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Survey

From Random Series to Random Integrals and Random Mappings*

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Dedicated to Professor Nguyen Duy Tien on the occasion of his 60th Birthday

Abstract. The aim of the paper is to give a brief survey on some directions of research in three closely related topics: Random series, Stochastic Integrals and Random mappings, which are of our interest and related to our work. The contributions of Prof. Nguyen Duy Tien on random series are placed into this context.

1. Introduction

It is well- known that the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

is divergent and the Leibniz series

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots$$

is convergent. Moreover, it is shown that the series

$$\sum_{k=1}^{\infty} \pm \frac{1}{k},$$

where the sign $\pm$ is chosen independently with the same probability converges almost surely (a.s.).

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Historically, the first random series is the series of the form

$$\sum_{i=1}^{\infty} \pm c_n, \tag{1}$$

where the real numbers c_n are given and the sign $\pm$ are random, independent and equiprobable. The series (1) is called a Rademakher series. Starting from the Rademakher series, Kolmogorov and Khinchin [13] made a systematic study of sums of independent random variables. This topic has been the subject of an extensive research and is essential for many areas of probability and analysis. We refer to the books [14, 50] for more information on this subject.

As a continuous analogue of random series, the stochastic integral of a function with respect to a random function was firstly introduced by Wiener [51], Levy [15] and has been developed by many authors (see [41] for more details). Several stochastic integrals such as the Ito stochastic integral, the Stratonovic stochastic integral are central in the modern theory of stochastic analysis and crucial for many applications in financial mathematics.

The concept of mapping plays a basic role in mathematics. A mapping from X into Y is a correspondence that associates with each element $x \in X$ an element $\Phi x \in Y$. In a random environment, however, the image Φx might not be known exactly. It should be a random variable with values in Y. Hence there arises the need of introducing a more realistic formulation of the concept of mapping. It is a concept of random mapping. A random mapping from X into Y is a rule that assigns to each element $x \in X$ a random variable Φx taking values in Y. The stochactic integral is an example of a random mapping, namely the correspondence that associates with each function $f(x) \in L_2[0,1]$ the random variable

$$\int_0^1 f(t)dW(t),$$

where $W(t)$ is a Brownian motion (or the Wiener process). Consequently, random mappings can be considered as a natural framework of stochastic integrals.

The purpose of the paper is to give a brief survey on some directions of research in three closely related topics: Random series, Stochastic integrals and Random mappings. The remainder of the paper consists of three sections, devoted respectively to the three topics under consideration. We do not attempt to present all aspects as well as the comprehensive history of the development of these topics but rather just some directions of research which are of our interest and related to our work.

2. Random Series

2.1. Rademakher, Gaussian and stable series

Let E be a Banach space and (x_n) be a sequence in E. Let (r_n) be a sequence of independent random variables taking two values ± 1 with the probability $1/2$

at each case. Such a sequence is called the Rademakher sequence. Consider the following random series

$$\sum_{n=1}^{\infty} \pm x_n = \sum_{n=1}^{\infty} r_n x_n. \tag{2}$$

Now the problem is to investigate the a.s. convergence of the Rademakher series (2) in the norm topology of E.

Parallel to the Rademakher series, the following series were also considered

$$\sum_{n=1}^{\infty} \xi_n x_n, \tag{3}$$

$$\sum_{n=1}^{\infty} \xi_n^{(p)} x_n, \tag{4}$$

where (ξ_n) is a standard Gaussian sequence (i.e. a sequence of Gaussian i.i.d. random variables with the distribution $N(0,1)$) and $\big(\xi_n^{(p)}\big)(0 < p < 2)$ is a standard p-stable sequence (i.e. a sequence of p-stable i.i.d. random variable with the unit lenght).

The series (3) and (4) are called the Gaussian series and p-stable series, respectively.

In particular, if E is a certain space of functions such as $C[0,1], L_p[0,1]$ then we have the random series of functions. Typical examples of these random series were the Fourier random series

$$\sum_{n=1}^{\infty} \xi_n \sqrt{2} \sin(n + \frac{1}{2})\pi t,$$

which represents the Brownian motion and the Taylor random series

$$\sum_{n=0}^{\infty} r_n z^n.$$

In the case E is a Hilbert space the following theorem gives a simple criterion of the a.s. convergence of the series (2), (3) (4).

Theorem 2.1. [5] *Let E be a Hilbert space. Then the Gaussian series and the Rademakher series converge a.s. if and only if*

$$\sum_{n=1}^{\infty} \|x_n\|^2 < \infty.$$

The p-stable series (4) converges if and only if

$$\sum_{n=1}^{\infty} \|x_n\|^p < \infty.$$

For a general Banach space it is rather hopeless to provide a simple characterization of the sequence $(x_n) \in E$ for which the series (2),(3) or (4) converge a.s. However, a usable necessary and sufficient condition for the a.s. convergence of the Gaussian and stable series can be obtained in the case $E = L_r(T.\mathcal{A}, \mu)$.

Theorem 2.2. [14, p. 46] *Let* $0 < r < \infty$ *and* $E = L_r(T.\mathcal{A}, \mu)$. *Then*
(1) *The Gaussian series* (2) *converges a.s. if and only if*

$$\int_T \left(\sum_{i=1}^{\infty} |x_i(t)|^2 \right)^{r/2} \mu(dt) < \infty.$$

(2) *The p-stable series* (3) *converges a.s. if and only if*

- *For* $r < p$

$$\int_T \left(\sum_{i=1}^{\infty} |x_i(t)|^p \right)^{r/p} \mu(dt) < \infty.$$

- *For* $r > p$

$$\sum_{i=1}^{\infty} \left(\int_T |x_i(t)|^r \right)^{p/r} \mu(dt) < \infty.$$

- *For* $r = p$

$$\sum_{i=1}^{\infty} \int_T |x_i(t)|^p \left(1 + \log^+ \frac{|x_i(t)|^p}{(\sum_{i=1}^{\infty} |x_i(t)|^p) \int_T |x_i(t)|^p \mu(dt)} \right) \mu(dt) < \infty.$$

Moreover, the a.s. convergence of the p-stable series implies its convergence in s-th mean for any $s < p$.

