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On Finite Groups Whose Every Normal Subgroup is a Union of the Same Number of Conjugacy Classes*

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Abstract. Let G be a finite group and $\mathcal{N}_G$ denote the set of non-trivial proper normal subgroups of G. An element K of $\mathcal{N}_G$ is said to be n-decomposable if K is a union of n distinct conjugacy classes of G.

In this paper, we investigate the structure of finite groups G in which G' is a union of three distinct conjugacy classes of G. We prove, under certain conditions, G is a Frobenius group with kernel G' and its complement is abelian. Furthermore, we investigate the structure of finite groups G in which $\mathcal{N}_G \neq \emptyset$ and every element of $\mathcal{N}_G$ is n-decomposable, for a given n. When G is solvable or $n = 2, 3, 4$, we determine the structure of such groups.

1. Introduction

Let G be a finite group and let $\mathcal{N}_G$ be the set of non-trivial proper normal subgroups of G. Following Shahryari and Shahabi [10], we say that a normal subgroup H of the group G is a small subgroup if $H = 1 \cup Cl_G(h)$, in which h is non-central and $Cl_G(h)$ denotes the G-conjugacy class containing h. It is easy to see that $H \leq G'$ and $|H|(|H|-1) \,||G|$. Moreover, H is an elementary abelian normal subgroup of G. In [10], Shahryari and Shahabi studied the structure of

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finite groups with a small subgroup. They proved that, under certain conditions, G is a Frobenius group with kernel H.

In this connection, one might ask about the structure of G, if G has a normal subgroup which is a union of three or four distinct conjugacy classes. For convenience, we say that a normal subgroup of G is n-decomposable if it is a union of n distinct conjugacy classes of G.

In [11], Shahryari and Shahabi studied the structure of finite groups G with a normal subgroup H which is 3-decomposable. They proved that H is either an elementary abelian subgroup, a metabelian p-group or a Frobenius group with elementary abelian kernel H'.

In [12], Riese and Shahabi determined the structure of finite groups G with a normal 4-decomposable subgroup H. In this case, they proved that the number of characteristic subgroups of G is at most 4 and H is either a p-group with $H'' = 1$, an alternating group of degree 5 with $G/C_G(H) \cong S_5$ or a subgroup of order $p^a q^b$, where p, q are distinct primes and a, b are positive integers. Also, they determined the structure of the subgroup H, when H is a subgroup of order $p^a q^b$, in which p, q are distinct primes and a, b are positive integers.

In this paper, as usual, G' denotes the derived subgroup of G, $Z(G)$ is the center of G, $\Phi(G)$ is the Frattini subgroup of G and $E(p^n)$ is an elementary abelian group of order p^n. Throughout this paper, all groups considered are assumed to be finite. Our notation is standard and taken mainly from [2, 4, 6].

2. Main Results and Theorems

Let h be a non-central element of a group G and let $H = 1 \cup Cl_G(h)$ be a small subgroup of G. In [10], Shahryari and Shahabi studied the structure of G with the additional condition that $G' = H$ and $Z(G) = 1$. With this condition, they proved that G is a Frobenius group with kernel H and its complement is abelian. Moreover, $|G| = |H|(|H|-1)$, $C_G(h) = H$, G has exactly one irreducible non-linear character χ with $\chi(1) = [G : H]$ and $\chi(h) = -1$.

In what follows, under certain condition, we improve this result to the case that G' is 3-decomposable.

