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Characterizations of Regular Ordered Semigroups by Quasi-Ideals

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Abstract. As a generalization of the concept of quasi-ideals of semigroups to ordered semigroup theory, the concept of quasi-ideals of ordered semigroups is introduced. Regular ordered semigroups are characterized by their quasi-ideals and the fact that for any regular ordered semigroup S, the set Q_S of all quasi-ideals of S, with multiplication defined by: $Q_1 \circ Q_2 = (Q_1Q_2]$, $\forall Q_1, Q_2 \in Q_S$, is a regular semigroup is obtained. Some special classes of regular ordered semigroups, in which the regular semigroups $(Q_S, \circ)$ are bands, left regular bands and semilattices, respectively, are considered.

1. Introduction and Preliminaries

An *ordered semigroup* (*po-semigroup*) $(S, \cdot, \leq)$ is a poset $(S, \leq)$ at the same time a semigroup $(S, \cdot)$ such that: for any $a, b, x \in S$, $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$. For $A, B \subseteq S$, let $AB := \{ab \mid a \in A \text{ and } b \in B\}$. Let T be a subsemigroup of S and let H be a nonempty subset of T. As in [5], we denote

$$(H]_T := \{x \in T \mid (\exists h \in H)\ x \leq h\}.$$

If $T = S$, then $(H]_T$ is denoted simply by $(H]$ (see [2]). We have $H \subseteq (H]_T \subseteq (H]$ and $A \subseteq B \Longrightarrow (A]_T \subseteq (B]_T$, for any nonempty subsets A, B of T. As in [2, 3], S is said to be *regular* (*intra-regular*) if: $a \in (aSa]$ ($a \in (Sa^2S]$), $\forall a \in S$. In this paper, S stands for an arbitrary ordered semigroup.

Let I be a nonempty subset of S. I is called a *left* (*righ*) *ideal* of S if: (i) $SI \subseteq I$ ($IS \subseteq I$) and, (ii) $(I] \subseteq I$. I is called an (*two-sided*) *ideal* of S if it is both a left and a right ideal of S. See [1]. Let X be a nonempty subset of S. We denote the least left (right) ideal of S containing X by $L(X)$ ($R(X)$). It is evident $L(X) = (SX \cup X] = (S^1X]$ ($R(X) = (X \cup XS] = (XS^1]$). If $X = \{a\}$,

$a \in S$, we denote $L(\{a\})$ $(R(\{a\}))$ by $L(a)$ $(R(a))$ and, $L(a) = (Sa \cup a] = (S^1a]$ $(R(a) = (a \cup aS] = (aS^1])$. In this paper, we denote

$$P_S = \{X \mid \emptyset \neq X \subseteq S \text{ and } (X] \subseteq X\},$$
$$L_S = \{L \mid L \text{ is a left ideal of } S\},$$
$$R_S = \{R \mid R \text{ is a right ideal of } S\},$$
$$I_S = \{I \mid I \text{ is a two-sided ideal of } S\},$$

and define a multiplication "$\circ$" on P_S by

$$(\forall X, Y \in P_S)\ X \circ Y = (XY].$$

For $\mathcal{A}, \mathcal{B} \subseteq P_S$, denote $\mathcal{A} \circ \mathcal{B} = \{A \circ B \mid A \in \mathcal{A},\ B \in \mathcal{B}\}$.

Lemma 1.1. *Let S be an ordered semigroup. Then*
(i) $(P_S, \circ, \subseteq)$ *is an ordered semigroup.*
(ii) $(L_S, \circ, \subseteq)$, $(R_S, \circ, \subseteq)$ *and* $(I_S, \circ, \subseteq)$ *are subsemigroups of* $(P_S, \circ, \subseteq)$.

Proof. (i) It is obvious that the multiplication "$\circ$" is well-defined. Let $A, B, C \in P_S$. By $AB \in (AB]$ we have $((AB)C] \subseteq ((AB]C]$. Further, from

$$(A \circ B) \circ C = (AB] \circ C = ((AB]C] \subseteq ((AB)C] = (ABC],$$

we obtain $(A \circ B) \circ C = (ABC]$. Similarly, we can prove $A \circ (B \circ C) = (ABC]$, so $(A \circ B) \circ C = A \circ (B \circ C)$. Thus $(P_S, \circ)$ is a semigroup. Let $A \subseteq B$. Then $A \circ C = (AC] \subseteq (BC] = B \circ C$ and $C \circ A = (CA] \subseteq (CB] = C \circ B$. Hence $(P_S, \circ, \subseteq)$ is an ordered semigroup.

