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Bounded Solutions and Periodic Solutions to Linear Differential Equations in Banach Spaces

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Abstract. We deal with linear inhomogeneous differential equations of the form $dx/dt = Ax(t) + f(t)$ in a Banach space $\mathbf{X}$, where A is the generator of a C_0-semigroup on $\mathbf{X}$ and f is a periodic function. In this paper, we present a Massera type theorem, a method to show the existence of bounded solutions and their structure of them. As results, we obtain criteria for the existence of quasi-periodic, periodic, asymptotically periodic solutions.

1. Introduction

Let $\mathbf{X}$ be a Banach space and $\mathbb{R}$ the real line. In this paper, we investigate criteria for the existence of bounded solutions and periodic solutions to linear inhomogeneous differential equations of the form

$$\frac{d}{dt}u(t) = Au(t) + f(t). \tag{1}$$

Throughout the present paper we make the following assumption.

Assumption : $A : \mathcal{D}(A) \subset \mathbf{X} \to \mathbf{X}$ is the generator of C_0-semigroup $U(t)$, and $f : \mathbb{R} \to \mathbf{X}$ is a τ-periodic function.

If $x(t)$ is a continuous function which satisfies the following equation

$$x(t) = U(t)x(0) + \int_0^t U(t-s)f(s)ds, \quad t \in \mathbb{R}_+ := [0, \infty), \tag{2}$$

it is called a (mild) solution to Equation (1).

The purpose of this paper is to give a survey of the Massera type theorem, criteria for the existence of bounded solutions and a structure of bounded solutions to Equation (1). As results, we can obtain criteria for the existence of quasi-periodic, periodic, asymptotically periodic solutions. The relationship between the existence of bounded solutions and the existence of τ-periodic solutions is characterized by the Massera type theorem.

2. A Survey of the Massera Type Theorem

In this section we give a survey of the Massera type theorem for the following linear equation in $\mathbf{X}$

$$\frac{d}{dt}u(t) = A(t)u(t) + f(t), \tag{3}$$

where $A(t)$ is a (unbounded) linear operator such that $A(t+\tau) = A(t), t \in \mathbb{R}$. Assume that its solution is expressed as follows

$$u(t) = U(t,0)u(0) + \int_0^t U(t,s)f(s)ds, \tag{4}$$

where $U(t,s)$ is a τ-periodic strongly continuous evolutionary process.

In 1950, Massera [16] showed the following result on the existence of periodic solutions to Equation (3) in $\mathbb{R}^n$.

Theorem 2.1. *If Equation* (3) *has a bounded solution on* $\mathbb{R}_+$*, then it has a* τ*-periodic solution.*

This result has been extended to various periodic linear equations in $\mathbb{R}^n$ or infinite dimensional Banach spaces, cf. [1, 4, 7, 9, 10, 15].

In 2000, Naito, Minh, Miyazaki and Shin [20] investigated the Massera type theorem and a structure of a bounded uniformly continuous solution on $\mathbb{R}$ to Equation (4) in Banach spaces and gave the following result. Let $BUC(\mathbb{R}, \mathbf{X})$ stand for the set of all bounded uniformly continuous functions from $\mathbb{R}$ to a Banach space $\mathbf{X}$. A function $u \in BUC(\mathbb{R}, \mathbf{X})$ is said to be a solution on $\mathbb{R}$ of Equation (4) if

$$u(t) = U(t,s)u(s) + \int_s^t U(t,s)f(s)ds$$

as long as $-\infty < s \le t < \infty$.

Theorem 2.2. *Assume that the monodromy operator* $U(1,0)$ *is compact. Then the following hold:*

(1) *If there is a solution* $u \in BUC(\mathbb{R}, \mathbf{X})$ *on* $\mathbb{R}$ *of Equation* (4)*, then Equation* (4) *has a* τ*-periodic solution.*

(2) *If* $u \in BUC(\mathbb{R}, \mathbf{X})$ *is a solution on* $\mathbb{R}$ *to Equation* (4)*, then it is expressed as*

$$u(t) = u_0(t) + \sum_{k=1}^{N} e^{i\lambda_k t}u_k(t),$$

where λ_k $(k = 1, 2, \cdots, N)$ *is a certain number,* u_0 *is a* τ*-periodic solution to Equation* (4)*,* u_k *is a* τ*-periodic solution to Equation* (4) *with* $f = -i\lambda_k u_k$.

Also refer to [21]. If the monodromy operator $P(t) := U(t, t-\tau)$ is norm continuous with respect to t and is compact for every $t \in \mathbb{R}$, then a bounded solution on $\mathbb{R}$ implies a bounded uniformly continuous solution on $\mathbb{R}$. However, if $P(t)$ is not compact, then this fact does not hold, in general.

For functional differential equations refer to [7, 9, 10]. In particular, Hatvani and Krisztin [8] obtained a necessary and sufficient condition for the existence of periodic solutions to a special equation.

More recently, interesting results on the existence of periodic or almost periodic solutions to functional differential equations have been given in Hino, Murakami and Minh [11] by using a decomposition technique of variation of constants formula on the phase space.

