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# A Stable Marching Difference Scheme for an Ill-Posed Cauchy Problem for the Three-Dimensional Laplace Equation

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**Abstract.** In this paper we prove some stability estimates of Hölder type for the solution (and all of its derivatives) of an ill-posed Cauchy problem for the three-dimensional Laplace equation and propose a marching difference scheme for solving the problem in a stable way.

## 1. Introduction

The Cauchy problems for the Laplace equation is well-known to be ill-posed: not for every Cauchy data there is a solution and if there is a solution, it may not depend continuously on the data (see, e.g. [3] and the references therein). The instability of the solution makes numerical methods for it difficult, since a small perturbation in the data may cause a very large error in the solution. There have been several methods for solving the Cauchy problem for the Laplace equation in a stable way (see, e.g. [3] and the references therein) and the most popular method is Tikhonov regularization. However, in order to solve the problem numerically one has to discretize the regularized problem somehow. It led to the idea that to discretize the problem directly and it was Chudov ([4]) who is the first to use the finite difference method for the Cauchy problem for the two-dimensional Laplace equation. The idea of Chudov has been developed further by himself and his collaborators (see, e.g. [5,10] and the references therein). Bakushinkii in [1,2] has used also the finite difference method for solving ill-posed abstract Cauchy problems, Bukhgeim ([3]) has used the Carleman estimates technique to prove some stability estimates for finite difference schemes for ill-posed problems as well as developed a convergence theory. Meanwhile, Samarskii and Vabishschevich ([9]) considered the finite difference method for solving ill-

posed problems with Tikhonov regularization from a quite different point of view. However, no marching difference scheme has been considered in these works. We note that marching difference schemes are very easy-to-implement and cheap, especially in multi-dimensional problems.
In this paper we consider the following Cauchy problem for the three-dimensional Laplace equation

$$u_{tt} + u_{xx} + u_{yy} = 0, \quad 0 < t < 1, \ (x, y) \in \mathbb{R}^2, \tag{1.1}$$

$$u(x, y, 0) = \varphi(x, y), \quad (x, y) \in \mathbb{R}^2, \tag{1.2}$$

$$u_t(x, y, 0) = 0, \quad (x, y) \in \mathbb{R}^2 \tag{1.3}$$

and suggest a stable marching difference scheme for it. We note that the non-homogeneous problem can be transformed to (1.1)-(1.3) via an appropriate well-posed boundary value problem. In the next section we shall give stability estimates for all derivatives of $u$ in $L_2$-norm. We note that such a kind of stability estimates for the solution of the Cauchy problem for elliptic equations has never appeared in the literature. In section 3 we shall use the mollification method of [6] for solving (1.1)–(1.3) in a stable way, we give also error estimates of Hölder type between the exact solution and its mollified solutions, as well as those of all their derivatives. Finally, in the last section we shall describe our easy-to-implement stable marching difference scheme for (1.1)–(1.3).

## 1. Stability Estimates

As we consider our problem in $L_2$-space, we assume

$$\varphi \in L_2(\mathbb{R}^2).$$

Furthermore, we suppose that the solution exists up to $t = 1$ and

$$u(\cdot, \cdot, 1) \in L_2(\mathbb{R}^2).$$

For simplicity, we denote $\|\cdot\|_{L_2(\mathbb{R}^2)} = \|\cdot\|$.

**Theorem 2.1.** *Suppose that* $\|\varphi\| \le \|u(\cdot,\cdot,1)\|$. *Then for* $t \in (0, 1)$, *we have*

(i) $\|u(\cdot,\cdot,t)\| \le \frac{3}{2}\|\varphi\|^{1-t}\|u(\cdot,\cdot,1)\|^t$, (2.1)

(ii) *for* $m = 0, 1, 2, \ldots$, $\ell = 0, 1, 2, \ldots$ *such that* $2m + \ell \ge 1$,

$$\left\| \frac{\partial^{2m+\ell} u(\cdot,\cdot,t)}{\partial x^m \partial y^m \partial t^\ell} \right\| \le \frac{1}{2^m} \left( \ln \left( e \frac{\|u(\cdot,\cdot,1)\|}{\|\varphi\|} \right) \right)^{2m+\ell} \times \left( e + \frac{1}{2e} \left( \frac{2m+\ell}{1-t} \right)^{2m+\ell} \right) \|\varphi\|^{1-t} \|u(\cdot,\cdot,1)\|^t. \tag{2.2}$$

