Vietnam Journal of Mathematics **30**:2 (2002) 149–165

Vietnam Journal
of
MATHEMATICS

Constructing Soliton Solutions of the Nonlinear Schrödinger Equation by Inverse Scattering and Hirota's Direct Methods

Pham Loi Vu and Nguyen Huy Hoang

Institute of Mechanics, 264 Doi Can Street, Hanoi, Vietnam

Received August 16, 2000

Abstract. We construct the exact complex-valued solutions of the nonlinear Schrödinger equation (NLS) in the class of nonscattering potentials, where the inverse problem associated with the NLS equation can be solved exactly. It is shown that in this class if the solution of the inverse problem satisfies the NLS equation and if the singular numbers of this problem satisfy some conditions, then the degree of normalization polynomials generated by the discrete spectrum must be zero and the polynomials are reduced to the corresponding normalization factors, which depend on time only. If the degree of normalization polynomials is zero, then the general N-soliton solution q of the NLS equation is given by the transform: $q = F/G$, where F and G are represented in the explicit forms in terms of the given scattering data.

1. Introduction

The problem associated with the nonlinear Schrödinger equation (NLS for short) on a half-line is the system of linear equations [5, 6]:

$$-iJ\Phi_x + C\Phi = \lambda\Phi, \quad \Phi = \big(\Phi_1(x,t,\lambda), \Phi_2(x,t,\lambda)\big), \quad (x,t) \in (0,\infty) \times (-\infty,\infty), \tag{1.1}$$

with the boundary condition:

* This work was supported in part by the National Basic Program in Natural Sciences, Vietnam.

$$\Phi_1(0,t,\lambda) = \Phi_2(0,t,\lambda), \tag{1.2}$$

where $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & c_1 \\ c_2 & 0 \end{pmatrix}$, λ is a parameter.

The potentials $c_1(x,t)$, $c_2(x,t)$ are complex valued measurable functions satisfying the estimate:

$$|c_k(x,t)| \le \widetilde{C}e^{-\varepsilon x}, \quad \widetilde{C} \text{ is a constant}, \quad \varepsilon > 0, \quad k = 1,2. \tag{1.3}$$

We begin by recalling the necessary results on the inverse scattering problem (1.1), (1.2) from the works [5, 6]. It is shown that the fundamental system of solutions of the system (1.1) is

$$e^{(1)}(x,t,\lambda) = \Big(e_1^{(1)}(x,t,\lambda), e_2^{(1)}(x,t,\lambda)\Big)$$

$$= \Big(e^{-i\lambda x} + e^{-i\lambda x}\int_0^\infty H_{11}(x, x+\xi; t)e^{-i\lambda\xi}d\xi,\ e^{-i\lambda x}\int_0^\infty H_{21}(x,x+\xi;t)e^{-i\lambda\xi}d\xi\Big), \tag{1.4}$$

$$e^{(2)}(x,t,\lambda) = \Big(e_1^{(2)}(x,t,\lambda), e_2^{(2)}(x,t,\lambda)\Big)$$

$$= \Big(e^{i\lambda x}\int_0^\infty H_{12}(x,x+\xi;t)e^{i\lambda\xi}d\xi,\ e^{i\lambda x} + e^{i\lambda x}\int_0^\infty H_{22}(x,x+\xi;t)e^{i\lambda\xi}d\xi\Big), \tag{1.5}$$

where λ is a real number and $H_{ij}(x, x+\xi; t)$, $i, j = 1, 2$ are elements of the matrix kernel $H(x, x+\xi; t)$ of the transformation operator usually used in scattering problems. In addition, $H_{ij}(x, s, t)$ satisfies the estimate analogous to the estimate (1.3):

$$|H_{ij}(x,s;t)| \le \widetilde{C}\exp\Big\{-\frac{1}{2}\varepsilon(x+s)\Big\}, \quad 0 \le x \le s, \quad i,j = 1,2. \tag{1.6}$$

We call

$$S(t,\lambda) = \frac{e_1^{(1)}(0,t,\lambda) - e_2^{(1)}(0,t,\lambda)}{e_2^{(2)}(0,t,\lambda) - e_1^{(2)}(0,t,\lambda)}$$

the scattering function for system (1.1). Due to the estimate (1.6) $e_1^{(1)}(0,t,\lambda) - e_2^{(1)}(0,t,\lambda)$ is holomorphic for $\mathrm{Im}\lambda < \varepsilon/2$ and $e_2^{(2)}(0,t,\lambda) - e_1^{(2)}(0,t,\lambda)$ is holomorphic for $\mathrm{Im}\lambda > -\varepsilon/2$, therefore they have a finite number of zeros : $\lambda_j^-, \mathrm{Im}\lambda_j^- < 0$, $j = 1,\ldots,\gamma$; λ_k^+, $\mathrm{Im}\lambda_k^+ > 0$, $k = 1,\ldots,\alpha$. We denote the multiplicities of these zeros by n_j and m_k, respectively. These zeros are called the singular numbers of the problem (1.1), (1.2). We put

$$f_k^+(-x,t) = \frac{1}{2\pi}\int_{c_k^+}\Big(e_2^{(2)}(0,t,\lambda) - e_1^{(2)}(0,t,\lambda)\Big)^{-1} \times$$

$$\Big[A_0 + A_1(\lambda - \lambda_k^+) + \ldots + \frac{A_{m_k-1}}{(m_k-1)!}(\lambda-\lambda_k^+)^{m_k-1}\Big]e^{i\lambda x}d\lambda,$$

$$k = 1,\ldots,\alpha, \tag{1.7}$$

$$g_j^-(x,t)=\frac{1}{2\pi}\int\limits_{c_j^-}\left(e_1^{(1)}(0,t,\lambda)-e_2^{(1)}(0,t,\lambda)\right)^{-1}\times$$
$$\left[B_0+B_1(\lambda-\lambda_j^-)+\ldots+\frac{B_{n_j-1}}{(n_j-1)!}(\lambda-\lambda_j^-)^{n_j-1}\right]e^{-i\lambda x}d\lambda,$$
$$j=1,\ldots,\gamma, \tag{1.8}$$

where c_k^+ and c_j^- stand for sufficiently small circles centered, respectively, at λ_k^+ and λ_j^-; $A_0,\ldots,A_{m_k-1}$ and $B_0,\ldots,B_{n_j-1}$ are the numbers that depend on the zeros λ_k^+ and λ_j^-, respectively [5].

We note that $f_k^+(-x,t)=p_k(x,t)e^{i\lambda_k^+x}$ and $g_j^-(x,t)=q_j(x,t)e^{-i\lambda_j^-x}$, where $p_k(x,t)$ and $q_j(x,t)$ are the normalization polynomials of x of degrees m_k-1 and n_j-1 respectively. We introduce in the sense of generalized functions the inverse Fourier transforms of the functions $S(t,\lambda)-1$ and $S^{-1}(t,\lambda)-1$:

$$f_S(x,t)=\frac{1}{2\pi}\int\limits_{-\infty+i\eta}^{+\infty+i\eta}[S(t,\lambda)-1]e^{-i\lambda x}d\lambda,$$
$$g_S(x,t)=\frac{1}{2\pi}\int\limits_{-\infty-i\eta}^{+\infty-i\eta}[S^{-1}(t,\lambda)-1]e^{-i\lambda x}d\lambda. \tag{1.9}$$

Here η is some number such that $0<\eta<\varepsilon_0$, $\varepsilon_0=\min\{\varepsilon/2,\mathrm{Im}\lambda_1^+,\ldots,\mathrm{Im}\lambda_\alpha^+,|\mathrm{Im}\lambda_1^-|,\ldots,|\mathrm{Im}\lambda_\gamma^-|\}$. Since $S(t,\lambda)-1$ and $S^{-1}(t,\lambda)-1$ are analytic in the strip $0<|\mathrm{Im}\ \lambda|<\varepsilon_0$, the values of the integrals (1.9) are independent of η. The functions $f_S(x,t)$, $g_S(x,t)$ and $f_k^+(-x,t)$, $g_j^-(x,t)$ characterize the problem (1.1), (1.2) on the continuous and on the point spectrum, respectively.

