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Generalized Rings of Linear Transformations Having the Intersection Property of Quasi-Ideals

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Abstract. Let V and W be vector spaces over a division ring R and $L_R(V,W)$ denote the set of all linear transformations $\alpha : V \to W$. For $\theta \in L_R(W,V)$, $(L_R(V,W), +, \theta)$ denotes the ring $L_R(V,W)$ under usual addition and the multiplication $*$ defined by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in L_R(V,W)$. In this paper, we prove that $(L_R(V,W), +, \theta)$ has the intersection property of quasi-ideals if and only if $\theta = 0$, or θ is a monomorphism, or θ is an epimorphism. Consequently, if $M_{m,n}(R)$ is the set of all $m \times n$ matrices over R and $(M_{m,n}(R), +, P)$ where $P \in M_{n,m}(R)$ is defined similarly, then $(M_{m,n}(R), +, P)$ has the intersection property of quasi-ideals if and only if either $P = 0$ or $\operatorname{rank} P = \min\{m, n\}$.

1. Introduction and Preliminaries

A subring Q of a ring R is said to be a *quasi-ideal* of R if $RQ \cap QR \subseteq Q$ where for nonempty subsets A, B of R, AB denotes the set of all finite sums of the form $\sum a_i b_i$ where $a_i \in A$ and $b_i \in B$. Then quasi-ideals are a generalization of left ideals and right ideals. The notion of quasi-ideal for rings was first introduced by Steinfeld in [4]. It is known that the intersection of a left ideal and a right ideal of a ring R is a quasi-ideal [5, p. 7]. However, a quasi-ideal of R need not be obtained in this way. Examples can be found in [2, 3], [5, p. 8]. We say that a quasi-ideal Q of R *has the intersection property* if Q is the intersection of a left ideal and a right ideal of R, and R is said to *have the intersection property of quasi-ideals* if every quasi-ideal of R has the intersection property. It is clearly seen that every commutative ring has the intersection property of quasi-ideals.

In particular, every zero ring has this property. The following fact will be quoted later.

Proposition 1.1. [5, p. 9] *If a ring R has a one-sided identity, then R has the intersection property of quasi-ideals.*

Moreover, every (von Neumann) regular ring has the intersection property of quasi-ideals [5, p. 73]. Weinert has characterized quasi-ideals of a ring which have the intersection property in [6]. In [3], Moucheng, Yuqun and Yonghau have given a characterization of rings having the intersection property of quasi-ideals as follows: Let $\mathbb{Z}$ denote the set of integers. For a nonempty subset X of R, $\mathbb{Z}X$ denotes the set of all finite sums of the form $\sum k_i x_i$ where $k_i \in \mathbb{Z}$ and $x_i \in X$.

Proposition 1.2. (Moucheng et al. [3]) *A ring R has the intersection property of quasi-ideals if and only if for any finite nonempty subset X of R,*

$$RX \cap (\mathbb{Z}X + XR) \subseteq \mathbb{Z}X + (RX \cap XR).$$

This known fact will be used. We note here that for any nonempty subset X of R, $\mathbb{Z}X + (RX \cap XR) = (X)_q$, the quasi-ideal of R generated by X (Weinert [6]), that is, $(X)_q$ is the intersection of all quasi-ideals of R containing X [5, p. 10].

In the remainder, let V and W be vector spaces over a division ring R and $L_R(V, W)$ denote the set of all linear transformations $\alpha : V \to W$. For $\theta \in L_R(W, V)$, we denote by $(L_R(V, W), +, \theta)$ the ring $L_R(V, W)$ under usual addition and the multiplication $*$ defined by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in L_R(V, W)$ (functions in this paper are written on the right). The aim of this paper is to give a characterization for the ring $(L_R(V, W), +, \theta)$ to have the intersection property of quasi-ideals.

Let $M_{m,n}(R)$ be the set of all $m \times n$ matrices over R. For $P \in M_{n,m}(R)$, let $(M_{m,n}(R), +, P)$ denote the ring $M_{m,n}(R)$ under usual addition and the multiplication $*$ defined by $A * B = APB$ for all $A, B \in M_{m,n}(R)$. As a consequence of the main result, the rings $(M_{m,n}(R), +, P)$ with the intersection property of quasi-ideals are characterized.

The following background on vector spaces, linear transformations and matrices will be used.

Assume that $\dim_R V = m$, $\dim_R W = n$, $B = \{v_1, v_2, \ldots, v_m\}$ is an ordered basis of V and $B' = \{w_1, w_2, \ldots, w_n\}$ is an ordered basis of W. For $\alpha \in L_R(V, W)$, let $[\alpha]_{B,B'}$ denote the $m \times n$ matrices (r_{ij}) where

$$\begin{aligned} v_1\alpha &= r_{11}w_1 + r_{12}w_2 + \ldots + r_{1n}w_n \\ v_2\alpha &= r_{21}w_1 + r_{22}w_2 + \ldots + r_{2n}w_n \\ &\vdots \\ v_m\alpha &= r_{m1}w_1 + r_{m2}w_2 + \ldots + r_{mn}w_n \end{aligned}$$

and the matrix $[\alpha]_{B,B'}$ is called *the matrix of α relative to the ordered bases B and B'* [1, p. 329]. Then

$$(L_R(V, W), +, \theta) \cong (M_{m,n}(R), +, [\theta]_{B',B})$$

by the map $\alpha \mapsto [\alpha]_{B,B'}$ ([1, pp. 329, 330]). Moreover, for every $\alpha \in L_R(V,W)$,

$$\operatorname{rank}\alpha = \operatorname{rank}[\alpha]_{B,B'}$$

([1, pp. 337, 339]). The following proposition is generally true for linear transformations and will be referred.