*Proof.* For a function $g \in L_2(\mathbb{R}^2)$ we denote its Fourier transform by

$$\hat{g}(\xi_1, \xi_2) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y)\, e^{-i(x\xi_1 + y\xi_2)} dx dy.$$

Taking the Fourier transform with respect to $x$ and $y$ in the both sides of (1.1)–(1.3) we have

$$\hat{u}_{tt}(\xi_1,\xi_2,t) - (\xi_1^2+\xi_2^2)\hat{u}(\xi_1,\xi_2,t) = 0, \quad 0<t<1,$$
$$\hat{u}(\xi_1,\xi_2,0) = \hat{\varphi}(\xi_1,\xi_2),$$
$$\hat{u}_t(\xi_1,\xi_2,0) = 0.$$

It follows that

$$\hat{u}(\xi_1,\xi_2,t) = \cosh\left(t\sqrt{\xi_1^2+\xi_2^2}\,\right)\hat{\varphi}(\xi_1,\xi_2).$$

Thus, for $m=0,1,2,\ldots$, $\ell=0,1,2,\ldots$, formally we have

$$\begin{aligned}\widehat{\frac{\partial^{2m+\ell}u(x,y,t)}{\partial x^m\partial y^m\partial t^\ell}} &= \frac{\partial^\ell}{\partial t^\ell}\,\widehat{\frac{\partial^{2m}u(x,y,t)}{\partial x^m\partial y^m}}\\ &= \frac{\partial^\ell}{\partial t^\ell}\Big((-i\xi_1)^m(-i\xi_2)^m\cosh(t\sqrt{\xi_1^2+\xi_2^2}\,)\hat{\varphi}(\xi_1,\xi_2)\Big)\\ &= (-i\xi_1)^m(-i\xi_2)^m\sqrt{\xi_1^2+\xi_2^2}^{\,\ell}\cosh(t\sqrt{\xi_1^2+\xi_2^2}\,)\hat{\varphi}(\xi_1,\xi_2).\end{aligned}$$

Noting that

$$\hat{\varphi}(\xi_1,\xi_2) = \frac{\hat{u}(\xi_1,\xi_2,1)}{\cosh\sqrt{\xi_1^2+\xi_2^2}},$$

$$\cosh(t\sqrt{\xi_1^2+\xi_2^2}\,) \le e^{t\sqrt{\xi_1^2+\xi_2^2}} \quad\text{and}\quad \cosh(\sqrt{\xi_1^2+\xi_2^2}) \ge \frac{1}{2}e^{\sqrt{\xi_1^2+\xi_2^2}}$$

we have, for some $L>0$,

$$\begin{aligned}&\left\|\frac{\partial^{2m+\ell}u(\cdot,\cdot,t)}{\partial x^m\partial y^m\partial t^\ell}\right\|^2 = \left\|\widehat{\frac{\partial^{2m+\ell}u(x,y,t)}{\partial x^m\partial y^m\partial t^\ell}}\right\|^2\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|(-i\xi_1)^m(-i\xi_2)^m\sqrt{\xi_1^2+\xi_2^2}^{\,\ell}\cosh(t\sqrt{\xi_1^2+\xi_2^2}\,)\hat{\varphi}(\xi_1,\xi_2)\right|^2 d\xi_1 d\xi_2\\ &= \iint\limits_{\xi_1^2+\xi_2^2\le L}\ldots + \iint\limits_{\xi_1^2+\xi_2^2\ge L}\ldots\\ &= \iint\limits_{\xi_1^2+\xi_2^2\le L}\left|(-i\xi_1)^m(-i\xi_2)^m\sqrt{\xi_1^2+\xi_2^2}^{\,\ell}\cosh(t\sqrt{\xi_1^2+\xi_2^2}\,)\hat{\varphi}(\xi_1,\xi_2)\right|^2 d\xi_1 d\xi_2\\ &\quad+\iint\limits_{\xi_1^2+\xi_2^2\ge L}\left|(-i\xi_1)^m(-i\xi_2)^m\sqrt{\xi_1^2+\xi_2^2}^{\,\ell}\frac{\cosh(t\sqrt{\xi_1^2+\xi_2^2}\,)}{\cosh\sqrt{\xi_1^2+\xi_2^2}}\hat{u}(\xi_1,\xi_2,1)\right|^2 d\xi_1 d\xi_2\\ &\le \iint\limits_{\xi_1^2+\xi_2^2\le L}|\xi_1|^{2m}|\xi_2|^{2m}(\xi_1^2+\xi_2^2)^\ell|\cosh(t\sqrt{\xi_1^2+\xi_2^2})|^2|\hat{\varphi}(\xi_1,\xi_2)|^2 d\xi_1 d\xi_2\\ &\quad+\iint\limits_{\xi_1^2+\xi_2^2\ge L}|\xi_1|^{2m}|\xi_2|^{2m}(\xi_1^2+\xi_2^2)^\ell\left|\frac{\cosh(t\sqrt{\xi_1^2+\xi_2^2})}{\cosh\sqrt{\xi_1^2+\xi_2^2}}\right|^2\cdot|\hat{u}(\xi_1,\xi_2,1)|^2 d\xi_1 d\xi_2\end{aligned}$$

