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Direct Sums of Type 2 $\mathcal{X}$-Extending Modules*

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Abstract. Let $\mathcal{X}$ be a class of left R-modules. It is proved that if type 2 $\mathcal{X}$-extending left R-modules M_1 and M_2 are relatively essentially $\mathcal{X}^e$-injective and M_1 is pseudoly M_2-$\mathcal{X}^e$-injective (or M_2 is pseudoly M_1-$\mathcal{X}^e$-injective) then $M_1 \oplus M_2$ is type 2 $\mathcal{X}$-extending. As applications, we characterize when the direct sum of two extending left R-modules is extending, when the direct sum of two CESS-modules is CESS, and when the direct sum of two uniform-extending left R-modules is uniform-extending.

Extending modules have been studied extensively in recent years and many generalizations have been considered by many authors (see, for examples, [1–3, 5, 12, 15, 16]). Dogruoz and Smith in [3] introduced the concepts of type 1 $\mathcal{X}$-extending modules and type 2 $\mathcal{X}$-extending modules relative to a given class $\mathcal{X}$ of left R-modules. In this paper we consider when the direct sum of two type 2 $\mathcal{X}$-extending modules is type 2 $\mathcal{X}$-extending. It is proved that if type 2 $\mathcal{X}$-extending left R-modules M_1 and M_2 are relatively essentially $\mathcal{X}^e$-injective and M_1 is pseudoly M_2-$\mathcal{X}^e$-injective (or M_2 is pseudoly M_1-$\mathcal{X}^e$-injective) then $M_1 \oplus M_2$ is type 2 $\mathcal{X}$-extending. As a corollary, we show that if extending modules M_1 and M_2 are relatively essentially injective and M_1 is pseudo-M_2-injective (or M_2 is pseudo-M_1-injective) then $M_1 \oplus M_2$ is extending. Also we characterize when the direct sum of two CESS-modules is CESS, and when the direct sum of two uniform-extending left R-modules is uniform-extending.

Throughout this paper we write $A \leq_e B(A|B)$ to denote that A is an essential submodule (a direct summand) of B.

A left R-module M is called extending if every submodule of M is essential

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in a direct summand of M. M is called quasi-continuous if it is extending and for any direct summands A and B of M with $A \cap B = 0$, $A \oplus B$ is a direct summand of M. M is called continuous if it is extending and every submodule of M which is isomorphic to a direct summand of M is itself a direct summand of M.

By a class of left R-modules we mean a collection of left R-modules containing the zero module and closed under isomorphisms. If $\mathcal{X}$ is a class of left R-modules and M is a left R-module then an $\mathcal{X}$-submodule of M will be a submodule N of M such that N belongs to $\mathcal{X}$.

Let $\mathcal{X}$ be a class of left R-modules. According to [3], a left R-module M is said to be type 2 $\mathcal{X}$-extending if for every $\mathcal{X}$-submodule N of M, every closure of N in M is a direct summand of M.

For the class $\mathcal{X} = \mathcal{U}$ of left R-modules with finite uniform dimension, type 2 $\mathcal{U}$-extending modules are discussed in [8–10] (where they are called modules with (1-C_1)), [4] and [5] (where they are called uniform extending modules). If $\mathcal{X} = \mathcal{S}$ is the class of semisimple left R-modules then type 2 $\mathcal{S}$-extending modules are considered in [1, 2, 15] (where they are called CESS-modules). If $\mathcal{X}$ is the class of left R-modules with finitely generated essential submodule then type 2 $\mathcal{X}$-extending modules are considered in [17] (where they are called ef-extending modules).

Note that it is not true in general that the direct sum of two type 2 $\mathcal{X}$-extending modules is type 2 $\mathcal{X}$-extending. For example, let $\mathbb{Z}$ denote the ring of integers, let p be any prime, let $M_1 = \mathbb{Z}/\mathbb{Z}p$ and $M_2 = \mathbb{Z}/\mathbb{Z}p^3$. Then M_1 and M_2 are type 2 $\mathcal{S}$-extending but $M = M_1 \oplus M_2$ is not (see [2]).

Let M be a left R-module. Define the family $\mathcal{X}(M)$ to be the set of all submodules N of M with $N \in \mathcal{X}$.

