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(h_0, h, M_0)-Uniform Stability Properties for Nonlinear Differential Systems*

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Abstract. This paper establishes some (h_0, h, M_0)-uniform stability criteria for nonlinear differential systems by the direct method of Lyapunov.

1. Introduction

In many concrete problems such as adaptive control systems, one needs to consider the stability of sets which are not invariant, so the notion of eventual stability [3] was introduced to deal with such situations. It is subsequently recognized that although the set is eventually stable is not invariant in the usual sense, it is so in the asymptotic sense. This observation leads to a new concept of asymptotically invariant sets, which form a special subclass of invariant sets. A natural generalization of the above concepts is the notion of M_0-stability [2], which describes a very general type of invariant set and its stability behavior.

In [4] Lakshmikantham and Liu introduced a very general type of stability called (h_0, h)-stability by combining the concepts of M_0-stability and (h_0, h, M_0)-stability [4] and presented a comparison result concerning (h_0, h, M_0)-uniform asymptotic stability. However, very little is known about (h_0, h, M_0)-uniform stability properties when the comparison principle fails. This paper is therefore devoted to the development of the basic theory of Lyapunov in terms of (h_0, h)-uniform asymptotic stability employing Lyapunov's direct method. In Sec. 2, we give some definitions and notations. We state and prove, in Sec. 3 our main results which establish some criteria for (h_0, h, M_0)-uniform stabilization. An example is also worked out which demonstrates the sharpness of conditions given in the theorems.

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2. Preliminaries

Consider the generalized initial value problem

$$x' = f(t,x),\ x(t_0) = \psi(t_0, x^*),\ t_0 \geq 0, \tag{2.1.1}$$

where $f, \psi \in C[R_+ \times R^n, R^n]$ and f is smooth enough to ensure the existence of solutions of (2.1.1).

Consider also the comparison equation

$$u' = g(t,u),\ u(t_0) = \varphi(t_0, u^*),\ t_0 \geq 0, \tag{2.1.2}$$

where $g \in C[R_+ \times R, R]$, $\varphi \in C[R_+ \times R, R_+]$.

For the reader's convenience, let us list the following classes of functions.

$$\Gamma = \{h \in C[R_+ \times R^n,\ R_+] : \inf h(t,x) = 0\}.$$
$$K = \{a \in C[R_+,\ R_+] : a(u) \text{ is strictly increasing in } u \text{ and } a(0) = 0\}.$$
$$CK = \{a \in C[R_+^2,\ R_+] : a(t,s) \in K \text{ for each } t\}.$$
$$KC = \{\sigma \in C[R_+,\ R_+] : \sigma \in K \text{ and } \sigma \text{ is convex}\}.$$
$$K\overline{C} = \{\sigma \in C[R_+,\ R_+] : \sigma \in K \text{ and } \sigma \text{ is concave}\}.$$
$$CKP = \{a \in C[R_+ \times R_+,\ R_+] : a \in CK \text{ and for every } \epsilon > 0, \text{ there exists } \delta(\epsilon)>0,\ \tau(\epsilon)>0\ (\tau(\epsilon)\to\infty \text{ as } \epsilon\to 0) \text{ such that } \int_{t_0}^{t_0+1} h_0(s,\psi(s,x^*))ds < \delta \text{ implies } \int_{t_0}^{t_0+1} a(s, h_0(s,\psi(s,x^*)))ds < \epsilon,\ t_0 \geq \tau(\epsilon)\}.$$

We also introduce the following notations.

$M = M(R_+^2, R^n)$ is the space of all measurable mappings from R_+^2 to R^n such that $x \in M$ if and only if $h(\eta, x(\eta, s))$ is locally integrable on R_+ and

$$\sup_{t>0} \int_t^{t+1} h(\eta, x(\eta,s))ds < \infty, \quad h \in \Gamma.$$

$M_0 = M_0(R_+^2, R^n)$ is the subspace of $M(R_+^2, R^n)$ consisting of all $x(\eta, s)$ such that

$$\int_t^{t+1} h(\eta, x(\eta,s))ds \to 0 \text{ as } t \to \infty.$$
$$M_0(h,\epsilon) = \{x \in M : \limsup_{t\to\infty} \int_t^{t+1} h(\eta, x(\eta,s))ds \leq \epsilon\}.$$
$$M_0^c(h,\epsilon) = \{x \in M : \limsup_{t\to\infty} \int_t^{t+1} h(\eta, x(\eta,s))ds > \epsilon\}.$$

$$S(h, M_0, \rho) = \{(\eta, x(\eta,s)) \in R_+ \times R^n : x(\eta,s) \in M_0(h,\rho),\ \rho > 0\}.$$

$$S^c(h, M_0, \rho) = \{(\eta, x(\eta,s)) \in R_+ \times R^n : x(\eta,s) \in M_0^c(h,\rho),\ \rho > 0\}.$$

For any function $V \in C[R_+ \times R^n,\ R_+]$, we define the functions

$$D^+V(t,x) = \lim_{\delta\to 0^+} \sup \frac{1}{\delta}\big[V(t+\delta,\, x+\delta f(t,x)) - V(t,x)\big], \tag{2.1.3}$$

$$D_-V(t,x) = \lim_{\delta\to 0^-} \inf \frac{1}{\delta}\big[V(t+\delta,\, x+\delta f(t,x)) - V(t,x)\big]. \tag{2.1.4}$$

Let us now give the following definitions. As usual, let $x(t) = x(t, s, \psi(s, x^*))$, $t \geq s$ represent a solution of (2.1.1) starting at $(s, \psi(s, x^*))$.

Definition 2.1. *With respect to system (2.1.1), the set A is said to be*

(M_1) *(h_0, h, M_0)-uniformly stable if for each $\epsilon > 0$, there exist $\delta_1(\epsilon), \delta_2(\epsilon) > 0$ and $\tau(\epsilon)$, $\tau(\epsilon) \to \infty$ as $\epsilon \to 0$, such that*

$$\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds < \epsilon \text{ for all } t \geq t_0 + 1.$$

whenever $x^* \in S(A, \delta_1)$ *and* $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \delta_2,\ t_0 \geq \tau(\epsilon)$;

(M_2) *(h_0, h, M_0)-uniformly attractive if for every $\epsilon > 0$, there exist positive numbers δ_{10}, δ_{20}, τ_0 and $T(\epsilon)$ such that*

$$\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds < \epsilon,\ t \geq t_0 + 1 + T(\epsilon),\ t_0 \geq \tau_0,$$

provided $x^* \in S(A, \delta_{10})$ *and* $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \delta_{20}$;

