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Survey

Option Pricing in Mathematical Financial Market with Jumps and Related Problems*

Situ Rong
Department of Mathematics, Zhongshan University
Guangzhou 510275, China

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Abstract. Option pricing is an important and interesting problem in the financial market. In this paper we will discuss how to attack this problem by probabilistic method and partial differential equation technique. Moreover, our market can be disturbed by jumps and it can be non-linear and non-Lipschitzian. The related problems are also discussed.

1. Option Pricing and BSDE

In the financial market one kind of the so-called "option" actually is a contract, which provides a right to the contract owner to do the following thing:

"At the given future time T (say, = Aug. 2002) by that contract one can buy a unit of some fixed thing (say, 1 ton of corn) at some place (say, some farm) with a fixed price K (say, = \$600)."

Suppose that the price of one unit of the fixed thing at time T is P_T. Then at the future time T the option will help its owner to earn money $x_T = (P_T - K)^+$. Therefore, if somebody wants to own this option now he has to pay for it. That is to say at the time t the option should have a price x_t. This raises an important and interesting problem: how can we price the option at the time t as x_t, which should be reasonable and fair to both sides: the seller and the buyer? To solve this problem we can imagine that if we put the money x_t in the market as follows: one part π_t^0 of it is deposited in the bank (with no risk), and another part π_t^1 of it

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is used to buy that fixed thing, or say, to buy the stock of it (whose price changes by some stochastic perturbation, so it is with risk). Then as time t evolves to T, if the option price x_t is a fair price, it should arrive at $x_T = (P_T - K)^+$ by the time T. So actually, we should require that

$$\begin{cases} x_t = \pi_t^0 + \pi_t^1, t \in [0,T), \\ x_T = (P_T - K)^+. \end{cases}$$

However, to deposit money in the bank can be seen as to buy some units η_t^0 of bond, and the bond's price is

$$\begin{cases} dP_t^0 = P_t^0 r_t dt, t \in [0,T], \\ P_0^0 = 1; \end{cases}$$

so $\pi_t^0 = \eta_t^0 P_t^0$. Similarly, we can assume that the price of the stock of that thing is

$$\text{(FSDE1)} \begin{cases} dP_t = P_t(b_t dt + \sigma_t dw_t + \rho_t d\widetilde{N}_t), t \in [0,T], \\ P_0 = P_0; \end{cases}$$

and $\pi_t^1 = \eta_t^1 P_t$, where w_t is an 1-dimensional standard Brownian motion (BM) and $\widetilde{N}_t$ is a centralized Poisson process such that $\widetilde{N}_t = N_t - \lambda_t t$, where N_t is a Poisson process. Here the stock price is disturbed by some continuous and jump stochastic perturbation. So $x_t = \pi_t^0 + \pi_t^1 = \eta_t^0 P_t^0 + \eta_t^1 P_t$. We call (π_t^0, π_t^1) a portfolio. Suppose now the portfolio is self-finance, i.e. in a short time dt we do not invest or withdraw money, in or from the market. We only let the money x_t change in the market while the market itself is operating, i.e. self-finance produces

$$dx_t = \eta_t^0 dP_t^0 + \eta_t^1 dP_t^1.$$

Then after simple evaluation we arrive at the following backward stochastic differential equation (BSDE), where the price x_t of option and the portfolio π_t^1 (actually it is the risk part of the portfolio) should satisfy:

$$\text{(BSDE1)} \begin{cases} dx_t = r_t x_t dt + \pi_t^1(b_t - r_t)dt + \pi_t^1 \sigma_t dw_t + \pi_t^1 \rho_t d\widetilde{N}_t \\ x_T = (P_T - K)^+, t \in [0,T]. \end{cases}$$

So solving the option pricing problem will lead to find the solution (x_t, π_t^1) of the above BSDE.

