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On Eventually and Asymptotically Lipschitzian Mappings*

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Abstract. In this note, first we establish a fixed point theorem for eventually lipschitzian mappings, next we prove a fixed point theorem for mappings of uniformly lipschitzian type and finally, we make some remarks about asymptotically lipschitzian and asymptotically nonlipschitzian mappings.

1. Introduction

Let C be a bounded closed convex subset of a Banach space $(X, \|.\|)$. A mapping $T : C \to C$ is called uniformly lipschitzian if there is a constant k such that

$$\|T^n x - T^n y\| \le k\|x - y\| \tag{1}$$

for all $n = 1, 2, ...$ and all x and y in C. Such mappings are originally considered by Goebel and Kirk [7] and then by Goebel, Kirk and Thele [8]. They showed that if the solution γ of the equation $\gamma[1 - \delta_X(1/\gamma)] = 1$ is greater than one and the characteristic of convexity $\varepsilon_0(X) < 1$ then every uniformly k-lipschitzian mapping has a fixed point in C whenever $k < \gamma$ (where δ_X stands for the modulus of convexity). Later, having introduced a characteristic $\kappa(M)$ of a metric space (M, d), Lifschitz showed that in a metric space setting, every uniformly k-lipschitzian mapping has a fixed point whenever $k < \kappa(M)$ [13]. This result was then generalized by Gornicki and Kruppel for mappings satisfying (1) only for n in a subset A of the set of all natural numbers N with the Banach density

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$\mu(A) > 1/2$ [10]. Here we employ the same method used in [13] to establish a fixed point theorem for mappings satisfying (1) for all $n \geq n_0$, where n_0 is an arbitrary fixed natural number. As corollaries we get some fixed point results for asymptotically lipschitzian mappings, asymptotically nonexpansive mappings in a metric space setting.

In [9] Kirk has introduced the notion of mappings of asymptotically nonexpansive type and established a fixed point theorem for such mappings. In Sec. 4 we generalize this result for a wider class of mappings so called mappings of uniformly lipschitzian type.

The last part of this note is concerned with compact metric spaces. In [4] Freudenthal and Hurewicz have proved that in such spaces every surjective nonexpansive mapping must be an isometry, and every expansive mapping must be a surjective isometry. Later in [8] Goebel, Kirk and Thele generalized the first result of Freudenthal and Hurewicz for asymptotically nonexpansive mappings. This result encourages us to introduce the notions of asymptotically expansive, asymptotically lipschitzian and nonlipschitzian mappings and establish similar results for such mappings.

2. Preliminaries

In this section we recall some definitions which we shall use below.

Definition 1. *Let (M, d) be a metric space. A mapping $T : M \to M$ is called asymptotically k-lipschitzian if there exists a sequence of positive numbers $\{k_n\}$ converging to $k > 0$ such that*

$$d(T^n x, T^n y) \leq k_n d(x, y)$$

for all $n = 1, 2, \ldots$ and all x, y in M.

In particular, if $k_n \equiv k$ for every $n \in \mathbb{N}$ we get the notion of uniformly k-lipschitzian mappings.

Definition 2. *If in Definition 1 we have $k = 1$ then we get the notion of asymptotically nonexpansive mappings.*

Definition 3. *If in Definitions 1 and 2 we have the converse inequalities then we get the notions of asymptotically k-nonlipschitzian and asymptotically expansive mappings, respectively.*

Definition 4. *A mapping $T : M \to M$ is called eventually k-lipschitzian if there is a positive number k and a natural number n_0 such that*

$$d(T^n x, T^n y) \leq k\, d(x, y) \tag{2}$$

for all $n \geq n_0$ and all $x, y \in M$.

Clearly, if $n_0 = 1$ this notion coincides with that of uniformly k-lipschitzian mappings.

Definition 5. *The Lifschitz characteristic of a metric space (M, d) is defined as follows*

$$\kappa(M) = sup\{\beta > 0 : \exists \alpha > 1 \text{ such that } \forall\, x, y \in M \text{ and } r > 0,\ d(x, y) > r$$
$$\Rightarrow \exists z \in M \text{ such that } B(x, \beta r) \cap B(y, \alpha r) \subset B(z, r)\},$$

where $B(z, r)$ denotes the closed ball of radius r centered at z.