On the other hand, the method using fixed point theorems is effective in showing the Massera type theorem for various equations. For the Massera type theorems based on Schauder's fixed point theorems, refer to [2, 12, 13].

In 1974, Chow and Hale [3] obtained the following simple fixed point theorem on an affine linear map from $\mathbf{X}$ to $\mathbf{X}$.

Theorem 2.3. *Let T be a bounded linear operator on $\mathbf{X}$ and $b(\neq 0) \in \mathbf{X}$ be fixed. Put $Vx = Tx + b, x \in \mathbf{X}$. Assume that the range $\mathcal{R}(I-T)$, I being the identity, is closed and that there is an $x_0 \in \mathbf{X}$ such that $\{V^n x_0\}_{n=0}^{\infty}$ is bounded. Then V has a fixed point in $\mathbf{X}$; that is, the equation $(I-T)x = b$ has a solution.*

Applying this fixed point theorem to functional differential equations with finite delay in $\mathbb{R}^n$, they obtained the Massera type theorem. The above fixed point theorem is very useful in the infinite dimensional case, because the compactness condition on T is not contained in it. However, it requires the closedness of the range $\mathcal{R}(I-T)$. This fact suggests the possibility of extending the Massera type theorem for noncompact periodic processes. In fact, the following result is obtained.

Theorem 2.4. *Assume that $\mathcal{R}(I - U(\tau, 0))$ is closed. If Equation (4) has a bounded solution on $\mathbb{R}_+$, then it has a τ-periodic solution.*

The theory (perturbation theory) of semi-Fredholm operators is useful in showing the closedness of the range for functional differential equations. Therefore, combining Theorem 2.4 and the theory of semi-Fredholm operators, many results on the Massera type theorem for functional differential equations in Banach spaces have been obtained in Shin and Naito [25] and Shin, Naito and Minh [27].

To complete the Massera type theorem, it is practically and theoretically important to show the existence of bounded solutions on $\mathbb{R}_+$.

3. The Existence of Bounded Solutions: General Theory

In this section, to obtain criteria for the existence of bounded solutions and periodic solutions to various equations, we give properties of affine maps. Let

$T : \mathbf{X} \to \mathbf{X}$ be a bounded linear operator and $b(\neq 0) \in \mathbf{X}$ be fixed. Put

$$Vx = Tx + b \tag{5}$$

and

$$S_n(T) = \sum_{k=0}^{n-1} T^k.$$

Our manner is based on the following fact : since $V^n b = S_{n+1}(T)b,\ n \in \mathbb{N}$, in Theorem 2.3, we have that

$$\limsup_{n\to\infty} \|S_n(T)b\| < \infty \tag{6}$$

if and only if $\{V^n b\}_{n=0}^{\infty}$ is bounded.

In order to obtain criteria for the existence of bounded solutions to Equation (1) or general equations, we will employ the above relation (6).

3.1. Results on Affine Maps

We give a property of the affine map V defined by (5), which is concerned with the existence of bounded solutions to Equation (1).

Theorem 3.1. *Let $\mathbf{Z}$ be a subset of $\mathbf{X}$. Assume that for any $x \in \mathbf{Z}$ there exists an $\alpha_x > 0$ such that $\|T^n x\| \le \alpha_x$ for all $n \in \mathbb{N}$. Then the following statements are equivalent.*

1) *There is an $x_0 \in \mathbf{Z}$ such that the sequence $\{V^n x_0\}_{n=0}^{\infty}$ is bounded.*
2) *For any $x \in \mathbf{Z}$, the sequence $\{V^n x\}_{n=0}^{\infty}$ is bounded.*
3) $$\limsup_{n\to\infty} \|S_n(T)b\| < \infty. \tag{7}$$

Proof. 1) $\Rightarrow$ 3). Since $V^n x_0 = T^n x_0 + S_n(T)b,\ x_0 \in \mathbf{Z}$, we have

$$\|S_n(T)b\| \le \|T^n x_0 + S_n(T)b\| + \|T^n x_0\| \le \|V^n x_0\| + \alpha_{x_0} < \infty.$$

This implies that $\limsup_{n\to\infty} \|S_n(T)b\| < \infty$.

3) $\Rightarrow$ 2). Since

$$\|V^n x\| \le \|T^n x\| + \|S_n(T)b\| \le \alpha_x + \|S_n(T)b\|$$

for every $x \in \mathbf{Z}$, the proof is obvious.

2) $\Rightarrow$ 1). It is obvious.

Therefore, the proof of the theorem is complete. ∎

The following result is directly derived from the proof of Theorem 3.1.