The problem (1.1), (1.2) is associated with the nonlinear matrix equation [5, 6]:

$$Q_t=-iQ_{xx}J+2iqrQJ,\qquad (x,t)\in(0,\infty)\times(-\infty,\infty),$$

where $\quad Q=\begin{pmatrix}0&q\\r&0\end{pmatrix}=\begin{pmatrix}0&ic_1\\-ic_2&0\end{pmatrix}.$

Note that, when $r=\mp\bar{q}$, the nonlinear matrix equation is reduced to the nonlinear Schrödinger (NLS) equation on the half-line:

$$iq_t=q_{xx}+2|q|^2q, \tag{1.10}$$
$$iq_t=q_{xx}-2|q|^2q,\qquad (x,t)\in(0,\infty)\times(-\infty,\infty). \tag{1.11}$$

It is known that if the matrix potential C in (1.1) is selfconjugate, then the problem (1.1), (1.2) has no discrete spectrum [7], and the NLS equation (1.11) can be solved by the inverse scattering method with a given initial condition: $q(x,t)\Big|_{t=0}=q(x,0)$, [6].

In the case: $r=-\bar{q}$, it follows from the integral equations for the kernels of the transformation operator that [5]:

$$H_{11}(x,s;t)=\overline{H}_{22}(x,s;t),\ H_{12}(x,x;t)=-\overline{H}_{21}(x,x;t),\ 0\le x\le s. \tag{1.12}$$

There exists a remarkable class of potentials, for which the inverse scattering problem can be solved exactly. These potentials are the nonscattering potentials, for which $f_S^+(x,t)$ and $g_S^-(x,t)$ defined by (1.9) are equal to zero, [6]. We assume that the scattering data for the NLS equation (1.10) belong to the class of nonscattering potentials and so we consider the set of the scattering data:

$$s(t)=\Big\{\lambda_k^+,\ \mathrm{Im}\lambda_k^+\ge\varepsilon_0,\ p_k(x,t);\ \lambda_k^-=\overline{\lambda_k^+},\\ \mathrm{Im}\lambda_k^-\le-\varepsilon_0,\ \bar{p}_k(x,t),\ k=1,\dots,N,\ \varepsilon_0>0\Big\}. \tag{1.13}$$

In the class of nonscattering potentials the kernels will be sought in the form:

$$\begin{aligned} H_{11}(x,x;t)&=\sum_{k=1}^{N}H_{11,k}(x;t)e^{i\lambda_k^+x},\\ H_{12}(x,x;t)&=\sum_{k=1}^{N}H_{12,k}(x;t)e^{-i\lambda_k^-x}. \end{aligned} \tag{1.14}$$

Due to (1.12), the functions $H_{11,k}(x;t)$ and $H_{12,k}(x;t)$ satisfy the following system [6]:

$$\begin{aligned} &H_{11,k}(x;t)-\sum_{j=1}^{N}H_{12,j}(x;t)\int_x^\infty p_k(x+\tau,t)e^{i(\lambda_k^+-\lambda_j^-)\tau}d\tau=0,\\ &H_{12,k}(x;t)+\sum_{j=1}^{N}H_{11,j}(x;t)\int_x^\infty \bar{p}_k(x+\tau,t)e^{i(\lambda_j^+-\lambda_k^-)\tau}d\tau=-\bar{p}_k(2x,t)e^{-i\lambda_k^-x}, \end{aligned} \tag{1.15}$$

where $\lambda_k^+=\overline{\lambda_k^-}\in s(t)$, $k=1,\dots,N$, $(x,t)\in(0,\infty)\times(-\infty,\infty)$.
We calculate $H_{12,k}(x;t)$, $k=1,\dots,N$, from the system (1.15), and then with the help of the formula (1.14) we find:

$$H_{12}(x,x;t)=\big(\det A\big)^{-1}\sum_{j=1}^{N}\Big(\det A^{(j+N)}\Big)e^{\frac12 k_{j+N}x}, \tag{1.16}$$

where $k_j=2i\lambda_j^+$, $k_{j+N}=\overline{k}_j=-2i\lambda_j^-$, the matrices A and $A^{(j+N)}$ are written in the block form. Namely,

$$A=\begin{pmatrix} I & M\\ \widetilde{M} & I\end{pmatrix},\quad I \text{ is the } N\times N \text{ unit matrix}, \tag{1.17}$$

$$M=\Big[-M_{lj}(x,t)e^{\frac12(k_l+k_{j+N})x}\Big]_{l,j=1}^{N},\quad \widetilde{M}=\Big[\overline{M}_{lj}(x,t)e^{\frac12(k_j+k_{l+N})x}\Big]_{l,j=1}^{N},$$

$$M_{lj}(x,t)e^{\frac12(k_l+k_{j+N})x}=\int_x^\infty p_l(x+\tau,t)e^{\frac12(k_l+k_{j+N})\tau}d\tau, \tag{1.18}$$

$A^{(j+N)}$ stands for the matrix obtained from the matrix A substituting the elements in its $(j+N)$-th column by the column of the right-hand side of the system (1.15).
The kernel (1.16) is related to the potential by the equality [6]:

$$2iH_{12}(x,x;t) = c_1(x,t) = -iq(x,t). \tag{1.19}$$

It follows from (1.16) and (1.19) that the general N-soliton solution of the NLS equation (1.10) has a complicated structure:

$$q(x,t) = \frac{G(x,t)}{F(x,t)}, \tag{1.20}$$

where $F(x,t) = \det A,\ G(x,t) = -2\sum_{j=1}^{N}\Big(\det A^{(j+N)}\Big)e^{\frac{1}{2}k_{j+N}x}$.

Our paper is constructed as follows. In Sec. 2 we find the representations of F and G. In Sec. 3 we assume that the singular numbers of the problem (1.1), (1.2) satisfy some conditions, then with the help of the obtained representations we prove that the normalization polynomials of (1.13) are reduced to the corresponding normalization factors. Further, in Sec. 4 using Hirota's method we show that the general N-soliton solution of the NLS equation (1.10) is given by the transform (1.20), wherein F and G are represented in the explicit forms in terms of the normalization factors.