The Lifschitz constant $\kappa_0(X)$ of a Banach space $(X, \|.\|)$ is defined to be the infimum of $\kappa(C)$ where C ranges over all nonempty closed bounded convex subsets of X.

Definition 6. *The modulus of convexity of a Banach space X is the function $\delta_X : [0, 2] \to [0, 1]$ defined by*

$$\delta_X(\varepsilon) = \inf\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1,\ \|y\| \le 1,\ \|x - y\| \ge \varepsilon\}.$$

The characteristic of convexity of X, $\varepsilon_0(X)$ is then defined to be

$$\sup\{\varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0\}.$$

It is known [5] that $\varepsilon_0(X) = 0$ if and only if X is uniformly convex, while if $\varepsilon_0(X) < 2$ then X is uniformly non-square [12] and isomorphic to a uniformly convex space [3], hence reflexive. It is also known [11, 15] that the function δ_X is strictly increasing on $[\varepsilon_0(X), 2]$ and continuous on $[0, 2)$, and moreover [16, 17] for any $d > 0$,

$$\|x\| \le d,\ \|y\| \le d,\ \|x - y\| \ge \varepsilon \Rightarrow \left\|\frac{x+y}{2}\right\| \le (1 - \delta_X(\varepsilon/d))\, d. \tag{3}$$

In [2] Downing and Turett proved that for every Banach space we have $\varepsilon_0(X) < 1$ if and only if $\kappa_0(X) > 1$.

Definition 7. *Let $\{x_n\}$ be a bounded sequence in a Banach space X, and C a closed convex subset of X. For each $x \in X$ we denote*

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|.$$

The asymptotic radius of $\{x_n\}$ with respect to C is defined by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\},$$

and the asymptotic center of $\{x_n\}$ with respect to C is

$$A(C, \{x_n\}) = \{z \in C : r(z, \{x_n\}) = r(C, \{x_n\})\}.$$

It is known that if C is weakly compact (resp., convex) then $A(C, \{x_n\})$ is nonempty (resp., convex). Moreover, if X is uniformly convex then $A(C, \{x_n\})$ is a singleton.

Definition 8. *Let C be a subset of a Banach space X. A mapping $T: C \to C$ is said to be of uniformly k-lipschitzian type if for each $x \in C$, the sequence $\{c_n(x)\}$ with*

$$c_n(x) = \max\{\sup_{y \in C}\{\|T^n x - T^n y\| - k\|x - y\|\}, 0\}$$

tends to 0 as $n \to \infty$.

If $k = 1$, this reduces to the notion of mappings of asymptotically nonexpansive type introduced by Kirk in [9]. It is clear that each uniformly k-lipschitzian mapping introduced by Goebel and Kirk in [7] is of uniformly k-lipschitzian type.

3. A Fixed Point Theorem for Eventually Lipschitzian Mappings

Now we employ the method used in [13] to prove our first result.

Theorem 1. *Let M be a complete bounded metric space and $T: M \to M$ an eventually k-lipschitzian mapping with $k < \kappa(M)$. Then T has a fixed point in M.*

Proof. For each $y \in M$ we define

$$r(y) = \inf\{R > 0, \exists x \in M \text{ such that } \{T^n x\}_{n \geq n_0} \subset B(y, R)\}$$

with n_0 mentioned in Definition 4. It is obvious that if $Ty = y$ then $r(y) = 0$. To prove the converse, let $r(y) = 0$ and $\varepsilon > 0$. Then there exists $x \in M$ such that $\{T^n x\}_{n \geq n_0} \subset B(y, \varepsilon)$. For every $n \geq n_0$ we have

$$d(T^n y, y) \leq d(T^n y, T^{2n} x) + d(T^{2n} x, y) \leq k\, d(y, T^n x) + \varepsilon \leq (k+1)\varepsilon.$$

Hence $T^n y = y$ for every $n \geq n_0$, from this $Ty = y$.