Corollary 3.2. *Assume that there is a positive number $\alpha > 0$ such that $\|T^n\| \le \alpha < \infty$ $(\forall n \in \mathbb{N})$ holds. Then the following statements are equivalent.*

1) *There is an* $x_0 \in \mathbf{X}$ *such that the sequence* $\{V^n x_0\}_{n=0}^{\infty}$ *is bounded.*
2) *For any* $x \in \mathbf{X}$, *the sequence* $\{V^n x\}_{n=0}^{\infty}$ *is bounded.*
3) *The relation* (7) *holds.*

3.2. General Criteria

In this subsection, we will apply Theorem 3.1 to Equation (1). Put

$$b_f = \int_0^\tau U(\tau - s)f(s)ds.$$

Theorem 3.3. *Let* $\mathbf{Z}$ *be a subset of* $\mathbf{X}$. *Assume that for any* $x \in \mathbf{Z}$ *there exists an* $\alpha_x > 0$ *such that* $\|U(n\tau)x\| \le \alpha_x$ *for all* $n \in \mathbb{N}$. *Then the following four statements are equivalent*;

1) *Equation* (1) *has a bounded solution* $x(t)$ *on* $\mathbb{R}_+$ *with the initial condition* $x(0) \in \mathbf{Z}$.
2) $\limsup\limits_{n\to\infty} \|\sum\limits_{k=0}^{n-1} U(k\tau)b_f\| < \infty$.
3) $\limsup\limits_{n\to\infty} \|\int\limits_0^{n\tau} U(s)f(-s)ds\| < \infty$.
4) $\limsup\limits_{t\to\infty} \|\int\limits_0^{t} U(t-s)f(s)ds\| < \infty$.

Proof. Since $S_n(U(\tau)) = \sum_{k=0}^{n-1} U(k\tau)$, the equivalence of the conditions 1) and 2) is proved by Theorem 3.1. The equivalence of the two conditions 1) and 4) is easily proved. Indeed, the solution $x(t)$ of Equation (1) with the condition $x(0) \in \mathbf{Z}$ is expressed as

$$x(t) = U(t)x(0) + \int_0^t U(t-s)f(s)ds,$$

and $\|U(n\tau)x(0)\| \le \alpha_{x(0)}$ holds, which implies the equivalence of the two conditions 1) and 4).

Now we will prove the equivalence of the two conditions 2) and 3). Since

$$\begin{aligned}\int_0^\tau U(m\tau - s)f(s)ds &= \int_{(m-1)\tau}^{m\tau} U(r)f(m\tau - r)dr\\ &= \int_{(m-1)\tau}^{m\tau} U(s)f(-s)ds,\end{aligned}$$

we have

$$\begin{aligned}\sum_{k=0}^{n-1} U(k\tau)\int_0^\tau U(\tau - s)f(s)ds &= \sum_{m=1}^{n}\int_0^\tau U(m\tau - s)f(s)ds\\ &= \int_0^{n\tau} U(s)f(-s)ds\end{aligned}$$

This implies the equivalence of the conditions 2) and 3). ∎

Slightly modifying the condition 1) in Theorem 3.3, we obtain the following result.

Corollary 3.4. *The following two statements are equivalent.*
1) *Equation* (1) has a bounded solution $x(t)$ on $\mathbb{R}_+$ with the initial condition $x(0) = 0$ or b_f.
2) *The condition* 2) *in Theorem* 3.3 *holds true.*

Corollary 3.5. *Assume that C_0-semigroup $U(t)$ is bounded. Then the following two statements are equivalent.*
1) *Equation* (1) *has a bounded solution on* $\mathbb{R}_+$.
2) *All solutions of Equation* (1) *are bounded on* $\mathbb{R}_+$.

Proof. The proof follows from Corollary 3.2 and Theorem 3.3. ∎

Finally, we see that in the case where A in Equation (1) is the generator of a bounded C_0-group $U(t),\ t \in \mathbb{R}$, a bounded solution on $\mathbb{R}_+$ of Equation (1) is extended to a bounded solution on $\mathbb{R}$ of Equation (1).

Proposition 3.6. *Assume that A in Equation* (1) *is the generator of a bounded C_0-group $U(t),\ t \in \mathbb{R}$. If Equation* (1) *has a bounded solution on $\mathbb{R}_+$, whose range is relatively compact, then Equation* (1) *has a bounded solution on* $\mathbb{R}$.

4. The Existence of Bounded Solutions

In this section, we give criteria for the existence of bounded solutions to Equation (1). We consider the condition 2) in Theorem 3.3 from the point of view of the spectrum of A in Equation (1).

First, we will check it for the case where $\mathbf{X} = \mathbb{C}^m, A = (a_{ij})$, an $m \times m$ matrix. Let the characteristic polynomial of A be factorized as follows:

$$\Phi(\lambda) := \det(\lambda I - A) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_\ell)^{m_\ell},$$

where $\lambda_1, \cdots, \lambda_\ell$ are the distinct roots of $\Phi(\lambda)$, and $m_1 + \cdots + m_\ell = m$. Put $\lambda_j =: a_j + ib_j, a_j, b_j \in \mathbb{R}$. Denote by $P_j : \mathbb{C}^m \to \mathbf{M}_j$ the projection corresponding to the direct sum decomposition $\mathbb{C}^m = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_\ell$, where $\mathbf{M}_j := \mathcal{N}((A - \lambda_j I)^{n_j})$ is the generalized eigenspace corresponding to λ_j.