2. The Representations of F and G

Lemma 1. *The value of the determinant F of the matrix A is real and F is represented in the form:*

$$F(x,t) = \sum_{\mu_j=0,1} D_1(\mu)a_\mu(x,t)\prod_{j=1}^{2N} e^{\mu_j k_j x}, \tag{2.1}$$

where the summation is taken over μ, $\mu = (\mu_1,\mu_2,\ldots,\mu_{2N})$, in addition $\mu_j = 0,1$; $a_\mu(x,t)$ is a polynomial of x with coefficients depending on t and

$$D_1(\mu) = \begin{cases} 1 & \text{when } \sum_{j=1}^{N}\mu_j = \sum_{j=1}^{N}\mu_{j+N}, \\ 0 & \text{otherwise.}\end{cases} \tag{2.2}$$

Proof. We write the expansion of $\det A$:

$$F = \det A = \sum_\sigma \mathrm{sign}\sigma A_{1\sigma(1)}A_{2\sigma(2)}\ldots A_{2N\sigma(2N)}, \tag{2.3}$$

where the summation is taken over all permutations σ of $2N$ numbers.
Consider a nonzero term corresponding to a permutation σ. Putting

$$A^0_\sigma = \{j : \sigma(j) = j\},\ A^1_\sigma = \{j : \sigma(j) \neq j \text{ and } j \le N\},$$
$$A^2_\sigma = \{j :\ \sigma(j) \neq j \text{ and } j > N\}, \tag{2.4}$$

and taking the structure of the matrix A into account we can verify that

$$\sigma(A^1_\sigma) = A^2_\sigma, \quad \sigma(A^2_\sigma) = A^1_\sigma. \tag{2.5}$$

Beside σ we introduce another permutation $\tilde\sigma$ depending on σ: $\tilde\sigma(j+N) = \sigma(j) - N$ for $j \in A^1_\sigma$; $\tilde\sigma(j-N) = \sigma(j) + N$ for $j \in A^2_\sigma$; $\tilde\sigma(j) = j$ otherwise, then

$$\begin{aligned} &A^1_{\tilde\sigma} = \{j : j+N \in A^2_\sigma\}; \quad A^2_{\tilde\sigma} = \{j : j-N \in A^1_\sigma\},\\ &\mathrm{sign}\tilde\sigma = \mathrm{sign}\sigma \text{ and } \tilde{\tilde\sigma} = \sigma. \end{aligned} \tag{2.6}$$

It follows from (1.17), (1.18) and (2.6) that

$$\begin{aligned} &\text{If } j \in A^1_{\tilde\sigma} \text{ then } k = j+N \in A^2_\sigma \text{ and}\\ &A_{j\tilde\sigma(j)} = A_{k-N\tilde\sigma(k-N)} = A_{k-N\sigma(k)+N} = -\overline{A}_{k\sigma(k)},\\ &\text{if } j \in A^2_{\tilde\sigma} \text{ then } k = j-N \in A^1_\sigma \text{ and}\\ &A_{j\tilde\sigma(j)} = A_{k+N\tilde\sigma(k+N)} = A_{k+N\sigma(k)-N} = -\overline{A}_{k\sigma(k)}. \end{aligned} \tag{2.7}$$

Due to (2.4)–(2.7) the nonzero terms corresponding to σ and $\tilde\sigma$ are

$$\mathrm{sign}\sigma A_{1\sigma(1)}A_{2\sigma(2)}\dots A_{2N\sigma(2N)} = \mathrm{sign}\sigma \prod_{j\in A^1_\sigma} A_{j\sigma(j)} \prod_{j\in A^2_\sigma} A_{j\sigma(j)}, \tag{2.8}$$

$$\begin{aligned} \mathrm{sign}\tilde\sigma A_{1\tilde\sigma(1)}A_{2\tilde\sigma(2)}\dots A_{2N\tilde\sigma(2N)} &= \mathrm{sign}\sigma \prod_{k\in A^2_\sigma}\left(-\overline{A}_{k\sigma(k)}\right)\prod_{k\in A^1_\sigma}\left(-\overline{A}_{k\sigma(k)}\right)\\ &= \mathrm{sign}\sigma \prod_{j\in A^1_\sigma}\overline{A}_{k\sigma(k)} \prod_{j\in A^2_\sigma}\overline{A}_{k\sigma(k)}. \end{aligned} \tag{2.9}$$

Hence, the right-hand sides of (2.8) and (2.9) are complex conjugate to each other.

We decompose permutations corresponding to nonzero terms of (2.3) into two sets.
The first set consists of permutations satisfying the condition: $\sigma = \tilde\sigma$, while all the others belong to the second one. Due to (2.7)–(2.9) the terms corresponding to permutations of the first set are real.

The permutations of the second set are grouped in pairs: $\{\sigma, \tilde\sigma\}$. By virtue of (2.7)–(2.9) two terms corresponding to every pair of permutations σ and $\tilde\sigma$ are complex conjugate to each other, therefore the sum of these terms is a real function. The determinant of A is the sum of real functions. Hence, the value of $\det A$ is real.

Using (1.17), (1.18), (2.5) and (2.6) we write the term (2.8) in the form:

$$\begin{aligned} &\prod_{j\in A^1_\sigma} A_{j\sigma(j)} \prod_{j\in A^2_\sigma} A_{j\sigma(j)}\\ &= \prod_{j\in A^1_\sigma}\left(-M_{j\sigma(j)-N}\right)\prod_{j\in A^2_\sigma}\overline{M}_{j-N\sigma(j)}\prod_{j\in A^1_\sigma} e^{\frac{k_j+k_{\sigma(j)}}{2}x}\prod_{j\in A^2_\sigma} e^{\frac{k_j+k_{\sigma(j)}}{2}x}\\ &= \prod_{j\in A^1_\sigma}\left(-M_{j\sigma(j)-N}\right)\prod_{j\in A^2_\sigma}\overline{M}_{j-N\sigma(j)}\prod_{j\in A^1_\sigma} e^{k_j x}\prod_{j\in A^2_\sigma} e^{k_j x}. \end{aligned} \tag{2.10}$$

Due to (2.10), the sum (2.3) can be rewritten as:

$$\sum_{\sigma} \mathrm{sign}\sigma A_{1\sigma(1)} A_{2\sigma(2)} \dots A_{2N\sigma(2N)} = \sum_{\sigma} \hat{a}(x,t) \prod_{j=1}^{2N} e^{\mu_j k_j x}, \tag{2.11}$$

where
$$\hat{a}(x,t) = \mathrm{sign}\sigma \prod_{j \in A_\sigma^1} \Big(- M_{j\sigma(j)-N} \Big) \prod_{j \in A_\sigma^2} \overline{M}_{j-N\sigma(j)},$$

$\hat{a}(x,t)$ is a polynomial of x with coefficients depending on t, $\mu = (\mu_1, \mu_2, \dots, \mu_{2N})$, $\mu_j = 1$ if $j \in A_\sigma^1 \cup A_\sigma^2$ and $\mu_j = 0$ if $j \in A_\sigma^0$.
After the sum (2.11) had been grouped in exponential terms together, the determinant F of A is represented in the form (2.1). The formula (2.2) is obtained from (2.4)–(2.5) and the coefficient $D_1(\mu)$ in (2.1) is defined by the formula (2.2). The lemma is proved.