Now let us work a little bit more to give out a pricing formula for the option. By the bond price equation one easily sees that $P_t^0 = e^{\int_0^t r_s ds}$. So $(P_t^0)^{-1} = e^{-\int_0^t r_s ds}$ can be seen as a discount factor contrasted to the bond price - a factor, which discounts anything from time t to time 0 (the initial time) contrasted to the bond price. By Ito's formula one easily sees that

$$\begin{cases} d(e^{-\int_0^t r_s ds} x_t) = e^{-\int_0^t r_s ds}[\pi_t^1(b_t - r_t)dt + \pi_t^1 \sigma_t dw_t + \pi_t^1 \rho_t d\widetilde{N}_t], \\ e^{-\int_0^T r_s ds} x_T = e^{-\int_0^T r_s ds}(P_T - K)^+, \ t \in [0,T]. \end{cases}$$

It means that the option price after discount will satisfy the above equation. This initiates us to simplify the above equation by Girsanov's transformation then to derive an option pricing formula. Let us make the following hypothesis:

(H1) *b_t, r_t and σ_t are all bounded, $(\sigma_t)^{-1}$ exists and is also bounded.*
Under assumption (H1) by Girsanov's theorem [29, Theorem 3.10 and Corollary 3.11] one knows that

$$\widetilde{w}_t = w_t + \int_0^t (\sigma_s)^{-1}(b_s - r_s)ds = w_t + \int_0^t \theta_s ds \quad (1)$$

is a new BM under a new probability measure $\widetilde{P}$, where

$$d\widetilde{P} = \exp[-\int_0^T \theta_s dw_s - \frac{1}{2}\int_0^T |\theta_s|^2 ds]dP; \quad (2)$$

and $\widetilde{N}_t = N_t - \lambda_t t$ is still a centralized Poisson process under the new probability measure $\widetilde{P}$.
Hence

$$\begin{cases} d(e^{-\int_0^t r_s ds} x_t) = e^{-\int_0^t r_s ds}(\pi_t^1 \sigma_t d\widetilde{w}_t + \pi_t^1 \rho_t d\widetilde{N}_t), \quad \widetilde{P}-a.s. \\ e^{-\int_0^T r_s ds} x_T = e^{-\int_0^T r_s ds}(P_T - K)^+, t \in [0,T]. \end{cases}$$

Therefore we have

$$e^{-\int_0^t r_s ds} x_t = E^{\widetilde{P}}[e^{-\int_0^T r_s ds}(P_T - K)^+ | \Im_t],$$

$$\text{i.e.} \qquad x_t = E^{\widetilde{P}}[e^{-\int_t^T r_s ds}(P_T - K)^+ | \Im_t].$$

This option pricing formula intuitively tells us that to get the option price by the time t one only needs to discount its future price at future time T to the time t and take its mean under some martingale measure $\widetilde{P}$. By this we can also derive the famous Black–Scholes formula for the price of option as follows:

Assume that $\lambda_t \equiv 0$, i.e. no jumps, and assume that $r = r_t, \sigma = \sigma_t$ are all constants. In this case the price of stock has the solution as: $P_t = P_0 \exp[\sigma \widetilde{w}_t - ((1/2)\sigma^2 - r)t], t \in [0,T]$. Hence

$$P_T = P_t \exp[\sigma(\widetilde{w}_T - \widetilde{w}_t) - \left(\frac{1}{2}\sigma^2 - r\right)(T-t)], t \in [0,T].$$

Note that $\widetilde{w}_T - \widetilde{w}_t$ given $\Im_t$ is normal distributed with mean 0 and variance $\sqrt{T-t}$ under probability measure $\widetilde{P}$. Hence it is not difficult to evaluate and obtain that [4]

$$\begin{aligned} x_t &= E^{\widetilde{P}}[e^{-\int_t^T r_s ds}(P_T - K)^+ | \Im_t] \\ &= P_t \Phi(\rho_+(T-t, P_t)) - Ke^{-r(T-t)}\Phi(\rho_-(T-t, P_t)), \text{ as } 0 \le t < T, y \ge 0, \end{aligned}$$

where

$$\rho_\pm(t,y) = \frac{1}{\sigma\sqrt{t}}[\log\frac{y}{K} + t(r \pm \frac{\sigma^2}{2})], \ \Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^y e^{-z^2/2}dz.$$

That is the so-called Black–Scholes formula.

The above idea can be easily generalized to price the contingent claim for a market with many different kinds of stocks.