Theorem 2.2 has a long history. We refer to the book [50] for more information on this subject. The most difficult case is the case $r = p$. The full proof of it appeared in [2].

The case of $E = C(T)$, the space of all continuous function on a compact space T, is especially important since these random series give series representation of stochastic processes with continuous sample paths. Unfortunately, this problem is as difficult as the general one because a general Banach space E can be embedded canonically into the space $C(T)$ for some compact space T. However, in the case of Gaussian series, there are some interesting results characterizing the convergence of the Gaussian series in terms of the entropy. Let us define a pseudometric $d(s,t)$ on the set T by

$$d(s,t) = \sqrt{\sum_{i=1}^{\infty} \big(x_i(t) - x_i(s)\big)^2}.$$

The entropy $N(\epsilon)$ is now defined as follows

$$N(\epsilon) = \inf\{n \in N : \exists t_1, \dots, t_n \in T \quad \text{such that} \quad T \subseteq \bigcup K(t_i, \epsilon)\},$$

where $K(t,\epsilon) = \{s \in T : d(t,s) < \epsilon\}$.

Theorem 2.3. [4]
1. *If*

$$\int_0^1 (\log N(\epsilon))^{1/2} d\epsilon < \infty.$$

then the Gaussian series (2) *converges a.s. in* $C(T)$.
2. *If the Gaussian series* (3) *converges a.s. in* $C(T)$ *then*

$$\int_0^1 (\log N(\epsilon))^r d\epsilon < \infty, \quad \forall r < 1/2.$$

3. *The Gaussian series* (3) *converges a.s. in* $C(T)$ *if and only if there exists a probability measure* m *on* T *such that*

$$\lim_{h\to 0} \sup_{t\in T} \int_0^h \left(\log \frac{1}{m(K(t,\epsilon))}\right)^{1/2} d\epsilon = 0.$$

More information on these and related subjects can be found in [16].

2.2. Relationship with Summing Operators

Let $T : l_2 \to E$ be a linear continuous operator and (e_n) be the orthogonal basis in l_2. Consider the random series (2), (3) where $x_n = Te_n$ i.e. the following random series

$$\sum_{n=1}^{\infty} \xi_n T e_n, \tag{5}$$

$$\sum_{n=1}^{\infty} r_n T e_n. \tag{6}$$

The problem is to give necessary and sufficient conditions on T to ensure the a.s. convergence of the series (5) or (6).

If E is a Hilbert space, from Theorem 2.1 it follows immediately that the Gaussian series (5) and the Rademakher series (6) converge a.s. if and only if T is a Hilbert-Schmidt operator. The extension of this result to Banach spaces was considered by many authors. The partition of linear operators into classes of p-summing operators introduced by Piesch [21] and the notion of type and cotype of Banach spaces turn out to be very helpful to this end.

Theorem 2.4. [18] *The following assertions are equivalent:*
1. E *is of cotype* 2
2. *The series* (5) *converges a.s. in* E *if and only if* T *is a* 2*-summing operator.*

Theorem 2.5. [3] *The following assertions are equivalent*
1. *E is of type* 2.
2. *The series* (5) *converges a.s. in E if and only if $T^*: E^* \to l_2$ is a* 2*-summing operator.*

Tien [28] obtained the same results for the Rademakher series (6) . He also remarked that there are results which are true for the Gaussian series but not true for the Rademakher series, namely if (x_n) and (y_n) are two sequences in E such that $\sum_n |(y_n, x^*)|^2 \leq \sum_n |(x_n, x^*)|^2$ for all $x^* \in E^*$ then the convergence a.s. of the Gaussian series $\sum_n \xi_n x_n$ in E implies the convergence a.s. of the Gaussian series $\sum_n \xi_n y_n$. However, an analogous assertion for the Rademakher series does not hold.

Analogous results for the stable series were considered by Thang, Tien [29, 31, 32, 35]. Let $1 < p < 2, 1/p + 1/q = 1$, $T : l_q \to E$ be a linear continuous operator and (e_n) be the standard basis in l_q. Consider the p-stable series

$$\sum_{n=1}^{\infty} \xi_n^{(p)} T(e_n). \tag{7}$$

Then we get the following

Theorem 2.6. [32] *The following assertions are equivalent*
1. *E is of s-cotype p and of p-stable.*
2. *The series* (7) *converges a.s. in E if and only if T is p-summing.*

Theorem 2.7. [31] *The following assertions are equivalent*
1. *E is of p-stable type and can be emmedded into some space L_p.*
2. *The series* (7) *converges a.s. in E if and only if $T^* : E^* \to l_p$ is p-summing.*

2.3. The Ito-Nisio Theorem and Three Series Theorem

Let (X_n) be a sequence of independent random variables with values in a Banach space E. Consider the random series

$$\sum_{n=1}^{\infty} X_n. \tag{8}$$

Obviously, the random series (2),(3) and (4) are special cases of the series (8). The following theorem, due to Ito and Nisio,is a pearl of probabilily theory. It asserts the equivalence of several types of the convergence of the series (8). Actually, the real line version of it was known from the early work of Levy.

Theorem 2.7. [9] *Let (X_n) be a sequence of independent random variables with values in a Banach space E. Then the following three conditions are equivalent:*
1. *The series* (8) *converges a.s.*
2. *The series* (8) *converges in probability.*

3. *The series* (8) *converges in distribution. If, additionally,* (X_n) *are symmetric then Conditions* (1) - (3) *are also equivalent to each of the following condition.*
4. *There exists a* E*-valued random variable* S *such that for each* $x^* \in E^*$ *the series* $\sum_{i=1}^{\infty}(X_i, x^*)$ *converges a.s. to* (S, x^*).
5. *There exists a probability measure* μ *on* E *such that for each* $x^* \in E^*$ *the distributions of* $\sum_{i=1}^{n}(X_i, x^*)$ *converge weakly to* (μ, x^*).

