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On Lifting LE-Modules

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Abstract. Let R be an associative ring with identity and M a right R–module. M is called a *lifting LE-module,* if M is lifting and $M = \bigoplus_{i\in I} M_i$, where M_i is a module with local endomorphism ring, for all $i \in I$. The purpose of this paper is to investigate some properties of these modules. Semiperfect rings such that any direct sum of a simple and a projective local module is lifting are characterized as semiprimary rings with Jacobson radical square-zero. Moreover we characterize the class of lifting LE-modules M such that $M \oplus S$ is lifting for all semisimple modules S as those modules that are direct sums of hollow LE-modules which are extensions of a semisimple by a simple module. Finally we show that this class coincides with the class of semisimple modules if and only if every extension of two simple modules splits.

1. Introduction and Preliminaries

Throughout this paper all rings are associative with identity and all modules will be unital right R–modules. Rad(M), Soc(M) and Jac(R) will denote the Jacobson radical of M, the socle of M and the radical of R, respectively. We will use the notation ACC (DCC) to indicate that a module M satisfies the ascending chain condition (descending chain condition).

We refer to [10] for basic terminology on lifting modules. Santa-Clara and Smith introduced the notion of *special extending modules* in [11]: Let M be a module. M is called a *special extending module* if M is extending and $M = \bigoplus_{i\in I} M_i$, where M_i is a module with local endomorphism ring, for all $i \in I$. Following this idea, in this paper, we introduce *lifting LE-modules* as a dual notion to special extending modules: Any module M is called *lifting LE* if M is lifting and M is a direct sum of modules M_i with local endomorphism rings for

all $i \in I$ (see [5]). Let M be a lifting right R-module. In this paper we prove that M is a lifting LE-module if one of the following cases is satisfied:
(1) M has ACC on small submodules;
(2) $\mathrm{Rad}(M)$ has ACC on direct summands and R is right artinian;
(3) $\mathrm{Rad}(M)$ has ACC on direct summands and R is a commutative max ring;
(4) M is finitely generated and R is commutative.
In this paper we also characterize rings for which every direct sum of a lifting LE-module and a semisimple module is lifting.

Let M_1 and M_2 be modules. M_1 is called *small M_2-projective* if every homomorphism $f : M_1 \longrightarrow M_2/A$, where A is a submodule of M_2 and $\mathrm{Im} f \ll M_2/A$, can be lifted to a homomorphism $\varphi : M_1 \longrightarrow M_2$. Let M and N be modules. If for all modules F, epimorphisms $f : M \longrightarrow F$ and homomorphisms $h : N \longrightarrow F$ there exists either $\varphi : N \longrightarrow M$ with $f\varphi = h$ or a non-zero direct summand M_1 of M and $\varphi : M_1 \longrightarrow N$ with $h\varphi = f_{|M_1}$, then N is called *almost M-projective.*

Let M be a module. M is said to have *finite Goldie dimension* if M contains a finite independent set of uniform submodules $\{N_1, \dots, N_n\}$ such that $\bigoplus_{i=1}^{n} N_i \leq_e M$. In this case n is the Goldie dimension of M and we denote n by $\dim(M)$. The module M is said to have *finite dual Goldie dimension* if there exists an epimorphism from M to a finite direct sum of n hollow factor modules with small kernel. In this case n is the dual Goldie dimension of M and we denote n by $\mathrm{codim}(M)$. Call a function $d : R\text{–Mod} \longrightarrow \mathbb{N} \cup \{\infty\}$ a *rank function on R–*Mod if for all $M, N \in R$–Mod: (R0) $d(M) = 0 \Leftrightarrow M = 0$ and (R1) $d(M \oplus N) = d(M) + d(N)$ holds. Note that if d is a rank function and M a module with $d(M) = 1$, then M is indecomposable. Clearly, $\dim(M)$ and $\mathrm{codim}(M)$ are rank functions.

