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A Wide Approximate Continuity

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Abstract. Introducing the notion of widely approximate continuous functions we study some basic properties of such functions.

1. Introduction and Definitions

Let $\mathbb{R}$ be the real line and m^*A , mA denote respectively the outer Lebesgue measure and Lebesgue measure of a set $A \subset \mathbb{R}$. We denote by $d^+(A,x) = \limsup_{y\to x+} m^*(A\cap(x,y))/(y-x)$, $d_+(A,x) = \liminf_{y\to x+} m^*(A\cap(x,y))/(y-x)$, $d^-(A,x) = \limsup_{y\to x-} m^*(A\cap(y,x))/(x-y)$ and $d_-(A,x) = \liminf_{y\to x-} m^*(A\cap(y,x))/(x-y)$ respectively the upper right, lower right, upper left and lower left density of the set A at $x \in \mathbb{R}$.

If these four extreme densities are equal to one another, their common value is the density of the set A at x and is denoted by $d(A,x)$.

Sarkhel and Dey [2] introduced the following definition.

Definition 1. *A set $E \subset \mathbb{R}$ is said to be sparse at a point $x \in \mathbb{R}$ on the right if there exists, for every $\varepsilon > 0$, a $k > 0$ such that every interval $(a,b) \subset (x, x+k)$ with $a - x < k(b-x)$, contains at least one point y such that $m^*\{E \cap (x,y)\} < \varepsilon(y-x)$.*

The family of sets sparse at x on the right is denoted by $S(x+)$ and analogously we define $S(x-)$. A set E is called sparse at x if $E \in S(x)$ where $S(x) = S(x+) \cap S(x-)$. We put

$$S_0(x+) = \{E : E \subset \mathbb{R} \text{ and } d^+(E,x) = 0\},$$
$$S_0(x-) = \{E : E \subset \mathbb{R} \text{ and } d^-(E,x) = 0\},$$

and

$$S_0(x) = S_0(x+) \cap S_0(x-).$$

Clearly $S_0(x+) \subset S(x+)$, $S_0(x-) \subset S(x-)$ and $S_0(x) \subset S(x)$ for every $x \in \mathbb{R}$. Also it is shown in [2] that these inclusions are proper.

We set

$$\underline{S}_0(x+) = \{E : E \subset \mathbb{R} \text{ and } d_+(E, x) = 0\},$$

$$\underline{S}_0(x-) = \{E : E \subset \mathbb{R} \text{ and } d_-(E, x) = 0\},$$

and

$$\underline{S}_0(x) = \underline{S}_0(x+) \cap \underline{S}_0(x+).$$

By [2, Corollary 3.1.1] it follows that $S(x+) \subset \underline{S}_0(x+), S(x-) \subset \underline{S}_0(x-)$ and $S(x) \subset \underline{S}_0(x)$.

The following example and its left analogue show that the above inclusions are proper.

Example 1. Since the outer Lebesgue measure is translation invariant, we may choose, without loss of generality, $x = 0$. Let

$$a_n = \frac{1}{n\exp(n^2+1)},\ b_n = \frac{1}{\exp(n^2)} \text{ for } n = 1, 2, 3, \ldots$$

and $E = \bigcup_{n=1}^{\infty}(a_n, b_n)$. Let $\varepsilon > 0$ be given. We fix a positive integer $m > 1$ such that $1/m < \varepsilon$. Since $b_{n+1}/a_n = n/\exp(2n) \to 0$ as $n \to \infty$, there exists a positive integer N such that $mb_{n+1} < a_n$ for $n \geq N$.

Now for $n \geq N$ and for $y \in [mb_{n+1}, a_n]$ we get

$$m^*\{E \cap (0, y)\} = m^*\left(E \cap (0, b_{n+1})\right) \leq \frac{1}{m}(mb_{n+1}) < \varepsilon y.$$

This shows that $d_+(E, 0) = 0$.

Again $m^*(E \cap (0, b_n)) \geq b_n - a_n = (1 - 1/ne)b_n$ for $n = 1, 2, 3, \ldots$ and so $d^+(E, 0) = 1$.

Since for a set $E_1 \in S(x+)$ we get $d^+(E_1, x) < 1$ (cf. [2, Corollary 3.1.1]), it follows that $E \notin S(0+)$.

Sinharoy [3] considered the following classes and their left analogues:

$$S_3(x+) = \{E : E \subset \mathbb{R} \text{ and } d_+(E, x) = 0 \text{ and } d^+(E, x) < 1\},$$

$$S_2(x+) = \{E : E \in S_3(x+) \text{ and } E \cup F \in S_3(x+) \text{ for all } F \in S_3(x+)\},$$

and

$$S_i(x) = S_i(x+) \cap S_i(x-) \text{ for } i = 2, 3.$$

Evidently $S_i(x+) \subset \underline{S}_0(x+)$, $S_i(x-) \subset \underline{S}_0(x-)$ and $S_i(x) \subset \underline{S}_0(x)$ for $i = 2, 3$. Example 1 shows that these inclusions are proper.

Let f be a real valued function defined on $I = [0, 1]$. The following definition was introduced in [2].