Another pearl of probability theory is the following Three Series Theorem obtained by Kolmogorov in the real line case. His proof also works in the case of Hilbert spaces.

Theorem 2.8. *Let* E *be a Hilbert space . Then the series* (8) *converges a.s. if and only if for some* $a > 0$ *(or, equivalently, for all* $a > 0$*) the following three deterministic series are convergent:*
1. $\sum_{i=1}^{\infty} P(\|X_i\| > a)$;
2. $\sum_{i=1}^{\infty} EX_i^a$;
3. $\sum_{i=1}^{\infty} E\|X_i^a - EX_i^a\|^2$,

where $X^a = XI_{\{\|X\| \leq a\}}$.

The extension of the Three Series Theorem to Banach spaces was considered by Tien [27].

The Ito-Nisio type Theorem is available in the case (X_n) is a martingale difference. Recall that the sequence (X_n) is a E-valued martingale difference if

$$E(X_n|X_1, \dots, X_{n-1}) = 0.$$

It should be noted that if (X_n) is a sequence of E-valued independent random variables with mean zero then it is a martingale difference. The following result is a martingale version of the Ito-Nisio Theorem.

Theorem 2.9. [5] *Let* (X_n) *be a sequence of* E*-valued martingale differences such that* $\sup_n E\|S_n\| < \infty$, *where* $S_n = \sum_{i=1}^{n} X_i$. *Then the following statements are equivalent:*
1. *The series* (8) *converges a.s.*
2. *The series* (8) *converges in probability.*
3. *The series* (8) *converges in distribution.*
4. *There exists a* E*-valued random variable* S *such that for each* $x^* \in E^*$ *the series* $\sum_{i=1}^{\infty}(X_i, x^*)$ *converges a.s. to* (S, x^*).
5. *There exists a probability measure* μ *on* E *such that for each* $x^* \in E^*$ *the distributions of* $\sum_{i=1}^{n}(X_i, x^*)$ *converge weakly to* (μ, x^*).

The three series-type theorem for martingale difference was due to Szulga [26].

Theorem 2.10. [26] *Let* E *be a Hilbert space and let* (X_n) *be a sequence of* E*-valued martingale difference such that the following three series converges a.s. for some* $a > 0$

1. $\sum_{i=1}^{\infty} P\big(\|X_i\| > a|\mathcal{F}_{i-1}\big)$;
2. $\sum_{i=1}^{\infty} E\left(X_i^a|\mathcal{F}_{i-1}\right)$;
3. $\sum_{i=1}^{\infty} E\left(\|X_i^a - E\left(X_i^a|\mathcal{F}_{i-1}\right)\|^2|\mathcal{F}_{i-1}\right)$.

Then the series (8) *converges a.s.*

3. Stochastic Integral

3.1. The Wiener Type Stochastic Integral

In many applications, there arises the need of constructing the stochastic integral of the form

$$\int_0^1 f(t)dX(t),$$

where $f(t)$ is a deterministic function and $X(t)$ is a stochastic process with the independent increaments. Roughly speaking, $\int_0^1 f(t)dX(t)$ can be defined as the limit in probability of the integral sums of the form

$$\sum_{i=0}^{n} f(t_i)\left(X(t_{i+1}) - X(t_i)\right)$$

when the gauge of the partition $0 = t_0 < t_1 < ... < t_n$ tends to zero. Hence the stochastic integral can be considered as a continuous analogue of weight sum of i.i.d. random variables of the form $\sum_{n=1}^{\infty} \xi_n x_n$.

Historically, the first stochastic integral of this type introduced by Wiener [51] for the case $X(t)$ is the Brownian motion. $W(t)$ is called the Wiener stochastic integral. It is shown that f is W-integrable if and only if f belongs to the space $L_2[0,1]$.

For the case $X(t)$ is a stochastic process with independent increament such that the distribution of $X(t) - X(s)$ is symmetric and depends only on $t - s$, it was shown by Urbanik and Woczynski [49] that the function f is X-integrable if and only if f belongs to certain Orlicz space $L_\Phi[0,1]$, where the Orlicz function Φ is defined from the Levy-Khinchin representation of the process $X(t)$.

The stochastic integral w.r.t. a stochastic process with independent increament is a special case of the stochastic integral w.r.t. a random measure. Let $(T, \mathcal{A})$ be a measurable space. A mapping $M : \mathcal{A} \longrightarrow L_0(\Omega)$ is called a random measure on $(T, \mathcal{A})$ if for every sequence (A_n) of disjoint sets from $\mathcal{A}$, the r.v.'s $M(A_n)$ are independent and

$$M\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} M(A_n) \quad \text{in} \quad L_0(\Omega).$$

The stochastic integral of real-valued deterministic functions w.r.t. random measures was defined in the following way. At first, if $f : T \to \mathbb{R}$ is a simple function, $f = \sum_{i=1}^{n} t_i I_{A_i}$ then the stochastic integral of f w.r.t. M is defined by

$$\int f dM = \sum_{i=1}^{n} t_i M(A_i).$$

Next, a function f is said to be M-integrable if there exists a sequence of simple functions (f_n) such that $\lim f_n(t) = f(t) \quad \mu$ - a.s. and the sequence $\{\int f_n dM\}$ converges in $L_0(\Omega)$. If f is M-integrable then we put

$$\int_T f\, dM = p\text{-}\lim \int_T f_n\, dM.$$

It can be shown that the above definition is well-defined, i.e., the limit $\int_T f\, dM$ does not depend on the choice of a particular approximating sequence (f_n). The set of M-integrable functions is denoted by $\mathcal{L}(M)$.