(M_3) *(h_0, h, M_0)-uniformly asymptotically stable if (M_1) and (M_2) hold together.*

Definition 2.2. *Let $h_0, h \in \Gamma$. Then h_0 is said to be integrally finer than h if for every $\epsilon > 0$, there exist $\delta(\epsilon) > 0$, $\tau(\epsilon) > 0$ ($\tau(\epsilon) \to \infty$ as $\epsilon \to 0$) such that $x^* \in A$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \delta$ implies $\int_{t_0}^{t_0+1} h(s, \psi(s, x^*))ds < \epsilon$, $t_0 \geq \tau(\epsilon)$.*

Definition 2.3. [1] *Let $\lambda : R_+ \to R_+$ be a measurable function. Then $\lambda(t)$ is said to be integrally positive if*

$$\int_I \lambda(s)ds = +\infty$$

whenever $I = \bigcup_{i=1}^{\infty}[\alpha_i, \beta_i]$, $\alpha_i < \beta_i < \alpha_{i+1}$ and $\beta_i - \alpha_i \geq \delta > 0$.

We need the following known result [5] before we can proceed to prove (h_0, h, M_0)-uniform stability criteria.

Lemma 2.4. (Jensen inequality) *Let φ be a convex (or concave) function and y integrable. Then*

$$\varphi\Big(\int y(t)dt\Big) \leq \int \varphi(y(t))dt \ \ (\text{or } \varphi\Big(\int y(t)dt\Big) \geq \int \varphi(y(t))dt).$$

3. Main Results

In this section we shall establish several results on (h_0, h, M_0)-uniform stability and (h_0, h, M_0)-uniform asymptotic stability. Let us begin with proving a result on (h_0, h, M_0)-uniform stability.

Theorem 3.1. *Assume that*
(i) $h_0, h \in \Gamma$ *and* h_0 *is integrally finer than* h*;*
(ii) $V \in C[R_+ \times R^n, R_+]$, $V(t,x)$ *is locally Lipschitzian in* x *and satisfies*

$$b(h(t,x)) \leq V(t,x) \leq a(t, h_0(t,x)), \ (t,x) \in S(h, M_0, \rho),$$

where $a \in CKP$, $b \in KC$*;*
(iii) $D^+V(t,x) \leq 0$, $(t,x) \in S(h, M_0, \rho)$.
Then the set A *is* (h_0, h, M_0)*-uniformly stable with respect to system* (2.1.1).

Proof. Let $0 < \epsilon < \rho$ and $t_0 \in R_+$ be given. By condition (i), there exist $\delta_1(\epsilon), \delta_2(\epsilon) > 0$ and $\tau_1(\epsilon)$, $\tau_1(\epsilon) \to \infty$ as $\epsilon \to 0$ such that

$$\int_{t_0}^{t_0+1} h(s, \psi(s, x^*))ds < \epsilon, \tag{3.1.1}$$

provided $x^* \in S(A, \delta_1)$, $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \delta_2$ and $t_0 \geq \tau_1(\epsilon)$.
In view of assumption (ii) and definition of a, we have, for some positive constants $\delta_3(\epsilon)$, $\delta_4(\epsilon)$ and $\tau_2(\epsilon)$, $\tau_2(\epsilon) \to \infty$ as $\epsilon \to 0$, the following inequality

$$\int_{t_0}^{t_0+1} a(s, h_0(s, \psi(s, x^*)))ds < b(\epsilon), \ t_0 \geq \tau_2(\epsilon). \tag{3.1.2}$$

when $x^* \in S(A, \delta_3)$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \delta_4$.

Let $\overline{\delta}_1(\epsilon) = \min\{\delta_1(\epsilon), \delta_3(\epsilon)\}$, $\overline{\delta}_2(\epsilon) = \min\{\delta_2(\epsilon), \delta_4(\epsilon)\}$ and $\overline{\tau}(\epsilon) = \max\{\tau_1(\epsilon), \tau_2(\epsilon)\}$. If we choose x^* such that $x^* \in S(A, \overline{\delta}_1)$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \overline{\delta}_2$, then we claim that

$$\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds < \epsilon, \ t \geq t_0 + 1, \ t_0 \geq \overline{\tau}(\epsilon), \tag{3.1.3}$$

where $x(t, s, \psi(s, x^*))$ is any solution of (2.1.1).

If this is false, then there exist $t_1 > t_0 + 1$, $t_0 \geq \overline{\tau}(\epsilon)$ such that

$$\int_{t_0}^{t_0+1} h(t_1, x(t_1, s, \psi(s, x^*)))ds = \epsilon,$$

and

$$\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds < \epsilon, \ t_0 + 1 \leq t < t_1, \ t_0 \geq \overline{\tau}(\epsilon). \tag{3.1.4}$$

It then follows from assumptions (ii) and (iii), relations (3.1.1), 3.1.2), (3.1.4) and Lemma 2.4, that

$$
\begin{aligned}
b(\epsilon) = b\Big(\int_{t_0}^{t_0+1} h(t_1, x(t_1, s, \psi(s, x^*)))ds\Big) \\
&\leq \int_{t_0}^{t_0+1} b(h(t_1, x(t_1, s, \psi(s, x^*)))ds \\
&\leq \int_{t_0}^{t_0+1} V(t_1, x(t_1, s, \psi(s, x^*)))ds \\
&\leq \int_{t_0}^{t_0+1} V(s, \psi(s, x^*))ds \\
&\leq \int_{t_0}^{t_0+1} a(s, h_0(s, \psi(s, x^*)))ds \\
&\leq b(\epsilon).
\end{aligned}
$$

This is a contradiction and the proof is complete.

If we utilize a family of Lyapunov functions instead of one, it is natural to expect that each member of the family has to satisfy weaker requirements. To illustrate this idea, we shall next give the following result which is an improvement of Theorem 3.1.

Theorem 3.2. *Assume that*

(i) *$h_0, h \in \Gamma$ and h_0 is integrally finer than h;*

(ii) *for every $\eta > 0$, there exists a function $V_\eta \in C[S(h, M_0, \rho) \cap S^c(h_0, M_0, \eta), R_+]$ such that $V_\eta(t, x)$ is locally Lipschitzian in x and satisfies*

$$b(h(t,x)) \leq V_\eta(t,x) \leq a(h_0(t,x)), \quad (t,x) \in S(h, M_0, \rho) \cap S^c(h_0, M_0, \eta),$$

where $a \in K\overline{C}$, $b \in KC$;

(iii) *$D^+V_\eta \leq 0$, $(t,x) \in S(h, M_0, \rho) \cap S^c(h_0, M_0, \eta)$.*

Then the set A is (h_0, h, M_0)-uniformly stable with respect to system (2.1.1).

