) $\|\varphi\|^* = \max\limits_{\eta \in G(k-1, \mathbb{R}^n)} |\overline{\varphi}(\eta)| := A,$

(2) $G(\varphi) = \{\frac{\overline{\varphi}(\eta)}{|\varphi(\eta)|} \wedge \eta \mid \overline{\varphi}(\eta) = A\}.$

(3) If $\xi \in G(\varphi)$, then $\operatorname{span}(\xi)$ is $\overline{\varphi}$-invariant, i.e. for all $x_1, x_2, \dots, x_{k-1} \in \operatorname{span}(\xi)$

$$\overline{\varphi}(x_1 \wedge x_2 \wedge \ldots \wedge x_{k-1}) \in \operatorname{span}(\xi).$$

Suppose $V$ is a $k$-dimensional $\overline{\varphi}$-invariant subspace with the orthonormal basis $\{e_1, e_2, \dots, e_k\}$, we have

$$\overline{\varphi}(e_1 \wedge e_2 \wedge \ldots \wedge \overset{\wedge}{e_i} \wedge \ldots \wedge e_k) = \lambda e_i \quad \text{if } n \text{ is even},$$
$$\overline{\varphi}(e_1 \wedge e_2 \wedge \ldots \wedge \overset{\wedge}{e_i}, \ldots \wedge e_k) = (-1)^{i+1} \lambda e_i \quad \text{if } n \text{ is odd},$$

A computation shows that, the matrix of $\overline{\tau}_X$ corresponding to the basis $\{E_{ij}\}_{i<j}$ is

$$\frac{1}{2}\begin{pmatrix} 0 & -a_4 & -a_5 & a_2 & a_3 & 0 \\ a_4 & 0 & -a_6 & -a_1 & 0 & a_3 \\ a_5 & a_6 & 0 & 0 & -a_1 & -a_2 \\ -a_2 & a_1 & 0 & 0 & -a_6 & a_5 \\ -a_3 & 0 & a_1 & a_6 & 0 & -a_4 \\ 0 & -a_3 & a_2 & -a_5 & a_4 & 0 \end{pmatrix},$$

and

$$\det(\overline{\tau}_X - \lambda I) = \lambda^2(\lambda^2 + f_1(X))(\lambda^2 + f_2(X)),$$

where

$$f_1(X) = \frac{1}{4}\Sigma_i a_i^2 + \frac{1}{2}(a_1 a_6 + a_3 a_4 - a_2 a_5);$$
$$f_2(X) = \frac{1}{4}\Sigma_i a_i^2 - \frac{1}{2}(a_1 a_6 + a_3 a_4 - a_2 a_5).$$

We have

$$\begin{aligned} f_1(X) &= \frac{1}{4}\Sigma_i a_i^2 + \frac{1}{2}(a_1a_6 + a_3a_4 - a_2a_5) \\ &\leq \frac{1}{2}\Sigma_i a_i^2 = \frac{1}{4}. \end{aligned}$$

The equality holds if and only if $a_1 = a_6, a_2 = -a_5, a_3 = a_4$. Therefore, $X$ must be of the form:

$$X = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ -b & -c & 0 & a \\ -c & b & -a & 0 \end{pmatrix}.$$

$$\begin{aligned} f_2(X) &= \frac{1}{4}\Sigma_i a_i^2 - \frac{1}{2}(a_1a_6 + a_3a_4 - a_2a_5) \\ &\leq \frac{1}{2}\Sigma_i a_i^2 = \frac{1}{4}. \end{aligned}$$

The equality holds if and only if $a_1 = -a_6, a_2 = a_5, a_3 = -a_4$. Hence $X$ must be of the form:

$$X = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & -c & b \\ -b & c & 0 & -a \\ -c & -b & a & 0 \end{pmatrix}.$$

Thus,

$$\|\tau\|^* = \frac{1}{2},$$

and

$$G(\tau) = \{L_1 \wedge L_2 \wedge L_3, R_1 \wedge R_2 \wedge R_3\},$$

where

$$L_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad , \quad R_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$L_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad , \quad R_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$L_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad R_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

By using the above result, a computation of the comass of $\tau$ on $SO(n)$ and a description of the face $G(\tau)$ has done successfully (see [33] for more detail).

A computation shows that the comass of $\tau$ is also equal to $1/2$ and $G(\tau)$ is just two $O(n)$-orbits

$$G(\tau) = \{Ad_T(\mathcal{L}_1 \wedge \mathcal{L}_2 \wedge \mathcal{L}_3), Ad_T(\mathcal{R}_1 \wedge \mathcal{R}_2 \wedge \mathcal{R}_3)\},$$

where

$$\mathcal{L}_1 = \begin{pmatrix} L_1 & O \\ O & O \end{pmatrix}, \quad \mathcal{R}_1 = \begin{pmatrix} R_1 & O \\ O & O \end{pmatrix},$$
$$\mathcal{L}_2 = \begin{pmatrix} L_2 & O \\ O & O \end{pmatrix}, \quad \mathcal{R}_2 = \begin{pmatrix} R_2 & O \\ O & O \end{pmatrix},$$
$$\mathcal{L}_3 = \begin{pmatrix} L_3 & O \\ O & O \end{pmatrix}, \quad \mathcal{R}_3 = \begin{pmatrix} R_3 & O \\ O & O \end{pmatrix}.$$

### 2.2. The Slag-Assoc Calibrations of Type $(k, l)$

In this subsection, we construct an operation $[.,.]$ on $\mathbb{H}^n$, which is bilinear, alternating and $\max\limits_{|x|=|y|=1} |[x, y]| = 1,\ x, y \in \mathbb{H}^n$. So, the 3-covector $\langle ., [., .]\rangle$ has comass one. It is called the multi-associative calibration. Restricting these calibrations on suitable subspaces, we get the so-called slag-assoc calibrations of type $(k, l)$. Their faces can be viewed as "a connection" between $k$ special Lagrangian faces and $l$ associative faces in some ways.

Let $\mathbb{H}$ be the quaternionic algebra. For each $X = (x_1, x_2, \ldots, x_n) \in \mathbb{H}^n$ and $Y = (y_1, y_2, \ldots, y_n) \in \mathbb{H}^n$, we define the product $XY$ to be

$$XY = Z = (z_1, z_2, \ldots, z_n),$$

where

$$\begin{aligned} z_1 &= x_1y_1 - \bar{y}_2x_2 - \bar{y}_3x_3 - \ldots - \bar{y}_nx_n, \\ z_2 &= y_2x_1 + x_2\bar{y}_1, \\ z_3 &= y_3x_1 + x_3\bar{y}_1, \\ &\ldots\ldots \\ z_n &= y_nx_1 + x_n\bar{y}_1. \end{aligned}$$

Denote by $1 = (1, 0, \ldots, 0)$ the unit element of $\mathbb{H}^n$. Let $\mathrm{Re}\mathbb{H}^n$ be the span of 1, and $\mathrm{Im}\mathbb{H}^n$ the orthogonal complement of $\mathrm{Re}\mathbb{H}^n$. Then each $X \in \mathbb{H}^n$ has the unique orthogonal decomposition

$$X = X_1 + X',$$

where $X_1 \in \mathrm{Re}\mathbb{H}^n$ and $X' \in \mathrm{Im}\mathbb{H}^n$.