Theorem 1. *Let G be a finite centerless group, $G' = 1 \cup Cl_G(g) \cup Cl_G(h)$, g, h be non-conjugate and non-central elements of G and $h^{-1} \in Cl_G(g)$. Then the following assertions holds:*

(i) *G is solvable and G' is the unique minimal normal subgroup of G,*
(ii) *G is a Frobenius group with kernel G' and its complement is cyclic,*
(iii) *G has exactly two irreducible non-linear character χ and ψ with $\chi(1) = \psi(1) = |G : G'|$,*
(iv) *$|G| = (1/2)p^a(p^a - 1)$, in which $p^a = |G'|$.*

Proof. It follows from [11, Proposition 1] and its proof that G' is elementary abelian and is a minimal normal subgroup of G. Suppose $1 \neq L \trianglelefteq G$. Since $Z(G) = 1$, $1 \neq [G, L] \subseteq L \cap G'$, and so $|L \cap G'| > 1$. By the minimality of G', we have $G' \subseteq L$. So G' is the unique minimal normal subgroup of G. Again, since

$Z(G)=1$, by Theorem 5.2.1 of [8], G is not nilpotent, and by Wielandt's theorem ([8]), $G' \nsubseteq \Phi(G)$. Therefore, there exists a maximal subgroup M of G such that $G' \nsubseteq M$. Now $G' \cap M \trianglelefteq M$ and so $M \leq N_G(G' \cap M)$. Since G' is abelian, $G' \cap M \trianglelefteq G'$, hence $G' \leq N_G(G' \cap M)$. This shows that $G = G'M \leq N_G(G' \cap M)$. Hence $G' \cap M$ is a normal subgroup of G. As G' is the unique minimal normal subgroup of G, $G' \cap M = 1$. This shows that M is an abelian subgroup of G and G is solvable.

Suppose $M \trianglelefteq G$. Since $M \cap G' = 1$ and $G = G'M$, by Theorem 2.5.2 of [3], $G \cong G' \times M$. So G is abelian, a contradiction. Assume $g \in G \backslash M$, then $M^g \neq M$. As M and M^g are abelian subgroups of G, they are contained in $N_G(M \cap M^g)$. Therefore $M \cap M^g$ is a normal subgroup of $\langle M, M^g \rangle = G$, G is a Frobenius group with kernel G' and its complement is abelian. As a Frobenius complement cannot contain any subgroup of type (p,p), any Frobenius complement of G is cyclic.

Since G is a Frobenius group with complement M, each irreducible character of M extends uniquely to an irreducible character of G containing G' in its kernel. So, by [7, Theorem 5.1], G' has two G-conjugacy classes of non-principal irreducible characters. Suppose η_1 and η_2 are two representatives of these classes. If $\chi = \eta_1^G$ and $\psi = \eta_2^G$ then χ and ψ are the only irreducible characters of G and $\chi(1) = \psi(1) = |G : G'|$. Furthermore, since G' is abelian, $(|G'|-1)/|M| = 2$. This completes the proof. ■

The following lemma, which improves Corollary 7 of [11], will be used later.

Lemma 1. *Let $H = 1 \cup Cl_G(g) \cup Cl_G(h)$, $h^{-1} \in Cl_G(h)$ and $(o(g), o(h)) = 1$. Then H is a Frobenius group of order $2^n p$, where $p = 2^n - 1$ is prime.*

Proof. Without loss of generality, we can assume that $gh, hg \in Cl_G(h)$. By Lemma 3 of [11], H is a Frobenius group of order $p^m q^n$, for some distinct primes p and q, and some positive integers n and m. By Lemma 5 of [11], H' is a small subgroup of H, and, by Lemma 4 of [11], $Z(H) = 1$. So, by Theorem 2.1 of [10], $|H| = |H'|(|H'|-1)$. On the other hand, by Lemma 6 of [11], $|H| = pq^n$. This shows that $p = q^n - 1$ and so $q = 2$. This concludes the proof of the lemma. ■

From now all, G is assumed to be a finite non-complete group, i.e. $G' \neq G$. We investigate the structure of the group G with the condition that every non-trivial proper normal subgroup of G is n-decomposable, for a given n. We denote the set of all such positive integers by Λ. In the following simple lemma, we determine the structure of abelian groups with the mentioned condition.

Lemma 2. *Let G be a finite abelian group in which any non-trivial proper normal subgroup is n-decomposable. Then n is a prime number and G has order n^2.*

Proof. Elementary. ■

The previous lemma shows that $p \in \Lambda$, for any prime p. In the following example, we show that $1 + (p-1)/q \in \Lambda$, in which p, q are primes and $q | p-1$.