$$\leq \iint_{\xi_1^2+\xi_2^2\leq L} \frac{1}{2^{2m}}\left(\xi_1^2+\xi_2^2\right)^{2m+\ell} e^{2t\sqrt{\xi_1^2+\xi_2^2}}|\hat{\varphi}(\xi_1,\xi_2)|^2 d\xi_1 d\xi_2$$

$$+\frac{1}{4}\iint_{\xi_1^2+\xi_2^2\geq L} \frac{1}{2^{2m}}\left(\xi_1^2+\xi_2^2\right)^{2m+\ell} e^{2(t-1)\sqrt{\xi_1^2+\xi_2^2}}|\hat{u}(\xi_1,\xi_2,1)|^2 d\xi_1 d\xi_2$$

$$\leq \frac{1}{2^{2m}} L^{2m+\ell} e^{2t\sqrt{L}}\|\varphi\|^2$$

$$+\frac{1}{4}\cdot\frac{1}{2^{2m}}\sup_{\xi_1^2+\xi_2^2\geq L}\left(\left(\xi_1^2+\xi_2^2\right)^{2m+\ell} e^{2(t-1)\sqrt{\xi_1^2+\xi_2^2}}\right)\|u(\cdot,\cdot,1)\|^2.$$

To estimate the second term in the last expression we need the following result.

**Lemma 2.2.** *Let $c>0$, $p>0$ and $\eta\geq 1$. We have*
(i) *if $p/c<1$, then*

$$\sup_{y\geq\eta}(e^{-cy}y^p)\leq e^{-c\eta}\eta^p$$

(ii) *if $p/c\geq 1$, then*

$$\sup_{y\geq\eta}(e^{-cy}y^p)\leq\left(\frac{p}{c}\right)^p e^{-c\eta}\eta^p.$$

The proof of this lemma is trivial and we omit it.

Now, if $m=\ell=0$, then

$$\sup_{\xi_1^2+\xi_2^2\geq L} e^{2(t-1)\sqrt{\xi_1^2+\xi_2^2}} = e^{2(t-1)\sqrt{L}}.$$

Thus

$$\|u(\cdot,\cdot,t)\|\leq e^{t\sqrt{L}}\|\varphi\|+\frac{1}{2}e^{(t-1)\sqrt{L}}\|u(\cdot,\cdot,1)\|.$$

Since $\|\varphi\|\leq\|u(\cdot,\cdot,1)\|$, we can take

$$\sqrt{L}=\ln\frac{\|u(\cdot,\cdot,1)\|}{\|\varphi\|}$$

and with this $L$ we arrive at (2.1).

If $2m+\ell\geq 1$, then for $L\geq 1$ we have

$$\sup_{\xi_1^2+\xi_2^2\geq L}\left(\left(\xi_1^2+\xi_2^2\right)^{2m+\ell} e^{2(t-1)\sqrt{\xi_1^2+\xi_2^2}}\right)\leq\left(\frac{2m+\ell}{1-t}\right)^{2(2m+\ell)} L^{2m+\ell}e^{2(t-1)\sqrt{L}}.$$

Thus, for any $L\geq 1$,

$$\left\|\frac{\partial^{2m+\ell}u(\cdot,\cdot,1)}{\partial x^m\partial y^m\partial t^\ell}\right\|\leq\frac{1}{2^m}\left(\sqrt{L}\right)^{2m+\ell}\times$$

$$\left(e^{t\sqrt{L}}\|\varphi\|+\frac{1}{2}\left(\frac{2m+\ell}{1-t}\right)^{2m+\ell}\right)e^{(t-1)\sqrt{L}}\|u(\cdot,\cdot,1)\|.$$

Taking $\sqrt{L} = \ln\left(e\frac{\|u(\cdot,\cdot,1)\|}{\|\varphi\|}\right)(\geq 1)$ we arrive at (2.2).