Now we take any β such that $k < \beta < \kappa(M)$. By definition of $\kappa(M)$, there is $\alpha > 1$ such that $d(u, v) > \rho \Rightarrow \exists w \in M$ such that

$$B(u, \beta\rho) \cap B(v, \alpha\rho) \subset B(w, \rho). \tag{4}$$

Choose $\lambda \in (0, 1)$ such that $\gamma = \min\{\alpha\lambda, \beta\lambda/k\} > 1$. We claim that there exists a sequence $\{y_m\} \subset M$ satisfying

$$r(y_{m+1}) \leq \lambda r(y_m) \text{ and } d(y_m, y_{m+1}) \leq (\lambda + \gamma) r(y_m) \tag{5}$$

for all $m = 1, 2, ...$

Take any $y_1 \in M$ and supposing that $y_1, ..., y_m$ are found, we shall define y_{m+1} as follows. If $r(y_m) = 0$ we put $y_{m+1} = y_m$. If $r(y_m) > 0$ then $\lambda r(y_m) < r(y_m)$. By definition of $r(y_m)$, for every $x \in M$ there is $n_1 \geq n_0$ such that

$$d(T^{n_1} x, y_m) > \lambda r(y_m).$$

In particular, for $x = y_m$ we get $d(T^{n_1} y_m, y_m) > \lambda r(y_m)$. On the other hand, since $\lambda r(y_m) > r(y_m)$, by definition of $r(y_m)$ again, there exists $x_0 \in M$ such that

$$\{T^n x_0\}_{n\geq n_0} \subset B(y_m, \gamma r(y_m)).$$

Putting $\tilde{x} = T^{n_1}x_0$ we consider the sequence $\{T^n\tilde{x}\}_{n\geq n_0}$. For every $n \geq n_0$ we have

$$d(T^n\tilde{x}, T^{n_1}y_m) = d(T^{n+n_1}x_0, T^{n_1}y_m) \leq k\, d(T^n x_0, y_m) < k\gamma r(y_m) \leq \beta\gamma r(y_m),$$

$$d(T^n\tilde{x}, y_m) = d(T^{n+n_1}x_0, y_m) \leq \gamma r(y_m) \leq \alpha\lambda r(y_m).$$

By putting $u = T^{n_1}y_m$, $v = y_m$, $\rho = \lambda r(y_m)$ and using (3) we get

$$\{T^n\tilde{x}\}_{n\geq n_0} \subset B(T^{n_1}y_m, \beta\lambda r(y_m)) \cap B(y_m, \alpha\lambda r(y_m)) \subset B(w, \lambda r(y_m)) \qquad (6)$$

with some $w \in M$. Defining $y_{m+1} = w$, from (6) we get immediately

$$r(y_{m+1}) \leq \lambda r(y_m).$$

Moreover, for any $n \geq n_0$ we have

$$d(y_m, w) \leq d(y_m, T^n\tilde{x}) + d(T^n\tilde{x}, w) \leq (\alpha+\gamma)r(y_m),$$

so y_{m+1} satisfies (5).

From (5) it is easy to see that $\{y_m\}$ is a Cauchy sequence in M, hence it converges to some point $y \in M$. Again from (5) we get $r(y) = 0$ and equivalently, $Ty = y$. The proof is complete. ∎

Remark 1. Since each uniformly k-lipschitzian mapping is eventually k-lipschitzian, Theorem 1 improves Lifschitz's theorem mentioned in Introduction. Moreover since each asymptotically k-lipschitzian mapping is eventually k'-lipschitzian with $k < k' < \kappa(M)$, we obtain

Corollary 1. *Let M be a complete bounded metric space and $T : M \to M$ an asymptotically k-lipschitzian mapping with $k < \kappa(M)$. Then T has a fixed point in M.*

In particular for $k = 1$ and $\kappa(M) > 1$ we get

Corollary 2. *Let M be a complete bounded metric space with $\kappa(M) > 1$ and $T : M \to M$ an asymptotically nonexpansive mapping. Then T has a fixed point in M.*

In particular, each nonexpansive mapping in such a space has a fixed point. As a direct consequence of Theorem 1 in a Banach space setting, we have

Corollary 3. *Let C be a bounded closed convex subset of a Banach space X with $\kappa_0(X) > 1$ and $T : C \to C$ be an eventually k-lipschitzian mapping with $k < \kappa_0(X)$. Then T has a fixed point in C.*

Note that there are in literature some results similar to ours in a Banach space setting (with $\varepsilon_0(X) < 1$ instead of $\kappa_0(X) > 1$) due to Goebel and Kirk [6], Lim and Xu [14], Casini and Maluta [1],...