(ii) It is evident that L_S, R_S and I_S are nonempty subsets of P_S. Let $J, K \in L_S$. It is obvious that $(J \circ K] = ((JK]] = (JK]$. Further, by

$$S(J \circ K) = S(JK] \subseteq (S(JK]] \subseteq ((SJ)K] \subseteq (JK] = J \circ K,$$

we conclude that $J \circ K$ is a left ideal of S, i.e., $J \circ K \in L_S$. Thus $(L_S, \circ, \subseteq)$ is a subsemigroup of $(P_S, \circ, \subseteq)$.

Dually, we can show that $(R_S, \circ, \subseteq)$ is a subsemigroup of $(P_S, \circ, \subseteq)$. By $I_S = L_S \cap R_S$ it follows that $(I_S, \circ, \subseteq)$ is a subsemigroup of $(P_S, \circ, \subseteq)$. ■

Definition 1.2. *Let S be an ordered semigroup. A nonempty subset Q of S is called a quasi-ideal of S if (i) $(QS] \cap (SQ] \subseteq Q$ and, (ii) $(Q] \subseteq Q$. Denote*

$$Q_S = \{Q \mid Q \text{ is a quasi-ideal of } S\}.$$

It is clear that $L_S \cup R_S \subseteq Q_S \subseteq P_S$, i.e., every one-sided ideal of an ordered semigroup S is a quasi-ideal of S.

In [4], regular semigroups (without order) is characterized by quasi-ideals. Quasi-ideals defined in Definition 1.2 is a generalization of the concept of quasi-ideals of semigroups (without order) to ordered semigroup theory. In this paper, we first consider the elementary properties of quasi-ideals of ordered semigroups. Then we characterize regular ordered semigroups by its quasi-ideals, left and right ideals and, prove that: an ordered semigroup S is regular if and only if

$(Q_S, \circ)$ is a regular subsemigroup of $(P_S, \circ)$. Finally, we characterize ordered semigroups S in which $(Q_S, \circ)$ are bands, left regular bands and semilattices respectively.

2. Elementary Properties of Quasi-ideals of Ordered Semigroups

Lemma 2.1. *Each quasi-ideal Q of an ordered semigroup S is a subsemigroup of S.*

In fact, $Q^2 \subseteq QS \cap SQ \subseteq (QS] \cap (SQ] \subseteq Q$. ■

Lemma 2.2. *For every right ideal R and left ideal L of an ordered semigroup S, $R \cap L$ is a quasi-ideal of S.*

Proof. Since $RL \subseteq SL \subseteq L$ and $RL \subseteq RS \subseteq R$, we have $RL \subseteq R \cap L$, so $R \cap L \neq \emptyset$. By $(R \cap L] \subseteq (R] \cap (L] \subseteq R \cap L$ and

$$((R \cap L)S] \cap (S(R \cap L)] \subseteq (RS] \cap (SL] \subseteq (R] \cap (L] \subseteq R \cap L,$$

it follows that $R \cap L$ is a quasi-ideal of S. ■

Lemma 2.3. *For every quasi-ideal Q of S, we have $Q = L(Q) \cap R(Q) = (SQ \cup Q] \cap (Q \cup QS]$.*

Proof. The inclusion $Q \subseteq (SQ \cup Q] \cap (Q \cup QS]$ is evident.

Conversely, let $a \in (SQ \cup Q] \cap (Q \cup QS]$. Then $a \leq q$, or $a \leq xu$ and $a \leq vy$ for some $q, u, v \in Q$ and $x, y \in S$. Since Q is a quasi-ideal of S, the first case implies $a \in (Q] \subseteq Q$ and the second case implies $a \in (SQ] \cap (QS] \subseteq Q$. Hence $(SQ \cup Q] \cap (Q \cap QS] = Q$. ■

Let X be a nonempty subset of an ordered semigroup S. We denote the least quasi-ideal of S containing X by $Q(X)$. If $X = \{a\}$, we denote $Q(\{a\})$ by $Q(a)$.

Corollary 2.4. *Let S be an ordered semigroup. Then*
(i) *For every $a \in S$, $Q(a) = L(a) \cap R(a) = (Sa \cup a] \cap (a \cup aS]$.*
(ii) *For every $\emptyset \neq X \subseteq S$, $Q(X) = L(X) \cap R(X) = (SX \cup X] \cap (X \cup XS]$.*

Proof.
(i) Let $a \in S$. By Lemma 2.2, we see that $L(a) \cap R(a)$ is a quasi-ideal of S containing a, so $Q(a) \subseteq L(a) \cap R(a)$. On the other hand, by Lemma 2.3 it follows that

$$\begin{aligned} L(a) \cap R(a) &= (Sa \cup a] \cap (a \cup aS] \\ &\subseteq (SQ(a) \cup Q(a)] \cap (Q(a) \cup Q(a)S] \\ &= Q(a). \end{aligned}$$

Thus $Q(a) = L(a) \cap R(a)$.