For any $x \in [0, 1)$

$$P^+f(x) = \inf\{r : r \in \mathbb{R} \text{ and } \{y : f(y) > r\} \in S(x+)\}$$

and

$$P_+f(x) = \sup\{r : r \in \mathbb{R} \text{ and } \{y : f(y) < r\} \in S(x+)\}$$

are called respectively the upper right hand proximal limit and lower right hand proximal limit of f at x. $P^-f(x)$ and $P_-f(x)$ are defined analogously and are called respectively the upper and lower left hand proximal limits of f at x.

If $P^+f(x) = f(x) = P_+f(x)$, f is said to be proximally continuous from the right at x. Similarly if $P^-f(x) = f(x) = P_-f(x)$, f is said to be proximally continuous from the left at x. When f is proximally continuous from both sides at x, it is called proximally continuous at x (with obvious modifications at the end points of I).

Using $\underline{S}_0(x+)$ and $\underline{S}_0(x-)$ we define in the paper the upper and lower unilateral limits and then a type of continuity which is wider than the ideas of approximate continuity and proximal continuity. We investigate some basic properties of functions which are continuous in this wide sense.

2. Wide Approximate Limit and Continuity

We start this section with the following definition.

Definition 2. *For $\xi \in [0,1)$ we put*

$$u^+(f,\xi) = \inf\{K : \{x : f(x) > K\} \in \underline{S}_0(\xi+)\},$$
$$l^+(f,\xi) = \sup\{K : \{x : f(x) < K\} \in \underline{S}_0(\xi+)\}.$$

Similarly we define $u^-(f,\xi)$ and $l^-(f,\xi)$ for $\xi \in (0,1]$.

Proposition 1. $P_+f(\xi) \le u^+(f,\xi) \le P^+f(\xi)$ *and* $P_+f(\xi) \le l^+(f,\xi) \le P^+f(\xi)$.

Proof. Since $S(\xi+) \subset \underline{S}_0(\xi+)$, it follows from the definition that $P_+f(\xi) \le l^+(f,\xi)$ and $u^+(f,\xi) \le P^+f(\xi)$. We claim that $P_1f(\xi) \le u^+(f,\xi)$.

Assume on the contrary that $u^+(f,\xi) < P_+f(\xi)$. Now we choose a number K such that $u^+(f,\xi) < K < P_+f(\xi)$. Then there exist two numbers K_1, K_2 with $K_1 < K < K_2$ such that $\{x : f(x) > K_1\} \in \underline{S}_0(\xi+)$ and $\{x : f(x) < K_2\} \in S(\xi+)$.

Let $A = \{x : f(x) \ge K\}$ and $B = I\backslash A$. Then $A \subset \{x : f(x) > K_1\}$ and $B \subset \{x : f(x) < K_2\}$ so that $A \in \underline{S}_0(\xi+)$ and $B \in S(\xi+)$ (cf. [2, Corollary 3.1.1]). So by Theorem 3.1 in [2] we get $d_+(A \cup B, \xi) = 0$ i.e., $d_+(I,\xi) = 0$, which is a contradiction. Hence $P_+f(\xi) \le u^+(f,\xi)$. Similarly we can prove $l^+(f,\xi) \le P^+f(\xi)$. This completes the proof. ∎

Proposition 2. $P_-f(\xi) \le u^-(f,\xi) \le P^-f(\xi)$ *and* $P_-f(\xi) \le l^-(f,\xi) \le P^-f(\xi)$.

The proof is omitted.

The following example shows that in some cases we may have $u^+(f,\xi) < l^+(f,\xi)$.

Example 2. Let E be the set considered in Example 1 and

$$f(x) = \begin{cases} 0 & \text{if } x \in E \\ 1 & \text{if } x \in I \backslash E. \end{cases}$$

Since $E \in \underline{S}_0(0+)$ and $I \backslash E \in \underline{S}_0(0+)$, it follows that $u^+(f,0) \le 0 < 1 \le l^+(f,0)$.

So we introduce the following definition.

Definition 3. *The numbers*

$$W^+f(\xi) = \max\{u^+(f,\xi), l^+(f,\xi)\}$$

and

$$W_+f(\xi) = \min\{u^+(f,\xi), l^+(f,\xi)\}$$

are called respectively the upper and lower wide approximate limits of f at $\xi \in [0,1)$ from the right.

Likewise we define $W^-f(\xi)$ and $W_-f(\xi)$, the upper and lower wide approximate limits of f at $\xi \in (0,1]$ from the left.