Now suppose that there are N different kinds of stocks in the financial market. Their price processes satisfy the following SDEs: $i = 1, 2, \dots, N$,

$$\text{(FSDE1)} \begin{cases} dP_t^i = P_t^i(b_t^i dt + \sigma_t^i dw_t + \rho_t^i d\widetilde{N}_t), t \in [0,T], \\ P_0^i = P_0^i; \end{cases}$$

where w_t is a d-dimensional BM, $\sigma_t^i = (\sigma_t^{i1}, \dots, \sigma_t^{id})$. If a small investor by the time t buys η_t^i units of the ith security, $i = 0, 1, \dots, N$, then his investment strategy is $(\eta_t^0, \eta_t^1, \dots, \eta_t^N)$ and the portfolio of his investment is $(\pi_t^0, \pi_t^1, \dots, \pi_t^N)$, where π_t^0 is the investment for the bond. In this case his wealth process in the market is

$$x_t = \sum_{i=0}^{N} \pi_t^i = \sum_{i=0}^{N} \eta_t^i P_t^i.$$

Suppose that this small investor wants his wealth arrives at $X \geq 0$ by the future time T, i.e. $x_T = X$. (It is natural to require that X is $\Im_T$-measurable. In the financial market X is also called a contingent claim). Then how should he invest and take a right portfolio now? As above we still assume that the portfolio is self-finance, i.e.

$$dx_t = \sum_{i=0}^{N} \eta_t^i dP_t^i.$$

After substituting the equations for P_t^i, $i = 0, 1, \dots, N$, one obtains that the investor's wealth process x_t and the portfolio process (actually it is the risk part of portfolio) π_t should satisfy the following BSDE:

$$\text{(BSDE2)} \begin{cases} dx_t = r_t x_t dt + \pi_t(b_t - r_t \underline{1})dt + \pi_t \sigma_t dw_t + \pi_t \rho_t d\widetilde{N}_t \\ x_T = X, t \in [0,T], \end{cases}$$

where $b_t^T = (b_t^1, \dots, b_t^N)$, $\pi_t = (\pi_t^1, \dots, \pi_t^N)$, $\underline{1}^T = (1, \dots, 1)$, and b_t^T is the transpose of b_t, etc. So the existence of solution to (BSDE2) will directly correspond to a right investment and right portfolio chosen by the investor for arriving his future target, or say, x_t will correspond to a right price at time t for the contingent claim. Moreover, under assumption (H1) and under the same calculation as above one has that the right price of the contingent claim X is $x_t = E^{\widetilde{P}}[e^{-\int_t^T r_s ds} X | \Im_t]$.

2. Non-Linear and Non-Lipschitzian Wealth Process and General BSDE

Suppose that the investor from time 0 up to time t also consumes money from the market. The cumulative consumption process may be assumed to be $C_t = \int_0^t c_s(\omega)ds$,where $c_s(\omega) \geq 0$. In the general case we may also assume that the consumption rate will depend on the wealth, i.e. $c_s = c(s, x_s)$; or, more general, depend on the wealth and portfolio both, i.e. $c_s = c(s, x_s, \pi_s)$. Moreover, suppose that the investor sometimes also borrows money from the bank. For this he has to pay back money with some rate. The money that he should pay back from time 0 up to time t may be assumed to be $\int_0^t (R_s - r_s)\left(x_s - \sum_{i=1}^N \pi_s^i\right)^- ds$. Under such assumption the investor's wealth process will satisfy the following BSDE:

$$\text{(BSDE3)} \begin{cases} dx_t = r_t x_t dt + \pi_t(b_t - r_t \underline{1})dt - (c(t,x_t,\pi_t) \\ + (R_t - r_t)\left(x_t - \sum_{i=1}^N \pi_t^i\right)^-)dt + \pi_t \sigma_t dw_t + \pi_t \rho_t d\widetilde{N}_t,\ t \in [0,T], \\ x_T = X, \end{cases}$$

where w_t is a d-dimensional BM, $\widetilde{N}_t$ is a centralized Poisson process such that $\widetilde{N}_t = N_t - \lambda_t t$. Obviously, this is a non-linear equation for x_t, and it also can be non-Lipschitzian for x_t, if the consumption rate $c(t,x_t,\pi_t)$ is so. This will happen when we discuss the optimal consumption problem in the financial market. The optimal consumption rate will be non-Lipschitzian in some cases [27, 34].