(ii) It can be proved similarly as (i). ■

By a *bi-ideal* B of an ordered semigroup S we shall mean a subsemigroup B of S such that $BSB \subseteq B$ and $(B] \subseteq B$.

Lemma 2.5. *Let J be a two-sided ideal of an ordered semigroup S and Q a quasi-ideal of J, then Q is a bi-ideal of S.*

Proof. Since Q is a quasi-ideal of J and $Q \subseteq J$, we have

$$QSQ \subseteq QSJ = Q(SJ) \subseteq QJ \subseteq (QJ] \subseteq (SJ] \subseteq (J] \subseteq J,$$

$$QSQ \subseteq JQS = (JS)Q \subseteq JQ \subseteq (JQ] \subseteq (JS] \subseteq (J] \subseteq J,$$

and

$$x \in (Q] \Longrightarrow (\exists q \in Q \subseteq J)\ x \leq q \Longrightarrow x \in (J] = J \,\&\, x \in (Q]$$
$$\Longrightarrow x \in J \cap (Q] = (Q]_J \subseteq Q,$$

whence

$$QSQ \subseteq (J \cap (JQ]) \cap (J \cap (QJ]) = (JQ]_J \cap (QJ]_J \subseteq Q$$

and $(Q] \subseteq Q$. These facts and Lemma 2.1 imply that Q is a bi-ideal of S. ∎

In view of Lemma 2.5, we see that quasi-ideals are special cases of bi-ideals of ordered semigroups.

3. Characterizations of Regular Ordered Semigroups by Quasi-Ideals

Lemma 3.1. *For any ordered semigroup S, the subsemigroup of $(P_S, \circ)$ generated by $(L_S, \circ)$ and $(R_S, \circ)$ is given by*

$$\langle L_S \cup R_S \rangle = L_S \cup R_S \cup (R_S \circ L_S).$$

Proof. It is clear that

$$\langle L_S \cup R_S \rangle = \{X_1 \circ \cdots \circ X_n \mid X_i \in L_S \text{ or } X_i \in R_S,\ i = 1, ..., n,\ n \in Z^+\}.$$

Let $X_i, X_{i+1} \in L_S \cup R_S$. Then we have the following cases: (i) $X_i, X_{i+1} \in L_S$. In this case, $X_i \circ X_{i+1} \in L_S$ by Lemma 1.1. (ii) $X_i, X_{i+1} \in R_S$. In this case, $X_i \circ X_{i+1} \in R_S$ by Lemma 1.1. (iii) $X_i \in L_S$ and $X_{i+1} \in R_S$. In this case, $X_i \circ X_{i+1} = (X_i X_{i+1}]$ is an ideal of S, so $X_i \circ X_{i+1} \in I_S = L_S \cap R_S$. (iv) $X_i \in R_S$ and $X_{i+1} \in L_S$. In this case, $X_i \circ X_{i+1} \in R_S \circ L_S$ in $(P_S, \circ)$. Thus for any $X_1, \ldots, X_n \in L_S \cup R_S$ with $n \in Z^+$, by (i)–(iv), there are three cases:

α) If $X_1 \in L_S$, then $X_1 \circ \cdots \circ X_n \in L_S$.

β) If $X_n \in R_S$, then $X_1 \circ \cdots \circ X_n \in R_S$.

γ) If $X_1 \in R_S$ and $X_n \in L_S$ with $n \geq 2$, then $X_1 \circ \cdots \circ X_n \in R_S \circ L_S$.

As stated above, we see that the assertion holds. ∎

Theorem 3.2. *The following conditions on an ordered semigroup S are equivalent:*

(i) *S is regular;*
(ii) *For every right ideal R and left ideal L of S,*
$$(RL] = R \cap L;$$
(iii) *For every right ideal R and left ideal L of S,*
(a) $(R^2] = R$,
(b) $(L^2] = L$,
(c) *$(RL]$ is a quasi-ideal of S;*
(iv) *$(L_S, \circ)$ and $(R_S, \circ)$ are bands (idempotent semigroups) and $(Q_S, \circ)$ is the subsemigroup of $(P_S, \circ)$ generated by $(L_S, \circ)$ and $(R_S, \circ)$;*
(v) *$(Q_S, \circ)$ is a regular subsemigroup of the semigroup $(P_S, \circ)$;*
(vi) *Every quasi-ideal Q of S has the form $Q = (QSQ]$;*
(vii) *$(Q_S, \circ, \subseteq)$ is a regular subsemigroup of the ordered semigroup $(P_S, \circ, \subseteq)$.*