If $A_+f(\xi)(A_-f(\xi))$ and $A^+f(\xi)(A^-f(\xi))$ denote respectively the lower and upper approximate limits of f at ξ from the right (left) then in view of Propositions 1 and 2 we get

$$A_+f(\xi) \le P_+f(\xi) \le W_+f(\xi) \le W^+f(\xi) \le P^+f(\xi) \le A^+f(\xi)$$

and

$$A_-f(\xi) \le P_-f(\xi) \le W_-f(\xi) \le W^-f(\xi) \le P^-f(\xi) \le A^-f(\xi).$$

Definition 4. *A function f is called widely approximate continuous at $\xi \in [0,1)$ ($\xi \in (0,1]$) from the right (left) if $W_+f(\xi) = f(\xi) = W^+f(\xi)(W_-f(\xi) = f(\xi) = W^-f(\xi))$. The function f is said to be widely approximate continuous at ξ if it is so from both the sides at ξ. The function f is called widely approximate continuous on I if it is so at every point of I (with the obvious modifications at the end points of I).*

We use WAC as the abbreviation of "widely approximate continuous". Also "a.e." stands for "almost everywhere".

3. Lemmas

In this section we present three lemmas which will be needed in the sequel.

Lemma 1. [2]

(i) *An arbitrary set E has density 0 or 1 a.e. on R.*

(ii) *If a subset E of a measurable set M has density 0 a.e. on $M \backslash E$ then E is measurable.*

The following lemmas give a convenient form of the unilateral wide approximate limits of a measurable function.

Lemma 2. *If f is measurable then for $\xi \in [0,1)$*

(i) $u^+(f,\xi) = \inf\{\limsup_{x\to\xi+, x\in A} f(x) : A \subset I$ *is measurable and* $d^+(A,\xi) = 1\}$,

(ii) $l^+(f,\xi) = \sup\{\liminf_{x\to\xi+, x\in A} f(x) : A \subset I$ *is measurable and* $d^+(A,\xi) = 1\}$.

Proof. Let $U^+(f,\xi)$ denote the right hand side of (i). Now we consider the following cases.

Case I: $-\infty < U^+(f,\xi) < +\infty$.

Let $\varepsilon > 0$ be arbitrary. Then there exists a measurable set $A \subset I$ with $d^+(A,\xi) = 1$ such that $\limsup_{x\to\xi+, x\in A} f(x) < U^+(f,\xi) + \varepsilon$. So there exists a $\delta > 0$ such that $f(x) < U^+(f,\xi) + \varepsilon$ for all $x \in A \cap (\xi, \xi+\delta)$. Therefore

$$\begin{aligned} A \cap \{x : f(x) > U^+(f,\xi) + \varepsilon,\ x > \xi\} &\subset C[A \cap (\xi,\xi+\delta)] \cap (\xi,\infty) \\ &= [CA \cap (\xi,\infty)] \cup [\xi+\delta,\infty), \end{aligned}$$

where CA denotes the complement of A. Since

$$\begin{aligned} \{x : x \in I \text{ and } f(x) > U^+(f,\xi) + \varepsilon,\ x > \xi\} \\ = [A \cap \{x : x \in I \text{ and } f(x) > U^+(f,\xi) + \varepsilon,\ x > \xi\}] \\ \cup [CA \cap \{x : x \in I \text{ and } f(x) > U^+(f,\xi) + \varepsilon,\ x > \xi\}] \\ \subset [\{CA \cap (\xi,\infty)\} \cup [\xi+\delta,\infty)] \cup CA \\ = CA \cup [\xi+\delta,\infty), \end{aligned}$$

the lower density from right of $\{x : x \in I \text{ and } f(x) > U^+(f,\xi) + \varepsilon,\ x > \xi\}$ at ξ is zero. So $u^+(f,\xi) \le U^+(f,\xi) + \varepsilon$ and so

$$u^+(f,\xi) \le U^+(f,\xi). \tag{1}$$

Let K be a real number such that $\{x : x \in I \text{ and } f(x) > K, x > \xi\}$ has lower density (from right) zero at ξ. Let $B = C\{x : x \in I \text{ and } f(x) > K, x > \xi\} \cap I$. Then $d^+(B,\xi) = 1$ and for all $x \in B$ we get $f(x) \le K$. So $U^+(f,\xi) \le \limsup_{x\to\xi+, x\in B} f(x) \le K$ and since K is arbitrary, it follows that

$$U^+(f,\xi) \le u^+(f,\xi). \tag{2}$$

In this case (i) follows from (1) and (2).

Case II: $U^+(f,\xi) = +\infty$.

Claim: $u^+(f,\xi) = +\infty$.

Assume on the contrary that $u^+(f,\xi) < +\infty$. Then there exists $K < +\infty$ such that $\{x : x \in I, f(x) > K, x > \xi\}$ has lower density (from right) zero at ξ. Let $D = C\{x : x \in I \text{ and } f(x) > K, x > \xi\} \cap I$. Then $d^+(D,\xi) = 1$ and $\limsup_{x\to\xi+, x\in D} f(x) \le K$ so that $U^+(f,\xi) \le K < +\infty$, a contradiction. Hence $u^+(f,\xi) = +\infty$.

Case III: $U^+(f,\xi) = -\infty$.