The conjugation $\overline{X}$ of $X$ is defined by

$$\overline{X} = X_1 - X'.$$

Some simple properties concerning the conjugation are:

$$\overline{\overline{X}} = X, \quad \overline{XY} = \overline{Y}\,\overline{X}, \quad X\overline{X} = \overline{X}X = |X|^2,$$
$$\langle X, Y\rangle = \mathrm{Re}X\overline{Y} = \frac{1}{2}(X\overline{Y} + Y\overline{X}).$$

Set

$$[X,Y] = -\frac{1}{2}(\overline{X}Y - \overline{Y}X) = \mathrm{Im}\overline{Y}X.$$

We have

**Theorem 1.**

(1) $[.,.]$ *is bilinear and alternating.*

(2) $[X,Y] \perp X, Y \quad \forall X, Y \in \mathbb{H}^n$.

(3) $\max\limits_{|X|=|Y|=1} |[X,Y]| = 1;\ X, Y \in \mathbb{H}^n$.

Therefore, we get the 3-covector

$$\Phi(X \wedge Y \wedge Z) = \langle X, [Y,Z] \rangle$$

on $\mathrm{Im}\,\mathbb{H}^n$ being a calibration, which is called the multi-associative calibration. Moreover, we have

$G(\Phi) = \{[X,Y] \wedge X \wedge Y \mid X, Y$ satisfying

(i) $\sum\limits_i x_i y_i = 0$,

(ii) $\overline{y}_2 x_2 \Uparrow \overline{y}_3 x_3 \Uparrow \ldots \Uparrow \overline{y}_n x_n$,

(iii) $|x_i y_j| = |x_j y_i| \quad i, j \geq 2, \quad i \neq j\}$,

where $a \Uparrow b$ means that $a = kb$ $(k \geq 0)$ (see [33] for more detail).

Let $V = \mathrm{Im}\mathbb{H} \times \underbrace{\mathrm{Im}\mathbb{H} \times \ldots \times \mathrm{Im}\mathbb{H}}_{k} \times \underbrace{\mathbb{H} \times \ldots \times \mathbb{H}}_{l} \times \{0\} \times \ldots \times \{0\} \cong \mathbb{R}^{3(k+1)+4l}$, then $\Phi_{|V}$ is also a calibration belonging to $F^*(SLAG)$. It is called the slag-assoc calibration of type $(k,l)$. The faces of these calibrations contain many $ASSOC$, $SLAG$, and $CP^k$ faces.

It is easy to see that, the double-slag and the double-assoc calibrations (two calibrations have described by Morgan [57]) are the slag-assoc calibrations of type $(2,0)$, and $(0,2)$, respectively.

### 2.3. Decomposition of a $k$-Covector with Respect to a Given Unit Vector

Suppose $\Phi$ is a $k$-covector on $\mathbb{R}^n$ with

$$\mathrm{span}(\Phi)^* := \{v \in \mathbb{R}^n \mid v \lrcorner\ \Phi = 0\}^\perp = \mathbb{R}^n,$$

and let $e$ be a unit vector on $\mathbb{R}^n$. Set $\varphi = e \lrcorner \Phi$, and $\psi = \Phi - e^* \wedge \varphi$. Then $\Phi$ has the decomposition with respect to $e$

$$\Phi = e^* \wedge \varphi + \psi.$$

Note that $\varphi$ is a $(k-1)$-covector and $\psi$ is a $k$-covector on $e^\perp$. We obtain

**Theorem 2.**

(1) $\|\Phi\|^* = \max\limits_{\eta \in G(k-1, e^\perp)} \{\sqrt{\varphi(\eta)^2 + |\overline{\psi}(\eta)|^2}\} := A.$

(2) $G(\Phi) = \{(\cos\alpha(\eta) e + \sin\alpha(\eta) f(\eta)) \wedge \eta \mid$

$$\eta \in G(k-1, e^\perp),\ \varphi(\eta)^2 + |\overline{\psi}(\eta)|^2 = A^2\},$$

*where* $f(\eta) = \frac{\overline{\psi}(\eta)}{|\overline{\psi}(\eta)|}$, $\cos\alpha(\eta) = \frac{\varphi(\eta)}{A}, \sin\alpha(\eta) = \frac{|\overline{\psi}(\eta)|}{A}$.

*Proof.* Suppose $\xi \in G(k, \mathbb{R}^n)$ has the canonical form with respect to the subspace $\mathrm{span}(e) = \{r.e \mid r \in \mathbb{R}\}$

$$\xi = (\cos\alpha e + \sin\alpha f) \wedge \eta,$$

where $e, f$ are orthonormal vectors; $\eta \in G(k-1, e^{\perp})$; $e \in \mathrm{span}(\eta)^{\perp}$; $f \in \mathrm{span}(\eta)^{\perp}$. Then

$$\begin{aligned}\Phi(\eta) &= \cos\alpha\varphi(\eta) + \sin\alpha\psi(f \wedge \eta)\\ &\leq \sqrt{\cos^2\alpha + \sin^2\alpha}.\sqrt{\varphi(\eta)^2 + \psi(f \wedge \eta)^2}\\ &\leq \sqrt{\varphi(\eta)^2 + |\overline{\psi}(\eta)|^2}.\end{aligned}$$

Therefore,

$$\|\Phi\|^* \leq \max_{\eta \in G(k-1, e^{\perp})} \{\sqrt{\varphi(\eta)^2 + |\overline{\psi}(\eta)|^2}\} = A.$$

Now suppose the equality $\sqrt{\varphi(\eta)^2 + |\overline{\psi}(\eta)|^2} = A$ holds for some $\eta \in G(k-1, e^{\perp})$. Let

$$f(\eta) = \frac{\overline{\psi}(\eta)}{|\overline{\psi}(\eta)|}, \quad \cos\alpha(\eta) = \frac{\varphi(\eta)}{A}, \quad \sin\alpha(\eta) = \frac{|\overline{\psi}(\eta)|}{A},$$

and

$$\xi = (\cos\alpha(\eta)e + \sin\alpha(\eta)f(\eta)) \wedge \eta.$$

We have

$$\begin{aligned}\Phi(\xi) &= \cos\alpha(\eta)\varphi(\eta) + \sin\alpha(\eta)|\overline{\psi}(\eta)|\\ &= \frac{\varphi(\eta)^2 + |\overline{\psi}(\eta)|^2}{A} = \frac{A^2}{A} = A.\end{aligned}$$

The first part is proved.

The proof of the second part is clear.

By using this theorem, we can compute the comass and describe the faces of many well-known calibrations including the special Lagrangian, the associative, the coassociative, the Cayley.... The computation here depends only on their any expression in terms of axis planes, but does not depend on the (complex or octionionic) structure associated with them.