Definition 1. *Let M, N be left R-modules. We say N is (essentially, pseudoly) M-$\mathcal{X}$-injective if for any submodule $A \in \mathcal{X}(M)$, any homomorphism $\phi : A \longrightarrow N$ (with $\operatorname{Ker}(\phi) \leq_e A$, $\operatorname{Ker}(\phi) = 0$, respectively) can be extended to a homomorphism $\psi : M \longrightarrow N$.*

Note that every M-injective left R-module is clearly M-$\mathcal{X}$-injective. But Example 1 shows that the converse is not true.

Lemma 1. *Let M, N be left R-modules and $L = N \oplus M$. Then the following conditions are equivalent:*

(1) *N is pseudoly M-$\mathcal{X}$-injective.*

(2) *For every submodule A of L with $A \in \mathcal{X}(L)$ and $A \cap M = A \cap N = 0$, there exists a submodule B of L such that $L = N \oplus B$ and $A \subseteq B$.*

Proof. Suppose first that N is pseudoly M-$\mathcal{X}$-injective. Denote the canonical projections $L \longrightarrow N$ and $L \longrightarrow M$ by π_N and π_M respectively. Let A be a submodule of L with $A \in \mathcal{X}(L)$ and $A \cap M = A \cap N = 0$. It is easy to see that $\pi_M|_A$ and $\pi_N|_A$ are monomorphisms. Thus there exists an isomorphism $\phi : \pi_M(A) \longrightarrow A$.

Since $A \in \mathcal{X}(L)$, it follows that $\pi_M(A) \in \mathcal{X}(M)$. Also $\pi_N|_A\phi : \pi_M(A) \longrightarrow N$

is a monomorphism. Thus by the pseudo M-$\mathcal{X}$-injectivity of N, there exists a homomorphism $\psi : M \longrightarrow N$ such that

$$\psi|_{\pi_M(A)} = \pi_N|_A\phi.$$

Set $B = \{m + \psi(m) \mid m \in M\}$. For any $a \in A$, $\psi\pi_M(a) = \pi_N|_A\phi(\pi_M(a)) = \pi_N(a)$, and so $a = \pi_N(a) + \pi_M(a) = \psi\pi_M(a) + \pi_M(a) \in B$. This means that $A \subseteq B$. It is easy to check that $B \cap N = 0$ and $N + M = N + B$. Thus $L = N \oplus B$.

Conversely suppose that $L = N \oplus M$ satisfies the condition that for every submodule A of L with $A \in \mathcal{X}(L)$ and $A \cap M = A \cap N = 0$ there exists a submodule B of L such that $L = N \oplus B$ and $A \subseteq B$. Let A be in $\mathcal{X}(M)$ and $\phi : A \longrightarrow N$ a homomorphism with $\mathrm{Ker}(\phi) = 0$.

Put $K = \{a - \phi(a) \mid a \in A\} \leq L$. Since ϕ is a monomorphism, it is easy to see that $K \cap M = K \cap N = 0$.

Since A is in $\mathcal{X}(M)$, and $K \cong A$, it follows that $K \in \mathcal{X}(L)$. By hypothesis, there exists a submodule B of L such that $L = N \oplus B$ and $K \subseteq B$. Let $\pi : L \longrightarrow N$ denote the projection with kernel B. Then $\pi|_M : M \longrightarrow N$ and for any $a \in A$, $\pi(a) = \pi(a - \phi(a)) + \pi(\phi(a)) = \phi(a)$. It follows that N is pseudoly M-$\mathcal{X}$-injective. ∎

Similarly we have

Lemma 2. *Let M, N be left R-modules and $L = N \oplus M$. Then the following conditions are equivalent:*

(1) *N is essentially M-$\mathcal{X}$-injective.*

(2) *For every submodule A of L with $A \in \mathcal{X}(L)$ and $A \cap M \leq_e A$, there exists a submodule B of L such that $L = N \oplus B$ and $A \subseteq B$.*

Proof. Note that in the proof of Lemma 1, $\pi_M|_A$ is a monomorphism and $\mathrm{Ker}(\pi_N|_A\phi) = \phi^{-1}(\mathrm{Ker}(\pi_N|_A)) = \phi^{-1}(A \cap M) \leq_e \phi^{-1}(A) = \pi_M(A)$.

Conversely let A be in $\mathcal{X}(M)$ and $\phi : A \longrightarrow N$ a homomorphism with $\mathrm{Ker}(\phi) \leq_e A$. Put $K = \{a - \phi(a) | a \in A\} \leq L$. For any $0 \neq a - \phi(a) \in K$, there exists $r \in R$ such that $0 \neq ra \in \mathrm{Ker}(\phi)$ since $\mathrm{Ker}(\phi) \leq_e A$. Thus $\phi(ra) = 0$, and so $0 \neq ra - \phi(ra) = ra \in K \cap A \cap R(a - \phi(a)) \leq K \cap M \cap R(a - \phi(a))$. This means that $K \cap M \leq_e K$. Now the result follows from the proof of Lemma 1.