Proof. Let $\epsilon \in (0, \rho)$ and $t_0 \in R_+$ be given. Assumption (i) implies that there exist $\delta_1(\epsilon), \delta_2(\epsilon) > 0$ and $\tau(\epsilon)$, $\tau(\epsilon) \to \infty$ as $\epsilon \to 0$ such that

$$\int_{t_0}^{t_0+1} h(s, \psi(s, x^*))ds < \epsilon, \quad t_0 \geq \tau(\epsilon), \tag{3.2.1}$$

provided $x^* \in S(A, \delta_1)$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \delta_2$.

We choose $\delta_3 = \delta_3(\epsilon) > 0$ such that $a(\delta_3) < b(\epsilon)$. Let $\overline{\delta}_2(\epsilon) = \min\{\delta_2(\epsilon), \delta_3(\epsilon)\}$. If we choose x^* such that $x^* \in S(A, \delta_1)$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \overline{\delta}_2$, then we claim that

$$\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds < \epsilon, \quad t \geq t_0 + 1, \ \ t_0 \geq \tau(\epsilon),$$

where $x(t, s, \psi(s, x^*))$ is any solution of (2.1.1).

If this is false, then there exist a solution $x(t, s, \psi(s, x^*))$ of (2.1.1) and t_1, t_2

satisfying $t_2 > t_1 > t_0 + 1$ such that

$$\begin{aligned}\int_{t_0}^{t_0+1} h_0(t_1, x(t_1, s, \psi(s, x^*)))ds = \overline{\delta_2},\\ \int_{t_0}^{t_0+1} h(t_2, x(t_2, s, \psi(s, x^*)))ds = \epsilon,\end{aligned} \tag{3.2.2}$$

and

$$(t, x(t)) \in S(h, M_0, \epsilon) \cap S^c(h_0, M_0, \overline{\delta_2}) \text{ for } t \in [t_1, t_2].$$

Hence, by letting $\eta = \bar{\delta}_2$ and condition (ii), there exists a $V_\eta(t, x)$ satisfying assumptions (ii) and (iii), which implies

$$\begin{aligned} b(\epsilon) &\le \int_{t_0}^{t_0+1} b(h(t_2, x(t_2, s, \psi(s, x^*))))ds \\ &\le \int_{t_0}^{t_0+1} V_\eta(t_2, x(t_2, s, \psi(s, x^*)))ds \\ &\le \int_{t_0}^{t_0+1} V_\eta(t_1, x(t_1, s, \psi(s, x^*)))ds \\ &\le \int_{t_0}^{t_0+1} a(h_0(t_1, x(t_1, s, \psi(s, x^*))))ds \\ &\le a\Big(\int_{t_0}^{t_0+1} h_0(t_1, x(t_1, s, \psi(s, x^*)))ds\Big) \\ &= a(\overline{\delta_2}) < b(\epsilon). \end{aligned}$$

This is absurd. Thus the proof is complete.

Let us next discuss a result on (h_0, h, M_0)-uniformly asymptotic stability that corresponds to Marachkov's result.

Theorem 3.3. *Assume that*

(i) $h_0, h \in \Gamma$ *and* h_0 *is integrally finer than* h;

(ii) $V \in C[R_+ \times R^n, R_+]$, $V(t, x)$ *is locally Lipschitzian in* x *and satisfies*

$$0 \le V(t, x) \le a(t, h_0(t, x)), \quad (t, x) \in S(h, M_0, \rho),$$

where $a \in CKP$, $b \in KC$;

(iii) $D^+V(t, x) \le -c(h, (t, x))$, $(t, x) \in S(h, M_0, \rho)$, $c \in KC$;

(iv) $h \in C^1[R_+ \times R^n, R_+]$ *and* $|h'(t, x)| \le M$, $(t, x) \in S(h, M_0, \rho)$, *where* $M > 0$ *and* $h'(t, x) = h_t(t, x) + h_x(t, x) \cdot f(t, x)$.

Then the set A *is* (h_0, h, M_0)*-uniformly asymptotically stable with respect to system* (2.1.1).

Proof. Let us first prove that the set A is (h_0, h, M_0)-uniformly stable with respect to (2.1.1). Let $\epsilon \in (0, \rho)$ and $t_0 \in R_+$ be given. In view of the definition of a, there exist positive constants $\delta_1(\epsilon)$, $\delta_2(\epsilon)$ and $\tau_1(\epsilon)$, $\tau_1(\epsilon) \to \infty$ as $\epsilon \to 0$, such that

$$\int_{t_0}^{t_0+1} a(s, h_0(s, \psi(s, x^*)))ds < \frac{\epsilon}{2M} C(\frac{\epsilon}{2}), \quad t_0 \geq \tau_1(\epsilon), \tag{3.3.1}$$

provided $x^* \in S(A, \delta_1)$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \delta_2$.

Assumption (i) implies that there exist $\delta_3(\epsilon), \delta_4(\epsilon) > 0$ and $\tau_2(\epsilon)$, $\tau_2(\epsilon) \to \infty$ as $\epsilon \to 0$, such that

$$\int_{t_0}^{t_0+1} h(s, \psi(s, x^*))ds \leq \frac{\epsilon}{2}, \quad t_0 \geq \tau_2(\epsilon), \tag{3.3.2}$$

provided $x^* \in S(A, \delta_3)$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \delta_4$.

Let $\overline{\delta}_1(\epsilon) = \min\{\delta_1(\epsilon), \delta_3(\epsilon)\}$, $\overline{\delta}_2(\epsilon) = \min\{\delta_2(\epsilon), \delta_4(\epsilon)\}$ and $\overline{\tau}(\epsilon) = \max\{\tau_1(\epsilon), \tau_2(\epsilon)\}$. If we choose x^* such that $x^* \in S(A, \overline{\delta}_1)$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \overline{\delta}_2$, then we claim that

$$\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds < \epsilon, \quad t \geq t_0 + 1, \ t_0 \geq \overline{\tau}(\epsilon), \tag{3.3.3}$$

where $x(t, s, \psi(s, x^*))$ is any solution of (2.1.1).