Proof. (i) $\Rightarrow$ (ii) Let R and L be right and left ideals of S, respectively, then
$$(RL] \subseteq R \cap L$$
always holds. Assume that S is regular, we have to show only that $R \cap L \subseteq (RL]$. Let $a \in R \cap L$. Since S is regular, we have $a \le axa$ for some $x \in S$, whence $a \in R$ and $xa \in L$, so $axa \in RL$. Thus $a \in (RL]$, so that $R \cap L \subseteq (RL]$.

(ii) $\Rightarrow$ (iii) The assumption (ii) and Lemma 2.2 imply that $(RL]$ is a quasi-ideal of S. Since the two-sided ideal of S generated by R is $(R \cup SR]$, from the assumption (ii), it follows that
$$R = R \cap (R \cup SR] = (R(R \cup SR]],$$
so $(R^2] \subseteq (R(R \cup SR]] = R$. Conversely, let $x \in (R(R \cup SR]]$. Then $x \le r_1 z$ for some $r_1 \in R$ and $z \in (R \cup SR]$. From $z \in (R \cup SR]$, we obtain $z \le w$, where $w = r_2 \in R$ or $w = sr_3$ for some $s \in S$ and $r_3 \in R$. Hence
$$x \le r_1 w = r_1 r_2 \in R^2 \text{ or } x \le r_1 w = r_1(sr_3) = (r_1 s) r_3 \in R^2,$$
so $x \in (R^2]$. Thus $R \subseteq (R^2]$, so that $(R^2] = R$.

The statement $(L^2] = L$ can be proved dually.

(iii) $\Rightarrow$ (iv) By Lemma 3.1, the conditions (a) and (b) in (iii) implies $(L_S, \circ)$ and $(R_S, \circ)$ is a band, respectively.

In view of (iii) (c), we have $R_S \circ L_S \subseteq Q_S$, so $\langle L_S \cup R_S \rangle \subseteq Q_S$ in $(P_S, \circ)$. Conversely, let $Q \in Q_S$. Then $(Q \cup SQ]$ is the left ideal of S generated by Q. By condition (iii) (b), we have
$$Q \subseteq (Q \cup SQ] = ((Q \cup SQ]^2] \subseteq (Q^2 \cup SQ^2 \cup QSQ \cup (SQ)^2] \subseteq (SQ].$$
Dually we can show $Q \subseteq (QS]$. These relations and Lemma 2.3 imply
$$Q \subseteq (SQ] \cap (QS] \subseteq (SQ \cup Q] \cap (Q \cup QS] = Q,$$
that is,
$$(\forall Q \in Q_S)\ Q = (SQ] \cap (QS]. \tag{1}$$
From the assumption (iii) (c) and from (1), it follows that
$$(\forall R \in R_S)(\forall L \in L_S)\ (RL] = (S(RL]] \cap ((RL]S]. \tag{2}$$

Furthermore, by condition (iii) (b) we have $S = (S^2]$ and

$$\begin{aligned}(SQ] = ((SQ]^2] &= ((SQ](SQ]] = ((SQ]((S^2]Q]]\\ &\subseteq (SQSSQ] \subseteq (S(QS](SQ]] \subseteq (S((QS](SQ]]],\\ &\subseteq (S(QS^2Q)] \subseteq (SQ],\end{aligned}$$

so $(SQ] = (S((QS](SQ]]]$. Dually, we have $(QS] = (((QS](SQ]]S]$. From these relations, by (1) and (2) it follows that

$$\begin{aligned}Q = (QS] \cap (SQ] = (((QS](SQ]]S] \cap (S((QS](SQ]]] &= ((QS](SQ]]\\ &= (QS] \circ (SQ] \in R_S \circ L_S \subseteq \langle L_S \cup R_S \rangle \qquad (3)\end{aligned}$$

by Lemma 3.1. Hence $Q_S \subseteq \langle L_S \cup R_S \rangle$. Therefore, $Q_S = \langle L_S \cup R_S \rangle$ in $(P_S, \circ)$.

(iv) $\Rightarrow$ (iii) It follows immediately from Lemma 3.1.