Lemma 3. *Let M, N be left R-modules and $L = N \oplus M$. Then the following conditions are equivalent:*

(1) *N is M-$\mathcal{X}$-injective.*

(2) *For every submodule A of L with $A \in \mathcal{X}(L)$ and $A \cap N = 0$, there exists a submodule B of L such that $L = N \oplus B$ and $A \subseteq B$.*

Let M, N be left R-modules. We say that N is pseudo-M-injective if for each submodule A of M and each monomorphism $f : A \longrightarrow N$, there exists a homomorphism $g : M \longrightarrow N$ such that $g|_A = f$.

Corollary 1. *Let M, N be left R-modules and $L = N \oplus M$. Then the following conditions are equivalent:*

(1) *N is pseudo-M-injective.*

(2) *For every submodule A of L with $A \cap M = A \cap N = 0$, there exists a submodule B of L such that $L = N \oplus B$ and $A \subseteq B$.*

Proof. Let $\mathcal{X}$ be the class of all left R-modules. Then $\mathcal{X}(L)$ and $\mathcal{X}(M)$ coincide with the set of all submodules of L, and the set of all submodules of M, respectively. Now the result follows from Lemma 1.

Let $\mathcal{X}$ be any class of left R-modules. Then $\mathcal{X}^e$ will denote the class of left R-modules which contain an essential $\mathcal{X}$-submodule. Note that $\mathcal{X} \subseteq \mathcal{X}^e$.

Lemma 4 [3, Proposition 3.1]. *For any class $\mathcal{X}$ of left R-modules, a left R-module M is type 2 $\mathcal{X}$-extending if and only if M is type 2 $\mathcal{X}^e$-extending.*

A left R-module M is called type 2 $\mathcal{X}$-quasi-continuous if it is type 2 $\mathcal{X}$ -extending and for any direct summands A and B of M with $A \in \mathcal{X}^e(M)$ and $A \cap B = 0$, $A \oplus B$ is a direct summand of M.

Corollary 2. *A left R-module M is type 2 $\mathcal{X}$-quasi-continuous if and only if M is type 2 $\mathcal{X}$-extending such that whenever $M = M_1 \oplus M_2$ is a direct sum of submodules then M_1 is M_2-$\mathcal{X}^e$-injective and M_2 is M_1-$\mathcal{X}^e$-injective.*

Proof. If M is type 2 $\mathcal{X}$-quasi-continuous, then clearly M is type 2 $\mathcal{X}$-extending. Now suppose that $M = M_1 \oplus M_2$ and A is a submodule of M such that $A \in \mathcal{X}^e(M)$ and $A \cap M_1 = 0$. By Lemma 4, M is type 2 $\mathcal{X}^e$-extending; hence there exists a direct summand L of M such that $A \leq_e L$. It is easy to see that $L \in \mathcal{X}^e(M)$ and $L \cap M_1 = 0$. Thus $M_1 \oplus L$ is a direct summand of M. Suppose that $M = M_1 \oplus L \oplus L'$. Then $A \leq L \leq L \oplus L'$. By Lemma 3, it follows that M_1 is M_2-$\mathcal{X}^e$-injective. ∎

Similarly M_2 is M_1-$\mathcal{X}^e$-injective.

Conversely suppose that $A \in \mathcal{X}^e(M)$ and A, N are direct summands of M with $A \cap N = 0$. Then there exist submodules N' and B such that $M = N \oplus N' = A \oplus B$. Since N is N'-$\mathcal{X}^e$-injective, by Lemma 3, there exists $L \leq M$ such that $M = N \oplus L$ and $A \leq L$. Thus

$$L = L \cap M = L \cap (A \oplus B) = A \oplus (L \cap B),$$

which implies that $M = N \oplus L = N \oplus A \oplus (L \cap B)$. Thus $N \oplus A$ is a direct summand of M, and so M is type 2 $\mathcal{X}$-quasi-continuous.

