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On Continuity Properties of the Solution Map in Linear Complementarity Problems

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Abstract. In an earlier paper [3] we have proved that, in a linear complementarity problem with a Q-matrix, the Lipschitzian continuity and the lower semicontinuity of the solution map are equivalent. In this paper, this fact is proved in the general case where the underlying matrix M of the problem need not have any prescribed special structure.

1. Introduction

For a given $M \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$, the linear complementarity problem corresponding to M and q is to find $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad Mx + q \geq 0, \quad x^T(Mx+q) = 0. \tag{1.1}$$

The solution set of (1.1) is denoted by $S_M(q)$. Thus, for a fixed M, S_M is a set-valued map from $\mathbb{R}^n$ into $\mathbb{R}^n_+$. It was known [1] that

$$\text{Dom}S_M = \bigcup_{\alpha\subseteq I} K_\alpha, \tag{1.2}$$

where $I = \{1, 2, \ldots, n\}$ and K_α is the complementarity cone corresponding to the index set α which is defined by setting

$$K_\alpha := \Big\{ \sum_{i\in\alpha} \lambda_i(-M^i) + \sum_{j\in I\backslash\alpha} \mu_j e_j \mid \lambda_i \geq 0, i \in \alpha;\ \mu_j \geq 0, j \in I\backslash\alpha \Big\}, \tag{1.3}$$

with M^i standing for the i^{th} column vector in M and e_j being the j^{th} unit vector in $\mathbb{R}^n$.

In Sec. 3 we shall prove that, for any $M \in \mathbb{R}^{n\times n}$ the solution map S_M is Lipschitz on its effective domain if and only if it is lower semicontinuous on the set. To this end, we first show that if S_M is lower semicontinuous on DomS_M then M is nondegenerate, and then, by utilizing results in [2, 4] we deduce that in this case S_M is also Lipschitz continuous on DomS_M.

From now on, let M be an $n \times n$-matrix with elements $a_{ij} \in \mathbb{R},\ 1 \leq i,j \leq n.$ For $\alpha \subseteq \{1, 2, \dots, n\}$, let M_α denote the submatrix of M with the elements a_{ij}, $i, j \in \alpha$. The determinants of these matrices are called the principal minors of M. A matrix is said to be nondegenerate if all of the principal minors are nonzero. If at least one of the principal minors is zero then M is a degenerate matrix. For abbreviation, we write M_k instead of $M_{\{1,2,\dots,k\}}$.

Recall that, a set-valued map F from $\mathbb{R}^n$ into $\mathbb{R}^n$ is said to be Lipschiptz on a subset $U \subset \mathbb{R}^n$ if there exists a constant number L such that

$$H(F(p), F(q)) \leq L\|p-q\|; \quad \forall p, q \in U, \tag{1.4}$$

where $H(.,.)$ denotes the Hausdorff distance. F is called lower semicontinuous (l.s.c. for short) at $\bar{q} \in \text{Dom}F$ if for any $\bar{x} \in F(\bar{q})$ and $\epsilon > 0$ there exists $\delta > 0$ such that $F(q) \cap B(\bar{x}, \epsilon) \neq \emptyset$ for all $q \in B(\bar{q}, \delta) \cap \text{Dom}F$. Or, equivalently, for any $\bar{x} \in F(\bar{q})$ and any sequence $(q^m) \subset \text{Dom}F$ converging to $\bar{q}$ there exists a sequence (x^m) such that $x^m \in F(q^m)$ for each $m \in \mathbb{N}$ and $x^m \to \bar{x}$. Finally, F is said to be l.s.c. if it is l.s.c. at every point of Dom$F$.

2. Lower Semicontinuity of S_M Implies Nondegeneracy of M

Theorem 2.1 below is one of the two main results of this paper. For the proof of that theorem we shall need the following lemma.

