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Short Communication

Methods for Finding Global Optimal Solutions to Linear Programs with Equilibrium Constraints*

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1. Introduction

Suppose that $f : \mathbb{R}^{n+m} \to \mathbb{R}$. $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ are given functions, $D \subseteq \mathbb{R}^{n+m}$ is a nonempty closed set, and $C : \mathbb{R}^n \to \mathbb{R}^m$ is a set-valued map with closed convex values, i.e, for each $x \in \mathbb{R}^n$, $C(x)$ is a (possibly empty) closed convex subset of $\mathbb{R}^m$. Consider the following mathematical program with equilibrium constraints, shortly MPEC

$$\min\{f(x,y) : \ (x,y) \in D\}, \tag{P}$$

where y solves the parametric variational inequality

$$\text{find} \quad y \in C(x) : \ (v-y)^T F(x,y) \geq 0, \ \text{for all } v \in C(x) \tag{VI(x)}$$

Material on the MPEC problem and its applications can be found in [5]. In what follows we assume that (P) is linear, i.e., D, $C(x)$ are polyhedral convex sets for every x, and f, F are affine functions.

The MPEC problem, even in linear case, is known to be a difficult multi extremal problem because of its nested structure. Few numerical methods have been proposed for solving the MPEC problem (P) (see e.g. [2-8]). All of the existing methods can compute only a local solution or a stationary point. To our knowledge, up to date no method has been developed for solving Problem (P) globally.

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In this short communication we present two methods for globally solving Problem (P). We reformulate the problem as an ordinary linear program with an additional complementarity constraint. For globally solving the latter problem we propose two branch-and-bound algorithms. The first algorithm uses a simplicial subdivision accompanied with a decoupling technique for bounding. The second one uses a binary tree defined according to the sign (zero or positive) of the dual variables appearing in the complementarity condition. The branching in the both algorithms takes place in the space of the dual variables whose dimension is just equal to the number of the constraints of the inner variational inequality. Preliminary computational experiences and results show that the algorithm using binary tree is more efficient than the simplicial subdivision algorithm. A lot of randomly generated problems up to twenty five - dual variables are solved by the binary tree algorithm on a PC Pentium II computer.

2. Preliminaries

Throughout the paper we suppose that

$$C(x) \equiv \{y \in R^m : g(x,y) := Ax + By + b \geq 0\}, \tag{2}$$

$$F(x,y) \equiv Px + Qy + q \tag{3}$$

where $b \in R^l$, $q \in R^m$ and A, B, P, Q are appropriate given matrices. By applying the Kuhn–Tucker theorem for the linear variational inequality (VI(x)) we can see that LMPEC problem is equivalent to the problem

$$f_* := \min f(x,y) \tag{CP}$$

subject to

$$(x,y) \in D, \tag{4}$$

$$Px + Qy + q - B^T\lambda = 0, \tag{5}$$

$$Ax + By + b \geq 0, \tag{6}$$

$$\lambda \geq 0, \lambda^T(Ax + By + b) = 0 \tag{7}$$

in the sense that if the pair (x,y) is a global minimizer of (P) then for any λ satisfying (5) and (7) the triple (λ, x, y) is a global minimizer of (CP); conversely, if the triple (λ, x, y) is a global minimizer of (CP), then the pair (x,y) is a global minimizer of (P). As usual we shall call λ dual variables and (x,y) primal variables.

We note that when $\lambda = 0$, Problem (CP) becomes a linear program. Thus we focus on the difficult case when $\lambda \neq 0$. In this case Problem (CP) takes the form

$$f_1 := \min f(x,y) \tag{CP1}$$

subject to

$$(x, y) \in D,\ Px + Qy + q - B^T\lambda = 0,$$

$$\lambda \geq 0,\ \lambda \neq 0,\ Ax + By + b \geq 0,\ \lambda^T(Ax + By + b) = 0.$$

3. Relaxation Bounding and Simplicial Subdivision

Let S_1 be the $\ell - 1$-simplex whose vertices are the unit vectors $e^1, \dots, e^\ell$ of R^ℓ. Let S be a fully dimensional subsimplex of S_1, and C_S be the polyhedral cone vertexed at the origin whose extreme edges are halflines passing the vertices of S. Consider Problem (CP1) restricted to this cone. That is

$$f(S) := \min f(x, y) \qquad \text{(CPS)}$$

subject to

$$(x, y) \in D,\ Ax + By + b \geq 0,$$

$$Px + Qy + q - B^T\lambda = 0,$$

$$\lambda^T(Ax + By + b) = 0,\ \lambda \in C_S,\ \lambda \neq 0$$

Clearly if $S = S_1$ Problems (CPS) and (CP1) coincide.