The Wiener-type stochastic integral $\int_T f\, dM$ under various hypotheses on the random measure has been investigated by many authors (see [49, 20, 23, 24]). The most general results, due to Rajput and Rosinski [23], were concerned with a systematic study of the case where M is an arbitrary infinitely divisible random measure. The deterministic characteristics of M were obtained and a necessary and sufficient condition for a function to be M-integrable is given in terms of these deterministic characteristics. Moreover, this stochastic integral is used to obtain the spectral representation of infinitely divisible processes. Namely, for any infinitely divisible process $X(s)$ there exist a infinitely divisible random measure M and a family of functions $(f_s(t))$ such that $X_s = \int f_s(t) dM(t)$.

Vector random measures arise naturally as a Banach space generalization of random measures, i.e., for each $A \in \mathcal{A}$, $M(A)$ are no longer a real-valued random variable but a random variable with values in a Banach space E. Some aspects of vector random measures and the random integral of real-valued functions with respect to vector random measures were considered by Thang [36–37, 48]. The following theorems are random analogues of the Pettis theorem and the Vitali-Hahn-Sacks theorem in the theory of vector measures.

Theorem 3.1. [36] *Let F be a vector symmetric random measure with values in a Banach space E and μ be a control measure for F (this means μ is a positive measure on T such that $\mu(A) = 0$ implies $F(A) = 0$). Then F is μ-continuous , i.e.*

$$\lim_{\mu(A)\to 0} F(A) = 0 \quad \text{in} \quad L_0^E(\Omega).$$

Theorem 3.2. [36] *Let (F_n) be a sequence of symmetric random measures with values in E and μ be a control measure for F_n for all n. Suppose that*

$$\lim_n F_n(A) = F(A) \quad in \quad L_0^E(\Omega)$$

exists for each $A \in \mathcal{A}$. Then the mapping $A \mapsto F(A)$ is also a symmetric random measure with values in E with the control measure μ .

Let Z be an E-valued symmetric Gaussian random measure with the characteristic measure Q. We notice that there exists a control measure for Z. Indeed, by Bartle-Dunford-Schwartz's theorem(see [6, Corollary 6]) there is a finite non-negative measure μ such that $Q(A) = 0$ whenever $\mu(A) = 0$. Clearly, μ is a control measure for Z.

Theorem 3.3. [48]

1. *A function $f : T \to R$ is Z-integrable if and only if the function $|f|^2$ is Q-integrable and*
$$\int |f|^2 \, dQ$$
is a Gaussian covariance operator.
2. *Suppose that E is a Banach space of type 2. Then the function f is Z-integrable if and only if $|f|^2$ is Q-integrable.*

Let Z_p be an E-valued symmetric p-stable random measure with the characteristic measure Q_p. It is easy to see that the variation $|Q_p|$ of the characteristic measure Q_p of Z_p is a control measure for Z_p.

Theorem 3.4. [48]

1. *A function $f : T \to R$ is Z_p-integrable if and only if the function $|f|^p$ is Q_p-integrable and*
$$\int |f|^p \, dQ_p$$
is a spectral measure on E. In this case, $\int |f|^p \, dQ_p$ is exactly the spectral measure of the E-valued symmetric p-stable r.v. $\int f dZ_p$.
2. *Suppose that E is of stable type p. Then the function f is Z_p-integrable if and only if the function $|f|^p$ is Q_p-integrable.*

Theorem 3.5. [37] *Let F be an E-valued symmetric infinitely divisible random measure with the characteristic function of $F(A)$ given by*
$$\Phi_A(x^*) = \exp\{\int_E (\cos(x, x^*) - 1) H(A, dx)\}$$
A function $f : T \to R$ is F-integrable if and only if the positive measure μ defined on E by
$$\mu(B) = H\{(t, x) \in T \times E : f(t)x \in B\}$$
is a Levy measure on E.

3.2. The Ito-Type Stochastic Integral

In order to provide a powerful method for the explicit construction of the paths of diffusion processes, Ito [8] introduced a very important generalization of the Wiener stochastic integral by omitting the restriction that the integrand was a deterministic function. He constructed the stochastic integral of the form

$$\int_0^1 u = u(t,\omega)dW,$$

where the integrand u is a random function adapted w.r.t. the Wiener process such that $\int_0^1 u^2(t,\omega)dt < \infty$ a.s. This stochastic integral can be defined as the limit in probability of Riemann integral sums of the form

$$\sum_{i=0}^{n-1} u(t_i)\left(W(t_{i+1}) - W(t_i)\right)$$

when $\max(t_{i+1} - t_i)$ of the partition $0 = t_0 < t_1 < \cdots < t_n$ tends to zero.

The Ito stochastic integral is subject to a strange calculus. For example, if f is a smooth function then we have the Ito formula

$$f(W_a) - f(W_b) = \int_a^b f'(W_t)dW(t) + \frac{1}{2}\int_a^b f"(W_t)dt,$$

often written instead in the differential form

$$d\left(f(W_t)\right) = f'(W_t)dW_t + \frac{1}{2}f"(W_t)dt.$$

The Ito stochastic integral is essential for the theory of stochastic analysis. Equipped with the notion of the Ito stochastic integral one can consider stochastic differential equations. For example, given smooth functions A, B with bounded derivatives and a random starting point ξ, the problem is to find a process $x(t) = x(t,\omega)$ satisfying

$$dx(t) = A\left(t, x(t)\right)dt + B(t, x(t))dW(t), \quad x(0) = \xi.$$

This is a stochastic differential equation for the process $x(t)$. It should be understood as an abbreviation for the stochastic integral equation

$$x(t) = \xi + \int_0^t A(s, x(s))ds + \int_0^t B(s, x(s))dW(s).$$

It is shown that the solution $x(t)$ is a Markov process with continuous sample paths, in fact a diffusion process. Conversely, every smooth diffusion process is a solution of a stochastic differential equation of this form. Hence stochastic differential equations provides an effective mean of constructing the paths of a diffusion process $x(t)$ from the paths of a Wiener process $W(t)$ and an initial value ξ.

