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Short Communication

On Multipoint Boundary-Value Problems for Linear Implicit Non-Autonomous Systems of Difference Equations

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1. Introduction

Linear implicit difference equations (LIDEs)

$$A_n x_{n+1} = B_n x_n + q_n \ (n \in \mathbb{N}), \tag{1}$$

where $A_n, B_n \in \mathbb{R}^{m\times m}$, $q_n \in \mathbb{R}^m$ are given and the matrices A_n are singular for all n, may be regarded as discrete analogues of linear differential-algebraic equations (DAEs), which have attracted much attention of researchers (see [3–5]). Recently, some concepts and techniques proposed in [5] for studying DAEs have been developed in our paper [6] on LIDEs.

The main results of [6] can be summarized as follows:

(i) Using singular-value decompositions (SVDs) of A_n, we have introduced the notion of index-1 LIDEs. It has been shown that the index of LIDEs does not depend on the choice of SVDs of A_n. The unique solvability of some initial-value problems (IVPs) for index-1 LIDEs has been proved.

(ii) Under some additional conditions, the solvability of the mentioned above IVPs for more general cases, where $\mathrm{Ker} A_{n+1} \subseteq \mathrm{Ker} A_n$ or $\mathrm{rank} A_{n+1} \geq \mathrm{rank} A_n$ for all $n \in \mathbb{N}$ has been established.

(iii) The multipoint boundary-value problem (MPBVP):

$$\begin{cases} A_i x_{i+1} = B_i x_i + q_i \ \ (i = 0, \dots, N-1), & (2) \\ \sum_{i=0}^{N} C_i x_i = \gamma, & (3) \end{cases}$$

when N becomes large, represents a large-scale system of $m(N+1)$ linear equations. If LIDE (2) has index-1, a regularity condition for MPBVP (2) and (3) has been introduced. It has been proved that the regularity of MPBVP (2), (3) is a necessary and sufficient condition for its unique solvability.

In this paper we derive a necessary and sufficient condition for the solvability of MPBVP (2), (3) when the regularity condition does not hold. Then combining our results in both regular and irregular cases, we arrive at a Fredholm alternative for special large-scale systems of linear equations (2), (3).

2. Preliminaries

According to [6], LIDE (2) is said to be of index-1, if:
(i) $\text{rank}A_n \equiv r$ $(n = 0, \ldots, N-1)$, where $0 < r < m$.
(ii) The matrices $G_n := A_n + B_n V_n Q V_{n+1}^T$ are nonsingular for $n = 0, \ldots, N-1$.
where $A_n := U_n \Sigma_n V_{n+1}^T$ is a SVD of A_n, $\Sigma_n := \text{diag}(\sigma_n^{(1)}, \ldots, \sigma_n^{(r)}, 0, \ldots, 0)$ is a diagonal matrix with singular values $\sigma_n^{(1)} \geq \sigma_n^{(2)} \geq \cdots \geq \sigma_n^{(r)} > 0$ on the main diagonal and U_n (V_{n+1}) are orthonormal matrices, whose columns are left (right) singular vectors of A_n $(n = 0, \ldots, N-1)$ respectively. Finally, $Q := diag(O_r, I_{m-r})$ where O_k and I_k $(k = 1, \ldots, m)$ stand for the $k \times k$ zero matrices and the $k \times k$ identity matrices respectively. For k = m, we simply denote $O := O_m$ and $I := I_m$.

In the rest of this paper, we assume that (2) is an index-1 LIDE. Let $P := I - Q$; $V_0 := I$; $Q_n := V_n Q V_n^T$; $P_n := I - Q_n$; $\tilde{P}_n := I - Q_n V_n V_{n+1}^T G_n^{-1} B_n$ $(0 \leq n \leq N-1)$;

$$M_{n-1}^{(n)} := \prod_{k=0}^{n-1} G_{n-1-k}^{-1} B_{n-1-k} \; (n = 1, \ldots, N)$$

and $R := \text{diag}(P, Q_N)$.
The fundamental solution of equations $A_n X_{n+1} = B_n X_n$ $(n = 0, \ldots, N-1)$ can be determined as $X_0 := \tilde{P}_0$; $X_n := \tilde{P}_n M_{n-1}^{(n)}$ $(n = 1, \ldots, N-1)$ and $X_N := P_N M_{N-1}^{(N)}$. Further, we define the matrix $(D|C_N Q_N)$, whose columns are the columns of the so-called shooting matrix $D = \sum_{n=0}^{N} C_n X_n$ and the matrix $C_N Q_N$. Obviously, $\text{Ker}R \subseteq \text{Ker}(D|C_N Q_N)$.

3. Main Result

Lemma 1. *The dimensions of* $\text{Ker}R$ *and* $\text{Ker}(D|C_N Q_N)$ *are independent on the choice of SVDs of* A_n $(n = 0, \ldots, N-1)$, *moreover,* $\dim\text{Ker}R = m$ *and* $\dim\text{Ker}(D|C_N Q_N) =: p \geq m$.