Then for arbitrary $M (> 0)$ there exists a measurable set $A \subset I$ with $d^+(A,\xi) = 1$ such that $\limsup_{x\to\xi+, x\in A} f(x) < -M$. So there exists a $\delta > 0$ such that $f(x) < -M$

for all $x \in A \cap (\xi, \xi+\delta)$. Since $A \cap \{x : f(x) > -M, x > \xi\} \subset [CA \cap (\xi, \infty)] \cup (\xi + \delta, \infty) = CA \cup [\xi+\delta, \infty)$, the lower density (from right) of $\{x : x \in I \text{ and } f(x) > -M,\ x > \xi\}$ at ξ is zero . Hence $u^+(f, \xi) \leq -M$ and so $u^+(f, \xi) = -\infty$.

From the above analysis the following cases are clear .

Case IV: If $u^+(f, \xi) = +\infty$ then $U^+(f, \xi) = +\infty$.

Case V: If $u^+(f, \xi) = -\infty$ then $U^+(f, \xi) = -\infty$.

Case VI: If $-\infty < u^+(f, \xi) < +\infty$ then $-\infty < U^+(f, \xi) < +\infty$ and $u^+(f, \xi) = U^+(f, \xi)$.

(ii) can be proved in a similar manner. This proves the lemma. ■

Lemma 3. *If f is measurable then for $\xi \in (0, 1]$.*

(i) $u^-(f, \xi) = \inf\{ \limsup_{x \to \xi-, x \in A} f(x) : A \subset I$ *is measurable and* $d^-(A, \xi) = 1\}$.

(ii) $l^-(f, \xi) = \sup\{ \liminf_{x \to \xi-, x \in A} f(x) : A \subset I$ *is measurable and* $d^-(A, \xi) = 1\}$.

The proof is omitted.

4. Theorems

Theorem 1. *If f is WAC from right(left) a.e. on I then f is measurable.*

Proof. Let f be WAC from right on $P \subset I$ where $m(I \backslash P) = 0$. For $r \in R$ we put

$$A = \{x : x \in P \text{ and } f(x) \leq r\},$$

$$B_n = \{x : x \in P \text{ and } f(x) > r + \frac{1}{n}\} \text{ for } n = 1, 2, 3, \ldots$$

Since f is WAC from right on P, it follows that $d_+(B_n, x) = 0$ for all $x \in A$ and $n = 1, 2, 3, \ldots$ So by Lemma 1 we see that $d(B_n, x) = 0$ for all $x \in A_{n,0}$, where $m(A \backslash A_{n,0}) = 0$ for $n = 1, 2, 3, \ldots$

Let $B = \bigcup_{n=1}^{\infty} B_n$ and $C = \bigcap_{n=1}^{\infty} A_{n,0}$. Then $m(A \backslash C) = 0$ and $d(B, x) = 0$ for all $x \in C$. Since $P \backslash A = B$, P is measurable and has density zero a.e. on $P \backslash B$, it follows by Lemma 1 that B is measurable and so A is measurable.

Let $A_0 = \{x : x \in I \text{ and } f(x) \leq r\}$. Since $A_0 = A \cup \{x : x \in I \backslash P \text{ and } f(x) \leq r\}$ and $m(I \backslash P) = 0$, it follows that A_0 is measurable. So f is measurable. This proves the theorem. ■

Corollary 1. *f is WAC from right (left) a.e. on I if and only if f is approximately continuous a.e. on I.*

Proof. The "if" part is obvious. If f is WAC from right (left) a.e. on I then f is measurable and so it is approximately continuous a.e. on I.

Corollary 2. *f is WAC from right(left) a.e. on I if and only if f is measurable.*

Remark 1. Considering Example 6.1 [2] we see that at a point a function may be WAC without being approximately continuous.

In [2] it is proved that if f is proximally continuous on I then $f(I)$ is connected and so f possesses the intermediate value property. Also as a consequence of a result in [3] it follows that a proximally continuous function is of Baire type 1. In the next two theorems we show that a WAC function also possesses these properties.

Theorem 2. *Let f be WAC on I. Then $f(I)$ is connected and so f possesses the intermediate value property.*

Proof Let f be non constant because otherwise the case is trivial. Let $a, b \in I$ be such that $f(a) < f(b)$. Without loss of generality we may choose $a = 0$ and $b = 1$. Let k be a number such that $f(0) < k < f(1)$. We show that $f(x) = k$ for some $x \in I$. Suppose on the contrary that $f(x) \neq k$ for any $x \in I$.

Let $A = \{x : x \in I \text{ and } f(x) < k\}$, $B = \{x : x \in I \text{ and } f(x) > k\}$. Then $I = A \cup B$. Since f is WAC on I, it follows that $d_+(B, x) = d_-(B, x) = 0$ for all $x \in A$ and $d_+(A, x) = d_-(A, x) = 0$ for all $x \in B$. This implies by Lemma 1 that $d(B, x) = 0$ for all $x \in A_0 \subset A$ and $d(A, x) = 0$ for all $x \in B_0 \subset B$ where $m(A \backslash A_0) = m(B \backslash B_0) = 0$. Also by Theorem 1, A and B are measurable.

Since $0 \in A$, it follows that $d_+(B, 0) = 0$ and so $d^+(A, 0) = 1$. This shows that $mA > 0$. Again since $1 \in B$, it follows similarly that $mB > 0$.