The Ito stochastic integral w.r.t. Wiener process is insufficient for application as well as for mathematical questions. It has been generalized in many directions. Generally speaking, the aim of these generalizations is to define stochastic integrals so that the class of integrators as well as the class of the integrands must be as wide as possible and, at the same time, the stochastic integral should enjoy many good properties. A general stochastic integral in which the integrator is a semimartingale has been developed in [10, 12]. Different definitions of stochastic integral in which the random functions to be integrated are not adapted have been proposed by several authors (see [19] and references therein).

Thang [43] constructed the Ito-type stochastic integral of the form

$$\int_0^1 u(t,\omega)dZ_p$$

in which Z_p is an E-valued symmetric p-stable random measure taking values in a sufficiently smoothable Banach space. Under the assumption that the Banach space E is q-smoothable, (where $q = 2$ if $p = 2$ and $q > p$ if $p < 2$) and the variation $|Q_p|$ of the characteristic measure Q_p is continuous, it is shown that the space of Z_p-integrable functions is precisely the class of adapted random functions in $L_p\left(|Q_p| \times P\right)$.

4. Random Mappings

Let (X, d) be a complete separable metric space and Y a separable Banach space. By definition, a deterministic mapping from X into Y is a rule that assigns to each element $x \in X$ a unique element $\Phi x \in Y$, which is called the image of x under Φ. Due to errors in the measurements and inherent randomness of the environment, the image Φx is not known exactly. Therefore, instead of considering Φx as an element of Y we have to think of it as a random variable with values in Y.

A family $\Phi = \{\Phi x\}_{x\in X}$ of Y-valued random variables indexed by the parameter set X is called a *random mapping* from X into Y. In other words, a random mapping Φ from X into Y is a rule that assigns to each element $x \in X$ a random variable Φx taking values in Y, i.e. a mapping $\Phi : X \to L_0^Y(\Omega)$, where $L_0^Y(\Omega)$ stands for the space of all Y-valued random variables.

Random series and stochastic integral provide a good mechanism to generate random mappings. Let us start by some examples.

Example 1. (Random series). Let $(f_n)_{n=1}^\infty$ be a sequence of deterministic measurable mappings from X into Y and $(\alpha_n)_{n=1}^\infty$ be a sequence of real-valued random variables. Assume that for each $x \in X$ the series

$$\sum_{n=1}^\infty \alpha_n f_n x$$

converges in probability. Put

$$\Phi x = \sum_{n=1}^\infty \alpha_n f_n x;$$

we have a random mapping Φ from X into Y.

Example 2. (The stochastic integral). Let $(W_t)(0 \le t \le 1)$ be the Brownian motion on [0,1]. For each function $x = x(t) \in L_2[0,1]$ we put

$$\Phi x(t) = \int_0^t x(s)dW(s).$$

$\Phi x(t)$ is a continuous random function on [0,1] so it can be regarded as a random variable with values in $C[0,1]$. Hence the correspondence $x \mapsto \Phi x$ defines a random mapping from $L_2[0,1]$ into $C[0,1]$.

More generally, let $(T, \mathcal{A})$ be a measurable space.Rosinski [22] has constructed a stochastic integral of the form

$$\int_T f(s)dM(s),$$

where $f : T \to Y$ is a Y-valued measurable function and M is a random measure on $(T, \mathcal{A})$. This stochastic integral is a Y-valued random variable. Now let $\{F(s,x)\}_{x\in X}$ be a family of M-integrable Y-valued functions indexed by the parameter set X. Then the rule that associates to each $x \in X$ a Y-valued r.v. Φx given by

$$\Phi x = \int_S F(s,x)dM(s),$$

produces a random mapping from X into Y.

4.1. Random Operators

Let X, Y be separable Banach spaces. A random mapping A from X into Y is said to be a random operator if the mapping $A : X \to L_0^Y(\Omega)$ is linear and continuous i.e.

- For each $x_1, x_2 \in X, \lambda_1, \lambda_2 \in \mathbb{R}$ we have
$$A(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 A(x_1) + \lambda_2 A(x_2) \quad \text{a. s.}$$
- $$\lim_{x\to x_0} Ax_n = Ax_0 \quad \text{in} \quad L_0^Y(\Omega).$$

The random operators can be regarded as a random generalization of deterministic (linear continuous) operators. From this point of view there arise naturally the desire for transfering basic notions and theorems of deterministic operators to random operators.

Theorem 4.1. [33] (The Principle of uniform boundedness for random operators)
Let $(A_i, i \in I)$ be a family of random operators from X into Y such that for each $x \in X$

$$\lim_{t\to\infty} \sup_{i\in I} P\{\|A_i x\| > t\} = 0.$$

Then we have

$$\lim_{t\to\infty} \sup_{\|x\|\leq 1} \sup_{i\in I} P\{\|A_i x\| > t\| = 0.$$

Theorem 4.2. [33] (The Banach-Steinhaus theorem for random operators) *Let (A_n) be a sequence of random operators from X into Y. Assume that for each $x \in X$*

$$\lim_{n\to\infty} A_n x \quad \textit{exists in} \quad L_0^Y(\Omega).$$

Then the random mapping $x \mapsto Ax$ given by

$$\lim_n A_n x = Ax$$

is a random operator from X into Y.

Let A be a random operator from X into Y. A random operator B from Y^* into X^* is called the adjoint of A if for every $x \in X, y^* \in Y^*$ we have

$$(Ax, y^*) = (x, By^*) \quad \text{a.s.}$$

The adjoint of a random operator, if it exists, is uniquely determined. However, a random operator need not admit the adjoint. So the problem is to find conditions ensuring the existence of the adjoint of a random operator.