Example 1. Let G be a non-abelian group of order pq, in which p and q are primes and $p>q$. It is well known that $q|p-1$ and G has exactly one normal subgroup. Suppose that $H=\langle a\rangle$ is the normal subgroup of G. Then H is $(1+(p-1)/q)$-decomposable. This shows that $1+(p-1)/q\in\Lambda$, for any pair of prime numbers p and q with $q|p-1$.

In the following theorem, we investigate the structure of a finite solvable group G with the condition that every normal subgroup of G is n-decomposable. In fact, we have:

Theorem 2. *Suppose that G is a non-abelian and every element of $\mathcal{N}_G$ is n-decomposable. We have:*

(i) *Every element of $\mathcal{N}_G$ is maximal and also minimal in $\mathcal{N}_G$,*
(ii) *G is centerless or n is a prime number and $|Z(G)|=n$,*
(iii) *If K and L are two distinct elements of $\mathcal{N}_G$, then $G=K\times L$,*
(iv) *If K is a solvable element of $\mathcal{N}_G$, then it is elementary abelian,*
(v) *If every element of $\mathcal{N}_G$ is solvable, then $\mathcal{N}_G$ consists of only one element,*
(vi) *G is solvable if and only if G' is abelian; in such a case, $\mathcal{N}_G=\{G'\}$, $G'\cong E(p^r)$ and is maximal in G, G is a Frobenius group with kernel G' and its complement is a cyclic group of prime order q with $p^r-1=(n-1)q$.*

Proof. (i), (ii) and (iii) are obvious. For (iv), we can see that K is characteristically simple. As K is solvable, it is elementary abelian. (iv) is then proved. We now assume that every element of $\mathcal{N}_G$ is solvable and K and L are two different elements of $\mathcal{N}_G$, then by (iii) and (iv), G is abelian, a contradiction. So (v) follows.

Finally, assume that G is solvable. By (v), $\mathcal{N}_G=\{G'\}$ and, by (i), G' is a maximal subgroup of G. This shows that $|G:G'|=q$ with q prime. Since G' is a minimal normal subgroup of G, G' is an elementary abelian subgroup of order, say p^r. Thus, $|G|=p^rq$. Since G is not abelian, $q\neq p$ and $C_G(x)=G'$, for any $x\in G'$, $x\neq 1$. Therefore, by [7, Theorem 1.2], G is a Frobenius group with kernel G'. Since G' is abelian, by [7, Theorem 5.1], $n-1=(|G'|-1)/q$. Therefore, $p^r-1=(n-1)q$, as desired. ■

Theorem 3. *Suppose that every proper non-trivial normal subgroup of G is small. Then one of the following holds:*

(a) *G is an abelian group of order 4,*
(b) *G is isomorphic to S_3, the symmetric group on three symbols,*
(c) *G is isomorphic to the semidirect product $Z_p\tilde{\times}E(2^n)$, in which $p=2^n-1$ is prime, and, for a given positive integer n and a prime number p such that $p=2^n-1$, there exists at most one such a group.*

Proof. By Lemma 2, we can assume that G is not abelian. According to Theorem 2.1 of [10], G' is the unique non-trivial proper normal subgroup of G and is elementary abelian. By Theorem 2, G is a semidirect product of an elementary abelian subgroup of order q^n by a cyclic group of order p with p prime, and $p=q^n-1$. Therefore, $q=2$ or $q=p+1$. If $q=p+1$, then $p=2$, $q=3$ and

G is isomorphic to S_3. Suppose $q = 2$. Then G is isomorphic to the semidirect product $Z_p \tilde{\times} E(2^n)$, in which $p = 2^n - 1$. It is well known that $\mathrm{Aut}(G') \cong GL(2,n)$ and $|GL(2,n)| = pm$, where $(p,m) = 1$. If $f : Z_p \longrightarrow GL(2,n)$ is a group homomorphism, then $o(f(1)) = 1$ or p. If $o(f(1)) = 1$ then G is abelian, a contradiction. Thus, $o(f(1)) = p$ and the image of Z_p is a Sylow subgroup of $GL(2,n)$, proving the theorem. ■