Let $x_0 \in A_0$. Then at least one of $[x_0, 1] \cap B_0$ and $[0, x_0] \cap B_0$ is non void. We suppose that $[x_0, 1] \cap B_0 \neq \phi$.

If x_0 is a limit point of B_0 from the right then we can choose $y_0 \in B_0$ such that
$$m\{B \cap (x_0, t)\} < t - x_0 \text{ for all } t \in [x_0, y_0],$$
because $d(B, x_0) = 0$.

If x_0 is not a limit point of B_0 from the right, there exists $p_0 > x_0$ such that $(x_0, p_0) \cap B_0 = \phi$ but $(x_0, y) \cap B_0 \neq \phi$ for every $y > p_0$. We choose $y_0 \in B_0$ such that $0 \leq y_0 - p_0 < p_0 - x_0$. Let $t \in [x_0, y_0]$. If $t \leq p_0$ then
$$m\{B \cap (x_0, t)\} \leq m\{B_0 \cap (x_0, p_0)\} + m\{(B \backslash B_0 \cap (x_0, p_0)\} = 0 < t - x_0,$$
and if $t > p_0$ then
$$\begin{aligned} m\{B \cap (x_0, t)\} &\leq m\{B_0 \cap (x_0, t)\} + m\{(B \backslash B_0) \cap (x_0, t)\} \\ &\leq m\{B_0 \cap (x_0, p_0)\} + m\{B_0 \cap (p_0, t)\} \\ &\leq t - p_0 \leq y_0 - p_0 < p_0 - x_0 < t - x_0. \end{aligned}$$

So for all $t \in [x_0, y_0]$ we get $m\{B \cap (x_0, t)\} < t - x_0$.

If y_0 is a limit point of A_0 from the left then we choose $x_1 \in A_0 \cap (x_0, y_0)$ such that
$$\{A \cap (t, y_0)\} < \frac{1}{2}(y_0 - t) \text{ for all } t \in [x_1, y_0],$$
because $d(A, y_0) = 0$.

Let y_0 be not a limit point of A_0 from the left. Then $m\{A_0 \cap (x_0, y_0)\} > 0$. For, otherwise $m\{B \cap (x_0, s)\} = s - x_0$ for any $s \in (x_0, y_0)$, which contradicts the fact that $d(B, x_0) = 0$. Since y_0 is a limit point of A_0 from the left, there exists $q_1 < y_0$ such that $(q_1, y_0) \cap A_0 = \phi$ but $(x, y_0) \cap A_0 \neq \phi$ for any $x < q_1$. We choose $x_1 \in A_0$ such that $0 \leq q_1 - x_1 < (1/2)(y_0 - q_1)$. Let $t \in [x_1, y_0]$. If $t \geq q_1$ then

$$m\{A\cap(t,y_0)\}\le m\{A_0\cap(q_1,y_0)\}+m\{(A\backslash A_0)\cap(q_1,y_0)\}$$
$$=0<\frac{1}{2}(y_0-t),$$

and if $t<q_1$ then

$$m\{A\cap(t,y_0)\}\le m\{A_0\cap(t,y_0)\}+m\{(A\backslash A_0)\cap(t,y_0)\}$$
$$\le m\{A_0\cap(t,q_1)\}+m\{A_0\cap(q_1,y_0)\}$$
$$\le q_1-t\le q_1-x_1<\frac{1}{2}(y_0-q_1)<\frac{1}{2}(y_0-t).$$

So for all $t\in[x_1,y_0]$ we get $m\{A\cap(t,y_0)\}<(1/2)(y_0-t)$.

If x_1 is a limit point of B_0 from the right then we can choose $y_1\in B_0\cap(x_1,y_0)$ such that

$$m\{B\cap(x_1,t)\}<\frac{1}{2}(t-x_1)\text{ for all }t\in[x_1,y_1],$$

because $d(B,x_1)=0$.

Let x_1 be not a limit point of B_0 from the right. Then $m\{B_0\cap(x_1,y_0)\}>0$, because otherwise $m\{A\cap(s,y_0)\}=y_0-s$ for any $s\in(x_1,y_0)$ which is a contradiction to the fact that $d(A,y_0)=0$. Since x_1 is not a limit point of B_0 from the right, there exists $p_1>x_1$ such that $B_0\cap(x_1,p_1)=\phi$ but $B_0\cap(x_1,y)\neq\phi$ for any $y>p_1$. We choose $y_1\in B_0$ such that $0\le y_1-p_1<(1/2)(p_1-x_1)$. Then for all $t\in[x_1,y_1]$ we get $m\{B\cap(x_1,t)\}<(1/2)(t-x_1)$.

If y_1 is a limit point of A_0 from the left then we can choose $x_2\in A_0\cap(x_1,y_1)$ such that

$$m\{A\cap(t,y_1)\}<\frac{1}{3}(y_1-t)\text{ for all }t\in[x_2,y_1],$$

because $d(A,x_1)=0$.