Proposition 4.1. *For $\tau > 0,\ b \in \mathbb{C}^m$, the vector sequence $\{S_n\}$, given as*

$$S_n := S_n(e^{\tau A})b = \sum_{k=0}^{n-1} e^{k\tau A} b,$$

is bounded if and only if for every $p = 1, \cdots, \ell$, the following conditions hold:

(i) *If $a_j > 0$, then $P_j b = 0$.*
(ii) *The case where $a_j = 0$:*
  (a) *if $\tau b_j \in 2\pi\mathbb{Z}$, then $P_j b = 0$.*
  (b) *if $\tau b_j \notin 2\pi\mathbb{Z}$, then $P_j b \in \mathcal{N}(A - \lambda_j I)$.*
(iii) *If $a_j < 0$, then $P_j b$ is arbitrary.*

To prove the proposition, the following lemma is needed.

Lemma 4.2. *Let $Q(t)$ be a vector in $\mathbb{C}^n$, whose component is a polynomial of t, and $\lambda = a + ib \in \mathbb{C}, a, b \in \mathbb{R}$. The vector sequence $\{R_n\}$, given as*

$$R_n = \sum_{j=1}^{n} e^{j\lambda} Q(j),$$

is bounded if an only if the following conditions hold:
(i) *In the case where $a > 0$, $Q(t) \equiv 0$.*
(ii) *In the case where $a = 0$, if $b \in 2\pi\mathbb{Z}$, then $Q(t) \equiv 0$: if $b \notin 2\pi\mathbb{Z}$, then $Q(t) = c$ (a constant vector).*
(iii) *In the case where $a < 0$, $Q(t)$ is arbitrary.*

Proof. Set $z = e^{\lambda}$. Then $R_n = \sum_{j=1}^{n} z^j Q(j)$. If $a < 0$, $\{R_n\}$ is always bounded. Consider the case where $a \geq 0$. If $\{R_n\}$ is bounded, the sequence $\{R_n - R_{n-1}\}_{n=2}^{\infty}$ is also bounded and $\|R_n - R_{n-1}\| = \|z^n Q(n)\| = e^{na}\|Q(n)\|$. Hence in the case where $a > 0$, $Q(t) \equiv 0$ if and only if $\{R_n\}$ is bounded. So, we see the case where $a = 0$. We note that $\|R_n - R_{n-1}\| = \|Q(n)\|$. From the definition of $Q(t)$ it follows that $\{Q(n)\}$ is bounded if and only if $Q(t) = c$ (a constant vector). If $b \in 2\pi\mathbb{Z}$, then $z = 1$, and so, $R_n = nc$. Namely, $c = 0$ if and only if $\{R_n\}$ is bounded. If $b \notin 2\pi\mathbb{Z}$, then $z \neq 1$. Hence we have

$$\|R_n\| = \|\frac{1 - z^{n+1}}{1 - z} c\| \leq \frac{2}{1 - z}\|c\|,$$

which implies that $\{R_n\}$ is bounded. Therefore the proof of the lemma is finished. ■

The proof of Proposition 4.1. $\mathbb{C}^m$ is decomposed as

$$\mathbb{C}^m = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_\ell.$$

Take a circle C_p centered at λ_p, whose radius is sufficiently small and its disk does not contain the other points $\lambda_q, q \neq p$. Then the projection P_p is expressed as

$$P_p = \frac{1}{2\pi} \int_{C_p} (\lambda I - A)^{-1} d\lambda. \tag{8}$$

Then P_p is a bounded operator having the following properties:

$$P_p \mathbb{C}^m = \mathbf{M}_p, \;\; AP_p = P_p A, \;\; P_p P_q = 0 \; (p \neq q), \;\; P_p^2 = P_p, \;\; P_1 + P_2 + \cdots + P_\ell = I.$$

Furthermore, e^{tA} is decomposed as follows:

$$e^{tA} = \sum_{p=1}^{\ell} e^{\lambda_p t} Q_p(t) P_p, \quad Q_p(t) = \sum_{k=0}^{n_p-1} \frac{t^k}{k!}(A - \lambda_p I)^k.$$

Using those facts, we have

$$S_n(e^{\tau A}) = \sum_{j=0}^{n-1} e^{j\tau A} = \sum_{j=0}^{n-1} \sum_{p=1}^{\ell} e^{j\tau\lambda_p} Q_p(j\tau) P_p = \sum_{p=1}^{\ell} \sum_{j=0}^{n-1} e^{j\tau\lambda_p} Q_p(j\tau) P_p.$$

Since $P_p P_q = 0 (p \neq q)$, $P_p^2 = P_p$, it follows that

$$P_p S_n(e^{\tau A}) b = \sum_{j=0}^{n-1} e^{j\tau\lambda_p} Q_p(j\tau) P_p b := R_n^p.$$

Hence, the sequence $\{S_n(e^{\tau A})b\}_{n\in\mathbb{N}}$ is bounded if and only if for every $p = 1, \cdots, \ell$, the sequence $\{R_n^p\}_{n\in\mathbb{N}}$ is bounded. Since

$$Q_p(j\tau) P_p b = \sum_{k=0}^{n_p-1} j^k \frac{\tau^k}{k!}(A - \lambda_p I)^k P_p b, \quad p \in \{1, 2, \cdots, \ell\},$$

we have, by Lemma 4.2, the following facts.

i) If $a_p > 0$, then $Q_p(j\tau)P_p b \equiv 0$, from which we have $P_p b = 0$.