Theorem 4. *Suppose that every proper non-trivial normal subgroup of G is a union of three conjugacy classes of G. Then one of the following holds:*
(a) *G is an abelian group of order 9,*
(b) *G is a group of order pq, p and q are primes and $q = (p-1)/2$,*
(c) *G is isomorphic to the semidirect product $Z_q \tilde{\times} E(3^n)$, in which $q = \frac{3^n-1}{2}$ is prime and, for a given positive integer n and a prime number q such that $q = \frac{3^n-1}{2}$, there exists at most one such a group.*

Proof. Suppose that G is non-abelian. Let H be an element of $\mathcal{N}_G$. As H is 3-decomposable, it follows from [11] that H is solvable. By Theorem 2, G' is the unique element of $\mathcal{N}_G$ and is elementary abelian. Again, by Theorem 2, G is either centerless or $|Z(G)| = p$. If $|Z(G)| = p$ then, since $G' = Z(G)$, G has order pq with q prime; such a non-abelian group is then centerless, a contradiction. So G is centerless. Suppose $|G'| = p^n$ and $|G|/|G'| = q$, q is prime. By Theorem 2, $p^n - 1 = 2q$. Since q is prime, $n = 1$ or $n > 1$ and $p = 3$. If $n = 1$, then G has order pq with p and q prime and $q = (p-1)/2$. If $n > 1$, then G is isomorphic to a semidirect product of the elementary abelian group $E(3^n)$ by a cyclic group of order $q = (3^n - 1)/2$ with q prime. A similar argument as in Theorem 3 shows that, if there exists such a group, it is unique. This completes the theorem. ■

Theorem 5. *Suppose that every proper non-trivial normal subgroup of G is 4-decomposable. Then one of the following holds:*
(a) *$G \cong S_5$, the symmetric group on five letters,*
(b) *G is a group of order pq, p and q are primes and $q = (p-1)/3$,*
(c) *G is isomorphic to the semidirect product $Z_q \tilde{\times} E(2^n)$, in which $q = (2^n-1)/3$ is prime, and, for a given positive integer n and a prime number q such that $q = (2^n - 1)/3$, there exists at most one such a group.*

Proof. By Lemma 2, G is not abelian. We first assume that $\mathcal{N}_G$ contains a non-solvable subgroup H of G. By Theorem 1 of [12], $H \cong A_5$, the alternating group of degree 5 and $G/C_G(H) \cong S_5$. Suppose $C_G(H) \neq 1$. If $C_G(H) = G$ then $H \subseteq Z(G)$, a contradiction. So we can assume that $1 \neq C_G(H) \neq G$. Since H is not abelian, $H \neq C_G(H)$, $G \cong H \times C_G(H)$ and $S_5 \cong G/C_G(H) \cong A_5$, which is impossible. Therefore, $C_G(H) = 1$ and $G \cong S_5$.

We next assume that every element of $\mathcal{N}_G$ is solvable. By Theorem 2, $\mathcal{N}_G = \{G'\}$ and G' is elementary abelian. This shows that G is a solvable group and, by Theorem 2, G is a centerless group of order $p^n q$ with p, q prime and $p^n - 1 = 3q$. Since p and q are primes, $n \leq 2$ or $n > 2$ and $p = 2$. If $n \leq 2$ then $n = 1$ and $|G| = pq$, in which p and $q = (p-1)/3$ are prime numbers. Thus, we can assume that $n > 2$ and $p = 2$. In this case, G is a semidirect product of an elementary abelian subgroup $E(2^n)$ by a cyclic group of order $q = (2^n - 1)/3$ with q prime. A similar argument as in Theorem 3 shows that there exists at most one such a group. This completes the proof. ■

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