Proposition 1.3. *Let $\alpha \in L_R(V,W)$.*

(i) *If α is a monomorphism, then there exists $\beta \in L_R(W,V)$ such that $\alpha\beta = 1_V$ where 1_V is the identity map on V.*

(ii) *If α is an epimorphism, then there exists $\beta \in L_R(W,V)$ such that $\beta\alpha = 1_W$.*

Proof.

(i) Let B be a basis of V. Since α is a monomorphism, we have $B\alpha$ is a basis of $\operatorname{Im}\alpha$ and $u\alpha \neq u'\alpha$ for all distinct $u, u' \in B$. Let B' be a basis of W containing $B\alpha$. Let $\beta \in L_R(W,V)$ be defined by

$$v\beta = \begin{cases} u & \text{if } v = u\alpha \quad \text{for some} \quad u \in B, \\ 0 & \text{if } v \in B' \setminus B\alpha. \end{cases}$$

Then $u\alpha\beta = u$ for all $u \in B$ and hence $\alpha\beta = 1_V$.

(ii) Let B be a basis of W. Since $\operatorname{Im}\alpha = W$, for each $v \in B$, there exists $v' \in V$ such that $v'\alpha = v$. Define $\beta \in L_R(W,V)$ by

$$v\beta = v' \text{ for all } v \in B.$$

Then $v\beta\alpha = v$ for all $v \in B$, so $\beta\alpha = 1_W$. ∎

2. Main Results

We first give the main theorem. Then we derive four remarkable corollaries by the basic knowledge introduced in Sec. 1.

Theorem 2.1. *For $\theta \in L_R(W,V)$, the ring $(L_R(V,W), +, \theta)$ has the intersection property of quasi-ideals if and only if θ satisfies one of the following conditions:*

(i) $\theta = 0$.

(ii) *θ is a monomorphism.*

(iii) *θ is an epimorphism.*

Proof. If $\theta = 0$, then $(L_R(V,W), +, \theta)$ is a zero ring, so it has the intersection property of quasi-ideals.

Assume that θ is a monomorphism. By Proposition 1.3 (i), $\theta\theta' = 1_W$ for some $\theta' \in L_R(V,W)$. It follows that $\alpha\theta\theta' = \alpha$ for all $\alpha \in L_R(V,W)$. This implies that θ' is a right identity of the ring $(L_R(V,W), +, \theta)$. We deduce from Proposition 1.1 that $(L_R(V,W), +, \theta)$ has the intersection property of quasi-ideals.

Next, assume that θ is an epimorphism. By Proposition 1.3 (ii), there is $\theta' \in L_R(V,W)$ such that $\theta'\theta = 1_V$. Then $\theta'\theta\alpha = \alpha$ for all $\alpha \in L_R(V,W)$, so θ' is a left identity of the ring $(L_R(V,W), +, \theta)$. Hence by Proposition 1.1, $(L_R(V,W), +, \theta)$ has the intersection property of quasi-ideals.

For the converse, assume that $\theta \neq 0$ and that θ is neither a monomorphism nor an epimorphism. It follows that

$$\{0\} \neq \operatorname{Ker}\theta \subsetneq W \text{ and } \{0\} \neq \operatorname{Im}\theta \subsetneq V.$$

Let $u \in \operatorname{Ker}\theta \setminus \{0\}$, $w \in W \setminus \operatorname{Ker}\theta$ and $z \in V \setminus \operatorname{Im}\theta$. Then $w\theta \in \operatorname{Im}\theta \setminus \{0\}$. Let B_1 be a basis of $\operatorname{Im}\theta$ containing $w\theta$. Since $z \in V \setminus \operatorname{Im}\theta$, $B_1 \cup \{z\}$ is linearly independent over R. Let B be a basis of V containing $B_1 \cup \{z\}$. Let $\alpha, \beta, \gamma \in L_R(V,W)$ be defined on B as follows:

$$v\alpha = \begin{cases} u & \text{if } v = w\theta, \\ w & \text{if } v = z, \\ 0 & \text{if } v \in B \setminus \{w\theta, z\}, \end{cases}$$

$$v\beta = \begin{cases} w & \text{if } v = w\theta, \\ 0 & \text{if } v \in B \setminus \{w\theta\} \end{cases}$$

and

$$v\gamma = \begin{cases} -w & \text{if } v = w\theta, \\ 0 & \text{if } v \in B \setminus \{w\theta\}. \end{cases}$$