Lemma 2. *The kernel* (1.16) *is represented in the form:*

$$H_{12}(x, x+t) = (\det A)^{-1} \sum_{\mu_j = 0,1} \Big(D_2(\mu) b_\mu(x,t) \prod_{j=1}^{2N} e^{\mu_j k_j x} \Big), \tag{2.12}$$

where the summation is taken over μ, $\mu = (\mu_1, \mu_2, \dots, \mu_{2N})$, *in addition* $\mu_j = 0, 1$; $b_\mu(x,t)$ *is a polynomial of* x *with coefficients depending on* t *and*

$$D_2(\mu) = \begin{cases} 1 & \textit{when} \quad 1 + \sum_{j=1}^{N} \mu_j = \sum_{j=1}^{N} \mu_{j+N}, \\ 0 & \textit{otherwise}. \end{cases} \tag{2.13}$$

Proof. Let $B^{(j+N)}$ be the matrix obtained from the matrix $A^{(j+N)}$ multiplying the elements of its $(j+N)$-th column by $e^{\frac{k_{j+N}}{2} x}$. Then the elements B_{lm} of the matrix $B^{(j+N)}$ are:

$$\begin{aligned} B_{lm} &= -M_{lm-N} e^{\frac{1}{2}(k_l + k_m)x} \text{ for } l \le N < m \text{ and } m \ne j+N, \\ B_{lm} &= \overline{M}_{l-Nm} e^{\frac{1}{2}(k_l + k_m)x} \text{ for } l > N \ge m, \quad B_{ll} = 1 \text{ for } l \ne j+N, \\ B_{lj+N} &= -\overline{p}_{l-N} e^{\frac{1}{2}(k_l + k_{j+N})x} \text{ for } l > N, \quad B_{lm} = 0 \text{ otherwise}. \end{aligned} \tag{2.14}$$

The expansion of $\det B^{(j+N)}$ is

$$\det B^{(j+N)} = \sum_{\sigma} \mathrm{sign}\sigma B_{1\sigma(1)} B_{2\sigma(2)} \dots B_{2N\sigma(2N)}, \tag{2.15}$$

where the summation is taken over all permutations σ of $2N$ numbers.
A nonzero term corresponding to a permutation σ of the sum (2.15) can be written in the form:

$$\mathrm{sign}\sigma B_{1\sigma(1)} B_{2\sigma(2)} \dots B_{2N\sigma(2N)} = \mathrm{sign}\sigma \prod_{l \in B_\sigma^0} B_{l\sigma(l)} \prod_{l \in B_\sigma^1} B_{l\sigma(l)} \prod_{l \in B_\sigma^2} B_{l\sigma(l)}, \tag{2.16}$$

where the sets B^0_σ, B^1_σ and B^2_σ are determined analogously to the sets A^0_σ, A^1_σ and A^2_σ defined by (2.4).
It follows from the matrix $B^{(j+N)}$ that

If $l \in B^1_\sigma$ then $\sigma(l) \in B^2_\sigma \setminus \{j+N\}$, i.e., $\sigma(B^1_\sigma) \subset B^2_\sigma \setminus \{j+N\}$, (2.17)

if $l \in B^2_\sigma$ then $\sigma(l) \in B^1_\sigma$ or $\sigma(l) = j+N$, i.e., $\sigma(B^2_\sigma) \subset B^1_\sigma \cup \{j+N\}$.(2.18)

There are two cases.

Case 1: $\sigma(j+N) = j+N$, i.e., $j+N \in B^0_\sigma$. In this case the relations (2.17) and (2.18) are: $\sigma(B^1_\sigma) \subseteq B^2_\sigma$ and $\sigma(B^2_\sigma) \subseteq B^1_\sigma$. Hence,

$$\sigma(B^1_\sigma) = B^2_\sigma, \quad \sigma(B^2_\sigma) = B^1_\sigma \text{ for } j+N \in B^0_\sigma. \tag{2.19}$$

The equalities (2.5) and (2.19) show that in the case 1 the permutation σ of B^1_σ has the same property as the permutation σ of A^1_σ. Hence, by an argument analogous to that used for the proof of equality (2.11) we get

$$\mathrm{sign}\sigma B_{1\sigma(1)} B_{2\sigma(2)} \dots B_{2N\sigma(2N)} = \hat{b}(x,t) e^{k_{j+N}x} \prod_{\substack{l=1 \\ l\neq j+N}}^{2N} e^{\mu_l k_l x}, \tag{2.20}$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_{2N})$, in addition $\mu_l = 0$ for $l \in B^0_\sigma \setminus \{j+N\}$, $\mu_{j+N} = 1, \mu_l = 1$ for $l \notin B^0_\sigma$; $\hat{b}(x,t)$ is a polynomial of x with coefficients depending on t.

Case 2: $\sigma(j+N) \neq j+N$, i.e., $j+N \in B^2_\sigma$. In this case we have

$$\sigma(B^1_\sigma) = B^2_\sigma \setminus \{j+N\} \text{ and } \sigma(B^2_\sigma) = B^1_\sigma \cup \{j+N\}. \tag{2.21}$$

Using (2.21) we obtain from (2.16):

$$\mathrm{sign}\sigma \prod_{l\in B^0_\sigma} B_{l\sigma(l)} \prod_{l\in B^1_\sigma} B_{l\sigma(l)} \prod_{l\in B^2_\sigma} B_{l\sigma(l)}$$

$$= \mathrm{sign}\sigma \left\{ \prod_{l\in B^1_\sigma} (-M_{l\sigma(l)-N}) e^{\frac{1}{2}(k_l + k_{\sigma(l)})x} \right\}$$

$$\times \left\{ \prod_{l\in B^2_\sigma \setminus \{\sigma^{-1}(j+N)\}} \overline{M}_{l-N\sigma(l)} e^{\frac{1}{2}(k_l+k_{\sigma(l)})x} \right\}$$

$$\times (-\overline{p}_{\sigma^{-1}(j+N)-N}) e^{\frac{1}{2}(k_{\sigma^{-1}(j+N)} + k_{j+N})x}$$

$$= \mathrm{sign}\sigma \left\{ \prod_{l\in B^1_\sigma} (-M_{l\sigma(l)-N}) e^{\frac{1}{2}k_l x} \right\} \left\{ \prod_{l\in\sigma(B^1_\sigma) = B^2_\sigma\setminus\{j+N\}} e^{\frac{1}{2}k_l x} \right\}$$

$$\times \left\{ \prod_{l\in B^2_\sigma \setminus \{\sigma^{-1}(j+N), j+N\}} \overline{M}_{l-N\sigma(l)} e^{\frac{1}{2}k_l x} \right\} \left\{ \prod_{l\in\sigma(B^2_\sigma)\setminus\{j+N\} = B^1_\sigma} e^{\frac{1}{2}k_l x} \right\}$$

$$\times \overline{M}_{j\sigma(j+N)} e^{\frac{1}{2}k_{j+N}x} \times (-\overline{p}_{\sigma^{-1}(j+N)-N}) e^{\frac{1}{2}(k_{\sigma^{-1}(j+N)}x} e^{\frac{1}{2}k_{j+N}x}$$

$$= \tilde{b}(x,t) e^{k_{j+N}x} \prod_{l\in B^1_\sigma} e^{k_l x} \prod_{l\in B^2_\sigma\setminus\{j+N\}} e^{k_l x}$$

$$= \tilde{b}(x,t) e^{k_{j+N}x} \prod_{\substack{l=1 \\ l\neq j+N}}^{2N} e^{\mu_l k_l x}, \tag{2.22}$$

where

$$\tilde{b}(x,t) = \operatorname{sign}\sigma(-\overline{p}_{\sigma^{-1}(j+N)-N}) \prod_{l\in B_\sigma^1} (-M_{l\sigma(l)-N}) \times \prod_{l\in B_\sigma^2\setminus\{\sigma^{-1}(j+N)\}} \overline{M}_{l-N\sigma(l)}, \tag{2.23}$$

and $\mu = (\mu_1, \mu_2, \ldots, \mu_{2N})$, $\mu_l = 0$ for $l \in B_\sigma^0$, $\mu_l = 1$ for $l \notin B_\sigma^0$.
Hence, the sum (2.15) is written as follows:

$$\det B^{(j+N)}(x,t) = \sum_\sigma b^*(x,t) e^{k_{j+N}x} \prod_{\substack{l=1\\ l\neq j+N}}^{2N} e^{\mu_l k_l x} = \sum_{\substack{\mu_l=0,1,\ l\neq j+N\\ \mu_{j+N}=1}} \left\{ D_2(\mu) b_{1\mu}(x,t) \prod_{l=1}^{2N} e^{\mu_l k_l x} \right\}. \tag{2.24}$$