*Remark.* When consider the $k$-covector of the following form

$$\Phi = e_1^* \wedge \varphi + e_2^* \wedge \psi,$$

where $\varphi, \psi$ are $(k-1)$-covector on $(e_1, e_2)^{\perp}$, Binh has showed that (by a similary proof as the proof of Theorem 2)

$$\|\Phi\|^* = \max_{\eta \in G(k-1, \mathbb{R}^n)} \sqrt{\varphi^2(\eta) + \psi^2(\eta)} := A,$$

and

$$G(\Phi) \supset \{(\cos\alpha e_1 + \sin\alpha e_2) \wedge \eta \mid \sqrt{\varphi^2(\eta) + \psi^2(\eta)} = \|\Phi\|^*\},$$

where

$$f(\eta) = \frac{\psi(\eta)}{|\psi(\eta)|}, \quad \cos\alpha(\eta) = \frac{\varphi(\eta)}{A}, \quad \sin\alpha(\eta) = \frac{|\psi(\eta)|}{A}.$$

2.4. General Associative and General Coassociative Calibrations.

Let $\{e_1, Je_1, e_2, Je_2, \ldots, e_{2n}, Je_{2n}\}$ be an orthonormal basis on $\mathbb{C}^{2n}$ corresponding to the complex structure $J$. The subspace $(e_n, Je_n)^\perp \simeq \mathbb{C}^{2n-1}$ inherits the induced complex structure, which also denote by $J$.

Let

$$\mathrm{Re}dZ = \mathrm{Re}(dz_1 \wedge dz_2 \wedge \ldots \wedge dz_{2n-1}),$$

and

$$\Omega_p = \frac{1}{(p)!}\Omega^p,$$

be the special Lagrangian calibration and the exterior power of the Kăhler form on $(e_n, Je_n)^\perp$, respectively.

By virtue of Theorem 6.11 in [29], we have

$$|\Omega_{n-1}(\eta)|^2 + |\overline{\mathrm{Re}(dZ)}(\eta)|^2 = |\Omega_{n-1}(\eta)|^2 + \sum_{|I|=2(n-1)} |dZ^I(\eta)|^2 \leq |\eta|^2 = 1,$$

and

$$|\mathrm{Im}dZ(\eta)|^2 + |\overline{\Omega_n}(\eta)|^2 \leq |\eta|^2 = 1.$$

Therefore, by virtue of Theorem 2 the $(2n-1)$-covector

$$\Phi_{GASSOC} = (Je_n)^* \wedge \Omega_{n-1} + \mathrm{Re}dZ,$$

and the $(2n)$-covector

$$\Phi_{GCOASSOC} = (Je_n)^* \wedge \mathrm{Im}dZ + \Omega_n,$$

on $\mathrm{span}(Je_n) \oplus \mathbb{C}^{2n-1} \simeq \mathbb{R}^{4n-1}$ have both comass one, i. e. calibrations. They are called the general associative and general coassociative calibrations, respectively.

We can see that:

(1) $G(\Phi_{GASSOC})$ is the set of all $\xi = (\cos\alpha(\eta)Je_n + \sin\alpha(\eta)f(\eta)) \wedge \eta$, where

$$\sum_{k=1}^{n-1} \sum_{|I|=2k}{}' |dZ^I \wedge \Omega_{n-k}(\eta)|^2 = 0,$$

$$f(\eta) = \frac{\overline{\mathrm{Re}\, dZ}(\eta)}{|\overline{\mathrm{Re}\, dZ}(\eta)|}; \quad \cos\alpha(\eta) = \Omega_{n-1}(\eta); \quad \sin\alpha(\eta) = |\overline{\mathrm{Re}\, dZ}(\eta)|.$$

(2) $G(\Phi_{GCOASSOC})$ is the set of all $\xi = (\cos\alpha(\eta)Je_n + \sin\alpha(\eta)f)\wedge\eta$, where

$$|\mathrm{Re}\, dZ(\eta)|^2 = 0; \quad \sum_{k=1}^{2(n-1)} \sum_{I=2k+1} |dZ^I \wedge \Omega_{2k+1}(\eta)|^2 = 0,$$

$$f(\eta) = \frac{\overline{\Omega_n}(\eta)}{|\overline{\Omega_n}(\eta)|}, \quad \cos\alpha(\eta) = \mathrm{Im}\, dZ(\eta); \quad \sin\alpha(\eta) = |\overline{\Omega_n}(\eta)|.$$

(3) When $n = 4$, the general associative and general coassociative calibrations are just the associative and coassociative calibrations, respectively.

2.5. Calibrations Invariant Under Transitive Actions

Let $\Phi$ be a $k$-covector on $\mathbb{R}^n$. Denote by

$$G_1(\Phi) = \bigcup_{\xi\in G(\Phi)} \mathrm{span}(\xi).$$

Suppose $\Phi$ has the decomposition with respect to the unit vector $e$

$$\Phi = e^* \wedge \varphi + \psi,$$

where $\varphi$ is a $(k-1)$-form and $\psi$ is a $k$-form on $e^\perp$.

**Theorem 3.**

$$\|\Phi\|^* = \|\varphi\|^* \quad \textit{if and only if} \quad e \in G_1(\Phi).$$
$$G(\Phi) = \bigcup_{\substack{e\in G_1(\Phi)\\ |e|=1}} (e \wedge G(\varphi)).$$

*Proof.* Lemma 2.3 in [34] shows that

$$\|\Phi\|^* \geq \max\{\|\varphi\|^*, \|\psi\|^*\}.$$

If $e \in G_1(\Phi)$, then there exists $\xi \in G(\Phi)$ such that $e \in \mathrm{span}(\xi)$ and hence $\xi = e \wedge \eta$, where $\eta \in G(k-1, e^\perp)$. We have

$$\|\Phi\|^* = \Phi(e\wedge\eta) = \varphi(\eta) \leq \|\varphi\|^*,$$

and therefore the equality holds. Conversely, if $\|\Phi\|^* = \|\varphi\|^*$, let $\eta \in G(\varphi)$ and $\xi = e\wedge\eta$ then

$$\Phi(\xi) = \varphi(\eta) = \|\varphi\|^* = \|\Phi\|^*.$$

Thus, $\xi = e\wedge\eta \in G(\Phi)$ and hence $e \in G_1(\Phi)$.

The proof of the second part is clear.

We consider below a simple application of Theorem 3, where the $k$-covector $\Phi$ is assumed invariant under a transitive action.

Let $\mathcal{G}$ be a subgroup of $SO(n)$, which acts transitively on $\mathbf{S}^{n-1}$ and $\Phi$ be a $\mathcal{G}$-invariant $k$-covector on $\mathbb{R}^n$

Since $\Phi$ is a $\mathcal{G}$-invariant form, $\mathcal{G}$ acts on both $G(\Phi)$ and $G_1(\Phi)$. For every $u \in \mathbf{S}^{n-1}$ and unit vector $v \in G_1(\Phi)$, there exists $g \in \mathcal{G}$ such that $g(v) = u$ (since $\mathcal{G}$ acts transitively on $\mathbf{S}^{n-1}$). This implies that $u \in G_1(\Phi)$ and hence

$$G_1(\Phi) = \mathbb{R}^n.$$

Then by Theorem 3

$$\|\Phi\|^* = \|\varphi\|^*, \quad \text{for every } e \in \mathbf{S}^{n-1},$$

and

$$G(\Phi) = \bigcup_{e \in \mathbf{S}^{n-1}} (e \wedge G(\varphi)) = \{g(e \wedge G(\varphi)) \mid \ g \in \mathcal{G}\}.$$

By using this fact, we can compute the comass and describe the face of some $SU(n)$-invariant $k$-covectors (including Kắhler, exterior power of the Kắhler, special Lagrangian, Cayley...) easily (see [35]).

