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On Martingales in the Limit and Their Classification

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Abstract. Martingales in the limit and mils would be regarded as two important generalizations of martingales. It is known that every L^1-bounded such a sequence of random variables converges a.s. Recently , the first author of this note has noted that this convergence result still holds for another essentially larger class, that is, the class of quasi-martingales in the limit. The main aim of this note is to give a complete classification of the latter class into an increasing family of subclasses whose smallest element is just the class of mils.

1. Notations and Definitions

In this note, let $(\Omega, \mathcal{A}, P)$ denote a complete probability space, N the set of all positive integers and $(\mathcal{A}_n)$ an increasing sequence of complete sub-σ-fields of $\mathcal{A}$ with $\mathcal{A}_n \uparrow \mathcal{A}$. Given a sub-$\sigma$-field $\mathcal{B}$ of $\mathcal{A}$ we denote by $L^1(\mathcal{B})$ the Banach space of all (equivalence classes of) random variables $X : \Omega \to R$ which are $\mathcal{B}$-measurable and $E(|X|) = \int_\Omega |X| dP < \infty$.

From now on, we shall consider only sequences (X_n) in $L^1(\mathcal{A})$, which are assumed to be adapted to $(\mathcal{A}_n)$, i.e , each X_n is $\mathcal{A}_n$-measurable. For other related notions, the reader is referred to [1]. For our purpose, we need to recall only the following:

Definition 1.1. *A sequence (X_n) is said to be*

a) *a martingale in the limit, if*

$$\lim_n \sup_{m \geq n} |E^n(X_m) - X_n| = 0, \text{ a.s.},$$

where given $n \in N$ and $X \in L^1(\mathcal{A})$, the function $E^n(X)$ is the conditional expectation of X given $\mathcal{A}_n$.

b) *a mil, if for every $\varepsilon > 0$, there exists $p \in N$ such that for all $n \geq p$ we have*

$$P\big(\sup_{p \leq q \leq n} |E^q(X_n) - X_q| > \varepsilon\big) < \varepsilon. \tag{1.1}$$

It is worth noting that martingales in the limit were first introduced and considered by Mucci [4], who proves that every L^1-bounded martingale in the limit converges, a.s. Later, this result was extended to the Banach space-valued case by Peligrad [5], under another additional condition. Next, by the upcrossing method, Talagrand [6] introduced the notion of mil, essentially more general than martingales in the limit and proves that every L^1-bounded mil in a Banach space can be written in the unique form:

$$X_n = M_n + P_n,$$

where (M_n) is (rather) a uniformly integrable martingale and (P_n) goes to zero a.s. But it is not completely satisfactory. Recently, to answer the interesting conjecture posed in [3], we have introduced the following:

Definition 1.2. *A sequence (X_n) is called a quasi-martingale in the limit (shortly, a quasi-mil), if for every $\varepsilon > 0$, there exists $p \in N$ such that for every $m \geq p$ there exists $p_m \in N$ with $p_m \geq m$ such that for all $n \geq p_m$, we have*

$$P\big(\sup_{p \leq q \leq m} |E^q(X_n) - X_q| > \varepsilon\big) > \varepsilon. \tag{1.2}$$

It is clear that by definition, every mil is a quasi-mil. In reality, the conjecture given in [2] says that there exists a real-valued quasi-mil which is not a mil. Unfortunately, we could not prove earlier the conjecture until 1997 when we luckly found a sequence of "locally dependent" random variables which justifies the truth of the conjecture. Further, we noted in [2] that together with a stopping time technique the upcrossing Doob's method would also be applied to get the above decomposition for L^1-bounded quasi-mils in Banach spaces. Thus, this note would be regarded as a natural continuation of the above mentioned considerations. Namely, as the main result we shall give in the next section a complete classification of the class of quasi-mils into an increasing family of subclasses for which the smallest element is just the class of mils.