2. Lifting LE-Modules

By [10, Lemma 4.7 and Corollary 4.9], any lifting LE-module has the decomposition $M = \bigoplus_{i \in I} H_i$, where H_i is hollow, for all $i \in I$. It is well-known from [10, Corollary 5.5] that any indecomposable discrete module has local endomorphism ring. From this fact, [10, Theorem 4.15, Lemma 4.7], discrete modules are also lifting LE-modules. The Prüfer p-group $\mathbb{Z}(p^\infty)$ is an example of a lifting LE-$\mathbb{Z}$-module that is not discrete. Note that the endomorphism ring of $\mathbb{Z}(p^\infty)$ is local by [8, Theorem 7.2.8] and $\mathbb{Z}(p^\infty)$ is not discrete because every nonzero factor module of $\mathbb{Z}(p^\infty)$ is isomorphic to $\mathbb{Z}(p^\infty)$. The following lemma is well-known:

Lemma 2.1. ([10, Lemma 5.1]) *A quasi-discrete module M is discrete if and only if every epimorphism $M \longrightarrow M$ with small kernel is an isomorphism.*

Recall that a module M is called *hopfian* if every surjective endomorphism of M is an isomorphism. Examples of hopfian modules are noetherian modules (see [2, 11.6]) and finitely generated modules over commutative rings (this is due to Vasconcelos [13]). By the above lemma we get:

Corollary 2.2. *Any hopfian quasi-discrete module is discrete. In particular a* *hollow module is discrete if and only if it is hopfian.*

Lemma 2.3. *Let M be a lifting right R-module. Then the following are equivalent:*
(a) *M is a finite direct sum of hollow modules.*
(b) *M satisfies the ACC on direct summands.*
(c) *M satisfies the DCC on direct summands.*
(d) *There exists a rank function d such that $d(M)$ is finite.*
(e) *M has finite dual Goldie dimension.*

Proof. The equivalences (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (e) follow from [12, Proposition 4.11] and the fact that supplements in lifting modules are direct summands (see also [10, Proposition 4.8]).

(a) $\Rightarrow$ (e) $\Rightarrow$ (d) are obvious.

We shall show (d) $\Rightarrow$ (a) by induction. Let M be a lifting module such that there exists a rank function d and $d(M) = 1$. Then M is indecomposable and hence hollow. Assume now $n \geq 1$ and assume that for every lifting module N such that there exists a rank function d with $d(N) < n$, N is a direct sum of hollow modules. Let M be a lifting module and d a rank function with $d(M) = n$. If M is indecomposable, then it is hollow and we are done. Otherwise M has a decomposition $M = N \oplus L$ with N and L non-zero. Thus by (R1), we have $n = d(N) + d(L)$ and hence $d(N)$ and $d(L)$ are less than n and as N and L are lifting modules as well, by hypothesis they are finite direct sums of hollows and so is M. ■

Before stating a general decomposition theorem for lifting modules we need the following lemma.

Lemma 2.4. ([9, Lemma 4.2.1]) *Let M be an R-module with essential radical. For every direct summands $D_1 \subseteq D_2$ of M we have $\mathrm{Rad}(D_1) = \mathrm{Rad}(D_2)$ if and only if $D_1 = D_2$. In particular if $\mathrm{Rad}(M)$ has ACC or DCC on direct summands, then so has M.*

Proof. Straightforward. ■

Theorem 2.5. *Any lifting module M has a decomposition $M = M_1 \oplus M_2 \oplus M_3$ such that*
- *M_1 is semisimple;*
- *M_2 is a lifting module with $\mathrm{Rad}(M_2) \ll M_2$ and $\mathrm{Rad}(M_2) \leq_e M_2$;*
- *M_3 is a lifting module with $\mathrm{Rad}(M_3) = M_3$.*

If $\mathrm{Rad}(M)$ satisfies the ACC or DCC on direct summands, then M_2 and M_3 are finite direct sums of hollow modules.

Proof. Let M be any lifting module. By [4, Proposition 2.9], we have a decomposition $M = M_1 \oplus N$ with M_1 semisimple and N lifting with $\mathrm{Rad}(M) = \mathrm{Rad}(N)$ essential in N. As N is lifting, $\mathrm{Rad}(M)$ contains a direct summand M_3 such that $N = M_2 \oplus M_3$ and $\mathrm{Rad}(M) \cap M_2 = \mathrm{Rad}(M_2) \ll M_2$. Clearly, $\mathrm{Rad}(M_2) \leq_e M_2$. As $M_3 \subseteq \mathrm{Rad}(M)$ and M_3 a direct summand of M, we have $\mathrm{Rad}(M_3) = M_3$.