(iii) $\Rightarrow$ (v) By proving (iii) $\Rightarrow$ (iv), we see that (2) and (3) hold. Let Q_1, Q_2 be two quasi-ideals of S. Then $(S(Q_1Q_2] \cup (Q_1Q_2]]$ is the least left ideal of S containing $(Q_1Q_2]$. By condition (iii) (b), we have

$$\begin{aligned}(Q_1Q_2] \subseteq (S(Q_1Q_2] \cup (Q_1Q_2]] &= ((S(Q_1Q_2] \cup (Q_1Q_2]]^2]\\ &\subseteq (S(Q_1Q_2]] = ((S^2](Q_1Q_2]] \subseteq (S(S(Q_1Q_2]]].\end{aligned}$$

Dually we can show $(Q_1Q_2] \subseteq ((Q_1Q_2] \cup (Q_1Q_2]S] \subseteq (((Q_1Q_2]S]S]$. These relations and (2) imply

$$\begin{aligned}(Q_1Q_2] &\subseteq (S(Q_1Q_2] \cup (Q_1Q_2]] \cap ((Q_1Q_2] \cup (Q_1Q_2]S]\\ &\subseteq (S(S(Q_1Q_2]]] \cap (((Q_1Q_2]S]S]\\ &= (((Q_1Q_2]S](S(Q_1Q_2]]] \subseteq ((Q_1(Q_2SS)Q_1)Q_2] \subseteq (Q_1Q_2].\end{aligned}$$

Thus $(Q_1Q_2] = (S(Q_1Q_2] \cup (Q_1Q_2]] \cap ((Q_1Q_2] \cup (Q_1Q_2]S]$ is a quasi-ideal of S by Corollary 2.4 (ii), so $Q_1 \circ Q_2 \in Q_S$. Hence $(Q_S, \circ)$ is a subsemigroup of $(P_S, \circ)$. For every $Q \in Q_S$, by (3) we have

$$Q = ((QS](SQ]] \subseteq (QS^2Q] \subseteq (QSQ] \subseteq Q,$$

whence $Q = (QSQ] = Q \circ S \circ Q$ with $S \in Q_S$. Thus $(Q_S, \circ)$ is a regular subsemigroup of $(P_S, \circ)$.

(v) $\Rightarrow$ (vi) Let Q be a quasi-ideal of S. By the assumption (iv), there exists a quasi-ideal X of S such that

$$\begin{aligned}Q = Q \circ X \circ Q = (QXQ] \subseteq (QSQ] &\subseteq (SQ] \cap (QS]\\ &\subseteq (SQ \cup Q] \cap (Q \cup QS] = Q\end{aligned}$$

by Lemma 2.3, and hence $Q = (QSQ]$.

(vi) $\Rightarrow$ (vii) Obvious.

(vii) $\Rightarrow$ (i). For every $a \in S$, by Corollary 2.4 (i), $R(a) \cap L(a)$ is a quasi-ideal of S containing a. From the assumption (vii), there exists $Q \in Q_S$ such that

$$\begin{aligned} a \in R(a) \cap L(a) &\subseteq (R(a) \cap L(a)) \circ Q \circ (R(a) \cap L(a)) \\ &= ((R(a) \cap L(a))Q(R(a) \cap L(a))] \subseteq (R(a)SL(a)] \\ &= ((a \cup aS]S(Sa \cup a]] \subseteq (aSa]. \end{aligned}$$

Thus S is a regular ordered semigroup. ∎

Lemma 3.3. *Every two-sided ideal J of a regular ordered semigroup S is a regular subsemigroup of S.*

Proof. Let $a \in J$. Since S is regular, there exists $x \in S$ such that

$$a \leq axa \leq axaxa = a(xax)a.$$

Since $xax \in SJS \subseteq J$, we see that $a \in (aJa]_J$. ∎

Corollary 3.4. *Let S be a regular ordered semigroup. Then the following assertions hold:*

(i) *every quasi-ideal Q of S can be written in the form*

$$Q = R \cap L = (RL],$$

where R (L) is the right (left) ideal of S generated by Q;

(ii) *if Q is a quasi-ideal of S, then $(Q^2] = (Q^3]$;*

(iii) *every bi-ideal of S is a quasi-ideal of S;*

(iv) *every bi-ideal of any two-sided ideal of S is a quasi-ideal of S;*

(v) *for every $L_1, L_2 \in L_S$ and $R_1, R_2 \in R_S$, we have*

$$L_1 \cap L_2 \subseteq (L_1L_2], \quad \textit{and} \quad R_1 \cap R_2 \subseteq (R_1R_2].$$

Proof. Since S is a regular ordered semigroup, then by Lemma 2.3 and Theorem 3.2, the assertion (i) holds.