Theorem 4.3. [40] *Let A be a random operator from $l_p, (p > 1)$ into Y. Then A admits the adjoint random operator if and only if for each $y^* \in Y^*$*

$$\sum_{n=1}^{\infty} |(Ae_n, y^*)|^q < \infty \quad a.s.$$

Theorem 4.4. [40] *Let A be a Gaussian random operator from H into H. Denote $\alpha_{ij} = (Ae_i, e_j)$.Then A admits the adjoint random operator if and only if*

1. *For each i, j the series*
$$\sum_{i=1}^{\infty} E(\alpha_{ik}\alpha_{ij}) = r_{kj}$$
converges,
2. *The matrix $R = [r_{kj}]$ represents a bounded linear operator on H with respect the basis (e_n).*
In particular the adjoint of A exists if

- *The matrix (α_{ij}) is symmetric, or*
- $\sum\sum r_{kj}^2 < \infty$.

Let X, Y, Z be three Banach spaces, A be a random operator from X into Y and let U be a deterministic linear operator from Z into X. Define the composition AU by

$$(AU)x(\omega) = A(Ux)(\omega).$$

It is easy to check that AU is a random operator from Z into Y and it admits the adjoint if A does but the converse does not hold. We have

Theorem 4.5. [42]
1. *Suppose that for some* $p > 1$ *we have* $E\,|(Ax, y^*)|^p < \infty$ *for all* $x \in X, y^* \in Y^*$*. Then if the adjoint* U^* *is a* p*-summing operator then* AU *admits the adjoint.*
2. *Suppose that* $E\,|(Ax, y^*)| < \infty$ *for all* $x \in X, y^* \in Y^*$*. Then if the adjoint* U^* *is a nuclear operator then* AU *admits the adjoint.*

For each fixed sample $\omega \in \Omega$ the mapping $A_\omega : X \to Y$ given by $x \mapsto Ax(\omega)$ is called a sample path of A. A is called continuous if for almost all ω the sample A_ω is a linear continuous operator. In this case, the random operator A can be understood as a family $\{A_\omega\}_{\omega\in\Omega}$ of deterministic linear continuous operators from X into Y.

More generally, let V be a subset of the Banach space $L(X, Y)$ of all linear continuous operators from X into Y. The random operator A is called a V-sample if there exists a mapping $T : \Omega \to V$ such that for each $x \in X$

$$P\{\omega : Ax(\omega) = T\omega(x)\} = 1.$$

In particular, if $V = L(X, Y)$ then a $L(X, Y)$-sample random operator is also called sample- continuous. Below we present some conditions for V-sample property of a random operator.

Theorem 4.6. [39, 44] *Let* $X = l_p, (p > 1)$*,* q *stands for the conjugate number of* $p, (1/p) + (1/q) = 1$ *and* (e_n) *the standard basis in* l_p*. Then*
1. *A random operator* A *from* X *into* Y *is sample continuous if*
$$\sum_{n=1}^{\infty} \|Ae_n\|^q < \infty \quad a.s.. \tag{9}$$
2. *For* A *to be sample continuous a necessary condition is that*
$$\sum_{n=1}^{\infty} |(Ae_n, y^*)|^q < \infty, \quad \forall y^* \in Y^*. \tag{10}$$
3. *If* Y *is finite dimensional then* (9) (*and* (10)) *are both necessary and sufficient. Otherwise, neither* (9) *nor* (10) *is necessary and sufficient.*
4. *In the case* $Y = L_r$ *where* $1 < q < r$ *for* $r < 2$ *or* $1 < q$ *for* $r = 2$*, the condition* (9) *is necessary and sufficient for* A *to be* V*-sample, where* V *is the space of all completely summing from* X *into* Y*.*
5. *Let* $X = l_1$*. Then* A *is sample continuous if and only if*
$$\sup_n \|A_n e_n\| < \infty \quad a.s.$$

Theorem 4.7. [39] *Assume that two random operators* A, B *are symmetric and independent of the sense that for all* $x_1, \ldots, x_n, x'_1, \ldots, x'_m$ *in* X*, two random vectors* $(Ax_1, \ldots, Ax_n)$ *and* $(Bx'_1, \ldots, Bx'_m)$ *are symmetric and independent.*

In addition suppose that V is separable. Then the random operator $A+B$ is V-sample if and only if both A and B are V-sample.

Other results on random operators in Hilbert spaces can be found in [25].

4.2. Regularity and Convergence of Random Mappings

Let Φ be a random mapping from X into Y. By definition, Φ can be identified as a mapping $(x,\omega) \mapsto \Phi x(\omega)$ from $X \times \Omega \to Y$ such that for each $x \in X$ the mapping $\omega \mapsto \Phi x(\omega)$ is measurable. A random mapping Ψ from X into Y is said to be a modification of Φ if for each $x \in X$ we have

$$P\{\omega : \Phi x(\omega) = \Psi x(\omega)\} = 1$$

Noting that the exceptional set can depend on x.

- For each fixed sample $\omega \in \Omega$, the mapping $x \mapsto \Phi x(\omega)$ is called a sample path (or sample mapping) of Φ. The random mapping Φ is called continuous if the sample path is continuous for almost all ω, and called measurable if the mapping $(x,\omega) \mapsto \Phi x(\omega)$ is measurable w.r.t. the product σ-algebra $\mathcal{B}(X) \otimes \mathcal{F}$.
- Φ is stochastically continuous if the mapping $\Phi : X \to L_0^Y(\Omega)$ is continuous.
- Φ is sample continuous if Φ has a continuous modification.

Theorem 4.8.[44] *Let Φ be a stochastically continuous random mapping. Then Φ has a measurable modification.*

The problem of the existence of continuous modification of stochastic processes and random fields has been studied intensively by many authors. On the basis of Kolmogorov's continuity theorem we obtain the following sufficient condition ensuring the sample continuity of a Y-valued Gaussian random mapping on a bounded set X of a finite-dimensional space.