2.6. Span of a 3-covector

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ and $\varphi \in \bigwedge^k(\mathbb{R}^n)^*$. Then $\varphi$ is expressed in terms of $\frac{n(n-1)(n-2)}{6}$ axis 3-planes. The variables may be unnecessary, so it is very convenient for computing the comass of $\varphi$ if we eliminate unnecessary variables. A subspace $V^*$ is said to envelope $\varphi$ if $\varphi \in \bigwedge^k(V)^* \subseteq \bigwedge^k(\mathbb{R}^n)^*$. It is easy to see that span$\varphi = \{\eta \lrcorner\ \varphi \mid \eta \in \bigwedge^{k-1}(\mathbb{R}^n)\}$, is a unique minimal subspace which envelopes $\varphi$ and $(\text{span}\varphi)^* = \{v \in \mathbb{R}^n \mid v \lrcorner\ \varphi = 0\}^\perp$.

For every $\varphi \in \bigwedge^2(\mathbb{R}^n)^*$ we can choose a suitable orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$, such that $\varphi$ can be written in the following canonical form:

$$\lambda_1 e_1^* \wedge e_2^* + \lambda_2 e_3^* \wedge e_4^* + \ldots + \lambda_k e_{2m-1}^* \wedge e_{2m}^*,$$

where $2m \leq n$ and $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_m > 0$.

We can see that $\|\varphi\|^* = \lambda_1, G(\varphi) = \mathbb{CP}^k$, where $k$ is the largest index such that $\lambda_1 = \lambda_2 = \ldots = \lambda_k$ and span$\varphi = (\mathbb{R}^{2m})^* = \text{span}(e_1, e_2, \ldots, e_{2m})^*$.

For $k$-covector $\varphi$, it is very difficult to determine the minimal subspace which envelopes it and to choose an orthonormal basis, so that $\varphi$ can be expressed in the simplest form. Theorem 4 below gives us a method for determining span$\varphi$ when $\varphi$ is a 3-covector. For $\varphi$ being a 3-covector and $x \in \mathbb{R}^n$, let ad be the linear mapping from $\mathbb{R}^n$ to $L(\mathbb{R}^n, \mathbb{R}^n)$ [33] defined as follows:

$$\text{ad}(x) = \overline{\varphi}_x.$$

**Theorem 4.**

$$(\text{kerad})^\perp = (\text{span}\varphi)^* = \{v \in \mathbb{R}^n \mid v \lrcorner\ \varphi = 0\}^\perp.$$

*Proof.* Suppose that $\{e_1, e_2, ..., e_n\}$ is an orthonormal basis on $\mathbb{R}^n$, then

$$\begin{aligned} \forall x \in \text{kerad} \Leftrightarrow \text{ad}_x = 0 &\Leftrightarrow [x, e_j] = 0, \forall j = 1, ..., n \\ &\Leftrightarrow \Sigma_{i=1}^n (e_i \lrcorner\ \varphi)(x, e_j) e_i = 0, \quad \forall j = 1, ..., n \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \varphi(e_i, x, e_j) = 0, \quad \forall i, j = 1, ..., n \\
&\Leftrightarrow -\varphi(x, e_i, e_j) = 0, \quad \forall i, j = 1, ..., n \\
&\Leftrightarrow \varphi(x, e_i, e_j) = 0, \quad \forall i, j = 1, ..., n \\
&\Leftrightarrow x \lrcorner\ \varphi(e_i, e_j) = 0, \quad \forall i, j = 1, ..., n \\
&\Leftrightarrow x \in \{v \in \mathbb{R}^n \mid v \lrcorner\ \varphi = 0\}.
\end{aligned}$$

Thus, $\text{kerad} = \{v \in \mathbb{R}^n \mid v \lrcorner\ \varphi = 0\}$, or $(\text{kerad})^\perp = \{v \in \mathbb{R}^n \mid v \ \lrcorner\ \varphi = 0\}^\perp$. The following is an application of Theorem 4.

**Conclusion 5.** $\varphi \in \bigwedge^3(\mathbb{R}^n)^*$ *is simple if and only if* $\dim \text{kerad} = n - 3$.

*Examples.* Direct computations show that:
1. If $\varphi = e^*_{123} + e^*_{456} (\in \lambda^3(\mathbb{R}^6))$, then $\text{span}\varphi = (\mathbb{R}^6)^*$ and hence $\varphi$ is not simple.
2. If $\varphi = 2e^*_{125} + e^*_{127} - 2e^*_{135} - e^*_{137} - 2e^*_{245} - e^*_{247} + 2e^*_{345} + e^*_{347} (\in \Lambda^3(\mathbb{R}^7))$, then $\text{span}\varphi$ has dimension 4. Thus, $\varphi$ is simple.
3. If $\varphi = 2e^*_{123} + e^*_{128} + 2e^*_{167} - e^*_{168} - e^*_{257} + e^*_{258} + e^*_{234} + e^*_{247} - e^*_{248} - e^*_{346} + 2e^*_{356} + e^*_{467} - e^*_{468} + e^*_{568} (\in \bigwedge^3(\mathbb{R}^8))$, then $\text{span}\varphi$ has dimension 6 and $\varphi$ can be expressed in the following form:

$$\varphi = 2\sqrt{2}\, \beta^*_{123} + 3\sqrt{2}\, \beta^*_{456},$$

where

$$\begin{aligned}
&\beta_1 = \frac{1}{\sqrt{2}}(1, 0, 0, 0, 1, 0, 0, 0), \quad \beta_2 = \frac{1}{\sqrt{2}}(0, 1, 0, 0, 0, 1, 0, 0), \\
&\beta_3 = \frac{1}{\sqrt{2}}(0, 0, 1, 0, 0, 0, 1, 0), \quad \beta_4 = \frac{1}{\sqrt{3}}(1, 0, 0, 1, -1, 0, 0, 0), \\
&\beta_5 = \frac{1}{\sqrt{2}}(0, 1, 0, 0, 0, -1, 0, 0), \quad \beta_6 = \frac{1}{\sqrt{3}}(0, 0, 1, 0, 0, 0, -1, 1).
\end{aligned}$$

## 3. Cartesian Products Problem [54]

Let $\varphi \in \bigwedge^k(\mathbb{R}^n)^*$ and $\psi \in \bigwedge^l(\mathbb{R}^m)^*$. Consider $\varphi \wedge \psi \in \bigwedge^{k+l}(\mathbb{R}^{n+m})^*$. The following problem was posed by Federer:

*Does the equality* $\|\varphi \wedge \psi\|^* = \|\varphi\|^* \|\psi\|^*$ *hold?*

If the equality holds, then it is sufficient (in the class of normal currents of geometric measure theory with the area replaced by mass) to give an affirmative answer to the following question:

*Is the Cartesian product of two area-minimizing surfaces area-minimizing?*

The inequality $\|\varphi \wedge \psi\|^* \geq \|\varphi\|^* \|\psi\|^*$ is trivial, but the equality has only been proved for a few cases.

Federer proved the equality when $\varphi$ or $\psi$ is simple and Morgan proved the equality when $l = 2$ or $k = 2, k = l = 3$ and $n - k = l - m = 3$ (for more detail see [14, 54]).