If this is false, then there exist t_1, t_2 satisfying $t_2 - t_1 > \frac{\epsilon}{2M}$ and $t_2 > t_1 > t_0 + 1$ such that

$$\int_{t_0}^{t_0+1} h(t_1, x(t_1, s, \psi(s, x^*)))ds = \frac{\epsilon}{2},$$

$$\int_{t_0}^{t_0+1} h(t_2, x(t_2, s, \psi(s, x^*)))ds = \epsilon,$$

$$\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds \geq \frac{\epsilon}{2}, \quad t \in [t_1, t_2], \tag{3.3.4}$$

and

$$\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds < \epsilon, \quad t \in [t_0 + 1, t_2).$$

It then follows from (3.3.1)–(3.3.4) that

$$\begin{aligned}
0 &\leq \int_{t_0}^{t_0+1} V(t_2, x(t_2, s, \psi(s, x^*)))ds \\
&< \int_{t_0}^{t_0+1} V(s, \psi(s, x^*))ds \\
&\quad + \int_{t_0}^{t_0+1} [V(t_2, x(t_2, s, \psi(s, x^*))) - V(t_1, x(t_1, s, \psi(s, x^*)))]ds \\
&\leq \int_{t_0}^{t_0+1} a(s, h_0(s, \psi(s, x^*)))ds + \int_{t_0}^{t_0+1} \Big(\int_{t_1}^{t_2} D^+V(t, x(t, s, \psi(s, x^*)))dt\Big)ds \\
&< \frac{\epsilon}{2M} C(\frac{\epsilon}{2}) - \int_{t_1}^{t_2} c\Big(\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds\Big)dt \leq 0,
\end{aligned}$$

which is a contradiction. Hence the set A is (h_0, h, M_0)-uniformly stable. Thus for $\epsilon = \rho$, there exist $\tau_1(\rho) > 0$, $\delta_1(\rho) > 0$ and $\delta_2(\rho) > 0$ such that

$$\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds < \rho, \;\; t \geq t_0 + 1, \;\; t_0 \geq \tau_1(\rho), \tag{3.3.5}$$

provided $x^* \in S(A, \delta_1)$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \delta_2$, where $x(t, s, \psi(s, x^*))$ is any solution of (2.1.1).

In view of assumption (ii) and the definition of a, we have, for some positive constants $\tau_2(\rho)$, $\delta_3(\rho)$ and $\delta_4(\rho)$, the following inequality

$$\int_{t_0}^{t_0+1} a(s, h_0(s, \psi(s, x^*)))ds < \rho, \quad t_0 \geq \tau_2(\rho), \tag{3.3.6}$$

whenever $x^* \in S(A, \delta_3)$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \delta_4$.

By condition (i), there exist $\tau_3(\rho) > 0$, $\delta_5(\rho) > 0$ and $\delta_6(\rho) > 0$ such that

$$\int_{t_0}^{t_0+1} h(s, \psi(s, x^*))ds < \rho, \quad t_0 \geq \tau_3(\rho), \tag{3.3.7}$$

provided $x^* \in S(A, \delta_5)$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \delta_6$.

Let $\delta_{10} = \min\{\delta_1, \delta_3, \delta_5\}$, $\delta_{20} = \min\{\delta_2, \delta_4, \delta_6\}$ and $\tau_0 = \max\{\tau_1(\rho), \tau_2(\rho), \tau_3(\rho)\}$. If we choose x^* such that $x^* \in S(A, \delta_{10})$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \delta_{20}$, then we claim that

$$\lim_{t \to \infty} \int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds = 0, \quad t_0 \geq \tau_0, \tag{3.3.8}$$

where $x(t, s, \psi(s, x^*))$ is any solution of (2.1.1).

If this is not true, then for some $\epsilon_0 > 0$, there exist a solution $x(t, s, \psi(s, x^*))$ with $x^* \in S(A, \delta_{10})$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \delta_{20}$ and a divergent sequence $\{t_k\}$ such that

$$\int_{t_0}^{t_0+1} h(t_k, x(t_k, s, \psi(s, x^*)))ds \geq \epsilon_0, \;\; k = 1, 2, \ldots .$$

It then follows from assumption (iv) that, on the intervals $t_k - \epsilon_0/2M \leq t \leq t_k + \epsilon_0/2M$, $k = 1, 2, \ldots$, we have

$$\begin{aligned}\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds &\geq \epsilon_0 + \int_{t_k}^{t} \int_{t_0}^{t_0+1} h'(m, x(m, s, \psi(s, x^*)))ds\, dm \\ &\geq \frac{\epsilon_0}{2}.\end{aligned}$$

We can assume that these intervals are disjoint and $t_1 - \epsilon_0/2M > t_0 + 1$ by taking, if necessary, a sequence of t_k. This, together with assumptions (iii) and (iv), implies

$$\int_{t_0}^{t_0+1} V(t_k + \frac{\epsilon_0}{2M}, x(t_k + \frac{\epsilon_0}{2M}, s, \psi(s, x^*)))ds$$
$$\leq \int_{t_0}^{t_0+1} V(t_0 + 1, x(t_0 + 1, s, \psi(s, x^*)))ds$$
$$- \int_{t_0}^{t_0+1} \int_{t_0+1}^{t_k+\frac{\epsilon_0}{2M}} c(h(t, x(t, s, \psi(s, x^*))))dt\, ds$$
$$< \int_{t_0}^{t_0+1} V(s, \psi(s, x^*))ds - \int_{t_0+1}^{t_k+\frac{\epsilon_0}{2M}} c\Big(\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds\Big)dt$$
$$< \int_{t_0}^{t_0+1} a(s, h_0(s, \psi(s, x^*)))ds - \frac{\epsilon_0}{M}c(\frac{\epsilon_0}{2})k$$
$$< \rho - \frac{\epsilon_0}{M}c(\frac{\epsilon_0}{2})k \to -\infty \text{ as } k \to \infty.$$

which is a contradiction. Thus the set A is (h_0, h, M_0)-uniformly asymptotically stable.

To obtain a smooth converse theorem for (h_0, h, M_0)-uniform asymptotic stability, we should assume large domain of attraction, that is we need to have a stronger concept than (h_0, h, M_0)-uniform asymptotic stability. The following result is a direct theorem of this type.