We have the following properties of α, β and γ:

$$v\alpha\theta = 0 \text{ for all } v \in \operatorname{Im}\theta, \tag{1}$$

$$\begin{aligned}(w\theta)(\alpha + \alpha\theta\gamma) &= (w\theta)\alpha + (w\theta)\alpha\theta\gamma = u, \\ (w\theta)\beta\theta\alpha &= (w\theta)\alpha = u, \\ z(\alpha + \alpha\theta\gamma) &= z\alpha + z\alpha\theta\gamma = w + (w\theta)\gamma = w - w = 0, \\ z(\beta\theta\alpha) &= 0\end{aligned}$$

and

$$v(\alpha + \alpha\theta\gamma) = 0 = v\beta\theta\alpha \text{ for all } v \in B \setminus \{w\theta, z\}.$$

Consequently, we have

$$\beta\theta\alpha = \alpha + \alpha\theta\gamma \in L_R(V,W)\theta\alpha \cap (\mathbb{Z}\alpha + \alpha\theta L_R(V,W)). \tag{2}$$

Suppose that $\beta\theta\alpha \in \mathbb{Z}\alpha + (L_R(V,W)\theta\alpha \cap \alpha\theta L_R(V,W))$. Then there exist $k \in \mathbb{Z}$ and $\lambda, \eta \in L_R(V,W)$ such that

$$\beta\theta\alpha = k\alpha + \lambda\theta\alpha = k\alpha + \alpha\theta\eta.$$

Thus

$$u = (w\theta)\beta\theta\alpha = (w\theta)(k\alpha + \alpha\theta\eta) = ku = (k1_R)u$$

where 1_R is the identity of R. This implies that $k1_R = 1_R$ since $u \neq 0$. Therefore $k\alpha = \alpha$ and so

$$\beta\theta\alpha = \alpha + \lambda\theta\alpha.$$

Hence

$$0 = z(\beta\theta\alpha) = z(\alpha + \lambda\theta\alpha) = w + (z\lambda\theta)\alpha$$

and so $(z\lambda\theta)\alpha = -w$. It then follows from this equality and (1), that

$$-(w\theta) = (z\lambda\theta)\alpha\theta = 0$$

which is a contradiction since $w\theta \neq 0$. This shows that

$$\beta\theta\alpha \notin \mathbb{Z}\alpha + (L_R(V,W)\theta\alpha \cap \alpha\theta L_R(V,W)). \tag{3}$$

The statements (2), (3) and Proposition 1.2 yield the result that the ring $(L_R(V,W),+,\theta)$ does not have the intersection property of quasi-ideals.

Hence the theorem is completely proved. ■

Corollary 2.2. *Assume that* $\dim_R V = m$, $\dim_R W = n$ *and* $\theta \in L_R(W,V)$. *Then the ring* $(L_R(V,W),+,\theta)$ *has the intersection property of quasi-ideals if and only if either* $\theta = 0$ *or* $\operatorname{rank}\theta = \min\{m,n\}$.

Proof. We clearly have that $\operatorname{rank}\theta = n \leqslant m$ if and only if θ is a monomorphism and $\operatorname{rank}\theta = m \leq n$ if and only if θ is an epimorphism. From these facts and Theorem 2.1, the Corollary is obtained. ■

The following corollary is an immediate consequence of Corollary 2.2.

Corollary 2.3. *Assume that* $\dim_R V = \dim_R W < \infty$ *and* $\theta \in L_R(W,V)$. *Then the ring* $(L_R(V,W),+,\theta)$ *has the intersection property of quasi-ideals if and only if either* $\theta = 0$ *or* θ *is an isomorphism.*

Corollary 2.4. *For* $P \in M_{n,m}(R)$, *the ring* $(M_{m,n}(R),+,P)$ *has the intersection property of quasi-ideals if and only if either* $P = 0$ *or* $\operatorname{rank}P = \min\{m,n\}$.

Proof. Let V and W be finite dimensional vector spaces over R such that $\dim_R V = m$ and $\dim_R W = n$. Let B and B' be respectively ordered bases of V and W. Then there exists $\theta \in L_R(W,V)$ such that $[\theta]_{B',B} = P$. Therefore

$$(L_R(V,W),+,\theta) \cong (M_{m,n}(R),+,P) \text{ by } \alpha \mapsto [\alpha]_{B,B'}. \tag{4}$$

Also, we have

$$\operatorname{rank}\theta = \operatorname{rank}[\theta]_{B',B} = \operatorname{rank}P. \tag{5}$$

Hence the theorem holds by Corollary 2.2, (4) and (5). ■

As an immediate consequence of Corollary 2.4, we obtain

Corollary 2.5. *For* $P \in M_n(R)$ *where* $M_n(R)$ *denotes* $M_{n,n}(R)$, *the ring* $(M_n(R),+,P)$ *has the intersection property of quasi-ideals if and only if either* $P = 0$ *or* P *is invertible.*

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