4. A Fixied Point Theorem for Mappings of Uniformly Lipschitzian Type

Modifying the method used by Kirk in [9] we can prove the following result.

Theorem 2. *Let C be a bounded closed convex subset of a Banach space X with $\varepsilon_0(X) < 1$ and let $T : C \to C$ be a continuous mapping of uniformly k-lipschitzian type with $k < \gamma$, where γ is the solution to the equation $\gamma(1 - \delta_X(1/Y)) = 1$. Then T has a fixed point.*

Proof. It is not difficult to show that $\varepsilon_0(X) < 1$ implies $\gamma > 1$ (see also [2]) so we may assume that $k \geq 1$. Now taking any x_0 in C we denote $x_n = T^n x_0$ for $n = 1, 2, \ldots$ Since $\varepsilon_0(X) < 1$, X is reflexive, hence C is weakly compact and $A(C, \{x_n\})$ is nonempty. Take any $z_1 \in A(C, \{x_n\})$ and denote $r_1 = r(z_1, \{x_n\})$. By Definition 8 we have

$$\|T^n x - T^n y\| \leq k\|x - y\| + c_n(x), \quad \forall x, y \in C, \ \forall n \geq 1.$$

So for any $m \geq 1$ we have

$$\begin{aligned}\limsup_{n\to\infty} \|T^n x_0 - T^m z_1\| &=\leq k \limsup_{n\to\infty} \|T^{n-m} x_0 - z_1\| + c_m(z_1)\\ &= kr_1 + c_m(z_1). \qquad (7)\end{aligned}$$

Since C is convex, it is clear that

$$r_1 \leq \limsup_{n\to\infty} \Big\|T^n x_0 - \frac{z_1 + T^m z_1}{2}\Big\|, \ \forall m \geq 1. \qquad (8)$$

On the other hand, for each $m \geq 1$, we have

$$\Big\|T^n x_0 - \frac{z_1 + T^m z_1}{2}\Big\| = \frac{1}{2}\big\|(T^n x_0 - z_1) + (T^n x_0 - T^m z_1)\big\|.$$

By definition of r_1 and from (7) we get

$$\limsup_{n\to\infty} \|T^n x_0 - z_1\| = r_1 \leq kr_1 + c_m(z_1),$$

$$\limsup_{n\to\infty} \|T^n x_0 - T^m z_1\| \leq kr_1 + c_m(z_1).$$

From (3) and (8) we get

$$\begin{aligned} r_1 &\leq \limsup_{m\to\infty} \limsup_{n\to\infty} \Big\|T^n x_0 - \frac{z_1 + T^m z_1}{2}\Big\|\\ &\leq \limsup_{m\to\infty} (kr_1 + c_m(z_1))\Big[1 - \delta_X\Big(\frac{\|z_1 - T^m z_1\|}{kr_1 + c_m(z_1)}\Big)\Big]\\ &= kr_1\Big[1 - \delta_X\Big(\limsup_{m\to\infty} \frac{\|z_1 - T^m z_1\|}{kr_1}\Big)\Big]. \qquad (9)\end{aligned}$$

If $r_1 = r(z_1, \{x_n\}) = 0$ then $T^n x_0 \to z_1$ and, since T is continuous, $Tz_1 = z_1$. Thus, we may assume $r_1 > 0$. From (9) we get

$$\frac{1}{k} \leq 1 - \delta_X\Big(\limsup_{m\to\infty} \frac{\|z_1 - T^m z_1\|}{kr_1}\Big),$$

hence

$$\delta_X\Big(\limsup_{m\to\infty} \frac{\|z_1 - T^m z_1\|}{kr_1}\Big) \leq 1 - \frac{1}{k}. \tag{10}$$