Therefore, (BSDE3) will lead us to discuss the general BSDE with non-Lipschitzian coefficients even in the n-dimensional case. Let us briefly mention the definition of a solution to BSDE. Consider the following BSDE:

$$\text{(BSDE4)}\ x_t = X + \int_t^T b(s,x_s,q_s,p_s,\omega)ds - \int_t^T q_s dw_s - \int_t^T p_s d\widetilde{N}_s,\ t \in [0,T],$$

where w_s and $\widetilde{N}_s$ are given in (BSDE3), $X \in R^n$ and it is $\Im_T = \Im_T^{w,\widetilde{N}}$-measurable. We say that (x_t,q_t,p_t) is a solution of (BSDE4), if it is $\Im_t$-adapted, where p_t is $\Im_t$-predictable, and such that

$$E\int_0^T (|x_t|^2 + |q_t|^2 + |p_t|^2 \lambda_t)dt < \infty,$$

and it satisfies (BSDE4). We can get a unique solution of (BSDE4) under condition that b is non-Lipschitzian continuous[19] and condition that b is even discontinuous when x_t is R^1-valued [27, 34]. However, from solutions of (BSDE4) returning to solutions of (BSDE3) we still need some assumption:

(H2) R_t, ρ_t and $c(t,x)$ are all bounded such that ρ_t^{-1} exists and is also bounded, and the consumption rate $c(t,x)$ does not depend on π, and $c(t,x)$ is continuous in x; moreover, $d = N-1$.

Theorem 1. *Under assumption* (H1)–(H2), (BSDE3) *has a solution* (x_t,π_t), *where* x_t *is* $\Im_t$*-adapted,* π_t *is* $\Im_t$*−predictable and* $E\int_0^T (|x_t|^2 + \sum_{i=1}^{N-1} |\pi_t^i|^2 + |\pi_t^N|^2 \lambda_t)dt < \infty$. *That is to say, under assumption* (H1)–(H2) *if for a small investor his wealth process satisfies* (BSDE3), *then his wealth still can arrive at his target* X *by the future time* T, *provided that at each time* t *he invests the money* x_t *and take the right portfolio* π_t, *where* (x_t,π_t) *is the solution of* (BSDE3).

Theorem 1 can be derived by [27, 34]. Indeed, for $\pi = (\pi^1,\ldots,\pi^{N-1},\pi^N)$ denote $\pi^{(N-1)} = (\pi^1,\ldots,\pi^{N-1})$, and let $(\pi^{(N-1)})^T = (\sigma^T)^{-1}q^T, \pi^N = \rho^{-1}p$. Then (BSDE3) becomes an equation as (BSDE4). By [27, 34] (BSDE4) has a solution (x_t,q_t,p_t). Hence (BSDE3) has a solution (x_t,π_t).

In the financial market if for any contingent claim X (BSDE3) has a solution (x_t,π_t), then it is called a complete financial market. From Theorem 1 and its proof it is seen that a financial market with jumps in general is not a complete market. Usually even if (BSDE4) has a solution (x_t, q_t, p_t), (BSDE3) still can have no solution (x_t, π_t). However, if the market is without jumps, then from that (BSDE4) has a solution (x_t, q_t) to derive that (BSDE3) has a solution (x_t, π_t) will be much more easier, and usually it is true if $N = d$, R_t is bounded and (H1) holds.

The study of (BSDE3) from a BSDE point of view has many advantages. For example, from the comparison theorem of solutions to BSDE it is easily derived that if the small investor wants his wealth to arrive at a bigger target X by the future time T, then he must invest much now, i.e. x_t should be bigger [27, 34]. Or if he wants to consume much from now to T, i.e. to take a bigger $c(t, x, \pi)$, and if $c(t, x) = c(t, x, \pi)$ does not depend on π, or $|c(t, x, \pi)| \leq 1$, then he also should invest much now [27, 34].