Theorem 4.9. [44] *Let Φ be a Y-valued centered Gaussian random mapping on a bounded set X and X can be isometrically embedded into a finite dimensional space. Assume that there exist constants $C > 0, \delta > 0$ and $r > 0$ such that*

$$E\|\Phi x_1 - \Phi x_2\|^r \le C\|x_1 - x_2\|^\delta, \quad \forall x_1, x_2 \in X.$$

Then Φ is sample continuous.

The above condition is not sufficient if X is the unit ball of the space l_1 as shown by a consequence of the following theorem.

Theorem 4.10. [44] *Let (α_n) be a sequence of random variables in $L_1^Y(\Omega)$ such that $\sup_n E\|\alpha_n\| = C < \infty$, X the unit ball of the space l_1 and let (e_n) be the standard basis of l_1. Then*

1. *For each $x \in X$ the series*

$$\Phi x = \sum_{n=1}^{\infty} \alpha_n (x, e_n)$$

converges in $L_1^Y(\Omega)$ *and defines a random mapping* Φ *from* X *into* Y *satisfying*

$$E\|\Phi x_1 - \Phi x_2\| \leq C\|x_1 - x_2\|.$$

2. Φ *is sample continuous if and only if*

$$\sup_n \|\alpha_n\| < \infty.$$

In particular, if (α_n) *is a sequence of Gaussian i.i.d. random variables then* Φ *is not sample continuous.*

The finite dimensional distributions of a random mapping are defined as follows. Let $x_1, x_2, \ldots, x_k$ be elements of X. We define the law of $(\Phi x_1, \ldots, \Phi x_k)$ by

$$P^{\Phi}_{x_1,\ldots,x_k}(A) = P\{\omega : (\Phi x_1, \ldots, \Phi x_k) \in A\}$$

for each $A \in \mathcal{B}(Y^k)$. The family of probability measures $\left(P^{\Phi}_{x_1,\ldots,x_k}\right)$ is called the finite-dimensional distribution of Φ.

For a sequence $\{\Phi_n\}$ of random mappings from X into Y we introduce two types of convergence as follows.

1. The sequence $\{\Phi_n\}$ is said to *converge in probability* if for each $x \in X$ the sequence $\{\Phi_n x\}$ converges in probability. In this case we can define a new random mapping Φ by

$$\Phi x = p\text{-}\lim_{n\to\infty} \Phi_n x.$$

Φ is called the limit in probability of $\{\Phi_n x\}$ and we write

$$\Phi = p\text{-}\lim_{n\to\infty} \Phi_n.$$

2. The sequence $\{\Phi_n\}$ is said to *converge in law* if for each positive integer k and for each finite set $x_1, x_2, \ldots, x_k$ in X the sequence $(P^{\Phi_n}_{x_1,\ldots,x_k})$ converges weakly as $n \to \infty$.In this case, it can be shown that there exists a random mapping Φ such that for each finite set $x_1, x_2, \ldots, x_k$ in X the sequence $\left(P^{\Phi_n}_{x_1,\ldots,x_k}\right)$ converges weakly to $\left(P^{\Phi}_{x_1,\ldots,x_k}\right)$. The random mapping Φ is called the limit in law of the sequence $\{\Phi_n\}$ and we write

$$\Phi = \mathcal{L}\text{-}\lim_{n\to\infty} \Phi_n.$$

Now we study the relation between two types of convergence. Clearly, if the sequence (Φ_n) converges in probability then it also converges in law. Under some assumption, the converse is true if the sequence (Φ_n) is replaced by an equivalent one.

Theorem 4.11. [46] *Let* $(\Phi_n)_{n=1}^{\infty}$ *be a sequence of stochastically continuous random mappings converging in law . In addition, the limit in law of the sequence* (Φ_n) *is a stochastical continuous random mapping* Φ_0*. Then there exist random mappings* $(\Psi_n)_{n=1}^{\infty}$ *and* Ψ_0 *such that for each* $n = 0, 1, \ldots,$ Φ_n *and* Ψ_n *have* *the same finite-dimensional distributions and the sequence* $(\Psi_n)_{n=1}^{\infty}$ *converges in probability to* Ψ_0.

In the case of random operators, it is interesting to note that the assertion of the above theorem holds without the assumption about the stochastical continuity of Φ_0. Namely we have the following

Theorem 4.12. [46] *Let* $(\Phi_n)_{n=1}^{\infty}$ *be a sequence of random operators converging in law. Then*

1. *The limit in law of the sequence* $(\Phi_n)_{n=1}^{\infty}$ *is again a random operator.*
2. *There exists a sequence of random operators* $(\Psi_n)_{n=1}^{\infty}$ *such that for each positive integer* n*,* Φ_n *and* Ψ_n *have the same finite-dimensional distributions and the sequence* $(\Psi_n)_{n=1}^{\infty}$ *converges in probability.*

4.3. Series and Integral Representations of Random Mappings

It is well-known that a second order stochastic process can be represented as the sum of orthogonal random variables (the Karhunen-Loeve expansion) and a stationary Gaussian stochastic process can be written as a integral of white noise (the spectral representation). For non-Gaussian processes similar spectral representations were obtained by several authors (Kuelbs for symmetric stable processes, Rajput [23] for symmetric semi stable processes and recently Rajput, Rosinski [24] for infinitely divisible processes). Such representations have proved exceedingly useful in the statistical analysis of processes under consideration.

In this section we establish some series and spectral representations for random mappings from X into Y.

Theorem 4.13. [45] *Let* Φ *be a stochastically continuous symmetric Gaussian random mapping from* X *into* Y*. Then* Φ *can be expressed as a random series of the form*

$$\Phi x = \sum_{n=1}^{\infty} \alpha_n f_n x,$$

where (α_n) *is a sequence of real-valued Gaussian independent random variables with distribution N(0,1)and* (f_n) *is a sequence of deterministic continuous mappings from* X *into* Y.

Gaussian random mappings belong to the class of stable random mappings. However the random series are not enough to represent symmetric stable random mappings in general. The random integral of Banach space valued functions w.r.t. random symmetric measure constructed by Rosinski [22]is used in studying the representation problems.