## 3. Mollification Method

Suppose that instead of the exact $\varphi$ we have only its approximation $\varphi^\varepsilon \in L_2(\mathbb{R}^2)$ such that

$$\|\varphi - \varphi^\varepsilon\| \leq \varepsilon. \tag{3.1}$$

It is desired to solve (1.1)–(1.3) with the approximate data $\varphi^\varepsilon$ in a stable way. In doing so we mollify $\varphi^\varepsilon$ by the Dirichlet kernel

$$\varphi^{\varepsilon,\nu}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varphi^\varepsilon(x,y)\frac{\sin\nu(x-\overline{x})}{x-\overline{x}}\cdot\frac{\sin\nu(y-\overline{y})}{y-\overline{y}}d\overline{x}d\overline{y} \tag{3.2}$$

for some positive $\nu$, and instead of considering (1.1)–(1.3) with $\varphi^\varepsilon$ we look for its mollified version

$$u_{tt}^{\varepsilon,\nu} + u_{xx}^{\varepsilon,\nu} + u_{yy}^{\varepsilon,\nu} = 0, \quad (x,y)\in\mathbb{R}^2, \quad t\in(0,1), \tag{3.3}$$

$$u^{\varepsilon,\nu}(x,y,0) = \varphi^{\varepsilon,\nu}(x,y), \quad (x,y)\in\mathbb{R}^2, \tag{3.4}$$

$$u_t^{\varepsilon,\nu}(x,y,0) = 0 \qquad (x,y)\in\mathbb{R}^2. \tag{3.5}$$

**Theorem 3.1.** *For any $\nu > 0$, the problem (3.3)–(3.5) is solvable and its solution is stable*

$$\left\|\frac{\partial^{m+n+\ell}u^{\varepsilon,\nu}(\cdot,\cdot,t)}{\partial x^m\partial y^n\partial t^\ell}\right\| \leq \left(\sqrt{2}\right)^\ell \nu^{m+n+\ell}e^{t\sqrt{2}\nu}\|\varphi^\varepsilon\| \;\; \forall m,n,\ell = 0,1,2,\ldots \tag{3.6}$$

*Furthermore, suppose that*

$$\|u(\cdot,\cdot,1)\| \leq E. \tag{3.7}$$

*Then for $\varepsilon$ small enough, with*

$$\nu = \nu^* = \frac{1}{\sqrt{2}}\ln\frac{E}{\varepsilon} \tag{3.8}$$

*we have, for $0 \leq t < 1$,*

$$\|u(\cdot,\cdot,t) - u^{\varepsilon\nu^*}(\cdot,\cdot,t)\| \leq \varepsilon^{1-t}E^t + \frac{1}{2}\varepsilon^{\frac{1-t}{\sqrt{2}}}E^{\frac{t}{\sqrt{2}}}E^{1-\frac{1}{\sqrt{2}}} \tag{3.9}$$

*and*

$$\left\|\frac{\partial^{2m+\ell}u(\cdot,\cdot,t)}{\partial x^m\partial y^m\partial t^\ell} - \frac{\partial^{2m+\ell}u^{\varepsilon,\nu^*}(\cdot,\cdot,t)}{\partial x^m\partial y^m\partial t^\ell}\right\| \leq \frac{1}{2^m}\left(\ln\frac{E}{\varepsilon}\right)^{2m+\ell}\varepsilon^{1-t}E^t$$
$$+\frac{1}{2^{2m+\ell/2+1}}\left(\frac{2m+\ell}{1-t}\right)^{2m+\ell}\left(\ln\frac{E}{\varepsilon}\right)^{2m+\ell}\varepsilon^{\frac{1-t}{\sqrt{2}}}E^{\frac{t}{\sqrt{2}}}E^{1-\frac{1}{\sqrt{2}}}. \tag{3.10}$$