Therefore, as conclusions of these results, one gets:

*The Catersian product of two area-minimizing surfaces is area-minimizing when one of the following is true*

1. *one of the surfaces is a $k$-plane,*
2. *one of the surfaces is of dimension or codimension at most two,*
3. *both surfaces are of dimension or codimension three.*

Below are some more results on this problem. Huan [41] proved the equality for the case $\varphi$ or $\psi$ is simply separable and Binh [2] for the case when one of factors is a torus form.

### 3.1. The Case When a Factor is Simply Separable

Let $e_1, e_2$ be unit vectors on $\mathbb{R}^n$, and let $\Phi$ be a $k$-covector of the following form

$$\Phi = e_1^* \wedge e_2^* \wedge \varphi + \psi,$$

where $\varphi$ is a $(k-2)$-covector and $\psi$ is a $k$-covector on $\mathbb{R}^{n-2} = \mathrm{span}(e_1, e_2)^\perp$.

A result of Dadok, Harvey and Morgan [11, Lemma 2.1] shows that

$$\|\Phi\|^* = \max\{\|\varphi\|^*, \|\psi\|^*\}.$$

Moreover, each $\xi \in G(\Phi)$ is of the following form

$$\xi = (\cos a e_1 + \sin a f_1) \wedge (\cos a e_2 + \sin a f_2) \wedge \eta,$$

where $f_1, f_2$ are orthnormal vectors in the orthogonal complement of $\mathrm{span}(e_1, e_2)$, and one of the following holds:
(1) $a = 0$ and $\varphi(\eta) = \|\varphi\|^* \geq \|\psi\|^*$,
(2) $a = \frac{\pi}{2}$ and $\psi(f_1 \wedge f_2 \wedge \eta) = \|\psi\|^* \geq \|\varphi\|^*$,
(3) $0 < a < \frac{\pi}{2}$ and $\varphi(\eta) = \psi(f_1 \wedge f_2 \wedge \eta) = \|\varphi\|^* = \|\psi\|^*$.

*Remark.* We can see that:
- If $\|\varphi\|^* > \|\psi\|^*$ $(a = 0)$, then $G(\Phi) = G(\varphi)$,
- If $\|\varphi\|^* < \|\psi\|^*$ $(a = \frac{\pi}{2})$, then $G(\Phi) = G(\psi)$,
- If $\|\varphi\|^* = \|\psi\|^*$ $(a \in [0, \frac{\pi}{2}])$, then $G(\Phi) = G(\varphi) \cup G(\psi) \cup A$, where $A$ is the set of all $\xi = (\cos a e_1 + \sin a f_1) \wedge (\cos a e_2 + \sin a f_2) \wedge \eta, 0 < a < \frac{\pi}{2}$, $\eta \in G(\varphi)$, and $f_1 \wedge f_2 \wedge \eta \in G(\psi)$.

A result of Huan [41, Theorem 3.2] shows that $A$ may be empty. For example, if $\varphi$ is of the form $\varphi = e_3^* \wedge \varphi'$, where $\varphi' \in \bigwedge^{k-3}(\mathrm{span}(e_1, e_2, e_3))^\perp$.

**Theorem 6.** [41, Theorem 3.10] *Let $\varphi$ be a simply separable $k$-covector with respect to $(V_1, V_2, \ldots, V_r)$ on $\mathbb{R}^n$, and $\psi$ a $l$-covector on $\mathbb{R}^m$. Consider $\varphi \wedge \psi \in \bigwedge^{k+l}(\mathbb{R}^{n+m})^*$, we have*

$$\|\varphi \wedge \psi\|^* = \|\varphi\|^* \|\psi\|^*,$$

*and*

$$G(\varphi \wedge \psi) = G(\varphi) \wedge G(\psi).$$

*Proof.* (For more detail, see [41])

If $r = 1$, then $\varphi$ is simple. The theorem is proved by a result of Federer [14].

Suppose the theorem is proved for $r = p-1$. We shall prove that the theorem is true for $r = p$. In fact, $\varphi$ can be expressed in the following form

$$\varphi = e_{V_1} \wedge \varphi_1 + \varphi_2,$$

where $\varphi_1, \varphi_2$ are simply separable covectors on $V_1^{\perp}$. Therefore

$$\|\varphi\| = \max\{\|\varphi_1\|, \|\varphi_2\|\},$$

and

$$\varphi \wedge \psi = e_{V_1} \wedge \varphi_1 \wedge \psi + \varphi_2 \wedge \psi.$$

By Theorem 3.2 in [41], we have

$$\begin{aligned}
\|\varphi \wedge \psi\|^* &= \max\{\|\varphi_1 \wedge \psi\|^*, \|\varphi_2 \wedge \psi\|^*\} \\
&= \max\{\|\varphi_1\|^*\|\psi\|^*, \|\varphi_2\|^*\|\psi\|^*\} \\
&= \max\{\|\varphi_1\|^*, \|\varphi_2\|^*\}\|\psi\|^* \\
&= \|\varphi\|^*\|\psi\|^*.
\end{aligned}$$

The proof of the second part is clear.

*Remark.* The complex line forms are simply separable, and the exterior powers of a Kắhler form are complex line forms, so the theorem also holds for $\varphi$ being a complex line form or a exterior power of a Kắhler form.

As a conlusion of the above result, we obtain:

**Conclusion 7.** *The Catersian product of two area-minimizing surfaces is area-minimizing when one of the surfaces is a complex surface.*

3.2. The Case When a Factor is Torus

Consider a $k$-covector $\varphi$ of the form

$$\Phi = e_1^* \wedge \varphi_1 + e_2^* \wedge \varphi_2,$$

where $\varphi_1$ and $\varphi_2$ are $(k-1)$-covector on $\text{span}(e_1, e_2)^{\perp}$. The such forms have been studied by Morgan [11] and Binh [2]. We can see that $\Phi$ is invariant under the action of $SO(2)$ on $\text{span}(e_1, e_2)$, the intersection of $G_1(\Phi)$ and $\text{span}(e_1, e_2)$ is not empty and hence

$$\|\Phi\| = \max_{\substack{v \in \text{span}(e_1, e_2) \\ |v|=1}} \|v \lrcorner\, \Phi\|^*.$$

**Theorem 8.** [2, Theorem 2.5] *Let $\varphi$ be a torus $k$-covector on $\mathbb{R}^{2k}$, and $\psi$ a $l$-covector on $\mathbb{R}^m$. Consider $\varphi \wedge \psi \in \bigwedge^{k+l}(\mathbb{R}^{2k+m})^*$, we have*

$$\begin{aligned}
&\|\varphi \wedge \psi\|^* = \|\varphi\|^*\|\psi\|^*, \\
&G(\varphi \wedge \psi) \supset G(\varphi) \wedge G(\psi).
\end{aligned}$$

*Remark.* The special Lagrangian calibrations are torus forms, so we get

**Conclusion 9.** *The Catersian product of two area-minimizing surfaces is area-minimizing when one of the surfaces is special Lagrangian.*

3.3. An Approach to Cartesian Product Problem

In this subsection we introduce an approach to Cartesian product problem by using the Reduction principle (see [31]) and the decomposition a $k$-covector with respect to a unit vector (see Sec. 2.3).