Lemma 2.1. *Let $M \in \mathbb{R}^{n\times n}$. For every $n \geq k \geq 2$ and $k \geq l \geq 1$ there exists a vector $v = (v_1, v_2, \cdots, v_k)^T \in \mathbb{R}^k$ such that*

$$v_l = \det\left(M_{\{1,\dots,k\}\setminus\{l\}}\right) \tag{2.1}$$

and

$$v^T M_k = \det(M_k).e_l^T. \tag{2.2}$$

Proof. For each $i = 1, \dots, k$ we define v_i as the cofactor of a_{il} in the matrix M_k. By M_k^j we denote the j-th column vector of M_k. From the theory of determinants it follows that

$$v^T M_k^j = \begin{cases} 0 & \text{if } j \neq l, \\ \det(M_k) & \text{if } j = l. \end{cases}$$

Or, $v^T M_k = \det(M_k) e_l^T$. Besides, $v_l = \det\left(M_{\{1,\cdots,k\}\setminus\{l\}}\right)$ by definition. The proof is complete. ∎

Theorem 2.1. *For any $M \in \mathbb{R}^{n\times n}$, if $S_M(\cdot)$ is l.s.c. then M is nondegenerate.*

Proof. We first consider the case $n = 1$. If M is degenerate then $M = (0)$ and

$$S_M(q) = \begin{cases} \mathbb{R}_+ & \text{if } q = 0, \\ 0 & \text{if } q > 0, \\ \emptyset & \text{if } q < 0. \end{cases}$$

So S_M is not l.s.c. at $q = 0 \in \text{Dom}S_M$.

Now, for the case $n \geq 2$, we suppose, by contrary, that S_M is l.s.c. and M is degenerate. Denote by M_α the singular submatrix of M having the property that all its proper principal minors are nonzero. Without loss of generality, we can assume that $\alpha = \{1, 2, \ldots, k\}$, $k \leq n$. So, $\det(M_k) = 0$ and, if $k > 1$, $\det\left(M_{\{1,\ldots,k\}\setminus\{l\}}\right) \neq 0$ for all $l \in \{1, \ldots, k\}$.

If $k = 1$ then $a_{11} = 0$. Choose $\bar{x} := (1, 0, \cdots, 0)^T \in \mathbb{R}^n$ and, for each $m \in N$, set

$$\bar{q} := \begin{pmatrix} 0 \\ r \\ \vdots \\ r \end{pmatrix} \in \mathbb{R}^n, \quad q^m := \begin{pmatrix} \frac{1}{m} \\ r \\ \vdots \\ r \end{pmatrix} \in \mathbb{R}^n, \tag{2.3}$$

where $r := \max\{|a_{21}|, |a_{31}|, \ldots, |a_{n1}|\} + 1 \geq 1$. It is not difficult to verify that $\bar{x} \in S_M(\bar{q})$, $0 \in S_M(q^m)$ for every $m \in N$. So $\bar{q} \in \text{Dom}S_M$ and $q^m \in \text{Dom}S_M$ for every $m \in N$. Furthermore, $q^m \to \bar{q}$. By the lower semicontinuity of $S_M(.)$ there exists a sequence (x^m) satisfying $x^m \in S_M(q^m)$ for all $m \in N$ and

$$\lim_{m\to\infty} x^m = \bar{x} = (1, 0, \ldots, 0)^T. \tag{2.4}$$

We have

$$\lim_{m\to\infty} (Mx^m + q^m) = M\bar{x} + \bar{q} = \begin{pmatrix} 0 \\ a_{21} + r \\ \vdots \\ a_{n1} + r \end{pmatrix} \geq \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{2.5}$$

It follows from (2.4) and (2.5) that for some m_0 large enough we have

$$\begin{cases} (Mx^{m_0} + q^{m_0})_j > 0; \quad \forall j \geq 2, \\ x_1^{m_0} > 0. \end{cases} \tag{2.6}$$

Since $x^{m_0} \in S_M(q^{m_0})$, from (2.6) we obtain

$$\begin{cases} x_j^{m_0} = 0; \quad \forall j \geq 2, \\ (Mx^{m_0} + q^{m_0})_1 = 0. \end{cases} \tag{2.7}$$

Using the first property in (2.7) and the assumption $a_{11} = 0$, one has

$$(Mx^{m_0} + q^{m_0})_1 = \sum_{j=1}^{n} a_{1j} x_j^{m_0} + q_1^{m_0} = \frac{1}{m_0} > 0.$$

This contradicts the second property in (2.7).