Following [6] we say that MPBVP (2), (3) is regular if

$$\text{Ker}R = \text{Ker}(D|C_N Q_N). \tag{4}$$

As a direct consequence of Lemma 1 we have

Corollary 1. *The MPBVP (2), (3) is regular if and only if $p = m$ and its regularity does not depend on the choice of SVDs of A_n.*

The following fact has been proved in [6]:

Theorem 1. *Suppose that the LIDE (2) is of index-1 then the regularity of MPBVP (2), (3) is a necessary and sufficient condition for its unique solvability.*

Now we turn to the irregular case, where $p > m$.
Let $\dim\mathrm{Ker}(D|C_N Q_N)^T =: q$. Denote by $\{w_i^0\}_{i=1}^m$ and $\{w_i\}_{i=1}^q$ certain bases of $\mathrm{Ker}R$ and $\mathrm{Ker}(D|C_N Q_N)^T$ respectively. We extend $\{w_i^0\}_{i=1}^m$ to a basis $\{w_i^0\}_{i=1}^p$ of $\mathrm{Ker}(D|C_N Q_N)$. Let $u_i^0 \in \mathbb{R}^m$ and $v_i^0 \in \mathbb{R}^m$ be the first and the second groups of components of w_i^0, i.e $w_i^0 = ({u_i^0}^T, {v_i^0}^T)^T$ $(i = 1, \dots, p)$. Define the column matrices

$$\Phi_n := (X_n u_{m+1}^0 \; X_n u_{m+2}^0 \dots X_n u_p^0) \in \mathbb{R}^{m\times(p-m)} \; (n = 0, \dots, N-1)$$

and

$$\Phi_N := (X_N u_{m+1}^0 + Q_N v_{m+1}^0 \; X_N u_{m+2}^0 + Q_N v_{m+2}^0 \dots X_N u_p^0 + Q_N v_p^0) \in \mathbb{R}^{m\times(p-m)}.$$

Further, let us consider a linear operator acting in $\mathcal{X} = \mathbb{R}^{m(N+1)}$, defined by

$$\mathcal{L}x := ((A_0 x_1 - B_0 x_0)^T, \dots, (A_{N-1} x_N - B_{N-1} x_{N-1})^T, (\sum_{n=0}^{N} C_n x_n)^T)^T$$

Lemma 2. $\mathrm{Ker}\mathcal{L} = \left\{((\Phi_0 a)^T, \dots, (\Phi_N a)^T)^T : a \in \mathbb{R}^{p-m}\right\}.$

Set

$$z_0 := -QV_1^T G_0^{-1} q_0; \; z_1 := \tilde{P}_1 G_0^{-1} q_0 - V_1 Q V_2^T G_1^{-1} q_1;$$

$$z_n := \tilde{P}_n(\sum_{i=0}^{n-2} M_{n-2-i}^{(n)} G_i^{-1} q_i + G_{n-1}^{-1} q_{n-1}) - V_n Q V_{n+1}^T G_n^{-1} q_n \; (n = 2, \dots, N-1);$$

$$z_N := P_N(\sum_{i=0}^{N-2} M_{N-2-i}^{(N)} G_i^{-1} q_i + G_{N-1}^{-1} q_{N-1}); \; \gamma^* := \gamma - \sum_{n=0}^{N} C_n z_n.$$

Let W be a row matrix whose rows are vectors w_i^T $(i = 1, \dots, q)$ of the basis of $\mathrm{Ker}(D|C_N Q_N)^T$.

Now we are able to state the main result of this paper.

Theorem 2. *Suppose that the LIDE (2), (3) is of index-1 and $p > m$. Then the MPBVP (2), (3) is solvable if and only if $W\gamma^* = 0$. Moreover a general solution of (2), (3) can be represented as:*

$$\begin{cases} x_n = X_n x_0^* + z_n + \Phi_n a \ (n = 0, \dots, N-1), \\ x_N = X_N x_0^* + z_N + Q_N \xi^* + \Phi_N a, \end{cases} \tag{5}$$

where $a \in \mathbb{R}^{p-m}$ *is an arbitrary vector;* $(x_0^{*T}, \ \xi^{*T})^T = (D|C_N Q_N)^+ \gamma^*$ *and* $(D|C_N Q_N)^+$ *is the generalized inverse in Moore-Penrose's sense of* $(D|C_N Q_N)$.

Combining Theorem 1 and Theorem 2 we come to the following.

Corollary 2. (Fredholm alternative) *Assume that* (2) *is an index-1 LIDE and let* $p := \dim\mathrm{Ker}(D|C_N Q_N)$. *Then:*

1/ *Either* $p = m$ *and the MPBVP* (2), (3) *is uniquely solvable for any data* $q_n \ (n = 0, \dots, N-1)$ *and* γ.

2/ *Or* $p > m$ *and the MPBVP* (2), (3) *is solvable if and only if*

$$W\gamma^* = 0 \tag{6}$$

Moreover, there holds the solution formula (5).