Assume now that $\mathrm{Rad}(M)$ has ACC or DCC on direct summands. By Lemma 2.4, M_2 and M_3 also satisfy those chain conditions and by Lemma 2.3 they are finite direct sums of hollow modules. ∎

Recall that a ring R is called a right *max ring* if every right R-module has a maximal submodule.

Corollary 2.6. *Let M be a lifting right R-module. Then M is lifting LE in each of the following cases:*
(1) M has ACC on small submodules;
(2) $\mathrm{Rad}(M)$ has ACC on direct summands and R is right artinian;
(3) $\mathrm{Rad}(M)$ has ACC on direct summands and R is a commutative max ring;
(4) M is finitely generated and R is commutative.

Proof. The first case follows from [1, Proposition 2], which states that $\mathrm{Rad}(M)$ is noetherian if M has ACC on small submodules, and from [7, Proposition 3.1] which gives a decomposition of M into a semisimple and a noetherian module. The second case follows from Theorem 2.5 and from the fact that hollow modules over artinian rings are noetherian and therefore hopfian. The third case follows from Theorem 2.5 and from the fact that every hollow module H over a max ring R is local and since R is commutative, H is also hopfian. The last case follows from Theorem 2.5 and from the fact that finitely generated modules over commutative rings are hopfian. ∎

Theorem 2.7. *Let R be a right artinian ring or a commutative max ring. Let $M = M_1 \oplus \ldots \oplus M_n$ be a direct sum of relatively projective quasi-discrete right R-modules M_i $(1 \leq i \leq n)$. Then M is a lifting LE-module .*

Proof. M is quasi-discrete by [6, Theorem 2.13]. Then we can write $M = \bigoplus_{i\in I} H_i$, where H_i is hollow for all $i \in I$ by [10, Theorem 4.15]. Thus by the above, M is lifting LE. ∎

3. A Class of Lifting Modules Satisfying (∗)

The following property was considered in [6]:

$M_1 \oplus M_2$ is a lifting right R-module whenever M_1 is a simple right R-module and M_2 is a lifting right R-module.

Let us say that a lifting LE-module M satisfies (∗) if any direct sum of M and a semisimple module is again a lifting module. Every semisimple module satisfies (∗). We shall show that the class of lifting LE-modules that satisfy (∗) are exactly the direct sums of semisimple modules and those hollow LE-modules that are non-split extensions of a semisimple by a simple module.

A family of modules $\{M_i\}_{i\in I}$ is called *locally semi-T-nilpotent (lsTn)* if for any infinitely countable set of non-isomorphisms $\{f_n : M_{i_n} \to M_{i_{n+1}}\}$ with all i_n distinct in I, and for any $x \in M_{i_1}$, there exists a positive integer k (depending on x) such that $f_k \cdots f_1(x) = 0$.

We have Baba and Harada's result:

Theorem 3.1. ([3, Theorem 2]) *Let $M = \bigoplus_{i\in I} M_i$ be an R-module where the M_i's are local modules with local endomorphism rings. Then the following are equivalent:*
(1) M is lifting.
(2) M_i is almost M_j-projective for all $i \neq j$ and $\{M_i\}_{i\in I}$ is lsTn.

Note that any non-zero module is almost S-projective for any simple module S. We first need a technical lemma regarding locally semi-T-nilpotent (lsTn) families of modules.

Lemma 3.2. *Let $\{M_i\}_{i\in I}$ be a family of R-modules and $\{S_j\}_{j\in J}$ a family of simple R-modules. If $\{M_i\}_{i\in I}$ is lsTn, then also the disjoint union $\{M_i\}_{i\in I} \cup \{S_j\}_{j\in J}$ is lsTn.*

Proof. Let us denote $\{L_k\}_{k\in K} := \{M_i\}_{i\in I} \cup \{S_j\}_{j\in J}$ the disjoint union of the two families, where K is the disjoint union of I and J. Let $\{f_n : L_{i_n} \to L_{i_{n+1}}\}$ be a countably infinite family of non-isomorphisms with all i_n pairwise distinct. We may also assume that all f_n are non-zero. Consider the chain of homomorphisms at a position $k > 1$:

$$\cdots L_{i_{k-1}} \xrightarrow{f_{k-1}} L_{i_k} \xrightarrow{f_k} L_{i_{k+1}} \cdots$$