*Proof.* As in the proof of Theorem 2.1 we have

$$\frac{\partial^{m+n+\ell} \widehat{u(x,y,t)}}{\partial x^m \partial y^n \partial t^\ell} = (-i\xi_1)^m(-i\xi_2)^n \sqrt{\xi_1^2+\xi_2^2}^{\,\ell} \times \cosh(t\sqrt{\xi_1^2+\xi_2^2}) \cdot \widehat{\varphi^{\varepsilon,\nu}}(\xi_1,\xi_2).$$

Since supp $\widehat{\varphi^{\varepsilon,\nu}} \subset [-\nu,\nu]\times[-\nu,\nu]$ (see, e.g. [8, p. 316–318]), we obtain (3.6).

Now we estimate the error $\left\| \frac{\partial^{2m+\ell} u(\cdot,\cdot,t)}{\partial x^m \partial y^m \partial t^\ell} - \frac{\partial^{2m+\ell} u^{\varepsilon,\nu}(\cdot,\cdot,t)}{\partial x^m \partial y^m \partial t^\ell} \right\|$ for $\ell, m = 0,1,2,\dots$ and $0 \le t < 1$. We have

$$\left\| \frac{\partial^{2m+\ell} u(\cdot,\cdot,t)}{\partial x^m \partial y^m \partial t^\ell} - \frac{\partial^{2m+\ell} u^{\varepsilon,\nu}(\cdot,\cdot,t)}{\partial x^m \partial y^m \partial t^\ell} \right\| \le \left\| \frac{\partial^{2m+\ell} u(\cdot,\cdot,t)}{\partial x^m \partial y^\ell \partial t^\ell} - \frac{\partial^{2m+\ell} u^{0,\nu}(\cdot,\cdot,t)}{\partial x^m \partial y^n \partial t^\ell} \right\| + \left\| \frac{\partial^{2m+\ell} u^{0,\nu}(\cdot,\cdot,t)}{\partial x^m \partial y^n \partial t^\ell} - \frac{\partial^{2m+\ell} u^{\varepsilon,\nu}(\cdot,\cdot,t)}{\partial x^m \partial y^m \partial t^\ell} \right\|.$$

The second term in the right-hand side can be estimated as follows

$$\begin{aligned}
&\left\| \frac{\partial^{2m+\ell} u^{0,\nu}(\cdot,\cdot,t)}{\partial x^m \partial y^m \partial t^\ell} - \frac{\partial^{2m+\ell} u^{\varepsilon,\nu}(\cdot,\cdot,t)}{\partial x^m \partial y^m \partial t^\ell} \right\| = \left\| \frac{\partial^{2m+\ell} \widehat{u^{0,\nu}(\cdot,\cdot,t)}}{\partial x^m \partial y^m \partial t^\ell} - \frac{\partial^{2m+\ell} \widehat{u^{\varepsilon,\nu}(\cdot,\cdot,t)}}{\partial x^m \partial y^m \partial t^\ell} \right\| \\
&= \Bigg( \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Big|(-i\xi_1)^m(-i\xi_2)^m \sqrt{\xi_1^2+\xi_2^2}^{\,\ell} \cosh\left(t\sqrt{\xi_1^2+\xi_2^2}\right) \\
&\qquad \times \chi_{[-\nu,\nu]}(\xi_1)\chi_{[-\nu,\nu]}(\xi_2)[\widehat{\varphi}(\xi_1,\xi_2) - \widehat{\varphi^\varepsilon}(\xi_1,\xi_2)]\Big|^2 d\xi_1 d\xi_2 \Bigg)^{1/2} \\
&= \Bigg( \int_{-\nu}^{+\nu}\int_{-\nu}^{+\nu} \Big|(-i\xi_1)^m(-i\xi_2)^m \sqrt{\xi_1^2+\xi_2^2}^{\,\ell} \cosh\left(t\sqrt{\xi_1^2+\xi_2^2}\right) \\
&\qquad \times \left(\widehat{\varphi}(\xi_1,\xi_2) - \widehat{\varphi^\varepsilon}(\xi_1,\xi_2)\right)\Big|^2 d\xi_1 d\xi_2 \Bigg)^{1/2} \\
&\le \Bigg( \int_{-\nu}^{+\nu}\int_{-\nu}^{+\nu} \frac{1}{2^{2m}} \left(\xi_1^2+\xi_2^2\right)^{2m} \left(\xi_1^2+\xi_2^2\right)^{\ell} e^{t\sqrt{\xi_1^2+\xi_2^2}} \left|\widehat{\varphi}(\xi_1,\xi_2) - \widehat{\varphi^\varepsilon}(\xi_1,\xi_2)\right|^2 d\xi_1 d\xi_2 \Bigg)^{1/2} \\
&\le \frac{1}{2^m}\left(2\nu^2\right)^{m+\ell/2} e^{t\nu\sqrt{2}} \|\varphi - \varphi^\varepsilon\| \le \frac{1}{2^m}\left(2\nu^2\right)^{m+\ell/2} e^{t\nu\sqrt{2}}\, \varepsilon.
\end{aligned}$$

