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Short Communication

Optimal Recovery of Periodic Functions Using Wavelet Decompositions

Dinh Dung[1] **and Mai Xuan Thao**[2]

[1] *Information Technology Institute, Vietnam National University*
144 Xuan Thuy Rd., Cau Giay, Hanoi, Vietnam
[2] *Hong Duc University, Thanh Hoa Province, Vietnam*

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1. Let $L_q := L_q(\mathbb{T}^d)$, $1 \le q \le \infty$, be the normed space of functions defined on the d-torus $\mathbb{T}^d = [-\pi, \pi]^d$, equipped with the usual q-integral norm $\|\cdot\|_q$. If $\{x^1, \ldots, x^n\} \subset \mathbb{T}^d$ is a selection of n points and $P_n(y_1, \ldots, y_n)$ a mapping from $\mathbb{R}^n$ into a linear manifold in L_q of dimension at most n. We can naturally consider approximate recovering a function $f \in L_q(\mathbb{T}^d)$ from its values $f(x^1), \ldots, f(x^n)$ by the element $g = P_n(f(x^1), \ldots, f(x^n))$. The recovery error is measured by $\|f - g\|_q$. Let W be a subset in L_q. We are interested in the optimal recovery of $f \in W$ over all such methods of approximate recovery. The error of this optimal recovery is given by

$$R_n(W, L_q) := \inf \ \sup_{f \in W} \|f - P_n(f(x^1), \ldots, f(x^n))\|_q,$$

where inf is taken over all selections of n points $\{x^1, \ldots, x^n\} \subset \mathbb{T}^d$ and mappings P_n from $\mathbb{R}^n$ into a linear manifold in L_q of dimension at most n.

In the present paper, we study optimal methods of recovery of functions from the Besov class of common smoothness $SB^{\omega}_{p,\theta}$ in terms of the quantity $R_n(SB^{\omega}_{p,\theta}, L_q)$ for $1 \le p, q \le \infty$. Its smoothness is defined via modulus of smoothness dominated by a function ω of modulus of smoothness type. With some restrictions on ω and p, q, we give the asymptotic order of this quantity when $n \to \infty$. An asymptotically optimal method of recovery is constructed using periodic wavelet decompositions of functions into the integer translates of the dyadic scales of multivariate de la Vallée Poussin kernels. Problems of recovery of periodic functions which are related to the present paper, were considered in [1–4].

2. Let us define smoothness Besov spaces of functions on $\mathbb{T}^d$. For a positive integer l, the symmetric difference operator Δ^l_h, $h \in \mathbb{T}^d$, is defined inductively

by $\Delta_h^l := \Delta_h^1 \Delta_h^{l-1}$, starting from $\Delta_h^1 f := f(\cdot + h/2) - f(\cdot - h/2)$.
Let $\omega_l(f,t)_p := \sup_{|h|<t} \|\Delta_h^l f\|_p$, $t \geq 0$, is the lth p-integral modulus of smoothness of f.

We introduce the class MS_l of functions ω of modulus of smoothness type as follows. It consists of all non-negative functions ω on $[0,\infty)$ such that:

(i) $\omega(0) = 0$,
(ii) $\omega(t) \leq \omega(t')$ if $t \leq t'$,
(iii) $\omega(kt) \leq k^l \omega(t)$, for $k = 1, 2 \ldots$,
(iv) ω satisfies Condition Z_l, that is, there exist a positive number $a < l$ and positive constant C_l such that

$$\omega(t)t^{-a} \geq C_l h^{-a} \omega(h),\ 0 \leq t \leq h,$$

(v) ω satisfies Condition BS, that is, there exist a positive number b and positive constant C such that

$$\omega(t)t^{-b} \leq C h^{-b} \omega(h),\ \ 0 \leq t \leq h \leq 1.$$

Let $1 \leq p \leq \infty$, $0 < \theta \leq \infty$ and $\omega \in MS_l$. The Besov space $B^\omega_{p,\theta}$ consists of all functions $f \in L_p$ for which the Besov semi-quasi-norm

$$|f|_{B^\omega_{p,\theta}} := \begin{cases} \left(\int_0^\infty \{\omega_l(f,t)_p/\omega(t)\}^\theta dt/t\right)^{1/\theta}, & \theta < \infty, \\ \sup_{t>0} \omega_l(f,t)_p/\omega(t), & \theta = \infty \end{cases} \tag{1}$$

is finite. We define the Besov quasi-norm by

$$\|f\|_{B^\omega_{p,\theta}} := \|f\|_p + |f|_{B^\omega_{p,\theta}}. \tag{2}$$

The definition of $B^\omega_{p,\theta}$ does not depend on l, i.e., for a given ω, (1)–(2) determine equivalent norms for all l such that $\omega \in MS_l$. The function $\omega(t) = t^r$, $r > 0$, belongs to the class MS_l for any $l > r$. The space $B^r_{p,\theta} := B^\omega_{p,\theta}$ with $\omega(t) = t^r$, $r > 0$, is the classical Besov space.

We say that ω satisfies Condition $R(\varepsilon)$ ($\varepsilon \geq 0$), if $\omega(t)t^{-\varepsilon}$ satisfies Condition BS. If ω satisfies Condition $R(d/p)$, then $B^\omega_{p,\theta}$ is compactly embedded into $C(\mathbb{T}^d)$.

The Besov class

$$SB^\omega_{p,\theta} := \{f \in B^\omega_{p,\theta} : \|f\|_{B^\omega_{p,\theta}} \leq 1\}$$

is defined as the unit ball of the space $B^\omega_{p,\theta}$.