If y_1 is not a limit point of A_0 from the left then $m\{A_0\cap(x_1,y_1)\}>0$ and there exists $q_2<y_1$ such that $(q_2,y_1)\cap A_0=\phi$ but $(x,y_1)\cap A_0\neq\phi$ for any $x<q_2$. We choose $x_2\in A_0$ such that $0\le q_2-x_2<(1/3)(y_1-q_2)$. Then for any $t\in[x_2,y_1]$ we get $m\{A\cap(t,y_1)\}<(1/3)(y_1-t)$.

Continuing this process we obtain two sequences $\{x_n\}\uparrow$ in A_0 and $\{y_n\}\downarrow$ in B_0 such that $x_n<y_n$ for $n=0,1,2,\dots$ and

$$[x_0,y_0]\supset[x_1,y_0]\supset[x_1,y_1]\supset[x_2,y_1]\supset[x_2,y_2]\supset\dots$$

Also for all $t\in[x_n,y_n]$,

$$m\{B\cap(x_n,t)\}<\frac{1}{n+1}(t-x_n), \tag{3}$$

and for all $t\in[x_{n+1},y_n]$,

$$m\{A\cap(t,y_n)\}<\frac{1}{n+2}(y_n-t), \tag{4}$$

for $n=0,1,2,\dots$.

Now from (3) and (4) we get

$$\begin{aligned} y_{n+1} - x_{n+1} &\le m\{A \cap (x_{n+1}, y_{n+1})\} + m\{B \cap (x_{n+1}, y_{n+1})\} \\ &< \frac{1}{n+2}(y_{n+1} - x_{n+1}) + m\{B \cap (x_{n+1}, y_n)\} \\ &< \frac{1}{n+2}\{(y_{n+1} - x_{n+1}) + (y_n - x_{n+1})\} \\ &\le \frac{2}{n+2} \to 0 \text{ as } n \to \infty. \end{aligned}$$

So there exists a point $c \in (0,1)$ such that $c = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. Also $c \in [x_n, y_n]$ and $c \in [x_{n+1}, y_n]$ for $n = 0, 1, 2, \ldots$ Since (3) and (4) hold for $t = c$ and for $n = 0, 1, 2, \ldots$, it follows that $d_-(B,c) = d_+(A,c) = 0$ and so $d^-(A,c) = d^+(B,c) = 1$.

If $[x_0, 1] \cap B_0 = \phi$ then $[0, x_0] \cap B_0 \ne \phi$ and applying a similar process to the left of x_0 we can find a point $c \in (0,1)$ such that $d_+(B,c) = d_-(A,c) = 0$ so that $d^+(A,c) = d^-(B,c) = 1$.

Hence by Theorem 1, Lemmas 2 and 3 it follows that $u^-(f,c) \le k \le l^+(f,c)$ or $u^+(f,c) \le k \le l^-(f,c)$. Since f is WAC at $x = c$, bilateral wide approximate limit exists at $x = c$ and so $W_-f(c) = W^-f(c) = W_+f(c) = W^+f(c) = k$. Since $f(c) \ne k$, this implies a contradiction and the theorem is proved. ∎

Theorem 3. *If f is WAC from the right (left) on I then f is of Baire type 1.*

Proof. We extend f to the right of 1 and to the left of 0 by setting

$$f(x) = \begin{cases} f(1) & \text{for } x > 1 \\ f(0) & \text{for } x < 0. \end{cases}$$

We now consider an open interval $I_1 = (\alpha, \beta)$ where $\alpha < 0 < 1 < \beta$ and we construct an interval function F converging to f in I_1 which by a result of Gleyzal (cf. [1, p.151]) proves the theorem.

Now for any interval $J = (a_1, b_1) \subset I_1$ for which $2b_1 - a_1 < \beta$, let $F(J)$ denote the infimum of the number k for which there exists a $\delta > 0$ such that

$$m^*\{x : x \in (a_1, t) \text{ and } f(x) \le k\} > \delta(t - a_1) \tag{5}$$

for all $t \in (2b_1 - a_1, \beta)$ and for some $t_1 \in (2b_1 - a_1, \beta)$

$$m^*\{x : x \in (a_1, t_1) \text{ and } f(x) \le k\} > (1 - \delta)(t_1 - a_1). \tag{6}$$

Let $c \in I_1$ and $\varepsilon (> 0)$ be given . We set

$$E_1 = \{x : x \in (c, \beta) \text{ and } f(x) \ge f(c) - \varepsilon\},$$
$$E_2 = \{x : x \in (c, \beta) \text{ and } f(x) \le f(c) + \varepsilon\},$$

and let $A_i = (c, \beta)\backslash E_i$ for $i = 1, 2$. Since f is WAC from right at c, we get $u^+(f,c) = f(c) > f(c) - \epsilon$ and $l^+(f,c) = f(c) < f(c) + \epsilon$ and so $d_+(A_i, c) = 0$ for $i = 1, 2$.

Claim: $d^+(A,c) < 1$.