Two left R-modules M_1 and M_2 are called relatively essentially $\mathcal{X}$-injective if M_1 is essentially M_2-$\mathcal{X}$-injective and M_2 is essentially M_1-$\mathcal{X}$-injective. By analogy with the proof of Corollary 2, we have

Corollary 3. *Let M be a type 2 $\mathcal{X}$-extending left R-module. Then the following conditions are equivalent:*

(1) *For every $A \in \mathcal{X}^e(M)$ with $A|M$, and every submodules N and N' with $M = N \oplus N'$, if $A \cap N' \leq_e A$, then $A \oplus N|M$.*
(2) *Whenever $M = M_1 \oplus M_2$ is a direct sum of submodules then M_1 and M_2 are relatively essentially $\mathcal{X}^e$-injective.*

Note that every type 2 $\mathcal{X}$-quasi-continuous module M satisfies the equivalent conditions of Corollary 3.

Let M, X be left R-modules. Define the family

$$\mathcal{A}(X, M) = \{A \subseteq M | \exists Y \subseteq X, \exists f \in \mathrm{Hom}(Y, M), f(Y) \leq_e A\}.$$

Consider the properties

$\mathcal{A}(X, M)$-(C_1): For all $A \in \mathcal{A}(X, M)$, $\exists A^*|M$, such that $A \leq_e A^*$.

$\mathcal{A}(X, M)$-(C_3): For all $A \in \mathcal{A}(X, M)$ and $B|M$, if $A|M$ and $A \cap B = 0$ then $A \oplus B|M$.

According to [12], M is said to be X-extending, or X-quasi-continuous, respectively, if M satisfies $\mathcal{A}(X, M)$-(C_1) and $\mathcal{A}(X, M)$-(C_3).

The following is a result of [12]. Recall that a ring R is called a left SI-ring if every singular left R-module is injective.

Lemma 5. *A ring R is a left SI-ring if and only if every left R-module is X-quasi-continuous for every singular left R-module X.*

Lemma 6. *Let $\mathcal{X}$ be a class of left R-modules. If a left R-module M is X-quasi-continuous for any $X \in \mathcal{X}$, then M is type 2 $\mathcal{X}$-quasi-continuous.*

Proof. Let N be in $\mathcal{X}(M)$ and A be a closure of N in M. It is easy to see that $A \in \mathcal{A}(N, M)$. Since M is N-extending, it follows that A is a direct summand of M. Thus M is type 2 $\mathcal{X}$-extending.

Let A and B be direct summands of M with $A \cap B = 0$ and $A \in \mathcal{X}^e(M)$. Then there exists an $\mathcal{X}$-submodule X of M such that $X \leq_e A$. Clearly $A \in \mathcal{A}(X, M)$. Since M is X-quasi-continuous, it follows that $A \oplus B$ is a direct summand of M. Hence M is type 2 $\mathcal{X}$-quasi-continuous. ■

It is easy to see that a type 2 $\mathcal{X}$-extending left R-module is not necessarily type 2 $\mathcal{X}$-quasi-continuous (for example, let $\mathcal{X}$ be the class of all left R-modules). The following example shows that a type 2 $\mathcal{X}$-quasi-continuous left R-module is not necessarily quasi-continuous, and that an M-$\mathcal{X}$-injective left R-module is not necessarily M-injective.

Example 1. Let F be any field. Set $R = T_2(F)$, the ring of all upper triangular 2×2 matrices with entries in F. Then, by [5, 13.6], R is a left SI-ring and a (left and right) hereditary artinian serial ring. Clearly $J(R)^2 = 0$. Thus by [5, 13.5], every left R-module is extending. By Lemma 5, it follows that every left R-module is X-quasi-continuous for any singular left R-module X. Let $\mathcal{X}$ be the class of all singular left R-modules. Then, by Lemma 6, every left R-module is type 2 $\mathcal{X}$-quasi-continuous. If all left R-modules are quasi-continuous, then for every left R-module M, $M \oplus E(M)$ is quasi-continuous, and so M is injective by [13, Lemma C], where $E(M)$ denotes the injective hull of M. Thus R is artinian semisimple, a contradiction. Hence there exists a left R-module M which is not quasi-continuous. By [5, Corollary 7.6], it follows that there exists a decomposition $M = M_1 \oplus M_2$ such that M_1 is not M_2-injective. But M_1 is M_2-$\mathcal{X}$-injective by Corollary 2.

Clearly every M-$\mathcal{X}$-injective module is pseudoly M-$\mathcal{X}$-injective. But the following example shows that the converse is not true. Recall that a left R-module M is said to be pseudo-injective if M is pseudo-M-injective.