2. Main Results

Since the first example of a quasi-mil which is not a mil, given in [2] was constructed in a purely nonatomic probability space, it would be useful to know such another example in a purely atomic situation. This suggests us to the following:

Example 2.1. *There exists an L^1-bounded quasi-mil in a purely atomic probability space which converges to zero a.s. But it is not a mil.*

Construction: Let $\Omega = [0,1]$. We take $Q_1 = \{[0,\frac{1}{4})\}$ and $\mathcal{A}_1 = \sigma - (Q_1)$. For $n \geq 2$, let $a_n = \sum_{j=2}^{n} 2^{-j}$; $Q_n = \{Q_{n-1}, [a_n, a_{n+1})\}$ and $\mathcal{A}_n = \sigma - (Q_n)$. Then the chosen stochastic basis $(\mathcal{A}_n)$ increases to a σ-field, denoted by $\mathcal{A}$. It is clear that $\mathcal{A}$ is purely atomic and the only one of its atoms, i.e., the interval $[\frac{1}{2}, 1)$ does not belong to any $\mathcal{A}_n$. Now we can construct the desired sequence (X_n) as follows: For $n \leq 2$, set $X_n \equiv 0$. Given $n \geq 3$ we choose

$$X_n = 2^n(1 - a_{n-1}) - 2^{n+1}(1 - a_{n-1})I_{[a_n, a_{n+1})}. \tag{2.1}$$

where I_A is the characteristic function of $A \in \mathcal{A}$. It is easily checked that $E(|X_n|) \leq 2a_{n-1}$, $n \in N$ with $a_0 = 0$. Then (X_n) is L^1-bounded. Further, for $n \geq 3$ we have

$$E^{n-1}(X_n) = 2^n(1 - a_{n-1})I_{[a_{n-1}, a_n} - \frac{1 - a_{n-1}}{1 - a_n} I_{[a_n, 1)}.$$

It follows that

$$E^q(X_n) = 0, \quad q \leq n - 2. \tag{2.2}$$

The fact that (X_n) converges to zero a.s., is evident. It implies that for any but fixed $\varepsilon > 0$, there exists $p \in N$ such that

$$P\big(\sup_{p \leq q < \infty} |X_q| > \varepsilon\big) < \varepsilon.$$

This together with (2.2) implies that for any $m \geq p$ and $n \geq p_m = m + 2$ we have

$$P\big(\sup_{p \leq q \leq m} |E^q(X_n) - X_q| > \varepsilon\big) = P\big(\sup_{p \leq q \leq m} |X_q| > \varepsilon\big) < \varepsilon.$$

Thus by Definition 1.2, (X_n) is a quasi-mil. However, for every $n \geq 3$ we also have

$$P(|E^{n-1}(X_n)| > \frac{1}{2}) \geq P\big(\{E^{n-1}(X_n) \neq 0\} \cap \{X_{n-1} = 0\}\big) = 1 - a_n > \frac{1}{2}.$$

It follows that (X_n) cannot be a mil. This completes the construction.

In order to establish a classification of quasi-mils, let G denote the set of all nondecreasing functions $f : N \to N$. For $f, g \in G$, let define $f =' g$ if and only if card $(\{f \neq g\}) < \infty$ and $f <' g$ if and only if card $(\{f > g\}) < \infty$ and card $(\{f < g\}) = \infty$. Then endowed with the partial order "$\leq$", G becomes a directed set. Thus the proof of the first part of the next main result is contained in the following characterization.

Theorem 2.2. *A sequence (X_n) is a quasi-mil if and only if there exists some $g \in G$ such that (X_n) is a mil of size g, i.e., for every $\varepsilon > 0$, there exists $p \in N$ such that for all $m, n \in N$ with $p \leq m < m + g(m) \leq n$, we have*

$$P\big(\sup_{p \leq q \leq m} |E^q(X_n) - X_q| > \varepsilon\big) < \varepsilon. \tag{2.3}$$

In particular, (X_n) is a mil if and only if it is a mil of size 1.