Now assume that $k > 1$. Since M_k is singular, k column vectors of M_k are linearly dependent. By Lemma 2.1 in [2] we can find $\lambda_1, \ldots, \lambda_k \geq 0$ such that at least one of them equals zero and

$$\sum_{j=1}^{k} M_k^j = \sum_{j=1}^{k} \lambda_j M_k^j. \tag{2.8}$$

Since all the columns of M_k have the same role in the sense that $M_{\{1,\dots,k\}\setminus\{l\}}$ is nonsingular for all $l \in \{1,\dots,k\}$, without loss of generality we can assume that $\lambda_k = 0$ and (2.8) can be rewritten as follows

$$\sum_{j=1}^{k} a_{ij} = \sum_{j=1}^{k-1} \lambda_j a_{ij}; \quad \forall i = 1,\dots,k. \tag{2.9}$$

Now let $\bar{x}$, x', $\bar{q}$, q^m $(m \in N)$ be the vectors in $\mathbb{R}^n$ defined by

$$\bar{x} := \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \quad (k^{\text{th}}) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad x' := \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{k-1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.10}$$

$$\bar{q} := \begin{pmatrix} -\sum_{j=1}^{k} a_{1j} \\ \vdots \\ -\sum_{j=1}^{k} a_{kj} \\ r \\ \vdots \\ r \end{pmatrix}, \quad q^m := \bar{q} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{m} \quad (k^{th}) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.11}$$

where

$$r := \max\Big\{ \Big|\sum_{j=1}^{k} a_{k+1,j}\Big|, \dots, \Big|\sum_{j=1}^{k} a_{nj}\Big|, \Big|\sum_{j=1}^{k-1} \lambda_j a_{k+1,j}\Big|, \dots, \Big|\sum_{j=1}^{k-1} \lambda_j a_{nj}\Big| \Big\} + 1.$$

Then

$$M\bar{x} + \bar{q} = \begin{pmatrix} \sum_{j=1}^{k} a_{1j} \\ \sum_{j=1}^{k} a_{2j} \\ \vdots \\ \sum_{j=1}^{k} a_{nj} \end{pmatrix} + \bar{q} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{j=1}^{k} a_{k+1,j} + r \\ \vdots \\ \sum_{j=1}^{k} a_{nj} + r \end{pmatrix} \geq \begin{pmatrix} 0 \\ \vdots \\ 0 \quad (k^{th}) \\ 1 \\ \vdots \end{pmatrix}. \tag{2.12}$$

Using (2.9)–(2.11), we obtain

$$Mx' + q^m = \begin{pmatrix} \sum_{j=1}^{k-1} \lambda_j a_{1j} \\ \sum_{j=1}^{k-1} \lambda_j a_{2j} \\ \vdots \\ \sum_{j=1}^{k-1} \lambda_j a_{nj} \end{pmatrix} + q^m$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{m} \\ \sum_{j=1}^{k-1} \lambda_j a_{k+1,j} + r \\ \vdots \\ \sum_{j=1}^{k-1} \lambda_j a_{nj} + r \end{pmatrix} \geq \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{m} \quad (k^{th}) \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{2.13}$$

Combining (2.12) and (2.13) with (2.10) it implies that $\bar{x} \in S_M(\bar{q})$ and $x' \in S_M(q^m)$; $\forall m \in N$. Furthermore, $q^m \longrightarrow \bar{q}$ as $m \longrightarrow \infty$. By the lower semicontinuity of S_M, there exists a sequence (x^m) converging to $\bar{x}$ and $x^m \in S_M(q^m)$, $m \in N$. Since $x^m \longrightarrow \bar{x}$ and $Mx^m + q^m \longrightarrow M\bar{x} + \bar{q}$, from (2.10) and (2.12) it follows that there exists m_0 large enough such that

$$\begin{cases} x_i^{m_0} > 0, & \forall i = 1, \dots, k, \\ (Mx^{m_0} + q^{m_0})_j > 0; & \forall j = k+1, \dots, n. \end{cases} \tag{2.14}$$