If $\delta_X\Big(\limsup_{m\to\infty} \frac{\|z_1 - T^m z_1\|}{kr_1}\Big) = 0$, then by definition of $\varepsilon_0(X)$ we get

$$\limsup_{m\to\infty} \|z_1 - T^m z_1\| \leq \varepsilon_0(X) kr_1. \tag{11}$$

We show that $\varepsilon_0(X)k < 1$, for this it suffices to prove that $\varepsilon_0(X)\gamma \leq 1$. Suppose on the contrary that $\varepsilon_0(X) > 1$, then we have

$$1 > \varepsilon_0(X) > \frac{1}{\gamma} = 1 - \delta_X(\frac{1}{\gamma}).$$

Consequently, $\delta_X(1/\gamma) > 0$ and hence, $1/\gamma \geq \varepsilon_0(X)$ therefore we get $1 \geq \varepsilon_0(X)\gamma$, a contradiction to the above assumption.

Now suppose that $\delta_X\Big(\limsup_{m\to\infty} \frac{\|z_1 - T^m z_1\|}{kr_1}\Big) > 0$. Then

$$\limsup_{m\to\infty} \frac{\|z_1 - T^m z_1\|}{kr_1} \geq \varepsilon_0(X).$$

Since the function δ_X restricted on $[\varepsilon_0(X), 2]$ is invertible and strictly increasing, from (10) we obtain

$$\limsup_{m\to\infty} \frac{\|z_1 - T^m z_1\|}{kr_1} \leq \delta_X^{-1}(1 - \frac{1}{k}) < \delta_X^{-1}(1 - \frac{1}{\gamma}) = \frac{1}{\gamma}.$$

This implies

$$\limsup_{m\to\infty} \|z_1 - T^m z_1\| < \frac{kr_1}{\gamma}. \tag{12}$$

Denoting $\alpha = \max\{\varepsilon_0(X)k, \frac{k}{\gamma}\} < 1$, from (11) and (12) in any case we have

$$\limsup_{m\to\infty} \|z_1 - T^m z_1\| \leq \alpha r_1. \tag{13}$$

Now take any $z_2 \in A(C, \{T^n z_1\})$ and denote $r_2 = r(z_2, \{T^n z_1\})$. So we have from (13)

$$r_2 = r(C, \{T^n z_1\}) \leq r(z_1, \{T^n z_1\}) = \limsup_{m\to\infty} \|z_1 - T^m z_1\| \leq \alpha r_1.$$

Continuing this process we get a sequence $\{z_m\} \subset C$ satisfying

(a) $z_m \in A(C, \{T^n z_{m-1}\})$,

(b) $r_m = r(z_m, \{T^n z_{m-1}\}) = r(C, \{T^n z_{m-1}\})$,

(c) $r_m \leq \alpha r_{m-1}$,

(d) $\limsup_{n\to\infty}\|z_m - T^n z_m\| \le \alpha r_m$.
From this we get

$$\begin{aligned}\|z_{m+1} - z_m\| &\le \limsup_{n\to\infty}\{\|T^n z_m - z_{m+1}\| + \|T^n z_m - z_m\|\} \\ &\le r_{m+1} + \alpha r_m \le 2\alpha r_m \le 2\alpha^m r_1 .\end{aligned}$$

Since $\alpha < 1$, $\{z_m\}$ is a Cauchy sequence, hence it converges to some point $z^* \in C$. From (c), (d) and the inequality

$$\begin{aligned}\|z^* - T^n z^*\| &\le \|z^* - z_m\| + \|z_m - T^n z_m\| + \|T^n z_m - T^n z^*\| \\ &\le (1+k)\|z^* - z_m\| + \|z_m - T^n z_m\| + c_n(z^*)\end{aligned}$$

we get

$$\limsup_{m\to\infty}\|z^* - T^n z^*\| = 0,$$

and so z^* is a fixed point of T. The proof is complete. ■

Remark 2. For $k = 1$, Theorem 1 reduces to a fixed point theorem of Kirk in [9] for mappings of asymptotically nonexpansive type. In particular, it implies a fixed point theorem of Goebel and Kirk for uniformly lipschitzian mappings in [7].