Proof. To prove the general equivalence, let (X_n) be a quasi-mil. Then by Definition 1.2, one can find a strictly increasing sequence $(p(n))$ such that for every $k \in N$, there exists a sequence $(p_n(k))$ with each $p_m(k) \geq p(k)$ such that if $n \geq p_m(k) \geq m \geq p(k)$, then we have

$$P(\sup_{p(k) \leq q \leq m} |E^q(X_n) - X_q| > 2^{-k}) < 2^{-k}. \tag{2.4}$$

Now let define the function $g : N \to N$ as follows: For $m < p(1)$, set $g(m) = 1$. For $p(k) \leq m < p(k+1)$ with some $k \in N$, set $g(m) = \sum_{i=1}^{m} \sum_{j=1}^{k} p_i(j) - m$. Then $g \in G$. We shall show that (X_n) is a mil of size g. To check this, let $\varepsilon > 0$ and $2^{-k} < \varepsilon$ with some $k \in N$. Then for all $n, m \in N$ with $m \geq p(k)$ and $n \geq m + g(m)$, by (2.4) we have

$$P(\sup_{p(k) \leq q \leq m} |E^q(X_n) - X_q| > \varepsilon) \leq P(\sup_{p(k) \leq q \leq m} |E^q(X_n) - X_q| > 2^{-k}) < 2^{-k} < \varepsilon,$$

since $n \geq p_m(k)$. Therefore, by definition, (X_n) is a mil of size g. Conversely, suppose that (X_n) is a mil of size g for some $g \in G$. Then by taking each $p_m = g(m) + m$, $m \in N$ we infer that the sequence (p_n) does not depend on ε and satisfies Definition 1.2 automatically. It means that (X_n) is a quasi-mil. This completes the proof of the first equivalence.

For the particular case, suppose first that (X_n) is a mil. Then for every $\varepsilon > 0$, $p, m, n \in N$ with $m \geq p$ and $n \geq m + 1$ we have

$$P(\sup_{p \leq q \leq m} |E^q(X_n) - X_q| > \varepsilon) \leq P(\sup_{p \leq q \leq n} |E^q(X_n) - X_q| > \varepsilon).$$

This together with (1.1) implies (2.3). Thus by definition, (X_n) is a mil of size 1. Conversely assume that (X_n) is a mil of size 1. Then by definition, given $\varepsilon > 0$ there exists $p \in N$ such that if $m, n \in N$ with $m \geq p$ and $n \geq m + 1$, (2.3) is satisfied. Consequently, given $n > p$, by taking $m = n - 1$, we get

$$P(\sup_{p \leq q \leq n} |E^q(X_n) - X_q| > \varepsilon) \leq P(\sup_{p \leq q \leq m} |E^q(X_n) - X_q| > \varepsilon) < \varepsilon.$$

It means that (X_n) is a mil, noting that for the case when $n = p$, (1.1) is always true. The proof of the theorem is complete. ∎

Now we are able to formulate our main result as follows

Theorem 2.3. *When g runs over the directed set $(G, \leq')$, the set of all quasi-mils is classified into the increasing family of the subclasses of mils of size g for which the smallest element, i.e., the subclass of mils of size 1, is just the class of mils. Furthermore, if $f, g \in G$ with $f <' g$, the class of mils of size f is strictly contained in the class of mils of size g.*

Proof. Clearly, the first part of the classification is contained in the previous theorem. Before proving the second part, we define first for any $h \in G$ the following:

$$a_m(h) = m + h(m), \; m \in N, \qquad b_n(h) = \max\{m : a_m(h) \leq n\}, \; n \geq a_1(h). \tag{2.5}$$

Then it is clear that

$$b_n(h) + h[b_n(h)] \leq n; \; n \geq a_1(h). \tag{2.6}$$

Furthermore, since $h \in G$, the sequence $(a_m(h))$ strictly increases to infinity when m runs to infinity. This implies that