Assume (on the contrary) that $d^+(A_1, c) = 1$. Since f is WAC from the right, it follows from Lemma 2 that $f(c) = u^+(f,c) \le \limsup_{x\to c+, x\in A_1} f(x) \le f(c) - \epsilon$, which is a contradiction. So $d^+(A_1, c) < 1$. Similarly we can prove that $d^+(A_2, c) < 1$. Since f is measurable, it follows that $d_+(E_i, c) > 0$ and $d^+(E_i, c) = 1$ for $i = 1, 2$. Then there exists $r \in (0, 1/2)$ such that for all $t \in (c, \beta)$

$$m^*\{E_i \cap (c,t)\} > r(t-c) \text{ for } i = 1,2$$

and there exist $c_1, c_2 \in (c, \beta)$ such that

$$m^*\{E_i \cap (c, c_i)\} > (1 - r_1)(c_i - c) \ for \ i = 1,2,$$

where $4r_1 = r$.
Now we choose a number h such that

$$0 < h < \min\left\{\frac{r_1(c_1 - c)}{1 - 2r_1}, \frac{r_1(c_2 - c)}{1 - 2r_1}\right\}.$$

We consider any $J = (a_1, b_1) \subset (c - h, c + h)$ with $c \in J$. Then for $i = 1, 2$

$$2b_1 - a_1 < c + 3h < c + \frac{3r_1(c_i - c)}{1 - 2r_1} < c_i$$

because $3r_1/(1 - 2r_1) < 1$.

Also for $i = 1, 2$ we get, because $h < r_1(c_i - c)/(1 - 2r_1)$,

$$\begin{aligned} m^*\{E_i \cap (a_1, c_i)\} &\geq m^*\{E_i \cap (c, c_i)\} > (1 - r_1)(c_i - c) \\ &> (1 - 2r_1)(c_i - c + h) > (1 - 2r_1)(c_i - a_1). \end{aligned} \tag{7}$$

Again for $t \in (2b_1 - a_1, \beta)$, it follows that for $i = 1, 2$

$$\begin{aligned} m^*\{E_i \cap (a_1, t)\} &\geq m^*\{E_i \cap (c,t)\} > r(t - c) \\ &= \frac{r}{2}(t - a_1) + \frac{r}{2}(t - 2c + a_1) \\ &> \frac{r}{2}(t - a_1) + r(b_1 - c) > 2r_1(t - a_1). \end{aligned} \tag{8}$$

Considering (7) and (8) for E_2 we see that $F(J) \leq f(c) + \epsilon$. Claim: $f(c) - \epsilon \leq F(J)$. Assume (on the contrary) that $F(J) < f(c) - \epsilon$. Then for some $k < f(c) - \epsilon$ there exists a $\delta > 0$ such that (5) and (6) hold. Now putting $t = c_1$ in (5) we see that

$$\begin{aligned} \delta(c_1 - a_1) &< m^*\{x : x \in (a_1, c_1) \text{ and } f(x) \leq k\} \\ &\leq m^*\{x : x \in (a_1, c_1) \text{ and } f(x) < f(c) - \epsilon\} \\ &= (c_1 - a_1) - m^*\{E_1 \cap (a_1, c_1)\} \\ &< (c_1 - a_1) - (1 - 2r_1)(c_1 - a_1) \\ &= 2r_1(c_1 - a_1) \end{aligned}$$

i.e.

$$\delta < 2r_1. \tag{9}$$

Also from (6) and (8) we get for $t = t_1$

$$\begin{aligned} (1 - \delta)(t_1 - a_1) &\leq m^*\{x : x \in (a_1, t_1) \text{ and } f(x) \leq k\} \\ &\leq m^*\{x : x \in (a_1, t_1) \ and \ f(x) < f(c) - \epsilon\} \\ &= (t_1 - a_1) - m^*\{E_1 \cap (a_1, t_1)\} \\ &< (t_1 - a_1) - 2r_1(t_1 - a_1) \end{aligned}$$

i.e.

$$2r_1 < \delta. \tag{10}$$

Since (9) and (10) imply a contradiction, we get $f(c) - \epsilon \leq F(J)$. Thus $| F(J) - f(c) | \leq \epsilon$ for all $J \subset (c-h, c+h)$ with $c \in J$. Therefore for every $c \in I_1$, $F(J) \to f(c)$ as $mJ \to 0$ with $c \in J$. Hence the extended f is of Baire type 1 on I_1 and so the given function f is of Baire type 1 on I. This proves the theorem. ∎

As an application of Theorem 3 we prove the following result.

Theorem 4. *If f is WAC from the right on I then for each $\alpha \in R$, $E^\alpha = \{x : x \in I$ and $f(x) < \alpha\}$ and $E_\alpha = \{x : x \in I$ and $f(x) > \alpha\}$ are of type F_σ and for each $x_0 \in E^\alpha$(or E_α) $d^+(E^\alpha, x_0) = 1$ (or $d^+(E_\alpha, x_0) = 1$).*

Proof. Since by Theorem 3 f is of Baire type 1, E^α and E_α are of type F_σ.

Let $x_0 \in E^\alpha$. Since f is WAC from the right at x_0, it follows that $f(x_0) = \inf\{K : \{x : f(x) > K\} \in \underline{S}_0(x_0+)\}$. Since $f(x_0) < \alpha$ and f is measurable, it follows that $\{x : x \in I$ and $f(x) > \alpha\} \in \underline{S}_0(x_0+)$ and so $d^+(\{x : x \in I$ and $f(x) \leq \alpha\}, x_0) = 1$.