Here the polynomial $b^*(x,t)$ is $\hat{b}(x,t)$ from (2.20) or $\tilde{b}(x,t)$ from (2.23) corresponding to the permutations σ from (2.20) or (2.22). After the sum in the left-hand side of (2.24) has been grouped in exponential terms together, we obtain the right-hand side of (2.24); wherein $b_{1\mu}(x,t)$ is a polynomial of x with coefficients depending on t. The formula (2.13) is obtained from (2.19) and (2.21) and the coefficient $D_2(\mu)$ in (2.24) is defined by (2.13). From (2.24) we have

$$\sum_{j=1}^N \sum_{\substack{\mu_l=0,1,\ l\neq j+N\\ \mu_{j+N}=1}} \left\{ D_2(\mu) b_{1\mu}(x,t) \prod_{l=1}^{2N} e^{\mu_l k_l x} \right\} = \sum_{\mu_j=0,1} \left\{ D_2(\mu) b_\mu(x,t) \prod_{j=1}^{2N} e^{\mu_j k_j x} \right\} \tag{2.25}$$

After the sum in the left-hand side of the equality (2.25) has been grouped in exponential terms together, we receive the right-hand side of one, wherein $b_\mu(x,t)$ is a polynomial of x with coefficients depending on t. Hence, the representation of the kernel (2.12) is obtained from (1.16) and (2.25), which is what we wished to prove.

Thereby, the general N-soliton solution $q(x,t)$ of (1.10) is presented by the transfom (1.20), wherein F is defined by (2.1) and G is

$$G(x,t) = -2\sum_{j=1}^N \det B^{(j+N)}$$
$$= -2\sum_{\mu_j=0,1} \left\{ D_2(\mu) b_\mu(x,t) \prod_{j=1}^{2N} e^{\mu_j k_j x} \right\},\ (x,t) \in (0,\infty)\times(-\infty,\infty).$$

3. On the Degree of the Normalization Polynomials

It is easy to prove by induction the following lemma.

Lemma 3. *Let $a_j(x)$, $j = 1,2,...,m$ be polynomials of x and $\alpha_1,...,\alpha_m$ be different complex numbers, then the following indentity*

$$a_1(x)e^{\alpha_1 x} + a_2(x)e^{\alpha_2 x} + \ldots + a_m(x)e^{\alpha_m x} \equiv 0$$

holds if only if $a_j(x) \equiv 0$, $j = 1,...,m$.

The potential $q(x,t)$ satisfying the NLS equation (1.10) belongs to the class of nonscattering potentials satisfying the estimate (1.3). Substituting (1.20) into (1.10) and using the reality of F and the operators D [1], we write the NLS equation in the form:

$$\begin{aligned} K(F,G) = F\{iG_tF - iGF_t - G_{xx}F + 2G_xF_x - GF_{xx}\} \\ + G\{2F_{xx}F - 2F_x^2 - 2G\overline{G}\} = 0, \end{aligned} \tag{3.1}$$

or

$$F(iD_t - D_x^2)G \circ F + G(D_x^2 F \circ F - 2G\overline{G}) = 0.$$

Let the scattering data (1.13) consist of only one pair of singular numbers: λ_1^+ and λ_1^-, $\lambda_1^+ = \overline{\lambda_1^-}$, $\operatorname{Im}\lambda_1^+ > 0$ and let n be the multiplicity of the given singular numbers. In this case F and G defined by (2.1) and (2.26) are

$$F = 1 + |M_{11}|^2 e^{(k_1+k_2)x}, \quad G = 2\overline{p}_1 e^{k_2 x}, \quad k_1 = 2i\lambda_1^+, \quad k_2 = -2i\lambda_1^-, \tag{3.2}$$

where M_{11} is defined by (1.18) and p_1 is the polynomial of degree $n-1$:

$$p(2x,t) = a_{n-1}x^{n-1} + \ldots + a_0.$$

Hence, the explicit form of the solution of (1.10) is

$$q(x,t) = \frac{2\overline{p}_1 e^{k_2 x}}{1 + |M_{11}|^2 e^{(k_1+k_2)x}}. \tag{3.3}$$

Substituting (3.2) into (3.1) and using Lemma 3 we obtain three equations:

$$i\overline{p}_{1t} - (k_2^2\overline{p}_1 + 2k_2\overline{p}_{1x} + \overline{p}_{1xx}) = 0, \tag{3.4}$$

$$\begin{aligned} - i\overline{p}_1(|M_{11}|^2)_t + 2(k_2\overline{p}_1 + \overline{p}_{1x})\Big[(|M_{11}|^2)_x + (k_1+k_2)|M_{11}|^2\Big] - 8\overline{p}_1^2 p_1 \\ + \overline{p}_1\Big[(|M_{11}|^2)_{xx} + 2(k_1+k_2)(|M_{11}|^2)_x + (k_1+k_2)^2|M_{11}|^2\Big] = 0, \end{aligned} \tag{3.5}$$

$$4\overline{p}_1 p_1 |M_{11}|^2 - \Big[(|M_{11}|^2)_x + (k_1+k_2)|M_{11}|^2\Big]^2 = 0. \tag{3.6}$$

Integrating the right-hand side of (1.18) by parts $(n-1)$ times successively, finally we receive

$$M_{11}(x,t) = \sum_{l=0}^{n-1}\left(-\frac{2}{k_1+k_2}\right)^{l+1}\left(\frac{d^l}{dx^l}p_1(x+\tau,t)\Big|_{\tau=x}\right). \tag{3.7}$$

Differentiating both sides of the equality (3.7) with respect to x we have

$$M_{11x}(x,t) = -(k_1+k_2)M_{11}(x,t) - 2p_1(2x,t).$$

Hence,

$$p_1(2x,t) = -\frac{1}{2}\left[(k_1+k_2)M_{11}(x,t) + M_{11x}(x,t)\right], \tag{3.8}$$

$$\overline{p}_1(2x,t) = -\frac{1}{2}\left[(k_1+k_2)\overline{M}_{11}(x,t) + \overline{M}_{11x}(x,t)\right]. \tag{3.9}$$

Substituting (3.9) into the left-hand side of (3.4) we obtain

$$i\overline{M}_{11t} - k_2^2\overline{M}_{11} - 2k_2\overline{M}_{11x} - \overline{M}_{11xx} = \alpha e^{-(k_1+k_2)x}, \quad \alpha \text{ is some constant.}$$

Since M_{11} is the polynomial of x, α is zero. Then

$$i\overline{M}_{11t} = k_2^2\overline{M}_{11} + 2k_2\overline{M}_{11x} + \overline{M}_{11xx}, \tag{3.10}$$

$$-iM_{11t} = k_1^2 M_{11} + 2k_1 M_{11x} + M_{11xx}. \tag{3.11}$$

Substituting (3.10) and (3.11) into the left-hand side of (3.5), we obtain

$$\begin{aligned}
&(k_1+k_2)^2(M_{11}\overline{M}_{11}\overline{M}_{11x} + \overline{M}_{11}^2 M_{11x}) + (k_1+k_2)(M_{11}\overline{M}_{11}\overline{M}_{11xx} \\
&+ \overline{M}_{11}^2 M_{11xx} + \overline{M}_{11}M_{11x}\overline{M}_{11x} + M_{11}\overline{M}_{11x}^2) \\
&+ (M_{11}\overline{M}_{11x}\overline{M}_{11xx} + \overline{M}_{11}M_{11xx}\overline{M}_{11x} + \overline{M}_{11}M_{11x}\overline{M}_{11xx}) = 0.
\end{aligned} \tag{3.12}$$