To estimate the first term we note that

$$\left\| \frac{\partial^{2m+\ell} u(\cdot,\cdot,t)}{\partial x^m \partial y^\ell \partial t^\ell} - \frac{\partial^{2m+\ell} u^{0,\nu}(\cdot,\cdot,t)}{\partial x^m \partial y^n \partial t^\ell} \right\|$$

$$= \left( \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Big| (-i\xi_1)^m (-i\xi_2)^m \sqrt{\xi_1^2+\xi_2^2}^{\,\ell} \cosh\left(t\sqrt{\xi_1^2+\xi_2^2}\right) \right.$$

$$\times \hat{\varphi}(\xi_1,\xi_2) - (-i\xi_1)^m(-i\xi_2)^m \sqrt{\xi_1^2+\xi_2^2}^{\,\ell}$$

$$\left. \times \cosh\left(t\sqrt{\xi_1^2+\xi_2^2}\right)\chi_{[-\nu,\nu]}(\xi_1)\chi_{[-\nu,\nu]}(\xi_2)\hat{\varphi}(\xi_1,\xi_2)\Big|^2 d\xi_1 d\xi_2 \right)^{1/2}$$

$$= \left( \int_{|\xi_1|\geq\nu}\int_{|\xi_2|\geq\nu} \Big| (-i\xi_1^m)(-i\xi_2)^m \sqrt{\xi_1^2+\xi_2^2}^{\,\ell} \right.$$

$$\left. \times \frac{\cosh(t\sqrt{\xi_1^2+\xi_2^2})}{\cosh\sqrt{\xi_1^2+\xi_2^2}} \hat{u}(\xi_1,\xi_2,1)\Big|^2 d\xi_1 d\xi_2 \right)^{1/2}$$

$$\leq \left( \int_{\xi_1^2+\xi_2^2\geq\nu^2} \frac{1}{2^{2m}} (\xi_1^2+\xi_2^2)^{2m+\ell} \frac{1}{4} e^{2(t-1)\sqrt{\xi_1^2+\xi_2^2}} |\hat{u}(\xi_1,\xi_2,t)|^2 d\xi_1 d\xi_2 \right)^{1/2}$$

$$\leq \frac{1}{2^{m+1}} \sup_{\xi_1^2+\xi_2^2\geq\nu^2} \left( (\xi_1^2+\xi_2^2)^{m+\ell/2} e^{(t-1)\sqrt{\xi_1^2+\xi_2^2}} \right) \|u(\cdot,\cdot,1)\|.$$

If $m=\ell=0$, then the last is bounded by

$$\frac{1}{2} e^{(t-1)\nu} E.$$

Thus

$$\|u(\cdot,\cdot,t) - u^{\varepsilon,\nu}(\cdot,\cdot,t)\| \leq e^{t\nu\sqrt{2}}\varepsilon + \frac{1}{2}e^{(t-1)\nu}E.$$

Taking $\nu = \nu^* = \frac{1}{\sqrt{2}} \ln\frac{E}{\varepsilon}$ (since $\varepsilon$ is small enough, $\ln\frac{E}{\varepsilon} > 0$), we arrive at (3.9).