First we have a result by using the decomposition a $k$-covector with respect to a unit vector.

**Theorem 10.** *Let* $\varphi \in \bigwedge^3(\mathbb{R}^6)^*$ *and* $\psi \in \bigwedge^k(\mathbb{R}^n)^*$*, then we have:*

$$\|\varphi \wedge \psi\|^* = \|\varphi\|^* \|\psi\|^*.$$

*Proof.* If $k = 1, 2, 3$, then Theorem 3 holds by results of Morgan [54].

If $k \geq 4$, let $\xi \in G(\varphi \wedge \psi)$, then we have span$\xi \cap \mathbb{R}^n \neq \{0\}$. Therefore, there exists a unit vector $e \in$ span$\xi \cap \mathbb{R}^n$.

By vitue of Theorem 3, we get

$$\|\varphi \wedge \psi\|^* = \|e\lrcorner\, \varphi \wedge \psi\|^* = \|\varphi \wedge (e\lrcorner\, \psi)\|^* = \|\varphi\| \|e\lrcorner\, \psi\|^* \leq \|\varphi\|^* \|\psi\|^*,$$

and hence

$$\|\varphi \wedge \psi\|^* = \|\varphi\|^* \|\psi\|^*.$$

**Lemma 11.** (The Reduction principle [31, 2.1]) *Let* $\Phi \in \bigwedge^k(\mathbb{R}^n)^*$*, and* $L$ *be a* $(k+1)$*-dimensional linear subspace of* $\mathbb{R}^n$*. For* $x \in \mathbb{R}^n$*, let* $\Phi[x]$ *denote the restriction of* $\Phi$ *to span*$(x)^\perp$*. Then* $\|\Phi\|^* \leq 1$ *if and only if*

$$\|\Phi[x]\|^* \leq 1, \quad \textit{for all } x \in L.$$

*If* $\|\Phi\|^* = 1$*, then*

$$G(\Phi) = \bigcup \{G(\Phi[x]) \mid \|\Phi[x]\|^* = 1, x \in L\}.$$

Let $\varphi \in \bigwedge^k(\mathbb{R}^n)^*$ and $\psi \in \bigwedge^l(\mathbb{R}^m)^*$. We consider the following cases:

Case 1. $k + l > n$ (or $k + l > m$). In this case, for $\xi \in G(\varphi \wedge \psi)$, we have span $\xi \cap \mathbb{R}^m \neq \{0\}$.

Therefore, there exists a unit vector $e \in \mathbb{R}^m$, such that

$$\|\varphi \wedge \psi\|^* = \|\varphi \wedge (e\lrcorner\, \psi)\|^*.$$

Note that, $\varphi \wedge (e\lrcorner\, \psi) \in \bigwedge^{k+l-1}(\mathbb{R}^{n+m})^*$.

Case 2. $k + l < n$ (or $k + l < m$). By virtue of the Reduction principle, there exists a unit vector $e \in \mathbb{R}^n$, such that

$$\|\varphi \wedge \psi\|^* = \|\varphi\|^* \|\psi[e]\|^*.$$

Note that, $\varphi \wedge (\psi[e]) \in \bigwedge^{k+l}(\mathbb{R}^{n+m-1})^*$.

The above facts lead us to the following theorem:

**Theorem 12.** *If the equality $\|\varphi \wedge \psi\|^* = \|\varphi\|^*\|\psi\|^*$ holds for the case $n = m$ and $k + l = n$ (i.e. $\varphi \wedge \psi$ is an $n$-covector on $\mathbb{R}^{2n}$), then the equality holds for all $n, m, k$ and $l$.*

*Proof.* The proof is omitted.

*Remark.*
1. Let $\varphi \in \bigwedge^k(\mathbb{R}^n)^*$ and $\psi \in \bigwedge^l(\mathbb{R}^m)^*$. If $\|\varphi \wedge \psi\|^* = \|\varphi\|^*\| \star \psi\|^*$, then the equality $\|\varphi \wedge \psi\|^* = \|\varphi\|^*\|\psi\|^*$ holds for $k = 3, n = 7, l = 4, m = 7$. Therefore it holds for $k = 3, n = 7, l = 4, m > 7$ by the reduction principle and then for $k = 3, n = 7, l > 4, m > 7$ by the decomposition a $k$-covector with respect to a unit vector. So we address the question:

*"Does the equality $\|\varphi \wedge \psi\|^* = \|\varphi\|^*\| \star \psi\|^*$ hold?"*

2. If $G_1(\varphi \wedge \psi) \cap \mathbb{R}^n \neq \emptyset$ or $G_1(\varphi \wedge \psi) \cap \mathbb{R}^m \neq \emptyset$ for all $k, n, l, m$ , then the equality $\|\varphi \wedge \psi\|^* = \|\varphi\|^*\|\psi\|^*$ holds for every $k, n, l, m$ by the decomposition a $k$-covector with respect to a unit vector. So we address the question:

*"Can we prove $G_1(\varphi \wedge \psi) \cap \mathbb{R}^n \neq \emptyset$ or $G_1(\varphi \wedge \psi) \cap \mathbb{R}^m \neq \emptyset$ for all $k, n, l, m$?"*

3. Let $\varphi \in \bigwedge^k(\mathbb{R}^n)^*$ and $\psi \in \bigwedge^l(\mathbb{R}^m)^*$, where $n = m, k+l = n$ and $\xi \in G(\varphi\wedge\psi)$, and let $\pi : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^n$ denote the orthogonal projection. Consider the bilinear form on span $\xi$ defined by $B(u, v) = \langle \pi(u), \pi(v)\rangle$. If $B$ has an eigenvalue 1, then we can see that $G_1(\varphi\wedge\psi)\cap\mathbb{R}^n \neq \emptyset$. Therefore the equality $\|\varphi\wedge\psi\|^* = \|\varphi\|^*\|\psi\|^*$ holds in this case. We hope that:

*"B has at least an eigenvalue 1".*

## 4. Classification Problem

### 4.1. Classification Problem

The exterior algebra $\bigwedge^k(\mathbb{R}^n)^*$ of $k$-covectors is an Euclidean space of dimension $C_n^k = \frac{n!}{k!(n-k)}$. By identifying the $k$-plane of oriented orthonormal basis $\{e_1, e_2, \ldots , e_k\}$ with the $k$-vector $e_1 \wedge e_2 \wedge \ldots \wedge e_k$, the Grassmannian $G(k, \mathbb{R}^n)$ may be viewed as a submanifold (of dimension $k(n - k)$) of the unit sphere in $\bigwedge^k(\mathbb{R}^n)^*$.

A face of $G(k, \mathbb{R}^n)$ is the set of the points of contact with a supporting hyperplane in $\bigwedge^k(\mathbb{R}^n)$. We can see that a face of $G(k, \mathbb{R}^n)$ is just the set of all maximal directions of a $k$-covector of comass one, i.e. a calibration. Thus by Principle of calibration, a face of $G(k, \mathbb{R}^n)$ defines a Calibrated geometry of area-minimizing surfaces.

A face of a single point ($k$-vector) associated with the geometry consists merely of portion of $k$-planes defined that point. But large faces produce rich Calibrated geometries.