Example 2. This example is due to Teply and appears in [11]. Let $A = \mathbb{Z}_2[x]$, $B = A/(x)$, and $C = A/(x^2)$. Let

$$R = \begin{pmatrix} B & B \\ 0 & C \end{pmatrix}, \qquad M = \begin{pmatrix} B \\ C \end{pmatrix}.$$

Then M is pseudo-injective, but is not extending; hence is not quasi-injective. Let $\mathcal{X}$ be the class of all left R-modules. Then $\mathcal{X}(M)$ coincides with the set of all submodules of M. Thus M is pseudoly M-$\mathcal{X}$-injective, but is not M-$\mathcal{X}$-injective.

Lemma 7. *Let $\mathcal{X}$ be a class of left R-modules which is closed under submodules. Let $M = M_1 \oplus M_2$. Then the following conditions are equivalent:*

(1) *M is type 2 $\mathcal{X}$-extending.*

(2) *Any closed submodule A of M with $A \in \mathcal{X}^e(M)$, and $A \cap M_1 = 0$ or $A \cap M_2 = 0$ is a direct summand of M.*

(3) *Any closed submodule A of M with $A \in \mathcal{X}^e(M)$, and $A \cap M_1 = A \cap M_2 = 0$, or $A \cap M_1 \leq_e A$ or $A \cap M_2 \leq_e A$ is a direct summand of M.*

Proof. The implications (1)$\Longrightarrow$(2) and (1)$\Longrightarrow$(3) follow from Lemma 6.

(2)$\Longrightarrow$(1). Let $L \in \mathcal{X}^e(M)$ be a closed submodule of M. Suppose that H is the maximal essential extension of $L \cap M_2$ in L. Then H is closed in L. Thus by [5, 1.10] H is closed in M. Since $\mathcal{X}$ is closed under submodules, it is easy to see that the class $\mathcal{X}^e$ is closed under submodules. Thus $H \in \mathcal{X}^e(M)$. Clearly $H \cap M_1 = 0$. Thus $M = H \oplus H'$. Now

$$L = L \cap M = L \cap (H \oplus H') = H \oplus (L \cap H').$$

Since $L \cap H'$ is closed in L, by [5, 1.10] again, $L \cap H'$ is closed in M. Clearly $(L \cap H') \cap M_2 = (L \cap M_2) \cap H' \subseteq H \cap H' = 0$. Also $L \cap H' \in \mathcal{X}^e(M)$. Thus by hypothesis, there exists a submodule N of M such that $M = (L \cap H') \oplus N$. Now

$$H' = H' \cap M = H' \cap ((L \cap H') \oplus N) = (L \cap H') \oplus (N \cap H').$$

Thus $M = H \oplus H' = H \oplus (L \cap H') \oplus (N \cap H') = L \oplus (N \cap H')$. This means that M is type 2 $\mathcal{X}$-extending.

(3)$\Longrightarrow$ (2). Let K be a closed submodule of M with $K \cap M_2 = 0$ and $K \in \mathcal{X}^e(M)$. Let L be a closed submodule of K such that $K \cap M_1 \leq_e L$. By [5, 1.10], L is closed in M. Clearly $L \cap M_1 = K \cap M_1 \leq_e L$. Since $\mathcal{X}^e(M)$ is closed under submodules, it follows that $L \in \mathcal{X}^e(M)$. By hypothesis, L is a direct summand of M. Suppose that $M = L \oplus L'$. Then $K = K \cap (L \oplus L') = L \oplus (K \cap L')$. Since $K \cap L'$ is closed in K, by [5, 1.10] again, $K \cap L'$ is closed in M. Obviously $K \cap L' \in \mathcal{X}^e(M)$, $(K \cap L') \cap M_1 = (K \cap M_1) \cap L' \leq L \cap L' = 0$, and $(K \cap L') \cap M_2 \leq K \cap M_2 = 0$. Thus, by hypothesis, $K \cap L'$ is a direct summand of M and, consequently, is also a direct summand of L'. Therefore, $K = L \oplus (K \cap L')$ is a direct summand of $L \oplus L' = M$.