Since the degree of M_{11} is equal to $n-1$, the left-hand side of (3.12) forms a polynomial of degree $3n-4$. Then using (3.7), from (3.12) we can verify that the coefficient of x^{3n-4} of the polynomial is

$$-16(n-1)(k_1+k_2)^{-1}a_{n-1}(\overline{a}_{n-1})^2 = 0, \quad k_1+k_2 = -4\,\mathrm{Im}\lambda_1^+ < 0.$$

If $n>1$, then it follows from (3.12) that $a_{n-1} = 0$, this contradicts the hypothesis that the degree of the polynomial p_1 is $n-1$. Hence, the degree of p_1 is zero, i.e., $n=1$, so we can put $p_1 = C(t)$. Substituting p_1 into (3.4) gives

$$p_1 = \widetilde{C}e^{ik_1^2 t}, \quad \widetilde{C} \text{ is a constant.} \tag{3.13}$$

Then due to (3.8):

$$M_{11} = -\frac{2p_1}{k_1+k_2}. \tag{3.14}$$

It is clear that M_{11} satisfies (3.6).
We can verify the truth of the converse assertion: if p_1 and M_{11} are defined by (3.13) and (3.14) respectively, then q_1 constructed by (3.3) satisfies the NLS equation (1.10). The results obtained above can be started in the following lemma.

Lemma 4. *The nonscattering potential* (1.19) *constructed from one pair of singular numbers:* $\lambda_1^+, \lambda_1^-, \lambda_1^+ = \overline{\lambda_1^-}, Im\,\lambda_1^+ > 0$, *of the system of linear equations*

(1.1) *satisfies the NLS equation* (1.10) *if and only if the polynomial* p_1 *generated by the pair of singular numbers is reduced to the normalization factor and evolves according to* (3.13).

We proceed now to consider the general case when the set (1.13) consists of N pairs of singular numbers. For this purpose we substitute (2.1) and (2.26) into the equation (3.1). Further, the terms of the obtained equation are grouped in exponential terms together, then the NLS equation (3.1) takes the form:

$$K(F,G) = \sum_{\mu_j=0,1,2,3} \hat{a}_\mu(x,t) \prod_{j=1}^{2N} e^{\mu_j k_j x} = 0, \tag{3.15}$$

where $\hat{a}_\mu(x,t)$ is a polynomial of x with coefficients depending on t.
We assume that for every pair of singular numbers: $\lambda_j^+ = \frac{1}{2i}k_j$, $\lambda_j^- = \frac{-1}{2i}k_{j+N}$, $j = 1,...,N$, the equality

$$\alpha_1 k_j + \alpha_2 k_{j+N} = \sum_{l=1}^{2N} \beta_l k_l \tag{3.16}$$

holds if and only if $\beta_l = 0$ for $l \neq j, j+N$ and $\alpha_1 = \beta_j$, $\alpha_2 = \beta_{j+N}$, where $\alpha_1, \alpha_2, \beta_l = 0,1,2,3$. We consider a particular case of the diffirential polynomial $K(F,G)$:

$$K\Big(1 + |M_{jj}|^2 e^{(k_j+k_{j+N})x},\ 2\overline{p}_j e^{k_{j+N}x}\Big), \quad j = 1,...,N. \tag{3.17}$$

Let a permutation σ in (2.7) be defined by the sets: $A_\sigma^1 = \{j\}$ and $A_\sigma^2 = \{j+N\}$, where the index j, $1 \le j \le N$, is fixed. Then, taking the structure of A into account we have

$$\mathrm{sign}\sigma A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{2N\sigma(2N)} = |M_{jj}|^2 e^{(k_j+k_{j+N})x}, \quad j = 1,...,N. \tag{3.18}$$

Under the assumption (3.16) there exists the term (3.18) in the representation (2.1), i.e., the term (3.18) can not be grouped in exponential terms with any other term of (2.3). Indeed, if there exists a term of the sum (2.3) corresponding to the permutation σ_1, which can be grouped in the exponential term together with the term (3.18), then

$$\prod_{l\in A_{\sigma_1}^1} e^{k_l x} \prod_{l \in A_{\sigma_1}^2} e^{k_l x} = e^{(k_j+k_{j+N})x} \ \text{ thus } k_j + k_{j+N} = \sum_{l=1}^{2N} \mu_l k_l, \tag{3.19}$$

where $\mu_l = 0$ for $l \in A_{\sigma_1}^0$ and $\mu_l = 1$ for $l \in A_{\sigma_1}^1 \cup A_{\sigma_1}^2$.
Due to (3.16) the equality (3.19) is fulfilled only in the following case:

$$\mu_j = \mu_{j+N} = 1 \ \text{ and } \mu_l = 0 \text{ for any } l \neq j,\ j+N.$$

Hence, $A_{\sigma_1}^1 = \{j\}$ and $A_{\sigma_1}^2 = \{j+N\}$, i.e., the permutation σ_1 coincides with the permutation σ. Therefore, the term (3.18) in (2.3) can not be grouped in exponential term together with any other term of (2.3).

Using the assumption (3.16) by an argument analogous to the previous one we can show that there exist in (2.1) the term 1 and in (2.26) the terms $2\overline{p}_j(2x,t)$. $e^{k_{j+N}x}$, $j = 1, \dots, N$. Further, in the same way we can prove the following lemma.

Lemma 5. *If the assumption* (3.16) *is fulfilled, then for every* j, $1 \le j \le N$, *the coefficients of the exponential functions:* $e^{(\alpha k_j + (\alpha+1)k_{j+N})x}$, $\alpha = 0, 1, 2$, *in* (3.15) *and* (3.17) *coincide with each other.*

We are now in a position to prove the following theorem.

Theorem 1. *Let the solution of the NLS equation* (1.10) *be constructed by the transform* (1.20) *in terms of the given scattering data* (1.13), *wherein every pair of singular numbers satisfies the assumption* (3.16). *Then the normalization polynomials of* (1.13) *are reduced to the corresponding normalization factors and evolve according to the evolutionary equations:*

$$p_j = C_j e^{ik_j^2 t}, \quad -\infty < t < \infty, \quad C_j \text{ is a constant}, \; j = 1, \dots, N. \tag{3.20}$$

Proof. Let G and F in (1.20) satisfy equation (3.15), wherein the coefficients $\hat{a}_\mu(x,t)$ of exponential functions are polynomials of x. Then in view of Lemma 3:

$$\hat{a}_\mu(x,t) \equiv 0. \tag{3.21}$$

Further, by virtue of Lemma 5 the coefficients of exponential functions in (3.17) are the same (3.21), i.e., the coefficients of exponential functions: $e^{k_{j+N}x}$, $e^{(k_j+2k_{j+N})x}$ and $e^{(2k_j+3k_{j+N})x}$ of (3.17) are identically zero. Hence, from (3.17) we obtain N equations for every fixed pair of singular numbers:

$$K\left(2\overline{p}_j e^{k_{j+N}x},\, 1 + |M_{jj}|^2 e^{(k_j+k_{j+N})x}\right) = 0, \quad j = 1, \dots, N. \tag{3.22}$$

Applying Lemma 4 to every equation of (3.22) we obtain the evolutionary equations (3.20). The theorem is proved.