Theorem 3.4. *Assume that*

(i) $h_0, h \in \Gamma$ *and* h_0 *is integrally finer than* h;

(ii) $V \in C[R_+ \times R^n, R_+]$, $V(t,x)$ *is locally Lipschitzian in* x *and there exist functions* $a \in CKP$ *and* $b \in KC$ *such that*

$$b(h(t,x)) \leq V(t,x) \leq a(t, h_0(t,x)),\ \ D^+V(t,x) \leq 0,\ \ (t,x) \in S(h, M_0, \rho);$$

(iii) $W \in C[R_+ \times R^n, R_+]$, $W(t,x)$ *is locally Lipschitzian in* x *and*

$$W(t,x) \leq N,\ \ D^+W(t,x) \leq -c(V(t,x)),\ \ (t,x) \in S(h, M_0, \rho),$$

where $c \in K$, $N > 0$;

(iv) *there exists a positive constant* $\gamma \in (0, \rho)$ *such that*

$$D_- \int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x*)))ds < 0$$

if

$$\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds = \gamma,\ \ t \geq t_0 + 1,$$

where $h(t,x)$ *is locally Lipschitzian in* x *for each* t, $x(t, s, \psi(s, x^*))$ *is any solution of system* (2.1.1) *and* t_0 *is sufficiently large.*

Then the set A *is* (h_0, h, M_0)*-uniformly stable and* (h, h, M_0)*-uniformly attractive with respect to* (2.1.1).

Proof. It follows from assumption (i)–(ii) that the set A is (h_0, h, M_0)-uniformly stable with respect to (2.1.1). By condition (i), there exist $\tau_1(\gamma) > 0$, $\delta_1(\gamma) > 0$ and $\delta_2(\gamma) > 0$ such that

$$\int_{t_0}^{t_0+1} h(s,\psi(s,x^*))ds < \gamma, \quad t_0 \geq \tau_1(\gamma), \tag{3.4.1}$$

provided $x^* \in S(A,\delta_1)$ and $\int_{t_0}^{t_0+1} h_0(s,\psi(s,x^*))ds < \delta_2$.

Let $x(t,s,\psi(s,x^*))$ be any solution of (2.1.1) with $x^* \in S(A,\delta_1)$ and $\int_{t_0}^{t_0+1} h_0(s,\psi(s,x^*))ds < \delta_2$, we shall first show that

$$\int_{t_0}^{t_0+1} h(s,\psi(s,x^*))ds < \gamma$$

implies $\int_{t_0}^{t_0+1} h(t,x(t,s,\psi(s,x^*)))ds < \gamma$, $t \geq t_0+1$, $t_0 \geq \tau_1(\gamma)$. (3.4.2)

If it is false, then there exist $t_1 > t_0+1$, $t_0 \geq \tau_1(\gamma)$ such that

$$\int_{t_0}^{t_0+1} h(t_1,x(t_1,s,\psi(s,x^*)))ds = \gamma \quad \text{and}$$

$$\int_{t_0}^{t_0+1} h(t,x(t,s,\psi(s,x^*)))ds < \gamma, \quad t \in [t_0+1,t_1).$$

Then

$$\begin{aligned} &D_- \int_{t_0}^{t_0+1} h(t_1,x(t_1,s,\psi(s,x^*)))ds \\ &= \liminf_{\delta\to 0^-} \frac{1}{\delta}\Big[\int_{t_0}^{t_0+1} h(t_1+\delta,x(t_1+\delta,s,\psi(s,x^*)))ds \\ &\qquad - \int_{t_0}^{t_0+1} h(t_1,x(t_1,s,\psi(s,x^*)))ds\Big] \geq 0, \end{aligned}$$

which contradicts assumption (iv). Thus the set $S(h,M_0,\gamma)$ is a positive invariant set of system (2.1.1).

Now let $\epsilon \in (0,\rho)$ be given. Set $\delta_{10} = \delta_1(\gamma)$, $\delta_{20} = \gamma$, $\tau_0 = \tau_1(\gamma) > 0$, $T(\epsilon) = N/c(b(\epsilon)) + 1$. If we choose x^* such that $x^* \in S(A,\delta_{10})$ and $\int_{t_0}^{t_0+1} h(s,\psi(s,x^*))ds < \delta_{20}$, then we claim that there exists a $t^* \in [t_0+1, t_0+1+T]$ such that

$$V(t^*,x(t^*,s,\psi(s,x^*))) < b(\epsilon), \tag{3.4.3}$$

where $x(t,s,\psi(s,x^*))$ is any solution of (2.1.1).

If this is not true, then there exists a solution $x(t,s,\psi(s,x^*))$ of (2.1.1) with $x^* \in S(A,\delta_{10})$ and $\int_{t_0}^{t_0+1} h(s,\psi(s,x^*))ds < \delta_{20}$ such that

$$V(t,x(t,s,\psi(s,x^*))) \geq b(\epsilon), \quad t \in [t_0+1,t_0+1+T].$$

It then follows from condition (iii) that

$$\begin{aligned}
&W(t_0+1+T, x(t_0+1+T, s, \psi(s,x^*))) \\
&\le W(t_0+1, x(t_0+1, s, \psi(s,x^*))) - \int_{t_0+1}^{t_0+1+T} c(V(m, x(m, s, \psi(s,x^*))))dm \\
&< N - c(b(\epsilon))T < 0,
\end{aligned}$$

which is a contradiction. Then by (ii), we get

$$\begin{aligned}
b\Big(\int_{t_0}^{t_0+1} h(t, x(t,s,\psi(s,x^*)))ds\Big) &\le \int_{t_0}^{t_0+1} b(h(t, x(t,s,\psi(s,x^*))))ds \\
&\le \int_{t_0}^{t_0+1} V(t, x(t,s,\psi(s,x^*)))ds \\
&\le \int_{t_0}^{t_0+1} V(t^*, x(t^*,s,\psi(s,x^*)))ds \\
&< b(\epsilon),
\end{aligned}$$

which implies

$$\int_{t_0}^{t_0+1} h(t, x(t,s,\psi(s,x^*)))ds < \epsilon, \quad t \ge t_0 + 1 + T(\epsilon).$$

Thus the set A is (h, h, M_0)-uniformly attractive and the proof is complete.

As we shall see, employing several Lyapunov functions offers a better mechanism to obtain results under much weaker assumptions.

Theorem 3.5. *Assume that*

(i) $h_0, h \in \Gamma$ *and* h_0 *is integrally finer than* h;

(ii) $V \in C[R_+ \times R^n, R_+]$, $V(t,x)$ *is locally Lipschitzian in* x *and satisfies*

$$b(h(t,x)) \le V(t,x) \le a(t, h_0(t,x)), \quad (t,x) \in S(h, M_0, \rho),$$

where $a \in CKP$, $b \in KC$;

(iii) $D^+V(t,x) \le -\lambda(t)c(h(t,x))$, $(t,x) \in S(h, M_0, \rho)$, *where* $\lambda(t)$ *is integrally positive and* $c \in KC$;

(iv) $W_1, W_2, \ldots, W_m \in C[R_+ \times R^n, R_+]$, *for each* $i = 1, 2, \ldots, m$, $W_i(t,x)$ *is locally Lipschitzian in* x, $D^+W(t,x)$ *is bounded from below on* $S(h, M_0, \rho)$ *and there exist functions* $a_1 \in K\overline{C}$, $b_1 \in KC$ *such that*

$$b_1(h(t,x)) \le \sum_{i=1}^{m} W_i(t,x) \le a(t, h_0(t,x)), \quad (t,x) \in S(h, M_0, \rho).$$

Then the set A *is* (h_0, h, M_0)*-uniformly asymptotically stable with respect to* (2.1.1).