Theorem 4.14. [45] *Let* Φ *be a symmetric p-stable random mapping from* X *into* Y *such that the real process* $\{\langle \Phi x, y^* \rangle, x \in X, y^* \in Y^*\}$ *is separable. Then there exist a symmetric p-stable random measures* M *on some measurable* *space* $(S, \mathcal{A}, \mu)$ *and a family* $\{g(s,x), x \in X\}$ *of* M*-integrable* Y*-valued functions indexed by the parameter set* X *such that*

$$\Phi x = \int_S g(s,x)\, dM(s).$$

In particular a symmetric Gaussian random mapping can be also written as a random integral w.r.t. a white noise.

Let Φ be a symmetric p-stable random mapping and $[\Phi]$ be the closed subspace of L_0 spanned by the random variables $\{(\Phi x, y^*)\}$. It is shown that $[\Phi]$ is a Frechet space and can be isometrically embedded into some space L_p. The following theorem provides a sufficient condition for a symmetric p-stable mapping to be represented as a random series.

Theorem 4.15. [45] *If* $[\Phi]$ *is isometric to* l_p *then* Φ *can be expressed as a random series of the form*

$$\Phi x = \sum_{n=1}^{\infty} \alpha_n^{(p)} f_n x,$$

where $(\alpha_n^{(p)})$ *is a sequence of real-valued* p*-stable i.i.d. random variables and* (f_n) *is a sequence of deterministic continuous mappings from* X *into* Y*.*

4.4. Strong Random Mappings

A random mapping Φ from X into Y may be considered as an action which transforms each deterministic input $x \in X$ into a random output Φx. The notion of strong random mappings arises in the case we have an action which acts on random outputs.

Definition. *Let* $\mathcal{D}(X)$ *be a subset of* $L_0^X(\Omega)$*. The mapping* $\Phi : \mathcal{D}(X) \to L_0^Y(\Omega)$ *is said to be a strong random mapping from* $\mathcal{D}(X)$ *into* Y*.*

The Ito stochastic integral provides a good example of strong random mappings. Let $X = L_2[0,1]$ and $\mathcal{D}(X)$ be the set of X-valued random variables $u(.,\omega)$ such that the random function $u(t,\omega)$ is adapted with respect to the Wiener process $W(t)$. Then the correspondence

$$u \mapsto \int_0^1 u(t,\omega) dW(t),$$

where the stochastic integral is the Ito stochastic integral defines a strong random mapping from $\mathcal{D}(X)$ into $\mathbb{R}$.

Theorem 4.16. [44] *Let* Φ *be a random mapping from* X *into* Y *admitting the series expansion*

$$\Phi x = \sum_{n=1}^{\infty} \alpha_n f_n x, \tag{11}$$

where (α_n) *is a sequence of real-valued independent random variables,* (f_n) *is a sequence of deterministic measurable mappings from* X *into* Y *and the series* (10) *converges in probability for each* $x \in X$.

1. *Suppose that* Y *is a Hilbert space,* $\mathcal{F}_n$ *is the* σ*-algebra generated by* $\alpha_1, \dots, \alpha_n$ *and* $\mathcal{D}(X)$ *denotes the set of* X*-valued random variables* u *such that* $f_n u$ *is* $\mathcal{F}_{n-1}$*- measurable for each* $n > 1$*. Then for each* $u \in \mathcal{D}(X)$ *the series*
$$\sum_{n=1}^{\infty} \alpha_n f_n(u(\omega))$$
converges in probability and so defines a strong random mapping from $\mathcal{D}(X)$ *into* Y *which is called an extension of* Φ.
2. *Suppose that* Y *is a* p*-smoothable Banach space and the sequence* (α_n) *is bounded in* L_p *(i.e.* $\sup_n E|\alpha_n|^p < \infty$*). Let* $\mathcal{A}(X)$ *denote the set of* X*-valued random variables* u *such that* $f_n u$ *is* $\mathcal{F}_{n-1}$*- measurable for each* $n > 1$*, and in addition,*
$$\sum_{n=1}^{\infty} \|f_n u\|^p < \infty \quad a.s.$$
Then for each $u \in \mathcal{A}(X)$ *the series*
$$\sum_{n=1}^{\infty} \alpha_n f_n(u(\omega))$$
converges in probability and so defines a strong random mapping from $\mathcal{A}(X)$ *into* Y *extending* Φ.

Theorem 4.17. [44]

1. *Let* Φ *be a random mapping from* X *into* Y *and let* $\mathcal{D}(X)$ *denotes the set of countably-valued random variables. Then* Φ *can be extended into a strong random mapping from* $\mathcal{D}(X)$ *into* Y *by the direct substitution*
$$\Phi u = \Phi(u(\omega), \omega).$$
2. *Let* Φ *be a measurable random mapping from* X *into* Y*. Then* Φ *can be extended into a strong random mapping from* $L_0^X(\Omega)$ *into* Y *by the direct substitution*
$$\Phi u = \Phi\,(u(\omega), \omega)$$
for each $u \in L_0^X(\Omega)$.
In particular a sample continuous random mapping can be extended in this way.

Theorem 4.18. *Let* Φ *be a strong random mapping from* $L_0^X(\Omega)$ *into* Y*. Suppose that there exist constants* $c, k \in (0,1)$ *such that for every two* X*-valued random variable* u, v*,* $\Phi u - \Phi v$ *is* (c,k)*-dominated by* $u - v$*, i.e.,*
$$P\{\|\Phi u - \Phi v\| > t\} \le cP\{k\|u - v\| > t\}$$

for all $t > 0$. Then there exists a unique X-valued random variable u satisfying $\Phi u = u$ i.e. Φ has a unique random fixed point.

The problem of establishing the existence, the uniqueness of random fixed point of strong random mappings would be very useful in the theory of random equations, stochastic differential equations and the study of stochastic approximation procedures.

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