4. Constructing the General N-Soliton Solution

In this section we construct the general N-soliton solution of the equation (3.1) in terms of the given scattering data (3.20). For this purpose we need to calculate the coefficients $a_\mu(x,t)$ and $b_\mu(x,t)$ in the sums (2.1) and (2.26). Indeed, let the normalization polynomials evolve according to (3.20), then due to (1.18):

$$M_{lj} = -\frac{2C_l}{k_l + k_{j+N}} e^{ik_l^2 t}, \qquad \overline{M}_{lj} = -\frac{2\overline{C}_l}{k_{l+N} + k_j} e^{-ik_{l+N}^2 t}. \tag{4.1}$$

Putting

$$\Omega_j = -ik_j^2, \; C_{j+N} = \overline{C}_j, \; \Omega_{j+N} = \overline{\Omega}_j, \; e_j = 2C_j e^{k_j x - \Omega_j t}, \; e_{j+N} = \overline{e}_j,$$
$$j = 1, \dots, N, \tag{4.2}$$

we write the nonzero elements A_{lj}, $l \neq j$, of the matrix A:

$$A_{lj} = \frac{2C_l}{k_l + k_j} e^{\frac{k_l+k_j}{2}x - \Omega_l t} = \frac{e_l e^{\frac{k_j - k_l}{2}x}}{k_l + k_j} \text{ for } l \leq N,$$
$$A_{lj} = -\frac{2C_l}{k_l + k_j} e^{\frac{k_l+k_j}{2}x - \Omega_l t} = \frac{e_l e^{\frac{k_j - k_l}{2}x}}{k_l + k_j} \text{ for } l > N. \tag{4.3}$$

By virtue of Theorem 1 a nonzero term of the sum (2.1) can be written as:

$$X_\mu = D_1(\mu) a_\mu \prod_{j=1}^{2N} e^{\mu_j k_j x}, \tag{4.4}$$

where $a_\mu = a_\mu(0,t)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_{2N})$.
We can assume that the fixed μ from (4.4) defines the following sets of indices:

$$\begin{aligned}
\mathcal{A}_0 &= \{j : 1 \leq j \leq 2N, \mu_j = 0\} = \{m_1, \ldots, m_k\},\ m_1 < \ldots < m_k, \\
\mathcal{A}_1 &= \{j : j \leq N, \mu_j = 1\} = \{j_1, \ldots, j_n\},\ j_1 < \ldots < j_n \leq N, \\
\mathcal{A}_2 &= \{j : j > N, \mu_j = 1\} = \{l_1, \ldots, l_n\},\ N < l_1 < \ldots < l_n,\ k + 2n = 2N.
\end{aligned} \tag{4.5}$$

Let a permutation σ from (2.11) correspond to the considered μ. The permutation σ possesses the properties (2.4)–(2.5). The set of all permutations σ corresponding to the considered μ is denoted by J. It is clear from (2.4) and (4.5) that if $\sigma \in J$, then $\mathcal{A}_p = A^p_\sigma$, $p = 0,1,2$, where A^p_σ, $p = 0,1,2$, are defined by (2.4). Conversely, if $\mathcal{A}_p = A^p_\sigma$, $p = 0,1,2$, then the permutation σ belongs to the set J. Hence, the set J consists of all permutations σ such that $\sigma(\mathcal{A}_1) = \mathcal{A}_2, \sigma(\mathcal{A}_2) = \mathcal{A}_1$ and $\sigma(j) = j$ for $j \in \mathcal{A}_0$. Using (4.3) and the properties of the set J we can write the nonzero term (4.4) in the form:

$$X_\mu = \sum_{\sigma \in J} \operatorname{sign}\sigma A_{j_1\sigma(j_1)} \cdots A_{j_n\sigma(j_n)} A_{l_1\sigma(l_1)} \cdots A_{l_n\sigma(l_n)} = \det \mathcal{A}, \tag{4.6}$$

where the matrix $\mathcal{A}$ is written in the block form:

$$\mathcal{A} = \begin{pmatrix} [0] & [\alpha]_{n,n} \\ [\beta]_{n,n} & [0] \end{pmatrix}, \quad [0] \text{ is the } n \times n \text{ zero matrix},$$
$$[\alpha]_{n,n} = [\alpha_{j_s l_r}]^n_{s,r=1}, \qquad [\beta]_{n,n} = [\beta_{l_r j_s}]^n_{r.s=1}, \tag{4.7}$$
$$\alpha_{j_s l_r} = e_{j_s} \frac{e^{\frac{k_{l_r} - k_{j_s}}{2}x}}{k_{l_r} + k_{j_s}}, \qquad \beta_{l_r j_s} = -e_{l_r} \frac{e^{\frac{k_{j_s} - k_{l_r}}{2}x}}{k_{j_s} + k_{l_r}}.$$

Using (4.7) we calculate

$$\begin{aligned}
\det \mathcal{A} &= \prod_{m=1}^n e_{j_m} e_{l_m} \prod_{1 \leq m < m' \leq n} (k_{j_m} - k_{j_{m'}})^2 (k_{l_m} - k_{l_{m'}})^2 \prod_{m,m'=1}^n (k_{j_m} + k_{l_{m'}})^{-2} \\
&= \prod_{j=1}^{2N} e_j^{\mu_j} \prod_{1 \leq l < j \leq 2N} \Phi_{lj}^{\mu_l \mu_j},
\end{aligned} \tag{4.8}$$

where

$$\Phi_{lj} = \Phi_{jl} = \begin{cases} (k_l + k_j)^{-2} & \text{for } l \le N < j \text{ or } j \le N < l, \\ (k_l - k_j)^2 & \text{for } l, j \le N \text{ or } l, j > N. \end{cases} \tag{4.9}$$

Hence, from (2.1), (2.2), (4.6) and (4.8) we obtain the explicit representation of F:

$$F(x,t) = \det A = \sum_{\mu_j=0,1} D_1(\mu) \Big\{ \prod_{j=1}^{2N} e_j^{\mu_j} \prod_{1 \le l < j \le 2N} \Phi_{lj}^{\mu_l \mu_j} \Big\}. \tag{4.10}$$

Further, we find the explicit representation of G. In view of Theorem 1 a nonzero term of the sum (2.26) can be written as:

$$Y_\mu = D_2(\mu) b_\mu \prod_{l=1}^{2N} e^{\mu_l k_l x}, \quad b_\mu = b_\mu(0,t). \tag{4.11}$$

According to the calculations (2.24)–(2.26) the term (4.11) can be represented in the form:

$$Y_\mu = \sum_{j=1}^{N} Y_{j+N}, \tag{4.12}$$

where $Y_{j+N} = D_2(\mu) b_{1\mu} \prod_{l=1}^{2N} e^{\mu_l k_l x}, \quad b_{1\mu} = b_{1\mu}(0,t).$

We can assume that the fixed μ from (4.11) defines the following sets of indices:

$$\begin{aligned} &\mathcal{B}_0 = \{j : 1 \le j \le 2N,\ \mu_j = 0\} = \{m_1, \ldots, m_k\},\ m_1 < \ldots < m_k, \\ &\mathcal{B}_1 = \{j : j \le N,\ \mu_j = 1\} = \{j_1, \ldots, j_n\},\ j_1 < \ldots < j_n \le N, \\ &\mathcal{B}_2 = \{j : j > N,\ \mu_j = 1\} = \{l_1, \ldots, l_{n+1}\},\ N < l_1 < \ldots < l_{n+1}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad k + 2n + 1 = 2N. \end{aligned} \tag{4.13}$$

Let the permutations σ from (2.20) and (2.22) correspond to the considered $\mu = (\mu_1, \ldots, \mu_{2N})$. Let $\mu_{j+N} = 1$, then there are two cases:

Case 1: $\sigma(j+N) = j+N$. In this case

$$B_\sigma^0 = \mathcal{B}_0 \cup \{j+N\}, \quad B_\sigma^1 = \mathcal{B}_1, \quad B_\sigma^2 = \mathcal{B}_2 \setminus \{j+N\}, \tag{4.14}$$

where the sets B_σ^p, $p = 0,1,2$, are determined in (2.16).