We note that $E^\alpha = \cup_{n=1}^{\infty}\{x : x \in I$ and $f(x) < \alpha - 1/n\} = \bigcup_{n=1}^{\infty}\{x : x \in I$ and $f(x) \leq \alpha - 1/n\}$. Since $x \in E^\alpha$, it follows that $x_0 \in \{x : f(x) < \alpha - 1/n_0\}$ for some positive integer n_0. So by the preceding paragraph we get $d^+(\{x : x \in I$ and $f(x) \leq \alpha - 1/n_0\}, x_0\} = 1$ and so $d^+(\bigcup_{n=1}^{\infty}\{x : x \in I$ and $f(x) \leq \alpha - 1/n\}, x_0) = 1$ i.e. $d^+(E, x_0) = 1$.

Since when f is WAC from the right on I, so is $-f$ and for each $\alpha \in R$, $E_\alpha = \{x : x \in I$ and $-f(x) < -\alpha\}$, it follows by above that $d^+(E_\alpha, x_0) = 1$ for each $x_0 \in E_\alpha$. This proves the theorem. ∎

Note 1. In a similar manner the left analogue of Theorem 4 can be proved.

The next theorem follows as a consequence of Theorem 1 and Theorem 4.

Theorem 5. *Let f be WAC on I. Then f is strictly increasing and so continuous on I if and only if for every $\xi \in [0, 1)$*
(i) *ξ is a limit point of $\{x : f(x) \geq f(\xi)\}$ from right* ,
(ii) *$\{x : f(x) = f(\xi)\}$ is a finite set.*

Proof. Since the "only if" part is straight forward, we prove the "if" part. Let x_1, x_2 be any two points such that $0 \leq x_1 < x_2 \leq 1$. By condition (ii) there exists a point α, $x_1 \leq \alpha < x_2$, such that $f(x_1) = f(\alpha)$ and $f(x) \neq f(x_1)$ for $\alpha < x < x_2$. Let $J = [\alpha, x_2]$ and

$$A = \{x : x \in J \text{ and } f(x) < f(\alpha)\},$$
$$B = \{x : x \in J \text{ and } f(x) > f(\alpha)\}.$$

Then $A \cup B \cup \{\alpha\} \cup \{x_2\} = J$. By conditions (i) and (ii) $B \neq \phi$. Claim: $A = \emptyset$. Assume (on the contrary) that $A \neq \phi$. Since f is WAC on I, by Theorem 4 we get $d^+(A, x) = 1$ for all $x \in A$ and $d^+(B, x) = 1$ for all $x \in B$ (in view of Note

1 we can consider for the point x_2 the left hand upper density) and so $mA > 0$, $mB > 0$. Since by Theorem 1 f is measurable, it follows that $d_+(B,x) = 0$ for all $x \in A$ and $d_+(A,x) = 0$ for all $x \in B$ (for the point x_2 we consider $d^-(A,x_2)$ or $d^-(B,x_2)$). So by Lemma 1 we see that $d(B,x) = 0$ for all $x \in A_0 \subset A$ and $d(A,x) = 0$ for all $x \in B_0 \subset B$ where $m(A \backslash A_0) = m(B \backslash B_0) = 0$.

Now proceeding in the line of Theorem 2 we can find a point c, $\alpha < c < x_2$ such that $d^-(A,c) = d^+(B,c) = 1$ or $d^+(A,c) = d^-(B,c) = 1$. So by Theorem 1, Lemmas 2 and 3 we get $u^-(f,c) \leq f(\alpha) \leq l^+(f,c)$ or $u^+(f,c) \leq f(\alpha) \leq l^-(f,c)$. Since f is WAC at $x = c$, we get $f(c) = f(\alpha)$, which is a contradiction. Therefore $A = \phi$ and so either $f(x_2) = f(\alpha) = f(x_1)$ or $f(x_2) > f(\alpha) = f(x_1)$. Claim: $f(x_2) \neq f(x_1)$. Assume on the contrary that $f(x_2) = f(x_1)$. Then by the condition (ii) we can find x_3 such that $x_1 < x_3 < x_2$ and $f(x_3) \neq f(x_1)$. Now by the above argument we see that $f(x_1) < f(x_3)$ and $f(x_3) < f(x_2)$, which is a contradiction. So $f(x_2) \neq f(x_1)$. Hence f is strictly increasing and so continuous on I. This proves the "if" part and the theorem. ■

Remark 2. The wide approximate continuity of f is necessary in Theorem 5 as we see in the following example.

Example 3. Let

$$f(x) = \begin{cases} x & \text{if } x \in I \text{ and } x \text{ is irrational} \\ \frac{x}{2} & \text{if } x \in I \text{ and } x \text{ is rational.} \end{cases}$$

Then f is not WAC on I and it is not strictly increasing on I but the conditions (i) and (ii) of Theorem 5 are clearly satisfied.

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