Since $x^{m_0} \in S_M(q^{m_0})$, (2.14) implies

$$\begin{cases} (Mx^{m_0} + q^{m_0})_i = 0; & \forall i = 1, \dots, k, \\ x_j^{m_0} = 0; & \forall j = k+1, \dots, n. \end{cases} \tag{2.15}$$

Thus, by setting $z := Mx^{m_0} + q^{m_0}$ one gets

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ z_{k+1} \\ \vdots \\ z_n \end{pmatrix} = z = Mx^{m_0} + q^{m_0}$$

$$= M \begin{pmatrix} x_1^{m_0} \\ \vdots \\ x_k^{m_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} -\sum_{j=1}^{k} a_{1_j} \\ \vdots \\ -\sum_{j=1}^{k} a_{k_j} \\ r \\ \vdots \\ r \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{m_0} \quad (k^{th}) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.16}$$

Noting that M_k^j is the j^{th} column vector of M_k, one derives from (2.16) that

$$\sum_{j=1}^{k} x_j^{m_0} M_k^j - \sum_{j=1}^{k} M_k^j + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{m_0} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^k. \tag{2.17}$$

By virtue of Lemma 2.1 we can find $v = (v_1, \dots, v_k)^T \in \mathbb{R}^k$ satisfying

$$v^T M_k = (0, \dots, 0)^T \in \mathbb{R}^k \tag{2.18}$$

and

$$v_k = det(M_{k-1}). \tag{2.19}$$

Taking the scalar product of both sides of the equality in (2.17) with v we have

$$\sum_{j=1}^{k} (x_j^{m_0} - 1) v^T M_k^j + \frac{1}{m_0} v_k = 0.$$

This together with (2.18) gives $\det(M_{k-1}) = v_k = 0$, a contradiction with the definition of M_k. So M is nondegenerate and the proof is complete. ∎

3. Equivalence Between the Two Continuity Properties

The next theorem is the second main result of this paper.

Theorem 3.1. *Let $M \in \mathbb{R}^{n \times n}$. Then S_M is Lipschitz on* DomS_M *if and only if it is lower semicontinuous.*

Proof. Obviously, we need only verify the sufficient condition. Assume that S_M is l.s.c. on DomS_M. By Theorem 2.1 M is nondegenerate, and hence, by [1] $S_M(q)$ is a finite set for every $q \in$ DomS_M. Besides, by virtue of [4, Proposition 1], S_M is uniformly locally upper Lipschitz on DomS_M. That is, with a certain positive number $\lambda > 0$, for all $\bar{q} \in$ DomS_M there exists $\delta(\bar{q}) > 0$ such that

$$S_M(q) \subset S_M(\bar{q}) + \lambda \|q - \bar{q}\| B(0, 1); \quad \forall q \in B(\bar{q}, \delta(\bar{q})). \tag{3.1}$$

The proof of the theorem now can be divided into three lemmas. ∎

Lemma 3.1. *For any $\bar{q} \in$ DomS_M there exists $\eta > 0$ such that*

$$H\left(S_M(q), S_M(\bar{q})\right) \le \lambda \|q - \bar{q}\|; \quad \forall q \in B(\bar{q}, \eta) \cap \text{Dom} S_M. \tag{3.2}$$

Proof. Take any $\bar{q} \in$ DomS_M and assume that $S_M(\bar{q}) = \{x^1, \dots, x^k\}$. We set

$$\epsilon := \min\left\{\|x^i - x^j\|, 1 \le i < j \le k\right\} > 0. \tag{3.3}$$

Since S_M is l.s.c. at $\bar{q}$ and $S_M(\bar{q})$ is finite, there exists $\delta_1 > 0$ such that

$$S_M(q) \cap B\left(x^i, \frac{\epsilon}{2}\right) \ne \emptyset; \ \forall q \in B(\bar{q}, \delta_1) \cap \text{Dom} S_M, \forall i = 1, 2, \dots, k. \tag{3.4}$$