ii) If $a_p < 0$, then $Q_p(j\tau)P_p b$ is arbitrary; that is, $P_p b$ is also arbitrary.

iii) Let $a_p = 0$. If $\tau b_p \in 2\pi\mathbb{Z}$, then $Q_p(j\tau)P_p b \equiv 0$; that is, $P_p b = 0$. If $\tau b_p \notin 2\pi\mathbb{Z}$, then $Q_p(j\tau)P_p b = c$(a constant vector). Notice that $(A - \lambda_p I)P_p b = 0$ if and only if $Q_p(j\tau)P_p b = P_p b$. This means that $c = P_p b$; namely, $P_p b \in \mathcal{N}(A - \lambda_p I)$. Hence we obtain the conclusion of the proposition. ∎

Lemma 4.3.

$$\limsup_{n\to\infty} \|e^{n\tau A} z\| < \infty$$

if and only if the $z \in \mathbb{C}^m$ *satisfies the following conditions*: *for every* $r \in \{1, \cdots, \ell\}$,

(i) *if* $a_r < 0$, *then* $P_r z$ *is arbitrary*;

(ii) *if* $a_r > 0$, *then* $P_r z = 0$;

(iii) *if* $a_r = 0$, *then* $P_r z \in \mathcal{N}(A - \lambda_r I)$.

Proof. Since

$$e^{n\tau A} P_r z = e^{n\tau\lambda_r} \sum_{k=0}^{n_r-1} n^k \frac{\tau^k}{k!}(A - \lambda_r I)^k P_r z$$

for every $r \in \{1, \cdots, \ell\}$, the proof of the lemma is obvious. ∎

The following theorem is an immediate result of Lemma 4.3.

Theorem 4.4. *Assume that* $\limsup_{n\to\infty} \|S_n(e^{\tau A})b\| < \infty$. *Then,*

$$\limsup_{n\to\infty} \|e^{n\tau A} z + S_n(e^{\tau A})b\| < \infty$$

if and only if the $z \in \mathbb{C}^m$ *satisfies the following conditions: for every* $r \in \{1, \cdots, \ell\}$

i) if $a_r < 0$*, then* $P_r z$ *is arbitrary;*

ii) if $a_r > 0$*, then* $P_r z = 0$;

iii) *if* $a_r = 0$*, then* $P_r z \in \mathcal{N}(A - \lambda_r I)$.

Next, we give criteria for the existence of bounded solutions to Equation (1) in an infinite dimensional Banach space.

Suppose that $\omega_e(U) := \lim_{t\to\infty} t^{-1} \log \alpha(U(t)) < 0$, where $\alpha(U(t))$ stands for the Kuratowski measure of $U(t)$ (cf. [29]). Then $\exp(t\omega_e(U)) < 1$, $(t > 0)$. This implies that there exists a $\gamma > 0$ such that $\sigma(U(t)) \cap \{z : |z| \ge e^{-\gamma t}\}$ and $\sigma(A) \cap \{z : \Re z \ge -\gamma\}$ consist of finite number of normal eigenvalues and that $\sigma(A) \cap \{z : -\gamma < \Re z < 0\} = \emptyset$ (cf. [17, 18]). Hence $\sigma(A) \cap \{z : \Re z > -\gamma\}$ consists of finite number of normal eigenvalues $\lambda_j, j = 1, 2, \cdots, r$, with nonnegative real parts. Set $a_j = \Re\lambda_j, b_j = \Im\lambda_j$. Assume that $a_j = 0$ for $1 \le j \le q$ $(\le r)$ and that $a_j > 0$ for $q < j \le r$. Thus 1 is a normal eigenvalue of $U(\tau)$. Set $\sigma_0(A) = \{\lambda_j : 1 \le j \le q\}$ and $\sigma_+(A) = \{\lambda_j : q+1 \le j \le r\}$. We understand that $\sigma_+(A) = \emptyset$ provided $q = r$. Let $\mathbf{M}_j$ be the generalized eigenspace of A corresponding to λ_j. Since λ_j is a normal eigenvalue of A, $n_j := \dim \mathbf{M}_j$ is finite and there exists a positive integer m_j such that $\mathbf{M}_j = \mathcal{N}((\lambda_j - A)^{m_j})$. The space $\mathbf{X}$ is decomposed as follows:

$$\mathbf{X} = \mathbf{Y} \oplus \mathbf{Z}, \quad \mathbf{Z} = \mathbf{M}_0 \oplus \mathbf{M}_+, \quad \mathbf{Y} = \bigcup_{j=1}^{r} \mathcal{R}((\lambda_j I - A)^{m_j}),$$
$$\mathbf{M}_0 = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_q, \quad \mathbf{M}_+ = \mathbf{M}_{q+1} \oplus \cdots \oplus \mathbf{M}_r.$$