Since $(Q^3] \subseteq (Q^2]$ always holds, we have to prove $(Q^2] \subseteq (Q^3]$. By Theorem 3.2, $(Q^2]$ is also a quasi-ideal of S, furthermore

$$(Q^2] = (Q^2SQ^2] = (Q(QSQ)Q] \subseteq (Q^3].$$

Let T be a bi-ideal of S. Then $(ST]$ is a left ideal and $(TS]$ is a right ideal of S. By Theorem 3.2, we have

$$(ST] \cap (TS] = ((TS](ST]] \subseteq (TST] \subseteq (T] \subseteq T.$$

Hence T is a quasi-ideal of S.

Let J be a two-sided ideal of S and let K be a bi-ideal of J. By Lemma 3.3 and the property (iii), K is a quasi-ideal of J, thus by Lemma 2.5, K is a bi-ideal of S. Again from the property (iii), it follows that K is a quasi-ideal of S.

Finally, let $L_1, L_2 \in L_S$. Since S is regular and $L_1 \cap L_2$ is a quasi-ideal of S, by Theorem 3.2 it follows that

$$L_1 \cap L_2 = ((L_1 \cap L_2)S(L_1 \cap L_2)] \subseteq (L_1(SL_2)] \subseteq (L_1L_2].$$

Dually, we have $R_1 \cap R_2 \subseteq (R_1R_2]$ for all $R_1, R_2 \in R_S$. ∎

4. Some Special Classes of Regular Ordered Semigroups

In this section, we shall consider ordered semigroups S in which the regular subsemigroup $(Q_S, \circ)$ of $(P_S, \circ)$ are bands, left regular bands (idempotent semigroups satisfying identity relation $efe = ef$) and semilattices (commutative idempotent semigroups), respectively.

Lemma 4.1. *The following conditions are equivalent on an ordered semigroup S:*

(i) *S is intra-regular;*

(ii) *For every right ideal R and left ideal L of S,*

$$R \cap L \subseteq (LR];$$

(iii) *For every quasi-ideal Q of S, $Q \subseteq (SQ^2S]$.*

Proof. (i) ⇒ (ii) Let $a \in R \cap L$. Since S is intra-regular, we have

$$a \in (Sa^2S] = ((Sa)(aS)] \subseteq ((SL)(RS)] \subseteq (LR].$$

Thus $R \cap L \subseteq (LR]$.

(ii) ⇒ (iii) Let (ii) hold and let Q be a quasi-ideal of S. In view of Lemma 2.3, we have

$$\begin{aligned} Q = L(Q) \cap R(Q) &\subseteq (L(Q)R(Q)] = ((S^1Q](QS^1]] \\ &= (S^1QQS^1] \subseteq (S^1Q(S^1QQS^1]S^1] = (S^1QS^1QQS^1S^1] \\ &\subseteq (S^1QS^1(S^1QQS^1]QS^1S^1] = ((S^1QS^1S^1)Q^2(S^1QS^1S^1)] \\ &\subseteq (SQ^2S]. \end{aligned}$$

(iii) ⇒ (i) Let (iii) hold and let $a \in S$. By Corollary 2.4 (i) we have $Q(a) = L(a) \cap R(a)$, whence

$$\begin{aligned} a \in Q(a) &\subseteq (SQ(a)Q(a)S] \subseteq (SL(a)R(a)S] \\ &= (S(S^1a](aS^1]S] = ((SS^1)a^2(S^1S)] \\ &\subseteq (Sa^2S], \end{aligned}$$

∎

Theorem 4.2. *The following conditions are equivalent on an ordered semigroup S:*

(i) *S is regular and intra-regular;*

(ii) *For every right ideal R and left ideal L of S,*

$$(RL] = R \cap L \subseteq (LR]; \tag{4}$$

(iii) *$(Q_S, \circ)$ is a band;*

(iv) *For every quasi-ideal Q of S, $(Q^2] = Q$.*

Proof. (i) $\Leftrightarrow$ (ii) It follows immediately from Theorem 3.2 and Lemma 4.1.

(ii) $\Rightarrow$ (iii) By Theorem 3.2, the first part of Condition (4) implies that $(Q_S, \circ)$ is a regular semigroup, so we have to show only that every quasi-ideal Q of S is idempotent in $(Q_S, \circ)$. It is evident that $(Q^2] \subseteq (Q] \subseteq Q$ by Lemma 2.1. Again by Theorem 3.2 and Lemma 4.1, the first and the last parts of Condition (4) implies $Q = (QSQ]$ and $Q \subseteq (SQ^2S]$ respectively, whence

$$Q = (QSQ] = (QS(QSQ]] \subseteq (QSQSQ] \subseteq (QS(SQ^2S]SQ]$$
$$\subseteq ((QS^2Q)(QS^2Q)] \subseteq ((QSQ)(QSQ)] \subseteq (Q^2].$$

So $Q = (Q^2] = Q \circ Q$. Thus $(Q_S, \circ)$ is a band.