Proof. By Theorem 3.1, the set A is (h_0, h, M_0)-uniformly stable with respect to (2.1.1). Furthermore, similar to Theorem 3.3, we see that (3.3.5)–(3.3.7) hold.

To prove (3.3.8), note that $b_1(h(t,x)) \le \sum_{i=1}^{m} W_i(t,x)$ for $(t,x) \in S(h, M_0, \rho)$, it is enough to show that

$$\lim_{t\to\infty} \int_{t_0}^{t_0+1} \sum_{i=1}^{m} W_i(t, x(t,s,\psi(s,x^*)))ds = 0, \quad t_0 \ge \tau_0. \tag{3.5.1}$$

If (3.5.1) is false, then there exists an i, $1 \le i \le m$, such that

$$\lim_{t\to\infty} \int_{t_0}^{t_0+1} W_i(t, x(t,s,\psi(s,x^*)))ds \ne 0.$$

Thus we can find a sequence $t_0 < t_1 < t_2 \ldots < t_k < \ldots$ with $t_k - t_{k-1} \ge \alpha > 0$ and $t_k \to \infty$ as $k \to \infty$ such that

$$\int_{t_0}^{t_0+1} W_i(t_k, x(t_k,s,\psi(s,x^*)))ds \ge l > 0. \tag{3.5.2}$$

Suppose that $D^+W_i(t,x) \ge -M$. Since

$$\int_{t_0}^{t_0+1} W_i(t, x(t,s,\psi(s,x^*)))ds =$$

$$\int_{t_0}^{t_0+1} W_i(t_k, x(t_k,s,\psi(s,x^*)))ds +$$

$$\int_{t_0}^{t_0+1} \int_{t_k}^{t} D^+W_i(m, x(m,s,\psi(s,x^*)))dm\,ds,$$

it follows from condition (iv) that there exists a constant δ, $0 < \delta < \min\{\alpha, l/2M\}$ such that

$$\int_{t_0}^{t_0+1} W_i(t, x(t,s,\psi(s,x^*)))ds \ge l - \delta M \ge \frac{l}{2}, \quad t \in [t_k - \delta, t_k]. \tag{3.5.3}$$

Since $\sum_{i=1}^{m} W_i(t,x) \le a_1(h,(t,x))$, we have from (3.5.3)

$$\int_{t_0}^{t_0+1} h(t, x(t,s,\psi(s,x^*)))ds \ge a_1^{-1}(\frac{l}{2}), \quad t \in [t_k - \delta, t_k],\ k = 1,2,\ldots \tag{3.5.4}$$

Let $I = \bigcup_{k=1}^{\infty} [t_k - \delta, t_k]$, then it follows from condition (iii) that

$$\begin{aligned}
\lim_{t\to\infty} & \int_{t_0}^{t_0+1} V(t, x(t,s,\psi(s,x^*)))ds \\
\le & \int_{t_0}^{t_0+1} V(t_0+1, x(t_0+1,s,\psi(s,x^*)))ds \\
& + \int_{t_0}^{t_0+1} \int_{t_0+1}^{\infty} D^+V(m, x(m,s,\psi(s,x^*)))dm\,ds \\
\le & \int_{t_0}^{t_0+1} V(s, \psi(s,x^*))ds \\
& - \int_{t_0}^{t_0+1} \int_{t_0+1}^{\infty} \lambda(m)c(h(m, x(m,s,\psi(s,x^*))))dm\,ds
\end{aligned}$$

$$\leq \int_{t_0}^{t_0+1} a(s, h_0(s, \psi(s, x^*)))ds$$
$$- \int_{t_0+1}^{\infty} \lambda(m)\Big(\int_{t_0}^{t_0+1} c(h(m, x(m, s, \psi(s, x^*))))ds\Big)dm$$
$$< \rho - \int_{t_0+1}^{\infty} \lambda(m)c\Big(\int_{t_0}^{t_0+1} h(m, x(m, s, \psi(s, x^*)))ds\Big)dm$$
$$< \rho - c(a_1^{-1}(\frac{l}{2})) \int_I \lambda(m)dm = -\infty,$$

which is a contradiction. Thus (3.3.8) is true and therefore the set A is (h_0, h, M_0)-uniformly asymptotically stable.

The following result which uses two Lyapunov-like functions is a special case of Theorem 3.5.

Corollary 3.6. *Assume that conditions* (i) *and* (ii) *of Theorem* 3.5 *hold. Suppose further that*

(iii) $W \in C[R_+ \times R^n, R_+]$, *$W(t, x)$ is locally Lipschitzian in x and there exist functions $c \in KC$ and $\lambda(t)$ which is integrally positive such that*
$$c(h(t, x)) \leq W(t, x), \;\; D^+V(t, x) \leq -\lambda(t)W(t, x), \;\; (t, x) \in S(h, M_0, \rho),$$
and $D^+W(t, x)$ is bounded from below on $S(h, M_0, \rho)$.

Then the set A is (h_0, h, M_0)-uniformly asymptotically stable with respect to system (2.1.1).

The following theorem offers a better conclusion.

Theorem 3.7. *Assume that*

(i) *$h_0, h \in \Gamma$ and h_0 is integrally finer than h;*

(ii) *$V_1, V_2 \in C[R_+ \times R^n, R_+]$, $V_1(t, x)$ and $V_2(t, x)$ are locally Lipschitzian in x and there exist functions $a \in CKP$, $b \in KC$ such that*
$$b(h(t, x)) \leq V_1(t, x), \;\; V_1(t, x) + V_2(t, x) \leq a(t, h_0(t, x)), \;\; (t, x) \in S(h, M_0, \rho),$$
$$D^+V(t, x) \leq -\lambda(t)c(V_1(t, x)), \quad (t, x) \in S(h, M_0, \rho),$$
where $V = V_1 + V_2$, $\lambda(t)$ is integrally positive and $c \in KC$;

(iii) *for every solution $x(t, s, \psi(s, x^*))$ of* (2.1.1), *the function*
$$\int_0^t \int_{t_0}^{t_0+1} [D^+V_2(m, x(m, s, \psi(s, x^*)))]_{\pm} \, ds \, dm$$
is uniformly continuous on R_+, where the symbol $[\cdot]_{\pm}$ means that either the positive part $[\cdot]_+$ or the negative part $[\cdot]_-$ is considered for all $m \in R_+$ and t_0 is sufficiently large.