If K is a closed submodule of M with $K \cap M_1 = 0$ and $K \in \mathcal{X}^e(M)$, then by analogy with the above proof, it follows that K is a direct summand of M. ■

Lemma 8. *Let M_1 be a type 2 $\mathcal{X}$-extending left R-module and M_2 be any left R-module. Set $M = M_1 \oplus M_2$. If M_2 is M_1-$\mathcal{X}$-injective (pseudoly M_1-$\mathcal{X}$-injective, essentially M_1-$\mathcal{X}$-injective), then every closed submodule K of M with $K \in \mathcal{X}(M)$ and $K \cap M_2 = 0$ (respectively, $K \cap M_1 = K \cap M_2 = 0$, $K \cap M_1 \leq_e K$) is a direct summand.*

Proof. Suppose that M_2 is pseudoly M_1-$\mathcal{X}$-injective, and let K be a closed submodule of M with $K \cap M_1 = K \cap M_2 = 0$ and $K \in \mathcal{X}(M)$. Then, by Lemma 1, there exists a submodule L of M such that $M = M_2 \oplus L$ and $K \leq L$. Clearly L is isomorphic to M_1, and so is type 2 $\mathcal{X}$-extending. Thus K, being a closed submodule of L and $K \in \mathcal{X}(L)$, is a direct summand of L. Hence K is also a direct summand of M.

From Lemmas 2 and 3, the results for M_2 being M_1-$\mathcal{X}$-injective and essentially M_1-$\mathcal{X}$-injective follow similarly.

The following result generalizes [6, Theorem 8], [7, Theorem 4.4], [14, Theorem 8(iii) and (iv)] and [18, Proposition 5.8].

Theorem 1. *Let $\mathcal{X}$ be a class of left R-modules which is closed under submodules. Let M_1 and M_2 be type 2 $\mathcal{X}$-extending left R-modules and let $M = M_1 \oplus M_2$. If one of the following conditions holds, then M is type 2 $\mathcal{X}$-extending.*

(1) *M_1 and M_2 are relatively essentially $\mathcal{X}^e$-injective and M_1 is pseudoly M_2-$\mathcal{X}^e$-injective.*

(2) *M_1 and M_2 are relatively essentially $\mathcal{X}^e$-injective and M_2 is pseudoly M_1-$\mathcal{X}^e$-injective.*

Proof. It follows from Lemmas 7 and 8.

Let M be a left R-module. A left R-module N is called essentially M-injective if for any submodule A of M, any homomorphism $\phi : A \longrightarrow N$ with $\mathrm{Ker}(\phi) \leq_e A$ can be extended to a homomorphism $\psi : M \longrightarrow N$. Two modules M and N are called relatively essentially injective if M is essentially N-injective and N is essentially M-injective.

Corollary 4. *Let M_1 and M_2 be extending left R-modules and let $M = M_1 \oplus M_2$. If M_1 and M_2 are relatively essentially injective and M_1 is pseudo-M_2-injective (or M_2 is pseudo-M_1-injective), then M is extending.*

Following [15], a left R-module M is called a CESS-module if every complement with essential socle is a direct summand, equivalently, every submodule with essential socle is essential in a direct summand of M. CESS-modules have been studied in [1, 2, 15].

Corollary 5. *Let $M = M_1 \oplus M_2$. Then the following conditions are equivalent:*

(1) *M is a CESS-module.*

(2) *Every closed submodule A of M with essential socle and $A \cap M_1 = 0$ or $A \cap M_2 = 0$ is a direct summand of M.*

(3) *Every closed submodule A of M with essential socle and $A \cap M_1 = A \cap M_2 = 0$, or $A \cap M_1 \leq_e A$ or $A \cap M_2 \leq_e A$ is a direct summand of M.*

Let $\mathcal{ES}$ be the class of left R-modules with essential socle. Then we have

Corollary 6. *Let $M = M_1 \oplus M_2$. If M_1 and M_2 are relatively essentially $\mathcal{ES}$-injective and M_1 is pseudoly M_2-$\mathcal{ES}$-injective (or M_2 is pseudoly M_1-$\mathcal{ES}$-injective), then M is a CESS-module if and only if M_1 and M_2 are CESS-modules.*

Proof. It follows from Theorem 1.

Note that Corollary 6 generalizes [2, Corollary 1.7].

Let $\mathcal{U}$ be the class of left R-modules with finite uniform dimension. Then we have

Corollary 7. *Let $M = M_1 \oplus M_2$. If M_1 and M_2 are relatively essentially $\mathcal{U}$-injective and M_1 is pseudoly M_2-$\mathcal{U}$-injective (or M_2 is pseudoly M_1-$\mathcal{U}$-injective), then M is uniform-extending if and only if M_1 and M_2 are uniform-extending.*

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