3. Option Pricing by PDE Technique. A New Feynman–Kac Formula

By using Girsanov's transformation as above we can rewrite (BSDE1) as follows:

$$\text{(BSDE1)'} \begin{cases} dx_t = r_t x_t dt + \pi_t^1 \sigma_t d\widetilde{w}_t + \pi_t^1 \rho_t d\widetilde{N}_t, \quad \widetilde{P} - a.s. \\ x_T = (P_T - K)^+, t \in [0, T]. \end{cases}$$

and rewrite (FSDE1) to be

$$\text{(FSDE1)'} \begin{cases} dP_t = P_t(r_t dt + \sigma_t d\widetilde{w}_t + \rho_t d\widetilde{N}_t), t \in [0, T], \ \widetilde{P} - a.s. \\ P_0 = P_0; \end{cases}$$

Now if we introduce a function $u(t, y) \in C^{1,2}$, and we require that $u(t, P_t) = x_t$, then by using Ito's formula to $u(t, P_t)$ on $t \in [0, T]$ with (FSDE1)' one easily derives that if $u(t, y)$ satisfies the following partial differential equation (PDE):

$$\text{(PDE1)} \begin{cases} -\frac{\partial u}{\partial t} + r_t u = \frac{1}{2}\sigma_t^2 y^2 \frac{\partial^2 u}{\partial y^2} + r_t y \frac{\partial u}{\partial y} \\ +\lambda_t [u(t, y + y\rho_t) - u(t, y) - \rho_t y \frac{\partial u(t,y)}{\partial y}], \\ u(T, y) = (y - K)^+, t \in [0, T], \end{cases}$$

then $u(t, P_t) = x_t$ holds true. This tells us that the option pricing problem can be solved by the PDE technique. In the following let us derive the famous Black–Scholes' formula for the option pricing again from the PDE point of view. Assume that $\lambda_t \equiv 0$, i.e. there are no jumps, and assume that $r = r_t, \sigma = \sigma_t$ are all constants. In this case it is easily checked that[10]

$$u(t, y) = \begin{cases} y\Phi(\rho_+(T - t, y)) - Ke^{-r(T-t)}\Phi(\rho_-(T - t, y)), & 0 \leq t < T, \ y \geq 0, \\ (y - K)^+, & t = T, \ y \geq 0, \end{cases}$$

with

$$\rho_\pm(t, y) = \frac{1}{\sigma\sqrt{t}}[\log \frac{y}{K} + t(r \pm \frac{\sigma^2}{2})], \ \Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-z^2/2} dz,$$

satisfies (PDE1). So $x_t = u(t, P_t)$ can be concretely evaluated. That is the so-called Black–Scholes' formula again.

In (FSDE1)' if we let the initial condition be $P_t = y$, and consider the FSDE for $s \in [t, T]$, then we get the solution of (PDE1) as $u(t, y) = x_t$. This means that the solution of PDE can be expressed by the solution of BSDE. This is so-called a new Feynman–Kac formula. Let us generalize such idea to the general PDE and BSDE with jumps.

Suppose that $D \subset R^d$ is a bounded open region, ∂D is its boundary, denote $D^c = R^d - D$. Consider the following PDE

$$\text{(PDE2)'} \begin{cases} \mathcal{L}_{b,\sigma,c}u(t,x) \equiv \frac{\partial}{\partial t}u(t,x) + \sum_{i=1}^d b_i(t,x)\frac{\partial}{\partial x_i}u(t,x) \\ +\frac{1}{2}\sum_{i,j=1}^d a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}u(t,x) + (u(t,x+\rho(t,x)) - u(t,x) \\ -\sum_{i=1}^d \rho_i(t,x)\frac{\partial}{\partial x_i}u(t,x)) \\ = f(t,x,u(t,x),u'_x(t,x)\cdot\sigma(t,x),u(t,x+\rho(t,x)) - u(t,x)), \\ u(T,x) = \phi(x), u(t,x)\,|_{D^c} = \psi(t,x), \psi(T,x) = \phi(x)\,|_{D^c}, \end{cases}$$

where $a = \sigma\sigma^T$. To derive a new Feynman-Kac formula for the solution of the above PDE let us introduce a forward SDE with Poisson jumps in R^d for any given $(t, x) \in [0, T] \times D$ as follows: as $t \le s \le T$,

$$\text{(FSDE2)'} \quad y_s = x + \int_t^s b(r, y_r)dr + \int_t^s \sigma(r, y_r)dw_r + \int_t^s \rho(r, y_{r-})d\tilde{N}_r.$$