The subspaces $\mathbf{Y}$ and $\mathbf{M}_j$ are closed in $\mathbf{X}$ and $\dim \mathbf{Z} = n_1 + n_2 + \cdots + n_r =: d$. If we define P_j as in (8), where $(\lambda I - A)^{-1}$ is understood as the resolvent operator $R(\lambda, A)$, then $P_j : \mathbf{X} \to \mathbf{M}_j$ are projections such that $P_j P_k = \delta_{jk} P_j$ and $AP_j x = P_j Ax$ for $x \in D(A)$. If we set $P = P_1 + P_2 + \cdots + P_r, P_0 = I - P$, then $P : \mathbf{X} \to \mathbf{Z}$ and $P_0 : \mathbf{X} \to \mathbf{Y}$ are projections. $\mathbf{Y}, \mathbf{M}_j$ and $\mathbf{Z}$ are invariant subspaces of $U(t)$. Since $U(t)x = U(t)P_0 x + U(t)Px$, we have

$$\|S_n(U(\tau))x\| \le \|S_n(U(\tau))P_0 x\| + \|S_n(U(\tau))Px\|.$$

It follows from [1, Proposition 4.15] that there are an $\varepsilon_0 > 0$ and a constant $K \ge 1$ such that

$$\|U(t)P_0 x\| \le Ke^{-\varepsilon_0 t}\|P_0 x\| \quad \text{for all} \quad x \in \mathbf{X},\ t \ge 0.$$

Hence we have

$$\|S_n(U(\tau))P_0 x\| \le K \sum_{k=0}^{n-1} e^{-\varepsilon_0 \tau k}\|P_0 x\| \le \frac{K}{1 - e^{-\varepsilon_0 \tau k}}\|P_0 x\| < \infty.$$

As a result, $\{\|S_n(U(\tau))x\|\}$ is bounded if and only if $\{\|S_n(U(\tau))Px\|\}$ is bounded.

Since $d = \dim \mathbf{Z} < \infty$, $A_{\mathbf{Z}}$, the restriction of A to $\mathbf{Z}$, is regarded as a $d \times d$ matrix with eigenvalues $\lambda_j, 1 \le j \le r$, and $U(t)Px = \exp(tA_{\mathbf{Z}})Px$ for all $x \in \mathbf{X}$. Thus we have the following result from Proposition 4.1.

Theorem 4.5. *Assume that $\sigma(U(t))$ and $\sigma(A)$ are as in the above. Then $S_n(U(\tau))x,\ n = 1, 2, \cdots,$ are bounded if and only if the following conditions hold*:

(i) *For $q < j \leq r$, $P_j x = 0$.*

(ii) *For $1 \leq j \leq q$,*

  (a) *if $\tau b_j \in 2\pi\mathbb{Z}, P_j x = 0$:*

  (b) *if $\tau b_j \notin 2\pi\mathbb{Z}, P_j x \in \mathcal{N}(A_{\mathbf{Z}} - \lambda_j I)$.*

Corollary 4.6. *Assume that $\sigma(U(t))$ and $\sigma(A)$ are as in the above. Then the solution $x(t)$ of Equation* (1) *such that $x(0) = b_f$ is bounded if and only if the following conditions hold*:

(i) *For $q < j \leq r$, $P_j b_f = 0$.*

(ii) *For $1 \leq j \leq q$,*

  (a) *if $\tau b_j \in 2\pi\mathbb{Z}, P_j b_f = 0$:*

  (b) *if $\tau b_j \notin 2\pi\mathbb{Z}, P_j b_f \in \mathcal{N}(A_{\mathbf{Z}} - \lambda_j I)$.*

Combining Theorem 4.5 with Theorem 3.3, we obtain the following result.

Corollary 4.7. *Suppose $U(t)$ is a bounded C_0-semigroup such that $\omega_e(U) < 0$. Then every solution of Equation* (1) *is bounded on $\mathbb{R}_+$ if and only if for $j = 1, \cdots, q$ the following conditions hold*:

(a) *If $\tau b_j \in 2\pi\mathbb{Z}$, then $P_j b_f = 0$;*

(b) *If $\tau b_j \notin 2\pi\mathbb{Z}$, then $P_j b_f \in \mathcal{N}(A_{\mathbf{Z}} - \lambda_j I)$.*

Using Corollary 4.6, we obtain the following result on the existence of a τ-periodic solution to Equation (1).

Theorem 4.8. *Assume that $\omega_e(U) < 0$ and that b_f satisfies the conditions* (i, ii) *in Corollary* 4.6. *Then Equation* (1) *has a τ-periodic solution.*

Proof. Since $\{S_n(U(\tau))b_f\}_n$ is bounded, it follows from Corollary 4.6 that the solution $x(t)$ of Equation (1) such that $x(0) = b_f$ is bounded. Since 1 is a normal point of $U(\tau)$, we see that $\mathcal{R}(I - U(\tau))$ is a closed subspace of $\mathbf{X}$. Therefore Theorem 2.4 implies that Equation (1) has a τ-periodic solution. ■

5. A Structure of Bounded Solutions

In this section we will give a structure of bounded solutions obtained in Sec. 4. Throughout this section, we assume the following conditions:

1) $\omega_e(U) < 0$,

2) 1 is a normal eigenvalue of $U(\tau)$,

3) $\limsup_{n\to\infty} \|S_n(U(\tau))b_f\| < \infty$.