Cases 2: $\sigma(j+N) \ne j+N$, then

$$B_\sigma^0 = \mathcal{B}_0, \quad B_\sigma^1 = \mathcal{B}_1, \quad B_\sigma^2 = \mathcal{B}_2. \tag{4.15}$$

We denote the set of all permutations σ corresponding to the considered μ by J^{j+N}. Hence, if $\sigma \in J^{j+N}$ then the condition (4.14) or (4.15) is fulfilled. Conbversely, if the condition (4.14) or (4.15) is fulfilled, then $\sigma \in J^{j+N}$. Hence, the set J^{j+N} consists of all permutations σ such that $\sigma(l) = l$ for $l \in \mathcal{B}_0$ and $\sigma(\mathcal{B}_1) = B_\sigma^2 = \mathcal{B}_2 \setminus \{j+N\}$.

Now we calculate the term Y_{j+N}. If $\mu_{j+N} = 0$, then it follows from (2.24), (2.25) that $Y_{j+N} = 0$. If $\mu_{j+N} = 1$, then due to (4.13)–(4.15): $j + N = l_s \in \mathcal{B}_2, s = 1, \ldots, n+1$ and the nonzero term Y_{j+N} is

$$Y_{j+N} = \sum_{\sigma \in J^{j+N}} \operatorname{sign}\sigma B_{j_1\sigma(j_1)} \dots B_{j_n\sigma(j_n)} B_{l_1\sigma(l_1)} \dots B_{l_{n+1}\sigma(l_{n+1})}$$
$$= \det \mathcal{D}^{(j+N)}, \tag{4.16}$$

where B_{lj} the elements of the matrix $B^{(j+N)}$ and the matrix $\mathcal{D}^{(j+N)}$ is obtained from the matrix:

$$\begin{pmatrix} [0]_{n,n} & [\alpha]_{n,n+1} \\ [\beta_{n+1,n} & [0]_{n+1,n+1} \end{pmatrix},$$

substituting the elements in its $(n+s)$-th column by

$$\left(0, \dots, 0, -\frac{1}{2} e_{l_1} e^{\frac{k_{l_s} - k_{l_1}}{2} x}, \dots, -\frac{1}{2} e_{l_{n+1}} e^{\frac{k_{l_s} - k_{l_{n+1}}}{2} x}\right)^T.$$

Here $[\alpha]_{n,n+1}$, $[\beta]_{n+1,n}$ are matrices of the sizes: $n \times (n+1)$, $(n+1) \times n$ respectively and the elements of $[\alpha]_{n,n+1}$ and $[\beta]_{n+1,n}$ are defined by (4.7).
By virtue of the representation (4.16) the term (4.12) is written as

$$Y_\mu = \sum_{j+N \in \mathcal{B}_2} \det \mathcal{D}^{(j+N)} = \det \mathcal{D},$$

where $\mathcal{D} = \begin{pmatrix} [0]_{n,n} & [\alpha]_{n,n+1} \\ [\beta]_{n+1,n} & [\gamma]_{n+1,n+1} \end{pmatrix}$ $\quad [\gamma]_{n+1,n+1} = \left[-\frac{1}{2} e_{l_r} e^{\frac{k_{l_j} - k_{l_r}}{2} x} \right]_{r,j=1}^{n+1}.$

Then using (4.9) we calculate

$$Y_\mu = -\frac{1}{2} \prod_{m=1}^{n} e_{j_m} \prod_{m=1}^{n+1} e_{l_m} \prod_{1 \le m < m' \le n} (k_{j_m} - k_{j_{m'}})^2$$
$$\times \prod_{1 \le m < m' \le n+1} (k_{l_m} - k_{l_{m'}})^2 \prod_{\substack{1 \le m \le n \\ 1 \le m' \le n+1}} (k_{j_m} + k_{l_{m'}})^{-2}$$
$$= -\frac{1}{2} \prod_{j=1}^{2N} e_j^{\mu_j} \prod_{1 \le l < j \le 2N} \Phi_{lj}^{\mu_l \mu_j}, \tag{4.17}$$

Due to (4.17) the formula (2.26) is represented in the explicit form:

$$G(x,t) = \sum_{\mu_j = 0,1} D_2(\mu) \left\{ \prod_{j=1}^{2N} e_j^{\mu_j} \prod_{1 \le l < j \le 2N} \Phi_{lj}^{\mu_l \mu_j} \right\}. \tag{4.18}$$

The representations of F and G by the formulas (4.10) and (4.18) are similar to the ones in the works [1,3]. We can prove that the constructed functions F and G satisfy the following Hirota equations [1]:

$$(iD_t - D_x^2) G \circ F = 0, \tag{4.19}$$
$$D_x^2 F \circ F = 2G\overline{G}. \tag{4.20}$$

Then F and G satisfies the NLS equation (3.1). The results obtained in this section can be stated as a converse assertion to Theorem 1. Namely

Theorem 2. *Let p_j, $j = 1, \dots, N$, from the set of scattering data* (1.13) *be the normalization factors* (3.20)*, then the functions F and G in the transform* (1.20) *are represented in terms of the given scattering data* (3.20) *in the explicit forms* (4.10) *and* (4.18) *respectively, and the transform* (1.20) *is the N-soliton solution of the NLS equation* (3.1).

References

1. M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform SIAM*, Studies in Applied Mathematics, Philadelphia, 1981.
2. W. Eckhaus and A. van Harten, *The inverse scattering Transformations and the Theory of Solitons. An Introduction*, Mathematics, Studies 50, Amsterdam: North-Holland, 1981.
3. R. Hirota, Direct methods in soliton theory, *Topics in Current Physics* **17**, R. Bullough and P. Caudrey eds., Spinger-Verlag, New York, 1980, 157–175.
4. M. Lakshmanan, Analytic methods for soliton systems, *J. of Bifurcation and Chaos* **3** (1993) 3–17.
5. L. P. Nizhnik and Pham Loi Vu, The inverse scattering problem on a half-line with a non-selfconjugate potential matrix, *Ukrainskii Math. J.* **26** (1974) 469–486 (Russian) (Tran. 1975, New York: Plenum).
6. Pham Loi Vu, Cauchy problems for a system of nonlinear equations and for the nonlinear Schrödinger equation, *J. Inverse Problems* **10** (1994) 415–429.
7. Pham Loi Vu, The inverse scattering problem for a system of Dirac equations on a half-line, *Linear Boundary Problems of Mathematical Physics*, Izdat. Inst. Matem. Acad. Nauk USSR, 1973 (Rusian).