It is known that under rather weak condition (FSDE2)' will have a unique solution $y_s, t \le s \le T$. [16] Consider now a BSDE with jumps as follows:

$$\text{(BDE2)'} \quad x_s = I_{\tau<T}\psi(\tau, y_\tau) + I_{\tau=T}\phi(y_T) - \int_{s\wedge\tau}^T f(r, y_r, x_r, q_r, p_r)dr - \int_{s\wedge\tau}^T q_r dw_r - \int_{s\wedge\tau}^T \rho_r d\tilde{N}_r, \text{ as } t \le s \le T,$$

where $\tau = \tau_{t,x} = \inf\{\, s > t : y_s^{t,x} \notin D \,\}$, and $\tau = \tau_{t,x} = T$, for $\inf\{\emptyset\}$. Then applying Ito's formula one can have a new Feynman-Kac formula: $u(t, x) = x_t$.[19] From this new Feynman-Kac formula we can even derive the existence of solution to (PDE2)' under weaker conditions as non-Lipschitzian conditions on f [19, 27, 34].

4. Other Related Problems

The above three simplest basic equations (FSDE1), (BSDE1), and (PDE1) actually motivate us a lot.

Many other important and interesting problems also can be discussed. For example, to solve (FSDE1) and (BSDE1) we can solve the first one then solve the second one, because P_t is not influenced by x_t. However, if x_t is the wealth process of a very big company, then its wealth process will influence its stock and other stock prices. Then we have to solve the F-B SDE with jumps together. That becomes a difficult and interesting topic. However, we can solve it even under some non-Lipschitzian condition [22, 26, 34]. Another problem is that we are concerned with some important wealth, say, it is run by the local government and it is very crucial. So, the local government does not allow it to be lower than some level. When such situation will occur, the local government will put some money in immediately to keep it still over that level. For such kind of wealth process we will meet the reflecting BSDE (RBSDE) with jumps. We still can solve it under some non-Lipschitzian conditions [23, 34].

Furthermore, the optimal consumption problem, the optimal recursive utility problem for the financial market with jumps and so on, can also be treated and solved by the BSDE with jumps point of view [24-27, 30, 34].

Finally, let us point out that under a mild modification the discussion on the relation between (FSDE2)', (BSDE2)' and (PDE2)' in Sec. 3 can be applied to treat some optimal control problems for some deterministic distributed parameter systems. For this we only need to assume that the coefficients b, σ, ρ in (PDE2)' all depend on some control $v \in V$, where $V \subset R^m$ is a compact set and it is called an admissible control set, i.e. $b = b(t, x, v)$, etc. Then the solution $u^v(t, x)$ of (PDE2)' in this case depends on v. Suppose now we want to maximize it, i.e., we want to find out the $\sup_{v \in V} u^v(t, x)$. By the approach in Sec. 3 one sees that $\sup_{v \in V} u^v(t, x) = \sup_{v(.) \in Œ} x_s^{t,x,v}|_{s=t}$, where $x_s^{t,x,v}$ is the solution of (BSDE2)', and in (FSDE2)' y_s is the solution of (FSDE2)' with coefficients $b(t, y, v)$, etc. and Œ $= \{v = v(t, \omega)$: it is $\Im_t$-predictable U-valued such that $E \int_0^T |v(t, \omega)|^2 \, dt < \infty\}$. Under mild conditions we can show that actually, $\overline{u(t, x)} = \sup_{v(.) \in Œ} x_s^{t,x,v}|_{s=t}$ (in the financial market we can also call it a generalized optimal recursive utility function) will be a viscosity solution of the following so-called Hamilton–Jacobi–Bellman equation: [25, 34].

$$\partial_t \overline{u(t, x)} + H(t, x, \overline{u(t, x)}, \overline{Du(t, x)}, \overline{D^2 u(t, x)}, \overline{u(t, \cdot)}) = 0,$$

where

$$\begin{aligned} H(t, x, r, \beta, A, \eta) = \sup_{v \in V} \{ & f(t, x, r, \sigma^T(t, x, v)\beta, \eta(x + \rho(t, x, v)) - \eta(x), v) \\ & + \langle \beta, b(t, x, v) \rangle + \frac{1}{2} Tr\, (\sigma(t, x, v)\sigma^T(t, x, v)A) \\ & + \eta(x + \rho(t, x, v)) - \eta(x) - \eta'(x)\rho(t, x, v) \}. \end{aligned}$$

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