The implication (iii) $\Rightarrow$ (iv) is evident.

(iv) $\Rightarrow$ (ii) Let R and L be right and left ideals of S, respectively. By Lemma 2.2, $R \cap L$ is a quasi-ideal of S. So assumption (iv) implies

$$R \cap L = ((R \cap L)^2] = ((R \cap L)(R \cap L)] \subseteq (RL],$$

$$R \cap L = ((R \cap L)(R \cap L)] \subseteq (LR].$$

Since $(RL] \subseteq R \cap L$ always holds, we get $(RL] = R \cap L$, so that the relation (4) holds. ∎

An ordered semigroup S is said to be *left (right) duo* if every left (right) ideal of S is a right (left) ideal of S; and S is said to be *duo* if S is both left and right duo.

Lemma 4.3. *An ordered semigroup S is left duo if and only if every quasi-ideal of S is a right ideal of S.*

Proof. $\Rightarrow$) Let Q be a quasi-ideal of S. By Lemma 2.3, there exist right ideal R and left ideal L of S such that $Q = R \cap L$. Since S is left duo, L is a right ideal of S, so that Q is a right ideal of S.

$\Leftarrow$) The assertion follows immediately from the fact that every left ideal of S is a quasi-ideal of S. ∎

Corollary 4.4. *An ordered semigroup S is duo if and only if every quasi-ideal of S is a two-sided ideal of S.*

Theorem 4.5. *Let S be an ordered semigroup. Then the following conditions are equivalent:*

(i) *S is a regular left duo ordered semigroup;*

(ii) *$(Q_S, \circ)$ is a left regular band;*

(iii) *For every right ideal R and left ideal L, L_1, L_2 of S,*

$$(RL] = (R \cap L] \text{ and } (L_1L_2] = L_1 \cap L_2;$$

(iv) $(L_S, \circ)$ *is a semilattice,* $(R_S, \circ)$ *is a band and* $(Q_S, \circ)$ *is the subsemigroup of the semigroup* $(P_S, \circ)$ *generated by* $(L_S, \circ)$ *and* $(R_S, \circ)$.

Proof. (i) ⇒ (ii) Let Q, T be quasi-ideals of S. Since S is left duo, Q, T are right ideals of S by Lemma 4.3. Since S is regular, by Theorem 3.2 and Lemma 2.1 we have

$$Q = (QSQ] = ((QS)Q] \subseteq (QQ] \subseteq (Q] \subseteq Q,$$

which implies $Q \circ Q = (Q^2] = Q$. Thus $(Q_S, \circ)$ is a band. Since T is also a right ideal of S, we have

$$(QTQ] = (Q(TQ)] \subseteq (Q(TS)] \subseteq (QT].$$

By Theorem 3.2, $(QT]$ is a quasi-ideal of S and hence

$$(QT] = ((QT)S(QT)] = (Q(TS)(QT)] \subseteq (QTQ].$$

Thus $(QT] = (QTQ]$, that is, $Q \circ T = Q \circ T \circ Q$ for all $Q, T \in (Q_S, \circ)$. Therefore, $(Q_S, \circ)$ is a left regular band.

(ii) ⇒ (iii) The first formula in (iii) follows immediately from Theorem 3.2 since a left regular band is a regular semigroup. Let L_1, L_2 be left ideals of S. Then $(L_1L_2] \subseteq (SL_2] \subseteq L_2$ and by the condition (ii) we have

$$(L_1L_2] = (L_1L_2L_1] = ((L_1L_2)L_1] \subseteq (SL_1] \subseteq (L_1] \subseteq L_1,$$

so $(L_1L_2] \subseteq L_1 \cap L_2$, from this by Corollary 3.4 (v) it follows that $(L_1L_2] = L_1 \cap L_2$.

(iii) ⇒ (iv) It follows immediately from Theorem 3.2 and the assumption (iii).