Then the set A is (h_0, h, M_0)-uniformly asymptotically stable with respect to system (2.1.1) *and $\int_{t_0}^{t_0+1} V_2(t, x(t, s, \psi(s, x^*)))ds$ has a finite limit as $t \to \infty$ for any solution $x(t) = x(t, s, \psi(s, x^*))$ of* (2.1.1) *such that $(t, x(t)) \in S(h, M_0, \rho)$.*

Proof. Because of assumptions (i)–(ii), the relations (3.3.5)–(3.3.7) hold. Define the functions

$$m_1(t) = \int_{t_0}^{t_0+1} V_1(t, x(t, s, \psi(s, x^*)))ds,$$

$$m_2(t) = \int_{t_0}^{t_0+1} V_2(t, x(t, s, \psi(s, x^*)))ds,$$

$$m(t) = \int_{t_0}^{t_0+1} V_1(t, x(t, s, \psi(s, x^*)))ds + \int_{t_0}^{t_0+1} V_2(t, x(t, s, \psi(s, x^*)))ds.$$

Since $b(h(t,x)) \leq V_1(t,x)$, $(t,x) \in S(h, M_0, \rho)$, the result $\lim_{t\to\infty} m_1(t) = 0$ suffices to prove (3.3.8). Clearly $\liminf_{t\to\infty} m_1(t) = 0$. For otherwise, we could have, in view of (ii), $m(t) \to -\infty$ as $t \to \infty$. Suppose now that $\limsup_{t\to\infty} m_1(t) \neq 0$, then there exists a $\gamma > 0$ such that

$$\limsup_{t\to\infty} m_1(t) > 3\gamma.$$

For definiteness, suppose that assumption (iii) holds with $[\cdot]_+$. Since $m_1(t)$ is continuous, we can choose a sequence

$$t_0 + 1 < \alpha_1 < \beta_1 < \ldots < \alpha_i < \beta_i < \ldots,$$

such that for $i = 1, 2, \ldots$,

$$m_1(\alpha_i) = 3\gamma, \quad m_1(\beta_i) = \gamma$$

and

$$\gamma \leq m_1(t) \leq 3\gamma, \quad t \in [\alpha_i, \beta_i]. \tag{3.6.1}$$

Assumption (iii) yields that $m(t)$ is nonincreasing and bounded from below, and therefore $\lim_{t\to\infty} m(t) = \sigma < \infty$. Thus there exists a $T > 0$ such that

$$\sigma \leq m(t) \leq \sigma + \gamma, \quad t \geq t_0 + 1 + T. \tag{3.6.2}$$

From (3.6.1) and (3.6.2), it is easy to see that

$$m_2(\alpha_i) \leq \sigma - 2\gamma, \quad m_2(\beta_i) \geq \sigma - \gamma. \tag{3.6.3}$$

It follows from (3.6.3) that

$$0 < \gamma \leq m_2(\beta_i) - m_2(\alpha_i) \leq \int_{t_0}^{t_0+1} \int_{\alpha_i}^{\beta_i} \left[D^+V_2(m, x(m, s, \psi(s, x^*)))\right]_+ dm\, ds$$

$$= \int_{\alpha_i}^{\beta_i} \int_{t_0}^{t_0+1} \left[D^+V_2(m, x(m, s, \psi(s, x^*)))\right]_+ ds\, dm,$$

which implies, in view of assumption (iii), that there exists a constant $d > 0$ such that

$$\beta_i - \alpha_i \geq d, \quad i = 1, 2, \ldots. \tag{3.6.4}$$

By (3.6.2)–(3.6.3) and condition (ii), we then get

$$\begin{aligned}\lim_{t\to\infty} m(t) &\le m(t_0+1) + \int_{t_0}^{t_0+1}\int_{t_0+1}^{\infty} D^+V(t, x(t,s,\phi(s,x^*)))dt\,ds \\ &< m(s) - \int_{t_0+1}^{\infty} \lambda(t)c(m_1(t))dt \\ &< \rho - c(\gamma)\int_I \lambda(t) = -\infty,\end{aligned}$$

where $I = \bigcup_{i=1}^{\infty}[\alpha_i, \beta_i]$. This contradiction implies that $\limsup_{t\to\infty} m_1(t) = 0$. Thus we conclude that the set A is (h_0, h, M_0)-uniformly asymptotically stable. To prove the last assertion of the theorem, note that $\lim_{t\to\infty} m(t) = \sigma$ and $\lim_{t\to\infty} m_1(t) = 0$, consequently the definition of $m(t)$ yields $\lim_{t\to\infty} m_2(t) = \sigma$. The proof is complete.

As we have seen, the use of comparison principle provides a unified approach and generalizes several stability results into one framework. However, a direct analysis of the right-hand side of the comparison equation can sometimes yield sharper results. This can be seen in the following theorem.

Theorem 3.8. *Assume that*

(i) $h_0, h \in \Gamma$ *and* h_0 *is integrally finer that* h;

(ii) $V \in C[R_+ \times R^n, R_+]$, $V(t,x)$ *is locally Lipschitzian in* x *and satisfies*

$$b(h(t,x)) \le V(t,x) \le a(h_0(t,x)),\ \ D^+V(t,x) \le g(t, V(t,x)),$$
$$(t,x) \in R_+ \times R^n,$$

where $a \in K\overline{C}$, $b \in KC$, $g \in C[R_+ \times R, R]$, $g(t,0) = 0$;

(iii) *for every pair of numbers* α, β, *such that* $0 < \alpha \le \beta$, *there exists constant* $\theta = \theta(\alpha,\beta) \ge 0$ *satisfying*

$$g(t,u) < 0,\ \ \alpha \le u \le \beta,\ \ t \ge \theta;$$

(iv) $h_0 \in C^1[R_+ \times R^n, R_+]$ *and for some function* $\lambda \in C[R_+, R_+]$,

$$\frac{\partial}{\partial t}h_0(t,x) + \frac{\partial}{\partial x}h_0(t,x).f(t,x) \le \lambda(t)h_0(t,x),\quad (t,x) \in R_+ \times R^n.$$

Then the set A *is* (h_0, h, M_0)*-uniformly stable with respect to* (2.1.1).