If $G(\varphi) \subset G(\psi)$, then we say that the Calibrated geometry associated with $\psi$ is richer than the one associated with $\varphi$ or $\psi$ is more effective than $\varphi$.

Often the area-minimizing surfaces are compact portions of sum of $k$-planes (viewed as area-minimizing intergral current). But that case is of interest in the study of singularities in $k$-dimensional area-minimizing intergral currents. The problem of classifying the faces of $G(k, \mathbb{R}^n)$ has been solved in a few cases:

When $k = 1$ or $k = n - 1$, $G(k, \mathbb{R}^n)$ is just the unit sphere of $\mathbb{R}^n$ and the faces are all single points.

When $k = 2$, by classiacal canonical form for an alternating 2-form, we can see that, each face of $G(2, \mathbb{R}^n)$ contains 2-planes which are complex lines in some $2m$-dimensional subspace (under some suitable complex structure) of $\mathbb{R}^n$ . A complementary classification holds when $k = n - 2$.

In 1982, the faces of $G(3, \mathbb{R}^6)$ were identified by the works of Dadok, Harvey and Morgan $[10, 30, 54]$. They are: singletons, doubletons, $\mathbb{CP}^1$'s and 5-dimensional submanifolds of special Lagrangian planes.

In 1986, Harvey and Morgan [31] classified the faces of 12-dimensional manifold $G(3, \mathbb{R}^7)$. There are ten types: five discrete types and five infinite families of types of faces. The faces of $G(3, \mathbb{R}^7)$ are: associative, special Lagrangian, $\mathbb{CP}^2$, $\mathbb{CP}^1$, double $\mathbb{CP}^1$, singleton, doubleton, $S^3, S^2$, and $S^1$.

For the case $k = 4, n = 8$, many faces have been identified by the work of Dadok, Harvey and Morgan [11].

The problem seems to be more difficult in the next cases, because in which the dimension of $\bigwedge^k(\mathbb{R}^n)^*$ is very large. A less complicated problem is:

*"What are all calibrations, whose faces contain the face of a given calibration."*

This problem is equivalent to the classification problem when the given calibration is simple.

In the next subsection we consider this problem for the given calibration being special Lagrangian on $\mathbb{R}^8$.

## 4.2. $F^*(SLAG)$ on $\mathbb{R}^8$

Let $F^*(SLAG) = \{\Phi \in \bigwedge^3(\mathbb{R}^8)^* \mid \|\Phi\|^* = 1 \text{ and } G(\Phi_{SLAG}) \subset G(\Phi)\}$. The first Cousin principle shows that each $\Phi \in F^*(SLAG)$ must be of the following form

$$\Phi(\lambda, a) = \Phi_{SLAG} + \lambda({e^*}_{14} + {e^*}_{25} + {e^*}_{36}) \wedge {e^*}_7 + a.{e^*}_1 \wedge {e^*}_{78}.$$

Then by using the decomposition of $\Phi(\lambda, a)$ with respect to the vector $e_8$ and Theorem 2, direct computations show that $\Phi(\lambda, a) \in F^*(SLAG)$ if and only if $a^2 + \lambda^2 \le 1$. Thus, each $\Phi(\lambda, a) \in F^*(SLAG)$ is one of the following four types:

1. $\Phi(\pm 1, 0) = \Phi_{ASSOC}$. Then $G(\Phi(\pm 1, 0)) = G(\Phi_{ASSOC})$.
2. $\Phi(0, \pm 1) = \Phi_{SLAG} \pm {e^*}_{178}$. Then $G(\Phi(0, \pm 1)) = G(\Phi_{SLAG}) \bigcup CP^2$.
3. $\Phi(\lambda, a)$ with $\lambda^2 + a^2 < 1$. Then $G(\Phi(\lambda, a)) = G(\Phi_{SLAG})$.
4. $\Phi(\lambda, a)$ with $\lambda \neq 0, a \neq 0$ and $\lambda^2 + a^2 = 1$. Then $G(\Phi(\lambda, a)) = G(\Phi_{SLAG}) \bigcup B$, where $B$ is the set of all 3-vectors of the form $(\cos \alpha e_8 + \sin \alpha f) \wedge \eta$, where $f = \frac{\overline{\psi}(\eta)}{|\overline{\psi}(\eta)|}$, and $\eta$ is of the form $e_1 \wedge (a_2 e_2 + a_3 e_3 + a_5 e_5 + a_6 e_6 + a_7 e_7)$ in which $a_7 \neq 0$.

*Remark.* The calibrations corresponding the case $\lambda^2 + a^2 = 1$ are all maximal calibrations, i.e. their faces are the lagest (under inclusion).

In the rest of this section we present some calibrations whose faces contain a special Lagrangian face.

4.3. The Complexification of a (real) $k$-covector (see [73])

Suppose $\{e_1, e_2, \dots, e_n, Je_1, Je_2, \dots, Je_n\}$ is a real orthonormal basis of $\mathbb{C}^n \cong \mathbb{R}^n + J\mathbb{R}^n$ with complex structure $J$. Let $\{e_1^*, e_2^*, ..., e_n^*, (Je_1)^*, (Je_2)^*, ..., (Je_n)^*\}$ be the dual basis of $\{e_1, e_2, \dots, e_n, Je_1, Je_2, \dots, Je_n\}$.

Suppose $\varphi$ is a (real) $k$-covector on $\mathbb{R}^n$ =span$\{e_1, e_2, \dots, e_n\}$ expressed in terms of axis $k$-planes as below

$$\varphi = \sum a_{i_1 \dots i_k} e_{i_1}^* \wedge e_{i_2}^* \wedge \dots \wedge e_{i_k}^*.$$

Set

$$\varphi^c = \sum a_{i_1 \dots i_k} (e_{i_1}^* + J(Je_{i_1})^*) \wedge (e_{i_2}^* + J(Je_{e_2})^*) \wedge \dots \wedge (e_{i_k}^* + J(Je_{i_k})^*).$$

We can see that, $\varphi^c$ does not depend on the chosen basis $\{e_1, e_2, \dots, e_n\}$. It is called the complex form induced by $\varphi$.

Let $dz_\xi$ denote the complex $k$-covector induced by the unit simple $k$-covector on span$(\xi) \in G(k, \mathbb{R}^n)$. we have the following theorem:

**Theorem 13.** *If* $\|\mathrm{Re}\,\varphi^c\|^* = \|\varphi\|^*$, *then*

$$\bigcup_{\xi \in G(\varphi)} G(\mathrm{Re}\, dz_\xi) \subset G(\mathrm{Re}\,\varphi^c).$$

*Remark.* The above theorem shows that if $\|\mathrm{Re}\varphi^c\|^* = \|\varphi\|^*$, then the face $G(\mathrm{Re}\varphi^c)$ must be larger than $G(\varphi)$ very much. In fact, $G(\mathrm{Re}\varphi^c)$ contains many special Lagrangian faces. For example, if $\varphi$ is simple then $\mathrm{Re}\varphi^c$ is the special Lagrangian calibration. The face of $\varphi$ is just the set of one point, but the face $G(\mathrm{Re}\varphi^c)$ is the $SU(n)$-orbit of $\xi_0$ (see Sec. 1).