We denote by $\lambda_j := a_j + ib_j, j = 1, \cdots, r$, the points in $\sigma(A) \cap \{z : \Re z \geq 0\}$ as in Sec. 4. Moreover, denote by $\mathbf{SP}$ and $\mathbf{SP_X}$ the set of all τ-periodic solutions of Equation (1) and the set of all solutions of the equation $(I - U(\tau))x = b_f$, respectively. They are affine spaces. If we take a vector $x_0 \in \mathbf{SP_X}$, then $\mathbf{SP_X} =$

$x_0 + \mathcal{N}(I - U(\tau))$. Let $\sigma(A) \cap \{z : \Re z = 0\} = \{ib_1, \cdots, ib_q\}$, and suppose that $\tau b_j \in 2\pi\mathbb{Z}$ for $1 \le j \le p(\le q)$ and that $\tau b_j \notin 2\pi\mathbb{Z}$ for $p+1 \le j \le q$. Set

$$\mathbf{N}_0 = \mathbf{N}_p \oplus \ \mathbf{N}_q \subset \mathbf{M}_0$$

where

$$\mathbf{N}_p = \mathcal{N}(A - ib_1 I) \oplus \cdots \oplus \mathcal{N}(A - ib_p I), \ \ \mathbf{N}_q = \mathcal{N}(A - ib_{p+1} I) \oplus \cdots \oplus \mathcal{N}(A - ib_q I).$$

Since 1 is a normal eigenvalue of $U(\tau)$, it follows that, $p \ge 1$ and

$$\mathcal{N}(I - U(\tau)) = \mathbf{N}_p \subset \mathbf{N}_0,$$

cf. [29, Proposition 4.13]. Hence $\mathbf{SP_X} = x_0 + \mathbf{N}_p$.

Proposition 5.1. *The following results hold.*
1) *$U(t)x$ is bounded for $t \ge 0$ if and only if $x \in \mathbf{Y} \oplus \mathbf{N}_0$; that is,*
  (i) $P_j x \in \mathcal{N}(A - \lambda_j I)$ for $1 \le j \le q$.
  (ii) *$P_j x = 0$ for $q+1 \le j \le r$.*
2) *$U(t)x$ is τ-periodic if and only if $x \in \mathbf{N}_p$; that is,*
  (iii) $P_0 x = 0$
  (iv) *$P_j x \in \mathcal{N}(A - ib_j I)$ for $1 \le j \le p$.*
  (v) *$P_j x = 0$ for $p+1 \le j \le r$.*

Proof. $U(t)x$ is bounded if and only if $P_0 U(t)x = U(t)P_0 x$ and $PU(t)x = U(t)Px$ are bounded. Since $P_0 U(t)x$ is bounded for all $x \in \mathbf{X}$, it suffices to check the boundedness of $U(t)Px$. Since $P = P_1 + \cdots + P_r$, $\{U(n\tau)Px\}_n$ is bounded if and only if $\{U(n\tau)P_j x\}_n, j = 1, 2, \cdots, r$, are bounded. On the other hand, since

$$\begin{aligned} \|U(t)P_j x\| &= \Big\| e^{(a_j + ib_j)t} \sum_{m=0}^{m_j - 1} \frac{t^m}{m!} (A - \lambda_j I)^m P_j x \Big\| \\ &= e^{a_j t} \Big\| \sum_{m=0}^{m_j - 1} \frac{t^m}{m!} (A - \lambda_j I)^m P_j x \Big\| \end{aligned}$$

for $1 \le j \le r$, the assertion 1) is easily derived from this relation. Similarly, $U(t)x$ is τ-periodic if and only if $P_0 U(t)x = U(t)P_0 x$ and $PU(t)x = U(t)Px$ are τ-periodic. If $U(t)P_0 x$ is τ-periodic, we have $P_0 x = U(n\tau)P_0 x$ for all $n = 1, 2, \cdots$. Since $U(t)P_0 x \to 0$ as $t \to \infty$, we have $P_0 x = 0$.

If $U(t)Px$ is τ-periodic, $U(t)P_j x$ is τ-periodic for $1 \le j \le r$. It follows at first that $P_j x \in \mathcal{N}(A - ib_j I)$ for $1 \le j \le q$ and that $P_j x = 0$ for $q+1 \le j \le r$. If $p+1 \le j \le q$, and if $P_j x \ne 0$, then $U(t)P_j x = e^{ib_j t} P_j x$ is not τ-periodic. Consequently, $x \in \mathbf{N}_p$. Clearly, if $x \in \mathbf{N}_p$, $U(t)x$ is τ-periodic. ■

Theorem 5.2. *A solution $x(t)$ of Equation* (1) *is bounded on $\mathbb{R}_+$ if and only if $x(0) \in \mathbf{Y} \oplus \mathbf{N}_0$.*

Proof. The solution $x(t)$ is written as Equation (2) in Introduction. Notice that the integral in this equation is bounded if and only if the condition 3) holds.

Hence $x(t)$ is bounded if and only if $U(t)x(0)$ is bounded. From Proposition 5.1 we have the result in the theorem. ∎

The following result follows from the condition 3) and Theorem 5.2.