Corresponding to (CPS) we consider the relaxed problem

$$\beta(S) := \min f(x, y) \qquad \text{(RCPS)}$$

subject to

$$(x, y) \in D, Ax + By + b \geq 0,$$

$$Px + Qy + q - B^T z = 0,\ z \in C_S,$$

$$\lambda^T(Ax + By + b) = 0,\ \lambda \in S,\ .$$

This bounding has the following properties.

Proposition 1. (i) *If (λ, x, y) is feasible for Problem* (CPS), *then $(\lambda/\sum_{i=1}^{\ell}\lambda_i, x, y, \lambda)$ is feasible for* (RCPS) *and $\beta(S) \leq f(S)$.*

(ii) *If $(\lambda^S, x^S, y^S, z^S)$ is optimal for* (RCPS) *and the condition*

$$(z^S)^T(Ax^S + By^S + b) = 0,\ \ z^S \neq 0, \qquad (*)$$

is satisfied, then (z^S, x^S, y^S) is optimal for (CPS) *and $\beta(S) = f(S)$.*

(iii) *If Problem* (RCPS) *is solvable, then there is an optimal solution (λ, x, y, z) such that $\lambda \in V(S)$ (the set of the vertices of S).*

Suppose that S is the simplex to be subdivided. Let $(v^1, \dots, v^\ell)$ be the vertices of S. Note that if the condition $(*)$ in Proposition 1 is fulfilled, then (z^S, x^S, y^S) is optimal for (CPS) and therefore $\beta(S) = f(S)$. In this case the simplex S can also be deleted. So we may assume that this condition does not hold. Let $\pi(z^S)$ be the point where the halfline connecting the origin and

z^S meets S. Since the condition (*) does not hold, $\lambda^S \neq \pi(z^S)$. Let $\omega^S := (\pi(z^S)+\lambda^S)/2$. Since $\omega^S \in S$, we have

$$\omega^S = \sum_{i=1}^{\ell} t_i v^i,\ \sum_{i=1}^{\ell} t_i = 1,\ t_i \geq 0\ (i = 1, ..., \ell).$$

Let $I(\omega^S) := \{i : t_i > 0\}$. We then subdivide S into subsimplices S_i, where S_i $(i \in I(\omega^S))$ is obtained from S by replacing the vertex v^i of S with ω^S. The points $\pi(z^S)$ and λ^S are called subdivision points for S (see e.g. [3]). Note that if $\pi(z^S)$ and λ^S are two adjacent vertices of S and the edge connecting $\pi(z^S)$ and λ^S is longest, then this is an exhaustive bisection process.

Having the above mentioned bounding and branching operations we can apply the prototype branch-and-bound scheme in [3] to solving Problem (CP1).

4. Branching and Bounding by Binary Tree

Let us define a binary tree as follows:

The tree is defined according to the sign of the dual variables $\lambda_1, ..., \lambda_\ell$. To each node of the tree we associate a dual variable by fixing it to be zero or positive. Every node has exactly two branches. The node corresponding to the variable λ_j has two branches: one corresponds to $\lambda_{j+1} = 0$, the other to $\lambda_{j+1} > 0$. The root has two branches corresponding to $\lambda_1 = 0$ and $\lambda_1 > 0$. Note that a variable λ_j may correspond to one or more nodes, but a node corresponds to exactly one variable. We agree to call a node *partition set*. The initial partition set T_1 is the root of the tree. Since the number of variables λ_j is finite, the binary tree is finite too, i.e., it has only a finite number of nodes.

Let $P(T)$ denote the path from the root T_1 to the node T, and let $J(T) \subset \{1, ..., \ell\}$ denote the set of indices that correspond to the nodes belonging to the path $P(T)$. Let

$$J_0(T) := \{j \in J(T) : \lambda_j = 0\},\ \ J_1(T) := \{j \in J(T) : \lambda_j > 0\}.$$

Since at the root the sign of every variable λ_j is free, $J_0(T_1) = J_1(T_1) = \emptyset$.

Let (CPT) denote the problem (CP) restricted to the partition set T, i.e.,

$$\alpha(T) := \min f(x,y) \tag{CPT}$$

subject to

$$(x,y) \in D,\ Px + Qy + q - B^T\lambda = 0,\ Ax + By + b \geq 0,$$

$$\lambda^T(Ax + By + b) = 0, \lambda \geq 0,\ \lambda_j = 0\ \ j \in J_0(T),\ \lambda_j > 0,\ j \in J_1(T).$$

Since $J_0(T_1) = J_1(T_1) = \emptyset$, Problem (CPT1) is just (CP).