4. Example

Consider MPBVP (2), (3) with the following data:

$$A_n = \begin{pmatrix} 1 & -n & n \\ 0 & 1 & -1 \\ n & 0 & 0 \end{pmatrix}; \quad B_n = \begin{pmatrix} n & -1 & 2n \\ 0 & 1 & n \\ 2n & n & n-1 \end{pmatrix};$$

$$q_n = \begin{pmatrix} n(1-2n) \\ -n \\ -2n^2 + n + 1 \end{pmatrix} \quad (n = 0, \dots, N-1) \tag{7}$$

and

$$C_n = \begin{pmatrix} n+1 & -n & 1 \\ -n & 3n & n-1 \\ 1 & 2n & n \end{pmatrix} \quad (n = 0, \dots, N); \quad \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \tag{8}$$

where $\gamma_i \in \mathbb{R}$ $(i = \overline{1,3})$ and $N \in \mathbb{N}$ is large enough.

Let $A_n = U_n \Sigma_n V_{n+1}^T \ (n = 0, \dots, N-1)$; be a SVD of A_n, where $\Sigma_n =$ $\mathrm{diag}(\sigma_n^{(1)}, \sigma_n^{(2)}, 0)$; $\sigma_n^{(1)} \geq \sigma_n^{(2)} > 0$ and $V_{n+1} = a_n^{-1} \begin{pmatrix} 2n & \frac{-b_n}{\sqrt{2}} & 0 \\ \frac{-b_n}{2} & -n\sqrt{2} & \frac{a_n}{\sqrt{2}} \\ \frac{b_n}{2} & n\sqrt{2} & \frac{a_n}{\sqrt{2}} \end{pmatrix}$,

where $a_n = (n^4 + 10n^2 + 1 + (n^2+1)(n^4 + 10n^2 + 1)^{1/2})^{1/2}$;
$b_n = n^2 + 1 + (n^4 + 10n^2 + 1)^{1/2}$.

Since

$$G_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \text{ and } G_n = \frac{1}{2}\begin{pmatrix} 2 & -1 & 4n-1 \\ 0 & n+3 & n-1 \\ 2n & 2n-1 & 2n-1 \end{pmatrix} \ (n = 1, \dots, N-1)$$

are nonsingular, the LIDE (2) with data (7), (8) is of index-1.

Further computation gives:

$$D = \begin{pmatrix} 1 & -N^2-1.5N-0.5 & 0 \\ 0 & 2N^2+0.5 & 0 \\ 1 & N^2-1.5N & 0 \end{pmatrix}; \; C_N Q_N = \frac{1}{2}\begin{pmatrix} 0 & 1-N & 1-N \\ 0 & 4N-1 & 4N-1 \\ 0 & 3N & 3N \end{pmatrix},$$

therefore $p := \dim\mathrm{Ker}(D|C_N Q_N) = 4 > m := 3$.

Since $\mathrm{Ker}(D|C_N Q_N)^T = \mathrm{Span}\{(-1,-1,1)^T\}$ it follows $W = (-1 \; -1 \; 1)$.

Besides, $z_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $z_n = \begin{pmatrix} n-1 \\ n \\ 1 \end{pmatrix}$ $(n = 1, \dots, N-1)$ and $z_N = \begin{pmatrix} N-1 \\ 0.5(N-1) \\ -0.5(N-1) \end{pmatrix}$.

Thus,

$$\begin{aligned} \gamma^* &:= \gamma - \sum_{n=0}^{N} C_n z_n \\ &= \left(\gamma_1 - \frac{N^2+1}{2}, \gamma_2 - \frac{4N^3-7N-3}{6}, \gamma_3 - \frac{4N^3+3N^2-7N}{6}\right)^T. \end{aligned}$$

Condition (6) implies that

$$\gamma_3 = \gamma_1 + \gamma_2 \tag{9}$$

Thus, relation (9) is a necessary and sufficient condition for the solvability of MPBVP (2), (3) with data (7), (8).

Now let, for example, $\gamma = (0.5, 0.5, 1)^T$. From (5) it follows that

$$x_0 = \frac{1}{4N-1}\begin{pmatrix} -\frac{1}{6}(4N^4+8N^3-10N^2+N+6)+(2N^3+7N^2)t \\ (4N-1)t \\ 4N-1 \end{pmatrix},$$

$$x_n = (n-1-t, \, n+t, \, 1)^T \;\; (n = 1, \dots, N-1) \;\; \text{and}$$

$$x_N = \frac{1}{4N-1}\begin{pmatrix} (4N-1)(N-1-t) \\ \frac{1}{6}(-4N^3+12N^2-8N+9)+(-2N^2+2N-1)t \\ \frac{1}{6}(-4N^3-12N^2+22N+3)-(2N^2+2N)t \end{pmatrix},$$

where $t \in \mathbb{R}$ is an arbitrary real number.

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