If $2m+\ell \geq 1$, then applying Lemma 2.2 for $\nu \geq 1$ we get

$$\sup_{\xi_1^2+\xi_2^2\geq\nu^2} \left( (\xi_1^2+\xi_2^2)^{m+\ell/2} e^{(t-1)\sqrt{\xi_1^2+\xi_2^2}} \right) \leq \left(\frac{2m+\ell}{1-t}\right)^{2m+\ell} e^{(t-1)\nu}\nu^{2m+\ell}.$$

Thus, in this case

$$\left\| \frac{\partial^{2m+\ell} u(\cdot,\cdot,t)}{\partial x^m \partial y^m \partial t^\ell} - \frac{\partial^{2m+\ell} u^{\varepsilon,\nu}(\cdot,\cdot,t)}{\partial x^m \partial y^m \partial t^\ell} \right\|$$

$$\leq \frac{1}{2^m}(2\nu^2)^{m+\ell/2} e^{t\nu\sqrt{2}}\varepsilon + \frac{1}{2^{m+1}}\left(\frac{2m+\ell}{1-t}\right)^{2m+\ell} e^{(t-1)\nu}\nu^{2m+\ell}E.$$

Again, with the assumption $\varepsilon$ is small enough we can take $\nu = \nu^* = \frac{1}{\sqrt{2}}\ln\frac{E}{\varepsilon}$ such that $\nu^* \geq 1$ and arrive at (3.10).

## 4. Stable Marching Difference Scheme

In order to solve our problem (1.1)–(1.3) numerically in an effective way we shall use a stable marching difference scheme for it based on our mollification method. In doing so we first mollify the Cauchy data $\varphi^\varepsilon$ with the mollification parameter $\nu$ according to Theorem 3.1, then we get a stable problem and we have error estimates (3.9) and (3.10). For simplicity, set

$$U := u^{\varepsilon,\nu}, \quad W := u_t^{\varepsilon,\nu}, \quad \Psi := \varphi^{\varepsilon,\nu}. \tag{4.1}$$

With the notation $\Delta U = u_{xx} + u_{yy}$ we have a Cauchy problem for a system of first-order differential equations for $U$ and $W$

$$U_t = W, \ \ t \in (0,1), \ \ (x,y) \in \mathbb{R}^2, \tag{4.2}$$
$$W_t = -\Delta U, \ \ t \in (0,1), \ \ (x,y) \in \mathbb{R}^2, \tag{4.3}$$
$$U(x,y,0) = \Psi(x,y), \ \ \ (x,y) \in \mathbb{R}^2, \tag{4.4}$$
$$W(x,y,0) = 0, \ \ \ (x,y) \in \mathbb{R}^2. \tag{4.5}$$

We introduce a uniform grid on $\mathbb{R}^2 \times [0,1]$ plane

$$\Big\{x_m = mh, \ y_n = nh, \ t_k = k\tau \,\Big|\, m,n = 0, \pm1, \pm2, \dots, \ k = 0,1,\dots,N, \ \tau = \frac{1}{N}\Big\}.$$

For a function $f(x,y,t)$ defined on $\mathbb{R}^2 \times [0,1]$ set

$$f^k_{m,n} = f(x_m, y_n, t_k).$$

We discretize (4.2)–(4.5) as follows

$$\frac{U^{k+1}_{m,n} - U^k_{m,n}}{\tau} = W^{k+1}_{m,n}, \ \ k = 0,1,\dots,N-1, \ \ m,n = 0,\pm1,\dots \tag{4.6}$$
$$\frac{W^{k+1}_{m,n} - W^k_{m,n}}{\tau} = -\Delta_h U^k_{m,n}, \ \ k = 0,1,\dots,N-1, \ \ m,n = 0,\pm1,\dots \tag{4.7}$$
$$U^0_{m,n} = \Psi_{m,n}, \ \ m,n = 0,\pm1,\dots \tag{4.8}$$
$$W^0_{m,n} = 0, \ \ m,n = 0,\pm1,\dots. \tag{4.9}$$

Here

$$\Delta_h U^k_{m,n} = \frac{U^k_{m+1,n} - 2U^k_{m,n} + U^k_{m-1,n}}{h^2} + \frac{U^k_{m,n+1} - 2U^k_{m,n} + U^k_{m,n-1}}{h^2}.$$