Remark 3. Since in Definition 8 we require only $c_n(x) \to 0$ as $n \to \infty$, the class of mappings of uniformly k-lipschitzian type contains also the class of eventually k-lipschitzian mappings, i.e. mappings satisfying the inequality

$$\|T^n x - T^n y\| \le k\|x - y\|, \ \forall x, y,$$

for all n greater than some n_0. This class in turn contains the class of asymptotically k'-lipschitzian mappings with $k' < k$, i.e. mappings satisfying

$$\|T^n x - T^n y\| \le k_n\|x - y\|, \ \forall x, y,$$

with $k_n \to k'$. So Theorem 1 also implies the corresponding results for the above mentioned classes of mappings, in particular for asymptotically nonexpansive mappings studied in [6].

Remark 4. If in the formula of $c_n(x)$ in Definition 8 we replace k by k_n converging to k then we get the definition of mappings of asymptotically k-lipschitzian type. Slightly modifying the proof of Theorem 2 we can get a similar result for mappings of this type.

5. Lipschitzian and Nonlipschitzian Mappings in Compact Metric Spaces

The first result of this section is a generalization of the first result of Freudenthal and Hurewicz to asymptotically lipschitzian mappings.

Theorem 3. *Let M be a compact metric space and $T : M \to M$ an asymptotically k-lipschitzian surjective mapping. Then T^{-1} exists and is a k-lipschitzian mapping with $k \geq 1$.*

Proof. Following the idea of Goebel, Kirk and Thele in the proof of Theorem 3.2 in [8] we construct two sequences $\{x_n\}$, $\{y_n\}$ with arbitrary x_0, y_0 in M by putting $Tx_{n+1} = x_n$, $Ty_{n+1} = y_n$ for $n = 0, 1, 2, \ldots$
By compactness of M for every $\varepsilon > 0$ there are m, n with $m > n$ such that

$$d(x_n, x_m) < \varepsilon \text{ and } d(y_n, y_m) < \varepsilon,$$

where $N = m - n$ can be assumed arbitrarily large. Then we have

$$d(x_n, x_m) \geq k_m^{-1} d(T^m x_m, T^m x_n) = k_m^{-1} d(x_0, T^N x_0).$$

Analogously we obtain

$$d(x_n, x_m) \geq k_m^{-1} d(y_0, T^N y_0).$$

Hence

$$\begin{aligned} d(Tx_0, Ty_0) &\geq k_{N-1}^{-1} d(T^N x_0, T^N y_0) \\ &\geq k_{N-1}^{-1} [d(x_0, y_0) - d(x_0, T^N x_0) - d(y_0, T^N y_0)] \\ &\geq k_{N-1}^{-1} [d(x_0, y_0) - 2\varepsilon k_m]. \end{aligned}$$

Since $k_N \to k > 0$ and ε is arbitrarily small, we get

$$d(x, y) \leq k\, d(Tx, Ty) \tag{14}$$

for all $x, y \in M$. From this it is easy to get $k \geq 1$, for if $k < 1$ then we get a contradiction to the compactness of M : $d(x, y) < d(Tx, Ty)$ for every $x, y \in M$, in particular for such x', y' that $d(x', y') = \operatorname{diam} M$, the diameter of M.
From (14) we see that T is injective, and being surjective, its inverse T^{-1} exists. Also from (14) we get

$$d(T^{-1}x, T^{-1}y) \leq k\, d(x, y)$$

for all x, y in M, i.e. T^{-1} is k-lipschitzian. The proof is complete. ■

Remark 5. If $k = 1$ then T^{-1} is nonexpansive and by the first result of Freudenthal and Hurewicz, T^{-1} is an isometry and so is T. Thus we get Theorem 3.2 of Goebel, Kirk and Thele in [8] for asymptotically nonexpansive mappings.

Remark 6. Applying Theorem 3 to the mapping T^n for every $n \in \mathbb{N}$ we get

$$d(T^{-n}x, T^{-n}y) \leq k\, d(x, y) \text{ for } n \in \mathbb{N},$$

so T^{-1} is also uniformly k-lipschitzian. Thus if $k < \kappa(M)$ then by Lifschitz's theorem, T^{-1} has a fixed point, and so does T. Note that the last conclusion can be obtained by directly applying our Theorem 1 because each asymptotically lipschitzian mapping is eventually lipschitzian.