Corollary 5.3. *The following assertions hold true.*
(1) $\mathbf{SP_X} \neq \emptyset, \mathbf{SP_X} = x_0 + \mathbf{N}_p \subset \mathbf{Y} \oplus \mathbf{N}_0$.
(2) $\mathbf{M}(b_f) \subset \mathbf{Y} \oplus \mathbf{N}_0$, *where* $\mathbf{M}(b_f)$ *is the linear space generated by* $\{U(n\tau)b_f\}_{n=0}^{\infty}$.

In the assertion (1) in Corollary 5.3, x_0 can be taken in $\mathbf{Y} \oplus \mathbf{N}_q$. Hence, it is a unique solution of the equation $(I - U(\tau))x = b_f$. Then x_0 is expressed as

$$x_0 = (I - U(\tau))^{-1} b_f \in \mathbf{Y} \oplus \mathbf{N}_q.$$

Therefore, the following result holds.

Proposition 5.4.

$$\mathbf{SP_X} = (I - U(\tau))^{-1} b_f + \mathbf{N}_p \subset \mathbf{Y} \oplus \mathbf{N}_0, \quad (I - U(\tau))^{-1} b_f \in \mathbf{Y} \oplus \mathbf{N}_q.$$

We are now in a position to state the main result in this section.

Theorem 5.5. *Take a τ-periodic solution $u_0(t)$ of Equation (1). Then the following statements are valid.*
(1) *Any bounded solution $x(t)$ of Equation (1) on $\mathbb{R}_+$ is written as*

$$x(t) = u_0(t) + \sum_{j=1}^{p} e^{ib_j t} x_j + \sum_{j=p+1}^{q} e^{ib_j t} x_j + U_{\mathbf{Y}}(t) y_0,$$

with some vectors $x_j \in \mathcal{N}(A - ib_j I), 1 \leq j \leq q$, and $y_0 \in \mathbf{Y}$.
(2) *Any τ-periodic solution $u(t)$ of Equation (1) is written as*

$$u(t) = u_0(t) + \sum_{j=1}^{p} e^{ib_j t} u_j,$$

with some vectors $u_j \in \mathcal{N}(A - ib_j I), 1 \leq j \leq p$.

Proof. Since $u_0(t)$ is the τ-periodic solution of Equation (1), $u_0(0) \in \mathbf{SP_X} \subset \mathbf{Y} \oplus \mathbf{N}_0$. Let $x(t)$ be a bounded solution on $\mathbb{R}_+$ of Equation (1). Then it follows from Theorem 5.2 that $x(0) \in \mathbf{Y} \oplus \mathbf{N}_0$. Therefore $x(0) - u_0(0) \in \mathbf{Y} \oplus \mathbf{N}_0$; it is expressed as

$$x(0) - u_0(0) = \sum_{j=1}^{q} x_j + y_0,$$

where $x_j \in \mathcal{N}(A - ib_j I)$ and $y_0 \in \mathbf{Y}$. Since $x(t) - u_0(t)$ is a solution of the homogeneous equation, we have

$$\begin{aligned} x(t) - u_0(t) &= U(t)[x(0) - u_0(0)] \\ &= U(t)\sum_{j=1}^{p} x_j + U(t)\sum_{j=p+1}^{q} x_j + U_{\mathbf{Y}}(t)y_0 \\ &= \sum_{j=1}^{p} e^{ib_j t}x_j + \sum_{j=p+1}^{q} e^{ib_j t}x_j + U_{\mathbf{Y}}(t)y_0, \end{aligned}$$

as required. The remainder is obvious. Therefore the proof of the theorem is complete. ∎

Corollary 5.6. *The following statements are valid.*

(1) *There is a τ-periodic solution to Equation* (1); dim $\mathbf{SP} = p$.
(2) *There is an asymptotically τ-periodic solution to Equation* (1).
(3) *If $p < q$, there is an asymptotically quasi-periodic solution to Equation* (1).
(4) *If $p = q$, every bounded solution is an asymptotically τ-periodic solution to Equation* (1).
(5) *If $p = q = r$, and if $R(\lambda, A) := (\lambda I - A)^{-1}$ has a pole of order* 1 *at $\lambda = \lambda_j, 1 \le j \le p$, then all τ-periodic solutions of Equation* (1) *are stable.*

Proof. Statements (1) - (4) are trivial. Assume that the conditions in (5) hold. Then we have $\mathbf{X} = \mathbf{Y} \oplus \mathbf{N}_0$. Hence there is a positive constant H such that $\|U(t)x\| \le H\|x\|$ for $t \ge 0, x \in \mathbf{X}$. Let $u_0(t)$ be any τ-periodic solution of Equation (1). Then for every solution $u(t)$, $u(t) - u_0(t)$ is a solution of the homogeneous equation. Hence $u(t) - u_0(t) = U(t)(u(0) - u_0(0))$, which implies $\|u(t) - u_0(t)\| \le H\|u(0) - u_0(0)\|$ for $t \ge 0$. Therefore the assertion (5) is valid. ∎

In a subsequent paper, we will consider the case where

$$\limsup_{n\to\infty} \|S_n(U(\tau))b_f\| = \infty$$

and the cases of Equation (3) with (4) as well as functional differential equations.

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