To obtain a lower bound for $\alpha(T)$ we consider the relaxed problem

$$\beta(T) := \min f(x,y) \tag{RCPT}$$

subject to

$$(x,y) \in D,\ Px + Qy + q - B^T\lambda = 0,\ (Ax + By + b)_j \geq 0,\ j \notin J_1(T)$$
$$\lambda \geq 0,\ \lambda_j = 0,\ j \in J_0(T),\ (Ax + By + b)_i = 0,\ \forall i \in J_1(T).$$

Since the feasible domain of (CPT) is contained in that of (RCPT), we have $\beta(T) \leq \alpha(T)$. In particular $\beta(T_1) \leq \alpha(T_1) = f_*$.

To obtain an upper bound for the optimal value of Problem (CP) we solve the following problem

$$\min f(x,y) \qquad \text{(UCPT)}$$

subject to

$$(x,y) \in D,\ Px + Qy + q - B^T\lambda = 0,\ (Ax + By + b)_j \geq 0,\ j \in J_0(T)$$
$$(Ax + By + b)_j = 0, \lambda_j \geq 0,\ j \notin J_0(T),\ \lambda_j = 0,\ j \in J_0(T).$$

Clearly, any solution of this problem is feasible for Problem (CP). A node (partition set) T is deleted (dead) if $\beta(T) \geq \alpha$ where α is an upper bound for the optimal value of (CP). If a node corresponding to some variable, say λ_j, is not deleted, it is branched (bisected) into two nodes by setting $\lambda_{j+1} = 0$ and $\lambda_{j+1} > 0$.

With this binary tree the algorithm can be described as follows.

ALGORITHM

Let the tolerance $\epsilon \geq 0$ be chosen in advance.

Compute the lower bound $\beta(T_1)$ by solving linear program (RCPT$_1$). Let (λ^1, x^1, y^1) be the obtained solution. If

$$\langle \lambda^1, Ax^1 + By^1 + b \rangle = 0,$$

then let $\alpha_1 = \beta_1 = \beta(S_1)$ and $(\xi^1, u^1, v^1) := (\lambda^1, x^1, y^1)$.

Otherwise, let $\alpha_1 = f(x^1, y^1)$ where (λ^1, x^1, y^1) is a feasible point of (CP1) known in advance.

Take

$$\Gamma_1 = \begin{cases} \{T_1\} & \text{if } \alpha_1 - \beta_1 > \epsilon(|\alpha_1| + 1) \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $k := 1$

Iteration k ($k = 1, 2 ...$)

a) If $\Gamma_k = \emptyset$, then terminate: (ξ^k, u^k, v^k) is an ϵ-global optimal solution to (CP).

b) If $\Gamma_k \neq \emptyset$, then choose T_k such that

$$\beta_k := \beta(T_k) = \min\{\beta(T) : T \in \Gamma_k\}.$$

Branch T_k into two nodes T_{k1} and T_{k2} by setting $\lambda_{i+1} = 0$ for the node T_{k1} and $\lambda_{i+1} > 0$ for T_{k2}, where λ_i is the variable corresponding to the node T_k.

Compute $\beta(T_{ki})$ by solving linear programs (RCPT$_{ki}$) $(i = 1, 2)$.

Compute upper bounds $\alpha(T_{k1})$ and $\alpha(T_{k2})$ by solving linear programs (UCPT$_{k1}$) and (UCPT$_{k2}$). Use $\alpha(T_{k1})$ and $\alpha(T_{k2})$ to update the incumbent $(\xi^{k+1}, u^{k+1}, v^{k+1})$ and the currently best upper bound $\alpha_{k+1} = f(u^{k+1}, v^{k+1})$.

Set

$$\Gamma'_{k+1} := (\Gamma_k \setminus T_k) \cup \{T_{k1}, T_{k2}\},$$

$$\Gamma_{k+1} := \{T \in \Gamma'_{k+1} : \alpha_{k+1} - \beta_{k+1} > \epsilon(|\alpha_{k+1}| + 1)\}.$$

Increase k by 1 and go to iteration k.

Since the binary tree has a finite number of nodes, this algorithm always terminates after a finite iterations yielding an ϵ- global optimal solution to Problem (CP).

In order to obtain a preliminary evaluation of the performance of the proposed algorithms, we have written computer codes that implement the algorithms. We use the code to solve hundred randomly generated problems. The computational results show that the binary tree algorithm is more efficient. In fact, it can solve problems up to twenty- dual variables. The number of the primal variables may be larger.

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