We conclude this note with a result generalizing the second result of Freudenthal and Hurewicz for asymptotically k-nonlipschitzian mappings.

Theorem 4. *Let M be a compact metric space and $T : M \to M$ an asymptotically k-nonlipschitzian mapping. Then T is a surjective lipschitzian mapping.*

Proof. First we prove that $T(M)$ is dense in M. Take any x in M, we construct the iterate sequence $\{x_n\}$ with $x_n = T^n x$, $n = 1, 2, \ldots$
By compactness of M there exists a convergent, hence Cauchy, subsequence $\{x_{n_j}\}$: for every $\varepsilon > 0$ there is $j \in \mathbb{N}$ such that $d(x_{n_i}, x_{n_k}) < \varepsilon$ for every $i, k \geq j$. Fixing such i, k and assuming $k > i$, putting $N = n_k - n_i$, we get

$$\varepsilon > d(x_{n_i}, x_{n_k}) = d(T^{n_i}x, T^{n_k}x) \geq k_{n_i}\, d(x, T^N x).$$

Since $k_{n_i} \to k > 0$ we may assume that $k_{n_i} \geq \alpha > 0$ for $i \geq j$. Consequently, $d(x, T^N x) < \varepsilon/\alpha$. Because x is arbitrary in X, ε is arbitrarily small, $T^N x \in T(M)$, we obtain $\overline{T(M)} = M$.

Now we take arbitrary x_0, y_0 in M and put $x_n = T^n x_0$, $y_n = T^n y_0$, $n = 1, 2, \ldots$ then construct a sequence $\{z_n\} \subset M \times M$ with $z_n = (x_n, y_n)$. By compactness of $M \times M$, there exists a convergent subsequence $\{z_{n_i}\}$: for every $\varepsilon > 0$ there is $j \in \mathbb{N}$ such that

$$d(x_{n_i}, x_{n_k}) < \frac{\varepsilon}{2}, \quad d(y_{n_i}, y_{n_k}) < \frac{\varepsilon}{2}$$

for all i, $k \geq j$. Defining N as above we have

$$\frac{\varepsilon}{2} > k_{n_i}\, d(x_0, T^N x_0) \text{ and } \frac{\varepsilon}{2} > k_{n_i}\, d(y_0, T^N y_0),$$

hence

$$\varepsilon > k_{n_i}[d(x_0, T^N x_0) + d(y_0, T^N y_0)].$$

From this we get

$$\begin{aligned}\varepsilon + k_{n_i}\, d(x_0, y_0) &> k_{n_i}[d(x_0, T^N x_0) + d(x_0, y_0) + d(y_0, T^N y_0)].\\ &\geq k_{n_i}\, d(T^N x_0, T^N y_0) \geq k_{n_i}\, k_{N-1} d(Tx_0, Ty_0).\end{aligned}$$

Since $k_{n_i} \to k$, $k_{N-1} \to k$ and ε is arbitrarily small, we have

$$d(Tx, Ty) \geq k^{-1}\, d(x, y)$$

for all x, y in M. Thus T is lipschitzian and hence continuous, consequently $T(M)$ is compact. Hence $T(M) = \overline{T(M)} = M$ and the proof is complete. ∎

Remark 7. If $k \geq 1$ then T is nonexpansive, hence by the first result of Freudenthal and Hurewicz, T is an isometry and we get again Theorem 3 for asymptotically expansive mappings.

Remark 8. Applying Theorem 4 to the mapping T^n for every $n \in \mathbb{N}$ we get that T is uniformly k^{-1}-lipschitzian. Thus if $k^{-1} < \kappa(M)$ then T has a fixed point by Lifschitz's theorem.

Remark 9. With $k = 1$, Theorem 4 reduces to the following result:

Let M be a compact metric space and $T : M \to M$ an asymptotically expansive mapping. Then T is a surjective isometry.

Indeed, in this case we have that T is a surjective nonexpansive mapping, so the conclusion follows from the first Freudenthal–Hurewicz's theorem. This is a generalization of the second result of Freudenthal and Hurewicz for asymptotically expansive mappings.

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