$$b_n(h) = m \text{ if and only if } a_m(h) \leq n < a_{m+1}(h). \tag{2.7}$$

Consequently,

$$b_{a_m(h)}(h) = m; \;\; m \in N. \tag{2.8}$$

Now let $f, g \in G$ with $f <' g$. We shall prove the second part of the theorem by constructing a mil (X_n) of size g which is not a mil of size f. Indeed, let $(\Omega, \mathcal{A}, P)$ be the usual Lebesgue probability space on $[0, 1)$. Given $n \in N$, let $a_n = \prod_{j=1}^{n} 2^j$, Q_n the partition of $[0, 1)$ in intervals of equal length and $\mathcal{A}_n$ the σ-field generated by Q_n. Let denote by k the first element $m \in N$ such that $g(m) > f(m)$ and $f(s) \leq g(s)$, $s \geq m$. Then, in particular, $g(k) \geq 2$. With all the additional notations we can define the desired sequence (X_n) as follows: For $n < a_k(g)$ set $X_n \equiv 0$. For any fixed $n \geq a_k(g)$, we define

$$X_n = \frac{a_n}{a_{b_n(g)+1}} \text{ or } X_n = -\frac{a_n}{a_{b_n(g)+1}}, \text{ resp.}$$

on the first interval of Q_n which is contained in the $(2p-1)$-th or $2p$-th interval of $Q_{b_n(g)+1}$, resp., with $1 \leq p \leq a_{b_n(g)+1}/2$. Then by the definition and (2.7) we have

$$b_n(g) \geq b_{a_k(g)}(g) = k, \quad n \geq a_k(g).$$

Consequently, for $n \geq a_k(g)$ we have $g[b_n(g)] \geq g(k) \geq 2$ since g is nondecreasing. This remark together with (2.6) guarantees that

$$n \geq b_n(g) + g[b_n(g)] \geq b_n(g) + 2, \; n \geq a_k(g).$$

Therefore,

$$P(\{Xn \neq 0\}) = \frac{a_{b_n(g)+1}}{a_n} = \prod_{j=b_n(g)+2}^{n} 2^{-j} \leq 2^{-n}. \tag{2.9}$$

This proves that

$$\text{the sequence } (X_n) \text{ converges to zero a.s.} \tag{2.10}$$

Further, for all $n \geq a_k(g)$ and $r = b_n(g) + 1$, also by the same construction we have

$$E^r(X_n) = 1 \text{ or } E^r(X_n) = -1, \text{ resp.,} \tag{2.11}$$

on the $(2p-1)$-th or $2p$-th interval of $Q_{b_n(g)+1}$ resp., with $1 \leq p \leq a_{b_n(g)+1}/2$. Consequently, for all $m \geq a_k(g)$, $n \geq m + g(m)$, we have $E^q(X_n) \equiv 0$ for all $q \leq m$ since, by (2.5), $m \leq b_n(g)$. This remark proves that

$$\sup_{a_k(g) \leq q \leq m} |E^q(X_n) - X_q| = \sup_{a_k(g) \leq q \leq m} |X_q| \text{ a.s.}$$

Thus by the property (2.10), the sequence (X_n) must be a mil of size g. To see that it is not a mil of size f, we set

$$V = \{v \geq a_k(g),\ a_v(f) < a_v(g)\}.$$

Then by (2.5),

$$V = \{v \geq a_k(g),\ f(v) < g(v)\}.$$

This together with the assumption that $f <' g$ implies that card $(V) = \infty$. But by (2.8), we have

$$b_{a_v(g)}(g) = v = b_{a_v(f)}(f), \quad v \in V.$$

It follows that

$$r = b_{a_v(f)}(g) < v, \quad v \in V \tag{2.12}$$

since by (2.7), $a_v(g)$ is the smallest $n \in N$ such that $b_n(g) = v$, while $a_v(f) < a_v(g)$. Thus, given $p \geq a_k(g)$, $m = v \geq p$, $n = m + f(m) = a_v(f)$, by (2.9) we get

$$P(\sup_{p \leq q \leq m} |E^q(X_n) - X_q| = 1) \geq P(|E^{r+1}(X_n) - X_r| = 1)$$
$$\geq P(\{|E^{r+1}(X_n)| = 1\} \cap \{X_r \neq 0\}) \geq 1 - 2^{-n}.$$

It implies that (X_n) cannot be a mil of size f. This completes the construction. ■

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