We denote $a_+ := \max\{a, 0\}$ and use the notation $F \asymp F'$ if $F \ll F'$ and $F' \ll F$, and $F \ll F'$ if $F \leq CF'$ with C an absolute constant. The main result of the present paper is the following

Theorem 1. *Let $1 \leq p, q \leq \infty$, $0 < \theta \leq \infty$. Assume that ω satisfies Condition $R(d/p)$. Then we have*

$$R_n(SB^\omega_{p,\theta}, L_q) \asymp \omega(d/n) n^{(d/p - d/q)_+}.$$

Theorem 1 was proved in [2] for the univariate classical Besov–Hölder class $SB^r_{p,\infty}$.

3. We now construct an asymptotically optimal method of recovery using periodic wavelet decompositions of functions into the integer translates of the dyadic scales of multivariate de la Vallée Poussin kernels. Let

$$V_m(t) := \frac{1}{3m^2}\sum_{k=m}^{2m-1} D_k(t) = \frac{\sin(mt/2)\sin(3mt/2)}{3m^2\sin^2(t/2)}$$

be the de la Vallée Poussin kernel of order m, where

$$D_m(t) := \sum_{|k|\le m} e^{ikt}$$

is the univariate Dirichlet kernel of order m. The multivariate de la Vallée Poussin kernel V_m is defined by

$$V_m(x) := V_m(x_1)V_m(x_2)\cdots V_m(x_d).$$

Next we let

$$v_0 := 1, \quad v_k := V_{2^{k-1}}, \ k = 1, 2, \ldots$$

be the periodic dyadic scaling functions, and the periodic wavelets

$$v_{k,s} := v_k(\cdot - h_{k,s}), \ s \in Q_k,$$

be defined as the integer translates of v_k, where

$$h_{k,s} := 2\pi s/2^k; \quad Q_k := \{s \in \mathbb{Z}^d : 0 \le s_j < 2^k, \ j = 0, \ldots, d\}.$$

Consider the recovery of functions on $\mathbb{T}^d$ from its values at the points

$$\{h_{k,s} : s \in Q_k\}$$

by the method S_k defined as follows

$$S_k(f) := \sum_{s\in Q_k} f(h_{k,s})v_{k,s}.$$

Notice that S_k is a linear operator and the number of the points $\{h_{k,s} : s \in Q_k\}$ is 2^{dk}. Let $\mathcal{T}_m$ denote the space of trigonometric polynomials of order at most m in each variable. Obviously, $S_k(f) \in \mathcal{T}_{2^k-1}$. Moreover, the method of recovery S_k is precise for trigonometric polynomials $\mathcal{T}_{2^k}$, i.e.,

$$S_k(f) = f, \quad f \in \mathcal{T}_{2^{k-1}}, \tag{3}$$

and S_k interpolates f at the points $h_{k,s}$, $s \in Q_k$, i.e.,

$$S_k(f, h_{k,s}) = f(h_{k,s}), \in Q_k.$$

From properties of de la Vallée Poussin kernels we easily prove that for any continuous function f on $\mathbb{T}^d$

$$\lim_{k\to\infty} \|f - S_k(f)\|_\infty = 0.$$

Hence, by use of (3) we derive that any continuous function f on $\mathbb{T}^d$ can be decomposed into the wavelets $v_{k,s}$ by the series

$$f(x) = \sum_{k=0}^{\infty}\sum_{s\in Q_k} \lambda_{k,s}v_{k,s},$$

converging uniformly on $\mathbb{T}^d$, where $\lambda_{k,s} = \lambda_{k,s}(\{f(h_{k,\nu})\}_{\nu\in Q_k})$ and $\lambda_{k,s}(\cdot)$ are linear functions defined on $\mathbb{R}^{2^{dk}}$.

For functions from $SB^\omega_{p,\theta}$, we proved the following

Theorem 2. *Let $1 \le p, q \le \infty$, $0 < \theta \le \infty$ and ω satisfy Condition $\mathrm{R}(d/p)$. For a given natural number n, let k be the largest non-negative integer such that $2^{dk} \le n$. Then we have for any continuous function f on $\mathbb{T}^d$*

$$\sup_{f\in SB^\omega_{p,\theta}} \|f - S_k(f)\|_q \asymp \omega(d/n)n^{(d/p-d/q)_+}.$$

In this way, under the assumptions of Theorem 1 the method of recovery S_k defined in Theorem 2, is asymptotically optimal for $R_n(SB^\omega_{p,\theta}, L_q)$. The lower bound of Theorem 2 was proved in [4] (see Theorem 6.2 of Chapter 1) for the univariate classical Besov–Hölder class $SB^r_{p,\infty}$. The upper bound of Theorem 2 is proved from well-known Stechkin's theorem of trigonometric approximation and the following

Theorem 3. *Let $1 \le q < \infty$. Then we have for any continuous function f on $\mathbb{T}^d$*

$$\|f - S_k(f)\|_q \le C_q \sum_{s=k-1}^{\infty} 2^{(s-k)d/q} E_{2^s}(f)_q,$$

where

$$E_m(f)_q := \inf_{g\in\mathcal{T}_m} \|f-g\|_q$$

is the best approximation to f by trigonometric polynomials of order at most m.

Theorem 3 was proved in [2] for the univariate case. A similar inequality earlier than that in [2], was proved in [3] for the univariate case and $1 \le q \le 2$.

Notice that for any continuous function f on $\mathbb{T}^d$, we have

$$\|f - S_k(f)\|_\infty \le C E_{2^{k-1}}(f)_\infty.$$

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