If $L_{i_k} \in \{S_j\}_{j\in J}$ then neither $L_{i_{k-1}}$ nor $L_{i_{k+1}}$ can be simple since otherwise the map f_{k-1} respectively f_k is zero or an isomorphism. Hence $L_{i_{k-1}}, L_{i_{k+1}} \in \{M_i\}_{i\in I}$. Moreover the composition $f_k f_{k-1}$ is a non-isomorphism. We refer to this observation by (†).
Let $\Lambda := \{n \in \mathbb{N} : L_{i_n} \in \{S_j\}_{j\in J}\}$. By (†) we know that $\mathbb{N} \setminus \Lambda$ is a (countably) infinite set. Denote the characteristic function of Λ by $\mathbf{1}_\Lambda$, that is $\mathbf{1}_\Lambda(n) = 1$ if $n \in \Lambda$ and otherwise $\mathbf{1}_\Lambda(n) = 0$. Define a bijection $\varphi : \mathbb{N} \to \mathbb{N} \setminus \Lambda$ recursively by setting

$$\varphi(1) := 1 + \mathbf{1}_\Lambda(1) \text{ and } \varphi(n+1) := (\varphi(n)+1) + \mathbf{1}_\Lambda(\varphi(n)+1) \text{ for all } n \geq 1.$$

(†) guarantees that φ defines a bijection, since the successor of each $k \in \Lambda$ lies in $\mathbb{N} \setminus \Lambda$.

$$\hat{f}_n := \begin{cases} f_{\varphi(n)} & \text{if } \varphi(n)+1 \notin \Lambda \\ f_{\varphi(n)+1} f_{\varphi(n)} & \text{otherwise} \end{cases}$$

By the remark above, property (†) guarantees that we always "jump" correctly over the modules in $\{S_j\}_{j\in J}$, and that all $\hat{f}_n$ are non-isomorphisms. By construction we get that $\{\hat{f}_n : L_{i_{\varphi(n)}} \to L_{i_{\varphi(n+1)}}\}$ is a countably infinite family of non-isomorphisms of modules in $\{M_i\}_{i\in I}$. Since $\{M_i\}_{i\in I}$ is lsTn, for every element $x \in L_{i_{\varphi(1)}}$ there exists a $k \geq 1$ such that $\hat{f}_k \ldots \hat{f}_1(x) = 0$. Hence for all $x \in L_{i_1}$ we also get a $k \geq 1$ such that $f_k \ldots f_1(x) = 0$. Thus $\{L_k\}_{k\in K}$ is lsTn. ∎

With the last lemma and Baba and Harada's Theorem at hand we easily deduce now the following criteria for a lifting LE-module to satisfy $(*)$:

Proposition 3.3. *Let $M = \bigoplus_{i\in I} M_i$ be a lifting LE-module with all M_i's local and let $S := \bigoplus_{j\in J} S_j$ be a semisimple module. Then the following are equivalent:*
(a) $M \oplus S$ is lifting.
(b) S_j is almost M_i-projective for all $j \in J$ and $i \in I$.

Proof. (a) $\Rightarrow$ (b) follows from Baba and Harada's Theorem 3.1.
(b) $\Rightarrow$ (a) Note that any module M is almost S-projective for all simple modules S (this is trivial). Since M was lifting, by Theorem 3.1 we have that M_i is almost M_j-projective for all $i \neq j$. By hypothesis all S_j are almost M_i-projective, hence the family $\{M_i\} \cup \{S_j\}$ is relatively almost-projective. Lemma 3.2 assures that this family is also lsTn since M was lifting and by Theorem 3.1 the family of the M_i's is lsTn. Thus applying once again Baba and Harada's Theorem we get that $M \oplus S$ is lifting. ∎

As an immediate consequence we can state:

Corollary 3.4. *Let $M = \bigoplus_{i\in I} M_i$ be a lifting LE-module with all M_i's local. Then the following statements are equivalent:*
(a) M satisfies $(*)$.
(b) $M \oplus S$ is lifting for any simple module S.
(c) Any simple module S is almost M_i-projective for all $i \in I$.