Proof. Let $\epsilon > 0$ and $t_0 \in R_+$ be given. By condition (i), there exist $\delta_1(\epsilon)$, $\delta_2(\epsilon) > 0$ and $\tau(\epsilon)$, $\tau(\epsilon) \to \infty$ as $\epsilon \to 0$ such that

$$\int_{t_0}^{t_0+1} h(s, \psi(s,x^*))ds < \epsilon,\quad t_0 \ge \tau(\epsilon), \tag{3.7.1}$$

provided $x^* \in S(A, \delta_1)$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s,x^*)) < \delta_2$.
We choose $\delta_3 = \delta_3(\epsilon) > 0$ such that $a(\delta_3) < b(\epsilon)$. Let $\theta = \theta(a(\delta_3), b(\epsilon)) > 0$, $\overline{\delta}_2 = \min\{\delta_2(\epsilon), \delta_3(\epsilon)\}e^{-N\theta}$, where

$$N = N(\theta) = \begin{cases} \sup\limits_{t_0+1 \le t \le \theta} \lambda(t), & \text{if } \theta \ge t_0+1; \\ \sup\limits_{\theta \le t \le t_0+1} \lambda(t), & \text{if } \theta \le t_0+1. \end{cases}$$

If we choose x^* such that $x^* \in S(A, \delta_1)$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))ds < \overline{\delta}_2$, then we claim that

$$\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds < \epsilon, \quad t \ge t_0+1, \ t_0 \ge \tau(\epsilon), \tag{3.7.2}$$

where $x(t, s, \psi(s, x^*))$ is any solution of (2.1.1).

Next we have two cases to discuss.

(1) If $\theta \le t_0+1$, we shall show

$$V(t, x(t, s, \psi(s, x^*))) < b(\epsilon), \quad t \ge t_0+1. \tag{3.7.3}$$

If this were false, there would exist t_1, t_2 satisfying $t_2 > t_1 \ge t_0+1$ such that

$$V(t_1, x(t_1, s, \psi(s, x^*))) = a(\overline{\delta}_2),$$
$$V(t_2, x(t_2, s, \psi(s, x^*))) = b(\epsilon),$$
$$a(\overline{\delta}_2) \le V(t, x(t, s, \psi(s, x^*))) \le b(\epsilon), \quad t \in [t_1, t_2]. \tag{3.7.4}$$

Hence at $t = t_1$, we would have

$$D^+V(t_1, x(t_1, s, \psi(s, x^*))) \ge 0. \tag{3.7.5}$$

On the other hand, as $t_1 \ge \theta$ and (3.7.4) holds, we would obtain, from condition (iii), the inequality

$$D^+V(t_1, x(t_1, s, \psi(s, x^*))) \le g(t_1, V(t_1, s, \psi(s, x^*))) < 0,$$

which would contradict (3.7.5). This proves that

$$\int_{t_0}^{t_0+1} V(t, x(t, s, \psi(s, x^*)))ds < b(\epsilon), \quad t \ge t_0+1. \tag{3.7.6}$$

(2) If $\theta \ge t_0+1$, we shall first show

$$V(t, x(t, s, \psi(s, x^*))) < a(\delta_3), \quad t_0+1 \le t \le \theta. \tag{3.7.7}$$

Defining $m(t) = h_0(t, x(t, s, \psi(s, x^*)))$, we obtain, by condition (iv),

$$m'(t) \le \lambda(t)m(t),$$

which implies, by the Gronwall's inequality,

$$\int_{t_0}^{t_0+1} h_0(t, x(t, s, \psi(s, x^*)))ds$$
$$\le \int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))\exp\Big[\int_s^t \lambda(m)dm\Big]ds. \tag{3.7.8}$$

By the definition of $\bar{\delta}_2$ and (3.7.8), we see that

$$\int_{t_0}^{t_0+1} h_0(t, x(t, s, \psi(s, x^*)))ds < \delta_3, \quad t_0 + 1 \le t \le \theta,$$

which implies

$$\int_{t_0}^{t_0+1} V(t, x(t, s, \psi(s, x^*)))ds < a(\delta_3) < b(\epsilon), \quad t_0 + 1 \le t \le \theta.$$

If $t \ge \theta$, the proof of (3.7.6) is similar to (1). Consequently, we get, from assumption (ii) and (3.7.7), that

$$\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))ds < \epsilon, \quad t \ge t_0 + 1.$$

Thus the proof is complete.

Example. Consider the nonliear differential system

$$\begin{cases} x_1' = -x_1 + 2x_2 + x_1 x_3^2 e^{-t}, \\ x_2' = -2x_1 - x_2 - x_2 x_4^2 e^{-t}, \\ x_3' = 2x_4 - x_1^2 x_3 e^{-t}, \\ x_4' = -2x_3 + x_2^2 x_4 e^{-t}, \\ x(t_0) = [x^*, x^* + \frac{1}{t_0}, x^* - \frac{1}{t_0}, \frac{1}{t_0}]'. \end{cases} \tag{3.7.9}$$

Let $V(t, x) = (1/2)(x_1^2 + x_2^2 + x_3^2 + x_4^2)$, $h(t, x) = x_1^2 + x_2^2$, $h_0(t, x) = x_1^2 + x_2^2 + x_3^2 + x_4^2$. Then we see that

$$\frac{1}{2} h(t, x) \le V(t, x) \le \frac{1}{2} h_0(t, x),$$

$$D^+ V(t, x) = -x_1^2 - x_2^2 \le -\frac{1}{2} h(t, x).$$

Let $W_1 = (1/2)x_1^2$ and $W_2 = (1/2)x_2^2$, then

$$D^+ W_1(t, x) = -x_1^2 + 2x_1 x_2 + x_1^2 x_3^2 e^{-t} \ge -3\rho^2,$$

$$\max_{1 \le i \le 2} |x_i| \le \rho,$$

$$D^+ W_2(t, x) = -2x_1 x_2 - x_2^2 - x_2^2 x_4^2 e^{-t} \ge -3\rho^2 - \rho^4,$$

$$\max_{1 \le i \le 2} |x_i| \le \rho$$

and

$$\frac{1}{8} h(t, x) \le W_1 + W_2 = \frac{1}{2}(x_1^2 + x_2^2) \le h(t, x).$$

Thus all conditions of Theorem 3.5 are satisfied and therefore we conclude that the set $x = 0$ is (h_0, h, M_0)-uniformly asymptotically stable with respect to (3.7.9).

References

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