Let $V$ be a $k$-dimensional subspace of $\mathbb{R}^n$. Denote by $dx_V$ the unit simple $k$-covector on $V$.

Suppose $\mathbb{R}^n = V_1 \oplus V_2 \oplus \dots \oplus V_m$. For any multi-index $I = (i_1, i_2, \dots, i_q)$, where $|I| = \sum_{i_j \in I} \mathrm{dim} V_{i_j} = k$, denote by $dx_I$ the $k$-covector $dx_{V_{i_1}} \wedge dx_{V_{i_2}} \wedge \dots \wedge dx_{V_{i_q}}$.

In the case, where $\mathrm{dim} V_i \geq 2$ for all $i \leq m$, the $k$-covector $\varphi = \sum_I a_I dx_I$ is called a simply separable $k$-covector (with respect to $V_1, V_2, \dots, V_m$) by Huan [41].

**Theorem 14.** [73, Theorem 2.5] *Suppose* $\varphi = \sum_I a_I dx_I$ *is a simply separable* $k$*-covector, then*

$$\|\mathrm{Re}\,\varphi^c\|^* = \|\varphi\|^* = \max_I \{|a_I|\}.$$

We can see that the complex line forms [11] are simply separable. So we have

**Conclusion 15.** *The real part of the complexification of each complex line form is a calibration.*

Since an exterior power of the Kăhler form is a complex line form, so we also have

**Conclusion 16.** *The real part of the complexification of any exterior power of the Kăhler form $\Omega_p$ is a calibration.*

Moreover, Thi and Binh [74] have shown that

$$G(\mathrm{Re}\Omega_p^c) = \bigcup SLAG(V),$$

where $V$ is a $p$-dimensional quaternionic subspace of $\mathbb{H}^n \cong \mathbb{C}^n \oplus \mathbb{C}^n$ with the suitably chosen quaternionic structure.

4.4. Some 3-Calibrations Whose Faces Contain a Special Lagrangian Face

Consider the special Lagrangian calibration of degree 3 on $\mathbb{C}^n \cong \mathbb{R}^{2n}$, i.e. for a complex structure on a suitable complex subspace of dimension 3, $\varphi$ can be expressed in the following form:

$$\varphi = \mathrm{Re}\, dZ = \mathrm{Re}\, dz_1 \wedge dz_2 \wedge dz_3.$$

And for a suitable orthonormal (real) basis, $\varphi$ can be expressed in terms of axis 3-planes as follows :

$$\varphi = e^*_{123} - e^*_{156} + e^*_{246} - e^*_{345} = e^*_1 \wedge (e^*_{23} - e^*_{56}) + e^*_4 \wedge (e^*_{35} - e^*_{26}).$$

Let $\psi$ be a 2-calibration on $(e_1, e_4)^\perp \cong \mathbb{C}^{n-1}$ whose face contains the face of $e^*_{35} - e^*_{26}$. A recent result of Hieu and Hanh [39] shows that the form:

$$\Phi = e^*_1 \wedge (e^*_{23} - e^*_{56}) + e^*_4 \wedge \psi$$

has comass one, i.e. a calibration. Its face contains a special Lagrangian face , some $\mathbb{CP}^1, \mathbb{CP}^2, \mathbb{CP}^3, \ldots$ faces.

The case, when $\psi$ is Kăhler, is more interesting. In this case, $\Phi$ is $SU_2 \times SU_{n-3}$-invariant and if $n = 4, \Phi$ is a maximal calibration on $\mathbb{C}^4$, (a calibration mentioned in 4.2 and [34]).

A recent result of Hieu [38] shows that $\Phi$ is not a maximal calibration on $\mathbb{C}^5 \cong \mathbb{R}^{10}$. A calibration whose face contains $G(\Phi)$ and obviously contains the special Lagrangian face is:

$$e^*_1 \wedge (e^*_{23} - e^*_{56} + e^*_{78} - e^*_{90}) + e^*_4 \wedge (e^*_{35} - e^*_{26} + e^*_{89} - e^*_{70}).$$

## 5. System of Calibrations with the Globally Length-Minimizing Steiner Networks

### 5.1. Steiner Networks

Let $M$ be a set of points in $\mathbb{R}^n$. The problem of finding a network of least length in the class of networks with fixed end $M$ is called Steiner problem. There are two approaches to this problem. The first requires that, the length-minimizing network is searched in the class of networks, whose vertices all belong to $M$, and the second allows the set of vertices may be larger than $M$. The vertices do not belong to $M$ are called Steiner points.

In this section we mention the second approach (for more detail see [55, 73]).

**Some definitions**

– A (simply) Steiner network in $\mathbb{R}^n$ is a connected complex of one-dimensional simplexes, whose vertices have degree at most three, and the boundary vertices are of degree one.

– A Steiner network is said to be oriented if its sides can be oriented so that two adjacent sides are oriented opposite to each other. It is easy to see that, each Steiner network has exactly two orientations.

– A path of a network is any continuous series of sides ( with orientation) joining two vertices. If these vertices are boundary points, then it is called a maximal path. A system of maximal paths is said to be independent if every one of them is not a combination of other paths from the system.

– A system of maximal paths $\{P_j\}$ in the network $N$ is called a basis of maximal paths if it satisfies the following conditions:

1. The union of all paths from $\{P_j\}$ overlaps $N$.
2. The system $\{P_J\}$ is independent.
3. Every maximal path in $N$ is a combination of paths from $\{P_j\}$.

**Theorem 17.** *Every oriented Steiner network $N$ with $k$ boundary points has a basis of maximal paths consisting of $(k-1)$ paths.*

### 5.2. Principle of System of Calibrations with Steiner Networks

Let $N$ and $N'$ be Steiner networks in $\mathbb{R}^n$ with the same boundary points $A_1$, $A_2, \dots, A_k$. We say that $N$ and $N'$ are of the same topological type if there is a homeomorphism $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $f(A_i) = A_i,\ i = 1, 2, \dots, k$ and $f(N) = N'$.

We have the so-call principle of system of calibrations

*Let $N$ be an oriented Steiner network with $k$ boundary points in $\mathbb{R}^n$ and a basis of maximal paths $\{P_1, P_2, \dots, P_k\}$. Suppose that there is a system of closed differential 1-forms $w_1, w_2, \dots, w_k$ on $\mathbb{R}^n$ such that*

1. $\|\sum\limits_{j\in J_a} \epsilon_j(a) w_j\| \leq 1,$
2. $\sum\limits_{j\in J_a} \epsilon_j(a) w_j(\vec{N_x}) = 1.$

*Here $J_a = \{j : a \in P_j\}$ for any side $a \in N$ and $\vec{N_x}$ is the unit tangent vector to $N$ at $x \in a$ with the same orientation as $a$. Then $N$ is length-minimizing network in the class of networks with fixed topological type.*

Such a system $\{w_j\}$ is called a system of calibrations on $N$.

5.3. Length-Minimizing Steiner Network

The classical results show that, every locally length-minimizing Steiner network has the following properties:
(1) The network consists of straightline segments.
(2) At every vertex the segments meet at angles of $120^0$.

By using the principle of system of calibrations, we obtain the following result.

**Theorem 18.** *Every locally length-minimizing Steiner network is also length-minimizing in the class of Steiner networks with the same topological type.*

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