Before we characterize when $(*)$ holds for all lifting LE-modules of the above type, we show that small M-projectivity and almost M-projectivity for some local module M coincide for simple modules:

Lemma 3.5. *Let M be a local right R-module and S a simple right R-module. Then S is M-projective if and only if S is almost small M-projective.*

Proof. Assume S is almost M-projective. Let $K \subseteq M$ and $f : S \to M/K$ be a non-zero homomorphism such that $f(S)$ is small in M/K. Hence f is monomorphic, but not epimorphic. By hypothesis, there exists either a map $g : S \to M$ such that $f = \pi_K g$ or a map $g : M \to S$ such that $fg = \pi_K$ (since M is indecomposable, M is the only non-zero direct summand). We shall rule out the second possibility. Assume there exists a $g : M \to S$ such that $fg = \pi_K$, then g is epimorphic and factors through $M/\mathrm{Rad}(M)$, that is there exists an isomorphism $\bar{g} : M/\mathrm{Rad}(M) \to S$ such that $g = \bar{g}\pi$ (where $\pi := \pi_{\mathrm{Rad}(M)}$). Hence we get the following commutative diagram:

$$\begin{array}{ccccc} M & \xrightarrow{\pi} & M/\mathrm{Rad}(M) & \longrightarrow & 0 \\ {\scriptstyle id_M}\downarrow & & \downarrow{\scriptstyle f\bar{g}} & & \\ M & \xrightarrow{\pi_K} & M/K & \longrightarrow & 0 \end{array}$$

Thus $\pi_K(\mathrm{Rad}(M)) = f\bar{g}\pi(\mathrm{Rad}(M)) = 0$ and as $\mathrm{Rad}(M)$ is the unique maximal

submodule, $K = \text{Rad}(M)$. But as $M/\text{Rad}(M)$ is simple, $f\bar{g}$ and hence f must be isomorphisms contradicting that $f(S)$ was small in M/K. Therefore we can not have this possibility. Thus S is small M-projective.
Now assume that S is small M-projective and assume we have the following diagram:

$$\begin{tikzcd} & S \arrow[d, "f"] & \\ M \arrow[r, "p"] & F \arrow[r] & 0 \end{tikzcd}$$

where p is an epimorphism and f is non-zero. If f is epimorphic, then f is an isomorphism and we have $g := f^{-1}p : M \longrightarrow S$ a non-zero map from a direct summand of M to S such that $fg = p$. If f is not epimorphic, then $f(S)$ must be small since F is local. As p factors through $M/\text{Ker}p$, there exists an isomorphism $\bar{p} : M/\text{Ker}p \to F$ such that $p = \bar{p}\pi$ (where $\pi := \pi_{\text{Ker}p}$). Hence $\bar{p}^{-1}f$ is a map from S to $M/\text{Ker}p$ with small image and by hypothesis there exists a map $g : S \to M$ such that $\pi g = \bar{p}^{-1}f$. Thus $pg = \bar{p}\pi g = \bar{p}\bar{p}^{-1}f = f$ shows that g is our desired map. Hence S is almost M-projective. ∎

Analyzing the proof of [6, Proposition 4.1] we also have:

Proposition 3.6. *Let M be a lifting right R-module with local endomorphism ring such that $M \oplus S$ is lifting for all simple subfactors S of M, then M is simple or local with* $\text{Rad}(M) = \text{Soc}(M)$.

Let $\mathcal{C}$ be the class of semisimple and $\mathcal{S}$ the class of simple modules. By $\text{Ext}(\mathcal{C}, \mathcal{S})$ we denote the isomorphism classes of extensions of $\mathcal{C}$ and $\mathcal{S}$, i.e the isomorphism classes of all R-modules M such that there exists an exact sequence

$$0 \to C \to M \to S \to 0 \ldots (\dagger)$$

with $C \in \mathcal{C}$ and $S \in \mathcal{S}$. If every exact sequence $(\dagger)$ splits, we will write $\text{Ext}(\mathcal{C}, \mathcal{S}) = 0$.