(iv) ⇒ (i). By Theorem 3.2 it follows that S is a regular ordered semigroup. For every left ideal L of S, since $(L_S, \circ)$ is a semilattice and $S \in L_S$, we have

$$LS \subseteq (LS] = L \circ S = S \circ L = (SL] \subseteq L,$$

which shows that L is a right ideal of S. Thus S is left duo. ∎

Theorem 4.6. *The following conditions are equivalent on an ordered semigroup* S:

(i) S *is a regular duo ordered semigroup;*

(ii) $(Q_S, \circ)$ *is a semilattice;*

(iii) *For any left ideals* L_1, L_2 *and right ideals* R_1, R_2 *of* S,

$$(L_1L_2] = L_1 \cap L_2 \quad \text{and} \quad (R_1R_2] = R_1 \cap R_2;$$

(iv) $(L_S, \circ)$ *and* $(R_S, \circ)$ *are semilattices and* $(Q_S, \circ)$ *is the subsemigroup of* $(P_S, \circ)$ *generated by* $(L_S, \circ)$ *and* $(R_S, \circ)$;

(v) *For any quasi-ideals* Q_1, Q_2 *of* S,

$$(Q_1Q_2] = Q_1 \cap Q_2;$$

(vi) *For every quasi-ideal* Q *of* S,

$$((R(Q))^2] = L(Q) \quad ((L(Q))^2] = R(Q);$$

(vii) *For every left ideal L and right ideal R of S,*

$$L \cap R = (LR].$$

Proof. (i) $\Rightarrow$ (ii) By Theorem 4.5 and its dual, the condition (i) implies that $(Q_S, \circ)$ is both a left and a right band. Hence $(Q_S, \circ)$ is a semilattice.

(ii) $\Rightarrow$ (iii) In view of the hypothesis, $(Q_S, \circ)$ is a band, from this by Theorem 4.2 it follows that S is regular. Since a semilattice is both a left and a right regular band, the assertion (iii) follows immediately from Theorem 4.5 and its dual.

(iii) $\Rightarrow$ (i) Let L and R be left and right ideals of S, respectively. By $S \in L_S$ and (iii) we have

$$LS \subseteq (LS] = L \cap S = L,$$

which shows that L is a right ideal of S. Symmetrically, the assumption implies that R is a left ideal of S. So L and R are two-sided ideals of S. Hence S is duo, and

$$S(RL] \subseteq (S(RL]] \subseteq ((SR)L] \subseteq (RL], \quad (RL]S \subseteq (R(LS)] \subseteq (RL],$$

which shows that $(RL]$ is a two-sided ideal of S. Again the condition (iii) implies $(L^2] = L$ and $(R^2] = R$. So S satisfies the condition (iii) in Theorem 3.2. Thus S is regular by Theorem 3.2.

(i) $\Leftrightarrow$ (iv) It follows immediately from Theorem 4.5 and its dual.

(i) $\Rightarrow$ (v) Let Q_1, Q_2 be quasi-ideals of S. Since S is duo, by Corollary 4.4 we see that Q_1, Q_2 are ideals of S, whence $(Q_1Q_2] \subseteq Q_1 \cap Q_2$. Since $Q_1 \cap Q_2$ is a two-sided ideal of S and S is regular, by Theorem 3.2 it follows that

$$Q_1 \cap Q_2 = ((Q_1Q_2)S(Q_1Q_2)] \subseteq (Q_1SQ_2] \subseteq (Q_1Q_2].$$

Hence $(Q_1Q_2] = Q_1 \cap Q_2$.

(v) $\Rightarrow$ (vi) Let Q be an arbitrary quasi-ideal of S. Then Condition (v) implies $(Q^2] = Q$, $QS \subseteq (QS] = Q \cap S = Q$ and $SQ \subseteq (SQ] = S \cap Q = Q$. Thus Q is a two-sided ideal of S. So $L(Q) = R(Q) = Q$, whence

$$((R(Q))^2] = (Q^2] = Q = (L(Q)], \quad ((L(Q))^2] = (Q^2] = Q = R(Q).$$

(vi) $\Rightarrow$ (i) Let L be a left ideal of S. Then the assumption (vi) implies

$$LS \subseteq (L \cup LS] = ((L \cup SL]^2] \subseteq (L^2] \subseteq (L] \subseteq L,$$

that is, L is also a right ideal of S. Dually we can prove that every right ideal of S is also a left ideal of S. So S is a duo ordered semigroup.

For any right ideal R and left ideal L of S, since S is duo, R, L are two-sided ideals of S. Thus $(RL]$ is a two-sided ideal of S and Condition (vi) implies that $(L^2] = L$ and $(R^2] = R$. By Theorem 3.2, we conclude that S is regular.

As stated above, we have proved that (i)–(vi) are equivalent.

The implication (v)$\Rightarrow$(vii) is evident.

(vii) $\Rightarrow$ (i) It can be proved similarly as that of proving (iii) $\Rightarrow$ (i). ∎

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