We now choose $\eta := \min\{\delta_1, \delta(\bar{q}), \epsilon/2\lambda\}$. Then for all $q \in B(\bar{q}, \eta) \cap \mathrm{Dom}S_M$ both (3.1) and (3.4) hold. For every $x^i \in S_M(\bar{q})$, by (3.4) there exists v such that

$$v \in S_M(q) \quad \text{and} \quad \|v - x^i\| < \frac{\epsilon}{2}. \tag{3.5}$$

By the definition of ϵ it follows that

$$v \notin x^j + \frac{\epsilon}{2} B(0,1), \quad \forall j \neq i, \tag{3.6}$$

hence, noting that $\lambda\|q - \bar{q}\| < \lambda\eta \leq \epsilon/2$ we have

$$v \notin x^j + \lambda\|q - \bar{q}\| B(0,1), \quad \forall j \neq i. \tag{3.7}$$

On the other hand, from (3.1) it follows that

$$v \in S_M(q) \subset \bigcup_{j=1}^{k} (x^j + \lambda\|q - \bar{q}\| B(0,1)). \tag{3.8}$$

Combining this with (3.7) we get

$$v \in x^i + \lambda\|q - \bar{q}\| B(0,1),$$

or,

$$x^i \in v + \lambda\|q - \bar{q}\| B(0,1) \subset S_M(q) + \lambda\|q - \bar{q}\| B(0,1).$$

Since this inclusion holds for every $x^i \in S_M(\bar{q})$, it follows that

$$S_M(\bar{q}) \subset S_M(q) + \lambda\|q - \bar{q}\| B(0,1)$$

which together with (3.1) yields (3.2). ■

Lemma 3.2. *For all $p, q \in \mathrm{Dom}S_M$ such that $[p, q] \subset \mathrm{Dom}S_M$ we have*

$$H(S_M(p), S_M(q)) \leq \lambda\|p - q\|, \tag{3.9}$$

where $[p, q]$ denotes the segment $\mathrm{co}\{p, q\}$.

Proof. This lemma can be derived from Lemma 3.1 and the compactness of the segment $[p, q]$. ■

Lemma 3.3. *There exists $L \geq 0$ such that*

$$H(S_M(p), S_M(q)) \leq L\|p - q\|, \quad \forall p, q \in \mathrm{Dom}S_M.$$

From this lemma the theorem follows.

Proof. Applying [2, Corollary 2.1] for the class of polyhedral convex cones $\{K_\alpha, \alpha \subseteq I\}$ there exists $\gamma > 0$ such that for all $p \in K_\alpha$, $q \in K_\beta$ with $\alpha \subseteq I$, $\beta \subseteq I$, there exists $u \in K_\alpha \cap K_\beta$ satisfying

$$\|p - q\| \geq \gamma(\|p - u\| + \|q - u\|). \tag{3.10}$$

Now we set $L := \lambda/\gamma$. For all $p, q \in \mathrm{Dom}S_M$ there are $\alpha \subseteq I$ and $\beta \subseteq I$ such that $p \in K_\alpha$ and $q \in K_\beta$. Denoting $u \in K_\alpha \cap K_\beta$ the vector satisfying (3.10) we have

$$[u,p] \subset K_\alpha \subset \mathrm{Dom}S_M, \ [u,q] \subset K_\beta \subset \mathrm{Dom}S_M.$$

From (3.9) one gets

$$H(S_M(p), S_M(u)) \le \lambda \|u - p\|,$$
$$H(S_M(u), S_M(q)) \le \lambda \|u - q\|.$$

Combining these two inequalities we obtain

$$\begin{aligned} H(S_M(p), S_M(q)) &\le H(S_M(p), S_M(u)) + H(S_M(u), S_M(q)) \\ &\le \lambda(\|u-p\| + \|u-q\|) \\ &\le \frac{\lambda}{\gamma}\|p-q\| = L\|p-q\|. \end{aligned}$$

The proof is complete. ■

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