Corollary 3.7. *A non-simple hollow LE-module M satisfies $(*)$ if and only if M is an extension of a semisimple and a simple module, i.e.* $M \in \text{Ext}(\mathcal{C}, \mathcal{S})$.

Proof. The necessity follows from 3.6. Assume that M is a non-simple hollow LE-module such that there exists a semisimple module $C \subseteq M$ with M/C simple. Since M is hollow and M/C simple, C is the unique maximal submodule of M. Hence M is local. Let S be a simple module. If $S \notin \sigma[M]$ then $\text{Hom}(S, F) = 0$ for any factor module F of M and hence S is almost M-projective. Now assume $S \in \sigma[M]$, then S is a submodule of a factor module M/N of M. Either $M/N = S$ and hence $N = C$. Or $N \subseteq C$ and there exists a direct summand $L \subseteq C$ such that $C = N \oplus L$. In the latter case $L \simeq C/N = \text{Soc}(M/N)$, as C/N is maximal in M/N. As $S \subseteq Soc(M/N)$, there exists an isomorphism $\phi : S \longrightarrow S' \subseteq L$. Hence either S is a factor module of M or S is isomorphic to a submodule of M. Thus S is almost M-projective. By 3.4 M satisfies $(*)$. ∎

Corollary 3.8. *A module M is a lifting LE–module satisfying $(*)$ if and only if $M = C \oplus \bigoplus_{i\in I} L_i$ with C semisimple, L_i hollow LE-modules that are extensions of a semisimple and a simple module such that L_i is almost L_j-projective for all i,j.*

Now we can characterize when $(*)$ holds for all lifting LE-modules that are direct sums of locals.

Proposition 3.9. *Let R be a ring. Then the following are equivalent:*

(a) *Any lifting LE right R-module $M = \bigoplus_{i\in I} M_i$ with all M_i's local satisfies $(*)$.*
(b) *Any finitely generated lifting LE right R-module satisfies $(*)$.*
(c) *Any direct sum of a simple right R-module and a local right R-module with local endomorphism ring is lifting.*
(d) *Any local right R-module M with local endomorphism ring is simple or* $\mathrm{Rad}(M) = \mathrm{Soc}(M)$.
(e) *Any simple right R-module is small M-projective for every local right R-module M with local endomorphism ring.*
(f) *Any simple right R-module is almost M-projective for every local right R-module M with local endomorphism ring.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is trivial.
(c) $\Rightarrow$ (d) follows from Proposition 3.6.
(d) $\Rightarrow$ (e) Let M be a local module with local endomorphism ring and S a simple module. Assume that there is a non-zero homomorphism $f : S \to M/K$ with $f(S)$ small in M/K and K a submodule of M. Since $\mathrm{Rad}(M) = \mathrm{Soc}(M)$ is the unique maximal submodule of M, we have $K \subseteq \mathrm{Soc}(M)$. Let L be a direct summand of K in $\mathrm{Soc}(M)$, then $L \cong (L\oplus K)/K = \mathrm{Soc}(M)/K = \mathrm{Rad}(M)/K = \mathrm{Rad}(M/K)$. Denote by $g : \mathrm{Rad}(M/K) \to L$ the natural isomorphism and denote by $\pi_K : M \to M/K$ the canonical projection. Then $\pi_K \circ g = id_{\mathrm{Rad}(M/K)}$ and hence $h := g\circ f : S \to M$ has the property that $\pi_K \circ h = \pi_K \circ g \circ f = id_{\mathrm{Rad}(M/K)} \circ f = f$. Hence S is small M-projective.
(e) $\Leftrightarrow$ (f) follows from Lemma 3.5.
(f) $\Rightarrow$ (a) follows from Corollary 3.4. ∎

Recall that a ring R is called *semilocal* if $R/\mathrm{Jac}R$ is semisimple. A semilocal ring with nilpotent Jacobson radical is called *semiprimary.* It is well-known that semiprimary rings are right and left perfect.