The system (4.6)–(4.9) is a marching difference scheme:

$$U^0_{m,n} = \Psi_{m,n}, \ \ m,n = 0,\pm1,\dots \tag{4.10}$$
$$W^0_{m,n} = 0, \ \ \ m,n = 0,\pm1,\dots \tag{4.11}$$
$$W^{k+1}_{m,n} = W^k_{m,n} + \tau\Delta_h U^k_{m,n}, \ \ k = 0,1,\dots,N-1, \tag{4.12}$$
$$U^{k+1}_{m,n} = U^k_{m,n} + \tau W^{k+1}_{m,n}, \ \ k = 0,1,\dots,N-1. \tag{4.13}$$

**Theorem 4.1.** *The difference scheme (4.6)–(4.9) approximates the problem (4.2)–(4.5) with a truncation error which behaves like $O(h^2+\tau^2)$. Furthermore if $h \le \pi/\nu$ ($\nu$ has been chosen in Theorem 3.1), then it is unconditionally stable.*

*Proof.* The first assertion is clear. We prove only the stability of the scheme. In doing so we need the notion of the discrete Fourier transform: for a function $f$ defined on the net $\{(mh, nh), m, n = 0, \pm 1, \dots\}$ we define its discrete Fourier transform by

$$\overset{\Delta}{f}(\omega,\eta) = \frac{h^2}{2\pi}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f_{mn} e^{-i(\omega x_m + \eta y_n)}, \quad -\frac{\pi}{h} \le \omega, \eta \le \frac{\pi}{h}$$

and its $\ell_2$-norm by

$$\|f\|_{\ell_2} = \sqrt{h^2 \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} |f_{mn}|^2}.$$

It is well-known that

$$\|f\|_{\ell_2} = \|\overset{\Delta}{f}\|_{L_2((-\frac{\pi}{2},\frac{\pi}{2})\times(-\frac{\pi}{2},\frac{\pi}{2}))}$$

and if $f \in L_2(\mathbb{R}^2)$, $\operatorname{supp}\hat{f}(\omega,\eta) \subset [-\nu,\nu]\times[-\nu,\nu]$, $\nu \le \pi/h$, then $\overset{\Delta}{f}(\omega,\eta) = \hat{f}(\omega,\eta)$ (see, e.g. [7, Appendix A]).

Now taking the discrete Fourier transform in the both sides of (4.12) and (4.13) we get for $k = 0, 1, \dots, N-1$

$$\overset{\Delta}{W}{}^{k+1} = \overset{\Delta}{W}{}^{k} + \tau \cdot 4\frac{\sin^2\frac{\omega h}{2} + \sin^2\frac{\eta h}{2}}{h^2}\overset{\Delta}{U}{}^{k},$$

$$\overset{\Delta}{U}{}^{k+1} = \overset{\Delta}{U}{}^{k} + \tau\overset{\Delta}{W}{}^{k+1}.$$

Since $\tau = 1/N \le 1$, taking the inequality

$$4\Big(\sin^2\frac{\omega h}{2} + \sin^2\frac{\eta h}{2}\Big)/h^2 \le \omega^2 + \eta^2$$

into account, we get

$$\max\Big\{|\overset{\Delta}{W}{}^{k+1}|, |\overset{\Delta}{U}{}^{k+1}|\Big\} \le (1+\tau+\tau(\omega^2+\eta^2))\max\Big\{|\overset{\Delta}{W}{}^{k}|, |\overset{\Delta}{U}{}^{k}|\Big\} \le \dots$$

$$\le (1+\tau+\tau(\omega^2+\eta^2))^{k+1}|\overset{\Delta}{\Psi}| \le e^{1+\omega^2+\eta^2}|\overset{\Delta}{\Psi}|.$$

Since $h \le \pi/\nu$, we have $\overset{\Delta}{\Psi} = \hat{\Psi}$. It follows that

$$\begin{aligned}\max\{\|W^k\|_{\ell_2}, \|U^k\|_{\ell_2}\} &= \max\Big\{\|\overset{\Delta}{W}{}^{k}\|_{L_2((-\frac{\pi}{h},\frac{\pi}{h})^2)}, \|\overset{\Delta}{U}{}^{k}\|_{L_2((-\frac{\pi}{h},\frac{\pi}{h})^2)}\Big\}\\ &\le \|e^{1+\omega^2+\eta^2}\hat{\Psi}(\omega,\eta)\|\\ &\le e^{1+2\nu^2}\|\Psi\|.\end{aligned}$$

Thus, our scheme is stable.

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