Theorem 3.10. *Let R be a ring. Then the following are equivalent:*

(a) *Every lifting LE right R-module satisfies $(*)$ and R is right perfect.*
(b) *Every finitely generated lifting LE right R-module satisfies $(*)$ and R is semiperfect.*
(c) *Any direct sum of a simple and a projective local right R-module is lifting and R is semiperfect.*
(d) $(\mathrm{Jac}R)^2 = 0$ *and R is semilocal.*
(e) *The left-version of* (a), (b) *or* (c) *hold.*

Proof. (a) $\Rightarrow$(b) $\Rightarrow$(c) is trivial.
(c) $\Rightarrow$ (d) As R is semiperfect $R = \bigoplus_{i=1}^{n} e_i R$ with e_i primitive idempotents. Let J denote the Jacobson radical of R. By hypothesis any direct sum of a simple and e_iR is lifting, hence by Proposition 3.6, e_iR is simple or $\mathrm{Rad}(e_iR) = e_iJ = \mathrm{Soc}(e_iR)$. Hence $J = \bigoplus_{i=1}^{n} e_iJ \subseteq \mathrm{Soc}(R_R) = l.ann(J)$ by [2, Proposition 15.17]. Thus $J^2 = 0$.
(d) $\Rightarrow$ (a) Semiprimary rings are right perfect. Moreover any hollow right R-module over a right perfect ring R is local. So all lifting LE-modules can be written as a direct sum of locals with local endomorphisms. Let L be a local, non-simple right R-module with local endomorphism ring. As R is semilocal $\mathrm{Rad}(L) = LJ$ and as $J^2 = 0$ it follows that $\mathrm{Rad}(L)J = 0$. Hence $\mathrm{Rad}(L) = \mathrm{Soc}(L)$. By Proposition 3.9, all lifting LE-right R-modules satisfy $(*)$.
Since (d) is left-right independent, we also get the equivalence of left versions of (a) - (c) with (d). ■

Finally we classify those rings for which every lifting LE-module that satisfies $(*)$ is semisimple. Let $\mathrm{Ext}(A, B) = \mathrm{Ext}(\{A\}, \{B\})$ denote the isomorphism classes of module extensions of the module A by the module B. Recall that $\mathrm{Ext}(A, B) = 0$ means that every extension of A by B splits.

Theorem 3.11. *The following statements are equivalent for a ring R:*
(a) *Every lifting LE-module that satisfies $(*)$ is semisimple.*
(b) *Every hollow LE-module that satisfies $(*)$ is simple.*
(c) *Every hollow module is either simple or has infinite length.*
(d) $\mathrm{Ext}(S, T) = 0$ *for any pair of simples S and T.*

Proof. (a) $\Rightarrow$ (b) is obvious.
(b) $\Rightarrow$ (c) Let M be a hollow module having length 2. Then M is hopfian by [2, 11.6] and by 2.2 M is discrete. Thus M has a local endomorphism ring by [10, 5.5]. Since M has length 2, M is non-simple and an extension of two simple modules. By 3.7, M satisfies $(*)$ and by hypothesis (b) M is simple – a contradiction to M having length 2. Thus there is no hollow module of length 2. Any hollow module of finite length ≥ 2 has a hollow factor module of length 2. Therefore there is also no hollow module of finite length ≥ 2 and every hollow module is either simple or has infinite length.
(c) $\Rightarrow$ (d) Assume M is an extension of S by T for some simples S and T:

$$0 \longrightarrow S \xrightarrow{\ f\ } M \xrightarrow{\ g\ } T \longrightarrow 0$$

We identify S with its image in M under f. If S is small in M, then M is a local module of length 2 what is impossible by (c). Therefore S is not small in M and hence a direct summand. Thus f splits and shows $\mathrm{Ext}(S, T) = 0$.
(d) $\Rightarrow$ (a) It is enough to consider local non-simple LE-modules M that satisfy $(*)$. By Corollary 3.7 M is an extension of a semisimple module C by a simple module T. Take a simple submodule S of C with $C = S \oplus X$. Then we have an exact sequence:

$$0 \longrightarrow S \longrightarrow M/X \longrightarrow T \longrightarrow 0$$

and $M/X \in \mathrm{Ext}(S,T)$. By (d) this sequence splits contradicting the fact that M/X is hollow. Hence every local LE-module that satisfies $(*)$ is simple and thus every lifting LE-module satisfying $(*)$ is semisimple. ∎

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